

1 The model

Two states of the world: policies x and y . Party A and B prefer x and y respectively. We can make this more explicit with

$$\begin{aligned} U_A(x) &> U_A(y) \\ U_B(y) &> U_B(x) \end{aligned}$$

For voters, $v_i > 0$ means voter i prefers x to y . All $i \in N$ vote for x or y . Simple majority rule determines which policy gets implemented. For each voter, each party sets a bribe schedule

$$\begin{aligned} a &\in (a_1, \dots, a_n) \in \mathbb{R}_+^n \\ b &\in (b_1, \dots, b_n) \in \mathbb{R}_+^n \end{aligned}$$

Solving through backward induction, given bribe schedules (a, b) , voter i prefers to vote for x if $a_i + v_i > b_i$ and for y otherwise. Since indifferent voters choose y , party B needs to only match bribes from A , adjusting for individual voters' preferences: $b_i = a_i + v_i$. Therefore, B solves

$$\min_C \left\{ \sum_{i \in C} \max\{0, a_i + v_i\} : |C| > \frac{n}{2} \right\}$$

As long as this sum is strictly less than W_B ; otherwise party B chooses to set $b_i = 0 : \forall i \in N$.

Following Banks (2000), we restrict our analysis to the set of equilibria in which party A wins, i.e. W_A is sufficiently large relative to \mathbf{v} and W_B so that policy x prevails over y . In other words, the following inequality must hold:

$$\sum_{i \in C} \max\{0, a_i + v_i\} \geq W_B$$

Let $U(v, W_b) \subseteq \mathbb{R}_+^n$ denote the set of unbeatable bribe schedules. Additionally, let $S(a) = \sum_i^n a_i$ denote the bribe schedule for party A . The above assumptions on W_A , W_B and v guarantee that there is an

$$\tilde{a} \in U(\mathbf{v}, W_B) : S(\tilde{a}) \leq W_A$$

For party A , the solution is

$$\min_a \{S(a) : a \in U(\mathbf{v}, W_B)\} \quad (1)$$

To fully describe the solution to equation 1, we note the following: for any $a \in \mathbb{R}_+^n$, let $C(a) = \{i \in N : a_i > 0\}$ denote the set of individuals who receive a bribe from A . One can show that there is a bribe schedule a' such that for any $i, j \in C(a)$, $a'_i + v_i = a'_j + v_j$. The intuition is that A has no incentive to make voters differentially bribed, because B will simply ignore the more expensive voters and target the weakest rings in the chain. Following Groseclose and Snyder (1996) we refer to this as a leveling schedule.

Let $U^l(\mathbf{v}, W_B) \subseteq U(\mathbf{v}, W_B)$ denote the set of unbeatable leveling schedules. These are bribe schedules such that $a_i + v_i = a_j + v_j \equiv t(a)$. The bribe $a_i = t(a) - v_i$ is the sum of two terms. The first is the common "transfer" among all voters in $C(a)$, the second $(-v_i)$ is individual specific. The latter term makes voters indifferent between x and y absent any bribe from B ; the former represents the per capita amount necessary to make $C(a)$, together with any unbribed voters, unaffordable for B .

To further simplify the analysis, Banks introduces the following sets of assumption:

$$A1 : v_{(n+1)/2} < 0$$

$$A2 : v_1 < 2W_B/(n+1)$$

$A1$ implies that absent any bribes by A , y will defeat x . Therefore A must bribe at least one voter. $A2$ implies that A must bribe at least a majority of voters, otherwise B will have sufficient resources to bribe $(n+1)/2$ voters and win. $A2$ also implies that for all $a \in U^l(\mathbf{v}, W_B)$, it must be that $t(a) \geq 2W_B/(n+1)$, otherwise B can bribe a majority from $C(a)$ itself and win the vote.

Banks then proceeds to show that there are monotonic bribing schedules contained within the solution for equation 1. For any $a \in \mathbb{R}_+^n$ let $k(a) = |C(a)|$. Suppose that $a \in U^l(\mathbf{v}, W_B)$ is such that $v_i \geq v_j$ and $j \in C(a)$ but $i \notin C(a)$. Then, under $A2$, there exists $a' \in U^l(\mathbf{v}, W_B)$ with $S(a') \leq S(a)$, $k(a') = k(a)$ and $i \in C(a')$ but $j \notin C(a')$ by simply swapping i for j : $a'_i = t(a) - v_i$, $a'_j = 0$ and $a'_m = a_m$ for all $m \notin i, j$.¹

Generalizing, and recalling that $v_1 \geq \dots \geq v_n$, we see that for all $a \in U^l(\mathbf{v}, W_b)$

¹Note that since $v_i \geq v_j$, we have that $t(a) - v_i \leq t(a) - v_j$, i.e. $a'_i \leq a_j$. $A2$ guarantees that a'_i and a_j are non-negative.

there exists a bribe schedule $a' \in U^l(\mathbf{v}, W_b)$ such that $S(a') \leq S(a)$ and $C(a') = \{1, \dots, k(a)\}$. Therefore, we can without loss of generality restrict attention to schedules a by A which bribe the first $k(a)$ voters. Call these monotonic leveling schedules and let $U_m^l \subseteq U(\mathbf{v}, W_B)$.

Therefore, when A2 holds,

$$\min\{S(a) : a \in U(\mathbf{v}, W_B)\} = \min\{S(a) : a \in U_m^l(\mathbf{v}, W_B)\}$$

We can now further simplify the total expenditure $S(a)$

$$S(a) = \sum_{i \in C(a)} a_i = k(a) \cdot t(a) - \sum_{i \leq k(a)} v_i$$

Note that the choice of $k(a)$ and $t(a)$ fully characterize any schedule $a \in U_m^l(\mathbf{v}, W_B)$. We can thus fully characterize the optimization problem of A as

$$\min_{k, t} k \cdot t - \sum_{i \leq k} v_i$$

subject to the constraint that the induced schedule $a \in U_m^l$. Banks then reformulates this as an unconstrained problem by noting the following. First, if $a(k, t, \mathbf{v})$ is unbeatable, it must be that $k \geq (n+1)/2$, so by A1 it must be that if $a_i(k, t, \mathbf{v}) = 0$, then $v_i < 0$. Therefore, B receives all non-bribed voters for free. For $a(k, t, \mathbf{v})$ to be unbeatable, then, it must be that B cannot afford the remaining $(n+1)/2 - (n-k) = k - (n-1)/2$ voters, or

$$t \cdot (k - (n-1)/2) \geq W_B$$

Solving this for equality yields the optimal transfer from A to members of $C(A) = \{1, \dots, k\}$, conditional on k :

$$t(k, W_B) = \frac{W_B}{k - (n-1)/2} \quad (2)$$

Defining minimal winning expenditures as

$$E(k, \mathbf{v}, W_B) = k \cdot t(k, W_B) - \sum_{i \leq k} v_i \quad (3)$$

we can state A 's problem as

$$\min_k \{E(k, \mathbf{v}, W_B) : k \in (n + 1/2), \dots, n\} \quad (4)$$

Denote the solution to expression 4 as $k^*(\mathbf{v}, W_B)$. This solution implicitly generates a solution to expression 1 through expression 3 and the induced bribe schedule above.

2 Results

First, characterize a solution for k^* . Because k is finite, calculus cannot be employed. Instead, we deploy a discrete version of these techniques. First let's define $\Delta(k) = E(k + 1) - E(k)$ as the difference in expenditure from adding another coalition member. Note that if $\Delta(k) \geq 0$ then A does not want to add another member to the coalition. Conversely, if $\Delta(k) < 0$, then A is strictly better off by adding the $k + 1$ th member of the coalition.

Next, suppose that $\Delta(k)$ is increasing in k . The following algorithm can then be used to identify k^* : if $\Delta((n + 1)/2) \geq 0$, then we know from $\Delta(k)$ increasing that A is better off by setting k^* to $(n + 1)/2$. If $\Delta((n + 1)/2) < 0$, then we know that k^* must be greater than $(n + 1)/2$, so we next solve for $\Delta((n + 3)/2)$, and so on.

We can therefore search for the optimal k^* with the following algorithm:

$$k^* = \begin{cases} (n + 1)/2 & \text{if } \Delta((n + 1)/2) \geq 0 \\ \max\{k : \Delta(k - 1) < 0\} & \text{otherwise} \end{cases} \quad (5)$$

Banks also shows that $\Delta(k)$ is indeed nondecreasing.

$$\Delta(k) = \left[\frac{(k + 1)W_B}{k + 1 - (n - 1)/2} - \sum_{i \leq k+1} v_i \right] \quad (6)$$

$$= \frac{-W_B(n - 1)}{2(k + 1 - (n - 1)/2)(k - (n - 1)/2)} - v_{k+1} \quad (7)$$

$$\equiv T(k, W_B) - v_{k+1} \quad (8)$$

Note that because $v_1 \geq \dots \geq v_n$, $-v_{k+1}$ is non-decreasing in k . Also, taking the first term as a continuous variable and differentiating w.r.t. k , we note that

it is also increasing in k .

Generating an explicit characterization of k is not straightforward. Instead, Banks opts to show the conditions which need to hold in order for the coalition to be universal or minimal. Using equation 5 and substituting in equation 7 we have the following.

Proposition 1. (a) $k^*(\mathbf{v}, W_B) = (n+1)/2$ if and only if $v_{(n+3)/2} \leq -W_B(n-1)/4$; (b) $k^*(\mathbf{v}, W_B) = n$ if and only if $v_n > -2W_B/(n+1)$.

Banks also identifies how the optimal coalition k^* respond to marginal changes in voter preference intensity. Given an arbitrary amount W_B and preference profile \mathbf{v}' , let $k' = k^*(\mathbf{v}', W_B)$. If $k' = (n+1)/2$, then we know that $k' \leq k^*(\mathbf{v}, W_B)$ for all \mathbf{v} , so suppose $k' > (n+1)/2$.

From equation 5 we infer that $\Delta(k-1, \mathbf{v}', W_B) < 0$, which from equations 7 and 8 is equivalent to $v'_k > T(k'-1, W_B)$. Now suppose that the preference profile changes from \mathbf{v}' to \mathbf{v} , and $v_{k'}$ is such that $v_{k'} \geq v'_{k'}$. Then, $v_{k'} > T(k'-1, W_B)$, and hence $\Delta(k'-1, \mathbf{v}, W_B) < 0$. But then from equation 5 it must be the case that $k^*(\mathbf{v}, W_B) \geq k'$. Therefore, the following holds:

Proposition 2. For all W_B , if \mathbf{v} and \mathbf{v}' are such that $v_{k'} \geq v'_{k'}$, where $k' = k^*(\mathbf{v}', W_B)$, then $k^*(\mathbf{v}, W_B) \geq k^*(\mathbf{v}', W_B)$

In words, if the preference intensity of the marginal bribed voter weakly increases, then the optimal coalition size also weakly increases. Substantively, the number of voters bribed by A weakly increases as the voter who receives the largest bribe finds A 's preferred alternative, x , more attractive. Similarly

Proposition 3. For all W_B , if \mathbf{v} and \mathbf{v}' are such that $v_{k'+1} \leq v'_{k'+1}$, where $k' = k^*(\mathbf{v}', W_B)$, then $k^*(\mathbf{v}, W_B) \leq k^*(\mathbf{v}', W_B)$

The "convexity" of E guarantees that local information is sufficient to generate comparative statistics regarding changes in preferences $\mathbf{v}' \rightarrow \mathbf{v}$. Here, this local information is summarized by the preferences of the marginal voter v_k and non-bribed voter v_{k+1} . In order to identify these voters, one needs to solve for the optimal coalition size. Instead, Banks takes an alternative route and characterizes a weaker comparative statistic result which holds globally. Given two preferences \mathbf{v} and \mathbf{v}' , write $\mathbf{v} \geq \mathbf{v}'$ if $v_i \geq v'_i$ for all $i \in N$.

Corollary 1. For all W_B , if $\mathbf{v} \geq \mathbf{v}'$ then $k^*(\mathbf{v}, W_B) \geq k^*(\mathbf{v}', W_B)$

Thus, the optimal number of voters bribed by A and hence the optimal size of A 's optimal coalition, weakly increases as voters find A 's preferred alternative, x , more attractive. Although A 's total expenditure will decrease as x becomes more attractive to the legislature, the optimal way to allocate this lower amount is to spread it more widely among voters.

We can also characterize the change in total expenditures as a result of a shift in voter preferences. From equation 3 we have

$$\begin{aligned}
E(k, \mathbf{v}, W_B) - E(k, \mathbf{v}', W_B) &= \\
&= k \cdot t(k, W_B) - \sum_{i \leq k} v_i - \left[k \cdot t(k, W_B) - \sum_{i \leq k} v'_i \right] \\
&= \sum_{i \leq k} (v'_i - v_i)
\end{aligned}$$

Since $v_i - v'_i \geq 0$, the difference in expenditure between a more favorable and unfavorable legislature is always non-positive.