First order optimization methods based on Hessian-driven Nesterov accelerated gradient flow *

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Abstract

A novel dynamical inertial Newton system, which is called Hessian-driven Nesterov accelerated gradient (H-NAG) flow is proposed. Convergence of the continuous trajectory are established via tailored Lyapunov function, and new first-order accelerated optimization methods are proposed from ODE solvers. It is shown that (semi-)implicit schemes can always achieve linear rate and explicit schemes have the optimal(accelerated) rates for convex and strongly convex objectives. In particular, Nesterov's optimal method is recovered from an explicit scheme for our H-NAG flow. Furthermore, accelerated splitting algorithms for composite optimization problems are also developed.

1 Introduction

In this paper, we introduce the Hessian-based Nesterov accelerated gradient (H-NAG) flow:

$$\gamma x'' + (\gamma + \mu)x' + (1 + \mu\beta + \gamma\beta')\nabla f(x) + \gamma\beta\nabla^2 f(x)x' = 0,$$
(1)

where x = x(t) is a V-valued function of time variable t and $(\cdot)'$ is the derivative taking respect to t, $f: V \to \mathbb{R}$ is a \mathcal{C}^2 and convex function defined on the Hilbert space V and the damping coefficient $\gamma(t)$ is dynamically changing by $\gamma' = \mu - \gamma$, $\mu \ge 0$. The additional damping coefficient $\beta(t)$ in front of the Hessian is nonnegative. Note that (1) belongs to the class of dynamical inertial Newton (DIN) system introduced recently in [5].

When choosing vanishing damping $\beta = 0$, (1) reduces to Nesterov accelerated gradient (NAG) flow proposed in our recent work [13]

$$\gamma x'' + (\gamma + \mu)x' + \nabla f(x) = 0, \tag{2}$$

or equivalently, the first-order ODE system

$$\begin{cases} x' = v - x, \\ \gamma v' = \mu(x - v) - \nabla f(x), \\ \gamma' = \mu - \gamma. \end{cases}$$
(3)

In [13], the presented numerical discretizations with an extra gradient step for NAG flow (3) lead to old and new accelerated schemes and can recover exactly Nesterov's optimal method [15, Chapter 2] for both convex ($\mu = 0$) and strongly convex cases ($\mu > 0$) in a unified framework. When applied to composite convex optimization, our methods can recover FISTA [11] for convex case and give new accelerated proximal gradient methods for strongly convex case. Compared to recent ODE models [20, 22, 23] for studying accelerated gradient methods which usually treat convex and strongly convex cases separately, our unified analysis in [13] is due to the introduction of the dynamic damping coefficient $\gamma' = \mu - \gamma$, which brings the effect of time rescaling.

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When $\beta > 0$, the added Hessian-driven damping in (1) will neutralize the possible transversal oscillation occurred in the accelerated gradient method; see [5, Figure 1] for illustration. Particularly, if $\beta \gg 1$, then the flow behaves like the continuous Newton's flow [4]. A direct discretization based on Hessian is restrictive and expensive since requiring $f \in \mathcal{C}^2$ and the cost to compute the Hessian matrix and its inverse.

Instead we will write (1) as a first order system

$$\begin{cases} x' = v - x - \beta \nabla f(x), \\ \gamma v' = \mu(x - v) - \nabla f(x), \\ \gamma' = \mu - \gamma, \end{cases}$$
(4)

in which Hessian disappears. This agrees with the most remarkable feature of the dynamical inertial Newton model discovered in [3]. Now (4) is well defined for $f \in \mathcal{C}^1$ and can be further generalized to non-smooth setting by replacing gradient with sub-gradient [8].

1.1 Main results

We first consider smooth and μ -convex ($\mu \ge 0$, cf. (9)) function f with L-Lipschitz gradient. Let $(x(t), v(t), \gamma(t))$ be the solution of (4) and denote by x^* a global minimum point of f. By introducing the Lyapunov function

$$\mathcal{L}(t) = f(x(t)) - f(x^*) + \frac{\gamma(t)}{2} \|v(t) - x^*\|^2, \quad t \geqslant 0,$$
 (5)

we shall first establish the exponential decay property

$$\mathcal{L}(t) + \int_0^t e^{s-t} \beta(s) \left\| \nabla f(x(s)) \right\|^2 ds \leqslant e^{-t} \mathcal{L}(0).$$
 (6)

Then we propose several implicit and explicit schemes for (4) to get a sequence of $\{(x_k, v_k, \gamma_k)\}$ and establish the convergence via the discrete analogue of (5)

$$\mathcal{L}_{k} = f(x_{k}) - f(x^{*}) + \frac{\gamma_{k}}{2} \|v_{k} - x^{*}\|^{2}, \quad k \geqslant 0.$$

For a semi-implicit scheme (proximal method), we shall prove

$$\mathcal{L}_k + \lambda_k \sum_{i=0}^{k-1} \frac{\alpha_i^2}{\lambda_i \gamma_i} \|\nabla f(x_{i+1})\|^2 \leqslant \lambda_k \mathcal{L}_0,$$

where the sequence $\{\lambda_k\}$ is defined by that

$$\lambda_0 = 1, \quad \lambda_k = \prod_{i=0}^{k-1} \frac{1}{1+\alpha_i}, \quad k \geqslant 1.$$
 (7)

We easily obtain the linear convergence rate as long as the time step size α_k is bounded below,

Proximal method relies on a fast solver of a regularized problem which may not be available. To be practical, we propose an explicit scheme (cf. (42)) for solving (4). This scheme has been rewritten in the following algorithm style.

Algorithm 1 HNAG method for minimizing *f*

Input: $\gamma_0 > 0 \text{ and } x_0, v_0 \in V.$

- 1: **for** $k = 0, 1, \dots$ **do**
- 2: Compute α_k, β_k by $\alpha_k = \sqrt{\frac{\gamma_k}{L}}, \quad \beta_k = \frac{1}{L\alpha_k}$.
- 3: Update $x_{k+1} = \frac{1}{1 + \alpha_k} [x_k + \alpha_k v_k \alpha_k \beta_k \nabla f(x_k)].$
- 4: Update $v_{k+1} = \frac{1}{\gamma_k + \mu \alpha_k} \left[\gamma_k v_k + \mu \alpha_k x_{k+1} \alpha_k \nabla f(x_{k+1}) \right].$
- 5: Update $\gamma_{k+1} = (\gamma_k + \mu \alpha_k)/(1 + \alpha_k)$.
- 6: end for

We shall prove the convergence result for Algorithm 1:

$$\mathcal{L}_k + \frac{1}{2L} \sum_{i=0}^{k-1} \frac{\lambda_k}{\lambda_i} \|\nabla f(x_i)\|^2 \leqslant \lambda_k \mathcal{L}_0, \tag{8}$$

where λ_k is introduce by (7) and has the estimate

$$\lambda_k \leqslant \min \left\{ 8L \left(2\sqrt{2L} + \sqrt{\gamma_0}k \right)^{-2}, \left(1 + \sqrt{\min\{\gamma_0, \mu\}/L} \right)^{-k} \right\}.$$

Note that the above rate of convergence is optimal in the sense of the optimization complexity theory [14, 15]. Furthermore (8) promises faster convergence rate for the norm of the gradient; see Remark 4.2.

In our recent work [13], we verified that NAG method can be recovered from an explicit scheme for NAG flow (3) with an extra gradient descent step which is not a discretization of the ODE (3). In this paper, we further show that NAG method is actually an explicit scheme for H-NAG flow (4) without extra gradient step. From this point of view, our H-NAG model (4) offers better explanation and understanding for Nesterov's accelerated gradient method than NAG flow (3) does.

We finally propose a new splitting method (cf. (65)) for composite convex optimization f = h + g. Here, the objective f is μ -convex with $\mu \ge 0$, h is a smooth convex function with L-Lipschitz gradient and g is convex but non-smooth.

Algorithm 2 HNAG method for minimizing f = h + g

Input: $\gamma_0 > 0$ and $x_0, v_0 \in V$.

- 1: **for** $k = 0, 1, \dots$ **do**
- 2: Compute α_k, β_k by $\alpha_k = \sqrt{\frac{\gamma_k}{L}}, \quad \beta_k = \frac{1}{L\alpha_k}$.
- 3: Set $z_k = \frac{1}{1 + \alpha_k} [x_k + \alpha_k v_k \alpha_k \beta_k \nabla h(x_k)].$
- 4: Update $x_{k+1} = \mathbf{prox}_{sq}(z_k)$ with $s = \alpha_k \beta_k / (1 + \alpha_k)$.
- 5: Set $p_{k+1} = \frac{1}{\beta_k} \left[v_k x_{k+1} \beta_k \nabla h(x_k) (x_{k+1} x_k) / \alpha_k \right] \in \partial g(x_{k+1}).$
- 6: Update $v_{k+1} = \frac{1}{\gamma_k + \mu \alpha_k} \left[\gamma_k v_k + \mu \alpha_k x_{k+1} \alpha_k \nabla h(x_{k+1}) \alpha_k p_{k+1} \right].$
- 7: Update $\gamma_{k+1} = (\gamma_k + \mu \alpha_k)/(1 + \alpha_k)$.
- 8: end for

Observe that Algorithm 2 is almost identical to Algorithm 1 except we use proximal operator for the non-smooth convex function g. For any $\lambda > 0$, the proximal operator $\mathbf{prox}_{\lambda g}$ is defined by

that [10, 18]

$$\mathbf{prox}_{\lambda g}(x) = \inf_{y \in V} \left(g(y) + \frac{1}{2\lambda} \|x - y\|^2 \right) \quad \forall x \in V.$$

For Algorithm 2, the following accelerated convergence rate has been established

$$\mathcal{L}_k \leqslant \mathcal{L}_0 \times \min \left\{ 8L \left(2\sqrt{2L} + \sqrt{\gamma_0 k} \right)^{-2}, \left(1 + \sqrt{\min\{\gamma_0, \mu\}/L} \right)^{-k} \right\},$$

and with an alternative choice for α_k and β_k , we can obtain faster convergence rate for the norm of (sub-)gradient; see Remark 5.1.

