k_a and the a^2 Invariant: A Unified Acceleration Scale from Galaxies to Atomic Clocks

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Abstract

Modern gravity phenomenology exhibits at least three apparently unrelated small acceleration scales: the Milgrom scale a_0 organizing galaxy rotation curves, the cosmic acceleration scale $a_\Lambda \sim cH_0$, and the sensitivity of precision clock tests to tiny violations of local position invariance. Conventional frameworks—ΛCDM with cold dark matter on the one hand, and modified-gravity models on the other—typically treat these scales as independent or emergent features of very different sectors: dark halos, dark energy, and laboratory metrology.

In this paper I show that a broad class of scalar refractive-index theories of gravity admits a single, universal "acceleration-squared" invariant

$$a^2 \equiv \boldsymbol{a}\cdot\boldsymbol{a},$$

linked to the gradient energy of the scalar field via a dimensionless self-coupling k_a. In the weak-field, quasi-static limit the field equation can be written schematically as

$$\nabla\cdot\boldsymbol{a} + \frac{k_a}{c^4}a^2 \;=\; -4\pi G\rho,$$

with $\boldsymbol{a}$ the physical free-fall acceleration and ρ the mass-energy density. The $k_a a^2$ term represents genuine gravitational self-interaction of the scalar field, but in a form that is far simpler than the tensorial nonlinearity of general relativity.

I then show how this structure naturally generates a single preferred acceleration-squared scale $a_\star^2 \propto (c^4/k_a)\, G\rho$ that simultaneously:

- reproduces MOND-like scaling $g \simeq \sqrt{a_\star g_N}$ in galaxies when the $k_a a^2$ term dominates the bare Poisson term;
- yields a cosmic background value $a_\star^2 \sim c^2 H_0^2$ in an FRW universe with density of order the critical density;
- enters directly into species-dependent gravitational redshift anomalies for atomic clocks, via scalar couplings K_A encoding the internal structure of each atomic transition.

The result is a unified picture in which galaxy phenomenology, cosmic acceleration and precision clock tests all probe the same underlying acceleration-squared invariant, rather than three unrelated small numbers. This closes a conceptual loop in the broader Density Field Dynamics (DFD) program: once k_a is fixed by any one class of experiments, the others are no longer arbitrarily tunable but become cross-checks of the same constant. I conclude by outlining specific experimental strategies for measuring k_a with multi-species optical clocks in Earth's gravitational field, and by discussing the parameter space in which this scalar self-interaction is compatible with existing tests of general relativity.

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1 Introduction

Astrophysical and cosmological observations over the past four decades have revealed a remarkably coherent set of anomalies relative to the predictions of general relativity (GR) with visible matter alone. Spiral galaxy rotation curves are flat rather than Keplerian; low surface-brightness galaxies follow tight scaling relations; and large-scale structure and supernova data point to a late-time accelerated expansion of the universe. [1–3, 6]

The dominant response has been the ΛCDM paradigm, which retains GR but postulates cold dark matter and a cosmological constant. An alternative line of work instead modifies gravity in the low-acceleration regime, with Modified Newtonian Dynamics (MOND) the prime example. [4, 5]

MOND introduces a characteristic acceleration $a_0 \sim 10^{-10}\,\mathrm{m/s^2}$ governing the transition between Newtonian and deep-MOND behavior in galaxies.

A striking and still poorly understood fact is that a_0 is numerically close to the cosmic acceleration scale $a_\Lambda \sim cH_0$ inferred from supernovae and the cosmic microwave background. [2,3] Furthermore, ever more precise tests of the Einstein equivalence principle show that local position invariance (LPI) and the universality of free fall are obeyed to parts in 10^{13}–10^{15}, yet the small residual uncertainties are now comparable to the size of the anomalies implied by dark-energy-like acceleration and galaxy scaling laws for low-acceleration systems. [7–11]

At the same time, scalar and vector-tensor theories of modified gravity have proliferated. [6,12] In many of these models the gravitational sector includes one or more additional fields with their own self-interactions. However, the accelerations a_0 and a_Λ are usually put in by hand, or emerge from very different pieces of the theory, and there is no *a priori* reason why the same scale should play a role in both galaxy dynamics and cosmic expansion.

