Theory

In this section, I use the canonical vote-buying model proposed by **(author?)** [2] and **(author?)** [1] to model mayor's strategy to engage in patronage in Brazil. For any given term, a mayor seeks to enact her preferred policy, but to do so needs to garner legislative support in the city council. The preferences of these legislators vary: some of these councilors are more inclined to supporting the government (*governista*), while others would rather oppose it. These different preferences translate into differential costs to sway their vote: city councilors aligned with the mayor are cheaper than their opposition counterparts.

In this set-up, the mayor competes with the opposition to buy votes. The strategy of the mayor focuses on how to pre-emptively coopt some of these councilors so that they ultimately vote in favor of the mayor's proposal. Building this winning coalition requires side payments of various forms, but my primary focus is on patronage appointments. City councilors who vote for the mayor's proposal are rewarded, as loyalists, with positions in government. This arrangement is mutually beneificial: as pointed out by **(author?)** [3], these appointments tie the councilor's fate to the patron. Because patronage appointments are discretionary, in order to maintain these appointments the city councilor must remain loyal to the government and vote in favor of the mayor's proposal.

The main prediction of the model is that, in equilibrium, the cost of passing the mayor's preferred policy decreases monotonically as the legislative support for the mayor increases. Substantively, the model predicts that municipalities in which the mayor has more (less) councilors favorable to her, we should observe less (more) patronage. This key comparative static is what guides us in the estimation exercises outlined in the empirical section. In a sense, patronage

In the remainder of this section, I outline the setting of the model and derive key comparative statics that guide the empirical estimation of this paper. I conclude with a discussion of the model and how it applies to the Brazilian municipal context.

The Setting

The government G and opposition O compete over legislative votes to enact their preferred policies. There are two possible outcomes: a policy x favored by the government, and the status quo, denote as y, preferred by the opposition. In order to implement her policy the mayor must pass a simple majority vote in the city council, comprised of an odd number N of legislators. The total amount of political resources available is W_G and W_O to win over votes, which for the mayor includes patronage appointments into the public sector.

Each city councilor is fully characterized by a publicly observed policy preference v_i for all $i \in N$, where $v_i > 0$ entails that the mayor's proposal x is preferred. Let $\mathbf{v} = (v_1, ..., v_n)$ describe the preference profile for the city council. In this setting, v_i measures the degree to which an individual city councilor supports the mayor, with higher values of v denoting stronger support for the mayor and vice versa. Payoff gets realized when city councilor i votes, independent of the outcome of the voting procedure.

We solve the game through backward induction. The timing of the game is as follows:

1. Government G offers a bribe schedule $a \in (a_1, ..., a_n) \in \mathbb{R}^n_+$.

2. Opposition O observes the bribe schedule m and makes a counter-offer $b \in (b_1, ..., b_n) \in \mathbb{R}^n_+$.

3. City councilors cast their votes at the end of bribing period.

4. Nature gathers votes and payoffs are realized.

Given a bribe schedule (a, b), councilor i prefers to vote for the mayor's proposal x if $a_i + v_i > b_i$ and the status quo y otherwise. Since indifferent councilors vote for the status quo, the opposition needs to only match bribes from M, adjusting for individual preferences, i.e. $b_i = a_i + v_i$. For the mayor, she needs to construct the cheapest winning coalition in order to defeat the opposition.

Following Groseclose and Snyder (1996) and Banks (2000) we focus our analysis on the set of equilibria in which the mayor wins.[1] In this context, the amount of patronage resources W_G is sufficiently large relative to W_O and $\mathbf{v}$ that the mayor's preferred policy x is implemented over y. Let $U(\mathbf{v}, W_O)$ denote the set of unbeatable patronage schedules for the mayor, and for any patronage schedule let $S(a) = \sum_{i=1}^n a_i$ be the total amount of patronage disbursed. The mayor then solves

$$\min_a \{S(a) : a \in U(\mathbf{v}, W_B)\} \tag{1}$$

Note that for any equilibrium strategy, it must be the case that mayor M uses a leveling schedule: every city councilor in her coalition C is equally expensive for the

[1] Since strategies for both players are symmetrical, any set of equilibria in which the mayor loses can be modeled as cases in which the the opposition loses.

opposition O to bribe. More formally, for any $a \in \mathbb{R}^n_+$, let $C(a) = \{i \in N : a_i > 0\}$ denote the set of individuals who receive a bribe from from the government G. One can show that there is a bribe schedule a' such that for any $i, j \in C(a)$, $a'_i + v_i = a'_j + v_j$. The intuition is that the mayor has no incentive to make voters differentially expensive, because the opposition O will simply ignore the more expensive voters and target the least favorable members of the coalition. Following G+S and BKS we refer to these strategies as leveling schedules.

We can characterize the set of equilibria in the game by introducing additional notation. Let $U^l(\mathbf{v}, W_O) \subseteq U(\mathbf{v}, W_O)$ denote the set of unbeatable leveling schedules. These are bribe schedules such that $a_i + v_i = a_j + v_j \equiv t(a)$. The bribe $a_i = t(a) - v_i$ is the sum of two terms. The first is the common "transfer" among all voters in $C(a)$, the second $(-v_i)$ is individual specific. The latter term makes voters indifferent between x and y absent any bribe from B; the former represents the per capita amount necessary to make $C(a)$, together with any unbribed voters, unaffordable for B.

I impose the following two assumptions:

$$A1 : v_{(n+1)/2} < 0$$
$$A2 : v_1 < 2W_B/(n+1)$$

$A1$ implies that absent any bribes by A, y will defeat x. Therefore A must bribe at least one voter. $A2$ implies that A must bribe at least a majority of voters, otherwise B will have sufficient resources to bribe $(n+1)/2$ voters and win. $A2$ also implies that for all $a \in U^l(\mathbf{v}, W_B)$, it must be that $t(a) \geq 2W_B/(n+1)$, otherwise B can bribe a majority from $C(a)$ itself and win the vote.

For any $a \in \mathbb{R}^n_+$ let $k(a) = |C(a)|$. Suppose that $a \in U^l(\mathbf{v}, W_B)$ is such that $v_i \geq v_j$ and $j \in C(a)$ but $i \notin C(a)$. Then, under $A2$, there exists $a' \in U^l(\mathbf{v}, W_B)$ with $S(a') \leq S(a)$, $k(a') = k(a)$ and $i \in C(a')$ but $j \notin C(a')$ by simply swapping i for j: $a'_i = t(a) - v_i$, $a'_j = 0$ and $a'_m = a_m$ for all $m \notin i, j$.[2] Generalizing, and recalling that $v_1 \geq ... \geq v_n$, we see that for all $a \in U^l(\mathbf{v}, W_b)$ there exists a bribe schedule $a' \in U^l(\mathbf{v}, W_b)$ such that $S(a') \leq S(a)$ and $C(a') = \{1, ..., k(a)\}$. Therefore, we can without loss of generality restrict attention to schedules a by A which bribe the first $k(a)$ voters. Call these monotonic leveling schedules and let $U^l_m \subseteq U(\mathbf{v}, W_B)$.

Therefore, when $A2$ holds,

[2]Note that since $v_i \geq v_j$, we have that $t(a) - v_i \leq t(a) - v_j$, i.e. $a'_i \leq a_j$. Also, because of $A2$ $a'_j \leq a_j$, since it guarantees a_j is non-negative.

$$\min\{S(a) : a \in U(\mathbf{v}, W_B)\} = \min\{S(a) : a \in U^l_m(\mathbf{v}, W_B)\}$$

We can now further simplify the total expenditure $S(a)$

$$S(a) = \sum_{i \in C(a)} a_i = k(a) \cdot t(a) - \sum_{i \leq k(a)} v_i$$

Note that the choice of $k(a)$ and $t(a)$ fully characterize any schedule $a \in U^l_m(\mathbf{v}, W_B)$. We can thus fully characterize the optimization problem of A as

$$\min_{k,t} k \cdot t - \sum_{i \leq k} v_i$$

subject to the constraint that the induced schedule $a \in U^l_m$. Banks then reformulates this as an unconstrained problem by noting the following. First, if $a(k, t, \mathbf{v})$ is unbeatable, it must be that $k \geq (n+1)/2$, so by $A1$ it must be that if $a_i(k, t, \mathbf{v}) = 0$, then $v_i < 0$. Therefore, B receives all non-bribed voters for free. For $a(k, t, \mathbf{v})$ to be unbeatable, then, it must be that B cannot afford the remaining $(n+1)/2 - (n-k) = k - (n-1)/2$ voters, or

$$t \cdot (k - (n-1)/2) \geq W_B$$

Solving this for equality yields the optimal transfer from A to members of $C(A) = \{1, ..., k\}$, conditional on k:

$$t(k, W_B) = \frac{W_B}{k - (n-1)/2} \tag{2}$$

Defining minimal winning expenditures as

$$E(k, \mathbf{v}, W_B) = k \cdot t(k, W_B) - \sum_{i \leq k} v_i \tag{3}$$

we can state A's problem as

$$\min_k \{E(k, \mathbf{v}, W_B) : k \in (n + 1/2), ..., n\} \tag{4}$$

Denote the solution to expression 4 as $k^*(\mathbf{v}, W_B)$. This solution implicitly generates a solution to expression 1 through expression 3 and the induced bribe schedule above. Therefore, characterizing the optimal k^* is sufficient to fully characterize the optimal behavior of the government G.

1 Results

First, characterize a solution for k^*. Because k is finite, calculus cannot be employed. Instead, we deploy a discrete version of these techniques. First let's define $\Delta(k) = E(k+1) - E(k)$ as the difference in expenditure from adding another coalition member. Note that if $\Delta(k) \geq 0$ then A does not want to add another member to the coalition. Conversely, if $\Delta(k) < 0$, then A is strictly better off by adding the $k+1$th member of the coalition.

Next, suppose that $\Delta(k)$ is increasing in k. The following algorithm can then be used to identify k^*: if $\Delta((n+1)/2) \geq 0$, then we know from $\Delta(k)$ increasing that A is better off by setting k^* to $(n+1)/2$. If $\Delta((n+1)/2) < 0$, then we know that k^* must be greater than $(n+1)/2$, so we next solve for $\Delta((n+3)/2)$, and so on.

We can therefore search for the optimal k^* with the following algorithm:

$$k^* = \begin{cases} (n+1)/2 & \text{if } \Delta((n+1)/2) \geq 0 \\ \max\{k : \Delta(k-1) < 0\} & \text{otherwise} \end{cases} \tag{5}$$

Banks also shows that $\Delta(k)$ is indeed nondecreasing.

$$\begin{aligned} \Delta(k) &= \left[\frac{(k+1)W_B}{k+1-(n-1)/2} - \sum_{i \leq k+1} v_i \right] && (6) \\ &= \frac{-W_B(n-1)}{2(k+1-(n-1)/2)(k-(n-1)/2)} - v_{k+1} && (7) \\ &\equiv T(k, W_B) - v_{k+1} && (8) \end{aligned}$$

Note that because $v_1 \geq ... \geq v_n$, $-v_{k+1}$ is non-decreasing in k. Also, taking the first term as a continuous variable and differentiating w.r.t. k, we note that it is also increasing in k.

Generating an explicit characterization of k is not straightforward. Instead, Banks opts to show the conditions which need to hold in order for the coalition to be universal or minimal. Using equation 5 and substituting in equation 7 we have the following.

Proposition 1. *(a)* $k^*(\mathbf{v}, W_B) = (n+1)/2$ *if and only if* $v_{(n+3)/2} \leq -W_B(n-1)/4$*;* *(b)* $k^*(\mathbf{v}, W_B) = n$ *if and only if* $v_n > -2W_B/(n+1)$*.*

Banks also identifies how the optimal coalition k^* respond to marginal changes in voter prefence intensity. Given an arbitrary amount W_B and preference profile $\mathbf{v}'$, let $k' = k^*(\mathbf{v}, W_B)$. If $k' = (n+1)/2$, then we know that $k' \leq k*(\mathbf{v}, W_B)$ for all $\mathbf{v}$, so suppose $k' > (n+1)/2$.

From equation 5 we infer that $\Delta(k-1, \mathbf{v}', W_B) < 0)$, which from equations 7 and 8 is equivalent to $v'_k > T(k'-1, W_B)$. Now suppose that the preference profile changes from $\mathbf{v}'$ to $\mathbf{v}$, and $v_{k'}$ is such that $v_{k'} \geq v'_k$. Then, $v_{k'} > T(k'-1, W_B)$, and hence $\Delta(k'-1, \mathbf{v}, W_B) < 0$. But then from equation 5 it must be the case that $k^*(\mathbf{v}, W_B) \geq k'$. Therefore, the following holds:

Proposition 2. *For all* W_B*, if* $\mathbf{v}$ *and* $\mathbf{v}'$ *are such that* $v_{k'} \geq v'_{k'}$*, where* $k' = k^*(\mathbf{v}', W_B)$*, then* $k^*(\mathbf{v}, W_B) \geq k^*(\mathbf{v}', W_B)$

In words, if the preference intensity of the marginal bribed voter weakly increases, then the optimal coalition size also weakly increases. Substantively, the number of voters bribed by A weakly increases as the voter who receives the largest bribe finds A's preferred alternative, x, more attractive. Similarly

Proposition 3. *For all* W_B*, if* $\mathbf{v}$ *and* $\mathbf{v}'$ *are such that* $v_{k'+1} \leq v'_{k'+1}$*, where* $k' = k^*(\mathbf{v}', W_B)$*, then* $k^*(\mathbf{v}, W_B) \leq k^*(\mathbf{v}', W_B)$

The "convexity" of E guarantees that local information is sufficient to generate comparative statistics regarding changes in preferences $\mathbf{v}' \to \mathbf{v}$. Here, this local information is summarized by the preferences of the marginal voter v_k and non-bribed voter v_{k+1}. In order to identify these voters, one needs to solve for the optimal coalition size. Instead, Banks takes an alternative route and characterizes a weaker comparative statistic result which holds globally. Given two preferences $\mathbf{v}$ and $\mathbf{v}'$, write $\mathbf{v}$ and $\mathbf{v}'$, write $\mathbf{v} \geq \mathbf{v}'$ if $v_i \geq v'_i$ for all $i \in N$.

Corollary 1. *For all* $\mathrm{W_B}$*, if* $\mathbf{v} \geq \mathbf{v}'$ *then* $\mathrm{k}^*(\mathbf{v}, \mathrm{W_B}) \geq \mathrm{k}^*(v', \mathrm{W_B})$

Thus, the optimal number of voters bribed by A and hence the optimal size of A's optimal coalition, weakly increases as voters find A's preferred alternative, x, more attractive. Although A's total expenditure will decrease as x becomes more attractive to the legislature, the optimal way to allocate this lower amount is to spread it more widely among voters.

We can also characterize the change in total expenditures as a result of a shift in voter preferences. From equation 3 we have

$$
\begin{aligned}
&E(k, \mathbf{v}, W_B) - E(k, \mathbf{v}', W_B) = \\
&\qquad = k \cdot t(k, W_B) - \sum_{i \leq k} v_i - \left[k \cdot t(k, W_B) - \sum_{i \leq k} v_i' \right] \\
&\qquad = \sum_{i \leq k} (v_i' - v_i)
\end{aligned}
$$

Since $v_i - v_i' \geq 0$, the difference in expenditure between a more favorable and unfavorable legislature is always non-positive.

References

[1] Jeffrey S Banks. Buying supermajorities in finite legislatures. *American Political Science Review*, pages 677–681, 2000.

[2] Tim Groseclose and James M Snyder Jr. Buying supermajorities. *American Political Science Review*, pages 303–315, 1996.

[3] James A. Robinson and Thierry Verdier. The political economy of clientelism. *The Scandinavian Journal of Economics*, 115(2):260–291, 2013.