Rapid Calculation of Radiative Heating Rates and Photodissociation Rates in Inhomogeneous Multiple Scattering Atmospheres

OWEN B. TOON, C. P. MCKAY, AND T. P. ACKERMAN[1]

Space Science Division, NASA Ames Research Center, Moffett Field, California

K. SANTHANAM

Computer Sciences Corporation, Mountain View, California

The solution of the generalized two-stream approximation for radiative transfer in homogeneous multiple scattering atmospheres is extended to vertically inhomogeneous atmospheres in a manner which is numerically stable and computationally efficient. It is shown that solar energy deposition rates, photolysis rates, and infrared cooling rates all may be calculated with simple modifications of a single algorithm. The accuracy of the algorithm is generally better than 10% so that other uncertainties, such as in absorption coefficients, may often dominate the error in calculation of the quantities of interest to atmospheric studies.

INTRODUCTION

Radiative transfer schemes are currently available which can calculate the radiation fields in plane parallel, vertically inhomogeneous, multiple scattering atmospheres with an accuracy of much better than 1% [*Lenoble*, 1985]. However, their heavy computational demands have made their use impractical for many applications. In particular, three-dimensional dynamical models require exceedingly rapid computation of net fluxes in order to calculate heating rates, and multidimensional photochemical models require fast computation of the mean intensity in order to find photolysis rates. Moreover, lack of knowledge of absorption coefficients, scattering properties, absorber concentrations, or atmospheric structure often limits the accuracy of the calculations independently of the numerical scheme used to calculate the radiation fields [*Ronnholm et al.*, 1980]. In response to these computational needs and accuracy limitations, many researchers have developed numerically efficient, approximate radiative transfer schemes. However, we find that three aspects of these approximate schemes have not been discussed thoroughly in the literature. Little attention has been given to techniques for solving the equations in vertically inhomogeneous atmospheres, to schemes for calculating photolysis rates in anisotropic multiple scattering atmospheres, or to methods of treating scattering in the presence of thermal emission.

Numerous approximate techniques have been developed to calculate the transfer of solar energy through a single homogeneous layer. Most of them can be classified within a simple consistent framework as two-stream techniques [*Meador and Weaver*, 1980]. In the literature concerning solar energy deposition in the atmosphere, the words "two-stream" have come to refer to a whole set of different approximations rather than to any specific scheme. Each

individual technique is identified by a specific name such as the quadrature technique or δ-Eddington technique.

Starting from the solution for the intensity at the boundaries of a single homogeneous layer, *Shettle and Weinman* [1970] showed that a matrix of equations could be derived for the intensities at the boundaries of multiple layers with different properties by matching the single-layer equations using the appropriate boundary conditions. *Wiscombe* [1977] discussed procedures for inverting the resulting pentadiagonal matrix and proposed solutions to several numerical problems that can occur, for example, when the optical depth is large or when the single scattering albedo is precisely equal to unity. *Lacis and Hansen* [1974] showed that when the solar forcing was treated as a boundary condition, the single-layer, two-stream solutions for reflectivity and transmissivity could be cast into an adding and doubling form for multiple layers, resulting in a tridiagonal matrix. The tridiagonal matrix equations can be solved without recourse to a complex matrix inversion, and the computational time needed for the solution is linear in the number of vertical levels. Later, *Hansen et al.* [1983] showed how these expressions could be extended for a quadrature form of the two-stream approximation when the solar zenith angle is included explicitly. *Harshvardhan et al.* [1987] presented a similar approach to treat the solar zenith angle.

In this work we derive a tridiagonal matrix solution for multiple layers valid for the entire class of two-stream equations [*Meador and Weaver*, 1980]. This scheme retains the simplicity, numerical stability, and computational speed of the *Lacis and Hansen* [1974], *Hansen et al.* [1983], and *Harshvardhan et al.* [1987] approaches without the restriction to a single type of two-stream approximation and in a somewhat more direct manner since we do not spend time calculating intermediate quantities such as reflection and transmission functions. We also build upon the work of *Wiscombe* [1977] in solving numerical problems by providing alternative solutions based upon the recent work by *Stamnes et al.* [1988].

The techniques used to obtain exact solutions to radiative transfer problems apply equally well for calculation of solar heating rates and calculation of photolysis rates. However,

the majority of approximate techniques that have been developed for calculating photolysis rates are significantly different than those developed to find solar heating rates. This difference has been obscured by indiscriminate use of the words "two-stream" in the photochemical literature. In the photochemical literature "two-stream" has come to mean any approximation in which only an upward and downward flux is concerned. Radically different techniques of treating these upward and downward streams generally have not been distinguished by having separate names. Indeed, most of the two-stream approximations described in the photochemical literature are not directly related to the schemes that have been found to be most successful by those studying solar energy deposition. Therefore one often encounters categorical statements such as "a two-stream model is in general not a good approximation . . ." [Anderson and Meier, 1979, p. 1959], when in fact the author can mean only that the specific scheme he used did not perform well.

Most photochemical schemes were developed for studies of the stratosphere at ultraviolet wavelengths, where Rayleigh scattering dominates, and nearly all of the existing approximate schemes assume that the phase function is isotropic [Boughner, 1986; Augustsson and Levine, 1982; Meier et al., 1982] or that the light scattered by an entire layer is scattered isotropically [Luther, 1980; Thompson, 1984]. Such approximations do not apply in the presence of strongly forward scattering clouds or aerosols. Another common practice in the photochemical literature is to judge the accuracy of an approximation by comparing exact and approximate calculations of a photorate. Although this is a sensible approach for some purposes, the altitude and wavelength integrations involved average over many different values of optical depth and single scattering albedo. Therefore there is the possibility that large errors, which may occur for some optical properties, are compensated by other errors or are diluted by precise answers at wavelengths for which the direct solar beam, which can be found exactly, dominates.

In this paper we show that the mean intensity, which is the radiative quantity needed to find the monochromatic photolysis rate, may be easily calculated using the same basic expressions developed to calculate solar energy deposition. Hence we make a connection between two major radiative transfer applications for approximate schemes. We also show that the magnitude of the errors for calculation of the mean intensity are similar to those for finding fluxes or heating rates (usually of the order of 10% or less) and that the errors have similar dependence on fundamental optical properties. Therefore some of the specific two-stream approximations are as accurate as the existing approximate photochemical schemes in the isotropic limit and unlike most approximate photochemical techniques are also quite useful when anisotropic scattering is considered.

The final area which we thought needed further discussion in the literature concerns the approximate treatment of scattering in the infrared. Very little work on this subject appears in the terrestrial literature, probably because most substances in Earth's atmosphere are absorbing in the infrared. However, scattering can be important in clouds in the 8-13-μm window region for Earth, and thermal scattering by dust is quite important in the atmosphere of Mars. We show that only a limited portion of the class of two-stream

solutions is useful in the infrared, because many of them are prone to yield physically incorrect results, such as emissivities greater than unity. These improper results occur because the flux and intensity from an isotropic source are related in a specific manner and many two-stream schemes implicitly assume incorrect relations.

We develop a technique in which solar and infrared radiation can be treated using the same basic expressions. The only change required is modification of the source term to account for an external source in the case of solar radiation and an internal source in the case of thermal radiation. Consequently, most calculations can be performed with a single algorithm, simplifying the code and enhancing computational speed.

Although the two-stream results are often sufficiently accurate, they do not produce exact results in the limit of no scattering at infrared wavelengths, a limit which is commonly encountered. Therefore we have augmented them with a second technique which utilizes the two-stream results to approximate the source function in the integral solution of the radiative transfer equation [Toon et al., 1977]. The resulting algorithm has improved accuracy in the limit of small single scattering albedo and adds little to the computational cost. This scheme can also be useful for calculating approximate intensities.

We first derive a general two-stream solution for the upward and downward flux for a single homogeneous layer. Then we derive a matrix solution for multiple homogeneous layers which approximate a vertically inhomogeneous atmosphere. After that we introduce the two-stream source function approximation. Following that we demonstrate the accuracy of the numerical technique using specific two-stream approximations for a number of idealized cases. Finally, we present timing tests and examples of calculations for realistic atmospheres.

DERIVATION OF THE TWO-STREAM EQUATIONS FOR A SINGLE LAYER

We use a two-stream solution of the radiative transfer equation to determine the upward and downward fluxes at the top and bottom of single homogeneous layers. The two-stream equations represent a class of solutions based upon varying approximations to the phase function and the angular integral of the intensity field. Meador and Weaver [1980] show that all of these solutions for incident solar radiation may be put into a general form. We expand upon their work by deriving solutions for other source functions, including the Planck function, and by deriving solutions in a form that is suitable for treating multiple layers.

The general equation of radiative transfer in a plane parallel scattering atmosphere is

$$\mu \frac{\partial I_v}{\partial \tau_v} (\tau_v, \mu, \phi) = I_v(\tau_v, \mu, \phi) - S_v(\tau_v, \mu, \phi) - \frac{\omega_{0v}}{4\pi}$$

$$\cdot \int_0^{2\pi} \int_{-1}^1 P_v(\mu, \mu', \phi, \phi') I_v(\tau_v, \mu', \phi') \, d\mu' \, d\phi' \quad (1)$$

Here μ is the cosine of the angle at which the intensity, I_v, is observed with the angle measured from the outward surface normal, τ is the optical depth measured along the zenith direction beginning at the top of the atmosphere, ω_0 is the

TABLE 1. Summary of Coefficients in Selected Two-Stream Approximations

Method*	γ_1	γ_2	γ_3†	μ_1
Eddington	$[7 - \omega_0(4 + 3g)]/4$	$-[1 - \omega_0(4 - 3g)]/4$	$(2 - 3g\mu_0)/4$	$1/2$
Quadrature	$(3)^{1/2}[2 - \omega_0(1 + g)]/2$	$\omega_0(3)^{1/2}(1 - g)/2$	$[1 - (3)^{1/2}g\mu_0]/2$‡	$1/(3)^{1/2}$
Hemispheric mean	$2 - \omega_0(1 + g)$	$\omega_0(1 - g)$	...	$1/2$

*The Eddington and quadrature schemes are discussed in detail by *Meador and Weaver* [1980]. The hemispheric mean scheme is derived by assuming that the phase function is equal to $1 + g$ in the forward scattering hemisphere and to $1 - g$ in the backward scattering hemisphere. The asymmetry parameter is g.

†$\gamma_4 = 1 - \gamma_3$.