1.2 Related work and main contribution

The most relevant works are [5, 19] where ODE models with Hessian driven damping are studied. We defer to §1.4 for a detailed literature review.

We follow closely to [5, 19]. Namely we first analyze the ODE using a Lyapunov function, then construct optimization algorithms from numerical discretizations of this ODE, and use a discrete Lyapunov function to study the convergence of the proposed algorithms.

Our main contribution is a relatively simple ODE model with dynamic damping coefficient γ which can handle both the convex case ($\mu = 0$) and strongly convex case ($\mu > 0$) in a unified way. Our continuous and discrete Lyapunov functions are also relatively simple so that most calculation is straightforward.

Another major contribution is a simplified Lyapunov analysis by introducing the strong Lyapunov property cf. (27), which simplifies the heavy algebraic manipulation in [5, 19, 21, 22]. We believe our translation of results from continuous-time ODE to discrete algorithms is more transparent and helpful for the design and analysis of existing and new optimization methods. For example, we successfully developed splitting algorithms for composite optimization problems not restricted to a special case as considered in [5].

1.3 Function class

Throughout this paper, assume V is equipped with the inner product (\cdot,\cdot) and the norm $\|\cdot\| = (\cdot,\cdot)^{1/2}$. We use $\langle\cdot,\cdot\rangle$ to denote the duality pair between V^* and V, where V^* is the dual space of V. Denote by \mathcal{F}^1_L the set of all convex functions $f \in \mathcal{C}^1$ with L-Lipschitz continuous gradient:

$$\|\nabla f(x) - \nabla f(y)\|_* \leqslant L\|x - y\| \quad \forall x, y \in V,$$

where $\|\cdot\|_*$ denotes the dual norm on V^* . We say that f is μ -convex if there exists $\mu \geqslant 0$ such that

$$f(x) - f(y) - \langle p, x - y \rangle \geqslant \frac{\mu}{2} ||x - y||^2 \quad \forall \, p \in \partial f(y), \tag{9}$$

for all $x, y \in V$, where the sub-gradient $\partial f(y)$ of f at $y \in V$ is defined by that

$$\partial f(y) := \{ p \in V^* : f(x) \geqslant f(y) + \langle p, x - y \rangle \quad \forall x \in V \}. \tag{10}$$

We use \mathcal{S}^0_{μ} to denote the set of all μ -convex functions. In addition, we set $\mathcal{S}^1_{\mu} := \mathcal{S}^0_{\mu} \cap \mathcal{C}^1$ and $\mathcal{S}^{1,1}_{\mu,L} := \mathcal{S}^1_{\mu} \cap \mathcal{F}^1_L$.

1.4 Literature review

We first review some dynamical models involving Hessian data. In [3], combining the well-known continuous Newton method [4] and the heavy ball system [17], Alvarez et al. proposed the so-called dynamical inertial Newton (DIN) system

$$x'' + \alpha x' + \beta \nabla^2 f(x) x' + \nabla f(x) = 0, \tag{11}$$

where $\alpha, \beta > 0$ are constants and $f \in C^2$ is bounded from below. Note that the Hessian term $\nabla^2 f(x)x'$ is nothing but the derivative of the gradient $(\nabla f(x))'$. Hence the DIN system (11) can be transferred into a first-order system without Hessian

$$\begin{cases} y' = -(\alpha - 1/\beta)x - y/\beta, \\ x' = -(\alpha - 1/\beta)x - y/\beta - \beta \nabla f(x). \end{cases}$$

For convex f, it has been proved [3, Theorem 5.1] that each trajectory of (11) weakly converges to a minimizer of f. Later on, in [8], Attouch et al. extended the DIN system (11) to the composite case f = h + g:

$$x'' + \alpha x' + \beta \nabla^2 h(x)x' + \nabla h(x) + \nabla g(x) = 0,$$

where $g \in C^1$ and $f \in C^2$ is convex such that f = h + g is convex. Like the DIN system (11), this model can also be rewritten as a first-order system

$$\begin{cases} x' = -(\alpha - 1/\beta) x - y/\beta - \beta \nabla h(x), \\ y' = -(\alpha - 1/\beta) x - y/\beta + \beta \nabla g(x), \end{cases}$$
(12)

based on which they generalized their model to nonsmooth case as well.

In [5] Attouch et al. added the Hessian term and time scaling to the ODE derived in [22] and obtained

$$x'' + \frac{\alpha}{t}x' + \beta \nabla^2 f(x)x' + b\nabla f(x) = 0, \quad t \geqslant t_0, \tag{13}$$

where $\alpha \geqslant 1$ is a constant, $f \in \mathcal{C}^2$ is convex and $\beta(t)$ is a nonnegative function such that

$$b(t) > \beta'(t) + \beta(t)/t, \quad t \geqslant t_0.$$

If b=1, then (13) reduces to the ODE consider in [9]. When $\beta(t)=0$, then (13) coincides with the rescaled ODE derived in [2]. When $\alpha=3$, $\beta(t)=\beta>0$ and $b(t)=1+1.5\beta/t$, then (13) recoveries the high resolution ODE (19). They derived the convergence result [5, Theorem 2.1]

$$t^{2}w(t)(f(x(t)) - f(x^{*})) + \int_{t_{0}}^{t} s^{2}\beta(s) \|\nabla f(x(s))\|^{2} ds \leqslant C,$$
(14)

provided that

$$w(t) = b(t) - \beta'(t) - \beta(t)/t, \quad tw'(t) \leqslant (\alpha - 3)w(t).$$

However, due to the above restriction on w, we have $w(t) \leq Ct^{\alpha-3}$ and the best decay rate they can obtain is $O(t^{1-\alpha})$. In [5], they also studied a DIN system for $f \in \mathcal{C}^2 \cap \mathcal{S}^1_{\mu}(\mu > 0)$:

$$x'' + 2\sqrt{\mu}x' + \beta\nabla f^{2}(x)x' + \nabla f(x) = 0,$$
(15)

where $\beta \ge 0$ is a constant. Note that the case $\beta = 0$ has been considered in [20, 23]. For $\beta > 0$, they established the result

$$f(x(t)) - f(x^*) + \beta^2 \int_0^t e^{\sqrt{\mu}(s-t)} \|\nabla f(x(s))\|^2 ds \leqslant Ce^{-t\sqrt{\mu}/2}.$$
 (16)

Recently, Shi et al. [19] derived two Hessian-driven models, which were called high-resolution ODEs. One requires $f \in \mathcal{C}^2 \cap \mathcal{S}_{\mu,L}^{1,1}$ with $\mu > 0$ and reads as follows

$$x'' + 2\sqrt{\mu}x' + \sqrt{\beta}\nabla f^{2}(x)x' + (1 + \sqrt{\mu}\beta)\nabla f(x) = 0,$$
(17)

where $0 < \beta \le 1/L$. This ODE interprets [15, Constant step scheme, III, Chapter 2] and achieves the exponential decay [19, Theorem 1 and Lemma 3.1]

$$f(x(t)) - f(x^*) + \sqrt{\beta} \int_0^t e^{(s-t)\sqrt{\mu}/4} \|\nabla f(x(s))\|^2 ds \leqslant Ce^{-t\sqrt{\mu}/4}.$$
 (18)

The second is for $f \in \mathcal{C}^2 \cap \mathcal{F}_L^1$:

$$x'' + \frac{3}{t}x' + \sqrt{\beta}\nabla^2 f(x)x' + (1 + t_0/t)\nabla f(x) = 0, \quad t \geqslant t_0 = 1.5\sqrt{\beta},$$
(19)

where $\beta > 0$. This model agrees with (13) in a special case that $\alpha = 3$, $\beta(t) = \sqrt{\beta}$ and $b(t) = 1 + 1.5\sqrt{\beta}/t$, and the convergence result (14) in this case has also been proved by [19, Lemma 4.1 and Corollary 4.2]. Compared with the dynamical systems derived in [5, 19], our H-NAG flow (4) uniformly treats $f \in \mathcal{S}^1_{\mu}$ with $\mu \geq 0$ and yields the convergence result (6) which, also gives the estimate for the gradient as what (14), (16) and (18) do.

Optimization methods based on differential equation solvers for those systems above are also proposed. Based on a semi-implicit scheme for (12), Attouch et al. [6] proposed an inertial forward-backward algorithm for composite convex optimization and established the weak convergence. In [12], Castera et al. applied the DIN system (11) to deep neural networks and presented an inertial Newton algorithm for minimizing the empirical risk loss function. Their numerical experiments showed that the proposed method performs much better than SGD and Adam in the long run and can reach very low training error. With minor change of (19), Shi et al. [19] developed a family of accelerated methods by explicit discretization scheme. Later in [21], for (17), they considered explicit and symplectic methods, among which only the symplectic scheme achieves the accelerated rate (20). More recently, Attouch et al. [5] proposed two explicit schemes for (13) and (15), respectively. However, only the discretization for (13) has accelerated rate $O(1/k^2)$; see [5, Theorem 3.3]. We emphasize that, our Algorithms 1 and 2 possess the convergence rate

$$O\left(\min\left\{1/k^2, \left(1+\sqrt{\mu/L}\right)^{-k}\right\}\right),\tag{20}$$

which is optimal for $f \in \mathcal{S}^1_{\mu}(\mu \geqslant 0)$ and accelerated for $f \in \mathcal{S}^0_{\mu}(\mu \geqslant 0)$. More methods that achieve the rate (20) are listed in Sections 4.2, 4.3 and 5.2.