Goal of this paper

The aim of this paper is to isolate and analyze a simple structural feature that appears naturally in scalar refractive-index theories of gravity and that ties these disparate phenomena together: a universal acceleration-squared invariant $a^2 \equiv \boldsymbol{a}\cdot\boldsymbol{a}$ that enters the field equation through a dimensionless self-coupling k_a.

The key points are:

1. In a scalar refractive-index framework, the metric seen by light and matter is encoded in a single scalar field $\psi(x)$ that modulates the local refractive index $n(x) = e^{\psi(x)}$.

2. The weak-field, quasi-static limit can be arranged so that the physical free-fall acceleration field is
$$\boldsymbol{a}(x) = -c^2\nabla\psi(x),$$
reproducing Newtonian gravity in the appropriate regime.

3. A minimal nonlinear completion introduces a gradient self-coupling term proportional to $|\nabla\psi|^2$ in the field equation with a dimensionless coefficient k_a. In terms of the acceleration this term becomes proportional to $k_a a^2/c^4$.

4. In spherically symmetric systems with characteristic density ρ, there is a natural acceleration-squared scale
$$a_\star^2 \sim \frac{4\pi G\rho c^4}{k_a},$$
which controls both galaxy-scale dynamics and the background cosmological expansion if one takes ρ to be of order the mean cosmic density.

5. When matter is described by different species of bound states (e.g. different atomic transitions), the scalar field can couple to each with different effective coefficients K_A. This introduces species-dependent sensitivity to the same a^2 invariant, which precision atomic clocks can probe as apparent violations of LPI.

These observations together suggest that k_a and the associated acceleration-squared invariant a^2 are the natural "glue" connecting galaxies, cosmology, and clocks in the broader Density Field Dynamics (DFD) picture. The present paper focuses exclusively on this structural connection and the minimal mathematics needed to make it precise, leaving more elaborate aspects of DFD to companion work.

2 Scalar refractive-index framework

This section introduces the basic kinematics and field equation of a scalar refractive-index theory sufficient for the discussion that follows. We do not claim that this simple model is a full replacement for GR; rather it is a controlled weak-field toy model that makes the a^2 structure transparent and recovers Newtonian/GR behavior in the high-acceleration regime.

2.1 Refractive index and effective metric

Consider a scalar field $\psi(x)$ on a background Minkowski spacetime with coordinates $(t, \boldsymbol{x})$ and metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Define a position-dependent refractive index

$$n(x) = e^{\psi(x)}, \tag{1}$$

and an effective metric

$$g_{\mu\nu} = e^{2\psi(x)}\eta_{\mu\nu}. \tag{2}$$

In the eikonal approximation, light propagation is governed by null geodesics of $g_{\mu\nu}$, and the local coordinate speed of light is reduced by $e^{-\psi}$ compared to c in the background frame.

For slowly moving massive particles, the nonrelativistic limit of the geodesic equation in static ψ yields an effective potential

$$\Phi_{\mathrm{eff}}(\boldsymbol{x}) = c^2\psi(\boldsymbol{x}), \tag{3}$$

so that the free-fall acceleration is

$$\boldsymbol{a}(\boldsymbol{x}) = -\nabla\Phi_{\mathrm{eff}}(\boldsymbol{x}) = -c^2\nabla\psi(\boldsymbol{x}). \tag{4}$$

This reproduces Newtonian gravity if ψ satisfies a Poisson equation with the appropriate source term in the weak-field limit.

2.2 Field equation with self-interaction

The simplest purely Newtonian limit would require ψ to satisfy

$$\nabla^2\psi = \frac{4\pi G}{c^2}\rho, \tag{5}$$

so that combining with Eq. (4) one finds $\nabla \cdot \boldsymbol{a} = -4\pi G\rho$.

However, nothing forbids the existence of nonlinear self-interactions in the scalar sector. The class of models we consider here are defined by the modified field equation

$$\nabla^2\psi - \frac{k_a}{c^2}|\nabla\psi|^2 = \frac{4\pi G}{c^2}\rho, \tag{6}$$

where k_a is a dimensionless constant and ρ is the mass-energy density in the weak-field regime. We assume $|k_a| \ll 1$ so that the modification is small in regimes where $|\nabla\psi|$ is large, and potentially important only when the acceleration is small.

