Q-3

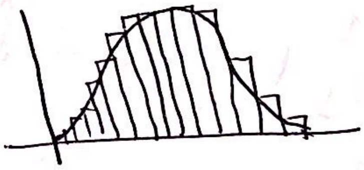
3

$$\hat{f}_0(x) = 0$$

$$\hat{f}_{i+1}(x) = \hat{f}_i(x) + g(h_{i+1}, a_{i+1}, b_{i+1}, x) \quad \ldots (1)$$

we have to construct a series of (1) with a fixed N, which satisfies $\|f - \hat{f}_{i+1}\| \leq \|f - \hat{f}_i\|$.

We can achieve this by incrementally adding N equal area parts' of the interval. The intuition is from the notion of integration, like we can get the area under a curve by splitting it into infinitesimal ~~pieces~~ rectangles and adding them up.



So, with a fixed n, we can generate new a_{i+1}, b_{i+1}, and h_{i+1}:

$$\left.\begin{aligned} a_{i+1} &= a_i + (\delta + \underbrace{0.00001}) \\ b_{i+1} &= a_i + \delta \end{aligned}\right\} \text{ where, } \delta = \frac{b-a}{N}$$

$\rightarrow$ to make sure that the splits are non-overlapping

$$h_{i+1} = (f(a_{i+1}) + f(b_{i+1}))/2$$

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Let's show that, $\|f - \hat{f}_{i+1}\| < \|f - \hat{f}_i\|$

$\|f - \hat{f}_i\|$

$\Rightarrow \int_{-1}^{1} |-x^2+1 - \hat{f}_{i-1} - g(h_i, a_i, b_i, x)\, dx$

$\Rightarrow \int_{-1}^{1} |-x^2+1 - \hat{f}_{i-1} - h_i|\, dx$

$\Rightarrow \left[-\frac{x^3}{3} + x - \hat{f}_{i-1} - h_i\right]_{-1}^{+1}$

$\Rightarrow \frac{4}{3} - \left[\hat{f}_{i-1} - h_i\right]_{-1}^{1}$ .... (2)

$\|f - \hat{f}_{i+1}\|$

$= \int_{-1}^{1} |-x^2+1 - \hat{f}_i - g(h_{i+1}, a_{i+1}, b_{i+1}, x)|\, dx$

$= \int_{-1}^{1} |-x^2+1 - \hat{f}_{i-1} - h_i - h_{i+1}|\, dx$

$= \left[-\frac{x^3}{3} + x - \hat{f}_{i-1} - h_i - h_{i+1}\right]_{-1}^{1}$

$= \frac{4}{3} - \left[\hat{f}_{i-1} - h_i\right]_{-1}^{1} - \left[h_{i+1}\right]_{-1}^{1}$ ... (3)

$> \frac{4}{3} - \left[\hat{f}_{i-1} - h_i\right]_{-1}^{1}$ [because, $\left[h_{i+1}\right]_{-1}^{1}$ evaluates to

$= \|f - \hat{f}_i\|$ positive : $h_{i+1} = \frac{(f(a_{i+1}) + f(b_{i+1}))}{2}$

~~both~~ both $f(a_{i+1})$ & $f(b_{i+1})$ are ~~positive~~ greater or equal to 0.

And, at least, one of them is greater than 0.