‡Only needed for solar wavelengths. However, the hemispheric mean is only useful for infrared wavelengths.

single scattering albedo, P is the scattering phase function, and v is the frequency.

For an emitting atmosphere,

$$S_{ve} = (1 - \omega_0)B_v(T) \qquad (2)$$

while for a purely external source at solar wavelengths,

$$S_{vs} = \frac{\omega_{0v}}{4} F_{sv}P_v(\mu, -\mu_0, \phi, \phi_0) \exp\left(-\tau_v/\mu_0\right) \qquad (3)$$

where πF_s is the solar flux, μ_0 is the incident direction of the solar beam, and $B_v(T)$ is the Planck function at temperature T. The use of expression (3) implies that an additional direct component of the radiation field given by $I_d = 0.5F_s \exp(-\tau_v/\mu)\delta(\mu - \mu_0)$ is present which must be added to I_v to find the total radiation field. Here δ is a delta function about the solar angle.

Equation (1) may be integrated over azimuth and zenith angle to yield

$$\frac{\partial F_v^{\pm}}{\partial \tau_v} = \pm \int_0^1 I_v^{\pm}(\tau_v, \mu)\, d\mu \mp \frac{1}{2} \int_0^1 \int_{-1}^1 P_v(\mu, \mu')$$

$$\cdot I_v^{\pm}(\tau_v, \mu')\, d\mu\, d\mu' \mp \int_0^{2\pi} \int_0^1 S_v(\tau_v)\, d\mu'\, d\phi' \qquad (4)$$

where the diffusive flux in the upward (downward) direction is

$$F_v^{\pm} = \int_0^1 \mu I_v^{\pm}(\tau_v, \mu)\, d\mu \qquad (5)$$

The azimuthally integrated intensity is

$$I_v^{\pm}(\tau, \mu) = \int_0^{2\pi} I_v(\tau, \pm\mu, \phi)\, d\phi \qquad (6)$$

while

$$\int_0^1 \int_0^{2\pi} S_{ve}(\tau)\, d\phi'\, d\mu' = 2\pi(1 - \omega_0)B_v(T) \qquad (7)$$

$$\int_0^1 \int_0^{2\pi} S_{vs}(\tau)\, d\phi'\, d\mu' = \pi F_v\omega_{0v}\beta_{0v} \exp\left(-\tau_v/\mu_0\right) \qquad (8)$$

$$\beta_{0v} = \frac{1}{2} \int_0^1 P_v(\mu_0, -\mu')\, d\mu' \qquad (9)$$

with

$$P(\mu, \mu') = \frac{1}{2\pi} \int_0^{2\pi} P(\mu, \mu', \phi, \phi')\, d\phi \qquad (10)$$

The two-stream solution consists of approximating $I_v(\tau, \mu)$ in (4) so that it is related to F_v. *Meador and Weaver* [1980] have shown that all of the two-stream expressions can be written in the following form (ignoring the frequency dependence):

$$\frac{\partial F_n^+}{\partial \tau_n} = \gamma_{1n}F_n^+ - \gamma_{2n}F_n^- - S_n^+ \qquad (11)$$

$$\frac{\partial F_n^-}{\partial \tau_n} = \gamma_{2n}F_n^+ - \gamma_{1n}F_n^- + S_n^+ \qquad (12)$$

Here γ_1 and γ_2 are coefficients which depend upon the particular form of the two-stream equation. Table 1 presents the values of γ_1 and γ_2 for some common two-stream approximations. We anticipate a multiple-layer solution by labeling the flux by n, the layer number.

For the solar beam,

$$S^+ = \gamma_3\pi F_s\omega_0 \exp\left[-(\tau_c + \tau)/\mu_0\right] \qquad (13)$$

$$S^- = \gamma_4\pi F_s\omega_0 \exp\left[-(\tau_c + \tau)/\mu_0\right] \qquad (14)$$

where γ_3 and γ_4 are coefficients which depend on the two-stream equations used [*Meador and Weaver*, 1980]. Some examples are given in Table 1. When multiple layers are used, τ_c is the cumulative optical depth of layers above layer n.

Equations (11) and (12) admit to simple solutions for any function S that can be written as any linear combination of τ, exponentials in τ, or sines and cosines of τ. The Planck function can often be approximated in such a fashion. Then one can solve (11) and (12) for the infrared with

$$S^+ = S^- = 2\pi(1 - \omega_0)B(\tau) \qquad (15)$$

where B is the Planck function expressed in terms of the layer optical depth.

Most applications of the two-stream equations seek either the net flux or the mean intensity. The net diffusive flux is defined as

$$F_{\text{net}} = \int I\mu\, d\Omega = F^+ - F^- \qquad (16)$$

where Ω is the solid angle and μ is the cosine of the zenith angle measured from the surface normal. The mean diffusive intensity is given by

$$J = \frac{1}{4\pi} \int I \, d\Omega = (F^+ + F^-)/4\pi\mu_1 \qquad (17)$$

Approximating the angular dependence of the intensity so that this integral can be calculated is central to the derivation of the two-stream approximation. Examination of the expressions in the work of *Meador and Weaver* [1980] for the mean intensities in the Eddington, quadrature, and hemispheric techniques yields

$$\mu_1 = (1 - \omega_0)/(\gamma_1 - \gamma_2) \qquad (18)$$

The parameter μ_1 is a constant such as $\frac{1}{2}$ or $(3)^{-1/2}$, rather than a frequency dependent property. Expression (18) is not general enough to apply to every expression examined by *Meador and Weaver* [1980].

The general solution to the two-stream equations (11) and (12) can be shown to be

$$F_n^+(\tau) = k_{1n} \exp(\lambda_n\tau) + \Gamma_n k_{2n} \exp(-\lambda_n\tau) + C_n^+(\tau) \qquad (19)$$

$$F_n^-(\tau) = \Gamma_n k_{1n} \exp(\lambda_n\tau) + k_{2n} \exp(-\lambda_n\tau) + C_n^-(\tau) \qquad (20)$$

These expressions, which again have a layer index n, apply for all cases except those in which ω_0 is exactly unity. In this case the homogeneous part of the solutions reduces to linear functions of τ. It can be shown that as ω_0 approaches unity, then (19) and (20), when coupled with the boundary conditions, approach these linear solutions.

The terms k_1 and k_2 in (19) and (20) are determined by boundary conditions, while λ and Γ depend upon the form of the two-stream equation used.

$$\lambda = (\gamma_1^2 - \gamma_2^2)^{1/2} \qquad (21)$$

$$\Gamma = \frac{\gamma_2}{(\gamma_1 + \lambda)} = \frac{\gamma_1 - \lambda}{\gamma_2} \qquad (22)$$

For solar radiation,

$$C^+(\tau) = \frac{\omega_0\pi F_s \exp\left[-(\tau_c+\tau)/\mu_0\right][(\gamma_1-1/\mu_0)\gamma_3+\gamma_4\gamma_2]}{(\lambda^2-1/\mu_0^2)} \qquad (23)$$

$$C^-(\tau) = \frac{\omega_0\pi F_s \exp\left[-(\tau_c + \tau)/\mu_0\right][(\gamma_1+1/\mu_0)\gamma_4+\gamma_2\gamma_3]}{(\lambda^2-1/\mu_0^2)} \qquad (24)$$

The terms γ_3 and γ_4 depend upon the two-stream expression used and are given in Table 1. The C^+ and C^- functions are indeterminate if λ equals $1/\mu_0$. In practice, if the equality happens to occur, this problem can be eliminated by simply choosing a slightly different value of μ_0. The functions C^+ and C^- depend upon the cumulative optical depth, τ_c, which is the optical depth of layers above layer n.

The C functions for infrared wavelengths can only be determined if one knows the optical depth dependence of the Planck function. We assume here that the Planck function

can be represented by the first term of a Taylor series expansion so that

$$B_n(\tau) = B_{0n} + B_{1n}\tau \qquad (25)$$

where B_{0n} is the Planck function evaluated at the temperature of the top of layer n, and B_{1n} is related to the Planck function evaluated at the temperature (T_{btmn}) of the bottom of layer n as

$$B_{1n} = [B(T_{btmn}) - B_{0n}]/\tau_n \qquad (26)$$

Wiscombe [1976] has examined the accuracy of this approximation for the Planck function and found that the errors are a function of the temperature, the temperature gradient across the layer, and of the wavelength.

Using this approximation of the Planck function,

$$C_n^+(\tau) = 2\pi\mu_1\{B_{0n} + B_{1n}[\tau + 1/(\gamma_{1n} + \gamma_{2n})]\} \qquad (27)$$

$$C_n^-(\tau) = 2\pi\mu_1\{B_{0n} + B_{1n}[\tau - 1/(\gamma_{1n} + \gamma_{2n})]\}$$

Other expressions for C^+ and C^- are easily derived for different assumed functional relations between B and τ. Note that in the infrared the C values depend only upon τ for the layer n. We have now introduced three different types of optical depths. The optical depth at any point within a layer τ, the total optical depth of the nth layer, τ_n, and the cumulative optical depth above layer n, τ_c. Of course, $0 < \tau < \tau_n$.

Although many two-stream approximations have been suggested, in this paper we will concentrate on only three as defined in Table 1. The Eddington and quadrature schemes developed by *Shettle and Weinman* [1970] and *Liou* [1974], respectively, have been shown to be reasonably accurate at solar wavelengths for a wide variety of conditions, especially when the "δ" scalings of *Joseph et al.* [1976] are used to replace ω_0, g, and τ, where g is the asymmetry factor. We show below, however, that these two schemes have undesirable properties when used with internal isotropic sources such as the Planck function and that the hemispheric mean approximation is preferable for that case.

Although the accuracy of a scheme can be judged by looking at the percentage errors from a wide variety of cases, we feel that certain limiting cases should be treated exactly for a scheme to be acceptable. For example, a scheme may be reasonably accurate and energy conserving but still produce unphysical results such as emissivities larger than unity. Clearly, it is desirable that the emissivity limit to unity. Table 2 lists the well-known limiting cases and indicates which schemes satisfy these exactly.

One of the most important limits at solar wavelengths is that when ω_0 equals unity the net flux should be conserved so that no energy is absorbed in the atmosphere. Using (11) and (12) for the derivative of the diffusive flux and adding the derivative of the direct flux, we find

$$0 = \frac{\partial F_{net}}{\partial \tau} = (\gamma_1 - \gamma_2)F^+ + (\gamma_1 - \gamma_2)F^-$$
$$- (S^+ + S^-) + \pi F_s \exp(-\tau/\mu_0) \qquad (28)$$

From Table 1 it can be seen that when $\omega_0 = 1$, then $\gamma_1 = \gamma_2$ and $S^+ + S^- = \pi F_s \exp(-\tau/\mu_0)$, so this limit is satisfied for all of the two-stream equations.

TABLE 2. Limiting Cases for Various Approaches

Case	δ-Quadrature	δ-Eddington	δ-Hemispheric Mean	Source Function
Solar				
$\omega_0 = 1$, flux conserved	yes	yes	yes	...*
$\omega_0 = 0$, flux down exact	yes	maybe†	yes	...
Infrared				
$\omega_0 = 0$, fluxes exact	no	no	no	yes
$\omega_0 = 0$, $\tau = \infty$, emissivity $= 1$	no	no	yes	yes
$\omega_0 = 1$, flux conserved	yes	yes	yes	yes

*The source function technique is not considered here at solar wavelengths.

†Maybe is exact if no reflecting surfaces are present.

Although the solutions to the two-stream equations (19) and (20) have the proper behavior as ω_0 approaches unity after the boundary conditions are applied, it is numerically difficult to obtain the correct solutions when ω_0 is precisely unity. Following *Wiscombe* [1977], we find that if $1 - \omega_0 = \varepsilon$, where $\varepsilon = 10\varepsilon_0$ and ε_0 is the smallest digit retained by the computer being used, then expressions (19) and (20) yield fluxes which do not differ significantly from those expected.

Another significant limit at solar wavelengths is that when $\omega_0 = 0$, the solar flux should be calculated exactly. When a reflecting surface is not present, this limit is satisfied for all the two-stream solutions because the diffuse radiation field vanishes leaving only the exact direct solar radiation term. However, when a reflecting surface is present, none of the two-stream solutions can compute the upward flux correctly because for isotropically reflecting surfaces, for example, it involves an exponential integral. The Eddington scheme alters the downward direct radiation so that its solution is no longer exact.

At infrared wavelengths, when $\omega_0 = 1$, no emission occurs and the solutions return to scattering solutions which satisfy (28) as for the solar case. However, when $\omega_0 = 0$, the two-stream results again do not behave well. When $\omega_0 = 0$, Γ in (19) and (20) becomes zero for the quadrature and hemispheric mean schemes. For a semi-infinite, isothermal layer without a reflecting boundary, $F^- + F^+ = C^-(\tau) + C^+(\tau) = 4\pi\mu_1 B_0$. Hence the correct result is obtained only if $\mu_1 = 0.5$ as is the case for the hemispheric mean scheme but not for the quadrature scheme. For the Eddington technique. Γ does not become zero and one finds $F^+ + F^- = \pi B_0(1 - \Gamma)$. Hence the Eddington technique does not yield unit emissivity in this limit. Indeed, the Eddington technique can yield emissivity greater than unity even for the case when ω_0 is not equal to zero.

The difficulty with the quadrature and Eddington schemes in the limit when $\omega_0 = 0$ at infrared wavelengths may be traced to the isotropy of the Planck function. The flux corresponding to an isotropic intensity is not related numerically to the intensity in the manner assumed by the quadrature or Eddington schemes which leads to systematic errors. The hemispheric mean technique is preferable at infrared wavelengths since it does assume the correct relation between flux and intensity and yields the proper emissivity in the $\omega_0 = 0$ limit for a semi-infinite atmosphere.

The final solution for F^+ and F^- is now obtained by applying the appropriate boundary conditions. We indicate in the next section how this may be done efficiently for a vertically inhomogeneous atmosphere.

SOLUTION OF THE TWO-STREAM EQUATIONS FOR MULTIPLE LAYERS

Equations (19) and (20) for a single homogeneous layer can be extended to an inhomogeneous atmosphere by dividing the atmosphere into a series of homogeneous layers. Applying boundary conditions yields a matrix which can then be inverted to find the fluxes throughout the atmosphere. *Shettle and Weinman* [1970] and *Wiscombe* [1977] directly inverted the matrices. We prefer to reorganize the equations so that they form a tridiagonal matrix and then utilize the standard tridiagonal technique for solving the equations. Tridiagonal solutions are computationally faster and numerically more stable than other matrix inversion techniques.

A subtle problem in solving (19) and (20) occurs when $\omega_0 = 0$ if the surface is not reflecting or the optical depth is large. In that case for many two-stream techniques, $\Gamma = 0$ so that F^+ and F^- become uncoupled. Numerical difficulties in the matrix inversion may then occur. Another difficulty with (19) and (20) is that they contain terms which grow exponentially with τ. *Stamnes et al.* [1988] showed that the positive exponentials could be eliminated by scaling them out of the equations. We therefore redefine the coefficients such that

$$Y_{1n} = [k_{1n} \exp(\lambda_n\tau_n) + k_{2n}]/2 \qquad (29)$$

$$Y_{2n} = [k_{1n} \exp(\lambda_n\tau_n) - k_{2n}]/2 \qquad (30)$$

Then

$$F_n^+(\tau) = Y_{1n}\{\exp[-\lambda_n(\tau_n - \tau)] + \Gamma_n \exp(-\lambda_n\tau)\}$$
$$+ Y_{2n}\{\exp[-\lambda_n(\tau_n - \tau)] - \Gamma_n$$
$$\cdot \exp(-\lambda_n\tau)\} + C_n^+(\tau) \qquad (31)$$

$$F_n^-(\tau) = Y_{1n}\{\Gamma_n \exp[-\lambda_n(\tau_n - \tau)] + \exp(-\lambda_n\tau)\}$$
$$+ Y_{2n}\{\Gamma_n \exp[-\lambda_n(\tau_n - \tau)] - \exp(-\lambda_n\tau)\} + C_n^-(\tau) \qquad (32)$$

In these expressions the exponentials only contain negative arguments.

The boundary conditions are

$$F_1^-(0) = F_0^-(0) \qquad (33)$$

$$F_N^+ (\tau_N) = R_{sfc}F_N^- (\tau_N) + S_{sfc} \qquad (34)$$

$$F_n^+ (\tau = \tau_n) = F_{n+1}^+(\tau = 0) \qquad (35)$$

$$F_n^- (\tau = \tau_n) = F_{n+1}^-(\tau = 0) \qquad (36)$$

The first boundary condition says that the downward diffuse flux at the top of the atmosphere equals any diffuse flux that may be incident. Usually, there is no incident diffuse flux. The second boundary condition states that the upward flux at the surface (layer N) is the sum of the reflected downward diffuse flux, where R_{sfc} is the reflectivity, and any energy from other sources, S_{sfc}. At solar wavelengths

$$S_{sfc} = R_{sfc}\mu_0 \exp (-\tau_c/\mu_0)\pi F_s \qquad (37)$$

while at infrared wavelengths

$$S_{sfc} = \varepsilon \pi B(T_{grnd}) \qquad (38)$$

where ε is the emissivity of the surface. The remaining boundary conditions express the continuity of the upward and downward fluxes at the internal boundaries.

If we write out (33)–(36) using (31) and (32), then it is found that they form a pentadiagonal matrix similar to the one studied by *Wiscombe* [1977]. However, it is also obvious by inspection that the equations contain some similar terms that may be eliminated. Linear combinations of (33)–(36) produce a new set of $2N$ equations which compose a tridiagonal matrix of the form

$$A_lY_{l-1} + B_lY_l + D_lY_{l+1} = E_l \qquad (39)$$

$$Y_l = Y_{1n} \qquad l = 2n - 1 = \text{odd}$$
$$\qquad \qquad \qquad \qquad \qquad \qquad (40)$$
$$Y_l = Y_{2n} \qquad l = 2n = \text{even}$$

The coefficients for odd $l = 1$ up to $2N - 1$ are

$$A_1 = 0 \qquad B_1 = e_{11} \qquad D_1 = -e_{21}$$

$$E_1 = F_0^- (0) - C_1^- (0)$$

$$A_l = e_{2n}e_{3n} - e_{4n}e_{1n}$$

$$B_l = e_{1n}e_{1n+1} - e_{3n}e_{3n+1} \qquad (41)$$

$$D_l = e_{3n}e_{4n+1} - e_{1n}e_{2n+1}$$

$$E_l = e_{3n}[C_{n+1}^+(0) - C_n^+ (\tau_n)] + e_{1n}[C_n^- (\tau_n) - C_{n+1}^-(0)]$$

The coefficients for even $l = 2$ up to $2N - 2$ are

$$A_l = e_{2n+1}e_{1n} - e_{3n}e_{4n+1}$$

$$B_l = e_{2n}e_{2n+1} - e_{4n}e_{4n+1}$$

$$D_l = e_{1n+1}e_{4n+1} - e_{2n+1}e_{3n+1} \qquad (42)$$

$$E_l = [C_{n+1}^+(0) - C_n^+ (\tau_n)]e_{2n+1} + [C_{n+1}^-(0) - C_n^- (\tau_n)]e_{4n+1}$$

and for $l = 2N$ the coefficients are defined as follows:

$$A_{2N} = e_{1N} - R_{sfc}e_{3N}$$
$$B_{2N} = e_{2N} - R_{sfc}e_{4N}$$
$$\qquad \qquad \qquad \qquad \qquad \qquad (43)$$
$$D_{2N} = 0$$
$$E_{2N} = S_{sfc} - C_N^+ (\tau_N) + R_{sfc}C_N^- (\tau_N)$$

In the foregoing expressions we use

$$e_{1n} = 1 + \Gamma_n \exp (- \lambda_n\tau_n)$$

$$e_{2n} = 1 - \Gamma_n \exp (- \lambda_n\tau_n)$$
$$\qquad \qquad \qquad \qquad \qquad \qquad (44)$$
$$e_{3n} = \Gamma_n + \exp (- \lambda_n\tau_n)$$

$$e_{4n} = \Gamma_n - \exp (- \lambda_n\tau_n)$$

To solve these equations, we use the standard tridiagonal solution method. Special fast tridiagonal solvers are available on many computers, but the general form is

$$AS_{2N} = A_{2N}/B_{2N} \qquad (45)$$

$$DS_{2N} = E_{2N}/B_{2N}$$

followed by a progression from $l = 2N - 1$ to $l = 1$:

$$X_l = (B_l - D_lAS_{l+1})^{-1}$$

$$AS_l = A_lX_l \qquad (46)$$

$$DS_l = (E_l - D_lDS_{l+1})X_l$$

followed by a progression from $l = 1$ to $2N$ of

$$Y_1 = DS_1$$

$$Y_l = DS_l - AS_lY_{l-1} \qquad (47)$$

We have found this solution technique to be numerically stable and computationally very efficient, especially for vector processing computers. *Wiscombe* [1977] pointed out that the matrix equation contains the solar and infrared source terms only on the right-hand side of the expression. That is, (39) explicitly contains the temperature or the solar zenith angle only in the E_l term. Hence for problems in which the temperature or solar flux are the only parameters which change, it is necessary to compute the coefficients A_l, B_l, and D_l only one time. Of course, optical depths and other parameters may depend upon temperature, lessening the utility of this approach.

In terms of the coefficients just found, the net flux and the mean intensity at the base of layer n are

$$F_{net} = Y_{1n}(e_{1n} - e_{3n}) + Y_{2n}(e_{2n} - e_{4n})$$
$$\qquad \qquad + C_n^+ (\tau_n) - C_n^- (\tau_n) - \text{direct} \qquad (48)$$

$$4\pi J_n = (1/\mu_1)[Y_{1n}(e_{1n} + e_{3n}) + Y_{2n}(e_{2n} + e_{4n})$$
$$\qquad \qquad + C_n^+ (\tau_n) + C_n^- (\tau_n)] + \text{direct}/\mu_0 \qquad (49)$$

where for solar radiation

$$\text{direct} = \mu_0\pi F_s \exp [- (\tau_c + \tau_n)/\mu_0] \qquad (50)$$

and for infrared radiation

$$\text{direct} = 0 \qquad (51)$$

THE TWO-STREAM SOURCE FUNCTION TECHNIQUE

The two-stream approximation is reasonably accurate for most uses. At solar wavelengths it reduces to an exact result in the limit that scattering does not occur and there is no reflecting surface present. At infrared wavelengths, however, the two-stream results are not exact in the limit of no scattering except for a semi-infinite atmosphere. Since the

TABLE 3. Parameters in Hemispheric Mean Two-Stream
Source Function Technique

Parameter	Expression
G	$(Y_1 + Y_2)(1/\mu_1 - \lambda)$
H	$(Y_1 - Y_2)\Gamma(\lambda + 1/\mu_1)$
J	$(Y_1 + Y_2)\Gamma(\lambda + 1/\mu_1)$
K	$(Y_1 - Y_2)(1/\mu_1 - \lambda)$
α_1	$2\pi\{B_0 + B_1[1/(\gamma_1 + \gamma_2) - \mu_1]\}$
α_2	$2\pi B_1$
σ_1	$2\pi\{B_0 - B_1[1/(\gamma_1 + \gamma_2) - \mu_1]\}$
σ_2	$2\pi B_1$

$P(\mu_1, \mu_2) = 1 + g$ in forward hemisphere, $P(\mu_1, \mu_2) = 1 - g$ in backward hemisphere, and $I_t^{\pm} = F^{\pm}/\mu_1$ used in (52).

no-scattering limit is of considerable significance at infrared wavelengths, we have sought a technique that would be exact in the limit of pure absorption but would retain the accuracy of the two-stream approach when scattering occurs.

Previously, *Toon et al.* [1977] showed that infrared intensities could be calculated with good accuracy by using the two-stream approximation to define the source function in the equation of transfer. That is, we write the source function contained in (1) as

$$S_{vt} = \frac{\omega_0}{4\pi} \int_0^{2\pi} \int_{-1}^{1} P_v(\mu, \mu', \phi, \phi') I_t \, d\mu' \, d\phi' + S_v \quad (52)$$

This expression differs from (1) in having the two-stream intensity I_t in place of the true intensity in the source function. Given choices for the phase function and the form of the two-stream approximation, S_{vt} can be found explicitly. We find that the source function for the upward and downward intensities, including (2), can be written in the form

$$S_{vt}^{+} = G \exp(\lambda\tau) + H \exp(-\lambda\tau) + \alpha_1 + \alpha_2\tau \quad (53)$$

$$S_{vt}^{-} = J \exp(\lambda\tau) + K \exp(-\lambda\tau) + \sigma_1 + \sigma_2\tau \quad (54)$$

Table 3 gives the values of G, H, J, K, α, and σ. In deriving Table 3, we used expressions for the phase function and for the angular dependence of the two-stream intensity appropriate for a hemispheric mean two-stream approximation. Other two-stream approximations would require different expressions.

With the source function specified, (1) can easily be solved to yield the azimuthally integrated intensity at an angle μ at the top ($\tau = 0$) and bottom ($\tau = \tau_n$) of a layer which comprises part of an inhomogeneous atmosphere as

$$I_n^{+}(0, \mu) = I_n^{+}(\tau_n) \exp(-\tau_n/\mu)$$

$$+ \int_0^{\tau_n} S_{vt}^{+} \exp(-t/\mu) \, dt/\mu$$

$$= I_n^{+}(\tau_n) \exp(-\tau_n/\mu) + \frac{G}{(\lambda\mu - 1)}$$

$$\cdot [\exp(-\tau_n/\mu) - \exp(-\tau_n\lambda)]$$

$$+ \frac{H}{(\lambda\mu + 1)} \{1 - \exp[-\tau_n(\lambda + 1/\mu)]\}$$

$$+ \alpha_1[1 - \exp(-\tau_n/\mu)] + \alpha_2$$

$$\cdot [\mu - (\tau_n + \mu) \exp(-\tau_n/\mu)] \quad (55)$$

$$I_n^{-}(\tau_n, -\mu) = I_n^{-}(0, -\mu) \exp(-\tau_n/\mu)$$

$$+ \int_0^{\tau_n} S_{vt}^{+} \exp\{-(\tau_n - t/\mu)\} \, dt/\mu$$

$$= I_n^{-}(0) \exp(-\tau_n/\mu) + \frac{J}{(\lambda\mu + 1)}$$

$$\cdot \{1 - \exp[-\tau_n(\lambda + 1/\mu)]\}$$

$$+ \frac{K}{(\lambda\mu - 1)} [\exp(-\tau_n/\mu) - \exp(-\tau_n\lambda)]$$

$$+ \sigma_1[1 - \exp(-\tau_n/\mu)]$$

$$+ \sigma_2[\mu \exp(-\tau/\mu) + \tau - \mu] \quad (56)$$

These expressions are used by picking an angle μ and working downward toward the surface finding the downward intensity, then applying the surface boundary condition and working back upward through the layers finding the upward intensity. Given the intensities in each layer at several angles, it is simple to compute the flux or mean intensity using quadrature.

The source function scheme just outlined has several advantages. It is computationally fast because no matrices need to be inverted. It yields accurate results for intensities in the infrared when $\omega_0 = 0$ even when reflecting surfaces are present. This approach is superior to the two-stream approximation in the infrared since the infrared is often dominated by absorption and since the approach maintains good accuracy even when scattering occurs. It is simple to show analytically that the scheme is energy conserving. For example, the sum of the emissivity and reflectivity equals unity for a single isothermal slab overlying an emitting surface. The scheme is not as useful for solar radiation, and we have therefore not presented the coefficients needed to treat solar radiation. At solar wavelengths the two-stream results are exact when $\omega_0 = 0$ so long as no reflecting ground is present so increased accuracy in this limit often is not needed. Also, the scattering limit is more common at solar wavelengths. Unfortunately, when $\omega_0 = 1$, the approach does not necessarily yield flux conservation at solar wavelengths (although flux is always conserved at infrared wavelengths). Moreover, the solar flux in the scattering limit is not calculated significantly more accurately with the source function technique than with the two-stream approach. We feel therefore that for the calculation of fluxes the source function technique is of value mainly in the infrared. In the infrared and solar the approach does yield a useful approximation to intensities and can be used to obtain quantities such as the geometric albedo that cannot be found with the two-stream approximation [*Toon et al.*, 1977, also unpublished manuscript, 1989]. *Davies* [1980] also explored a similar source function scheme at solar wavelengths and concluded it could be a useful approximation for intensities.

For cases in which solar and infrared emission overlap, two choices are possible. The standard two-stream technique could be used with the sum of the infrared and solar source functions. Or, since light waves do not interact in atmospheric situations, the two beams could be treated separately: the solar with the two-stream approach and the infrared with the source function technique.

ACCURACY REQUIREMENTS

Even if it were computationally feasible to employ exact radiative transfer schemes in complex dynamical or photochemical models, there would be little advantage in doing so because of other error sources. Our criteria for determining the accuracy required of the radiative code is that the radiative code should produce an error which is smaller than that produced by other error sources. These error sources include uncertainties in input parameters to the radiative transfer schemes and approximations to the structure of the atmosphere. It is also possible that so little energy may be available that photolysis or heating do not occur at a significant rate so that large fractional errors in the radiation field calculation are not important. Determining when the heating or photolysis is small enough so the the errors may be ignored is not always simple and may depend upon the time scale of the calculation. Several authors have attempted to treat this problem by evaluating the errors as a fraction of the incident energy. This error is referred to as the relative error. When either the absorption, reflection, or transmission is small, the maximum relative error in the small quantity is always small. Unfortunately, such definitions can produce very misleading error estimates. For example, if only 10% of the solar energy were absorbed, and the radiation calculation indicated that 20% were absorbed, then the fractional error calculated this way would be only 10%. If 10% of the solar energy incident were absorbed in a 100-mbar-thick layer it would result in an instantaneous heating rate of about 10 K per day. Clearly, if even 1% of the solar energy is absorbed, it is quite adequate to produce significant heating. Since small fractions of the incident energy can be important, we prefer to define errors normalized upon the exact result, not upon the incident flux. That is, if A_e is the exact value and A_c is the calculated value then we base our errors on $(A_e - A_c)/A_e$. We refer to this error as the fractional error. We expect such errors to be moderate even for relatively small values of A_e.

Using the criteria outlined above, we estimate the accuracy required for calculating the mean intensity by noting that the photolysis rate is evaluated by multiplying together the cross section, the quantum yield, the species concentration, and the mean intensity. According to a recent compilation of cross sections and quantum yields [DeMore et al., 1985], the combined uncertainty of cross sections and quantum yields was at least 10%. The abundances of the species being photolyzed can rarely be measured to 10%. Given these two error sources, a radiative transfer scheme whose basic accuracy for mean intensities is about 10% should be sufficiently accurate for calculating photorates.

Ronnholm et al. [1980] compared the errors in solar energy deposition calculations due to using approximate radiative transfer schemes with the errors caused by uncertainties in the scattering properties (single scattering albedo, asymmetry parameter, and optical depth) and with the errors caused by using an average value of the property of interest.

Ronnholm et al. [1980] found that the largest errors were generated by replacing a horizontally variable cloud or aerosol field with a field having an average optical depth. Particularly for clouds having a few holes, the errors in using the average optical depth were very large, exceeding several hundred percent in some cases. Even for horizontal variation in the optical depth of a few percent, errors in the radiation field of 5% may occur for small optical depths, and errors of up to 30% or more may occur for large cloud optical depths.

Ronnholm et al. [1980] also showed that 1% errors in the single scattering albedo, optical depth, or asymmetry parameter may be significant. For example, a nearly conservatively scattering cloud of either large or small optical depth can have a several hundred percent error in heating rates for a 1% error in the single scattering albedo. One percent errors in optical depth lead to a 1% error in heating rate for a variety of single scattering albedos. Since observational uncertainty in optical depth of 10% is common for aerosols, clouds, and most gases, similar sized errors in calculating the solar radiation field are acceptable.

Several authors have estimated the acceptable error for computing the infrared cooling rate in cloud free atmospheres. Rodgers and Walshaw [1966] find that random errors in temperature and dew point of 1°C were common in radiosonde data and that such errors led to ±5% errors in cooling rates. Ellingson and Gille [1978] felt that 0.4°C temperature errors and 10% humidity errors are typical. Such errors are found to cause 1–2% errors in the flux leaving the atmosphere and in the flux reaching the surface. Random errors of 10% in humidity led to 10% errors in cooling rate while random temperature errors of 0.4°C led to 3% errors in cooling rates. Such studies suggest that errors of a few percent for clear sky radiances are desirable to achieve.

Little work has been done on the errors arising from scattering at infrared wavelengths since terrestrial clouds are predominantly absorbers. Since scattering is completely ignored by most modelers, we believe that calculating the infrared fluxes and heating rates when scattering occurs to about 5% should be adequate.

Generally speaking, the error margins at infrared wavelengths are smaller than at solar wavelengths. In the lower atmosphere the infrared cooling rate is several times larger than the solar heating rate. Hence to maintain a comparable contribution to the error in net heating, the infrared cooling rate must be determined more precisely than the solar heating rate.

Although the accuracy of the radiation scheme can be judged by examining the errors from a variety of cases, it is also desirable that certain limiting cases be treated exactly so that results which are not physically possible do not occur. For example, no absorption should occur when $\omega_0 = 1$; when the optical depth is infinite and $\omega_0 = 0$, no reflection or transmission should occur. We have developed a radiative transfer scheme which satisfies exactly the well-known limiting cases (Table 2).

TESTS OF ACCURACY

There are several ways to judge the accuracy of a radiative transfer technique. One popular method is to compare approximate and accurate calculations of the radiative pa-

TABLE 4. Errors in Diffusively Transmitted and Reflected Boundary Fluxes

	Case Number				
	1	2	3	4	5
Parameters for Case					
μ_0	1	1	0.5	1	1
τ	1	1	1	64	64
ω_0	1	0.9	0.9	1	0.9
g	0.794	0.794	0.794	0.848	0.848
Results for Case					
$F^+(0)$ exact	0.173	0.124	0.226	2.662	0.376
$F^+(0)$ approximate	0.174	0.133	0.221	2.686	0.376
% error (F^+)	0.6	7	2	0.9	0
$F^-(\tau)$ exact	1.813	1.516	0.803	0.480	0.0000
$F^-(\tau)$ approximate	1.812	1.522	0.864	0.455	0.000
% error (F^-)	0	0.3	8	5	0

In all of the cases, $R_{sfc} = 0$, and $F_s = 1$ so that the incident solar flux equals π. Exact values are from *Wiscombe* [1977] or *Lenoble* [1985].

rameter of interest, such as the wavelength-integrated solar heating rate. One problem with this approach is that the two calculations may differ for many reasons, making it difficult to understand the origins of errors. They may use different absorption coefficients or wavelength intervals, for example. Also, wavelength integration may average intervals with large, compensating errors to yield a final result with only a small error thus causing errors to be overlooked. Therefore we prefer the alternate approach of contrasting exact and approximate calculations both using well-defined optical properties in which no averaging has taken place.

The accuracy of the single-layer two-stream approximation for the calculation of monochromatic absorption, reflectivity, and transmission has been extensively examined at solar wavelengths [*Meador and Weaver*, 1980; *King and Harshvardhan*, 1986; *Cuzzi et al.*, 1982]. We shall not present detailed results for this case due to the large amount of previous study. In general, two-stream results such as

those from the δ-Eddington or δ-quadrature technique are accurate to 10% or better. However, these schemes do occasionally produce larger errors for special cases such as high solar zenith angles or for small values of the quantity being considered. The two-stream approximation for a single layer is thus generally, but not always, suitable.

One property of the matrix solution for multiple layers is that the emergent flux from a set of layers, identical except for their optical depths, is the same as the emergent flux from a single layer having an optical depth which equals the sum of the optical depths of the layers. Therefore simulations for single-layer solutions give an indication of the errors in multiple-layer simulations of the emergent flux.

Wiscombe [1977] and *Lenoble* [1985] have performed simulations to look at the errors in the internal fields using multiple-layer solutions. Tables 4 and 5 indicate the errors (noted in parentheses in Table 5) for cases representing typical hazes and clouds. In the approximate calculations we

TABLE 5. Errors in Net Flux and Flux Divergence for Multiple Layers

τ Level	Case 1		Case 2		Case 3		Case 4		Case 5	
	Flux	Divergence	Flux	Divergence	Flux	Divergence	Flux	Divergence	Flux	Divergence
$\tau = 0$	2.968	0	3.018	0.0017	1.345	0.019	0.480	0	2.776	1.161
$\tau = 0$	2.967	0	3.009	0.0017	1.350	0.018	0.455	0	2.766	1.161
$\tau = 0$	(0.03)	(0)	(0.3)	(0)	(0.3)	(5)	(5)	(0)	(0)	(0)
$\tau/20$	2.968	0	3.001	0.0018	1.326	0.020	0.480	0	1.605	0.775
$\tau/20$	2.967	0	2.992	0.0017	1.332	0.017	0.455	0	1.605	0.783
$\tau/20$	(0.03)	(0)	(0.3)	(5)	(0.5)	(15)	(5)	(0)	(0)	(1)
$\tau/10$	2.968	0	2.983	0.0035	1.306	0.040	0.480	0	0.830	0.634
$\tau/10$	2.967	0	2.975	0.0034	1.315	0.032	0.455	0	0.822	0.639
$\tau/10$	(0.03)	(0)	(0.3)	(3)	(0.7)	(20)	(5)	(0)	(1)	(1)
$\tau/5$	2.968	0	2.948	0.1060	1.266	0.109	0.480	0	0.196	0.194
$\tau/5$	2.967	0	2.941	0.1010	1.283	0.088	0.455	0	0.183	0.182
$\tau/5$	(0.03)	(0)	(0.2)	(4)	(1)	(19)	(5)	(0)	(7)	(6)
$\tau/2$	2.968	0	2.842	0.0870	1.157	0.078	0.480	0	0.0022	0.0021
$\tau/2$	2.967	0	2.840	0.0820	1.195	0.064	0.455	0	0.0013	0.0013
$\tau/2$	(0.03)	(0)	(0.1)	(6)	(1)	(13)	(5)	(0)	(40)	(39)
$3\tau/4$	2.968	0	2.755	0.0840	1.079	0.063	0.480	0	0.0001	0.0001
$3\tau/4$	2.967	0	2.758	0.0810	1.131	0.054	0.455	0	0.00002	0.00002
$3\tau/4$	(0.03)	(0)	(0.1)	(4)	(5)	(14)	(5)	(0)	(80)	(80)
τ	2.968		2.671		1.016		0.480		0.0000	
τ	2.967		2.677		1.077		0.455		0.0000	
τ	(0.03)		(0.2)		(6)		(5)		(0)	

First value is exact, second value is δ-quadrature result, and percent error is in parentheses. Net flux is at optical depth indicated. Divergence is for layer between adjacent optical depth levels. Cases and τ are defined in Table 4.

TABLE 6. Mean Intensities for Conservative Rayleigh Scatter

μ_0	τ	$R_{sfc} = 0$		% Error	$R_{sfc} = 0.25$		% Error	$R_{sfc} = 0.80$		% Error
		Exact	Approximate		Exact	Approximate		Exact	Approximate	
					$J(0)/\pi F_s$					
0.1	0.02	1.045	1.015	3	1.089	1.054	3	1.187	1.141	4
0.1	0.25	1.170	1.08	7	1.189	1.104	7	1.239	1.153	7
0.1	1.00	1.212	1.119	8	1.220	1.127	7	1.247	1.156	7
0.4	0.02	1.047	1.017	3	1.235	1.184	4	1.653	1.556	6
0.4	0.25	1.284	1.164	9	1.402	1.278	9	1.707	1.570	8
0.4	1.00	1.534	1.374	10	1.584	1.422	10	1.778	1.591	10
0.92	0.02	1.040	1.017	2	1.477	1.406	5	2.453	2.274	7
0.92	0.25	1.279	1.191	7	1.597	1.493	6	2.404	2.266	6
0.92	1.00	1.691	1.572	7	1.851	1.727	7	2.398	2.268	5
					$J(\tau)/\pi F_s$					
0.10	0.02	0.864	0.834	3	0.912	0.875	4	1.018	0.964	5
0.10	0.25	0.192	0.156	18	0.224	0.183	18	0.307	0.253	17
0.10	1.0	0.057	0.054	4	0.082	0.077	6	0.168	0.156	7
0.40	0.02	0.998	0.968	3	1.203	1.141	5	1.661	1.526	8
0.40	0.25	0.787	0.693	12	0.988	0.856	13	1.502	1.274	15
0.40	1.00	0.385	0.344	10	0.540	0.476	12	1.071	0.938	12
0.92	0.02	1.018	0.996	2	1.495	1.398	6	2.561	2.296	10
0.92	0.25	1.028	0.950	7	1.560	1.382	11	2.928	2.491	15
0.92	1.00	0.881	0.822	7	1.384	1.245	12	3.109	2.726	12

used the δ-quadrature technique. *Wiscombe* [1977] presents the errors for the δ-Eddington scheme which are of the same magnitude as those presented here. The errors in the internal heating rates and fluxes are similar to the errors for the emergent flux in single-layer cases. That is, the errors are typically less than 10% unless the flux or heating rate is 1% or less than the incident flux. For small heating rates or fluxes the fractional error can be large.

Table 6 presents a series of intercomparisons of the mean intensity for a conservative Rayleigh scattering atmosphere calculated using the δ-quadrature approach and calculated exactly. The error in the mean intensity is generally less than about 10%, although for high solar zenith angles and moderate optical depths the error rises to almost 20%. These tests were used by *Yung* [1976] for comparison with a technique he devised which could in principle be made exact for this limiting case. It should be noted that our approach as well as most approximations (other than Yung's) used in

TABLE 7. Maximum Errors in Calculating the Mean Intensity

Asymmetry Parameter	Optical Depth	Single Scattering Albedo			
		0.99	0.9	0.6	0.2
0	0.03125	4/4	3/4	2/2	1/1
0	0.25	9/15	7/13	5/8	1/2
0	1.00	9/9	7/6	4/20	1/33
0	8	7/3	6/3	3/50	1/-
0	32	7/-	6/-	3/-	1/-
0.75	1	14/10	12/10	8/15	2/8
0.75	4	13/16	12/17	7/14	2/34
0.75	8	10/18	12/21	7/25	2/-

The largest fractional error expressed in percent at any value of μ_0 in the range 0.1–1.0 is presented. The surface is assumed to have no reflectivity. There is a downward solar component whose mean intensity is unity. The exact values were taken from *Van de Hulst* [1980]. The first value is the maximum error in the mean intensity at the top of the layer, and the second value is the maximum error in the mean intensity at the bottom of the layer. Hence 4/4 means 4% error at top and 4% error at bottom.
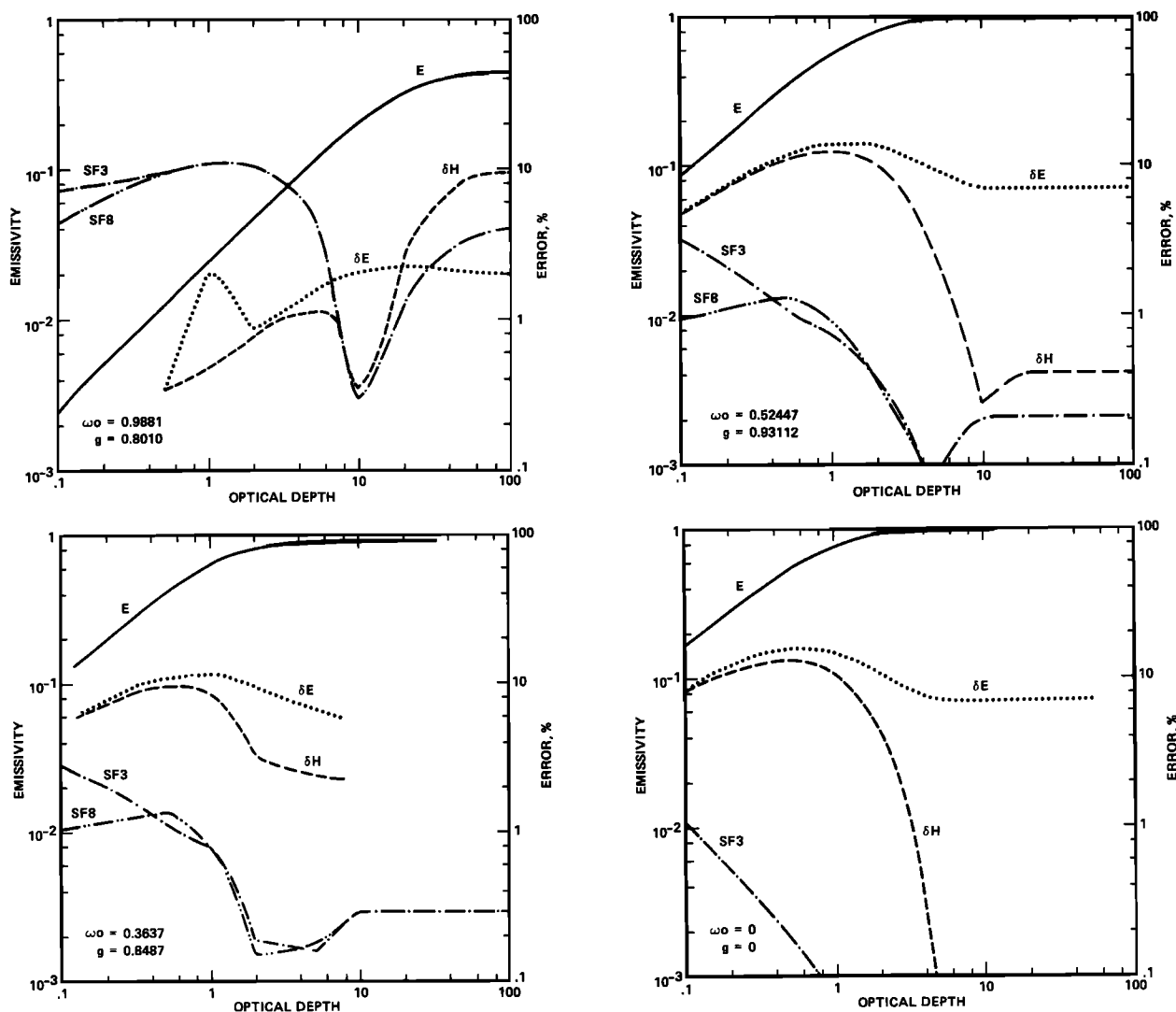
photochemical models treat Rayleigh scattering as unpolarized isotropic scattering. Such an approximation can never be expected to be much better than about 10%.

Table 7 illustrates comparisons of the mean intensity calculated using the δ-quadrature technique and calculated exactly [*Van de Hulst*, 1980] for isotropic and for strong forward scattering. Table 7 presents the largest fractional error at any solar zenith angle for a range of optical depths and single scattering albedos. As in Table 6, one occasionally finds errors for small optical depths that exceed 10%. These errors occur at large solar zenith angles. For large optical depths the transmitted mean intensity becomes small and again errors begin to grow. For example, for an optical depth of 1, $\omega_0 = 0.6$, isotropic scattering and $\mu_0 = 0.1$, the transmitted mean intensity is 1.6% of the incident mean intensity, and a 20% error occurs in the approximate technique. However, when the zenith angle increases to 0.5, the transmitted mean intensity has increased to 25% of the incident mean intensity and the error is about 1%.

We conclude that the two-stream technique can be used for the calculation of mean intensities and that it has similar errors to those found in using the two-stream method for finding fluxes. That is, the error is generally less than 10% but may exceed 10% if the solar zenith angle is large or if the quantity being calculated is small.

The two-stream technique can also be used to find fluxes in the infrared. *Hunt* [1973] examined a number of cases chosen to represent highly forward scattering isothermal water clouds. Figure 1 illustrates the emissivity of three of these cases which span a range of single scattering albedos. We illustrate the δ-hemispheric mean, the δ-Eddington, and the source function technique using a δ-hemispheric mean two-stream scattering intensity. As we pointed out previously, the δ-Eddington approach can result in physically unrealistic results such as emissivities larger than 1. However, the magnitude of its error is not large. The worst errors for the source function technique are of the order of 11% and occur for the largest values of ω_0 when the emissivity itself

Fig. 1. Shown in each segment of the figure are the emissivity (E) for an isothermal multiple-scattering layer as calculated precisely [*Hunt*, 1973] as well as the errors in calculating the emissivity with the δ-hemispheric mean technique (δH), the δ-Eddington technique (δE), and the source function technique in which the δ-hemispheric mean approach is used to obtain the two-stream approximation to the source function. For the source function technique, results are shown for Gauss quadratures using both three (SF3) and eight (SF8) Gauss points to obtain the fluxes. The left-hand scale applies to the emissivity and the right-hand scale to the error (in percent). The various panels cover a range of values for the asymmetry parameter and the single scattering albedo. The sign of the error is a function of both the technique and the optical depth. In order to emphasize the magnitude of the error in a clear manner, only the absolute value of the error is shown.

is less than 10%. For this case the δ-hemispheric mean and especially the δ-Eddington approach are more accurate than the source function technique with the δ-Eddington technique having errors that are always less than 3%. As the single scattering albedo drops, the error in the source function technique rapidly decreases to a few percent or less, while the error in the δ-hemispheric mean and δ-Eddington schemes rise into the range of 10–15%. Figure 1 illustrates the error in the emissivity for the source function technique when three and eight Gauss quadrature points are used to determine the flux from the intensity. Good accuracy is obtained with only three Gauss points.

We have performed similar tests for isotropic scattering with the same basic conclusions. For single scattering albedos larger than 0.5 the δ-hemispheric mean and the δ-Eddington solution are more accurate, while for ω_0 less the

0.5 the source function technique is more accurate. For large single scattering albedo when the emissivity is only a few percent, the source function technique has its largest error of order 11%. Generally, the source function technique is accurate to a few percent, and for pure absorption it is arbitrarily precise depending upon the number of Gauss points used. The use of three Gauss points is usually sufficient for accuracy of a few percent.

In addition to the emissivity one may also be interested in the reflectivity or transmissivity of a layer. For infrared calculations, unlike solar ones, any reflected or transmitted light may be overwhelmed by emitted radiation. Hence to understand the magnitude of the actual error, one must examine the accuracy with which the total upward or downward radiation can be calculated. Since the total radiation depends upon the magnitude of the incident light, this error
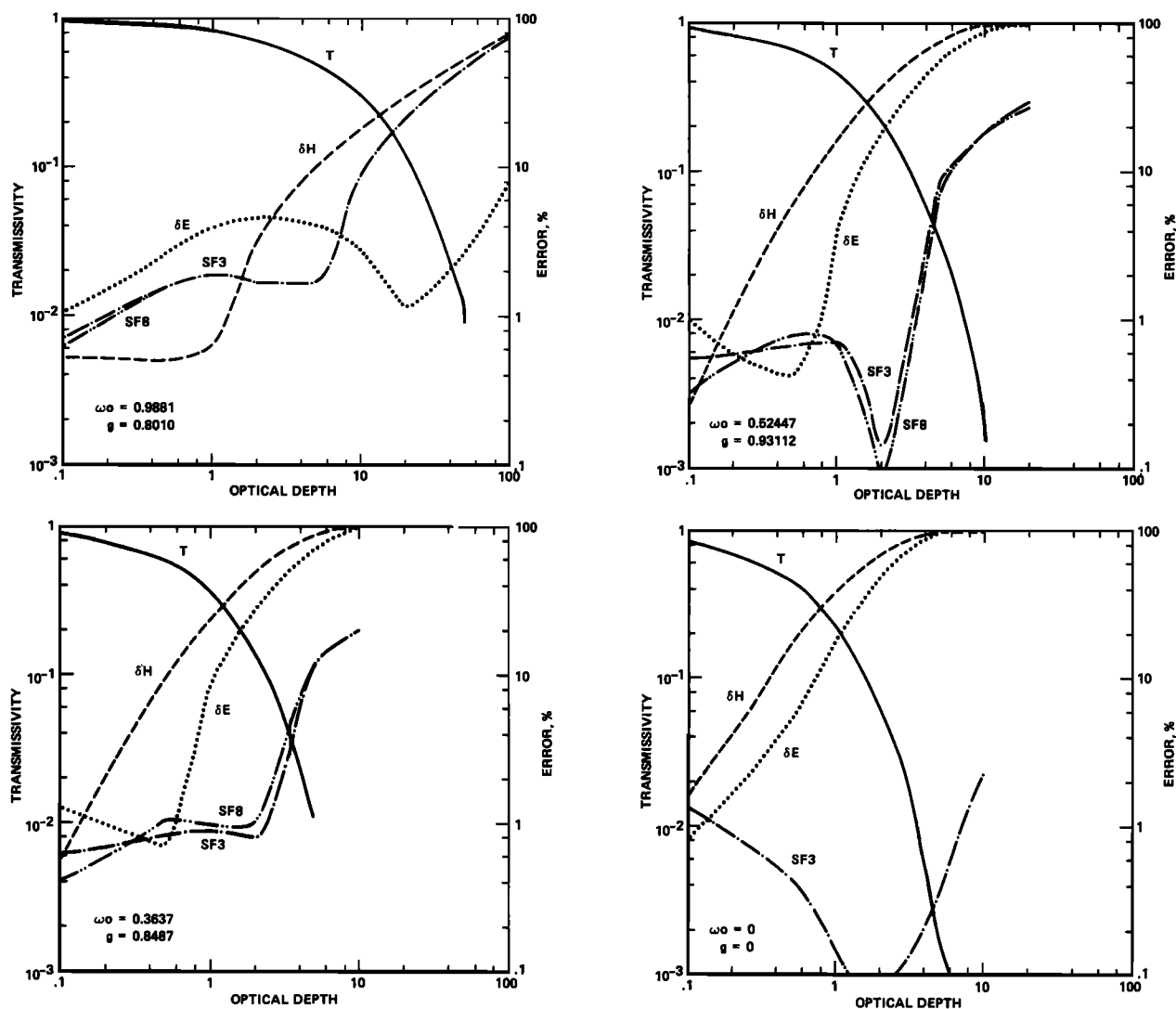
Fig. 2. The transmissivities corresponding to the same conditions described for Figure 1. When the single scattering albedo is set to zero and eight Gauss points are used, the error in finding the transmissivity is less than 0.1% and so is not illustrated. The exact transmissivity is noted by T, δ-Eddington error is noted by δE, δ-hemispheric mean error is noted by δH, source function error with three Gauss points is marked by SF3, and source function error with eight Gauss points is marked with SF8.

will depend upon the circumstances considered. Figures 2 and 3 illustrate the transmission and reflection, respectively, from the same cloud layers for which the emissivity was illustrated in Figure 1. Again, the δ-hemispheric mean or δ-Eddington techniques are more accurate for the most highly scattering cloud. However, for more absorbing clouds the source function technique is consistently more accurate.

We conclude from these tests that the two-stream methods can generally meet the accuracy requirements for calculating the heating rate, reflectivity, transmissivity, or mean intensity at visible and infrared wavelengths. At infrared wavelengths one must use the hemispheric mean approach, as opposed, for example, to the δ-Eddington technique, to avoid physically unrealistic answers, and in general, it is best to use the source function technique.

APPLICATIONS OF THE ALGORITHMS

We developed these algorithms with a number of diverse applications in mind. *Toon et al.* [1987] employed the δ-

quadrature method in order to calculate the photorates in a one-dimensional photochemical model used to investigate sulfur chemistry of the marine troposphere. We have found the source function technique to be useful for calculation of infrared intensities for remote sensing of planetary atmospheres [*Toon et al.*, 1977] and for calculating quantities related to the solar intensity, such as the geometric albedo of a planet, which cannot be found using the two-stream approximation (O. B. Toon et al., unpublished manuscript, 1989). *Pollack and McKay* [1985] used the scheme to calculate the heating rates in polar stratospheric clouds including scattering in the infrared. *Ackerman et al.* [1988] employed the infrared scheme with scattering included to investigate the energy balance of cirrus clouds. We are also using the two-stream approximation at solar and infrared wavelengths in simulations of the energy balance of Mars, in a three-dimensional model of the Earth's stratospheric circulation, and in a three-dimensional model of the Earth's regional scale circulation.
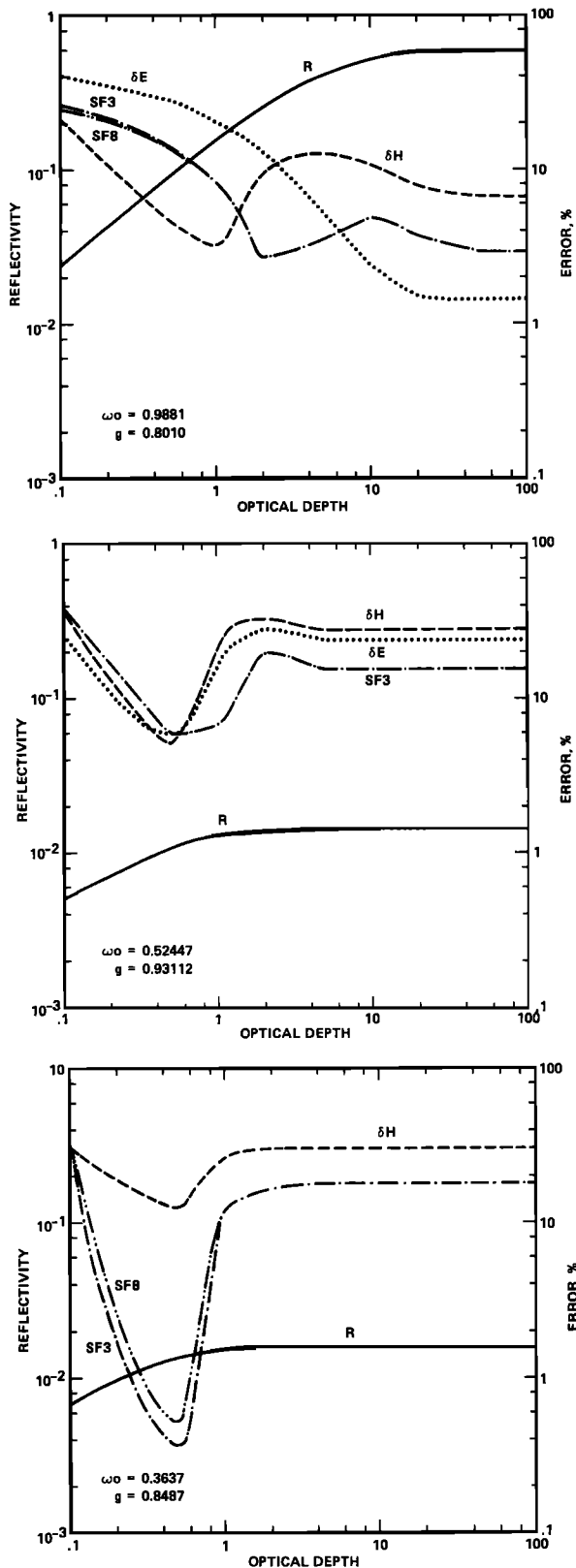
Fig. 3. The reflectivities corresponding to the same conditions described for Figure 1. Note that the reflectivity is exactly zero when the single scattering albedo is zero, so no results are shown for that case. For the case with $\omega_0 = 0.3637$, the error in finding the reflectivity with the δ-Eddington approach is greater than 100% and so is not illustrated. The exact reflectivity is noted by R, δ-Eddington error is noted by δE, δ-hemispheric mean error is noted by δH, source function error with three Gauss points is marked by SF3, and source function error with eight Gauss points is marked with SF8.

TABLE 8. Timing of Sample Problem

Case	Computation Time on Cray-XMP, ms			
	Optical Properties*	Two-Stream†	Source Function ‡	Total
Solar only	1	3.2	⋯	4.2
Solar + IR absorption	3.8	3.2	1.2	8.5
IR absorption	2.3	⋯	1.3	3.6
IR scattering	2.3	2.2	2.3	6.8

*This portion of the code computes the optical properties of the layers, such as optical depth, and the Planck function.

†This portion of the code computes the coefficients for the two-stream solutions and solves the tridiagonal matrix.

‡This portion of the code makes use of the source function technique at infrared wavelengths.

Each application of our algorithms requires different absorption coefficients and other optical properties. The choices of these parameters and others, such as the vertical resolution, all impact the ultimate accuracy of the results. Here we shall not focus on the accuracy of the results, which we examined in a general way in the previous section, but rather on the numerical efficiency of the algorithm for performing the calculations.

In order to indicate the efficiency of the algorithm, we consider one sample application of the code to calculate the solar and infrared energy deposition in the Earth's atmosphere. In this calculation the model atmosphere was divided into 18 vertical levels. We used 26 wavelength intervals in the solar and 18 in the infrared. Within each wavelength interval an appropriate number of terms in an exponential sum formulation was used to represent the gaseous absorption and to treat overlap [*Hansen et al.*, 1983] resulting in 77 solar intervals and 71 infrared intervals. Therefore in total we performed 148 spectral calculations for each vertical column in order to obtain the heating rate. In the infrared we used three Gauss quadrature points to find the fluxes from the intensities in the source function technique.

Table 8 presents the timing of the calculations on a Cray-XMP computer for a variety of cases. A photochemical calculation involving 77 spectral intervals would require similar computational time to that used in the solar-only case in Table 8. Since most dynamical models require several seconds of computer time per time step, one could afford to perform three-dimensional photorate calculations, even with this relatively large number of spectral intervals, on grids having 10^3 grid points before the computation time for the radiation became comparable to that for the dynamics. Most general circulation models run a radiation code involving solar radiation (usually without scattering) and infrared without scattering which is comparable to the second case described in Table 8. *Stephens* [1984] states that the European Centre for Medium Range Weather Forecasts model with nine vertical levels and five spectral intervals, requires 2.5 ms per time step per grid point on a Cray 1 computer. Our model with the same number of vertical levels would require about 4.2 ms but with much higher spectral resolution. We are not aware of timings of other such codes on the same computers. Accurate timing comparisons are difficult to make unless they are performed upon a single computer. The important point is simply that our code is competitive in

speed with existing, often less flexible, approximations that are used in three-dimensional models.

Table 8 indicates that in general a significant portion of the total computation time is spent in computing the optical depths and scattering parameters such as single scattering albedo. Further, the time spent in the infrared portion of this code is comparable to that spent in the solar part. Treating scattering in the infrared doubles the amount of time needed to compute the infrared energy deposition.

We have run this radiation code in a 14 vertical level version of the Pennsylvania State University/National Center for Atmospheric Research three-dimensional regional scale model with a high-resolution boundary layer scheme in an application similar to that described by *Westphal et al.* [1988]. In these dynamical calculations we transported aerosols as well as calculating their optical properties and impact on the radiation field once each hour. During a 24-hour simulation the calculation of the radiation heating rates consumed about 10% of the total computational time, and the calculation of the optical properties consumed about 20% of the computational time. The optical properties calculation was time consuming since Mie scattering calculations over the aerosol size distribution were performed at each of the 44 wavelengths and 14 vertical levels. In this calculation the ratio of the number of calls to the radiation routines to the number of calls to the dynamical routines was about 1/22. Some dynamical simulations use a larger time step and may require a higher frequency of radiation calculations relative to dynamical calculations. We could have called the radiation code with a full recalculation of the optical properties every eighth dynamical time step or without a full recalculation of the optical properties every third dynamical time step and then spent 50% of the total computational time on radiation. Of course, we could also have reduced the amount of spectral resolution or simplified the calculation of the optical properties to have reduced the computational time.

SUMMARY

We have described an algorithm suitable for computing photorates, solar energy deposition rates, or infrared energy deposition rates in vertically inhomogeneous, multiple-scattering atmospheres. The code makes use of the generalized two-stream solutions of *Meador and Weaver* [1980] and can therefore be used with any of the standard two-stream approaches such as the δ-Eddington or δ-quadrature schemes. The algorithms are numerically stable, and we have developed simple techniques to solve difficulties which occur, for example, with large optical depths. The algorithm uses a tridiagonal solution technique. Therefore the computational time is linear in the number of grid points used making it especially attractive for use in models with large numbers of vertical levels.

We show that the accuracy of the model for multiple layers is usually better than 10% for heating rates or photorates which is sufficient for most atmospheric applications. However, as has been pointed out many times by others for single homogeneous layers, there are cases, especially when the solar zenith angle is large or when the quantity being computed is of small magnitude, for which errors can rise above 10%. Numerous comments in the photochemical literature concerning the lack of accuracy of two-stream solutions seem to be biased by a failure to distinguish

between the many different types of two-stream solutions. We show that the mean intensity (needed to find the photorate) may be found using the same standard techniques that are employed in computing solar energy deposition, such as the δ-Eddington approximation, with an accuracy that is competitive with other approximations used in the photochemical literature. However, the two-stream technique has the advantage of being able to treat anisotropic scattering while most of the photochemical approximations are limited to isotropic scattering.

Infrared radiation is not well treated by some two-stream techniques, such as δ Eddington or δ quadrature, because they can produce emissivities greater than unity. The δ-hemispheric mean two-stream approach is preferable since it does not yield emissivities that exceed unity. Infrared radiation is best treated using the two-stream source function technique which we describe because that technique is exact in the often encountered limit of pure absorption yet can also produce results which are as accurate as those found using two-stream approximations when scattering occurs.

REFERENCES

Ackerman, T. P., K. N. Liou, F. P. J. Valero, and L. Pfister, Heating rates in tropical anvils, *J. Atmos. Sci.*, *45*, 1606–1623, 1988.

Anderson, D. E., Jr., and R. R. Meier, Effects of anisotropic multiple scattering on solar radiation in the troposphere and stratosphere, *Appl. Opt.*, *18*, 1955–1960, 1979.

Augustsson, T. R., and J. S. Levine, The effects of isotropic multiple scattering and surface albedo on the photochemistry of the troposphere, *Atmos. Environ.*, *16*, 1373–1380, 1982.

Boughner, R. E., A rapid and accurate method for the calculation of stratospheric photolysis rates with molecular scattering, *J. Geophys. Res.*, *91*, 13,187–13,194, 1986.

Cuzzi, J. N., T. P. Ackerman, and L. C. Hemle, The delta-four stream approximation for radiative transfer, *J. Atmos. Sci.*, *39*, 917–929, 1982.

Davies, R., Fast azimuthally dependent model of the reflection of solar radiation by plane-parallel clouds, *Appl. Opt.*, *19*, 250–255, 1980.

DeMore, W. B., J. J. Margitan, M. J. Molina, R. T. Watson, D. M. Golden, R. F. Hampson, M. J. Kurylo, C. J. Howard, and A. R. Ravishankara, Chemical kinetic and photochemical data for use in stratospheric modeling: Evaluation number 7, *Publ. 85-37, Jet Propul. Lab.*, Calif. Inst. of Technol., Pasadena, 1985.

Ellingson, R. G., and J. C. Gille, An infrared radiative transfer model, I, Model description and comparison of observations with calculations, *J. Atmos. Sci.*, *35*, 523–545, 1978.

Hansen, J. E., G. Russell, D. Rind, P. Stone, A. Lacis, S. Lebedeff, R. Ruedy, and L. Travis, Efficient three-dimensional global models for climate studies: Models I and II, *Mon. Weather Rev.*, *111*, 609–662, 1983.

Harshvardhan, R. Davies, D. A. Randall, and T. G. Corsetti, A fast radiation parameterization for atmospheric circulation models, *J. Geophys. Res.*, *92*, 1009–1016, 1987.

Hunt, G. E., Radiative properties of terrestrial clouds at visible and infrared thermal wavelengths, *Q. J. R. Meteorol. Soc.*, *99*, 346–369, 1973.

Joseph, J. H., W. J. Wiscombe, and J. A. Weinman, The delta-Eddington approximation for radiative transfer, *J. Atmos. Sci.*, *33*, 2452–2459, 1976.

King, M. D., and Harshvardhan, Comparative accuracy of selected multiple scattering approximations, *J. Atmos. Sci.*, *43*, 784–801, 1986.

Lacis, A. A., and J. E. Hansen, A parameterization for the absorption of solar radiation in the Earth's atmosphere, *J. Atmos. Sci., 31*, 118–133, 1974.

Lenoble, J., *Radiative Transfer in Scattering and Absorbing Atmospheres: Standard Computational Procedures*, 300 pp., A. Deepak, Hampton,Va., 1985.

Liou, K. N., Analytic two-stream and four-stream solutions for radiative transfer, *J. Atmos. Sci., 31*, 1473–1475, 1974.

Luther, F. M., Annual report of Lawrence Livermore Laboratory to the FAA on the High Altitude Pollution Program-1980, *Publ. UCRL-50042-80*, Lawrence Livermore Lab., Livermore, Calif., 1980.

Meador, W. E., and W. R. Weaver, Two-stream approximations to radiative transfer in planetary atmospheres: A unified description of existing methods and a new improvement, *J. Atmos. Sci., 37*, 630–643, 1980.

Meier, R. R., D. E. Anderson, Jr., and M. Nicolet, Radiation field in the troposphere and stratosphere from 240–1000 nm, I, General analysis, *Planet. Space Sci., 30*, 923–933, 1982.

Pollack, J. B., and C. P. McKay, The impact of polar stratospheric clouds on the heating rates of the winter polar stratosphere, *J. Atmos. Sci., 42*, 245–262, 1985.

Rodgers, C. D., and C. D. Walshaw, The computation of infrared cooling rates in planetary atmospheres, *Q. J. R. Meteorol. Soc., 92*, 67–92, 1966.

Ronnholm, K., M. B. Baker, and H. Harrison, Radiative transfer through media with uncertain or variable parameters, *J. Atmos. Sci., 37*, 1279–1290, 1980.

Shettle, E. P., and J. A. Weinman, The transfer of solar irradiance through inhomogeneous turbid atmospheres evaluated by Eddington's approximation, *J. Atmos. Sci., 27*, 1048–1055, 1970.

Stamnes, K., S. Tsay, W. Wiscombe, and K. Jayaweera, Numerically stable algorithm for discrete-ordinate-method radiative transfer in multiple scattering and emitting layered media, *Appl. Opt., 27*, 2502–2509, 1988.

Stephens, G. L., The parameterization of radiation for numerical weather prediction and climate models, *Mon. Weather Rev., 112*, 826–867, 1984.

Thompson, A. M., The effects of clouds on photolysis rates and ozone formation rates in the unpolluted troposphere, *J. Geophys. Res., 89*, 1341–1349, 1984.

Toon, O. B., J. B. Pollack, and C. Sagan, Physical properties of particles composing the great Martian dust storm of 1971–1972, *Icarus, 30*, 663–696, 1977.

Toon, O. B., J. F. Kasting, R. P. Turco, and M. S. Liu, The sulfur cycle in the marine atmosphere, *J. Geophys. Res., 92*, 943–963, 1987.

Van de Hulst, H. C., *Multiple Light Scattering: Tables, Formulas, and Applications*, vols. 1 and 2, 739 pp., Academic, San Diego, Calif., 1980.

Westphal, D. L., O. B. Toon, and T. N. Carlson, A case study of transport and mobilization of Saharan dust, *J. Atmos. Sci., 45*, 2145–2175, 1988.

Wiscombe, W. J., Extension of the doubling method to inhomogeneous sources, *J. Quant. Spectrosc. Radiat. Transfer, 16*, 477–489, 1976.

Wiscombe, W. J., The delta-Eddington approximation for a vertically inhomogeneous atmosphere, *Tech. Note TN-121 + STR*, Natl. Cent. for Atmos. Res., Boulder, Colo., 1977.

Yung, Y. L., A numerical method for calculating the mean intensity in an inhomogeneous Rayleigh scattering atmosphere, *J. Quant. Spectrosc. Radiat. Transfer, 16*, 755–761, 1976.

T. P. Ackerman, Department of Meteorology, Pennsylvania State University, University Park, PA 16802.

C. P. Mckay and O. B. Toon, Space Science Division, NASA Ames Research Center, M.S. 245-3, Moffett Field, CA 94035.

K. Santhanam, Computer Sciences Corporation, 700 East El Camino Real, Mountain View, CA 94040.