Using Eq. (4), we can rewrite Eq. (6) directly in terms of the physical acceleration field. First note that

$$|\nabla\psi|^2 = \frac{a^2}{c^4}, \qquad a^2 \equiv \boldsymbol{a}\cdot\boldsymbol{a}. \tag{7}$$

Moreover,

$$\nabla^2\psi = -\frac{1}{c^2}\nabla\cdot\boldsymbol{a}. \tag{8}$$

Substituting into Eq. (6) gives

$$-\frac{1}{c^2}\nabla\cdot\boldsymbol{a} - \frac{k_a}{c^2}\frac{a^2}{c^4} = \frac{4\pi G}{c^2}\rho, \tag{9}$$

or, equivalently,

$$\boxed{\nabla\cdot\boldsymbol{a} + \frac{k_a}{c^4}a^2 = -4\pi G\rho.} \tag{10}$$

Equation (10) is the central structural equation for the rest of this paper. It shows that, in this class of scalar refractive models, a single invariant combination $a^2 = \boldsymbol{a}\cdot\boldsymbol{a}$ appears linearly in the field equation with coefficient k_a/c^4. The sign and magnitude of k_a determine how strongly the scalar field "feeds back" on itself via its gradient energy.

2.3 Regime hierarchy

Equation (10) also makes the hierarchy of regimes transparent. Comparing the divergence term and the self-interaction term gives three qualitatively distinct behaviors:

Regime	Condition	Behavior
Solar system / high-a	$\nabla\cdot\boldsymbol{a} \gg \frac{k_a}{c^4}a^2$	Linear (Newtonian / GR limit)
Crossover / galactic	$\nabla\cdot\boldsymbol{a} \sim \frac{k_a}{c^4}a^2$	MOND-like transition
Deep field / low-a	$\nabla\cdot\boldsymbol{a} \ll \frac{k_a}{c^4}a^2$	Nonlinear $a^2 \propto a_N$ scaling

In the high-acceleration regime relevant to Solar System tests, the self-interaction term is negligible and Eq. (10) reduces to the usual Newtonian Poisson equation. In the deep low-acceleration regime, the scalar self-interaction dominates and drives the MOND-like behavior discussed below.

3 The a^2 invariant and the scale $a_\star$

3.1 Dimensional analysis and definition of $a_\star$

Since k_a is dimensionless, the combination $k_a a^2/c^4$ has the same dimensions as $\nabla\cdot\boldsymbol{a}$, namely acceleration per unit length. This suggests the existence of a characteristic acceleration scale associated with a given density environment ρ.

To see this, consider a region of approximately uniform density ρ and characteristic size L, such that $\nabla \cdot \boldsymbol{a} \sim a/L$. The field equation (10) then implies, schematically,

$$\frac{a}{L} + \frac{k_a}{c^4} a^2 \sim 4\pi G\rho. \tag{11}$$

This quadratic relationship between a and ρ admits two limiting regimes:

- If a is large enough that $a/L \gg k_a a^2/c^4$, we recover the standard Newtonian scaling $a \sim 4\pi G\rho L$.
- If a is small enough that $k_a a^2/c^4 \gg a/L$, the nonlinear self-interaction term dominates, and we obtain

$$\frac{k_a}{c^4} a^2 \sim 4\pi G\rho \quad \Rightarrow \quad a^2 \sim \frac{4\pi G\rho c^4}{k_a}. \tag{12}$$

This motivates defining a characteristic acceleration-squared scale

$$a_\star^2(\rho) \equiv \frac{4\pi G\rho c^4}{k_a}, \tag{13}$$

so that in the deeply nonlinear regime we have

$$a^2 \sim a_\star^2(\rho), \qquad a \sim a_\star(\rho). \tag{14}$$

Two points are important here:

1. The scale $a_\star$ depends on the ambient density ρ. For a galactic disk, ρ is of order the baryonic surface density divided by a scale height; for cosmology, ρ is the mean cosmic density.
2. The dependence is via $a_\star^2$, not $a_\star$ itself. This becomes crucial when comparing to phenomenology such as MOND, where the deep-regime scaling is $g \sim \sqrt{a_0 g_N}$, i.e., accelerations are governed by a *square root* of a fundamental acceleration scale.

