

# 1 The model

Two states of the world: policies  $x$  and  $y$ . Party  $A$  and  $B$  prefer  $x$  and  $y$  respectively. We can make this more explicit with

$$\begin{aligned} U_A(x) &> U_A(y) \\ U_B(y) &> U_B(x) \end{aligned}$$

For voters,  $v_i > 0$  means voter  $i$  prefers  $x$  to  $y$ . All  $i \in N$  vote for  $x$  or  $y$ . Simple majority rule determines which policy gets implemented. For each voter, each party sets a bribe schedule

$$\begin{aligned} a &\in (a_1, \dots, a_n) \in \mathbb{R}_+^n \\ b &\in (b_1, \dots, b_n) \in \mathbb{R}_+^n \end{aligned}$$

Solving through backward induction, given bribe schedules  $(a, b)$ , voter  $i$  prefers to vote for  $x$  if  $a_i + v_i > b_i$  and for  $y$  otherwise. Since indifferent voters choose  $y$ , party  $B$  needs to only match bribes from  $A$ , adjusting for individual voters' preferences:  $b_i = a_i + v_i$ . Therefore,  $B$  solves

$$\min_C \left\{ \sum_{i \in C} \max\{0, a_i + v_i\} : |C| > \frac{n}{2} \right\}$$

As long as this sum is strictly less than  $W_B$ ; otherwise party  $B$  chooses to set  $b_i = 0 : \forall i \in N$ .

Following Banks (2000), we restrict our analysis to the set of equilibria in which party  $A$  wins, i.e.  $W_A$  is sufficiently large relative to  $\mathbf{v}$  and  $W_B$  so that policy  $x$  prevails over  $y$ . In other words, the following inequality must hold:

$$\sum_{i \in C} \max\{0, a_i + v_i\} \geq W_B$$

Let  $U(v, W_b) \subseteq \mathbb{R}_+^n$  denote the set of unbeatable bribe schedules. Additionally, let  $S(a) = \sum_i^n a_i$  denote the bribe schedule for party  $A$ . The above assumptions on  $W_A$ ,  $W_B$  and  $v$  guarantee that there is an

$$\tilde{a} \in U(\mathbf{v}, W_B) : S(\tilde{a}) \leq W_A$$

For party  $A$ , the solution is

$$\min_a \{S(a) : a \in U(\mathbf{v}, W_B)\} \tag{1}$$

To fully describe the solution to equation 1, we note the following: for any  $a \in \mathbb{R}_+^n$ ,

let  $C(a) : i \in N : a_i > 0$  denote the set of individuals who receive a bribe from  $A$ . One can show that there is a bribe schedule  $a'$  such that for any  $i, j \in C(a)$ ,  $a'_i + v_i = a'_j + v_j$ . The intuition is that  $A$  has no incentive to make voters differentially bribed, because  $B$  will simply ignore the more expensive voters and target the weakest rings in the chain. Following Groseclose and Snyder (1996) we refer to this as a leveling schedule.

Let  $U^l(\mathbf{v}, W_B) \subseteq U(\mathbf{v}, W_B)$  denote the set of unbeatable leveling schedules. These are bribe schedules such that  $a_i + v_i = a_j + v_j \equiv t(a)$ . The bribe  $a_i = t(a) - v_i$  is the sum of two terms. The first is the common "transfer" among all voters in  $C(a)$ , the second ( $-v_i$ ) is individual specific. The latter term makes voters indifferent between  $x$  and  $y$  absent any bribe from  $B$ ; the former represents the per capita amount necessary to make  $C(a)$ , together with any unbribed voters, unaffordable for  $B$ .

To further simplify the analysis, Banks introduces the following sets of assumption:

$$\begin{aligned} A_1 : v_{(n+1)/2} &< 0 \\ A_2 : v_1 &< 2W_B/(n+1) \end{aligned}$$

$A_1$  implies that absent any bribes by  $A$ ,  $y$  will defeat  $x$ . Therefore  $A$  must bribe at least one voter.  $A_2$  further implies that  $A$  must bribe at least a majority of voters, otherwise  $B$  will have sufficient resources to bribe  $(n+1)/2$  voters and win.