The rest of this paper is organized as follows. In Section 2 we focus on the continuous problem (4). Then, in Sections 3 and 4 we consider (semi-)implicit and explicit schemes sequentially. Then, we deal with the composite case f = h + g in Section 5. Finally, we give conclusion and future work in Section 6.

2 Continuous Problem

In this section, we study our H-NAG flow for $f \in \mathcal{S}^1_{\mu}(\mu \geqslant 0)$ and establish the minimizing property of the trajectory.

2.1 Notation

To move on and for later use, throughout this paper, we define the Lyapunov function $\mathcal{L}: V \to \mathbb{R}_{\geq 0}$ by that

$$\mathcal{L}(x) = \mathcal{L}(x, v, \gamma) := f(x) - f(x^*) + \frac{\gamma}{2} \|v - x^*\|^2.$$
 (21)

where $\boldsymbol{x}=(x,v,\gamma)\in V\times V\times \mathbb{R}_+:=\boldsymbol{V}$, and x^* is a global minimum point of f. When $\boldsymbol{x}(t)=(x(t),v(t),\gamma(t))$ is a \boldsymbol{V} -valued function of time variable t on $[0,\infty)$, we also introduce the abbreviated notation

$$\mathcal{L}(t) := \mathcal{L}(\boldsymbol{x}(t)) = \mathcal{L}(x(t), v(t), \gamma(t)), \quad t \geqslant 0.$$
(22)

Note that \mathcal{L} is convex with respect to (x, v) and linear in γ . Moreover, \mathcal{L} is whenever smooth in respect of (v, γ) and it is trivial that

$$\nabla_{x} \mathcal{L} = \nabla f(x),$$

$$\nabla_{v} \mathcal{L} = \gamma(v - x^{*}),$$

$$\nabla_{\gamma} \mathcal{L} = \frac{1}{2} \|v - x^{*}\|^{2}.$$

Above, ∇_{\times} means the partial derivative of $\times = x$, v or γ . For any $\beta \in \mathbb{R}_+$ and $\boldsymbol{x} = (x, v, \gamma) \in \boldsymbol{V}$, we introduce the flow field \mathcal{G}

$$\mathcal{G}(\boldsymbol{x},\beta) := (\mathcal{G}^{x}(\boldsymbol{x},\beta), \mathcal{G}^{v}(\boldsymbol{x}), \mathcal{G}^{\gamma}(\boldsymbol{x})), \tag{23}$$

where the three components are defined as follows

$$\mathcal{G}^{x}(\boldsymbol{x}, \beta) = v - x - \beta \nabla f(x),$$

$$\mathcal{G}^{v}(\boldsymbol{x}) = \frac{\mu}{\gamma}(x - v) - \frac{1}{\gamma} \nabla f(x),$$

$$\mathcal{G}^{\gamma}(\boldsymbol{x}) = \mu - \gamma.$$

Our H-NAG system (4) can be simply written as

$$\mathbf{x}'(t) = \mathcal{G}(\mathbf{x}(t), \beta(t)), \tag{24}$$

where $\boldsymbol{x}(t) = (x(t), v(t), \gamma(t))$. We find that $\boldsymbol{x}^* = (x^*, x^*, \mu)$ is a candidate of the equilibrium point to the dynamic system (24).

The well-posedness of (24) is standard. Indeed, if f has Lipschitz continuous gradient, then apply the classical existence and uniqueness results of ODE (see [1, Theorem 4.1.4]) yields that the ODE system (24) admits a unique solution $\mathbf{x} = (x, v, \gamma)$ with $x \in \mathcal{C}^2([0, \infty); V)$ and $v \in \mathcal{C}^1([0, \infty); V)$.

2.2 Strong Lyapunov property

Originally the Lyapunov function is used to study the stability of an equilibrium point of a dynamical system. The function $\mathcal{L}(\boldsymbol{x})$ defined by (21) is called a Lyapunov function of the vector field $\mathcal{G}(\boldsymbol{x},\beta)$ (23) near an equilibrium point \boldsymbol{x}^* if $\mathcal{L}(\boldsymbol{x}^*)=0$ and

$$-\nabla \mathcal{L}(\boldsymbol{x}) \cdot \mathcal{G}(\boldsymbol{x}, \beta) \text{ is locally positive near } \boldsymbol{x}^*. \tag{25}$$

To obtain the convergence rate, we need a stronger condition than merely $-\nabla \mathcal{L}(\boldsymbol{x}) \cdot \mathcal{G}(\boldsymbol{x}, \beta)$ is locally positive definite. We introduce the strong Lyapunov property: there exist a positive function $c(\boldsymbol{x}) > 0$, and a function $q(\boldsymbol{x}) : \boldsymbol{V} \to \mathbb{R}$ such that

$$-\nabla \mathcal{L}(\mathbf{x}) \cdot \mathcal{G}(\mathbf{x}, \beta) \geqslant c(\mathbf{x})\mathcal{L}(\mathbf{x}) + q^2(\mathbf{x}) \quad \forall \mathbf{x} \in \mathbf{V}.$$
 (26)

Next we will show the Lyapunov function (21) satisfies the strong Lyapunov property.

Lemma 2.1. Assume $f \in \mathcal{S}^1_{\mu}(\mu \geqslant 0)$. For any $\beta \in \mathbb{R}_+$ and $\boldsymbol{x} = (x, v, \gamma) \in \boldsymbol{V}$, we have

$$-\nabla \mathcal{L}(\boldsymbol{x}) \cdot \mathcal{G}(\boldsymbol{x}, \beta) \geqslant \mathcal{L}(\boldsymbol{x}) + \beta \|\nabla f(\boldsymbol{x})\|^2 + \frac{\mu}{2} \|\boldsymbol{x} - \boldsymbol{v}\|^2.$$
 (27)

Proof. Indeed, observing the identity

$$2\langle x - v, v - x^* \rangle = \|x - x^*\|^2 - \|x - v\|^2 - \|v - x^*\|^2$$

and using the convexity of f

$$\langle \nabla f(x), x - x^* \rangle \geqslant f(x) - f(x^*) + \frac{\mu}{2} \|x - x^*\|^2,$$

a direct computation gives

$$-\nabla \mathcal{L}(\boldsymbol{x}) \cdot \mathcal{G}(\boldsymbol{x}, \beta) = -\mu \langle x - v, v - x^* \rangle + \langle \nabla f(x), x - x^* \rangle$$

$$+\beta \|\nabla f(x)\|^2 + \frac{\gamma - \mu}{2} \|v - x^*\|^2$$

$$\geqslant \mathcal{L}(\boldsymbol{x}) + \beta \|\nabla f(x)\|^2 + \frac{\mu}{2} \|x - v\|^2.$$

This finishes the proof of this lemma.

When f is nonsmooth, we introduce the notation $\partial \mathcal{L}(x, p) = (p, \nabla_v \mathcal{L}(x), \nabla_\gamma \mathcal{L}(x))$ and $\mathcal{G}(x, \beta, p)$ by replacing $\nabla f(x)$ in \mathcal{G} with some $p \in \partial f(x)$. Namely we substitute $\nabla f(x)$ in $\nabla_x \mathcal{L}$ and \mathcal{G} with some $p \in \partial f(x)$, where the sub-gradients $\partial f(x)$ of f is defined in (10). Then we can easily generalize Lemma 2.1 to the non-smooth version.

Lemma 2.2. Assume $f \in \mathcal{S}^0_{\mu}(\mu \geqslant 0)$. Then for any $\beta \in \mathbb{R}_+$, $\boldsymbol{x} = (x, v, \gamma) \in \boldsymbol{V}$ and $p \in \partial f(x)$, we have

 $-\partial \mathcal{L}(\boldsymbol{x}, p) \cdot \mathcal{G}(\boldsymbol{x}, \beta, p) \geqslant \mathcal{L}(\boldsymbol{x}) + \beta \|p\|^2 + \frac{\mu}{2} \|x - v\|^2.$ (28)

2.3 Minimizing property

The crucial inequality (27) implies that \mathcal{G} is a descent direction for minimizing \mathcal{L} and thus \mathcal{L} and $\|\nabla f\|$ decrease along the trajectory defined by (24). Indeed, we have the following theorem that depicts this.

Theorem 2.1. Let $x(t) = (x(t), v(t), \gamma(t))$ be the solution of (24), then for any $t \ge 0$,

$$\mathcal{L}(t) + \int_0^t e^{s-t} \beta(s) \|\nabla f(x(s))\|^2 ds \le e^{-t} \mathcal{L}(0).$$
 (29)

Proof. By the chain rule $\mathcal{L}'(t) = \nabla \mathcal{L}(\boldsymbol{x}(t)) \cdot \mathcal{G}(\boldsymbol{x}(t), \beta(t))$ and the key estimate (27), we have the inequality

$$\mathcal{L}'(t) \leqslant -\mathcal{L}(t) - \beta(t) \left\| \nabla f(x(t)) \right\|^2 - \frac{\mu}{2} \left\| x(t) - v(t) \right\|^2 \leqslant -\mathcal{L}(t).$$

This yields the exponential decay rate $\mathcal{L}(t) \leq e^{-t}\mathcal{L}(0)$. Moreover, we find that

$$\mathcal{L}'(t) + \mathcal{L}(t) + \beta(t) \|\nabla f(x(t))\|^2 \leqslant 0.$$

Multiplying both sides by e^t and integrating over (0,t) gives

$$\int_0^t d\left(e^s \mathcal{L}(s)\right) + \int_0^t e^s \beta(s) \left\|\nabla f(x(s))\right\|^2 ds \leq 0,$$

which also implies

$$e^{t}\mathcal{L}(t) + \int_{0}^{t} e^{s}\beta(s) \|\nabla f(x(s))\|^{2} ds \leqslant \mathcal{L}(0), \quad t \geqslant 0.$$

This proves (29) and establishes the proof of this theorem.

Remark 2.1. We do not have to give the explicit form of $\beta(t)$, which is acceptable as long as it is positive, i.e., $\beta(t) > 0$ for all t > 0. In the discretization level, however, to obtain optimal rate of convergence, we shall choose special coefficient β_k , which is positive and computable (cf. Theorem 3.1 and Theorem 4.1).

Remark 2.2. As discussed in [13, section 2.2], the exponential decay (29) may be sped or slowed down if we introduce the time rescaling. In our model (4), such rescaling is automatically encoded in the damping parameter γ governed by the equation $\gamma' = \mu - \gamma$ which allow us to handle $\mu > 0$ and $\mu = 0$ in a unified way.

3 A Semi-implicit Scheme

In this section, we consider a semi-implicit scheme for our H-NAG flow (4), where $f \in \mathcal{S}^1_{\mu}$ with $\mu \geq 0$. We will see that in the discrete level, rescaling effect and exponential decay can be inherit by (semi-)implicit scheme which has no restriction on step size; see Theorem 3.1, [2, Theorem 3.1] and [13, Theorem 1].

Our scheme reads as follows

$$\begin{cases} \frac{x_{k+1} - x_k}{\alpha_k} = v_k - x_{k+1} - \beta_k \nabla f(x_{k+1}), \\ \frac{v_{k+1} - v_k}{\alpha_k} = \frac{\mu}{\gamma_k} (x_{k+1} - v_{k+1}) - \frac{1}{\gamma_k} \nabla f(x_{k+1}), \\ \frac{\gamma_{k+1} - \gamma_k}{\alpha_k} = \mu - \gamma_{k+1}. \end{cases}$$
(30)

If we set

$$y_k := \frac{x_k + \alpha_k v_k}{1 + \alpha_k}, \quad s_k := \frac{\alpha_k \beta_k}{1 + \alpha_k},$$

then the update for x_{k+1} is equivalent to

$$x_{k+1} = y_k - s_k \nabla f(x_{k+1}) = \mathbf{prox}_{s_k, f}(y_k).$$

After obtaining x_{k+1} , v_{k+1} is obtained through the second equation of (30). To characterize the convergence rate, denote by

$$\lambda_0 = 1, \quad \lambda_k = \prod_{i=0}^{k-1} \frac{1}{1+\alpha_i}, \quad k \geqslant 1.$$
 (31)

We introduce the discrete Lyapunov function

$$\mathcal{L}_{k} := \mathcal{L}(\boldsymbol{x}_{k}) = f(x_{k}) - f(x^{*}) + \frac{\gamma_{k}}{2} \|v_{k} - x^{*}\|^{2},$$
(32)

where $\boldsymbol{x}_k = (x_k, v_k, \gamma_k)$, and

$$\mathcal{R}_0 = 0, \quad \mathcal{R}_k := \frac{\lambda_k}{2} \sum_{i=0}^{k-1} \frac{\alpha_i \beta_i}{\lambda_i} \left\| \nabla f(x_{i+1}) \right\|^2, \quad k \geqslant 1.$$
 (33)

Furthermore, for all $k \ge 0$, we set

$$\mathcal{E}_k = \mathcal{L}_k + \mathcal{R}_k. \tag{34}$$

In the following, we present the convergence result for our semi-implicit scheme (30).

Theorem 3.1. Assume β_k satisfies $\beta_k \gamma_k = \alpha_k$, then for the semi-implicit scheme (30) with any step size $\alpha_k > 0$, we have

$$\mathcal{E}_{k+1} \leqslant \frac{\mathcal{E}_k}{1 + \alpha_k} \quad \forall \, k \geqslant 0. \tag{35}$$

Consequently, for all $k \ge 0$, it holds that

$$\mathcal{L}_k + \frac{\lambda_k}{2} \sum_{i=0}^{k-1} \frac{\alpha_i^2}{\lambda_i \gamma_i} \left\| \nabla f(x_{i+1}) \right\|^2 \leqslant \lambda_k \mathcal{L}_0. \tag{36}$$

Proof. We first split the difference as

$$\mathcal{L}_{k+1} - \mathcal{L}_k = \mathcal{L}(x_{k+1}, v_k, \gamma_k) - \mathcal{L}(x_k, v_k, \gamma_k)$$

$$+ \mathcal{L}(x_{k+1}, v_{k+1}, \gamma_k) - \mathcal{L}(x_{k+1}, v_k, \gamma_k)$$

$$+ \mathcal{L}(x_{k+1}, v_{k+1}, \gamma_{k+1}) - \mathcal{L}(x_{k+1}, v_{k+1}, \gamma_k)$$

$$:= I_1 + I_2 + I_3.$$

The last item I_3 is the easiest one as \mathcal{L} is linear in γ

$$I_3 = \langle \nabla_{\gamma} \mathcal{L}(\boldsymbol{x}_{k+1}), \gamma_{k+1} - \gamma_k \rangle = \alpha_k (\nabla_{\gamma} \mathcal{L}(\boldsymbol{x}_{k+1}), \mathcal{G}^{\gamma}(\boldsymbol{x}_{k+1})). \tag{37}$$

For item I₂, we use the fact $\mathcal{L}(x_{k+1},\cdot,\gamma_k)$ is γ_k -convex and the discretization (30) to get

$$I_{2} \leqslant \langle \nabla_{v} \mathcal{L}(x_{k+1}, v_{k+1}, \gamma_{k}), v_{k+1} - v_{k} \rangle - \frac{\gamma_{k}}{2} \|v_{k+1} - v_{k}\|^{2}$$

$$= \alpha_{k} \langle \nabla_{v} \mathcal{L}(\boldsymbol{x}_{k+1}), \mathcal{G}^{v}(\boldsymbol{x}_{k+1}) \rangle - \frac{\gamma_{k}}{2} \|v_{k+1} - v_{k}\|^{2}.$$

$$(38)$$

In the last step, as γ_k is canceled in the product, we can switch the argument γ_k to γ_{k+1} . By the convexity of f, it is clear that

$$I_1 = f(x_{k+1}) - f(x_k) \leqslant \langle \nabla f(x_{k+1}), x_{k+1} - x_k \rangle$$

= $\alpha_k \langle \nabla_x \mathcal{L}(x_{k+1}), \mathcal{G}^x(x_{k+1}, \beta_k) \rangle + \alpha_k \langle \nabla f(x_{k+1}), v_k - v_{k+1} \rangle$.

Observing the negative term in (38), we bound the second term as follows

$$\alpha_k \|\nabla f(x_{k+1})\| \|v_k - v_{k+1}\| \le \frac{\alpha_k^2}{2\gamma_k} \|\nabla f(x_{k+1})\|^2 + \frac{\gamma_k}{2} \|v_k - v_{k+1}\|^2.$$

Now, adding all together and using the strong Lyapunov property (27), we get

$$\mathcal{L}_{k+1} - \mathcal{L}_{k} \leqslant \alpha_{k} \left(\nabla \mathcal{L}(\boldsymbol{x}_{k+1}), \mathcal{G}(\boldsymbol{x}_{k+1}, \beta_{k}) \right) + \frac{\alpha_{k}^{2}}{2\gamma_{k}} \|\nabla f(\boldsymbol{x}_{k+1})\|^{2}$$

$$\leqslant -\alpha_{k} \mathcal{L}_{k+1} + \left(\frac{\alpha_{k}^{2}}{2\gamma_{k}} - \alpha_{k} \beta_{k} \right) \|\nabla f(\boldsymbol{x}_{k+1})\|^{2}$$

$$= -\alpha_{k} \mathcal{L}_{k+1} - \frac{\alpha_{k} \beta_{k}}{2} \|\nabla f(\boldsymbol{x}_{k+1})\|^{2}. \tag{39}$$

Finally, by definition $\lambda_{k+1} - \lambda_k = -\alpha_k \lambda_{k+1}$, it is evident that

$$2\mathcal{R}_{k+1} - 2\mathcal{R}_{k} = \lambda_{k+1} \sum_{i=0}^{k} \frac{\alpha_{i}\beta_{i}}{\lambda_{i}} \|\nabla f(x_{i+1})\|^{2} - \lambda_{k} \sum_{i=0}^{k-1} \frac{\alpha_{i}\beta_{i}}{\lambda_{i}} \|\nabla f(x_{i+1})\|^{2}$$

$$= \alpha_{k}\beta_{k} \|\nabla f(x_{k+1})\|^{2} + (\lambda_{k+1} - \lambda_{k}) \sum_{i=0}^{k} \frac{\alpha_{i}\beta_{i}}{\lambda_{i}} \|\nabla f(x_{i+1})\|^{2}$$

$$= \alpha_{k}\beta_{k} \|\nabla f(x_{k+1})\|^{2} - \alpha_{k}2\mathcal{R}_{k+1}. \tag{40}$$

Now combining the relation $\beta_k \gamma_k = \alpha_k$ with (39) and (40) implies (35) and thus concludes the proof of this theorem.

With carefully designed parameter $\beta_k = \alpha_k/\gamma_k$, the semi-implicit scheme (30) can always achieve linear convergence rate as long as the step size α_k is chosen uniformly bounded below $\alpha_k \geqslant \widehat{\alpha} > 0$ for all k > 0 and larger α_k yields faster convergence rate. Observing the update of γ_{k+1} , we conclude that, if $\gamma_0 \geqslant \mu$, then $\gamma_k \geqslant \gamma_{k+1} \geqslant \mu$, and if $0 < \gamma_0 < \mu$, then $\gamma_k < \gamma_{k+1} < \mu$. Hence, it follows that

$$\min\{\gamma_0, \mu\} \leqslant \gamma_k \leqslant \max\{\gamma_0, \mu\},\tag{41}$$

and from (36) we can get fast convergence for the norm of the gradient.

