Math of Mobiles

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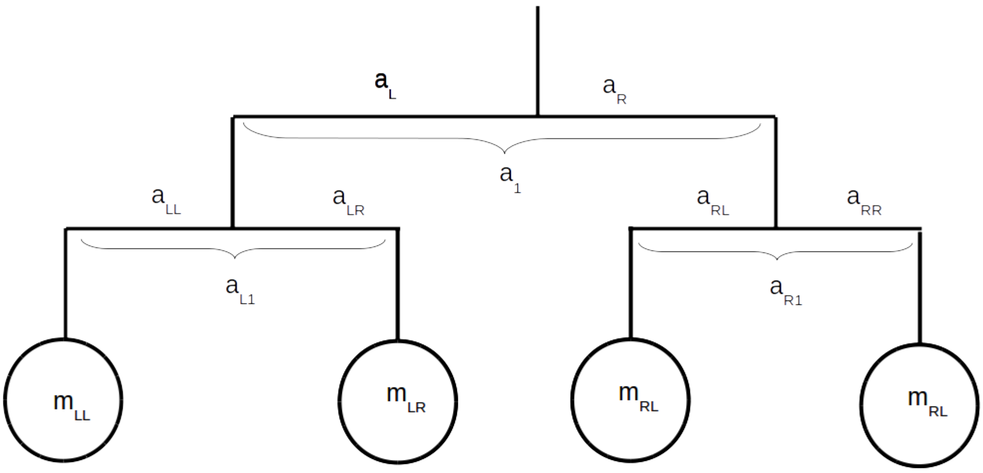
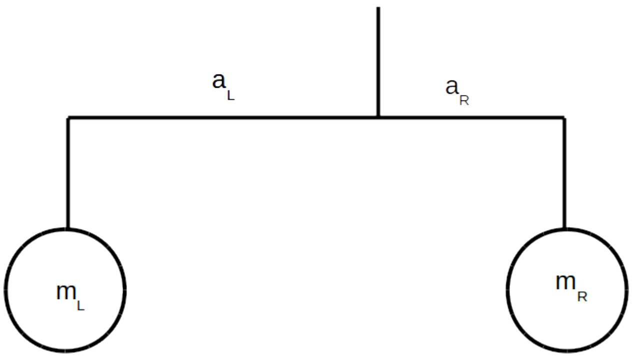




Starting with a simple mobile, the mass of the entire mobile equals the sum of the left and right masses:
(a slightly idealised mobile where we ignore arm and string weights)

$m_0 = m_L + m_R$

But since it is balanced, we also have -
Left torque has to equal right torque

$a_L m_L = a_R m_R$

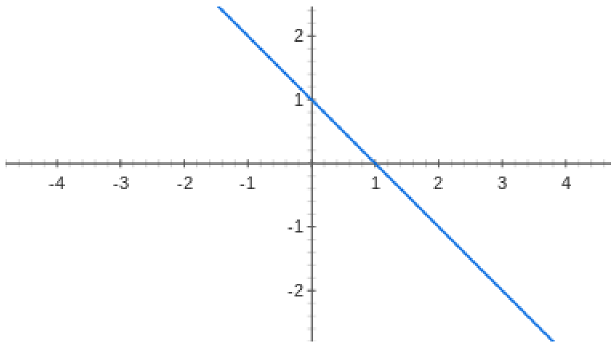
Finally, we would like to constrain the length of the arm:

$a_L + a_R = a_1$
This is a system of 3 equations and 6 unknowns, giving us 3 degrees of freedom.
However, if we set the total mass m_0 to a unit mass, and a_1 to a unit length, we get
$m_0 = 1$
and
$a_1 = 1$.
This gives us 5 equations and 6 unknowns... a line!
Since
$a_L + a_R = a_1 = 1$
yields
$a_L = 1 - a_R$

and
$m_L + m_R = m_0 = 1$
yields
$m_L = 1 - m_R$
$a_L m_L = a_R m_R$
yields $(1 - a_R)(1 - m_R) = a_R m_R$
$1 - m_R - a_R + a_R m_R = a_R m_R$
so