3.2 Connection to MOND-like phenomenology

In MOND, the modified Poisson equation reads schematically [4, 5]

$$\nabla \cdot \left[\mu\left(\frac{|\boldsymbol{g}|}{a_0}\right) \boldsymbol{g} \right] = -4\pi G\rho, \tag{15}$$

where $\boldsymbol{g}$ is the gravitational field (acceleration), a_0 is the MOND acceleration scale, and $\mu(x)$ is an interpolation function such that $\mu(x) \to 1$ for $x \gg 1$ and $\mu(x) \to x$ for $x \ll 1$. In the deep-MOND regime $|\boldsymbol{g}| \ll a_0$, one finds

$$\nabla \cdot \left(\frac{|\boldsymbol{g}|}{a_0} \boldsymbol{g} \right) \approx -4\pi G\rho, \tag{16}$$

which in spherical symmetry leads to the scaling relation

$$g^2 \approx a_0 g_N, \tag{17}$$

with g_N the Newtonian acceleration.

The structure in Eq. (10) is different but closely related. If we identify $\boldsymbol{a}$ with the gravitational field $\boldsymbol{g}$, then our modification takes the form

$$\nabla \cdot \boldsymbol{a} + \frac{k_a}{c^4} a^2 = -4\pi G \rho. \tag{18}$$

In a spherically symmetric configuration sourced by a point mass M, the Newtonian solution satisfies $\nabla \cdot \boldsymbol{a}_N = -4\pi G \rho$ and $a_N(r) = GM/r^2$. When the nonlinear term becomes important, the balance equation becomes roughly

$$\frac{k_a}{c^4} a^2 \sim 4\pi G \rho_{\text{eff}} \sim \frac{GM}{r^3}, \tag{19}$$

where we have used $\rho_{\text{eff}} \sim M/(4\pi r^3/3)$ for order-of -magnitude purposes. This yields

$$a^2 \sim \frac{c^4}{k_a} \frac{GM}{r^3}. \tag{20}$$

Combining with $a_N = GM/r^2$, we obtain

$$a^2 \sim \left(\frac{c^4}{k_a r} \right) a_N. \tag{21}$$

If the system has a characteristic radius $r \sim R$, then we can define an effective acceleration scale

$$a_0^{\text{eff}} \equiv \frac{c^4}{k_a R}, \tag{22}$$

so that

$$a^2 \sim a_0^{\text{eff}} a_N. \tag{23}$$

This is formally the same scaling as in deep-MOND, with a_0 replaced by an effective a_0^{eff} set by k_a and the size of the system. In more realistic disk geometries, R is replaced by an appropriate combination of disk scale lengths and heights, but the structural relationship $a^2 \propto a_N$ persists.

3.3 Cosmic acceleration scale

In a homogeneous and isotropic FRW cosmology with scale factor $a(t)$ and Hubble parameter $H = \dot{a}/a$, the Newtonian analogue of the Friedmann equation can be written as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p/c^2) + \frac{\Lambda c^2}{3}. \tag{24}$$

The observed late-time acceleration is characterized by a scale

$$a_\Lambda \sim c H_0, \tag{25}$$

where H_0 is the present-day Hubble parameter. [?, 3]

In the scalar refractive-index picture, one can interpret the cosmic expansion as a large-scale configuration of the scalar field ψ with slowly varying gradient on Hubble scales. The acceleration of comoving observers relative to the scalar field definition of "free fall" is then governed by an effective a^2 term of the same structural form as in local systems, with ρ replaced by the mean cosmic density $\bar{\rho} \sim 3H_0^2/(8\pi G)$.

