Pairwise distance probabilities in a disc

Galen Wilkerson

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1 $P(\|\mathbf{p_1} - \mathbf{p_2}\|)$ with uniform radius distribution

The probability of random variable X being at distance r from the center of a disc is given by

$$P(r \leq X < r + \delta r) = \frac{2\pi r \delta r}{\pi 1^2} = 2r\delta r,$$

which gives the probability density function

$$f(r) = \begin{cases} 2r & \text{if } 0 \leq r < 1 \\ 0 & \text{otherwise.} \end{cases}$$

At a particular radius, the probability of being at a particular location on a circle is

$$P(\theta \leq X < \theta + \delta\theta) = \frac{\theta\delta\theta}{2\pi},$$

thus

$$g(\theta) = \begin{cases} \frac{\theta}{2\pi} & \text{if } 0 \leq \theta < 2\pi \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the probability density function of a point being at a particular location in a circle is

$$h(r, \theta) = f(r)g(\theta) = \begin{cases} \frac{r\theta}{\pi} & \text{if } 0 \leq r < 1 \text{ and } 0 \leq \theta < 2\pi \\ 0 & \text{otherwise.} \end{cases}$$

The probability distribution of the distances between the two points is therefore

$$P(\|\mathbf{p_1} - \mathbf{p_2}\| \leq X < \|\mathbf{p_1} - \mathbf{p_2}\| + \delta\|\mathbf{p_1} - \mathbf{p_2}\|) = h(r_1, \theta_1)h(r_2, \theta_2)\|\mathbf{p_1} - \mathbf{p_2}\|$$

$$= \frac{r_1\theta_1}{\pi}\frac{r_2\theta_2}{\pi}\sqrt{(r_1 \cos\theta_1 - r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1 - r_2 \sin\theta_2)^2},$$

where $\mathbf{p_1} = (r_1, \theta_1)$, and $\mathbf{p_2} = (r_2, \theta_2)$.

To express this as a function of $\|\mathbf{p_1} - \mathbf{p_2}\|$, we need to use the *Method of Transformations* to change variables from $r_1, \theta_1, r_2, \theta_2$ to $\|\mathbf{p_1} - \mathbf{p_2}\|$.

2 $P(\|\mathbf{p_1} - \mathbf{p_2}\|)$ with normal radius distribution

We now use a normal distribution for radius r, to give a 2-d normal distribution of points centered at $(0, 0)$,

$$f_n(r) = P(r \leq X < r + \delta r) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r-\mu)^2}{2\sigma^2}},$$

with $\mu = 0$ and $\sigma = 1$.

$$h_n(r,\theta) = f_n(r)g(\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r-\mu)^2}{2\sigma^2}} \frac{\theta}{2\pi}.$$

Similarly, this gives a distance distribution

$$P(\|\mathbf{p_1} - \mathbf{p_2}\| \leq X < \|\mathbf{p_1} - \mathbf{p_2}\| + \delta\|\mathbf{p_1} - \mathbf{p_2}\|) = h_n(r_1,\theta_1)h_n(r_2,\theta_2)\|\mathbf{p_1} - \mathbf{p_2}\|$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r_1-\mu)^2}{2\sigma^2}} \frac{\theta_1}{2\pi} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(r_2-\mu)^2}{2\sigma^2}} \frac{\theta_2}{2\pi} \sqrt{(r_1\cos\theta_1 - r_2\cos\theta_2)^2 + (r_1\sin\theta_1 - r_2\sin\theta_2)^2}$$

Again, using the method of transformations, we express this in terms of $\|\mathbf{p_1} - \mathbf{p_2}\|$.