Banks then proceeds to show that there are monotonic bribing schedules contained within the solution for equation 1. For any  $a \in \mathbb{R}_+^n$  let  $k(a) = |C(a)|$ . Suppose that  $a \in U^l(\mathbf{v}, W_B)$  is such that  $v_i \geq v_j$  and  $j \in C(a)$  but  $i \notin C(a)$ . Then, under  $A_2$ , there exists  $a' \in U^l(\mathbf{v}, W_B)$  with  $S(a') \leq S(a)$ ,  $k(a') = k(a)$  and  $i \in C(a')$  but  $j \notin C(a')$  by simply swapping  $i$  for  $j$ .<sup>1</sup>

Generalizing, and recalling that  $v_1 \geq \dots \geq v_n$ , we see that for all  $a \in U^l(\mathbf{v}, W_b)$  there exists a bribe schedule  $a' \in U^l(\mathbf{v}, W_b)$  such that  $S(a') \leq S(a)$  and  $C(a') = \{1, \dots, k(a)\}$ . Therefore, we can without loss of generality restrict attention to schedules  $a$  by  $A$  which bribe the first  $k(a)$  voters. Call these monotonic leveling schedules and let  $U_m^l \subseteq U(\mathbf{v}, W_B)$ .

Therefore, when  $A_2$  holds,

$$\min\{S(a) : a \in U(\mathbf{v}, W_B)\} = \min\{S(a) : a \in U_m^l(\mathbf{v}, W_B)\}$$

We can now further simplify the total expenditure  $S(a)$

$$S(a) = \sum_{i \in C(a)} a_i = k(a) \cdot t(a) - \sum_{i \leq k(a)} v_i$$

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<sup>1</sup>Note that since  $v_i \geq v_j$ , we have that  $t(a) - v_i \leq t(a) - v_j$ , i.e.  $a'_i \leq a_j$ .  $A_2$  guarantees that  $a'_i$  and  $a_j$  are non-negative.

Note that the choice of  $k(a)$  and  $t(a)$  fully characterize any schedule  $a \in U_m^l(\mathbf{v}, W_B)$ . We can thus fully characterize the optimization problem of  $A$  as

$$\min_{k,t} k \cdot t - \sum_{i \leq k} v_i$$

subject to the constraint that the induced schedule  $a \in U_m^l$ . Banks then reformulates this as an unconstrained problem by noting the following. First, if  $a(k, t, \mathbf{v})$  is unbeatable, it must be that  $k \geq (n+1)/2$ , so by  $A_1$  it must be that if  $a_i(k, t, \mathbf{v}) = 0$ , then  $v_i < 0$ . Therefore,  $B$  receives all non-bribed voters for free. For  $a(k, t, \mathbf{v})$  to be unbeatable, then, it must be that  $B$  cannot afford the remaining  $(n+1)/2 - (n-k) = k - (n-1)/2$  voters, or

$$t * (k - (n-1)/2) \geq W_B$$

Solving this for equality yields the optimal transfer from  $A$  to members of  $C(A) = \{1, \dots, k\}$ , conditional on  $k$ :

$$t(k, W_B) = \frac{W_B}{k - (n-1)/2} \quad (2)$$

Defining minimal winning expenditures as

$$E(k, \mathbf{v}, W_B) = k \cdot t(k, W_B) - \sum_{i \leq k} v_i \quad (3)$$

we can state  $A$ 's problem as

$$\min_k \{E(k, \mathbf{v}, W_B) : k \in (n+1/2), \dots, n\} \quad (4)$$

Denote the solution to expression 4 as  $k^*(\mathbf{v}, W_B)$ . This solution generates a solution to expression 1

## 2 Results

First, characterize a solution for  $k^*$ . Because  $k$  is finite, calculus cannot be employed. Instead, we deploy a discrete version of these techniques. First let's define  $\Delta(k) = E(k+1) - E(k)$  as the difference in expenditure from adding another coalition member. Note that if  $\Delta(k) \geq 0$  then  $A$  does not want to add another member to the coalition.

Conversely, if  $\Delta(k) < 0$ , then  $A$  is strictly better off by adding the  $k + 1$ th member of the coalition.