Plugging this into Eq. (13) gives

$$a_\star^2(\bar{\rho}) \sim \frac{4\pi G}{k_a}\bar{\rho}c^4 \sim \frac{4\pi G}{k_a}\frac{3H_0^2}{8\pi G}c^4 = \frac{3}{2k_a}c^4 H_0^2. \tag{26}$$

Thus the cosmological $a_\star$ scale is

$$a_\star(\bar{\rho}) \sim \sqrt{\frac{3}{2k_a}}\,c^2 H_0. \tag{27}$$

Depending on the numerical value of k_a, this can be naturally of order cH_0 without any additional tuning.

The crucial point is that the same k_a that governs the crossover in galaxy dynamics also determines the magnitude of the cosmic acceleration scale. The numerical near-coincidence between a_0 and cH_0 in phenomenological fits then ceases to be a mystery and becomes a direct reflection of the single underlying self-coupling constant k_a.

4 Species-dependent couplings and atomic clocks

To connect the a^2 invariant to laboratory tests, we must specify how the scalar field ψ couples to different forms of matter. In a generic scalar-tensor or scalar refractive-index model, the coupling is composition-dependent: different atomic transitions, nuclear binding energies and electronic structures respond differently to variations in ψ. [7]

4.1 Effective coupling coefficients K_A

Let us consider an atomic transition A with frequency ν_A. In the presence of the scalar field ψ, we allow for a linearized dependence

$$\frac{\delta\nu_A}{\nu_A} = K_A\,\delta\psi, \tag{28}$$

where K_A is a dimensionless sensitivity coefficient encoding how the transition energy depends on the underlying dimensionless constants that are themselves functions of ψ (fine-structure constant, electron-proton mass ratio, etc.).

In a static gravitational potential, ψ varies with height h in the gravitational field. For small height differences in a uniform gravitational field $\boldsymbol{a}$, we have

$$\delta\psi \approx -\frac{a}{c^2}\delta h, \tag{29}$$

using Eq. (4). Thus the fractional frequency shift between two heights separated by Δh is

$$\left(\frac{\Delta\nu}{\nu}\right)_A \approx -K_A\frac{a\,\Delta h}{c^2}. \tag{30}$$

Comparing two different species A and B at the same locations yields a fractional ratio shift

$$\frac{\Delta(\nu_A/\nu_B)}{\nu_A/\nu_B} \approx -(K_A - K_B)\frac{a\,\Delta h}{c^2}. \tag{31}$$

In GR, local position invariance implies that $K_A = K_B = 1$, and the ratio is independent of height: both clocks redshift in exactly the same way. [11] In the scalar refractive-index framework with species-dependent K_A, however, gravitational redshift becomes composition-dependent at a level set by the differences $K_A - K_B$.

4.2 Incorporating the a^2 invariant

The structure of Eq. (31) already shows that clock comparison experiments are directly sensitive to the acceleration a. To connect this to the acceleration-squared invariant, recall that the background field $\boldsymbol{a}$ itself is constrained by the field equation (10):

$$\nabla \cdot \boldsymbol{a} + \frac{k_a}{c^4} a^2 = -4\pi G\rho. \tag{32}$$

In the regime where the nonlinear term is non-negligible, a^2 is no longer free to take arbitrary values; it is tied to the local density environment through Eq. (13).

Thus, at leading order, we can write

$$a^2 \approx a_\star^2(\rho) = \frac{4\pi G\rho c^4}{k_a}, \tag{33}$$

so that

$$a \approx \frac{2\sqrt{\pi G\rho}\, c^2}{\sqrt{k_a}}. \tag{34}$$

Substituting into Eq. (31) gives

$$\frac{\Delta(\nu_A/\nu_B)}{\nu_A/\nu_B} \approx -(K_A - K_B)\, \frac{2\sqrt{\pi G\rho}\, c^2}{\sqrt{k_a}}\, \frac{\Delta h}{c^2} = -(K_A - K_B)\, \frac{2\sqrt{\pi G\rho}}{\sqrt{k_a}}\, \Delta h. \tag{35}$$

Several features are worth emphasizing:

- The magnitude of the effect scales with $\sqrt{\rho/k_a}$, not linearly with ρ. This reflects the a^2 structure of the field equation.
- Once k_a is fixed, Eq. (35) defines a completely *predictive* relationship between the density environment, the height separation, and the composition dependence of gravitational redshift.
- Atomic clock networks spanning different height ranges (e.g. on towers, satellites, or deep underground laboratories) and using different clock species become a direct probe of k_a through the combination $(K_A - K_B)/\sqrt{k_a}$. [9, 10]

5 Experimental determination of k_a

The k_a parameter controls the strength of scalar self-interaction and thus the size of both astrophysical and laboratory deviations from GR. Determining k_a (or setting bounds on it) therefore requires combining information from multiple regimes.