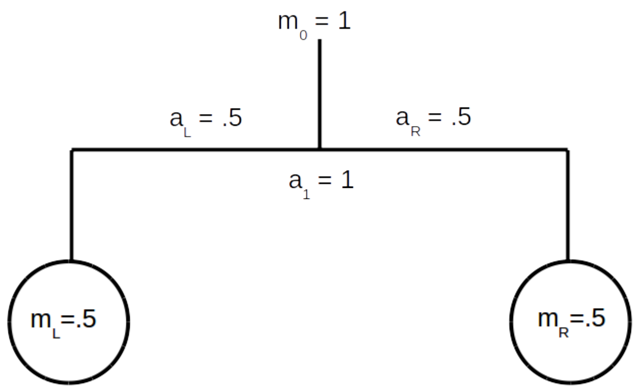
$1 - m_R - a_R = 0$
and
$m_R = -a_R + 1$

A very simple line! (Remember point-slope form of the line. $y = mx + b$):

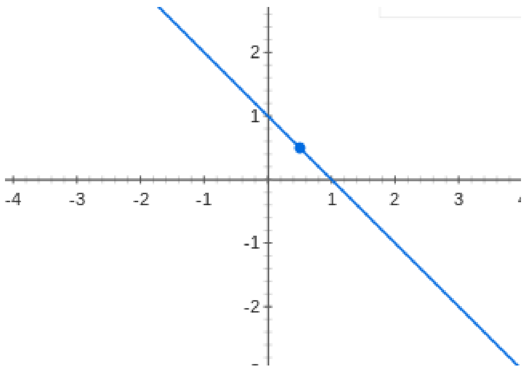


This is the solution-space of the set of equations above, so any particular mobile will lie somewhere on this line. (Of course, $m_0 > 0$ and $a > 0$.)

... making a depth-1 mobile a simple physical linear computer!



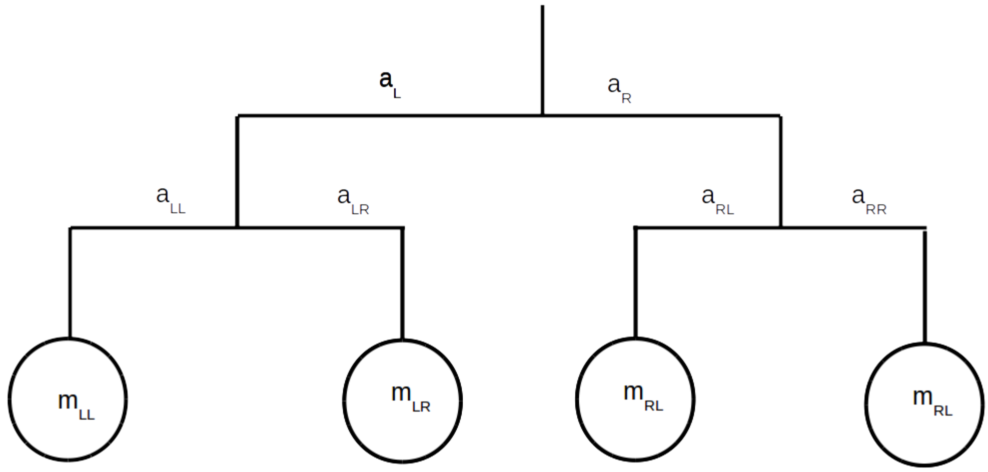
So for this particular mobile, with $m_0 = 1$, $a_1 = 1$ and symmetrically balanced, with $m_R = 0.5$ and $a_R = 0.5$, we have



Remember that the y-intercept at 1 was the product $a_1 m_0$, of the arm having unit length ($a_1 = 1$) and the mobile have total unit mass ($m_0 = 1$), giving us:

$m_R = -a_R + a_1 m_0$

So, the line simply moves up and down as we scale the total mass m_0 and the total arm length a_1.



If we add one level to the mobile, we have a set of equations, since each small mobile must balance:

Torque: (balance)

$a_L m_L = a_R m_R$

$a_{LL} m_{LL} = a_{LR} m_{LR}$

$a_{RL} m_{RL} = a_{RR} m_{RR}$

But since the middle masses m_L and m_R are just the sums of the masses below...