Next, suppose that  $\Delta(k)$  is increasing in  $k$ . The following algorithm can then be used to identify  $k^*$ : if  $\Delta((n + 1)/2) \geq 0$ , then we know from  $\Delta(k)$  increasing that  $A$  is better off by setting  $k^*$  to  $(n + 1)/2$ . If  $\Delta((n + 1)/2) < 0$ , then we know that  $k^*$  must be greater than  $(n + 1)/2$ , so we next solve for  $\Delta((n + 3)/2)$ , and so on.

We can therefore search for the optimal  $k^*$  with the following algorithm:

$$k^* = \begin{cases} (n + 1)/2 & \text{if } \Delta((n + 1)/2) \geq 0 \\ \max\{k : \Delta(k - 1) < 0\} & \text{otherwise} \end{cases} \quad (5)$$

Banks also shows that  $\Delta(k)$  is indeed nondecreasing.

$$\Delta(k) = \left[ \frac{(k + 1)W_B}{k + 1 - (n - 1)/2} - \sum_{i \leq k+1} v_i \right] \quad (6)$$

$$= \frac{-W_B(n - 1)}{2(k + 1 - (n - 1)/2)(k - (n - 1)/2)} - v_{k+1} \quad (7)$$

$$\equiv T(k, W_B) - v_{k+1} \quad (8)$$

Note that because  $v_1 \geq \dots \geq v_n$ ,  $-v_{k+1}$  is non-decreasing in  $k$ . Also, taking the first term as a continuous variable and differentiating w.r.t.  $k$ , we note that it is also increasing in  $k$ .

Generating an explicit characterization of  $k$  is not straightforward. Instead, Banks opts to show the conditions which need to hold in order for the coalition to be universal or minimal. Using equation 5 and substituting in equation 7 we have the following.

**Proposition 1.** (a)  $k^*(\mathbf{v}, W_B) = (n + 1)/2$  if and only if  $v_{(n+3)/2} \leq -W_B(n - 1)/4$ ; (b)  $k^*(\mathbf{v}, W_B) = n$  if and only if  $v_n > -2W_B/(n + 1)$ .

Banks also identifies how the optimal coalition  $k^*$  respond to marginal changes in voter preference intensity. Given an arbitrary amount  $W_B$  and preference profile  $\mathbf{v}'$ , let  $k' = k^*(\mathbf{v}, W_B)$ . If  $k' = (n + 1)/2$ , then we know that  $k' \leq k^*(\mathbf{v}, W_B)$  for all  $\mathbf{v}$ , so suppose  $k' > (n + 1)/2$ .

From equation 5 we infer that  $\Delta(k - 1, \mathbf{v}', W_B) < 0$ , which from equations 7 and 8 is equivalent to  $v'_k > T(k' - 1, W_B)$ . Now suppose that the preference profile changes from  $\mathbf{v}'$  to  $\mathbf{v}$ , and  $v_{k'}$  is such that  $v_{k'} \geq v'_k$ . Then,  $v_{k'} > T(k' - 1, W_B)$ , and hence  $\Delta(k' - 1, \mathbf{v}, W_B) < 0$ . But then from equation 5 it must be the case that  $k^*(\mathbf{v}, W_B) \geq k'$ . Therefore, the following holds:

**Proposition 2.** *For all  $W_B$ , if  $\mathbf{v}$  and  $\mathbf{v}'$  are such that  $v_{k'} \geq v'_{k'}$ , where  $k' = k^*(\mathbf{v}', W_B)$ , then  $k^*(\mathbf{v}, W_B) \geq k^*(\mathbf{v}', W_B)$*

In words, if the preference intensity of the marginal bribed voter weakly increases, then the optimal coalition size also weakly increases. Substantively, the number of voters bribed by  $A$  weakly increases as the voter who receives the largest bribe finds  $A$ 's preferred alternative,  $x$ , more attractive. Similarly

**Proposition 3.** *For all  $W_B$ , if  $\mathbf{v}$  and  $\mathbf{v}'$  are such that  $v_{k'+1} \leq v'_{k'+1}$ , where  $k' = k^*(\mathbf{v}', W_B)$ , then  $k^*(\mathbf{v}, W_B) \leq k^*(\mathbf{v}', W_B)$*