Name: Asadullah Hill Galib

CSE 847

HW 2

Linear Algebra

(1)

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Here, $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(2-3\lambda+\lambda^2-0) - 1(1-\lambda-0) + 0(0-0) = 0$$

$$\Rightarrow (4-8\lambda+5\lambda^2-\lambda^3) - 1(1-\lambda) = 0$$

$$\Rightarrow -\lambda^3+5\lambda^2-7\lambda+3 = 0$$

$$\Rightarrow -(\lambda-1)(\lambda^2-4\lambda+3) = 0$$

$$\Rightarrow -(\lambda-1)^2(\lambda-3) = 0$$

$$\Rightarrow \lambda - 1 = 0 \quad \text{or} \quad \lambda - 3 = 0$$

$$\Rightarrow \lambda = 1 \quad \text{or} \quad \lambda = 3$$

$$\therefore \lambda = 1, 3$$

For $\lambda = 1$,

$$A - \lambda I = A - I = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

P.T.O.

Using Gaussian Elimination processe:

$$A-\lambda I = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [R_2 \leftarrow R_2 - R_1]$$

So, $$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + x_2 = 0 \quad \Rightarrow x_1 = -x_2$

Eigenvector, $v = \begin{bmatrix} -x_2 \\ x_2 \\ x_3 \end{bmatrix}$

Let, $x_2 = 0$, $x_3 = 1$, then $v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Let, $x_2 = 1$, $x_3 = 0$, then $v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

for $\lambda = 3$

$$A - \lambda I = A - 3.I = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Using Gaussian elimination process:

$$A - \lambda I = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad [R_1 \leftarrow R_1 / (-1)]$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad [R_2 \leftarrow R_2 - R_1]$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad [\text{swapping } R_2 \Leftrightarrow R_3]$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad [R_2 \leftarrow R_2 / (-2)]$$

So,
$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 - x_2 = 0$ and $x_3 = 0$

$\Rightarrow x_1 = x_2$

Eigenvector, $v = \begin{bmatrix} x_2 \\ x_2 \\ 0 \end{bmatrix}$

Let, $x_2 = 1$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ (Ans).

$\lambda = 1, 3$

$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ (Ans.)

(2) $v_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$

v_1, v_2 & v_3 form an orthogonal set, iff, their standard euclidian in product is equal to zero

$$\Rightarrow v_1^T . v_2 = v_2^T . v_3 = v_3^T . v_1 = 0$$

$$v_1^T . v_2 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}^T . \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = 2\times 0 + 0(-1) + (-1)\times 0 = 0$$

$$v_2^T . v_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}^T . \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} = 0\times 2 + (-1)0 + 0(4) = 0$$

$$v_3^T v_1 = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}^T . \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = 2\times 2 + 0\times 0 + 4(-1) = 0$$

So, they form orthogonal set

But, they don't form an orthonormal set as

$$\|v_1\|_2 = \sqrt{2^2 + 0^2 + (-1)^2} = \sqrt{5} \neq 1,$$

$$\|v_2\|_2 = \sqrt{0^2 + (-1)^2 + 0^2} = 1, \quad [\text{it is okay}]$$

$$\|v_3\| = \sqrt{2^2 + 0^2 + 4^2} = \sqrt{20} \neq 1$$

~~$\|v\|$~~ We can convert/form them into a set of vectors that will form an orthonormal set of vectors under the standard euklidian inner product for $\mathbb{R}^3$ by normalizing,

$$u_1 = \frac{v_1}{\|v_1\|_2} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 0 \\ -1/\sqrt{5} \end{bmatrix}$$

$$u_2 = \frac{v_2}{\|v_2\|_2} = \frac{v_2}{1} = v_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$u_3 = \frac{v_3}{\|v_3\|_2} = \frac{1}{\sqrt{20}}\begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{20} \\ 0 \\ 4/\sqrt{20} \end{bmatrix} \quad [\sqrt{20} = 4\sqrt{5}]$$

Now, u_1, u_2, u_3 form an orthonormal set of vectors as $\|u_1\|_2 = \|u_2\|_2 = \|u_3\|_2 = 1$

Linear Algebra

3. $\underset{n \times m}{A}$ has linearly independent columns.

$$A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$$

$$x_1 v_1 + x_2 v_2 + \cdots x_n v_n = 0$$

$\Rightarrow Ax = 0$ ...... (1) $\quad [\, x = [x_1, \dots x_n]^T \,]$

$\Rightarrow x = 0$

So, $\mathcal{N}(A) = \{\vec{0}\}$ [as A's columns are linearly independent]

P.T.O.

Let, $x \in \mathcal{N}(A^TA)$, then,

$\Rightarrow (A^TA)x = 0$

$\Rightarrow x^T A^T A x = 0$

$\Rightarrow (Ax)^T Ax = 0$

$\Rightarrow ||Ax||^2 = 0$

$\Rightarrow Ax = 0 \Rightarrow x = 0$

or $x \in N(A)$, ~~So x is also~~

So, if $x \in \mathcal{N}(A^TA)$ then $x \in \mathcal{N}(A)$

$\cancel{\Rightarrow x = \vec{0}}$

$\therefore \mathcal{N}(A^TA) = \mathcal{N}(A) = \{0\}$

Therefore, the columns of A^TA are linearly independent and $\underbrace{A^TA}_{\underbrace{m\times n \;\; n\times m}_{m\times m}}$ is a square matrix.

$\therefore A^TA$ is invertible.