Remark 3.1. If f is nonsmooth, we use the proximal operator $\mathbf{prox}_{s_k f}$ to rewrite the implicit scheme (30) as follows

$$\begin{cases} x_{k+1} = \mathbf{prox}_{s_k f}(y_k), & y_k = \frac{x_k + \alpha_k v_k}{1 + \alpha_k}, & s_k = \frac{\alpha_k \beta_k}{1 + \alpha_k}, \\ p_{k+1} = \frac{1}{\beta_k} \left(v_k - x_{k+1} - \frac{x_{k+1} - x_k}{\alpha_k} \right), \\ \frac{v_{k+1} - v_k}{\alpha_k} = \frac{\mu}{\gamma_k} (x_{k+1} - v_{k+1}) - \frac{1}{\gamma_k} p_{k+1}, \\ \frac{\gamma_{k+1} - \gamma_k}{\alpha_k} = \mu - \gamma_{k+1}. \end{cases}$$

Note that $p_{k+1} \in \partial f(x_{k+1})$. We just simply replace $\nabla f(x_{k+1})$ by p_{k+1} . In addition, thanks to Lemma 2.2, proceeding as the proof of Theorem 3.1, we can derive

$$\mathcal{L}_k + \frac{\lambda_k}{2} \sum_{i=0}^{k-1} \frac{\alpha_i \beta_i}{\lambda_i} \| p_{i+1} \|^2 \leqslant \lambda_k \mathcal{L}_0.$$

4 Explicit Schemes with Optimal Rates

This section assumes $f \in \mathcal{S}_{\mu,L}^{1,1}$ with $\mu \geqslant 0$ and considers several explicit schemes including Algorithm 1. All of those methods have optimal convergence rates in the sense of Nesterov [15, Chapter 2].

4.1 Analysis of Algorithm 1

It is straightforward to verify that the Algorithm 1 is equivalent to the following explicit scheme

$$\begin{cases}
\frac{x_{k+1} - x_k}{\alpha_k} = v_k - x_{k+1} - \beta_k \nabla f(x_k), \\
\frac{v_{k+1} - v_k}{\alpha_k} = \frac{\mu}{\gamma_k} (x_{k+1} - v_{k+1}) - \frac{1}{\gamma_k} \nabla f(x_{k+1}), \\
\frac{\gamma_{k+1} - \gamma_k}{\alpha_k} = \mu - \gamma_{k+1},
\end{cases} (42)$$

where

$$\alpha_k = \sqrt{\frac{\gamma_k}{L}}, \quad \beta_k = \frac{1}{L\alpha_k}.$$
 (43)

Given (x_k, v_k, γ_k) , we can solve the first equation to get x_{k+1} and with known x_{k+1} , we can get v_{k+1} from the second equation. Moreover, the sequence $\{v_k\}$ can be further eliminated to get an equation of (x_{k+1}, x_k, x_{k-1})

$$\gamma_k \cdot \frac{\frac{x_{k+1} - x_k}{\alpha_k} - \frac{x_k - x_{k-1}}{\alpha_{k-1}}}{\alpha_k} + (\mu + \gamma_k) \cdot \frac{x_{k+1} - x_k}{\alpha_k} + \gamma_k \beta_k \cdot \frac{\nabla f(x_k) - \nabla f(x_{k-1})}{\alpha_k} + (1 + \mu \beta_k) \nabla f(x_k) + \gamma_k \cdot \frac{\beta_k - \beta_{k-1}}{\alpha_k} \cdot \nabla f(x_{k-1}) = 0,$$

which is an explicit scheme for (1) since the unknown x_{k+1} is not in the gradient. Note that Hessian term $\nabla^2 f$ is not present as the action $\nabla^2 f(x)x'$ can be discretized by the quotient of the gradient.

For the convergence analysis, we need the following tighter bound on the function difference; see [15, Theorem 2.1.5].

Lemma 4.1 ([15]). If $f \in \mathcal{F}_L^1$, then

$$f(y) - f(x) \le \langle \nabla f(y), y - x \rangle - \frac{1}{2L} \| \nabla f(y) - \nabla f(x) \|^2 \quad \forall x, y \in V.$$

For the explicit scheme, we modify the definition (33) of \mathcal{R}_k slightly as

$$\mathcal{R}_0 = 0, \quad \mathcal{R}_k := \frac{\lambda_k}{2} \sum_{i=0}^{k-1} \frac{\alpha_i \beta_i}{\lambda_i} \|\nabla f(x_i)\|^2, \quad k \geqslant 1,$$

and we also set $\mathcal{E}_k := \mathcal{L}_k + \mathcal{R}_k$, where λ_k and \mathcal{L}_k are defined in (31) and (32), respectively. Similar to the derivation of (40), we have

$$\mathcal{R}_{k+1} - \mathcal{R}_k = -\alpha_k \mathcal{R}_{k+1} + \frac{\alpha_k \beta_k}{2} \left\| \nabla f(x_k) \right\|^2. \tag{44}$$

Theorem 4.1. For Algorithm 1, we have

$$\mathcal{E}_{k+1} \leqslant \frac{\mathcal{E}_k}{1 + \alpha_k} \quad \forall \, k \geqslant 0.$$
 (45)

Consequently, for all $k \ge 0$, it holds that

$$\mathcal{L}_k + \frac{1}{2L} \sum_{i=0}^{k-1} \frac{\lambda_k}{\lambda_i} \left\| \nabla f(x_i) \right\|^2 \leqslant \lambda_k \mathcal{L}_0. \tag{46}$$

Above, λ_k is bounded above by the optimal convergence rate

$$\lambda_k \leqslant \min \left\{ 8L \left(2\sqrt{2L} + \sqrt{\gamma_0 k} \right)^{-2}, \left(1 + \sqrt{\min\{\gamma_0, \mu\}/L} \right)^{-k} \right\}. \tag{47}$$

Proof. Following the proof of Theorem 3.1, we first split the difference $\mathcal{L}_{k+1} - \mathcal{L}_k$ along the path $\mathbf{x}_k = (x_k, v_k, \gamma_k)$ to (x_{k+1}, v_k, γ_k) to $(x_{k+1}, v_{k+1}, \gamma_k)$ and finally to $\mathbf{x}_{k+1} = (x_{k+1}, v_{k+1}, \gamma_{k+1})$:

$$\mathcal{L}_{k+1} - \mathcal{L}_k = \mathcal{L}(x_{k+1}, v_k, \gamma_k) - \mathcal{L}(x_k, v_k, \gamma_k)$$

$$+ \mathcal{L}(x_{k+1}, v_{k+1}, \gamma_k) - \mathcal{L}(x_{k+1}, v_k, \gamma_k)$$

$$+ \mathcal{L}(x_{k+1}, v_{k+1}, \gamma_{k+1}) - \mathcal{L}(x_{k+1}, v_{k+1}, \gamma_k)$$

$$:= I_1 + I_2 + I_3.$$

Note that we still have (37) and (38):

$$I_{3} = \alpha_{k}(\nabla_{\gamma}\mathcal{L}(\boldsymbol{x}_{k+1}), \mathcal{G}^{\gamma}(\boldsymbol{x}_{k+1})),$$

$$I_{2} \leqslant \alpha_{k} \langle \nabla_{v}\mathcal{L}(\boldsymbol{x}_{k+1}), \mathcal{G}^{v}(\boldsymbol{x}_{k+1}) \rangle - \frac{\gamma_{k}}{2} \|v_{k+1} - v_{k}\|^{2}.$$

$$(48)$$

We now use Lemma 4.1 to estimate I_1

$$I_1 \leqslant \langle \nabla_x \mathcal{L}(\boldsymbol{x}_{k+1}), x_{k+1} - x_k \rangle - \frac{1}{2L} \|\nabla f(x_{k+1}) - \nabla f(x_k)\|^2.$$

In the first step, we can switch (x_{k+1}, v_k, γ_k) to \boldsymbol{x}_{k+1} because $\nabla_x \mathcal{L}$ is independent of (v, γ) . Then we use the discretization (42) to replace $x_{k+1} - x_k$ and compare with the flow evaluated at \boldsymbol{x}_{k+1} :

$$\langle \nabla_x \mathcal{L}(\boldsymbol{x}_{k+1}), x_{k+1} - x_k \rangle = \alpha_k \langle \nabla_x \mathcal{L}(\boldsymbol{x}_{k+1}), \mathcal{G}^x(\boldsymbol{x}_{k+1}, \beta_k) \rangle + \alpha_k \beta_k (\nabla f(x_{k+1}), \nabla f(x_{k+1}) - \nabla f(x_k)) + \alpha_k \langle \nabla f(x_{k+1}), v_k - v_{k+1} \rangle.$$

Observing the bound (48) for I_2 , we use Cauchy–Schwarz inequality to bound the last term as follows

$$\alpha_k \|\nabla f(x_{k+1})\| \|v_k - v_{k+1}\| \leqslant \frac{\alpha_k^2}{2\gamma_k} \|\nabla f(x_{k+1})\|^2 + \frac{\gamma_k}{2} \|v_k - v_{k+1}\|^2.$$
(49)

We use the identity for the second term

$$\alpha_k \beta_k (\nabla f(x_{k+1}), \nabla f(x_{k+1}) - \nabla f(x_k))$$

$$= -\frac{\alpha_k \beta_k}{2} \|\nabla f(x_k)\|^2 + \frac{\alpha_k \beta_k}{2} \|\nabla f(x_{k+1})\|^2 + \frac{\alpha_k \beta_k}{2} \|\nabla f(x_{k+1}) - \nabla f(x_k)\|^2.$$

Adding all together and applying Lemma 2.1 yield that

$$\mathcal{L}_{k+1} - \mathcal{L}_k \leqslant -\alpha_k \mathcal{L}_{k+1} - \frac{\alpha_k \beta_k}{2} \|\nabla f(x_k)\|^2$$

$$+ \frac{1}{2} \left(\alpha_k \beta_k - \frac{1}{L}\right) \|\nabla f(x_{k+1}) - \nabla f(x_k)\|^2$$

$$+ \frac{1}{2} \left(\frac{\alpha_k^2}{\gamma_k} - \alpha_k \beta_k\right) \|\nabla f(x_{k+1})\|^2.$$

$$(50)$$

Additionally, in view of the choice of parameters α_k and β_k (cf. (43)), we have

$$\alpha_k \beta_k - \frac{1}{L} = 0, \quad \frac{\alpha_k^2}{\gamma_k} - \alpha_k \beta_k = 0,$$

which implies

$$\mathcal{L}_{k+1} - \mathcal{L}_k \leqslant -\alpha_k \mathcal{L}_{k+1} - \frac{\alpha_k \beta_k}{2} \|\nabla f(x_k)\|^2.$$

This together (43) and (44) gives the desired estimates (45) and (46).

Next, let us study the asymptotic behavior of λ_k . The formula of γ_k yields

$$\frac{1}{1+\alpha_k} = \frac{\gamma_{k+1}}{\gamma_k + \mu \alpha_k} \leqslant \frac{\gamma_{k+1}}{\gamma_k},$$

and it follows from (31) that

$$\lambda_k \leqslant \frac{\gamma_k}{\gamma_0} = \frac{L\alpha_k^2}{\gamma_0}.\tag{51}$$

Using the lower bound of α_k implied by (51), we get

$$\frac{1}{\sqrt{\lambda_{k+1}}} - \frac{1}{\sqrt{\lambda_k}} \geqslant \frac{\lambda_k - \lambda_{k+1}}{2\lambda_k \sqrt{\lambda_{k+1}}} = \frac{\alpha_k}{2\sqrt{\lambda_k (1 + \alpha_k)}} \geqslant \frac{1}{2} \sqrt{\frac{\gamma_0}{2L}},$$

which implies

$$\frac{1}{\sqrt{\lambda_k}} \geqslant \frac{k}{2} \sqrt{\frac{\gamma_0}{2L}} + 1.$$

Therefore, we have

$$\lambda_k \leqslant 8L \left(2\sqrt{2L} + \sqrt{\gamma_0}k\right)^{-2}.\tag{52}$$

Note that this sublinear rate holds for $\mu \ge 0$. If $\mu > 0$, then by (41) it is evident that

$$\alpha_k^2 = \frac{\gamma_k}{L} \geqslant \frac{1}{L} \min\{\gamma_0, \mu\},\tag{53}$$

so we have that

$$\lambda_k \leqslant \left(1 + \sqrt{\min\{\gamma_0, \mu\}/L}\right)^{-k}$$
.

This together with (52) implies (47) and concludes the proof.

Remark 4.1. As we see, unlike the semi-implicit scheme (30), explicit scheme (42) has restriction on step size α_k . When $\mu > 0$, namely f is strongly convex, it is allowed to choose non-vanishing step size (cf. (53)) which promises (accelerated) linear rate. For convex f, i.e., $\mu = 0$, (51) becomes equality which gives vanishing step size $\alpha_k = O(1/k)$ and results in accelerated sublinear rate $O(1/k^2)$.

Remark 4.2. Note that (46) gives the optimal convergence rate under an oracle model of optimization complexity [15]. However, the explicit schemes proposed in [5, 21] for strongly convex case ($\mu > 0$) haven't achieved acceleration. In addition, we also have faster rate for the norm of gradient. Indeed, by (46), we have

$$\sum_{i=0}^{\infty} \frac{1}{\lambda_i} \|\nabla f(x_i)\|^2 \leqslant 2L\mathcal{L}_0.$$

This yields that

$$\min_{0 \leqslant i \leqslant k} \left\| \nabla f(x_i) \right\|^2 \leqslant \frac{2L\mathcal{L}_0}{\sum_{i=0}^k 1/\lambda_i},$$

and asymptotically, we have $\|\nabla f(x_k)\|^2 = o(2L\mathcal{L}_0\lambda_k)$. On the other hand, thanks to the Lemma 4.1, we have the bound

$$\frac{1}{2L} \left\| \nabla f(x_k) \right\|^2 \leqslant f(x_k) - f(x^*) \leqslant \mathcal{L}_k,$$

which yields the uniform estimate

$$\|\nabla f(x_k)\|^2 \leqslant 2L\mathcal{L}_0 \lambda_k. \tag{54}$$

4.2 HNAG method with one extra gradient step

Based on (42), we propose an explicit scheme with one extra gradient step:

$$\begin{cases}
\frac{y_k - x_k}{\alpha_k} = v_k - y_k - \beta_k \nabla f(x_k), \\
\frac{v_{k+1} - v_k}{\alpha_k} = \frac{\mu}{\gamma_k} (y_k - v_{k+1}) - \frac{1}{\gamma_k} \nabla f(y_k), \\
\frac{\gamma_{k+1} - \gamma_k}{\alpha_k} = \mu - \gamma_{k+1}, \\
x_{k+1} = y_k - \frac{1}{L} \nabla f(y_k),
\end{cases} (55)$$

where α_k and β_k are chosen from the relation

$$L\alpha_k^2 = \gamma_k(2 + \alpha_k), \quad \beta_k = \frac{1}{L\alpha_k}.$$
 (56)

Below, we present this scheme in the algorithm style.

Algorithm 3 HNAG Method with extra gradient step

Input: $\gamma_0 > 0$ and $x_0, v_0 \in V$.

1: **for** $k = 0, 1, \dots$ **do**

2: Compute α_k, β_k by $L\alpha_k^2 = \gamma_k(2 + \alpha_k), \quad \beta_k = \frac{1}{L\alpha_k}$.

3: Set
$$y_k = \frac{1}{1 + \alpha_k} [x_k + \alpha_k v_k - \alpha_k \beta_k \nabla f(x_k)].$$

4: Update
$$v_{k+1} = \frac{1}{\gamma_k + \mu \alpha_k} [\gamma_k v_k + \mu \alpha_k y_k - \alpha_k \nabla f(y_k)].$$

5: Update
$$x_{k+1} = y_k - \frac{1}{L} \nabla f(y_k)$$
.

6: Update $\gamma_{k+1} = (\gamma_k + \mu \alpha_k)/(1 + \alpha_k)$.

7: end for

Define

$$\widehat{\mathcal{L}}_k := f(y_k) - f(x^*) + \frac{\gamma_{k+1}}{2} \|v_{k+1} - x^*\|^2.$$

Proceeding as the proof of Theorem 4.1, we still have (50), i.e.,

$$\widehat{\mathcal{L}}_k - \mathcal{L}_k \leqslant -\alpha_k \widehat{\mathcal{L}}_k - \frac{\alpha_k \beta_k}{2} \|\nabla f(x_k)\|^2$$

$$+ \frac{1}{2} \left(\alpha_k \beta_k - \frac{1}{L}\right) \|\nabla f(y_k) - \nabla f(x_k)\|^2$$

$$+ \frac{1}{2} \left(\frac{\alpha_k^2}{\gamma_k} - \alpha_k \beta_k\right) \|\nabla f(y_k)\|^2.$$

We then use our choice of parameters (56) to obtain

$$\widehat{\mathcal{L}}_k - \mathcal{L}_k \leqslant -\alpha_k \widehat{\mathcal{L}}_k + \frac{1 + \alpha_k}{2L} \|\nabla f(y_k)\|^2 - \frac{1}{2L} \|\nabla f(x_k)\|^2, \tag{57}$$

Recalling the standard gradient descent result (cf. [15, Lemma 1.2.3])

$$f(y - \nabla f(y)/L) - f(y) \leqslant -\frac{1}{2L} \|\nabla f(y)\|^2 \quad \forall y \in V,$$

we get the inequality

$$\mathcal{L}_{k+1} - \widehat{\mathcal{L}}_k = f(x_{k+1}) - f(y_k) = f(y_k - \nabla f(y_k)/L) - f(y_k) \leqslant -\frac{1}{2L} \|\nabla f(y_k)\|^2.$$

By (57), it follows that

$$\mathcal{L}_{k+1} - \mathcal{L}_k \leqslant -\alpha_k \mathcal{L}_{k+1} - \frac{1}{2L} \|\nabla f(x_k)\|^2.$$
 (58)

Hence, using the same notation as that in Theorem 4.1, we have the following result.

Theorem 4.2. For Algorithm 3, we have

$$\mathcal{E}_{k+1} \leqslant \frac{\mathcal{E}_k}{1 + \alpha_k} \quad \forall \, k \geqslant 0. \tag{59}$$

Hence, for all $k \ge 0$, it holds that

$$\mathcal{L}_k + \frac{1}{2L} \sum_{i=0}^{k-1} \frac{\lambda_k}{\lambda_i} \left\| \nabla f(x_i) \right\|^2 \leqslant \lambda_k \mathcal{L}_0, \tag{60}$$

where λ_k is defined by (31) and still has the optimal upper bound

$$\lambda_k \leqslant \min \left\{ 4L \left(2\sqrt{L} + \sqrt{1.5\gamma_0} \, k \right)^{-2}, \, \left(1 + \sqrt{2\min\{\gamma_0, \mu\}/L} \right)^{-k} \right\}. \tag{61}$$

Proof. Note that (59) and (60) have been derived from (44) and (58). The estimate (61) for λ_k follows from the procedure in Theorem 4.1 so we omit it here.

Remark 4.3. Note that the optimal convergence rate (61) is slightly better than (47) due to an extra gradient step in Algorithm 3. However, two gradient $\nabla f(x_k)$ and $\nabla f(y_k)$ should be computed in one iteration. In Algorithm 1, although there are still two gradient $\nabla f(x_k)$ and $\nabla f(x_{k+1})$, the later one can be re-used in the next iteration and thus essentially only one gradient is computed in one iteration. In most applications, evaluation of gradient is the dominant cost and thus Algorithm 1 is still more efficient than Algorithm 3.

4.3 Equivalence to methods from NAG flow

In this section, we shall show some explicit schemes that are supplemented with one gradient descent steps for NAG flow (3) can be viewed as explicit discretizations for H-NAG flow (4).

Recall that, in [13], we present two explicit schemes for NAG flow (3). The first one reads as follows

$$\begin{cases}
\frac{y_k - x_k}{\alpha_k} = v_k - y_k, \\
\frac{v_{k+1} - v_k}{\alpha_k} = \frac{\mu}{\gamma_k} (y_k - v_{k+1}) - \frac{1}{\gamma_k} \nabla f(y_k), \\
x_{k+1} = y_k - \frac{1}{L} \nabla f(y_k), \\
\gamma_{k+1} = \gamma_k + \alpha_k (\mu - \gamma_{k+1}).
\end{cases} (62)$$

Let us represent x_k from the first equation

$$x_k = y_k + \alpha_k (y_k - v_k),$$

and put this into the third equation to obtain

$$y_{k+1} + \alpha_{k+1}(y_{k+1} - v_{k+1}) = y_k - \frac{1}{L} \nabla f(y_k).$$

Now reorganizing (62) yield that

$$\begin{cases} \frac{v_{k+1} - v_k}{\alpha_k} = \frac{\mu}{\gamma_k} (y_k - v_{k+1}) - \frac{1}{\gamma_k} \nabla f(y_k), \\ \frac{y_{k+1} - y_k}{\alpha_{k+1}} = v_{k+1} - y_{k+1} - \beta_{k+1} \nabla f(y_k), \end{cases}$$

where $\beta_{k+1} = 1/(L\alpha_{k+1})$. This is nothing but an explicit scheme for (4). In addition, writing the previous iteration for y_k before v_{k+1} and replacing y_k with x_{k+1} yield

$$\begin{cases} \frac{x_{k+1} - x_k}{\alpha_k} = v_k - x_{k+1} - \beta_k \nabla f(x_k), \\ \frac{v_{k+1} - v_k}{\alpha_k} = \frac{\mu}{\gamma_k} (x_{k+1} - v_{k+1}) - \frac{1}{\gamma_k} \nabla f(x_{k+1}), \end{cases}$$

which is identical to the scheme (42) but with slightly different choice of parameters. If $L\alpha_k^2 = \gamma_k(1 + \alpha_k)$, then by [13, Theorem 2], we have the optimal convergence rate

$$\mathcal{L}_k \leqslant \mathcal{L}_0 \times \min \left\{ 4L \left(2\sqrt{L} + \sqrt{\gamma_0} k\right)^{-2}, \left(1 + \sqrt{\min\{\gamma_0, \mu\}/L}\right)^{-k} \right\},$$

where \mathcal{L}_k is defined by (32).

The second scheme is listed below

$$\begin{cases}
\frac{y_k - x_k}{\alpha_k} = \frac{\gamma_k}{\gamma_{k+1}} (v_k - y_k), \\
\frac{v_{k+1} - v_k}{\alpha_k} = \frac{\mu}{\gamma_{k+1}} (y_k - v_k) - \frac{1}{\gamma_{k+1}} \nabla f(y_k), \\
x_{k+1} = y_k - \frac{1}{L} \nabla f(y_k), \\
\gamma_{k+1} = \gamma_k + \alpha_k (\mu - \gamma_k),
\end{cases} (63)$$

which recoveries Nesterov's optimal method [15, Chapter 2] constructed by estimate sequence. Proceeding as before, we can eliminate $\{x_k\}$ and rearrange (63) by that

$$\begin{cases} \frac{v_{k+1} - v_k}{\alpha_k} = \frac{\mu}{\gamma_{k+1}} (y_k - v_k) - \frac{1}{\gamma_{k+1}} \nabla f(y_k), \\ \frac{y_{k+1} - y_k}{\alpha_{k+1}} = \frac{\gamma_{k+1}}{\gamma_{k+2}} (v_{k+1} - y_{k+1}) - \beta_{k+1} \nabla f(y_k), \\ \gamma_{k+1} = \gamma_k + \alpha_k (\mu - \gamma_k), \end{cases}$$

where $\beta_{k+1} = 1/(L\alpha_{k+1})$. This is also an explicit scheme for our H-NAG flow (4). If $L\alpha_k^2 = \gamma_{k+1}$, then by [13, Theorem 3], we have the optimal convergence rate

$$\mathcal{L}_k \leqslant \mathcal{L}_0 \times \min \left\{ 4L \left(2\sqrt{L} + \sqrt{\gamma_0} k \right)^{-2}, \left(1 - \sqrt{\min\{\gamma_0, \mu\}/L} \right)^k \right\},$$

which indicates the decay of the norm of gradient, i.e.,

$$\|\nabla f(x_k)\|^2 \leqslant 2L\mathcal{L}_0 \times \min\left\{4L\left(2\sqrt{L} + \sqrt{\gamma_0}\,k\right)^{-2}, \left(1 - \sqrt{\min\{\gamma_0, \mu\}/L}\right)^k\right\}. \tag{64}$$

We conclude that H-NAG flow offers us a better explanation and understanding for Nesterov's optimal method [15, Chapter 2] than NAG flow (3) does and in view of Remark 4.2 and (64), algorithms based on H-NAG yields faster decay for the norm of the gradient.

5 Splitting Schemes with Accelerated Rates

In this section, we consider the composite case f = h + g and assume that $f \in \mathcal{S}^0_\mu$ with $\mu \geqslant 0$, $h \in \mathcal{F}^1_L$ is the smooth part and the nonsmooth part g is convex and lower semicontinuous. Note that this assumption on f is more general than that in [13, 16, 20]. To utilize the composite structure of f, we shall consider splitting schemes that are explicit in h and implicit in g and prove the accelerated convergence rates.

5.1 Analysis of Algorithm 2

It is easy to show Algorithm 2 can be written as a splitting scheme

$$\begin{cases} \frac{x_{k+1} - x_k}{\alpha_k} \in v_k - x_{k+1} - \beta_k \nabla h(x_k) - \beta_k \partial g(x_{k+1}), \\ \frac{v_{k+1} - v_k}{\alpha_k} = \frac{\mu}{\gamma_k} (x_{k+1} - v_{k+1}) - \frac{1}{\gamma_k} (\nabla h(x_{k+1}) + p_{k+1}), \\ \frac{\gamma_{k+1} - \gamma_k}{\alpha_k} = \mu - \gamma_{k+1}, \end{cases}$$
(65)

where α_k and β_k are chosen from (43), i.e.,

$$\alpha_k = \sqrt{\frac{\gamma_k}{L}}, \quad \beta_k = \frac{1}{L\alpha_k},$$
(66)

and the term p_{k+1} is defined as follows

$$p_{k+1} := \frac{1}{\beta_k} \left(v_k - x_{k+1} - \beta_k \nabla h(x_k) - \frac{x_{k+1} - x_k}{\alpha_k} \right) \in \partial g(x_{k+1}).$$

If we introduce

$$y_k := \frac{x_k + \alpha_k v_k}{1 + \alpha_k}, \quad s_k := \frac{\alpha_k \beta_k}{1 + \alpha_k}, \quad z_k := y_k - s_k \nabla h(x_k),$$

then the update of x_{k+1} in (65) is equivalent to

$$x_{k+1} = \underset{y \in V}{\operatorname{argmin}} \left(h(x_k) + \langle \nabla h(x_k), y - x_k \rangle + g(y) + \frac{1}{2s_k} \|y - y_k\|^2 \right)$$
$$= \mathbf{prox}_{s_k q} (y_k - s_k \nabla h(x_k)).$$

Theorem 5.1. For Algorithm 2, we have

$$\mathcal{L}_{k+1} \leqslant \frac{\mathcal{L}_k}{1 + \alpha_k} \quad \forall \, k \geqslant 0, \tag{67}$$

where \mathcal{L}_k is defined in (32), and it holds that

$$\mathcal{L}_k \leqslant \mathcal{L}_0 \times \min \left\{ 8L \left(2\sqrt{2L} + \sqrt{\gamma_0 k} \right)^{-2}, \left(1 + \sqrt{\min\{\gamma_0, \mu\}/L} \right)^{-k} \right\}.$$
 (68)

Proof. Based on the equivalent form (65), the proof is almost identical to a combination of that of Theorem 3.1 and Theorem 4.1. Let us start from the difference

$$\mathcal{L}_{k+1} - \mathcal{L}_k = \mathcal{L}(x_{k+1}, v_k, \gamma_k) - \mathcal{L}(x_k, v_k, \gamma_k)$$

$$+ \mathcal{L}(x_{k+1}, v_{k+1}, \gamma_k) - \mathcal{L}(x_{k+1}, v_k, \gamma_k)$$

$$+ \mathcal{L}(x_{k+1}, v_{k+1}, \gamma_{k+1}) - \mathcal{L}(x_{k+1}, v_{k+1}, \gamma_k)$$

$$:= I_1 + I_2 + I_3,$$

where the estimates for I_2 and I_3 keep unchanged

$$I_{3} = \alpha_{k}(\nabla_{\gamma}\mathcal{L}(\boldsymbol{x}_{k+1}), \mathcal{G}^{\gamma}(\boldsymbol{x}_{k+1})),$$

$$I_{2} \leqslant \alpha_{k} \langle \nabla_{v}\mathcal{L}(\boldsymbol{x}_{k+1}), \mathcal{G}^{v}(\boldsymbol{x}_{k+1}) \rangle - \frac{\gamma_{k}}{2} \|v_{k+1} - v_{k}\|^{2}.$$

Observing that

$$I_1 = \mathcal{L}(x_{k+1}, v_k, \gamma_k) - \mathcal{L}(x_k, v_k, \gamma_k) = g(x_{k+1}) - g(x_k) + h(x_{k+1}) - h(x_k),$$

we use Lemma 4.1 and the fact $p_{k+1} \in \partial g(x_{k+1})$ to estimate I_1

$$I_{1} \leqslant \langle \nabla h(x_{k+1}) + p_{k+1}, x_{k+1} - x_{k} \rangle - \frac{1}{2L} \| \nabla h(x_{k+1}) - \nabla h(x_{k}) \|^{2}.$$
 (69)

For simplicity, set $q_{k+1} = p_{k+1} + \nabla h(x_{k+1}) \in \partial f(x_{k+1})$. We use the discretization (65) to replace $x_{k+1} - x_k$ and compare with the flow evaluated at $\mathbf{x}_{k+1} = (x_{k+1}, v_{k+1}, \gamma_{k+1})$:

$$\langle q_{k+1}, x_{k+1} - x_k \rangle = \alpha_k \langle q_{k+1}, \mathcal{G}^x(\boldsymbol{x}_{k+1}, \beta_k) \rangle + \alpha_k \beta_k \langle q_{k+1}, \nabla h(x_{k+1}) - \nabla h(x_k) \rangle + \alpha_k \langle q_{k+1}, v_k - v_{k+1} \rangle.$$

The last term is estimated in the same way as (49), namely,

$$\alpha_k \|q_{k+1}\| \|v_k - v_{k+1}\| \le \frac{\alpha_k^2}{2\gamma_k} \|q_{k+1}\|^2 + \frac{\gamma_k}{2} \|v_k - v_{k+1}\|^2.$$

Thanks to the negative term in (69), we bound the second term by that

$$\alpha_k \beta_k \langle q_{k+1}, \nabla h(x_{k+1}) - \nabla h(x_k) \rangle \leqslant \frac{1}{2L} \|\nabla h(x_{k+1}) - \nabla h(x_k)\|^2 + \frac{L\alpha_k^2 \beta_k^2}{2} \|q_{k+1}\|^2.$$

We now get the estimate for I_1 as follows

$$I_{1} \leqslant \alpha_{k} \langle q_{k+1}, \mathcal{G}^{x}(\boldsymbol{x}_{k+1}, \beta_{k}) \rangle + \frac{\gamma_{k}}{2} \|v_{k} - v_{k+1}\|^{2} + \left(\frac{L\alpha_{k}^{2}\beta_{k}^{2}}{2} + \frac{\alpha_{k}^{2}}{2\gamma_{k}}\right) \|q_{k+1}\|^{2}.$$

Putting all together and using Lemma 2.2 implies

$$\mathcal{L}_{k+1} - \mathcal{L}_{k} \leqslant \alpha_{k} (\partial \mathcal{L}(\boldsymbol{x}_{k+1}, q_{k+1}), \mathcal{G}(\boldsymbol{x}_{k+1}, \beta_{k}, q_{k+1}))$$

$$+ \left(\frac{L\alpha_{k}^{2}\beta_{k}^{2}}{2} + \frac{\alpha_{k}^{2}}{2\gamma_{k}}\right) \|q_{k+1}\|^{2}$$

$$\leqslant -\alpha_{k}\mathcal{L}_{k+1} + \left(\frac{L\alpha_{k}^{2}\beta_{k}^{2}}{2} + \frac{\alpha_{k}^{2}}{2\gamma_{k}} - \alpha_{k}\beta_{k}\right) \|q_{k+1}\|^{2}$$

$$= -\alpha_{k}\mathcal{L}_{k+1},$$

$$(70)$$

where in the last step we used the fact (66). This establishes (67) and yields that $\mathcal{L}_k \leq \lambda_k \mathcal{L}_0$. Note the bound (47) for λ_k still holds here and (68) follows directly. We finally conclude the proof of this theorem.

Remark 5.1. To control the sub-gradient, we can choose

$$\alpha_k = \sqrt{\frac{\gamma_k}{4L}}, \quad \beta_k = \frac{1}{2L\alpha_k}.$$

Plugging this into (70) indicates

$$\mathcal{L}_{k+1} - \mathcal{L}_k \leqslant -\alpha_k \mathcal{L}_{k+1} - \frac{\alpha_k \beta_k}{2} \|q_{k+1}\|^2.$$

$$(71)$$

By slight modification of the proof, it follows that

$$\mathcal{L}_k + \frac{1}{4L} \sum_{i=0}^{k-1} \frac{\lambda_k}{\lambda_i} \|q_{i+1}\|^2 \leqslant \lambda_k \mathcal{L}_0.$$
 (72)

Following the estimate for λ_k in Theorem 4.1, we can derive that

$$\lambda_k \leqslant \min \left\{ 32L \left(4\sqrt{2L} + \sqrt{\gamma_0}k \right)^{-2}, \left(1 + 0.5\sqrt{\min\{\gamma_0, \mu\}/L} \right)^{-k} \right\}.$$

Therefore, (72) yields fast convergence for the norm of (sub-)gradient. However, the convergence bound is slightly worse than that of (68).

5.2 Methods using gradient mapping

In [13], using the gradient mapping technique [16], we presented two explicit schemes (supplemented with one gradient descent step) for NAG flow (3) in composite case f = h + g, where $h \in \mathcal{S}_{\mu,L}^{1,1}$ with $\mu \geq 0$, g is convex and lower-semicontinuous. Following the discussion in §4.3, we show that those two schemes can also be viewed as explicit discretizations for H-NAG flow (4).

Since the argument of those two methods are analogous, we only consider the following algorithm [13, Algorithm 2]

$$\begin{cases}
\frac{y_k - x_k}{\alpha_k} = v_k - y_k, \\
\frac{v_{k+1} - v_k}{\alpha_k} = \frac{\mu}{\gamma_k} (y_k - v_{k+1}) - \frac{1}{\gamma_k} \widehat{\nabla} f(y_k), \\
x_{k+1} = y_k - \frac{1}{L} \widehat{\nabla} f(y_k), \\
\gamma_{k+1} = \gamma_k + \alpha_k (\mu - \gamma_{k+1}).
\end{cases} (73)$$

Above, the gradient mapping $\widehat{\nabla} f(y_k)$ is defined as follows

$$\widehat{\nabla}f(y_k) := L\Big(y_k - \mathbf{prox}_{sg}\big[y_k - \nabla h(y_k)/L\big]\Big)$$

$$\in \nabla h(y_k) + \partial g\Big(\mathbf{prox}_{g/L}\big[y_k - \nabla h(y_k)/L\big]\Big). \tag{74}$$

If $L\alpha_k^2 = \gamma_k(1+\alpha_k)$, then by [13, Theorem 4], we have the accelerated convergence rate

$$\mathcal{L}_k \leqslant \mathcal{L}_0 \times \min \left\{ 4L \left(2\sqrt{L} + \sqrt{\gamma_0} k\right)^{-2}, \left(1 + \sqrt{\min\{\gamma_0, \mu\}/L}\right)^{-k} \right\},$$

where \mathcal{L}_k is defined in (32). With a similar simplify process as that in Section 4.3, we can eliminate the sequence $\{x_k\}$ and obtain the following equivalent form of (73):

$$\begin{cases} \frac{v_{k+1} - v_k}{\alpha_k} = \frac{\mu}{\gamma_k} (y_k - v_{k+1}) - \frac{1}{\gamma_k} \widehat{\nabla} f(y_k), \\ \frac{y_{k+1} - y_k}{\alpha_{k+1}} = v_{k+1} - y_{k+1} - \beta_{k+1} \widehat{\nabla} f(y_k), \end{cases}$$

where $\beta_{k+1} = 1/(L\alpha_{k+1})$. This is indeed an explicit scheme for H-NAG flow (4). Writing the previous iteration for y_k before v_{k+1} and replacing y_k with x_{k+1} yield

$$\begin{cases} \frac{x_{k+1} - x_k}{\alpha_k} = v_k - x_{k+1} - \beta_k \widehat{\nabla} f(x_k), \\ \frac{v_{k+1} - v_k}{\alpha_k} = \frac{\mu}{\gamma_k} (x_{k+1} - v_{k+1}) - \frac{1}{\gamma_k} \widehat{\nabla} f(x_k), \end{cases}$$

which is almost identical to (65). The difference is that the gradient mapping uses

$$\widehat{\nabla} f(x_k) = \nabla h(x_k) + \partial g(\widehat{x}_k),$$

where $\hat{x}_k = \mathbf{prox}_{g/L} [x_k - \nabla h(x_k)/L]$, while the scheme (65) considers

$$\nabla h(x_k) + \partial q(x_{k+1})$$
 and $\nabla h(x_{k+1}) + \partial q(x_{k+1})$.

6 Conclusion and Future Work

In this paper, for convex optimization problem, we present a novel DIN system, which is called Hessian-driven Nesterov accelerated gradient flow. Convergence of the continuous trajectory and algorithm analysis are established via tailored Lyapunov functions satisfying the strong Lyapunov property (cf. (26)). It is proved that explicit schemes posses the optimal(accelerated) rate

$$O\left(\min\left\{1/k^2, \left(1+\sqrt{\mu/L}\right)^{-k}\right\}\right),$$

and fast control of the norm of gradient is also obtained. This together with our previous work in [13], has already positively answered the fundamental question addressed in [21], that we can systematically and provably obtain accelerated methods via the numerical discretization of ordinary differential equations.

In future work, we plan to extend our results along this line and develop a systematic framework of developing and analyzing first-order accelerated optimization methods.

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