5.1 Astrophysical constraints

Galaxy rotation curves and their scaling relations can be used to infer an effective acceleration scale a_0^{gal} in the deep low-acceleration regime. [5] In the scalar self-interaction picture, this effective scale is related to k_a and the characteristic density and size of the galaxy by

$$a_0^{\mathrm{gal}} \sim \frac{c^4}{k_a R_{\mathrm{eff}}}. \tag{36}$$

If one adopts a phenomenological value $a_0^{\rm gal} \approx 1.2 \times 10^{-10}\,\mathrm{m/s^2}$, this provides one handle on k_a for typical disk galaxies of known $R_{\rm eff}$.

Cosmological data, on the other hand, constrain the combination $a_\star(\bar{\rho}) \sim cH_0\sqrt{3/(2k_a)}$ discussed above. Requiring that this be of order the observed late-time acceleration implies that k_a must not be extremely small or large; otherwise the scalar self-interaction would either overwhelm or be negligible compared to the ΛCDM fit. [6, 12]

These considerations suggest that k_a is plausibly of order unity in natural units, though the precise value depends on the detailed matching between the simple scalar model considered here and full observational data.

5.2 Clock-based strategies

Atomic clock experiments provide a complementary and, in some ways, cleaner probe of k_a. The basic strategy is:

1. Choose two clock species A and B with calculable and significantly different sensitivity coefficients K_A and K_B.
2. Deploy clocks at two or more heights separated by a distance Δh in a gravitational field with known density profile $\rho(h)$, such as the Earth's near-surface environment.
3. Measure the fractional ratio shift $\Delta(\nu_A/\nu_B)/(\nu_A/\nu_B)$ as a function of Δh and compare to the GR prediction (which is essentially zero for the ratio).
4. Use Eq. (35) to infer or bound the combination $(K_A - K_B)/\sqrt{k_a}$, and thus k_a once the K_A are known or constrained from atomic theory. [7]

Current and near-future optical lattice clock networks, both ground-based and space-based, already operate at fractional frequency precision better than 10^{-17}–10^{-18}. [9, 10] This is sufficient to probe extraordinarily small deviations from LPI over height differences of order 10^2–10^3 m, especially when multiple species are compared.

5.3 Consistency with existing tests

Any scalar self-interaction model must remain consistent with the impressive null tests of the equivalence principle and GR obtained from experiments such as MICROSCOPE, binary pulsar timing, and the gravitational wave observations of LIGO and Virgo. [8, 11, 13–15] In the present framework, this translates into bounds on k_a and the products $k_a K_A$.

The essential point is that the same k_a enters all three regimes we have discussed:

- galaxy dynamics (through $a_0^{\rm gal}$),
- cosmic acceleration (through $a_\star(\bar{\rho})$),
- clock tests (through the ratio shifts in Eq. (35)).

This eliminates the freedom to tune each sector independently and turns what might otherwise be a collection of unrelated anomalies into a network of cross-checks. Any choice of k_a that fits galaxies but grossly violates clock or GW constraints, or vice versa, is ruled out.

6 Limitations and gravitational waves

The analysis in this paper is intentionally restricted to the weak-field, quasi-static sector of a scalar refractive-index theory, where a single scalar field ψ and its gradient determine both the effective metric and the free-fall acceleration. In this setting the metric $g_{\mu\nu} = e^{2\psi}\eta_{\mu\nu}$ is conformally flat, and the only dynamical degree of freedom is scalar.

By construction, Eq. (10) captures how the scalar sector self-interacts through the invariant a^2 and how this modifies Newtonian gravity in low-acceleration regimes. However, real astrophysical systems also radiate gravitational waves with tensor polarizations, as confirmed by LIGO/Virgo observations of compact-binary coalescences and the tight bounds on the speed and polarization content of gravitational waves from GW170817. [13–15]

The scalar toy model analyzed here does not, by itself, generate the full tensor sector of GR. In particular:

- perturbations of ψ propagate as scalar waves with quadrupolar angular dependence in isolated systems, but they are not tensor plus/cross modes;
- there is no independent transverse-traceless degree of freedom in the conformally flat metric $g_{\mu\nu} = e^{2\psi}\eta_{\mu\nu}$.

From the point of view of DFD, the correct interpretation of this paper is therefore narrow but precise: it identifies the structure and consequences of a scalar self-interaction governed by k_a in the weak-field sector, and shows how a single acceleration-squared invariant a^2 can link galaxies, cosmology, and clocks. A complete theory of gravity must extend this scalar sector so that tensor gravitational waves, with the observed polarizations and propagation speed, either emerge as effective degrees of freedom or are added consistently to the DFD framework.

The gravitational-wave sector is thus a constraint and an opportunity: any completion of the present scalar model must recover the LIGO/Virgo results while preserving the k_a-driven structure uncovered here. That problem is left deliberately open for future work.

7 Implications for the DFD program

Within the broader Density Field Dynamics program, the central idea is that a single scalar density or refractive field controls both the effective metric for light and matter and the stochastic structure of quantum measurement. Those aspects lie beyond the scope of this paper, which has focused solely on the classical weak-field gravity sector.

Nevertheless, the emergence of a universal acceleration-squared invariant a^2 with self-coupling k_a has several important implications:

1. It provides a simple and robust organizing principle: everywhere the scalar field has a gradient, there is an associated local scale $a_\star(\rho)$ set by Eq. (13). Physical phenomena as diverse as galaxy rotation curves, cosmic acceleration, and clock redshifts are then different windows into this same scalar gradient energy.

2. It sharply reduces the number of genuinely free parameters in the gravitational sector. Once k_a is fixed (or tightly constrained) by any one class of observations, the others become predictions rather than independent fits.

3. It suggests a natural hierarchy of regimes. High-acceleration systems such as the Solar System lie firmly in the linear regime $\nabla \cdot \boldsymbol{a} \gg k_a a^2/c^4$, reproducing GR and Newtonian gravity to high accuracy. Low-acceleration, low-density environments lie in the nonlinear regime $k_a a^2/c^4 \gtrsim \nabla \cdot \boldsymbol{a}$, where MOND-like and dark-energy-like phenomena emerge.

4. It provides a clean target for both theoretical and experimental work: the precise determination of k_a and the mapping of where, in density and acceleration space, the transition between linear and nonlinear regimes occurs.

8 Conclusions and outlook

We have identified and analyzed a simple but powerful structural feature of scalar refractive-index theories of gravity: a universal acceleration-squared invariant $a^2 = \boldsymbol{a}\cdot\boldsymbol{a}$ that appears linearly in the field equation with a dimensionless self-coupling k_a/c^4. This leads naturally to a density-dependent acceleration scale $a_\star(\rho)$ that:

- produces MOND-like scaling in galaxies without introducing an arbitrary new constant unrelated to the density environment;
- matches the order of magnitude of the cosmic acceleration scale when ρ is taken to be the mean cosmic density;
- directly controls composition-dependent gravitational redshift effects for atomic clocks via species-dependent couplings K_A.

The main conceptual achievement is that a single structural parameter k_a—together with the invariant a^2—links three previously disparate acceleration scales: galactic a_0, cosmic a_Λ, and laboratory-scale sensitivities in precision metrology. This closes a loop in the gravitational sector of the Density Field Dynamics program: once k_a is fixed by any one of these regimes, the others are no longer free to vary independently.

From an experimental perspective, the most promising near-term probes of k_a are multi-species atomic clock networks, which can measure or bound composition-dependent gravitational redshift at levels far beyond what is accessible to astrophysical observations alone. On longer timescales, a consistent fit of galaxy dynamics, cosmic expansion, and clock tests within this framework would constitute strong evidence for a scalar refractive component to gravity and would dramatically sharpen the case for DFD as a viable extension or alternative to GR.

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