Mass:

$m_L = m_{LL} + m_{LR}$ $m_R = m_{RL} + m_{RR}$

Thus $a_L m_L = a_R m_R$

yields

$a_L(m_{LL} + m_{LR}) = a_R(m_{RL} + m_{RR})$

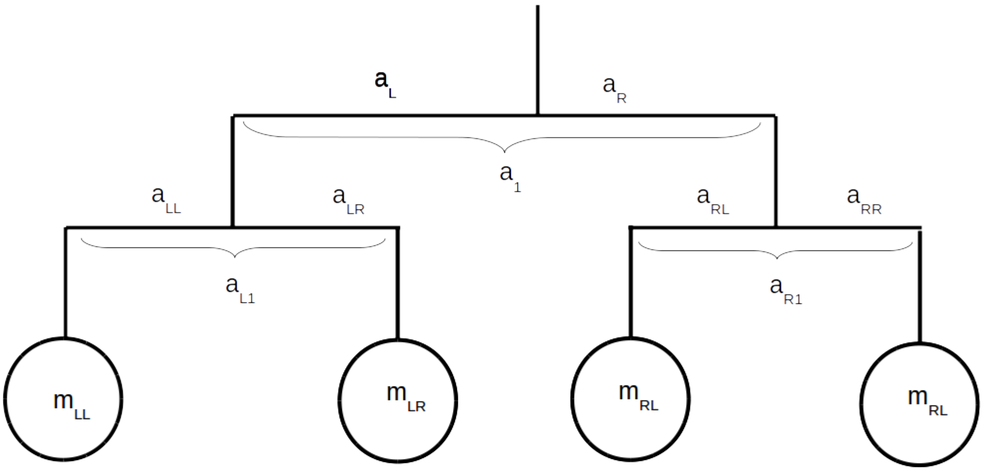
$a_{LL} m_{LL} = a_{LR} m_{LR}$

$a_{RL} m_{RL} = a_{RR} m_{RR}$

and since the arm lengths are related

$a_1 = a_l + a_r$

Finally, for a depth 2 mobile, we have a system of 9 equations and 16 unknowns, leaving 7 degrees of freedom:

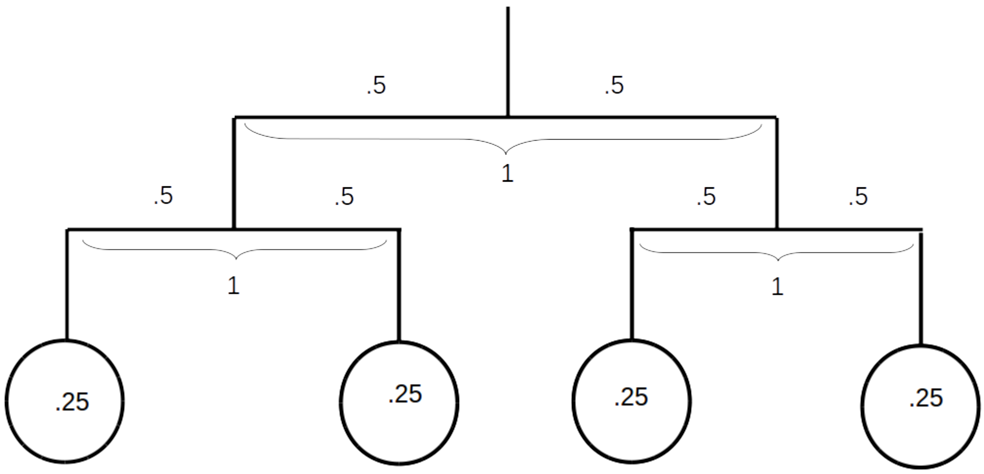


Mass:
$m_0 = m_L + m_R$
$m_L = m_{LL} + m_{LR}$
$m_R = m_{RL} + m_{RR}$

Torque:
$a_L m_L = a_R m_R$
$a_{LL} m_{LL} = a_{LR} m_{LR}$
$a_{RL} m_{RL} = a_{RR} m_{RR}$

Arms:
$a_L + a_R = a_1$
$a_{LL} + a_{LR} = a_{2L}$
$a_{RL} + a_{RR} = a_{2R}$

If we look at an example instance of this depth-2 mobile, we see that we can also again fix the total mass and the arm lengths (NOT DRAWN TO SCALE – for now we ignore collisions when the sub-mobiles rotate)



So, once again, the top level of the mobile can be plotted in the same way: