Computational Learning Theory

Lecture Notes for CS 582

Spring Semester, 1991

Sally A. Goldman
Department of Computer Science
Washington University
St. Louis, Missouri 63130

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Preface

This manuscript is a compliation of lecture notes from the graduate level course CS 582, "Computational Learning Theory," I taught at Washington University in the spring of 1991. Students taking the course were assumed to have background in the design and analysis of algorithms as well as good mathematical background. Given that there is no text available on this subject, the course material was drawn from recent research papers. I selected the first twelve topics and the remainder were selected by the students from a list of provided topics. This list of topics is given at the end of these notes.

These notes were mostly written by the students in the class and then reviewed by me. However, there are likely to be errors and omissions, particularly with regard to the references. Readers finding errors in the notes are encouraged to notify me by electronic mail (`sg@cs.wustl.edu`) so that later versions may be corrected.

Acknowledgements

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I thank Mike Kearns and Umesh Vazirani for providing me with a draft of the scribe notes from their Computational Learning Theory course taught at University of California at Berkeley in the Fall of 1990. Some of my lectures were prepared using their notes. Also most of the homework problems which I gave came from the problems used by Ron Rivest for his Machine Learning course at MIT taught during the Falls of 1989 and 1990. I thank Ron for allowing me to include these problems here.

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Topic 1: Introduction

Lecturer: Sally Golman

Scribe: Ellen Witte

1.1 Course Overview

Building machines that learn from experience is an important research goal of artificial intelligence, and has therefore been an active area of research. Most of the work in machine learning is empirical research. In such research, learning algorithms typically are judged by their performance on sample data sets. Although these *ad hoc* comparisons may provide some insight, it is difficult to compare two learning algorithms carefully and rigorously, or to understand in what situations a given algorithm might perform well, without a formally specified learning model with which the algorithms may be evaluated.

Recently, considerable research attention has been devoted to the theoretical study of machine learning. In *computational learning theory* one defines formal mathematical models of learning that enable rigorous analysis of both the predictive power and the computational efficiency of learning algorithms. The analysis made possible by these models provides a framework in which to design algorithms that are provably more efficient in both their use of time and data.

During the first half of this course we will cover the basic results in computational learning theory. This portion will include a discussion of the distribution-free (or PAC) learning model, the model of learning with queries, and the mistake-bound (or on-line) learning model. The primary goal is to understand how these models relate to one another and what classes of concepts are efficiently learnable in the various models. Thus we will present efficient algorithms for learning various concept classes under each model. (And in some cases we will consider what can be done if the computation time is not restricted to be polynomial.) In contrast to these positive results we present hardness results for some concept classes indicating that no efficient learning algorithm exists. In addition to studying the basic noise-free versions of these learning models, we will also discuss various models of noise and techniques for designing algorithms that are robust against noise. Finally, during the second half of this course we will study a selection of topics that follow up on the material presented during the first half of the course. These topics were selected by the students, and are just a sample of the types of other results that have been obtained. We warn the reader that this course only covers a small portion of the models, learning techniques, and methods for proving hardness results that are currently available in the literature.

1.2 Introduction

In this section we give a very basic overview of the area of computational learning theory. Portions of this introduction are taken from Chapter 2 of Goldman's thesis [18]. Also see Chapter 2 of Kearns' thesis [27] for additional definitions and background material.

1.2.1 A Research Methodology

Before describing formal models of learning, it is useful to outline a research methodology for applying the formalism of computational learning theory to "real-life" learning problems. There are four steps to the methodology.

1. Precisely define the problem, preserving key features while simplifying as much as possible.
2. Select an appropriate formal learning model.
3. Design a learning algorithm.
4. Analyze the performance of the algorithm using the formal model.

In Step 2, selecting an appropriate formal learning model, there are a number of questions to consider. These include:

- What is being learned?
- How does the learner interact with the environment? (e.g., Is there a helpful teacher? An adversary?)
- What is the prior knowledge of the learner?
- How is the learner's hypothesis represented?
- What are the criteria for successful learning?
- How efficient is the learner in time, data and space?

It is critical that the model chosen accurately reflect the real-life learning problem.

1.2.2 Definitions

In this course, we consider a restricted type of learning problem called *concept learning*. In a concept learning problem there are a set of *instances* and a single *target concept* that classifies each instance as a positive or a negative instance. The *instance space* denotes the set of all instances that the learner may see. The *concept space* denotes the set of all concepts from which the target concept can be chosen. The learner's goal is to devise a hypothesis of the target concept that accurately classifies each instance as positive or negative.

For example, one might wish to teach a child how to distinguish chairs from other furniture in a room. Each item of furniture is an instance; the chairs are positive instances and all other items are negative instances. The goal of the learner (in this case the child) is to develop a rule for the concept of a chair. In our models of learning, one possibility is that the rules are Boolean functions of features of the items presented, such as has-four-legs or is-wooden.

Since we want the complexity of the learning problem to depend on the "size" of the target concept, often we assign to it some natural size measure n. If we let X_n denote the set of instances to be classified for each problem of size n, we say that $X = \bigcup_{n \geq 1} X_n$ is the *instance space*, and each $x \in X$ is an *instance*. For each $n \geq 1$, we define each $C_n \subseteq 2^{X_n}$ to be a *family of concepts* over X_n, and $C = \bigcup_{n \geq 1} C_n$ to be a *concept class* over X. For $c \in C_n$ and $x \in X_n$, $c(x)$ denotes the classification of c on instance x. That is, $c(x) = 1$ if and only if $x \in c$. We say that x is a positive instance of c if $c(x) = 1$ and x is a negative instance of c if $c(x) = 0$. Finally, a *hypothesis* h for C_n is a rule that given any $x \in X_n$ outputs in polynomial time a prediction for $c(x)$. The *hypothesis space* is the set of all hypotheses h that the learning algorithm may output. The hypothesis must make a prediction for each $x \in X_n$. For the learning models which we will study here, it is not acceptable for the hypothesis to answer "I don't know" for some instances.

To illustrate these definitions, consider the concept class of monomials. (A monomial is a conjunction of literals, where each literal is either some boolean variable or its negation.) For this concept class, n is the number of variables. Thus $|X_n| = 2^n$ where each $x \in X_n$ represents an assignment of 0 or 1 to each variable. Observe that each variable can be placed in the target concept unnegated, placed in the target concept negated, or not put in the target concept at all. Thus, $|C_n| = 3^n$. One possible target concept for the class of monomials over five variables is $x_1 \overline{x_4} x_5$. For this target concept, the instance "10001" is a positive instance and "00001" is a negative instance.

We will model the learning process as consisting of two phases, a training phase and a performance phase. In the training phase the learner is presented with labeled examples (i.e., an example is chosen from the instance space and labeled according to the target concept). Based on these examples the learner must devise a hypothesis of the target concept. In the performance phase the hypothesis is used to classify examples from the instance space, and the accuracy of the hypothesis is evaluated. The various models of learning will differ primarily in the way they allow the learner to interact with the environment during the training phase and how the hypothesis is evaluated.

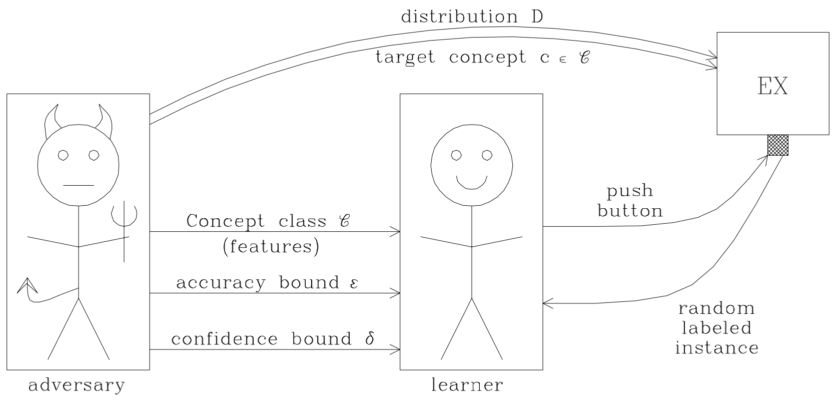

Figure 1.1: The PAC learning model.

1.3 The Distribution-Free (PAC) Model

The first formal model of machine learning we shall consider is the distribution-free or PAC model introduced by Valiant [43] in 1984. (This work initiated the field of computational learning theory.) In this model, an adversary chooses a concept class C, a target concept $c \in C$ and an arbitrary distribution D on the instance space. (We note that absolutely *no* restrictions are placed on D.) The learner is presented with the concept class C, an accuracy bound ϵ and a confidence bound δ. The learner is required to formulate a hypothesis h of the target concept based on labeled examples drawn randomly from the distribution D (which is unknown to the learner). See Figure 1.1.[1]

The PAC model requires the learner to produce a hypothesis which meets a certain error criteria. We can define the *error of the hypothesis* h on target concept c under distribution D to be:

$$error_D(h) = \Pr[c \oplus h] = \sum_{c(x) \neq h(x)} \Pr[x]$$

where $c \oplus h$ is the symmetric difference between c and h (i.e., the instances for which c and h differ). The error is the sum of the weight under distribution D placed on the examples for which c and h differ.

The goals of the learner are as follows:

1. With high probability ($\geq 1 \Leftrightarrow \delta$) the hypothesis must give a good approximation ($error_D(h) \leq \epsilon$) of the target concept.

2. The time and sample complexity of the learning algorithm must be polynomial in the size of the target concept, $1/\epsilon$ and $1/\delta$. (The sample complexity is the number of labeled examples needed by the algorithm.) Observe that as ϵ and δ go down, the algorithm is allowed more time and samples to produce a hypothesis.

[1]Compliments of Andy Fingerhut.

This model is called distribution-free because the distribution on the examples is unknown to the learner. Because the hypothesis must have high **p**robability of being **a**pproximately **c**orrect, it is also called the PAC model.

1.4 Learning Monomials in the PAC Model

We would like to investigate various concept classes that can be learned efficiently in the PAC model. The concept class of monomials is one of the simplest to learn and analyze in this model. Furthermore, we will see that the algorithm for learning monomials can be used to learn more complicated concept classes such as k-CNF.

A monomial is a conjunction of literals, where each literal is a variable or its negation. In describing the algorithm for learning monomials we will assume that n, the number of variables, is known. (If not, then it can be determined simply by counting the number of bits in the first example.)

We now describe the algorithm given by Valiant [43] for learning monomials. The algorithm is based on the idea that a positive example gives significant information about the monomial being learned. For example, if $n = 5$, and we see a positive example "10001", then we know that the monomial does *not* contain $\overline{x_1}$, x_2, x_3, x_4 or $\overline{x_5}$. A negative example does not give us near as much information since we do not know which of the bits caused the example to violate the target monomial.

Algorithm Learn-Monomials(n,ϵ,δ)

1. Initialize the hypothesis to the conjunction of all $2n$ literals.

$$h = x_1\overline{x_1}x_2\overline{x_2}\cdots x_n\overline{x_n}$$

2. Make $m = 1/\epsilon(n \ln 3 + \ln 1/\delta)$ calls to EX.
   - For each positive instance, remove x_i from h if $x_i = 0$ and remove $\overline{x_i}$ from h if $x_i = 1$.
   - For each negative instance, do nothing.

3. Output the remaining hypothesis.

In analyzing the algorithm, there are three measures of concern. First, is the number of examples used by the algorithm polynomial in n, $1/\epsilon$ and $1/\delta$? The algorithm uses $m = 1/\epsilon(n \ln 3 + \ln 1/\delta)$ examples; clearly this is polynomial in n, $1/\epsilon$ and $1/\delta$. Second, does the algorithm take time polynomial in these three parameters? The time taken per example is constant, so the answer is yes. Third is the hypothesis sufficiently accurate by the criteria of the PAC model? In other words, is $\Pr[error_D(h) \leq \epsilon] \geq (1 \Leftrightarrow \delta)$?

To answer the third question we first show that the final hypothesis output by the algorithm is consistent with all m examples seen in training. That is, the hypothesis correctly classifies all of these examples. We will also show that the hypothesis logically implies the target concept. This means that the hypothesis does not classify any negative example as positive. We can prove this by induction on the number of examples seen so far. We initialize the hypothesis to the conjunction of all $2n$ literals, which is logically equivalent to classifying every example as false. This is trivially consistent with all examples seen, since initially no examples have been seen. Also, it trivially implies the target concept since false implies anything. Let h_i be the hypothesis after i examples. Assuming the hypothesis h_k is consistent with the first k examples and implies the target concept, we show the hypothesis h_{k+1} is consistent with the first $k+1$ examples and still implies the target concept. If example $k+1$ is negative, $h_{k+1} = h_k$. Since h_k implies the target concept, it does not classify any negative example as positive. Therefore h_{k+1} correctly classifies the first $k+1$ examples and implies the target concept. If example $k+1$ is positive, we alter h_k to include this example within those classified as positive. Clearly h_{k+1} correctly classifies the first $k+1$ examples. In addition, it is not possible for some negative example to satisfy h_{k+1}, so this new hypothesis logically implies the target concept.

To analyze the error of the hypothesis, we define an ϵ-*bad* hypothesis $h^{'}$ as one with $error(h^{'}) > \epsilon$. We satisfy the PAC error criteria if the final hypothesis (which we know is consistent with all m examples seen in training) is not ϵ-bad. By the definition of ϵ-bad,

$$\Pr[\text{an } \epsilon\text{-bad hyp is consistent with 1 ex}] \leq 1 \Leftrightarrow \epsilon$$

and since each example is taken independently,

$$\Pr[\text{an } \epsilon\text{-bad hyp is consistent with m exs}] \leq (1 \Leftrightarrow \epsilon)^m.$$

Since the hypothesis comes from C, the maximum number of hypotheses is $|C|$. Thus,

$$\Pr[\exists \text{ an } \epsilon\text{-bad hyp consistent with m exs}] \leq |C|(1 \Leftrightarrow \epsilon)^m.$$

Now, we require that $\Pr[h \text{ is } \epsilon\text{-bad}] \leq \delta$, so we must choose m to satisfy

$$|C|(1 \Leftrightarrow \epsilon)^m \leq \delta.$$

Solving for m,

$$m \geq \frac{\ln |C| + \ln 1/\delta}{\Leftrightarrow \ln(1 \Leftrightarrow \epsilon)}.$$

Using the Taylor series expansion for e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots > 1 + x,$$

letting $x = \Leftrightarrow \epsilon$ and taking ln of both sides we can infer that $\epsilon < \Leftrightarrow \ln(1 \Leftrightarrow \epsilon)$. Also, $|C| = 3^n$ since each of the variables may appear in the monomial negated, unnegate, or not appear at all. So if $m \geq 1/\epsilon(n \ln 3 + \ln 1/\delta)$ then $\Pr[error(h) > \epsilon] \leq \delta$.

It is interesting to note that only $O(n \ln 3)$ examples are required by the algorithm (ignoring dependence on ϵ and δ) even though there are 2^n examples to classify. Also, the analysis of the number of examples required can be applied to any algorithm which finds a hypothesis consistent with all the examples seen during training. Observe, this is only an upper bound. In some cases a tighter bound on the number of examples may be achievable.

1.5 Learning k-CNF and k-DNF

We now describe how to extend this result for monomials to the more expressive classes of k-CNF and k-DNF. The class *k-Conjunctive Normal Form*, denoted k-CNF$_n$ consists of all Boolean formulas of the form $C_1 \wedge C_2 \wedge \ldots \wedge C_l$ where each clause C_i is the disjunction of at most k literals over $x_1, \ldots, x_n$. We assume k is some constant. We will often omit the subscript n. If $k = n$ then the class k-CNF consists of all CNF formulas. The class of monomials is equivalent to 1-CNF.

The class *k-Disjunctive Normal Form*, denoted k-DNF$_n$ consists of all Boolean formulas of the form $T_1 \vee T_2 \vee \ldots \vee T_l$ where each term T_i is the conjunction of at most k literals over $x_1, \ldots, x_n$.

There is a close relationship between k-CNF and k-DNF. By DeMorgan's law

$$f = C_1 \wedge C_2 \wedge \ldots \wedge C_l \Rightarrow \overline{f} = T_1 \vee T_2 \ldots \vee T_l$$

where T_i is the conjunction of the negation of the literals in C_i. For example if

$$f = (a \vee \overline{b} \vee c) \wedge (\overline{d} \vee e)$$

then by DeMorgan's law

$$\overline{f} = \overline{(a \vee \overline{b} \vee c)} \vee \overline{(\overline{d} \vee e)}.$$

Finally, applying DeMorgan's law once more we get that

$$\overline{f} = (\overline{a} \wedge b \wedge \overline{c}) \vee (d \wedge \overline{e}).$$

Thus the classes k-CNF and k-DNF are duals of each other in the sense that exchanging $\wedge$ for $\vee$ and complementing each variable gives a transformation between the two representations. So an algorithm for k-CNF can be used for k-DNF (and vice versa) by swapping the use of positive and negative examples, and negating all the attributes.

We now describe a general technique for modifying the set of variables in the formula and apply this technique to generalize the monomial algorithm given in the previous section to learn k-CNF. The idea is that we define a new set of variables, one for each possible clause. Next we apply our monomial algorithm using this new variable set. To do this we must compute the value for each new variable given the value for the original variables. However, this is easily done by just evaluating the corresponding clause on the given input. Finally, it is straightforward to translate the hypothesis found by the monomial algorithm into a k-CNF formula using the correspondence between the variables and the clauses.

We now analyze the complexity of the above algorithm for learning k-CNF. Observe that the number of possible clauses is upper bounded by:

$$\sum_{i=1}^{k} \binom{2n}{k} = O(n^k).$$

This overcounts the number of clauses since it allows clauses containing both x_i and $\overline{x}_i$. The high order terms of the summation dominate. Since $\binom{2n}{k} < (2n)^k$, the number of clauses is $O(n^k)$ which is polynomial in n for any constant k. Thus the time and sample complexity of our k-CNF algorithm are polynomial.

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Topic 2: Two-Button PAC Model

Lecturer: Sally Goldman *Scribe: Ellen Witte*

2.1 Model Definition

Here we consider a variant of the single button PAC model described in the previous notes. The material presented in this lecture is just one portion of the paper, "Equivalence of Models for Polynomial Learnability," by David Haussler, Michael Kearns, Nick Littlestone, and Manfred K. Warmuth [21]. Rather than pushing a single button to receive an example drawn randomly from the distribution D, a variation of this model has two buttons, one for positive examples and one for negative examples. (In fact, the this is the model originally introduced by Valiant.) There are two distributions, one on the positive examples and one on the negative examples. When the positive button is pushed a positive example is drawn randomly from the distribution D^+. When the negative button is pushed a negative example is drawn randomly from the distribution D^-. A learning algorithm in the two-button model is required to be accurate within the confidence bound for both the positive and negative distributions. Formally, we require

$$\Pr[error^+(h) \equiv \Pr_{D^+}(c \oplus h) \leq \epsilon] \geq 1 \Leftrightarrow \delta$$

$$\Pr[error^-(h) \equiv \Pr_{D^-}(c \oplus h) \leq \epsilon] \geq 1 \Leftrightarrow \delta.$$

Offering two buttons allows the learning algorithm to choose whether a positive or negative example will be seen next. It should be obvious that this gives the learner at least as much power as in the one button model. In fact, we will show that the one-button and two-button models are equivalent in the concept classes they can learn efficiently.

2.2 Equivalence of One-Button and Two-Button Models

In this section we show that one-button PAC-learnability is equivalent to two-button PAC-learnability.

Theorem 2.1 *One-button PAC-learnability (1BL) is equivalent to two-button PAC-learnability (2BL).*

Proof:
Case 1: 1BL $\Rightarrow$ 2BL.

Let $c \in C$ be a target concept over an instance space X that we can learn in the one-button model using algorithm A_1. We show that there is an algorithm A_2 for learning c in the two-button model. Let D^+ and D^- be the distributions over the positive and negative examples respectively. The algorithm A_2 is as follows:

1. Run A_1 with inputs n, $\epsilon/2$, δ.
2. When A_1 requires an example, flip a fair coin. If the outcome is heads, push the positive example button and give the resulting example to A_1. If the outcome is tails, push the negative example button and give the resulting example to A_1.
3. When A_1 terminates, return the hypothesis h found by A_1.

The use of the fair coin to determine whether to give A_1 a positive or negative example results in a distribution D seen by A_1 defined by,

$$D(x) = 1/2D^+(x) + 1/2D^-(x).$$

Let e be the error of h on D, e^+ be the error of h on D^+ and e^- be the error of h on D^-. Then

$$e = e^+/2 + e^-/2.$$

Since A_1 is an algorithm for PAC learning, it satisfies the error criteria. That is,

$$\Pr[e \leq \epsilon/2] \geq 1 \Leftrightarrow \delta.$$

Since $e \geq e^+/2$ and $e \geq e^-/2$ we can conclude that

$$\Pr[e^+/2 \leq \epsilon/2] \geq 1 \Leftrightarrow \delta$$

$$\Pr[e^-/2 \leq \epsilon/2] \geq 1 \Leftrightarrow \delta.$$

Therefore, $e^+ \leq \epsilon$ and $e^- \leq \epsilon$ each with probability $\geq 1 \Leftrightarrow \delta$, and so algorithm A_2 satisfies the accuracy criteria for the two-button model. Notice that we had to run A_1 with an error bound of $\epsilon/2$ in order to achieve an error bound of ϵ in A_2.

Case 2: 2BL $\Rightarrow$ 1BL.

Let $c \in C$ be a target concept over instance space X that we can learn in the two-button model using algorithm A_2. We show that there is an algorithm A_1 for learning c in the one-button model.

The idea behind the algorithm is that A_1 will draw some number of examples from D initially and store them in two bins, one bin for positive examples and one bin for negative examples. Then A_1 will run A_2. When A_2 requests an example, A_1 will supply one from the appropriate bin. Care will need to be exercised because there may not be enough of one type of example to run A_2 to completion. Let m be the total number of examples (of both types) needed to run A_2 to completion with parameters n, ϵ, $\delta/3$. Algorithm A_1 is as follows:

1. Make q calls to EX and store the positive examples in one bin and the negative examples in another bin, where
$$q = \max\left\{\frac{2}{\epsilon}m, \frac{8}{\epsilon}\ln\frac{3}{\delta}\right\}.$$

2. If the number of positive examples is $< m$ then output the hypothesis false. (This hypothesis classifies all instances as negative.)

3. Else if the number of negative examples is $< m$ then output the hypothesis true. (This hypothesis classifies all instances as positive.)

4. Else there are enough positive and negative examples, so run A_2 to completion with parameters (n,min$(\epsilon, 1/2)$,$\delta/3$). Output the hypothesis returned by A_2.

For now this choice of q seems to have appeared out of thin air. The rationale for the choice will become clear in the analysis of the algorithm. Before continuing the analysis, we need an aside to discuss Chernoff Bounds, which will be needed in the proof.

Aside: Chernoff Bounds

Chernoff Bounds are formulas which bound the area under the tails of the binomial distribution. We now describe some of the bounds cited in the literature. Much of this discussion is taken directly from Sloan [40]. Normally we are trying to say that if we run m Bernoulli trials each with probability of success p, then the chance of getting a number of successes very much different from pm is exponentially vanishing.

Formally, let $X_1, X_2, \ldots X_m$ be independent Boolean random variables each with probability of p ($0 \leq p \leq 1$) of being 1. We now define a random variable $S = \sum_{i=1}^{m} X_i$. Clearly the expectation of S is pm.

Define $LE(p, m, r) = \Pr[S \leq r]$ (i.e. the probability of at most r successes in m independent trials of a Bernoulli random variable with probability of success p). Let $GE(p, m, r) = \Pr[S \geq r]$ (i.e. the probability of at least r successes in m independent trials of a Bernoulli random variable with probability of success p). So $LE(p, m, r)$ bounds the area in the tail at the low end of the binomial distribution, while $GE(p, m, r)$ bounds the area in the tail at the high end of the binomial distribution.

Hoeffding's Inequality [22] states that:

$$\begin{aligned} \Pr[S \geq pm + t] &\leq e^{-2mt^2} \quad (2.1) \\ \Pr[S \geq \alpha m], \Pr[S \leq \alpha m] &\leq e^{-2m(\alpha - p)^2} \quad (2.2) \end{aligned}$$

where it is understood that in Equation (2.2) the first α must be at least p and the second α must be at most p.

The above bound is as good or better than any of the others in the literature except for the case when $p < 1/4$. In this case the following bounds given by Angluin and Valiant [8] are better:

$$LE(p, m, (1 \Leftrightarrow \alpha)pm) \leq e^{-\alpha^2 mp/2} \tag{2.3}$$

$$GE(p, m, (1 + \alpha)pm) \leq e^{-\alpha^2 mp/3}. \tag{2.4}$$

where $0 \leq \alpha \leq 1$. Note that in Equation (2.3), $r = (1 \Leftrightarrow \alpha)pm \leq pm$. And in Equation (2.4), $r = (1 + \alpha)pm \geq pm$.

We now return to the analysis of algorithm A_1. Let p^+ denote the probability that we draw a positive example from D, and p^- denote the probability that we draw a negative example from D. To analyze the accuracy of algorithm A_1 we consider four cases based on the probabilities p^+ and p^-. For each case we will show that $\Pr[error(h) \leq \epsilon] \geq 1 \Leftrightarrow \delta$.

Subcase A: $p^+ \geq \epsilon$, $p^- \geq \epsilon$.

There are three things which may happen in algorithm A_1. First, we may not have enough of one type of example and thus will output a hypothesis of true or false. Notice that if this occurs then $\Pr[error_D(h) \leq \epsilon] = 0$ since $p^+, p^- \geq \epsilon$. Second, we may have enough examples to run A_2 to completion and receive an ϵ-bad hypothesis from A_2. Third, we may have enough examples to run A_2 to completion receive an ϵ-good hypothesis from A_2. The first two outcomes are undesirable. We must make sure they occur with probability at most δ.

Let us determine the probability that we do not have enough of one type of example. Consider the positive examples. The probability of drawing a positive example is $p^+ \geq \epsilon$. We are interested in the probability that we draw fewer than m positive examples in q calls to EX. This probability can be bounded by the Chernoff bound of Equation (2.3).

$$\Pr[< m \text{ positive exs in } q \text{ calls to EX}] \leq LE(\epsilon, q, m)$$

From Equation (2.3) we know that

$$LE(p, m', (1 \Leftrightarrow \alpha)m'p) \leq e^{-\alpha^2 m' p/2}.$$

So, with $p = \epsilon$,$m' = q = \frac{2}{\epsilon}m$ and $\alpha = 1/2$ we obtain

$$LE(\epsilon, q, m) \leq e^{-m/4}.$$

Finally, we require that this bad even occurs with probability at most $\delta/3$. That is, we must have

$$e^{-m/4} \leq \delta/3.$$

Solving for m yields

$$m \geq 4 \ln \frac{3}{\delta} \Rightarrow q \geq \frac{8}{\epsilon} \ln \frac{3}{\delta}$$

which is satisfied by the choice of q in the algorithm.

The same analysis holds for bounding the probability that there are fewer than m negative examples in q calls to EX. That is, we know

$$\Pr[< m \text{ negative exs in } q \text{ calls to EX}] \leq \delta/3$$

$$\Pr[< m \text{ positive exs in } q \text{ calls to EX}] \leq \delta/3.$$

This implies that

$$\Pr[\geq m \text{ positive exs and } \geq m \text{ negative exs in } q \text{ calls to EX}] \geq 1 \Leftrightarrow 2\delta/3.$$

If we have enough positive and negative examples then we will run algorithm A_2 to completion. With probability $\geq 1 \Leftrightarrow \delta/3$ algorithm A_2 will return a hypothesis with $e^+ \leq \epsilon$ and $e^- \leq \epsilon$. This implies that the hypothesis h returned by A_2 satisfies:

$$\begin{aligned} error_D(h) &= p^+e^+ + p^-e^- \\ &= p^+e^+ + (1 \Leftrightarrow p^+)e^- \\ &\leq p^+\epsilon + (1 \Leftrightarrow p^+)\epsilon \\ &= \epsilon \end{aligned}$$

We have determined the following probabilities of each of the two bad outcomes that can occur when running A_1.

- With probability $\leq 2\delta/3$ we do not have enough of one type of example to run A_2.
- The probability that we run A_2 and it returns a bad hypothesis is given by the product of the probability that we have enough examples and the probability A_2 returns a bad hypothesis. This probability is at most $\left(1 \Leftrightarrow \frac{2\delta}{3}\right)\frac{\delta}{3} \leq \delta/3$.

In all other cases, the hypothesis output will be ϵ-good. Combining these probabilities yields the following expression for the probability that the good outcome occurs.

$$\Pr[error_D(h) \leq \epsilon] \geq 1 \Leftrightarrow \left(\frac{2\delta}{3} + \frac{\delta}{3}\right) = 1 \Leftrightarrow \delta$$

Subcase B: $p^+ < \epsilon$, $p^- \geq \epsilon$.

In this case, we have a good chance of getting more than m negative examples and less than m positive examples. If this occurs algorithm A_1 will output a hypothesis of false. Since $p^+ < \epsilon$, a hypothesis h of false satisfies $\Pr[error_D(h) \leq \epsilon] = 1$. This is good. It is also possible (although less likely) that we will not get enough negative examples, or that we will get enough of each kind of example to run A_2. In running A_2 we may get a hypothesis which has error $> \epsilon$. We must ensure that these bad things occur with probability at most δ.

More formally, we consider two possible bad situations. First, we may draw less than m negative examples. In Subcase A we showed that this will occur with probability $\leq \delta/3$. In this situation we will draw enough positive examples and thus output a hypothesis h of true. For this hypothesis, $\Pr[error_D(h) > \epsilon] = 1$. In the second situation, we draw at least m negative examples. This occurs with probability $\geq 1 \Leftrightarrow \delta/3$. In the worst case we also draw at least m positive examples and thus run A_2. We know that the hypothesis h returned by A_2 will satisfy $\Pr[error_D(h) \leq \epsilon] \geq 1 \Leftrightarrow \delta/3$. This implies that $\Pr[error_D(h) > \epsilon] \leq \delta/3$. Combining these two results we have

$$\Pr[error_D(h) > \epsilon] \leq \delta/3 + \delta/3$$

which implies that

$$\Pr[error_D(h) \leq \epsilon] \geq 1 \Leftrightarrow 2\delta/3.$$

Subcase C: $p^- < \epsilon$, $p^+ \geq \epsilon$.

This case is the same as Subcase B but with the roles of positive and negative examples interchanged. By a similar analysis,

$$\Pr[error_D(h) \leq \epsilon] \geq 1 \Leftrightarrow 2\delta/3.$$

Subcase D: $p^+ < \epsilon$, $p^- < \epsilon$.

Since $\epsilon \leq 1/2$, this case cannot occur.

These four subcases cover the behavior of algorithm A_1 for all possible values of p^+ and p^-. Thus, 2BL $\Rightarrow$ 1BL. Taken with Case 1 proving 1BL $\Rightarrow$ 2BL we have shown that one-button PAC learnability is equivalent to two-button PAC learnability.

■

CS 582T Computational Learning Theory **Spring Semester, 1991**

Topic 3: Learning k-term-DNF

Lecturer: Sally Goldman *Scribe: Weilan Wu*

3.1 Introduction

In this lecture, we shall discuss the learnability of k-term-DNF formulas. Most of the material presented in this lecture comes from the paper "Computational Limitations on Learning from Examples," by Leonard Pitt and Leslie Valiant [33].

When introducing the PAC model, we showed that both k-CNF and k-DNF are PAC learnable using a hypothesis space of k-DNF and k-CNF respectively. Does a similar result hold for k-term-DNF and k-clause-CNF? We first show that k-term-DNF is not PAC learnable by k-term-DNF in polynomial time, unless RP=NP. In fact, this hardness result holds even for learning monotone k-term-DNF by k-term-DNF. Likewise, k-clause-CNF is not PAC learnable by k-clause-CNF in both the monotone and unrestricted case. Contrasting these negative results, we then describe an efficient algorithm to learn k-term-DNF by k-CNF. As a dual result, we can learn k-clause-CNF using the hypothesis space of k-DNF. Finally, we explore the representational power of the concept classes that we have considered so far and the class of k-decision-lists [35].

Before describing the representation-dependent hardness result, we first give some basic definitions. The concept class of k-term-DNF is defined as follows:

Definition 3.1 *For any constant k, the class of k-term-DNF formulas contains all disjunctions of the form $T_1 \vee T_2 \vee \ldots \vee T_k$, where each T_i is monomial.*

Up to now we have assumed that in the PAC model the hypothesis space available to the learner is equivalent to the concept class. That is, each element of the hypothesis space corresponds to the representation of an element of the concept space. However, in general one can talk about a concept class C being PAC learnable by H (possibly different from C). Formally, we say that C is PAC learnable by H, if there exists a polynomial-time learning algorithm A such that for any $c \in C$, any distribution D and any ϵ, δ, A can output with probability at least $1 \Leftrightarrow \delta$ a hypothesis $h \in H$ such that h has probability at most ϵ of disagreeing with c on a randomly drawn instance from D.

3.2 Representation-dependent Hardness Results

In this section, we will show that for $k \geq 2$, k-term-DNF is not PAC learnable using a hypothesis from k-term-DNF. This type of hardness result is *representation-dependent* since it only holds if the learner's hypothesis class (or representation class) is restricted to be a certain class. Note that when $k = 1$, the class of k-term-DNF formulas is just the class of monomials, which we know is learnable using a hypothesis of a monomial.

We prove this hardness result by by reducing the learning problem to k-NM-Colorability, a generalization of the Graph k-Colorability problem. Before defining this problem we first describe two known NP-complete problems: Graph k-colorability (NP-complete for $k \geq 3$) and Set Splitting. These descriptions come from Garey and Johnson [14].

Problem: Graph k-colorability
Instance: For a graph $G = (V, E)$, with positive integer $k \leq |V|$.
Question: Is G k-colorable? That is, does there exist a function $f : V \rightarrow \{1, \dots, k\}$, such that $f(u) \neq f(v)$ whenever $\{u, v\} \in E$?

Problem: Set Splitting
Instance: Collection C of subsets of a finite set S.
Question: Is there a partition of S into two subsets S_1, S_2, such that no subset in C is entirely contained in either S_1 or S_2?

We now generalize both of these problems to obtain the k-NM-Colorability problem which we use for our reduction.

Problem: k-NM-Colorability
Instance: A finite set S and a collection $C = \{c_1, \dots, c_m\}$ of constraints $c_i \subseteq S$.
Question: Is there a k-coloring χ of the elements of S, such that for each constraints $c_i \in C$, the elements of c_i are not MONOCHROMATICALLY colored (i.e. $\forall c_i \in C$, $\exists x, y \in c_i$, such that $\chi(x) \neq \chi(y)$)?

We now argue that k-NM-Colorability is NP-complete. Clearly k-NM-Colorability is in NP. Note that if every $c_i \in C$ has size 2, then the k-NM-Colorability problem is simply the Graph-k-Colorability problem. Since the Graph-k-Colorability is NP-complete for $k \geq 3$, we only need to show that 2-NM-Colorability is NP-hard. However, note that 2-NM-Colorability is exactly the Set Splitting problem which is NP-complete. Thus it follows that k-NM-Colorability is NP-complete.

We now prove the main results of this section that k-term-DNF is not PAC learnable by k-term-DNF.

Theorem 3.1 *For all integers $k \geq 2$, k-term DNF is not PAC learnable (in polynomial time) using a hypothesis from k-term-DNF unless RP=NP.*

Proof: We reduce k-NM-coloring to the k-term-DNF learning problem. Let (S, C) be an instance of k-NM-coloring, we construct a k-term-DNF learning problem as follows: Each instance (S, C) will correspond to a particular k-term-DNF formula to be learned. If $S = \{s_1, \dots, s_n\}$, then we will create n variables $\{x_1, \dots, x_n\}$ for the learning problem. We now describe the positive and negative examples, as well as the distributions D^+ and D^-.

- The positive examples are $\{\vec{p}_i\}_{i=1}^{n}$, where $\vec{p}_i$ is the vector with x_i=0, and for $j \neq i$,

x_j=1. Thus there are n positive examples. Finally, let D^+ be uniform over these positive examples (i.e. each has weight $1/n$).

- The negative examples are $\{\vec{n}_i\}_{i=1}^{|C|}$, where for each constraint $c_i \in C$, if $c_i = \{s_{i_1}, \ldots, s_{i_m}\}$, then $\vec{n}_i = \vec{0}_{i_1,i_2,\ldots,i_m}$ (all elements of S in c_i are 0, the others are 1). For example, if the constraint c_i is $\{s_1, s_3, s_8\}$, then the vector corresponding to is: $\vec{n}_i =< 0101111011\ldots >$. Finally, let D^- be uniform over these negative examples (so each has weight $1/|C|$).

We now show that a k-term-DNF formula is consistent with all the positive and negative examples defined above if and only if (S, C) is k-NM-colorable. Then we use this claim to show that the learning problem is solvable in polynomial time if and only if RP= NP.

Claim 3.1 *There is a k-term-DNF formula consistent with all positive and negative examples defined above if and only if (S, C) is k-NM-colorable.*

Proof of Claim:
($\Longleftarrow$) Without loss of generality, assume that (S, C) is k-NM-colorable by a coloring $\chi : S \to \{1, 2, \ldots, k\}$, which uses every color at least once. Let f be the k-term-DNF formula $T_1 \vee T_2 \vee \ldots \vee T_k$, where

$$T_i = \bigwedge_{\chi(s_j)\neq i} x_j.$$

In other words, T_i is the conjunction of all x_j corresponding to s_j that are not colored with i.

We now show that f is consistent with positive examples. The positive example $\vec{p}_j$ ($x_j = 0$, $x_i = 1$ for all $i \neq j$) clearly satisfies the term T_i, where $\chi(s_j) = i$. Thus, f is true for all positive examples.

Finally, we show that f is consistent with the negative examples. Suppose some negative example, say $\vec{n}_i = \vec{0}_{i_1,\ldots,i_m}$ satisfies f, then $\vec{n}_i$ satisfies some term, say T_j. Then every element of constraint $c_i = \{s_{i_1}, \ldots, s_{i_m}\}$ must be colored j, (They are 0 in $\vec{n}_i$ and thus must not be in T_j, hence they are colored with j). But then c_i is monochromatic, giving a contradiction.

($\Longrightarrow$) Suppose $T_1 \vee \ldots \vee T_k$ is a k-term-DNF formula consistent with all positive examples and no negative examples. We now show that, without loss of generality, we can assume for all i, T_i is a conjunction of positive literals.

Case 1: T_i contains at least two negated variables. However, all positive examples have a single 0, so none could satisfy T_i. Thus just remove T_i.

Case 2: T_i contains 1 negated variable $\overline{x_j}$. Then T_i can only be satisfied by the single positive example $\vec{p}_j$. In this case, replace T_i by $T_i' = \bigwedge_{j\neq i} x_j$, which is satisfied only by the vectors $\vec{p}_j$ and $\vec{1}$, neither of which are negative examples.

Thus we now assume that all terms are a conjunction of positive literals. Now color the elements of S by the function: $\chi : S \to \{1, \ldots, k\}$, defined by $\chi(s_i) = \min\{j : x_i \text{ does not occur in } T_j\}$.

Now we show χ is well defined. Since each positive example p_i satisfies $T_1 \vee \ldots \vee T_k$, it must satisfy some term T_j. But each term is a conjunct of unnegated literals. Thus for some j, x_i must not occur in T_j. Thus each element of S receives a color (which is clearly unique).

Finally we show that χ obeys the constraints. Suppose χ violates constraint c_i, then all of the elements in c_i are colored by the same color, say j. By the definition of χ, none of the literals corresponding to elements in c_i occur in term T_j, so the negative example $\vec{n}_i$ associated with c_i satisfies T_j. This contradicts the assumption that none of the negative examples satisfy the formula $T_1 \vee \ldots \vee T_k$. This completes the proof of the claim.

We now complete the proof of the theorem. Namely, we show how a learning algorithm for k-term-DNF can be used to decide k-NM-colorability in random polynomial time. First we give the definition of complexity class RP.

Definition 3.2 *A set S is accepted in the random polynomial time (i.e. S is in RP) if there exists a randomized algorithm A such that on all inputs, A is guaranteed to halt in polynomial time and, if $x \notin S, A(x) =$ "no", if $x \in S, \Pr[A(x) = \text{"yes"}] \geq 1/2$.*

Now we show that if there is a PAC learning algorithm for k-term-DNF, it can be used to decide k-NM-colorability in randomized polynomial time. Given instance (S, C), let D^+ and D^- be defined as above. Choose $\delta < \frac{1}{2}$, $\epsilon < min\{\frac{1}{|S|}, \frac{1}{|C|}\}$.

If (S, C) is k-NM-Colorable, then by the above claim there exists a k-term-DNF formula consistent with the positive and negative examples, so with probability at least $1 \Leftrightarrow \delta$, our learning algorithm will be able to find it.

Conversely, if (S, C) is not k-NM-Colorable, by the Claim, there does not exist a consistent k-term-DNF formula, and the learning algorithm must either fail to produce a hypothesis in the allotted time, or produce one that is not consistent with at least one example. In either case, this can be observed, and we can determine that no legal k-NM-coloring is possible. ■

Thus we have shown that for $k \geq 2$, k-term-DNF is not PAC learnable by k-term-DNF. Furthermore, note that the target function f created in this proof is monotone and thus this result holds even if the concept class is monotone k-term-DNF and the hypothesis class is k-term-DNF. Finally, a dual hardness result applies for learning k-clause-CNF by k-clause-CNF.

3.3 Learning Algorithm for k-term-DNF

Although k-term-DNF is not PAC learnable by k-term-DNF, we now show that it is learnable using a hypothesis from k-CNF.

Let $f = T_1 \vee T_2 \vee \ldots \vee T_k$ be the target formula, where

$$T_1 = y_1^{(1)} \wedge y_2^{(1)} \wedge \ldots \wedge y_{m_1}^{(1)}$$

$$\begin{aligned} T_2 &= y_1^{(2)} \wedge y_2^{(2)} \wedge \ldots \wedge y_{m_2}^{(2)} \\ &\vdots \\ T_k &= y_1^{(k)} \wedge y_2^{(k)} \wedge \ldots \wedge y_{m_k}^{(k)} \end{aligned}$$

and $y_i^{(j)}$ represents one of the n variables.

By distributing, we can rewrite f as:

$$f = \bigwedge_{i_1, i_2, \ldots, i_k} (y_{i_1}^{(1)} \vee y_{i_2}^{(2)} \vee \ldots \vee y_{i_k}^{(k)}).$$

Now to learn f using a hypothesis form k-CNF, we introduce $O(2n^k)$ variables $\alpha_1, \alpha_2, \ldots, \alpha_m$, representing all disjunctions of the form: $y_{i_1}^{(1)} \vee y_{i_2}^{(2)} \vee \ldots \vee y_{i_k}^{(k)}$. Learning the conjunction over the α's is equivalent to learning the original disjunction. Note, however, we may not (in general) transform the conjunction we obtain to a k-term-DNF formula, thus we must output it as a k-CNF formula.

3.4 Relations Between Concept Classes

In this section, we briefly study the containment relations between various concept classes.

We have already seen that k-term-DNF is a subclass of k-CNF. We now show that k-term-DNF is properly contained in k-CNF by exhibiting a k-CNF formula which can not be expressed as a k-term-DNF formula. The k-CNF formula, $(x_1 \vee x_2) \wedge (x_3 \vee x_4) \wedge \ldots (x_{2k-1} \vee x_{2k})$, has 2^k terms when we unfold it into the form of DNF formula and thus cannot be represented by a k-term-DNF formula.

As a dual result, it is easily shown that k-clause-CNF is properly contained in k-DNF. Thus it follows that, k-CNF $\cup$ k-DNF $\cup$ k-term-DNF $\cup$ k-clause-CNF is *PAC* learnable.

We now consider the concept class k-DL as defined by Rivest [35]. Consider a Boolean function f that is defined on $\{0,1\}^n$ by a nested **if–then–else** statement of the form:

$$f(x_1, x_2, \ldots, x_n) = \textbf{if } l_1 \textbf{ then } c_1 \textbf{ elseif } l_2 \textbf{ then } c_2 \cdots \textbf{ elseif } l_k \textbf{ then } c_k \textbf{ else } c_{k+1}$$

where the l_j's are literals (either one of the variables or their negations), and the c_j's are either **T** (true) or **F** (false). Such a function is said to be computed by a *simple decision list*. The concept class k-decision lists (k-DL) is just the extension of a simple decision list where the condition in each **if** statement may be the conjunction of up to k literals, for some fixed constant k. We leave it as a simple exercise to show that k-CNF $\cup$ k-DNF is properly contained in k-DL. Thus the first problem of Homework 1 provides an even stronger positive result by showing that k-DL is PAC learnable using a hypothesis from k-DL.

CS 582T Computational Learning Theory **Spring Semester, 1991**

Topic 4: Handling an Unknown Size Parameter

Lecturer: Sally Goldman

4.1 Introduction

We have seen in earlier notes how to PAC learn a k-CNF formula. Recall that the algorithm used for learning k-CNF created a new variable corresponding to each possible term of at most k literals and then just applied the algorithm for learning monomials. Observe that this algorithm assumes that the learner knows k a priori. Can we modify this algorithm to work when the learner does not have prior knowledge of k? Here we consider a variant of the PAC model introduced in Topic 1 in which there is an unknown size parameter. The material presented in this lecture is just one portion of the paper, "Equivalence of Models for Polynomial Learnability," by David Haussler, Michael Kearns, Nick Littlestone, and Manfred Warmuth [21].

4.2 The Learning Algorithm

In this section we outline a general procedure to convert a PAC-learning algorithm A that assumes a known size parameter s (e.g. the k in k-CNF) to a PAC-learning algorithm B that works without prior knowledge of s. As an example application, this procedure will enable us to convert our algorithms for learning k-CNF, k-CNF, k-term-DNF, k-clause-CNF, or k-DL into corresponding algorithms that do not have prior knowledge of k. Observe, that while the learner does not know s, as one would expect the running time and sample complexity of B will still depend on s. The most optimistic goal would be to have the time and sample complexity of B match that of A.

The basic idea of this conversion is as follows. Algorithm B will run algorithm A with an estimate $\hat{s}$ for s such that this estimate is gradually increased. The key question is: How does algorithm B know when its estimate for s is sufficient? The technique of *hypothesis testing* used to solve this problem is a general technique which is also useful in other situations.

Aside: Hypothesis Testing

We now describe a technique to test if a hypothesis is good. More specifically, given a hypothesis h, an error parameter ϵ, and access to an example oracle EX we would like to determine with high probability if h is an ϵ-good hypothesis. Clearly it is not possible to distinguish a hypothesis with error ϵ from one with error just greater than ϵ, however, we can distinguish an $\epsilon/2$-good hypothesis from an ϵ-bad one. As we shall see this is sufficient to know when our estimate for the size parameter is large enough.

We now formally describe the hypothesis testing algorithm.

Algorithm Test(h,n,ϵ,δ)

1. Make $m = \left\lfloor \frac{32}{\epsilon}(n \ln 2 + \ln 2/\delta) \right\rfloor$ call to EX.
2. Accept h if it misclassifies at most $\frac{3\epsilon}{4}$ of the examples. Otherwise, reject h.

We now prove that this hypothesis testing procedure achieves the goal stated above.

Lemma 4.1 *The procedure Test when called with parameters h, n, ϵ, and δ has the property that:*

1. *If $error(h) \geq \epsilon$, then* Prob[h is accepted] $\leq \frac{\delta}{2^{n+1}}$
2. *If $error(h) \leq \epsilon/2$, then* Prob[h is rejected] $\leq \frac{\delta}{2^{n+1}}$

Proof Sketch:

We first sketch the proof showing the first property holds. Let p be the error of hypothesis h. Then,

$$\text{Prob}[h \text{ is accepted}] \leq LE\left(p, m, \frac{3}{4}\epsilon m\right).$$

Finally, since $p \geq \epsilon$ it follows that

$$LE\left(p, m, \frac{3}{4}\epsilon m\right) \leq LE\left(p, m, (1 \Leftrightarrow \frac{1}{4})mp\right) \leq e^{-\frac{m\epsilon}{32}}.$$

Plugging in the value of m used in Test, we get the stated result.

For the second property we know that $p \leq \epsilon/2$ and thus the probability of rejecting h is bounded above by:

$$GE\left(p, m, \frac{3}{4}\epsilon m\right) \leq GE\ paren\frac{\epsilon}{2}, m, (1 + \frac{1}{2})m\frac{\epsilon}{2} \leq e^{-\frac{m\epsilon}{24}}.$$

Again this gives the desired bound. ■

Finally, we note that a two-oracle version of this hypothesis testing procedure can be constructed by running the one oracle version Test twice (replacing δ by $\delta/2$), once using the positive oracle and once using the negative oracle. The two-oracle testing procedure accepts h if an only if both of the above calls to Test accept h. The above lemma also holds for this two-oracle testing procedure.

We now return to the problem of handling an unknown size parameter by describing how algorithm B (unknown s) can be implemented using Algorithm A (known s) as a subroutine.

Let $p(n, s, 1/\epsilon) = \max\{S_A(n, s, \epsilon, 1/2), T_A(n, s, \epsilon, 1/2)\}$ where S_A is the sample complexity of algorithm A and T_A is the time complexity of algorithm A. We know describe algorithm B.

Algorithm $B(n, \epsilon, \delta)$

```
1    $i \leftarrow 0$
2    UNTIL $h$ is accepted by Test$(h, n, \epsilon, \delta)$ DO
3        $i \leftarrow i + 1$
4        $\hat{s} \leftarrow \left\lfloor 2^{(i-1)/\ln(\frac{2}{\delta})} \right\rfloor$
5        $h_i \leftarrow$ hypothesis output by $A(n, \hat{s}, \epsilon/2, 1/2)$
6    Output $h = h_i$
```

Theorem 4.1 *Let h be the hypothesis output by algorithm B as described above. Then* $\text{Prob}[error(h) \leq \epsilon] \geq 1 \Leftrightarrow \delta$.

Proof Sketch:

Observe that algorithm B's estimate $\hat{s} \geq s$ at the ith repetition for all $i \geq \left\lceil 1 + \ln \frac{2}{\delta} \log_2 s \right\rceil$. Since the size parameter is only an upperbound on the allowable size and algorithm A is a PAC-learning algorithm we know that for any iteration in which $\hat{s} \geq s$, the $\text{Prob}[h_i \leq \epsilon/2] \geq 1/2$. In such a case, the $\text{Prob}[h_i$ is accepted by Test$] \geq 3/4$. So if $\hat{s} \geq s$ then the $\text{Prob}[B$ halts with hyp. of error $\leq \epsilon/2] \geq 3/8$.

Let $j = \lfloor (\ln 2/\delta)/(\ln 8/5) \rfloor$. Then

$$\text{Prob}[B \text{ fails to halt after } j \text{ iterations with } \hat{s} \geq s] \leq \left(\frac{5}{8}\right)^j \leq \delta/2.$$

Thus with probability at least $1 \Leftrightarrow \delta/2$, B will halt after at most

$$j' = \left\lceil \ln \frac{2}{\delta} \log_2 s \right\rceil + \left\lceil \frac{\ln 2/\delta}{\ln 8/5} \right\rceil$$

iterations.

Also the probability is at most $\delta/2$ that any call to Test will accept a hypothesis with error greater than ϵ. Thus with probability $\geq 1 \Leftrightarrow \delta$, algorithm B will halt after at most j' repetitions with an ϵ-good hypothesis.

Finally, one can verify that the time an sample complexity after j' iterations is still polynomial. We refer the reader to the paper by Haussler et al. [21] for the details. ■

CS 582T Computational Learning Theory **Spring Semester, 1991**

Topic 5: Learning with Noise

Lecturer: Sally Goldman *Scribe: Gadi Pinkas*

5.1 Introduction

Although up to now we have assumed that the data provided by the example oracle is noise-free, in real-life learning problems this assumption is almost never valid. Thus we would like to be able to modify our algorithms so that they are robust against noise. Before considering learning with noise in the PAC model, we must first formally model the noise. Although we will only focus on one type of noise here, we first describe the various formal models of noise that have been considered.

In all cases we assume that the usual noise free examples pass through a noise oracle before being seen by the learner. Each noise oracle represents some noise process being applied to the examples from EX. The output from the noise process is all the learner can observe. The "desired," noiseless output of each oracle would thus be a correctly labeled example (x, s), where x is drawn according to the unknown distribution D. We now describe the actual outputs from the following noise oracles:

Random Misclassification Noise [7]: This noise oracle models a benign form of misclassification noise. When it is called, it calls EX to obtain some (noiseless) (x, s), and with probability $1 \Leftrightarrow \eta$, it returns (x, s). However, with probability η, it returns $(x, \overline{s})$.

Malicious Noise [44]: This oracle models the situation where the learner usually gets a correct example, but some small fraction η of the time the learner gets noisy examples and the nature of the noise is unknown or unpredictable. When this oracle is called, with probability $1 \Leftrightarrow \eta$, it does indeed return a correctly labeled (x, s) where x is drawn according to D. With probability η it returns an example (x, s) about which no assumptions whatsoever may be made. In particular, this example may be maliciously selected by an adversary who has infinite computing power, and has knowledge of the target concept, D, η, and the internal state of the learning algorithm.

Malicious Misclassification Noise [41]: This noise oracle models a situation in which the only source of noise is misclassification, but the nature of the misclassification is unknown or unpredictable. When it is called, it also calls EX to obtain some (noiseless) (x, s), and with probability $1 \Leftrightarrow \eta$, it returns (x, s). With probability η, it returns (x, l) where l is a label about which no assumption whatsoever may be made. As with malicious noise we assume an omnipotent, omniscient adversary; but in the case the adversary only gets to choose the label of the example.

Uniform Random Attribute Noise [41]: This noise oracle models a situation where the attributes of the examples are subject to noise, but that noise is as benign as possible. For example, the attributes might be sent over a noisy channel. We consider this oracle only when the instance space is $\{0,1\}^n$ (i.e., we are learning Boolean functions). This oracle calls EX and obtains some $(x_1 \cdots x_n, s)$. It then adds noise to this example by independently flipping each bit x_i to $\bar{x}_i$ with probability η for $1 \leq i \leq n$. Note that the label of the "true" example is never altered.

Nonuniform Random Attribute Noise [17]: This noise oracle provides a more realistic model of random attribute noise than uniform random attribute noise.[2] This oracle also only applies when we are learning Boolean functions. This oracle calls EX and obtains some $(x_1 \cdots x_n, s)$. The oracle then adds noise by independently flipping each bit x_i to $\bar{x}_i$ with some fixed probability $\eta_i \leq \eta$ for each $1 \leq i \leq n$.

In this paper we focus on the situation in which there is random misclassification noise. The material presented here comes from the paper "Learning from Noisy Examples," by Dana Angluin and Phil Laird [7]. We show that the hypothesis that minimizes disagreements (i.e. the hypothesis that misclassifies the fewest training examples) meets the PAC correctness criterion when the examples are corrupted by random misclassification noise. Unfortunately, this technique is most often computationally intractable. However, for k-CNF formulas we describe an efficient PAC learning algorithm that works against random misclassification noise. Both positive results need only assume that the noise rate η is less than one half.

Before describing these results, we briefly review what is known about handling the other forms of noise. Sloan [41] has extended the above results to the case of malicious labeling noise. On the other hand, Kearns and Li [25] have shown that the method of minimizing disagreements can only tolerate a small amount of malicious noise. We will study this result in Topic 15.

Unlike the results for labeling noise, in the case of uniform random attribute noise, if one uses the minimal disagreement method, then the minimum error rate obtainable (i.e. the minimum "epsilon") is bounded below by the noise rate [41]. Although the method of minimizing disagreements is not effective against random attribute noise, there are techniques for coping with uniform random attribute noise. In particular, Shackelford and Volper [38] have an algorithm that tolerates large amounts of random attribute noise for learning k-DNF formulas. That algorithm, however, has one very unpleasant requirement: it must be given the *exact* noise rate as an input. Goldman and Sloan [17] describe an algorithm for learning monomials that tolerates large amounts of uniform random attribute noise (any noise rate less than 1/2), and only requires some upper bound on the noise rate as an input. Finally, for nonuniform random attribute noise, Goldman and Sloan [17] have shown that the minimum error rate obtainable is bounded below by one-half of the noise rate, regardless of the technique (or computation time) of the learning algorithm.

[2]Technically, this oracle specifies a family of oracles, each member of which is specified by n variables, $\eta_1, \ldots, \eta_n$, where $0 \leq \eta_i \leq \eta$.

5.2 Learning Despite Classification Noise

Let EX_η be the random misclassification noise oracle with a noise noise rate of η. Thus the standard noise free oracle is EX_0. Also we will use the notation that $C = \{L_1, L_2, \ldots, L_N\}$ where L_* is the target concept. So we only consider the simple case of a finite set of hypothesis.

In this section we will study the random misclassification model and give an algorithm to PAC learn any finite concept space with polynomial sample size (not necessarily with polynomial time). In the next section we show that for $\eta < 1/2$, the class of k-CNF formulas is PAC learnable from EX_η. Observe that the learning problem is not feasible for $\eta \geq 1/2$. If $\eta = 1/2$ the noise distorts all the information and clearly no learning is possible, and when $\eta > 1/2$, we actually learn the complement concept with $\eta < 1/2$.

For now we assume that the learner has an upper bound η_b on the noise rate. That is, $\eta \leq \eta_b < 1/2$. Later we show how to remove this assumption. As one would expect if η_b is very close to $1/2$, we must allow the learner more time and data. In fact, we will require that the time and sample complexity of the learner are polynomial in $1/(1 \Leftrightarrow 2\eta_b)$. Observe that this quantity is inversely proportional to how close η_b is to $1/2$.

5.2.1 The Method of Minimizing Disagreements

Our goal in this section is to study the sample complexity for PAC learning under random misclassification noise. For the noise-free case we have seen that if L_i agrees with at least $m \geq (1/\epsilon)(\ln|C| + \ln(1/\delta)) = (1/\epsilon)\ln(N/\delta)$ samples drawn from EX_0 then $\Pr[error(L_i) \geq \epsilon] \leq \delta$. How much more data is needed when there is random misclassification noise?

In the presence of noise the above approach will fail because there is no guarantee that any hypothesis will be consistent with all the examples. However, if we replace the goal of consistency with that of minimizing the number of disagreements with the examples and permit the sample size to depend on η_b, we get an analogous result for the noisy case.

We shall use the following notation in formally describing the method of minimizing disagreements.

- Let σ be the sequence of examples drawn from EX_η.
- Let $F(L_i, \sigma)$ be the number of times L_i disagrees with σ on an example in σ, where L_i disagrees with σ on an example $(\vec{x}, l) \in \sigma$ if and only if L_i classifies $\vec{x}$ differently from l.

Theorem 5.1 *If we draw a sequence σ of*

$$m \geq \frac{2}{\epsilon^2(1 \Leftrightarrow 2\eta_b)^2} \ln\left(\frac{2N}{\delta}\right)$$

samples from EX_η and find any hypothesis L_i that minimizes $F(L_i, \sigma)$ then

$$\Pr[error(L_i) \geq \epsilon] \leq \delta$$

.

Proof: We shall use the following notation in the proof. Let d_i be the $error(L_i)$. That is, d_i is the probability that L_i misclassifies a randomly drawn example. Let p_i be the probability that an example from EX_η disagrees with L_i. Observe that p_i is the probability that L_i misclassifies a correctly labeled example $(d_i(1 \Leftrightarrow \eta))$, plus the probability that L_i correctly classifies the example but the example has been improperly labeled by the noise oracle $((1 \Leftrightarrow d_i)\eta)$. Thus

$$p_i = d_i(1 \Leftrightarrow \eta) + (1 \Leftrightarrow d_i)\eta = \eta + d_i(1 \Leftrightarrow 2\eta)$$

Note that for the right hypothesis $(L_i = L_*)$, $d_i = 0$ and therefore $p_i = \eta$ (i.e., disagreements are only caused by noise). Since $\eta < 1/2$, it follows that for any hypothesis $p_i \geq \eta$. So all hypothesis have an expected rate of disagreement of at least η.

Let an ϵ-bad hypothesis be one for which $d_i \geq \epsilon$. Then for any ϵ-bad hypothesis L_i, we have

$$p_i \geq \eta + \epsilon(1 \Leftrightarrow 2\eta).$$

Thus we have a separation of at least $\epsilon(1 \Leftrightarrow 2\eta)$ between the disagreement rates of the correct and an ϵ-bad hypothesis. Although η_b is not known, we know that $\eta \leq \eta_b < 1/2$ thus the minimum separation (or gap) is at least $\epsilon(1 \Leftrightarrow 2\eta_b)$. We take advantage of this gap in the following manner. We will draw enough examples from EX_η to guarantee with high probability that no ϵ-bad hypothesis has a observed disagreement rate greater than $\eta + \epsilon(1 \Leftrightarrow 2\eta_b)/2$. Similarly, we will draw enough examples from EX_η to guarantee with high probability that the correct hypothesis has on observed disagreement rate less than $\eta + \epsilon(1 \Leftrightarrow 2\eta_b)/2$. Thus it follows that with high probability L$_*$ will have a lower observed rate of disagreement than any ϵ-bad hypothesis. Thus by selecting the hypothesis with the lowest observed rate of disagreement, the learner knows (with high probability) that this hypothesis has error at most ϵ.

We now formalize this intuition. We will draw m examples from EX_η and compute an empirical estimate for all p_i. That is, we compute $F(L_i, \sigma)$ for every L_i in the hypotheses space. The hypothesis output will be the hypothesis L_i that has the minimum estimate for p_i. What is the probability that L_i is ϵ-bad? Let $s = \epsilon(1 \Leftrightarrow 2\eta_b)$. In order for some ϵ-bad hypothesis L_i to minimize $F(L_i, \sigma)$ either the correct hypothesis must have a high disagreement rate

$$F(L_*, \sigma)/m \geq \eta + s/2$$

or an ϵ-bad hypothesis must have a low disagreement rate $(\leq \eta + s/2)$. Finally, assuming that neither of these bad events occur, since we select the hypothesis L_i that minimizes the disagreement rate we know that:

$$F(L_i, \sigma)/m < \eta + s/2$$

and thus L_i has error of at most ϵ.

Applying Chernoff bounds for the probability that a good hypothesis has high disagreement:

$$\Pr[F(L_*,\sigma)/m \geq \eta + s/2] = GE(\eta, m, m(\eta + s/2) < \delta/(2N) < \delta/2.$$

And if L_i is ϵ-bad then its probability to have low disagreement is:

$$\Pr[F(L_*,\sigma)/m \leq \eta + s/2] = LE(\eta + s, m, m(\eta + s/2) \leq \delta/(2N).$$

Thus the probability that any ϵ-bad hypothesis L_i has $F(L_i,\sigma)/m \leq \eta + s/2$ is at most $\delta/2$. (There are at most $N \Leftrightarrow 1$ hypothesis that are ϵ-bad.) Putting these two equalities together, the probability that some ϵ-bad hypothesis minimizes $F(L_i,\sigma)$ is at most δ. ■

Thus we know that by using the method of minimizing disagreements one can tolerate any noise rate strictly less than 1/2. Furthermore, if the hypothesis minimizing disagreements can be found in polynomial time then we obtain an efficient PAC algorithm for learning when there is random misclassification noise.

5.2.2 Handling Unknown η_b

Until now we have assumed the learner is given an upperbound η_b on the noise rate. What if such an upperbound is not known? We can solve this problem using the technique described in Topic 4 for handling an unknown size parameter. That is, just treat η as the unknown size parameter. The only detail we need to worry about here is how to perform the hypothesis testing. The basic idea is as follows, we draw some examples and estimate the failure probability of each of the hypotheses $L_1, ..., L_N$. The smallest estimate is compared to the current value of η_b. If the estimate $\hat{p_i}$ is less than the current value of η_b, we halt; otherwise we increase η_b and repeat. For details see the Angluin, Laird paper [7].

5.2.3 How Hard is Minimizing Disagreements?

How hard is to find an hypothesis L_i that minimizes $F(L_i,\sigma)$? Unfortunately, the answer is that is usually quite hard. For example consider the domain of all conjunctions of positive literals (monotone monomials). To find a monotone monomial that minimizes the disagreement is a NP-hard problem.

Theorem 5.2 *Given a positive integer n and c and a sample σ. The problem of determining if is there a monotone monomial ψ over n variables such that $F(\pi,\sigma) \leq c$ is NP-complete.*

The result indicates that even for a very simple domain, the approach of directly trying to minimize the disagreement is unlikely to be computationally feasible. However, in the next section we show that for some concept classes, we can bypass the minimization problem (which is hard) and efficiently PAC learn the concepts from noisy examples.

5.3 Learning k-CNF Under Random Misclassification Noise

In the previous section we described how to PAC learn any finite concept space if we remove the condition that our algorithm runs in polynomial time. However, we would like to have efficient algorithms for dealing with noise. In this section we use some of the ideas suggested by the method of minimizing disagreements to get a polynomial time algorithm for PAC learning k-CNF formulas using examples from EX_η.

5.3.1 Basic Approach

Instead of searching for the k-CNF formula with the fewest disagreements, we will test all potential clauses individually, and include those that are rarely false in a positive example. Of course, if a clause is false yet the example is reported as positive then either the clause is not in the target formula, or the label has been inverted by the noise process. Thus if a clause is false on a significant number of positive examples, then we do not want to include the clause in our hypothesis. Observe that we will not be solving an NP-complete problem, the k-CNF formula that is chosen may not minimize the disagreements with the examples, but it will (with probability $\geq 1 \Leftrightarrow \delta$) have an error that is less then ϵ.

We now give some notation that we use in this section.

- Let M be the number of possible clauses of at most k literals ($M \leq (2n+1)^k$), and let C be any such clause.
- Let ϕ_* be the target k-CNF formula
- For all clauses C, let $P_{00}(C) = \text{Prob}[C \text{ is false and } \phi_* \text{ is false}]$.
- For all clauses C, let $P_{01}(C) = \text{Prob}[C \text{ is false but } \phi_* \text{ is true}]$.
- For all clauses C, let $P_0(C) = P_{00}(C) + P_{01}(C) = \text{Prob}[C \text{ is false}]$.

Finally, we need the following two definitions. We say that a clause C is *important* if and only if $P_0(C) \geq Q_I = \epsilon/(16M^2)$. We say a clause C is *harmful* if and only if $P_{01}(C) \geq Q_H = \epsilon/(2M)$. Note that $Q_H \geq Q_I$, so every harmful clause is important. Also, no clause contained in ϕ_* can be harmful, since if ϕ_* classifies an example as positive, the example satisfies all its clauses and C must be true (i.e., $P_{01}(C) = 0$).

The algorithm to PAC learn k-CNF formulas works as follows. We must construct an hypothesis h that is ϵ-good with high probability. To achieve this goal the hypothesis h must have all the important clauses that are not harmful. A non-important clause is almost always assigned the value "true" (by the examples in EX_η), and thus it does not mater if it is included in h or not. On the other hand, a harmful clause must not be included in h, since it is very likely to be falsified by a positive example.

Thus our goal is to find an hypothesis h that includes all important clauses and does not include any harmful clause. We first prove that if we find such a hypothesis h then it is ϵ-good. Then we show how to efficiently construct such a hypothesis with high probability.

Lemma 5.1 *Let D be a fixed unknown distribution, and let ϕ_* be a fixed unknown target. Let ϕ be any conjunction (product) of clauses that contains every important clause in ϕ_* and no harmful clauses. Then $error(\phi) \leq \epsilon$.*

Proof: The probability that ϕ misclassifies an example from D is equal to the probability that ϕ_* classifies the example as positive but ϕ classifies it as negative, or vice versa (the example is truly negative but ϕ classifies it as positive).
The probability of an error on a positive example is equal to the probability that any clause in $\phi \Leftrightarrow \phi_*$ is falsified by positive example; therefore,

$$\text{Prob}[\phi_* = 1 \wedge \phi = 0] \leq \sum_{C \in \phi - \phi_*} P_{01}(C) < MQ_H = \epsilon/2$$

since there are at most M clauses in ϕ and none are harmful.

The probability of error on a negative example is equal to the probability that a clause in $\phi_* \Leftrightarrow \phi$ is falsified and all clauses in ϕ are true. This probability is less or equal to the probability that a clause in $\phi_* \Leftrightarrow \phi$ is falsified. Since ϕ contains all the important but not harmful clauses and ϕ_* contains no harmful clauses, then $\phi_* \Leftrightarrow \phi$ contains only non-important clauses. Therefore,

$$\text{Prob}[\phi = 1 \wedge \phi_* = 0] \leq \sum_{C \in \phi_* - \phi} P_0(C) < MQ_I = \epsilon/(16M) < \epsilon/2$$

Thus,

$$error(\phi) \leq \text{Prob}[\phi = 1 \wedge \phi_* = 0] + \text{Prob}[\phi = 0 \wedge \phi_* = 1] \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

■

To complete the algorithm for learning k-CNF under noise, we must construct an efficient algorithm to find a formula ϕ that contains all the important clauses and no harmful clauses (with high probability). Observe that we have no direct information about whether a clause is important or harmful. More specifically, we cannot directly compute $P_{00}(C)$ or $P_{01}(C)$, but rather must rely on the examples received from EX_η. However, since EX_η only modifies the label and not the values of the attributes in the example, $P_0(C)$ can be directly estimated by drawing a sample from EX_η, and calculating the fraction of the assignments that assign "false" to C. Thus we can accurately estimate $P_0(C)$ for every possible clause and thus decide (with high probability) which clauses are important. Let I be the set of clauses that are determined to be important. The only thing left, is to identify the harmful clauses in I and eliminate them.

Harmful clauses are those that are falsified by positive examples (examples that satisfy ϕ_*), but since the classification of the example is subject to noise we cannot directly estimate

$P_{01}(C)$. Let $P_{0+}(C)$ be the probability that a a clause is falsified by an example and that EX_η reports that the example is positive. This probability can be directly estimated by counting the number of examples from the sample that are reported as positive and falsify the clause C. We perform this estimate for every important clause. For the estimate of $P_{0+}(C)$, an example is counted if and only if the example is positive and no error occurred in the reporting, or the example is negative and there was an error in the reporting. Thus,

$$\begin{aligned} P_{0+}(C) &= (1 \Leftrightarrow \eta)P_{01}(C) + \eta P_{00}(C) \\ &= \eta(P_{00}(C) + P_{01}(C)) + (1 \Leftrightarrow 2\eta)P_{01}(C) \\ &= \eta P_0(C) + (1 \Leftrightarrow 2\eta)P_{01}(C) \end{aligned}$$

If $P_0(C) \neq 0$, then the proportion of examples falsifying C that are reported as positive is:

$$\frac{P_{0+}(C)}{P_0(C)} = \eta + \frac{P_{01}(C)}{P_0(C)}(1 \Leftrightarrow 2\eta)$$

and since $\eta < 1/2$, we get:

$$\frac{P_{0+}(C)}{P_0(C)} \geq \eta.$$

We would like to separate between two cases: one is the case where C is a desired clause ($C \in \phi_*$) and the other is when C is harmful. If $C \in \phi_*$ then

$$\frac{P_{0+}(C)}{P_0(C)} = \eta,$$

while if C is harmful ($P_{01}(C) \geq Q_H$):

$$\frac{P_{0+}(C)}{P_0(C)} \geq \eta + Q_H(1 \Leftrightarrow 2\eta) \geq \eta + \frac{\epsilon}{2M}(1 \Leftrightarrow 2\eta).$$

Thus, we have a separation of at least $s = Q_H(1 \Leftrightarrow 2\eta)$ between the clauses that are included in ϕ_* and the harmful clauses. Since $\eta \leq \eta_b$, the separation is bounded bellow by $s = Q_H(1 \Leftrightarrow 2\eta_b)$.

If we knew the value of η, we could estimate P_{0+}/P_0 for all clauses and delete from I all the clauses with an estimate that is greater than $\eta + s/2$. However, we do not know η and thus we do not know where the cutoff is.

How can we estimate η? If I contains any clause C that is also in ϕ_* then the estimate of $P_{0+}(C)/P_0(C)$ will be close to η. So we can take the minimum of this value over all the clauses in I as a first estimate for η:

$$\hat{\eta} = \min_{c \in I} \frac{P_{0+}(C)}{P_0(C)}.$$

If no clause in I is contained also in ϕ_*, then the estimate $\hat{\eta}$ calculated above may not be a good estimate of η. However, since I contains *all* the important clauses it must be that

ϕ_* contains no important clauses. This means that most of the examples drawn from D satisfy these non-important clauses and therefore satisfy ϕ_*. In this case most examples are positive, and thus the observed overall rate of negative examples is sufficiently close to η. Thus, the estimate of η is taken to be the minimum of the two estimates:

$$\hat{\eta} = \min\left\{\text{fraction of negative examples}, \min_{c\in I}\left\{\frac{P_{0+}(C)}{P_0(C)}\right\}\right\}.$$

We use therefore $\hat{\eta} + s/2$ to decide which are the harmful clauses.

5.3.2 Details of the Algorithm

We now put these ideas together to get an efficient algorithm for PAC learning k-CNF formulas under random misclassification noise.

Algorithm Learn-Noisy-k-CNF(n,k,ϵ,δ,η_b)

1. $m = \lceil \frac{2^{10}M^4}{\epsilon^3(1-\eta_b)^2}\ln(\frac{6M}{\delta})\rceil$, where M is the number of possible clauses
2. $Q_I = \epsilon/(16M^2); Q_H = \epsilon/2M; s_b = Q_H(1 \Leftrightarrow 2\eta_b)$
3. We draw m examples from the oracle EX_η, and set: $\hat{P}_- =$ number of negative examples in the sample
4. For each possible clause C, we compute:
   (a) $\hat{P}_0(C) =$ the number of examples that falsify the clause C
   (b) $\hat{P}_{0+}(C) =$ the number of positive (reported) examples that falsify C
   (c) If $P_0(C) \neq 0$ then $h(C) = \hat{P}_{0+}(C)/\hat{P}_0(C)$.
   (d) $\eta_1 = \hat{P}_-/m$, the observed fraction of negative examples
5. Form the set I by including all the important clauses C; i.e., $\hat{P}_0(C)/m \geq Q_I/2$
6. $\eta_2 = \min_{c\in I}\{h(C)\}$
7. $\hat{\eta} = \min\{\eta_1, \eta_2\}$
8. The final output ϕ is the conjunction (product) of all those clauses $C \in I$ such that $h(C) \leq \hat{\eta} + s_b/2$

To prove that the algorithm is correct, we need to prove that ϕ contains all important clauses that are not harmful (with probability greater than $1 \Leftrightarrow \delta$). Only in the following cases the algorithm could go wrong:

- Some important clauses might not be selected for inclusion in I.

- the estimate $\hat{\eta}$ could be too large ($\hat{\eta} \geq \eta + s/4$) or too small ($\hat{\eta} \leq \eta \Leftrightarrow s/4$).
- Some harmful clauses may be included in ϕ.
- Some correct clauses may be excluded from ϕ.

The proof uses Chernoff bounds, showing that the second case has a probability of at most $\delta/2$, while the other cases, each has a probability of at most $\delta/6$. Therefore, the toatl probability of error is at most δ, and by the previous lemma, the output formula is ϵ-good with high probability. For the details of the proof, see the Angluin, Laird paper[7].

CS 582T Computational Learning Theory **Spring Semester, 1991**

Topic 6: Occam's Razor

Lecturer: Sally Goldman *Scribe: Nilesh Jain*

"Entities should not be multiplied unnecessarily"
William of Occam, c. 1320

6.1 Introduction

The above quote is better known to us today as Occam's Razor or the Principle of Ontological Parsimony. Over the centuries, people from different fields have given it different interpretations. One interpretation used by the experimental scientists is: *given two explanations of the data, all other things being equal, the simpler explanation is preferable.*

This principle is consistent with the goal of machine learning: to discover the simplest hypothesis that is consistent with the sample data. However, the question still remains: why should one assume that the simplest hypothesis based on past examples will perform well on future examples. After all, the real value of a hypothesis is in predicting the examples that it has not yet seen.

It will be shown that, under very general assumptions, Occam's Razor produces hypotheses that with high probability will be predictive of future observations. As a consequence, when hypotheses of minimum or near minimum complexity can be produced in time polynomial in the size of the sample data, it leads to a polynomial PAC-learning algorithm. The material presented in the lecture comes from the paper "Occam's Razor," by Anselm Blumer, Andrzej Ehrenfeucht, David Haussler, and Manfred Warmuth [11]. Portion of the notes are taken from this paper.

6.2 Using the Razor

In this section, we define an Occam-algorithm and show that the existence of an Occam-algorithm implies polynomial learnability. First, we prove the *uniform convergence lemma.*

Lemma 6.1 (Uniform Convergence Lemma) *Given any function f in a hypothesis class of r hypotheses, the probability that any hypothesis with error larger than ϵ is consistent with a sample of size m is less than $(1 \Leftrightarrow \epsilon)^m r$.*

Proof: Let h_i be any hypothesis with error larger than ϵ. The probability that h_i is consistent with one observation of f is less than $(1 \Leftrightarrow \epsilon)$. Since all the m samples of f are drawn independent of each other, the probability that h_i is consistent with all m observations of f is less than $(1 \Leftrightarrow \epsilon)^m$. Finally, since there are r hypothesis, the probability that any hypothesis with error larger than ϵ is consistent with all m observations of f is less than $(1 \Leftrightarrow \epsilon)^m\, r$. ■

For an infinite hypothesis class, the learning algorithm would have to choose a hypothesis carefully. Occam's Razor suggests that the learning algorithm should choose its hypothesis among those that are consistent with the sample and have the minimum complexity. But this may sometimes be intractable. For example, finding a minimum length DNF expression consistent with a sample of a Boolean function and finding a minimum size DFA consistent with a set of positive and negative examples of a regular language are known to be NP-hard [14]. To obtain polynomial algorithms, the criterion is weakened as follows.

Definition 6.1 *An Occam-algorithm for H with constant parameters $c \geq 1$ and compression factor $0 \leq \alpha < 1$ is a learning algorithm that given a sample of size m which is labeled according to some hypothesis $h \in H$:*

1. *produces a hypothesis consistent with the data, that is, all the observations can be explained by the hypothesis,*
2. *produces a hypothesis of complexity at most $n^c m^\alpha$ where n is the complexity of h, and*
3. *runs in time polynomial in n and m.*

We now show that the existence of an Occam-algorithm for H implies polynomial learnability.

Theorem 6.1 *Given independent observations of any function in H of complexity at most* n*, an Occam-algorithm with parameters $c \geq 1$ and $0 \leq \alpha < 1$ produces a hypothesis of error at most ϵ with probability at least $1 \Leftrightarrow \delta$ using sample size polynomial in $n, 1/\epsilon$, and $1/\delta$, independent of the function and of the probability distribution. The sample size required is*

$$O\left(\frac{1}{\epsilon}\lg\left(\frac{1}{\delta}\right) + \left(\frac{n^c}{\epsilon}\right)^{1/(1-\alpha)}\right).$$

Proof: Let the size of the hypothesis space be r. We will show that the Occam-algorithm produces a good hypothesis after drawing a sample of size

$$m \geq \max\left\{\frac{2\lg(1/\delta)}{\Leftrightarrow \lg(1 \Leftrightarrow \epsilon)}, \left(\frac{2n^c}{\Leftrightarrow \lg(1 \Leftrightarrow \epsilon)}\right)^{1/(1-\alpha)}\right\}$$

Since the hypothesis under consideration are given by binary strings of length at most $n^c m^\alpha$, the size of the hypothesis space, r, is at most $2^{n^c m^\alpha}$.

We first consider the second lower bound on m,

$$m \geq \left(\frac{2n^c}{\Leftrightarrow \lg(1 \Leftrightarrow \epsilon)}\right)^{1/(1-\alpha)}.$$

Raising both sides to the power of $(1 \Leftrightarrow \alpha)$ yields

$$m^{1-\alpha} \geq \frac{2n^c}{\Leftrightarrow \lg(1 \Leftrightarrow \epsilon)}.$$

Simplifying further we obtain

$$n^c m^\alpha \leq \Leftrightarrow \frac{1}{2} m \lg(1 \Leftrightarrow \epsilon).$$

Finally, raising both sides to the power of 2 gives

$$2^{n^c m^\alpha} \leq (1 \Leftrightarrow \epsilon)^{-m/2}.$$

Recall that $r \leq 2^{n^c m^\alpha}$ and thus it follows from above that:

$$r \leq (1 \Leftrightarrow \epsilon)^{-m/2}.$$

From the uniform convergence lemma, we get that:

$$\begin{aligned}\Pr[\text{any } \epsilon \Leftrightarrow \text{bad hyp. is consistent with } m \text{ exs.}] &\leq (1 \Leftrightarrow \epsilon)^m r \\ &\leq (1 \Leftrightarrow \epsilon)^{-m/2}(1 \Leftrightarrow \epsilon)^m \\ &= (1 \Leftrightarrow \epsilon)^{m/2}. \qquad (6.1)\end{aligned}$$

Finally, we consider the first lower bound on m,

$$m \geq \frac{2\lg(1/\delta)}{\Leftrightarrow \lg(1 \Leftrightarrow \epsilon)}.$$

Multiplying both sides by $\lg(1 \Leftrightarrow \epsilon)$ which is negative, we get

$$\frac{m}{2} \lg(1 \Leftrightarrow \epsilon) \leq \Leftrightarrow \lg(1/\delta) = \lg \delta.$$

Raising both sides to the power of 2 gives

$$(1 \Leftrightarrow \epsilon)^{m/2} \leq \delta \qquad (6.2)$$

We complete the proof by combining Equations (6.1) and (6.2) gives that the probability that an ϵ-bad hypothesis is consistent with all the m examples is at most δ. Thus the Occam-algorithm produces a good hypothesis after drawing m examples where

$$m = O\left(\frac{1}{\epsilon} \lg\left(\frac{1}{\delta}\right) + \left(\frac{n^c}{\epsilon}\right)^{1/(1-\alpha)}\right)$$

By definition of an Occam-algorithm, it runs in time polynomial in n and m. Thus, we have a PAC-learning algorithm. ■

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Topic 7: The Vapnik-Chervonenkis Dimension

Lecturer: Sally Goldman *Scribe: Nilesh Jain*

7.1 Introduction

In the notes on Occam's Razor, we defined the condition for uniform convergence on finite concept classes. We can state the general condition for uniform convergence as follows:

$$\Pr_{S \in D^m}\left[\exists h \in \mathcal{C} \mid error_D(h) > \epsilon \wedge h \text{ is consistent with } S\right] \leq \delta$$

where S is a set of m examples from the instance space X having probability distribution D. To determine whether a concept class $\mathcal{C}$ is uniformly learnable, we have to find out if there exists a learning function $\mathcal{A}$ satisfying the above condition. If so, we say that $\mathcal{C}$ is uniformly learnable and has sample complexity m.

For finite concept classes, such uniform convergence proofs are easy because we know $|\mathcal{C}|$. However, when the concept class is infinite, we need some other measure to replace $|\mathcal{C}|$. Such a measure is a combinatorial parameter known as the *Vapnik-Chervonenkis (VC) dimension*. The VC dimension of a concept class $\mathcal{C}$ is a measure of the complexity of the class.

In the next section, we will define the VC dimension and some other concepts needed to define the VC dimension. We shall then measure the VC dimension of some concept classes. The following section will contain the theorems relating the VC dimension to uniform learnability and the sample complexity. Finally, we will see some relations on the VC dimension. The material presented in the lecture comes from the paper "Learnability and the Vapnik-Chervonenkis Dimension," by Anselm Blumer, Andrzej Ehrenfeucht, David Haussler, and Manfred Warmuth [12]. Portions of the notes are taken from this paper.

7.2 VC Dimension Defined

The definition of VC dimension uses the definition of a *shattered* set. We now give two equivalent definitions for a shattered set. Let X be the instance space and $\mathcal{C}$ the concept class.

Definition 7.1 A finite set $S \subseteq X$ is *shattered* by $\mathcal{C}$ if for each subset $S' \subseteq S$, there is a concept $c \in \mathcal{C}$ which contains all of S' and none of $S \Leftrightarrow S'$.

In order to give the alternate definition of shattering, we first need the following definition.

Definition 7.2 Given $S \subseteq X$, $\Pi_{\mathcal{C}}(S)$ denotes the set of all subsets of S that can be obtained by intersecting S with a concept in $\mathcal{C}$. Thus,

$$\Pi_{\mathcal{C}}(S) = \{S \cap c : c \in \mathcal{C}\}.$$

For any integer $m > 0$, $\Pi_{\mathcal{C}}(m) = \max(|\Pi_{\mathcal{C}}(S)|)$ over all $S \subseteq X$ where $|S| = m$.

Definition 7.3 Given $S \subseteq X$, if $\Pi_{\mathcal{C}}(S) = 2^S$, the power set of S, then S is *shattered* by $\mathcal{C}$.

We can now define the VC dimension.

Definition 7.4 *VC dimension* of $\mathcal{C}$, denoted as $\textsc{vcd}(\mathcal{C})$, is the smallest d for which no set of $d+1$ instances is shattered by $\mathcal{C}$.

Definition 7.5 Equivalently, $\textsc{vcd}(\mathcal{C})$ is the cardinality of the largest finite set of points $S \subseteq X$ that is shattered by $\mathcal{C}$ (i.e., the largest integer d such that $\Pi_{\mathcal{C}}(d) = 2^d$).

7.3 Example Computations of VC Dimension

Consider a concept class $\mathcal{C}$ with a finite VC dimension. To show the lower bound on $\textsc{vcd}(\mathcal{C})$, (i.e., $\textsc{vcd}(\mathcal{C}) \geq d$), we have to show that there exists a set of size d that is shattered by $\mathcal{C}$. To show the upper bound on $\textsc{vcd}(\mathcal{C})$, (i.e., $\textsc{vcd}(\mathcal{C}) \leq d$), we have to show that no set of size $d+1$ is shattered by $\mathcal{C}$. To show that $\textsc{vcd}(\mathcal{C}) = d$, we have to show that there exists a set of size d that is shattered by $\mathcal{C}$, and no set of size $d+1$ is shattered by $\mathcal{C}$. Keeping this in mind, let us compute $\textsc{vcd}(\mathcal{C})$ for some concept classes.

Example 7.1 *Intervals on the real line*

The concepts are intervals on the real line. Points lying on or inside the interval are positive, and points lying outside the interval are negative.

We first show that there exists a set of size two that can be shattered by $\mathcal{C}$.

Consider Figure 7.1. Let $S = \{x_1, x_2\}$ be a subset of the instance space X. Consider the concepts $c_1 = [0, r_1], c_2 = [r_1, r_2], c_3 = [r_2, r_3], c_4 = [0, 1]$. Let the concept class $\mathcal{C} = \{c_1, c_2, c_3, c_4\}$. Then, we have $c_1 \cap S = \phi, c_2 \cap S = \{x_1\}, c_3 \cap S = \{x_2\}$ and $c_4 \cap S = S$. Thus, S is shattered by $\mathcal{C}$.

Finally, we must show that no set of size three can be shattered by $\mathcal{C}$.

Consider Figure 7.2. Let $S = \{x_1, x_2, x_3\}$ be a subset of the instance space X. There is no concept which contains x_1 and x_3 and does not contain x_2. Thus, S is not shattered by $\mathcal{C}$.

Thus, $\textsc{vcd}(\mathcal{C}) = 2$.

Example 7.2 *Axis-parallel rectangles in* $\mathbb{R}^2$

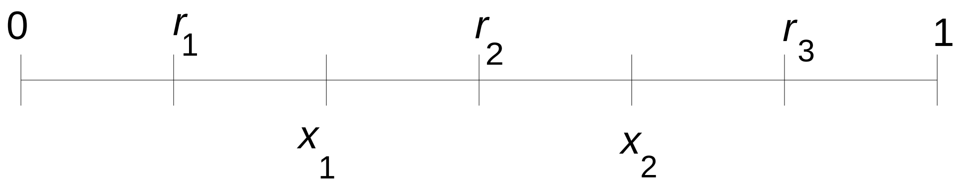

Figure 7.1: A line of reals with some instances and concepts.

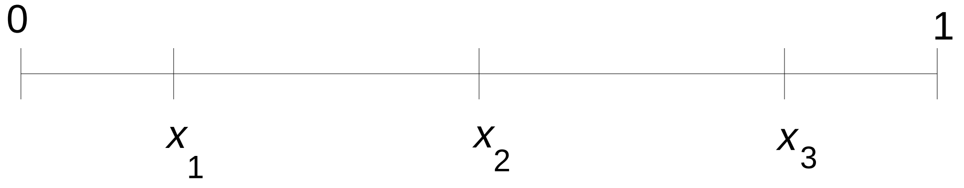

Figure 7.2: A line of reals with some instances.

The concepts are axis-parallel rectangles in $\mathbb{R}^2$. Points lying on or inside the rectangle are positive, and points lying outside the rectangle are negative.

We first show that there exists a set of size four that can be shattered by $\mathcal{C}$. Consider any four points, no three of which are collinear. Clearly, these points can be shattered by concepts from $\mathcal{C}$. Finally, we must show that no set of size five can be shattered by $\mathcal{C}$. Given any five points, one of the following two cases occur.

1. At least three of the points are collinear. In this case, there is no rectangle which contains the two extreme points, but does not contain the middle points. Thus clearly, the five points cannot be shattered.

2. No three of the points are collinear. In this case, consider the bounding rectangle formed by taking the maximum x-coordinate, maximum y-coordinate, minimum x-coordinate, and minimum y-coordinate. Clearly, one of the five points (possibly more) will be contained in this bounding rectangle. Note that there is no concept containing the points on this rectangle but not the internal points. Thus the five points cannot be shattered.

Thus we have demonstrated that $\text{VCD}(\mathcal{C}) = 4$.

Generalizing to axis-parallel rectangles in $\mathbb{R}^d$, we get $\text{VCD}(\mathcal{C}) = 2d$.

Example 7.3 *Half-spaces in* $\mathbb{R}^2$

The concepts are half-spaces in $\mathbb{R}^2$ formed by a line dividing $\mathbb{R}^2$ into two half-spaces. Points lying in one of the half-spaces or on the dividing line are positive, and points lying in the other half-space are negative.

We first show that there exists a set of size three that can be shattered by $\mathcal{C}$. Consider any three non-collinear points. Clearly, they can be shattered by concepts from $\mathcal{C}$. Finally we must show that no set of size four can be shattered by $\mathcal{C}$. Given any four points, one of the following two cases occur.

1. At least three of the points are collinear. In this case, there is no half-space which contains the two extreme points, but does not contain the middle points. Thus clearly the four points cannot be shattered.

2. No three of the points are collinear. In this case, the points form a quadrilateral. There is no half-space which labels one pair of diagonally opposite points positive, and the other pair of diagonally opposite points negative. Thus clearly the four points cannot be shattered.

Thus we have demonstrated that $\text{VCD}(\mathcal{C}) = 3$.

Generalizing to half-spaces in $\mathbb{R}^d$, we get $\text{VCD}(\mathcal{C}) = d + 1$.

Example 7.4 *Closed sets in* $\mathbb{R}^2$

The concepts are closed sets in $\mathbb{R}^2$. All points lying in the set or on the boundary of the set are positive, and all points lying outside the set are negative.

Any set can be shattered by $\mathcal{C}$. This is because the concepts can assume any shape in $\mathbb{R}^2$. Thus, the largest set that can be shattered by $\mathcal{C}$ is infinite.

Thus, $\text{VCD}(\mathcal{C}) = \infty$.

Example 7.5 *Convex d-gons in $\mathbb{R}^2$*

The concepts are convex polygons in $\mathbb{R}^2$ having d sides. All points lying in the convex d-gon or on the sides of the convex d-gon are positive, and all points lying outside the convex d-gon are negative.

We first show that there exists a set of $2d + 1$ points that can be shattered. Consider $2d+1$ points evenly spaced around a circle. We claim that given any labeling of these points one can find a d-gon consistent with the labeling. If there are more negative points then use the positive points as the vertices of the d-gon. If there are more positive points use the tangents to the negative points as the edges.

Finally, we informally argue that no set of size $2d + 2$ points can be shattered. If the points are not in a circular arrangement then clearly they can't be shattered. And if there are in a circular arrangement switching between positive and negative points as one goes around the circle produces a labeling that cannot be obtained by any d-gon.

So, for this concept class, $\text{VCD}(\mathcal{C}) = 2d + 1$.

Generalizing to convex polygons in $\mathbb{R}^2$, we get $\text{VCD}(\mathcal{C}) = \infty$.

7.4 VC Dimension and Sample Complexity

In this section, we relate the VC dimension to the sample complexity required for PAC learning. First, we need a definition.

Definition 7.6 A concept class $\mathcal{C} \subseteq 2^X$ is *trivial* if $\mathcal{C}$ consists of one concept, or two disjoint concepts c_1 and c_2 such that $c_1 \cup c_2 = X$.

When $\mathcal{C}$ is trivial, it is clear that a sample size of at most 1 is required to learn $\mathcal{C}$. Now to the more general case.

Theorem 7.1 *Let $\mathcal{C}$ be a non-trivial, well-behaved* [3] *concept class.*

1. *$\mathcal{C}$ is PAC learnable if and only if the VC dimension of $\mathcal{C}$ is finite* [4]

[3] This relatively benign measure-theoretic assumption holds of all the concept classes we have seen. It is discussed in detail in the appendix of [12].

[4] This assumes that the learner is limited to static sampling. It also does not consider the time or sample complexity of the learner.

2. if $\text{VCD}(\mathcal{C}) = d$*, where* $d < \infty$*, any hypothesis from* $\mathcal{C}$ *that is consistent with*

$$m \geq \max\left(\frac{2}{\epsilon}\log\frac{2}{\delta}, \frac{8d}{\epsilon}\log\frac{13}{\epsilon}\right)$$

examples is ϵ*-good with probability* $\geq 1 \Leftrightarrow \delta$*. The sample complexity is*

$$O\left(\frac{1}{\epsilon}\ln\frac{1}{\delta} + \frac{\text{VCD}(\mathcal{C})}{\epsilon}\ln\frac{1}{\epsilon}\right).$$

We will not prove this theorem here. It is given in detail in Blumer et al. [12]. However, we can note a consequence of the theorem. If $\text{VCD}(\mathcal{C})$ is finite, then $\mathcal{C}$ is PAC learnable with sample size $O\left(\frac{1}{\epsilon}\ln\frac{1}{\delta} + \frac{\text{VCD}(\mathcal{C})}{\epsilon}\ln\frac{1}{\epsilon}\right)$, and if $\text{VCD}(\mathcal{C})$ is infinite, then $\mathcal{C}$ is not PAC learnable at all. (See Topic 14 for a discussion of when dynamic sampling can be used to learn concept classes with infinite VC dimension.)

We now describe an information-theoretic lower bound that demonstrates that the above upper bound is almost tight.

Theorem 7.2 *For any concept class* $\mathcal{C}$ *with finite VC dimension, finding an* ϵ*-good hypothesis with probability* $\geq 1 \Leftrightarrow \delta$ *requires*

$$\Omega\left(\frac{1}{\epsilon}\ln\frac{1}{\delta} + \frac{\text{VCD}(\mathcal{C})}{\epsilon}\right)$$

examples.

Proof: We begin by proving that $\Omega\left(\frac{1}{\epsilon}\ln\frac{1}{\delta}\right)$ examples are needed.

Consider a portion of X having ϵ-weight in the distribution D_X on X. The probability of not seeing an instance from that portion in one drawing is at most $(1 \Leftrightarrow \epsilon)$. Thus the probability of not seeing an instance from that portion of X in m drawings is at most $(1 \Leftrightarrow \epsilon)^m$. We want this probability to be at most δ. Thus we require that

$$(1 \Leftrightarrow \epsilon)^m \leq \delta$$

Taking the natural logarithm of both sides, we get that

$$m \ln(1 \Leftrightarrow \epsilon) \leq \ln\delta$$

Dividing by $\ln(1 \Leftrightarrow \epsilon)$ which is negative, gives

$$m \geq \frac{\ln\delta}{\ln(1 \Leftrightarrow \epsilon)}$$

Finally, since $\ln(1 \Leftrightarrow \epsilon) < \Leftrightarrow \epsilon$ we can conclude that

$$\begin{aligned} m &\geq \Leftrightarrow \frac{\ln\delta}{\epsilon} \\ &= \frac{1}{\epsilon}\ln\frac{1}{\delta} \end{aligned}$$

Thus, $\Omega\left(\frac{1}{\epsilon}\ln\frac{1}{\delta}\right)$ examples are needed so that we get an example from any portion of X having ϵ-weight with probability $\geq (1 \Leftrightarrow \delta)$.

We now want to prove that $\Omega\left(\frac{\mathrm{VCD}(\mathcal{C})}{\epsilon}\right)$ examples are needed.

Let $\mathrm{VCD}(\mathcal{C}) = d$. We shall first prove that additional $\Omega(d)$ examples are needed. Then, we will improve it to $\Omega(\frac{d}{\epsilon})$.

Since $\mathrm{VCD}(\mathcal{C}) = d$, there exists a shattered set of size d. Let $S = \{x_1, \ldots, x_d\}$ be such a set, and let D_S be a uniform distribution over S. Assume without loss of generality that $|\mathcal{C}| = 2^d$.

Run the following experiment.

1. Draw sample $Y = \{y_1, \ldots, y_m\}$ from D_S where $m \leq \frac{d}{2}$. Assume without loss of generality that $y_1 = x_1, y_2 = x_2, \ldots, y_m = x_m$.

2. Choose target c by flipping d fair coins. (Let $b_1, \ldots, b_d$ be the outcomes).

3. Run the PAC algorithm on the m pairs $(x_1, b_1), \ldots, (x_m, b_m)$ to output hypothesis h.

4. Measure the error of h.

Consider the following modified experiment.

1. Draw sample $Y = \{y_1, \ldots, y_m\}$ from D_S where $m \leq \frac{d}{2}$. Assume without loss of generality that $y_1 = x_1, y_2 = x_2, \ldots, y_m = x_m$.

2. Choose target c by flipping m fair coins. (Let $b_1, \ldots, b_m$ be the outcomes).

3. Run the PAC algorithm on the m pairs $(x_1, b_1), \ldots, (x_m, b_m)$ to output hypothesis h.

4. Flip $d \Leftrightarrow m$ coins to determine the rest of c.

5. Measure the error of h.

Both the experiments have the same results. However, it is easier to measure the expected error in the second experiment. On each point not in the sample Y, the probability of h being correct is $\frac{1}{2}$. Thus, the total expected error $\geq (\frac{d}{2})(\frac{1}{2}) = \frac{d}{4}$. This implies that ϵ and δ can no longer be chosen arbitrarily because we know that the total expected error is at least $\frac{d}{4}$. Thus, the PAC algorithm needs at least $\frac{d}{2}$ examples. This shows that we need an additional $\Omega(d)$ examples.

We now modify the upper bound to prove the $\Omega(\frac{d}{\epsilon})$ bound. Let $S' = \{x_1, \ldots, x_{d-1}\}$. Let $S = S' \cup \{x_0\}$ and D_S be the distribution on S. Let D_S put weight $1 \Leftrightarrow 2\epsilon$ on x_0 and weight $\frac{2\epsilon}{d-1}$ on each of $x_1, \ldots, x_{d-1}$. Let $\mathcal{C}'$ be the concept class obtained by taking the concepts that shatter S' and let x_0 be positive. So $|\mathcal{C}'| = 2^{d-1}$.

Any PAC algorithm will quickly learn that x_0 is positive, but it cannot ignore S' since $\sum_{x \in S'} D_S(x) = 2\epsilon > \epsilon$. So, the PAC algorithm must see at least half the examples in S'. With probability $\frac{2\epsilon}{d-1}$ of seeing any of the examples in S', the PAC algorithm would need at least $\Omega(\frac{d}{\epsilon})$ examples to see $\frac{d}{2}$ of the examples in S'. This is because we want $\frac{d}{2}$ successful Bernoulli trials, each with $\frac{2\epsilon}{d-1}$ chance of success.

Thus combining this with the first part of the proof, we get that the PAC algorithm needs

$$\Omega\left(\frac{1}{\epsilon}\ln\frac{1}{\delta}+\frac{\textsc{vcd}(\mathcal{C})}{\epsilon}\right)$$

examples to find an ϵ-good hypothesis with probability at least $(1 \Leftrightarrow \delta)$. ∎

7.5 Some Relations on the VC Dimension

In this section, we will see some inequalities describing the relationships between the VC dimension of two or more concept classes, the relationships between the VC dimension of a concept class and the size of the concept class, etc. These bounds are very useful to compute the VC dimension of complicated concept classes that are constructed with simpler concept classes for which the VC dimension is known.

If $\mathcal{C}_1 \subseteq \mathcal{C}_2$, then clearly

$$\textsc{vcd}(\mathcal{C}_1) \leq \textsc{vcd}(\mathcal{C}_2).$$

Next we consider the relation between the VC dimension of $\mathcal{C}$ and the cardinality of $\mathcal{C}$. Given $\textsc{vcd}(\mathcal{C})$ points, we need $2^{\textsc{vcd}(\mathcal{C})}$ concepts to shatter them. Thus,

$$|\mathcal{C}| \geq 2^{\textsc{vcd}(\mathcal{C})}$$

Taking the base-2 logarithm of both sides

$$\lg|\mathcal{C}| \geq \textsc{vcd}(\mathcal{C})$$

For finite $\mathcal{C}$, we had earlier shown that

$$m \geq \frac{1}{\epsilon}\left(\ln|\mathcal{C}|+\ln\frac{1}{\delta}\right)$$

examples were sufficient to learn $\mathcal{C}$. Note that using $\textsc{vcd}(\mathcal{C})$ instead of $\ln|\mathcal{C}|$ gives a better bound.

Consider the complement $\bar{\mathcal{C}}$ of the concept class $\mathcal{C}$, defined as $\bar{\mathcal{C}} = \{X \Leftrightarrow c : c \in \mathcal{C}\}$. Then

$$\textsc{vcd}(\bar{\mathcal{C}}) = \textsc{vcd}(\mathcal{C}).$$

Consider the union of two concept classes. That is, let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ where $\mathcal{C}$, $\mathcal{C}_1$ and $\mathcal{C}_2$ are defined over the same instance space. Then,

$$\textsc{vcd}(\mathcal{C}) \leq \textsc{vcd}(\mathcal{C}_1)+\textsc{vcd}(\mathcal{C}_2)+1.$$

Consider the concept class $\mathcal{C}_s$ formed by the union (or intersection) of up to s concepts from $\mathcal{C}$ where $\text{VCD}(\mathcal{C}) = d$. For the union, $\mathcal{C}_s = \{\cup_{i=1}^s c_i : c_i \in \mathcal{C}\}$, and similarly for intersection. Then, for all $s \geq 1, \text{VCD}(\mathcal{C}_s) \leq 2ds \lg(3s)$. Thus,

$$\text{VCD}(\mathcal{C}_s) = O(s \ln(s) \text{VCD}(\mathcal{C})).$$

Earlier, we have defined $\Pi_{\mathcal{C}}(S)$ as the distinct ways that $\mathcal{C}$ can classify the instances in S. We shall now bound $|\Pi_{\mathcal{C}}(S)|$ by $|S|$ and $\text{VCD}(\mathcal{C})$.

Definition 7.7 For all $d \geq 0, m \geq 0$,

$$\begin{aligned} \Phi_d(m) &= \textstyle\sum_{i=0}^d \binom{m}{i} && \text{if } m \geq d \\ &= 2^m && \text{otherwise} \end{aligned}$$

For concept class $\mathcal{C}$, let $\text{VCD}(\mathcal{C}) = d$ and let S be a set of m distinct instances, then

$$|\Pi_{\mathcal{C}}(S)| \leq \Phi_d(m).$$

There are some inequalities relating $\Phi_d(m)$ using combinatorial arguments and Stirling's approximation.

$$\Phi_d(m) = \Phi_{d-1}(m \Leftrightarrow 1) + \Phi_d(m \Leftrightarrow 1).$$

For all $d \geq 0$ and $m \geq 0$,

$$\Phi_d(m) \leq m^d + 1.$$

For all $d \geq 2$ and $m \geq 2$,

$$\Phi_d(m) \leq m^d.$$

For all $m \geq d \geq 1$,

$$\Phi_d(m) \leq 2\left(\frac{m^d}{d!}\right) \leq \left(\frac{em}{d}\right)^d.$$

Thus, whenever $\text{VCD}(\mathcal{C})$ is finite, then $\Pi_{\mathcal{C}}(m)$ grows only polynomially in m.

CS 582T Computational Learning Theory **Spring Semester, 1991**

Topic 8: Representation Independent Hardness Results

Lecturer: Sally Goldman

8.1 Introduction

In Topic 3 we discussed a hardness result for learning k-term-DNF when the learner is restricted to use a hypothesis class of k-term-DNF. However, we then showed that k-term-DNF is learnable using the hypothesis class of k-CNF. While such *representation-dependent hardness results* provide some information, what one would really like to obtain is a hardness result for learning a concept class using any reasonable (i.e. polynomially evaluatable) hypothesis class. In this lecture we will briefly introduce a representation-independent hardness result for learning several simple concept classes such as Boolean formulas, deterministic finite automata, and constant-depth threshold circuits (a simplified form of "neural networks"). These hardness results are based on assumptions regarding the intractability of various cryptographic schemes such as factoring Blum integers and breaking the RSA function. The material presented here is just a brief introduction to the paper, "Cryptographic Limitations on Learning Boolean Formulae and Finite Automata," by Michael Kearns and Leslie Valiant [26]. We only give the intuition behind the hardness results—no details are described here. For the complete proofs we refer to reader to the Kearns and Valiant paper. Also a more complete discussion of these results is contained in Chapter 7 of Kearns' thesis [27].

8.2 Previous Work

Before describing the results of Kearns and Valiant we first briefly review the only previously known representation-independent hardness result. This previous result follows from the work of Goldreich, Goldwasser and Micali [19]. Let $\text{CKT}_n^{p(n)}$ denote the class of Boolean circuits over n inputs with at most $p(n)$ gates and let $\text{CKT}^{p(n)} = \bigcup_{n \geq 1} \text{CKT}_n^{p(n)}$. Goldreich, Goldwasser, and Micali showed that if there exists a one-way function, then for some polynomial $p(n)$, $\text{CKT}^{p(n)}$ is not polynomially learnable by *any* polynomially evaluatable hypothesis class.

Observe that the most powerful (polynomially evaluatable) hypothesis that can be output by any polynomial-time learning algorithm is a hypothesis that is itself a polynomial-time algorithm (or equivalently a polynomial-size Boolean circuit). Thus we cannot find efficient learning algorithms for concept classes that do not have small circuits—no learning algorithm even has time to just write down the representation. Schapire [37] has formally shown that any representation class that is not polynomially evaluatable cannot be learned in polynomial time. With this in mind, observe that the result of Goldreich, et al. shows that not everything with a small representation is efficiently learnable (assuming the existence of one-way functions). However, there is a large gap between the computational power of $\text{CKT}^{p(n)}$ and the classes for which we have efficient learning algorithms. Thus we would like to prove representation-independent hardness results for less powerful concept class such as the classes of Boolean formulas.

8.3 Intuition

In this section we describe the intuition behind the hardness results of Kearns and Valiant. We begin by very informally describing the type of cryptographic scheme on which this hardness result is based.

Suppose that Alice and Bob wish to communicate privately inspite of an eavesdropper Eve who has bugged the communication line between Alice and Bob. (We make no assumptions about Eve except that she has a polynomial bound on her computing resources.) Furthermore, we want a scheme which does not require Alice and Bob to meet privately ahead of time to set up their encoding scheme. Such encryption scheme can be devised using a *trapdoor function*. Informally, a trapdoor function is one that can be computed in polynomial time (i.e., it is easy to compute $f(x)$ on input x) but cannot be inverted in polynomial time (i.e., it is hard to compute x on input $f(x)$) unless one is the "creator" of the function and thus possesses a piece of "trapdoor" information that makes inversion possible in polynomial time.

Cryptographic schemes based on such trapdoor functions are known as *public-key cryptographic systems*. In such a scheme each user creates a trapdoor function f and publishes a program for computing f. (This program must reveal no information about f^{-1}.) Then anyone can send messages to the given user over the nonsecure communication line and only that user can decode such messages. So Alice and Bob can communicate with each other using such a scheme where they have both created their own trapdoor function. We say that this system is secure if Eve cannot do noticeably better than random guessing in trying to decode a bit that has been encoded using this encryption scheme. More formally, if

$$\Pr[\text{Eve predicts correct decoded bit}] \geq \frac{1}{2} + \frac{1}{p(n)}$$

for some polynomial $p(n)$ then the system is not secure.

We now describe how we can use the existence of such a public-key cryptographic system, such as RSA, to obtain a representation-independent hardness result for learning. For the hardness result view Eve as a learning algorithm. Since a program for f is available she can create any polynomially number of pairs of the form $(f(x), x)$ that she likes by simply choosing x and then computing $f(x)$. If we set $y = f(x)$ observe that such pairs have the form $(y, f^{-1}(y))$ and thus can be viewed as labeled examples of f^{-1}. Thus public-key cryptography assumes the existence of functions that are not learnable from examples. In fact, unlike the PAC learning model in which we ask the learner to be able to make arbitrarily good predications, in the cryptographic scheme we only ask the learner, Eve, to make predictions that are noticeably better than random guessing. (This "weak learning" model will be discussed further in the next topic.)

Observe that in the learning problem described above f^{-1} is "simple" in the sense that it has a small circuit (determined by the trapdoor used for decoding.) So the theory of cryptography provides simple functions that are difficult to learn. Kearns and Valiant show how to refine the functions provided by cryptography to find the simplest functions that are difficult to learn. Using this basic approach (of course, with lots of details which we will not discuss here) they show that polynomial-sized Boolean formulas, and constant-depth threshold circuits are not even weakly learnable by any polynomially evaluatable hypothesis class. Then using a prediction-preserving reduction (described below) of Pitt and Warmuth [34] it follows that deterministic finite automata are also not learnable.

8.4 Prediction Preserving Reductions

Given that we now have a representation-independent hardness result (assuming the security of the various cryptographic schemes) one would like a "simple" way to prove that other problems are hard in a similar fashion as one proves a desired algorithm is intractable by reducing a known NP-complete problem to it. Such a complexity theory for predictability has been provided by Pitt and Warmuth [34]. They formally define a *prediction-preserving* reduction ($A \trianglelefteq B$) that enables a prediction algorithm for concepts of type B to solve a prediction problem for concepts of type A. This reduction consists of the following two mapping:

1. A polynomial time computable function f that maps unlabeled examples of A to unlabeled examples of B.

2. A function g that maps representations of A to representations of B. Furthermore, this mapping g need not be computable—it is only required to be length preserving within a polynomial.

Then given a polynomial prediction algorithm for concepts of type B one can use it to obtain a polynomial prediction algorithm for concepts of type A in the obvious manner. Thus if A is known not to be learnable then B also cannot be learnable. They describe several such reductions in their paper, one of which is a reduction from Boolean formula predictability to DFA predictability (Boolean formula $\trianglelefteq$ DFA). Thus since Kearns and Valiant have shown that Boolean formula are not predictable (under cryptographic assumptions) it immediately follows that DFA are not predictable.

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Topic 9: Weak Learning

Lecturer: Sally Goldman *Scribe: Andy Fingerhut*

9.1 Introduction

In the first lecture we introduced the *probably approximately correct* (PAC) or *distribution free* model of learning. In this model, the learner is presented with examples which are randomly and independently drawn from an unknown but fixed distribution. The learner must, with arbitrarily high probability, produce a hypothesis that is arbitrarily close to the target concept.

As we saw in the last lecture, the representation-independent hardness results based on the security of various cryptographic schemes motivated the study of learning algorithms which do not attain arbitrarily high accuracy, but do just slightly better than random guessing (by an inverse polynomial in the size of the problem). This leads to the notion of *weak learning* which is the subject of this lecture. In particular we describe an algorithm to convert any weak learning algorithm into a PAC-learning algorithm. The material presented here comes from the paper, "The Strength of Weak Learnability," by Robert Schapire [37].

9.2 Preliminaries

We begin by formally defining the weak learning model. As in the PAC model, for a given size parameter n, there is a set of instances X_n. A *concept* c is a Boolean function $c : X_n \to \{0, 1\}$. A *concept class* $C_n \subseteq 2^{X_n}$ is a set of concepts. The learner has access to a source EX of labeled examples. Each time EX is called, an example is drawn randomly and independently according to a fixed but unknown distribution D on X_n, and returned in unit time.

After drawing some examples and running for a time, the learning algorithm must output an *hypothesis* h. This is a polynomially (in n) evaluatable hypothesis which, when given an instance $v \in X_n$, returns a prediction. The hypothesis may be randomized (this is important for weak learning).

We write $\text{Prob}[\pi(v)]$ to denote the probability that the predicate π holds for a particular instance v. This probability may be between 0 and 1 because of randomness in evaluating $\pi(v)$. For example, think of $\pi(v)$ as the event that $h(v) = 1$ for some randomized hypothesis h. By $\text{Prob}_{v \in D}[\pi(v)]$ we denote the probability that, after drawing v randomly from distribution D, $\pi(v)$ holds. Thus, assuming that these two random events (i.e., choice of v and evaluation of $\pi(v)$) are independent, we have $\text{Prob}_{v \in D}[\pi(v)] = \sum_{v \in X_n} D(v)\text{Prob}[\pi(v)]$ where $D(v)$ denotes the probability of instance v being chosen under distribution D.

Finally, we can define the *error* of an hypothesis h on c under distribution D as

$$error_D(h) = \text{Prob}_{v \in D}[h(v) \neq c(v)].$$

If the $error_D(h) \leq \epsilon$, we say h is ϵ-close to c under D. The *accuracy* of h is one minus its error.

We say that a concept class C is *strongly learnable* (called simply learnable or PAC-learnable elsewhere) if there is an algorithm A such that, for all $n \geq 1$, for all target concepts $c \in C_n$, for all distributions D and parameters $0 < \delta, \epsilon \leq 1$, algorithm A, given n, δ, ϵ, and access to oracle EX, outputs a hypothesis h such that, with probability $\geq 1 \Leftrightarrow \delta$, $error_D(h)$ is at most ϵ. Algorithm A should run in time polynomial in n, $1/\epsilon$, and $1/\delta$.

To be *weakly learnable*, there must be a polynomial p in addition to the algorithm A such that, for all $n \geq 1$, $c \in C_n$, for all distributions D, and for all $0 < \delta \leq 1$, algorithm A, given n, δ, and access to oracle EX, outputs a hypothesis h such that, with probability $\geq 1 \Leftrightarrow \delta$, $error_D(h)$ is at most $(1/2 \Leftrightarrow 1/p(n))$. Algorithm A should run in time polynomial in n and $1/\delta$.

9.3 The Equivalence of Strong and Weak Learning

It should be clear that if C is strongly learnable, then it is weakly learnable—just fix $\epsilon = 1/4$ (or any constant less than 1/2.) The converse (weak learnability implying strong learnability) is not at all obvious. In fact, if one restricts the distributions under which the weak learning algorithm runs then weak learnability *does not* imply strong learnability. In particular, Kearns and Valiant [26] have shown that under a uniform distribution monotone Boolean functions are weakly, but not strongly, learnable. Thus it will be important to take advantage of the requirement that the weak learning algorithm must work for all distributions. The remainder of this section will be devoted to proving the main result of these notes.

Theorem 9.1 *If concept class C is weakly learnable, then it is strongly learnable.*

Note that weak learning, even though it may be "weak", is not necessarily easy to do. Consider the simple-minded algorithm which draws and memorizes a polynomial $q(n, 1/\delta)$ number of examples. It then outputs a hypothesis h which looks up known results and otherwise flips a fair coin to determine the answer. Suppose that $|X_n| = 2^n$ and the distribution D is the uniform distribution. Then the $error_D(h) = (1/2 \Leftrightarrow q(n, 1/\delta)/2^n)$, which is larger than $(1/2 \Leftrightarrow 1/p(n))$ for any polynomial p. Thus it is non-trivial to devise an algorithm that weakly learns a concept class.

9.3.1 Hypothesis Boosting

Proving that weak learnability implies strong learnability has also been called the *hypothesis boosting problem*, because a way must be found to boost the accuracy of slightly-better-than-half hypotheses to be arbitrarily close to 1.

Suppose that we have a learning algorithm A which produces hypotheses with error at most α (for $\alpha < 1/2$). Simply running A twice to produce hypotheses h_1 and h_2 may not help at all—they may err in a very similar way in their classification. After obtaining h_1, we need a way to force the next hypothesis output to be more accurate. The answer lies in the power of A to work given any distribution D on the instances. If we give A a distribution D' such that the error of h_1 is exactly $1/2$, then the hypothesis h_2 produced must have an error $\leq \alpha$ on it, which means that h_2 has learned something about the target concept which h_1 did not. Unfortunately, the probability that h_1 and h_2 classify a random instance the same (their "overlap") may be very low; in this case, it would seem that h_1 and h_2 learned almost completely different parts of the D. To handle this, we ask A to learn a "tie-breaker" hypothesis h_3. This time, A is given a distribution D'' on the instances on which h_1 and h_2 disagree.

Thus to improve the accuracy of the weak learning algorithm A, the algorithm A' simulates A on distributions D, D', and D'' and outputs a hypothesis that takes the majority vote of the three hypothesis obtained. Of course, algorithm A' must use D to simulate the distributions D' and D''. We now describe how this task is achieved. These two modified distributions are created by *filtering* the original distribution. In other words, A' will draw samples from D and discard undesirable samples to obtain the desired distribution. While this requires additional samples (thus increasing both the time and sample complexity) it can be made to run in polynomial time. The key observation is that we will only apply the filter if the probability of obtaining an example which passes through the filter is $\Omega(\epsilon)$. The distribution D' given to learn h_2 is produced by the procedure EX_2 defined below. Likewise, the distribution D'' for learning h_3 is produced by EX_3.

```
$EX_2(EX, h_1)$
    flip fair coin
    if heads then return the first instance $v$ from $EX$ for which $h_1(v) = c(v)$
              else  return the first instance $v$ from $EX$ for which $h_1(v) \neq c(v)$

$EX_3(EX, h_1, h_2)$
    return the first instance $v$ from $EX$ for which $h_1(v) \neq h_2(v)$
```

Thus we can now formally state the hypothesis boosting procedure as follows:

1. Run the weak learning algorithm using EX. Let h_1 be the hypothesis output.

2. Run the weak learning algorithm using $EX_2(EX, h_1)$. Let h_2 be the hypothesis output.

3. Run the weak learning algorithm using $EX_2(EX, h_1, h_2)$. Let h_3 be the hypothesis output.

4. Return $h = \text{MAJORITY}(h_1, h_2, h_3)$.

An important question to address is: How much better is the accuracy of h than the accuracy of h_1? Suppose that the given weak learning algorithm is guaranteed to output a

Figure 9.1: A graph of the function $g(x) = 3x^2 \Leftrightarrow 2x^3$.

α-close hypothesis. (So $error_D(h_1) \leq \alpha$.) We shall show that for the hypothesis h output by the above hypothesis boosting procedure, $error_D(h) \leq g(\alpha) = 3\alpha^2 \Leftrightarrow 2\alpha^3$. See Figure 9.1 for a graph of $g(\alpha)$. Observe that both 0 and 1/2 are fixed points for the function g. Thus, as must be the case, for this boosting procedure to improve the error of the hypothesis, $\alpha < 1/2$. Note that for small α, $error_D(h) = O(\alpha^2) << \alpha$ and thus the new hypothesis is much better than the initial hypothesis. On the other hand, as α nears 1/2 the improvement is not as significant. However, as we shall show the number of iterations required to improve the error to be at most ϵ is polynomial in n and $1/\epsilon$.

9.3.2 The Learning Algorithm

In this section we describe the complete learning algorithm for converting a weak learning algorithm into a strong learning algorithm. The algorithm is shown in Figure 9.2. The basic idea is to recursively boost the hypothesis. The lowest level of recursion only requires the accuracy that `WeakLearn` (the weak learning algorithm given) can provide. Each higher level produces hypotheses with higher accuracy, until it has been boosted to be at least $1 \Leftrightarrow \epsilon$.

A technical difficulty which arises is that we may get lucky early and obtain an ϵ-close hypothesis for h_1 or h_2. If this happens, we are in trouble since either EX_2 or EX_3 would take too long. Therefore, the algorithm performs some hypothesis testing to prevent this from happening.

9.3.3 Correctness

We now prove that the algorithm of Figure 9.2 will produce a hypothesis meeting the PAC criterion.

```
Learn($\epsilon, \delta, EX$)

Input:
    $\epsilon$ - allowed error
    $\delta$ - confidence desired
    $EX$ - oracle for examples
    $n$ - (implicit) size
Output:
    $h$ - hypothesis that with probability $\geq 1 \Leftrightarrow \delta$ is $\epsilon$-close to target
Procedure:
    if $\epsilon \geq (1/2 \Leftrightarrow 1/p(n))$ then return WeakLearn($\delta, EX$)
    else $\alpha := g^{-1}(\epsilon)$
        $h_1 :=$ Learn($\alpha, \delta/5, EX$)
        estimate $error_D(h_1)$ to within $\epsilon/3$ of true value with confidence $\geq 1 \Leftrightarrow \delta/5$
        if $h_1$ is $\epsilon$-good then return $h_1$
        else construct $EX_2$
            $h_2 :=$ Learn($\alpha, \delta/5, EX_2$)
            estimate $error_D(h_2)$ to within $(1 \Leftrightarrow 2\alpha)\epsilon/8$ of true value with confidence $\geq 1 \Leftrightarrow \delta/5$
            if $h_2$ is $\epsilon$-good then return $h_2$
            else construct $EX_3$
                $h_3 :=$ Learn($\alpha, \delta/5, EX_3$)
                return $h = MAJ(h_1, h_2, h_3)$
            endif
        endif
    endif
```

Figure 9.2: An algorithm to convert a weak learning algorithm into a strong learning algorithm.

Theorem 9.2 *If $0 < \epsilon < 1/2$ and $0 < \delta \leq 1$, then with probability $\geq 1 \Leftrightarrow \delta$, the hypothesis returned by calling* `Learn`(ϵ, δ, EX) *is ϵ-close to the target c under D.*

Proof: Define a *good run* of `Learn` to be one in which everything goes right. That is, every call of `WeakLearn` produces a hypothesis with the required accuracy, and every error estimation is within its prescribed tolerance of being correct. We first show that the probability of having a good run is $\geq 1 \Leftrightarrow \delta$. We then show that whenever there is a good run, the hypothesis returned is ϵ-close to the target. These two facts imply the theorem.

We now argue by induction on the depth of the recursion that the probability of having a good run is at least $1 \Leftrightarrow \delta$. The base case is depth $= 0$, which occurs when $\epsilon \geq (1/2 \Leftrightarrow 1/p(n))$. In this case, the allowed error is high enough that `WeakLearn` can handle the situation. It produces a good run with probability $\geq 1 \Leftrightarrow \delta$. The induction step occurs if $\epsilon < (1/2 \Leftrightarrow 1/p(n))$. In this case, there are (at most) three recursive calls to `Learn` and (at most) two error estimations. In the worst case, all of the calls are performed. Since each of the calls and estimations succeeds (independently) with probability $\geq 1 \Leftrightarrow \delta/5$ (the calls succeed with this probability by the induction hypothesis), the probability that they all succeed is $\geq 1 \Leftrightarrow \delta$.

From now on, we assume that we have a good run, and our goal is to prove that the returned hypothesis is ϵ-good. We again proceed by induction on the depth of recursion. The base case is trivial, assuming a good run, since the call to `WeakLearn` produces an ϵ-good hypothesis.

The inductive step is trickier. In his paper Schapire gives a very detailed and formal proof—what follows here is a simpler but less formal (i.e. important details are ignored) proof.

If either h_1 or h_2 is tested to be ϵ-good, then it is returned and the Theorem is proved. If $h = MAJ(h_1, h_2, h_3)$, then there are two cases for h to be incorrect.

1. Both h_1 and h_2 give the wrong answer. Since this is a good run, $error_{D'}(h_2) \leq \alpha$. (Recall that D' is the distribution of examples given by EX_2). To make the probability of $h_1 = h_2 \neq c$ as high as possible, all of this error should coincide with h_1's error. With the filtering that occurs to get from D to D', the portion on which h_1 is incorrect expands from probability α to $1/2$. If h_2 has all of its error on that $1/2$ portion, it shrinks by multiplying it by 2α when translated back into D. Therefore h_2's error on D is $\leq 2\alpha(\alpha) = 2\alpha^2$.

2. The hypotheses h_1 and h_2 give different answers, and h_3 breaks the tie incorrectly. Since h_3 is the deciding vote, it should be wrong as often as possible to make the overall error large. Therefore we should make the distribution D'' (produced by EX_3) "cover" as much of the original distribution D as possible. This is done by making h_1 and h_2 disagree as often as possible. So, make h_2 wrong on the portion α of D' on which h_1 is correct. When translating back to D, this portion expands by a factor of $2(1 \Leftrightarrow \alpha)$. Therefore, h_2's error on D is $\leq 2(1 \Leftrightarrow \alpha)\alpha = 2\alpha \Leftrightarrow 2\alpha^2$, and the portion of D on which exactly one of h_1 and h_2 is wrong is $\leq 2\alpha \Leftrightarrow 2\alpha^2 + \alpha = 3\alpha \Leftrightarrow 2\alpha^2$. Thus h_3 can be wrong on a fraction α of that, so the error of h_3 on D is $\leq 3\alpha^2 \Leftrightarrow 2\alpha^3$.

The above two cases are mutually exclusive, and they exhaust all possibilities that h is wrong. Case (1) covers when h_1 and h_2 agree as often as possible, and case (2) covers when they disagree as often as possible. Thus h's real error will be somewhere in between these values. The larger value (when $0 \leq \alpha \leq 1/2$) is $g(\alpha) = 3\alpha^2 \Leftrightarrow 2\alpha^3 = \epsilon$, proving the theorem. ■

9.3.4 Analysis of Time and Sample Complexity

Although in the previous section we proved that `Learn` is correct, we have said nothing about its time or sample complexity. To complete the proof that strong and weak learning are equivalent, we must prove that `Learn` has time and sample complexity that are polynomial in n, $1/\epsilon$, and $1/\delta$.

First, consider the shape of the recursion tree obtained by the recursive calls occurring during `Learn`'s execution. If there are no early returns caused by serendipitously discovering an ϵ-good hypothesis, then the recursion tree is a rooted full ternary tree. Every node has either zero or three children, and every leaf occurs at the same depth. Denote this depth by $B(\epsilon, p(n))$, since it depends only on the arguments given.

If early returns do occur, then a node at depth less than B will have either one or two children and their corresponding subtrees "clipped". It is simpler to ignore such cases when determining worst-case time complexity, since the run time is greatest when there is no clipping.

The number of leaves (which correspond with calls to `WeakLearn`) is exactly 3^B. Therefore we would like to bound B to be logarithmic in n, $1/\epsilon$, and $1/\delta$. That is the result of Lemma 9.1 below.

At each level of recursion, ϵ is replaced by $g^{-1}(\epsilon)$, with $\epsilon \geq (1/2 \Leftrightarrow 1/p(n))$ at the bottom level. Therefore

$$B(\epsilon, p(n)) = \min_{i \geq 0} \left\{ g^i \left(\frac{1}{2} \Leftrightarrow \frac{1}{p(n)} \right) \leq \epsilon \right\}.$$

Note that $g^i(\alpha)$ is monotonically increasing with α for $0 \leq \alpha \leq 1/2$ for all i, and $g(\alpha) \leq \alpha$ on the same range.

Lemma 9.1 $B(\epsilon, p(n)) = O(\log(p(n)) + \log\log(1/\epsilon))$

Proof: Note that if $g^b(1/2 \Leftrightarrow 1/p(n)) \leq d$ and $g^c(d) \leq \epsilon$ for some values of b, c, and d, then $B(\epsilon, p(n)) \leq a + b$.

For $0 \leq x$, it is clear that $g(x) \leq 3x^2$. From this it can be proved easily by induction on i that $g^i(x) \leq (3x)^{2^i}$ for all $0 \leq x$. Therefore $g^c(1/4) \leq \epsilon$ if $c \geq \lg \log_{4/3}(1/\epsilon)$.

If $1/4 \leq x \leq 1/2$, then $1/2 \Leftrightarrow g(x) = (1/2 \Leftrightarrow x)(1 + 2x \Leftrightarrow 2x^2) \geq (11/8)(1/2 \Leftrightarrow x)$ (Note that $11/8$ is the minimum of $(1 + 2x \Leftrightarrow 2x^2)$ on the given interval). It can be proved by induction on i that $g^i(x) \geq (11/8)^i(1/2 \Leftrightarrow x)$, as long as x, $g(x)$, $\ldots$, $g^{i-1}(x)$ are all at least $1/4$. Therefore $g^b(1/2 \Leftrightarrow 1/p(n)) \leq 1/4$ if $b \geq \log_{11/8}(p(n)/4)$. ■

Quantities which are relevant to the time and sample complexity of `Learn` are $T(\epsilon, \delta)$, the expected running time of `Learn`(ϵ, δ, EX), $U(\epsilon, \delta)$, the time needed to evaluate a hypothesis returned by `Learn`, and $M(\epsilon, \delta)$, the expected number of samples needed by `Learn`. There are corresponding quantities $t(\delta)$, $u(\delta)$, and $m(\delta)$ for `WeakLearn`(δ, EX). All of these functions also depend (implicitly) on n.

These functions can be bounded quite effectively by the use of recurrence relations. These recurrences can be obtained by examination of `Learn`, with the base case being `WeakLearn` and its complexity. The final results will be mentioned here as lemmas, with proofs omitted. See Schapire's paper for the complete proofs.

Lemma 9.2 *The time to evaluate a hypothesis returned by* `Learn`(ϵ, δ, EX) *is* $U(\epsilon, \delta) = O(3^B \cdot u(\delta/5^B))$.

Lemma 9.3 *Let r be the expected number of examples drawn from EX by any oracle EX_i simulated by* `Learn` *on a good run when asked to provide a single example. Then $r \leq 4/\epsilon$.*

Lemma 9.4 *On a good run, the expected number of examples $M(\epsilon, \delta)$ needed by* `Learn`(ϵ, δ, EX) *is*

$$O\left(\frac{36^B}{\epsilon^2} \cdot (p^2 \log(5^B/\delta) + m(\delta/5^B))\right)$$

Lemma 9.5 *On a good run, the expected execution time of* `Learn`(ϵ, δ, EX) *is given by*

$$T(\epsilon, \delta) = O\left(3^B \cdot t(\delta/5^B) + \frac{108^B \cdot u(\delta/5^B)}{\epsilon^2} \cdot (p^2 \log(5^B/\delta) + m(\delta/5^B))\right)$$

The fact that nearly all of the expected values above are only taken over good runs of `Learn` can be taken into account by "borrowing" some of the confidence δ for that purpose.

9.4 Consequences of Equivalence

There are many very interesting consequences that follow from this main result (Theorem 9.1. We shall just briefly mention a few here. We refer the reader to Schapire's paper for more details on these consequences and a discussion of other interesting corollaries.

One of the more notable corollaries of Theorem 9.1 is a partial converse of Occam's Razor (see Topic 6). That is, if a concept class is learnable, then any sample can be compressed with high probability. More specifically, if C is PAC learnable then there exists a polynomial time algorithm which with confidence $1 \Leftrightarrow \delta$ outputs a consistent hypothesis of size polynomial in n and $/logm$ for any sample of size m labeled according to a concept c/inC. This corollary leads to the result that any representation class that is not polynomially evaluatable cannot be learned in polynomial.

Another interesting result discussed is an improved version of the learning algorithm given here, which achieves asymptotic performance which is poly-logarithmic in terms of $1/\epsilon$. Thus using this improved version of `Learn` an existing strong learning algorithm performance can be made to be poly-logarithmic in terms of 1ϵ by first fixing the error of the given strong learning algorithm at, say, $\epsilon = 1/4$ to obtain a weak learning algorithm. Then use this improved version of `Learn` to convert the weak learning algorithm back into a strong learning algorithm.

CS 582T Computational Learning Theory **Spring Semester, 1991**

Topic 10: Learning With Queries

Lecturer: Sally Goldman *Scribe: Kevin Ruland*

10.1 Introduction

In these notes we study the model of learning with queries. The material presented in this lecture comes from the paper, "Queries and Concept Learning," by Dana Angluin [3]. In the model of learning with queries, the learner must identify an unknown hypothesis L_* from some countable concept space of subsets of a universal set of instances. Unlike in the PAC model, the algorithm is not given a stochastically generated sequence of labeled instances. It is instead allowed to actively explore its environment by asking questions of an *oracle*.

Throughout these notes we use the following notation.

- $\mathcal{C} = \{L_1, L_2, \ldots\}$ is the set of concepts
- L_* is the target concept
- U is the instance space

Angluin [3] defines many different kinds of queries. We now define the types of queries we will consider here.

Membership Query The oracle will respond to input $x \in U$ "yes" if $x \in L_*$, otherwise it replies "no".

Equivalence Query The oracle will respond to input hypothesis $h \in \mathcal{C}$ "yes" if $h = L_*$, otherwise it will return some counterexample $x \in h \oplus L_*$. The counterexample returned by the oracle is selected by an all-powerful adversary, and no assumptions can be made about it.

Restricted Equivalence Query The oracle will respond to input hypothesis $h \in \mathcal{C}$ "yes" if $h = L_*$, otherwise it responds "no." It does *not* return a counterexample.

Generalized Equivalence Query The oracle will respond as an equivalence query to any input hypothesis that is a subset of U.

Unlike the PAC model in which the learner need only output a hypothesis that is a good approximation to the target concept, in the model of learning with queries, the learner must achieve *exact identification*. We say an algorithm *exactly identifies* the target concept L_* with access to certain types of queries if it always halts and outputs a concept equivalent to L_*. That is, for each element of $x \in U$ the concept output classifies x as L_* does.

10.2 General Learning Algorithms

In this section we explore some generic algorithms for learning with membership and equivalence queries. Then in the next section, we explore the relationship between this learning model and the PAC model. Finally, we will consider learning algorithms for particular concept classes.

10.2.1 Exhaustive Search

The most trivial algorithm that uses only equivalence queries is exhaustive search where each concept in $\mathcal{C}$ is tried in turn until the correct hypothesis is found. This method clearly requires, in the worst case, $|\mathcal{C}| \Leftrightarrow 1$ equivalence queries. It is interesting to note that for some classes this algorithm is no worse than any other. For example, consider the *singletons*, where for fixed instance space U, $\mathcal{C} = \{\, \{x\} \mid x \in U \,\}$. In this class, an adversary can respond to an equivalence query by returning the element in the hypothesis as the counterexample. This strategy allows the learner to eliminate at most one concept for each equivalence query. Also, $\mathcal{C}$ even requires $|\mathcal{C}| \Leftrightarrow 1$ membership queries because the adversary could reply "no" to each query.

This observation is generalized in the following lemma.

Lemma 10.1 *Suppose $\mathcal{C} = \{L_1, \ldots, L_N\}$, and there exists a set $L_\cap \notin \mathcal{C}$ such that*

$$L_i \cap L_j = L_\cap, \qquad i \neq j$$

Then any algorithm that achieves exact identification for any $L_* \in \mathcal{C}$ must make at least $N \Leftrightarrow 1$ equivalence and membership queries in the worst case.

Proof: We must exhibit an adversary which answers each query in such a way that only a single concept can be eliminated. Keep in mind that the adversary can choose the target concept as the learning session progresses. Namely, the adversary will keep a list of target concepts that are consistent with all previous queries. The learner cannot achieve exact identification until only one concept remains in the adversary's list.

For a membership query on instance x, if $x \in L_\cap$ then reply "yes," otherwise reply "no." In the first case, no concepts from the adversary's list are eliminated because x is in every concept. In the second, at most one concept, the concept containing x, is eliminated—if two were eliminated, L_i and L_j, then $L_i \cap L_j \supseteq L_\cap \cup \{x\}$, a contradiction.

For an equivalence query on hypothesis $L \neq L_*$, reply "no" and return any $x \in L \cap L_\cap$. Such an x exists since $L_\cap \notin \mathcal{C}$, so $L \cap L_\cap \neq \emptyset$. Also, if $x \in L_\cap$ then no concepts are eliminated, and if $x \notin L_\cap$ then at most one concept is eliminated. ∎

There is a dual result for union.

10.2.2 The Halving Algorithm

If the learner is allowed a more powerful equivalence query, better results are obtainable. Recall that the *generalized equivalence query* is similar to the equivalence query but allows as input any subset of U. Using such a query, any finite concept class can be learned in no more than $\lfloor \lg |\mathcal{C}| \rfloor$ queries using the following algorithm.

```
Halving Algorithm($\mathcal{C}$)
If $\mathcal{C}$ contains one element $L$, then $L = L_*$. Halt.
Else
  Let $M_{\mathcal{C}} = \{ x \mid x \in \mathcal{C}$ for at least $\frac{|\mathcal{C}|}{2}$ concepts $\}$.
  Make generalized equivalence query with hypothesis $M_{\mathcal{C}}$
  If "yes" then $M_{\mathcal{C}}$ is equivalent to $L_*$. Halt and output $M_{\mathcal{C}}$.
  Else let $x$ be the counterexample.
    If $x \in M_{\mathcal{C}}$ then $\mathcal{C}' = \mathcal{C} \setminus \{ L \in \mathcal{C} \mid x \in L \}$
    Else $\mathcal{C}' = \{ L \in \mathcal{C} \mid x \in L \}$
    Halving Algorithm($\mathcal{C}'$)
```

Theorem 10.1 *The number of queries performed by the halving algorithm is at most* $\lfloor \lg |\mathcal{C}| \rfloor$.

Proof: A query is made if $|\mathcal{C}| > 1$ and every query eliminates at least half of the concepts. ■

The upper bound is often not tight. For example, for the class of singletons the halving algorithm makes only one mistake versus the upper bound of $\lg n$ given in Theorem 10.1. Littlestone [29] gives an algorithm whose performance on any class is shown to be optimal. (This algorithm is discussed in Topic 12.)

A major drawback of the halving algorithm is that it is often not computationally feasible—often exponential computation time is needed to construct $M_{\mathcal{C}}$. However, it can sometimes be efficiently implemented. As an example note that Valiant's algorithm for learning k-CNF [43] indirectly implements the halving algorithm. Let h be defined as the conjunction of all clauses that have been true in all counterexamples. (Recall that if $h \subseteq L_*$, then h is consistent with all negative examples and thus for any counterexample x to h, $L_*(x) = +$.) Now let $\mathcal{C}' = \{$ concepts consistent with all counterexamples $\}$, so $\mathcal{C}'$ consists of every k-CNF formula that is a conjunction of some subset of the clauses in h. The hypotheses h constructed in this manner predicts according to the halving algorithm. If for a particular instance all clauses in h are true, the majority is true. Similarly, if for an instance some clause l in h is false, for every $L \in \mathcal{C}'$ that is true, there is some other hypothesis, $L' \in \mathcal{C}'$ (namely $L \wedge l$), that is false. So, if ties are false, the majority is false.

See Topic 20 for a discussion on how approximate counting schemes can be used to efficiently implement a variant of the halving algorithm.

10.3 Relationship Between Exact Identification and PAC Learnability

In this section we describe how an algorithm that uses equivalence queries can be converted to a PAC-learning algorithm. We then give an example to demonstrate the converse is not true.

Theorem 10.2 *Any algorithm that uses equivalence queries to achieve exact identification can be modified to achieve the PAC criterion (i.e.,* $\Pr[\text{error}(h) \geq \epsilon] \leq \delta$*) using calls to EX instead of equivalence queries. Furthermore, if the algorithm for learning with equivalence queries runs in polynomial time, then so will the PAC-learning algorithm.*

Proof: We need to simulate an equivalence query with calls to EX. For the i^{th} equivalence query, make $q_i = \left\lceil \frac{1}{\epsilon}(\ln\frac{1}{\delta} + i\ln 2)\right\rceil$ calls to EX. If h_i (the i^{th} hypothesis) is not consistent with a particular example, then that is the counterexample. Given that h_i is consistent with all q_i examples,

$$\Pr[\text{error}(h_i) > \epsilon] \leq (1 \Leftrightarrow \epsilon)^{q_i}$$

Let h be the final hypothesis. So

$$\begin{aligned}
\Pr[\text{error}(h) > \epsilon] &\leq \sum_{i=1}^{\infty}(1 \Leftrightarrow \epsilon)^{q_i} \\
&\leq \sum_{i=1}^{\infty} e^{-\epsilon q_i} \\
&\leq \sum_{i=1}^{\infty} \frac{\delta}{2^i} \\
&= \delta
\end{aligned}$$

where the second inequality holds because for all x, $1 \Leftrightarrow x \leq e^{-x}$. Finally, observe that the number of calls to EX made by the simulation is polynomial in the number of equivalence queries made by the original algorithm. ■

One can consider a variation of the PAC model in which the learner can either ask for a random example from EX or ask a membership query. We say a concept class is PAC-learnable with membership queries if the leaner can meet the PAC-criterion using a polynomial number of calls to EX and polynomial number of membership queries. As an immediate corollary to the above theorem we obtain.

Corollary 10.1 *A concept class that is learnable with equivalence and membership queries is PAC learnable with membership queries. Furthermore, if the algorithm for learning with equivalence and membership queries uses polynomial time, then so will the PAC-learning algorithm.*

The converse to the preceding theorem is false if efficiency must be preserved. Consider the class of singletons defined on the instance space of all binary strings of length n. We have already seen that the class of singletons requires $|\mathcal{C}| \Leftrightarrow 1 = 2^n \Leftrightarrow 1$ equivalence queries, but it only requires $\left\lceil \frac{1}{\epsilon}(n \ln 2 \Leftrightarrow \ln \delta)\right\rceil$ calls to EX in the PAC model.

10.4 Examples of Exactly Identifiable Classes

In this section we will exhibit examples, some with proofs, of classes that can be exactly identified by membership and equivalence queries.

10.4.1 k-CNF and k-DNF Formulas

As we saw in the last section, Valiant's algorithm for learning k-CNF can be used to efficiently implement the halving algorithm. Thus it follows that this class is learnable using a polynomial number of equivalence queries. There is a logically dual method for k-DNF formulas. Littlestone [29] describes an algorithm for learning these classes that may use significantly fewer queries. (This algorithm is described in the notes for Topic 12.) Finally, we note that by applying Lemma 10.1 to the class of 1-CNF formulas one gets that 2^{n-1} restricted equivalence and membership queries are needed in the worst case. A similar hardness result can be obtained for 1-DNF.

10.4.2 Monotone DNF Formulas

In this section we describe an efficient algorithm for learning any monotone DNF formula using membership and equivalence queries.

Theorem 10.3 *The class of monotone DNF formulas over n variables is exactly identifiable by equivalence and membership queries in time polynomial in n and the number of terms in the target concept.*

Before we prove this we give a definition and some observations.

Definition 10.1 *A prime implicant t of a propositional formula ϕ is a satisfiable product of literals such that t implies ϕ but no proper subterm of t implies ϕ.*

Example 10.1 *$\phi = ac \vee b\bar{c}$ has prime implicants ac, $b\bar{c}$, and ab.*

The number of prime implicants of a general DNF formula may be exponentially larger than the number of terms in the formula. However, for monotone DNF, the number of prime implicants is not greater than the number of terms in the formula.

Proof: [of Theorem 10.3] In the following let $\{y_1, y_2, \ldots, y_n\}$ be the variables.

```
Learn-Monotone-DNF
$\phi \leftarrow$ FALSE
Forever
        Make equivalence query with $\phi$
        If "yes," output $\phi$; halt.
        Else Let $x = b_1 b_2 \cdots b_n$ be the counterexample
             Let $t = \bigwedge_{b_i=1} y_i$
             For $i = 1, \ldots, n$
                  If $b_i = 1$ perform membership query on $x$ with $i^{\text{th}}$ bit flipped
                  If "yes," $t \leftarrow t \setminus \{y_i\}$ and $x = b_1 \cdots \bar{b_i} \cdots b_n$
             Let $\phi \leftarrow \phi \vee t$
```

We need to show that at each iteration, the term t is a new prime implicant of ϕ_*, the target concept. The proof will proceed by induction. For notation, let ϕ_i be the value of ϕ after the i^{th} iteration. Firstly, $\phi_0 = \text{FALSE}$ is trivially a prime implicant of ϕ_*. Assuming that ϕ_i is a prime implicant, the counterexample produced by the equivalence query is an instance x for which $\phi_*(x) = 1$ and $\phi_i(x) = 0$. So t is an implicant of ϕ_* but not of ϕ_i. Clearly the "trimming" procedure leaves a prime implicant.

Finally, the loop iterates exactly once for each prime implicant, and as stated above, this is bounded above by the number of terms in ϕ_*. Exactly n membership queries are performed during each iteration. So, assuming queries take polynomial time, the algorithm runs in time polynomial in both n and the number of terms in ϕ_*. ■

We now show that the counterexamples returned are essential to learn monotone DNF formulas efficiently.

Theorem 10.4 *For any $n > 0$ there is a class $\mathcal{D}$ of monotone DNF formulas with $2n$ variables and $n+1$ terms such that any algorithm that exactly identifies every formula in $\mathcal{D}$ using restricted equivalence queries and membership queries must make $2^n \Leftrightarrow 1$ queries in the worse case.*

Proof: Given $n > 0$, let $\phi_n = x_1y_1 + x_2y_2 + \cdots + x_ny_n$ and define $\mathcal{D}$ to be the 2^n formulas of the form $T + \phi_n$, where $T = \ell_1 \cdot \ell_2 \cdots \ell_n$ and $\ell_i = x_i$ or y_i.

Now, since for any formula ϕ in $\mathcal{D}$ there is exactly one assignment that satisfies ϕ but does not satisfy ϕ_n, any pair of distinct formulas $\phi_1, \phi_2 \in \mathcal{D}$ the assignment that satisfies both formulas are exactly those that satisfy ϕ_n. That is, $\phi_i \cap \phi_j = \phi_n \notin \mathcal{D}$. By Lemma 10.1 and $|\mathcal{D}| = 2^n$, at least $2^n \Leftrightarrow 1$ queries are needed. ■

We now consider the question: Is the class of monotone DNF formulas learnable with a polynomial number of equivalence queries? To prove that a concept class is not efficiently learnable by equivalence queries alone, Angluin [4] developed the "Method of Approximate Fingerprints". We now briefly describe this technique and apply it to the class of monotone DNF formulas.

The goal is to generalize the hardness result obtained for the class of singletons. Recall that for this class the adversary can reply to each equivalence query in such a way that only one concept is eliminated. This phenomenon is too much to ask, but also it is stronger than needed to prove a superpolynomial number of equivalence queries are needed.

Definition 10.2 *The instance w_n is an approximate fingerprint for hypothesis h in the concept class $\mathcal{C}$ if few (a superpolynomial fraction) concepts in $\mathcal{C}$ classify w_h the same as h.*

Approximate fingerprints can be used by the adversary to generate uninformative counterexamples to equivalence queries. Given any $h \in \mathcal{C}$, $h \neq L_*$, the adversary would respond "no", and return w_h, eliminating at most a superpolynomial fraction of the concept class. Thus the learner would require a superpolynomial number of equivalence queries in order to correctly eliminate all concepts but the target.

Angluin [4] has proven that monotone DNF formulas have approximate fingerprints, thus proving that exact identification cannot be achieved with a polynomial number of equivalence queries. The key property used to prove this result is that for every monotone DNF formula ϕ there is an assignment with few 1's that satisfies ϕ or an assignment with few 0's that falsifies ϕ. Moreover, not many formulas share the value of ϕ on this assignment. This assignment serves as the approximate fingerprint. See Angluin's paper for the complete proof.

Combining these two hardness results, we get that the both equivalence and membership queries are required to learn DNF formulas. Thus the algorithm described in the beginning of the section, is the best one can expect.

10.4.3 Other Efficiently Learnable Classes

We now briefly mention a few of the many other concept classes that are learnable using membership and equivalence queries.

1. **k-Term DNF, k-Clause CNF.** Angluin [1] gives an algorithm that using equivalence and membership queries identifies any k-term DNF formula using time polynomial in n^k. A dual result holds for k-clause CNF formulas. Furthermore, note that a simple modification of the proof of Pitt and Valiant [33] that k-term DNF is not PAC-learnable by k-term DNF can be used to show that for $k > 1$, the class of k-term DNF or k-clause CNF formulas cannot be exactly identified by any algorithm that uses just equivalence queries and runs in time polynomial in n^k unless $P = NP$. Thus membership queries appear to be essential to Angluin's algorithm.

2. **Regular Sets.** Angluin [2] shows that if the minimum deterministic finite automaton that generates a given regular set has n states, then the DFA can be determined in time polynomial in n and in the length of the longest counterexample. This algorithm it covered in Topic 13.

3. **Read-once Formulas.** Angluin, Hellerstein, and Karpinski [6] give an efficient algorithm to exactly identify any read-once formula using membership and equivalence queries. (A read-once formula is one in which every literal appears at most once.) Furthermore, they show that monotone read-once formulas can be learned using only membership queries.

4. **Conjunction of Horn Clauses.** A *Horn clause* is a disjunction of literals at most one of which is negated. A *Horn sentence* is a conjunction of Horn clauses. Angluin, Frazier, and Pitt [5] show that Horn sentences are efficiently learnable using membership and equivalence queries. This algorithm is described in the notes for Topic 11. Note that the class of monotone DNF formulas is properly contained in the class of Horn sentences, thus it follows that neither equivalence queries nor membership queries alone suffice.

CS 582T Computational Learning Theory **Spring Semester, 1991**

Topic 11: Learning Horn Sentences

Lecturer: Sally Goldman

11.1 Introduction

The material presented in this lecture comes from the paper, "Learning Conjunctions of Horn Clauses," by Dana Angluin, Michael Frazier, and Leonard Pitt [5]. We will present an algorithm the uses both membership and equivalence queries for learning the class of Boolean formulas that are expressible as conjunctions of Horn clauses. (A Horn clause is a disjunction of literals, all but at most one of which is a negated variable.) The computation time used by the algorithm is polynomial in the number of variables and the number of clauses in the target formula.

11.2 Preliminaries

We begin by defining the concept class of Horn sentences. Let $V = v_1, \ldots, v_n$ be the set of variables. A Horn sentence is defined as follows:

Definition 11.1 *A* Horn clause *is a disjunction of literals in which at most one literal is unnegated. A* Horn sentence *is a conjunction of Horn clauses.*

For example suppose we had the variables, a, b, c, d. Then one possible Horn sentence is $(\overline{a} \vee \overline{b} \vee c) \wedge (\overline{d} \vee b)$. Observe that an alternate way to represent a Horn clause is as an implication in which the consequent contains at most one variable. Thus the above Horn sentence could be represented as: $(a \wedge b \Rightarrow c) \wedge (d \Rightarrow b)$. We will use this representation throughout the remainder of these notes.

What is the representational power of Horn sentences with respect to some of the other Boolean concept classes we have considered? It is easily seen that the class of Horn sentences over variable set V is a proper subset of the class of Boolean formulas over V. Furthermore, observe that by applying DeMorgan's Law it can be shown that the class of monotone DNF formulas over variable set V is a proper subset of the class of Horn sentences over V. Thus it follows from the work of Angluin [3, 4] that either membership or equivalence queries alone are insufficient for polynomial-time learning of Horn sentences.

Observe that the dual of the class of Horn sentences is the class of "almost monotone" DNF formulas—a disjunct of terms, where each term is a conjunct of literals, at most one of which is negated. Thus the algorithm presented in the next section can easily be modified to learn this class. Finally, note that the problem of determining whether two Horn sentences are equivalent (and producing a counterexample if they are not) is solvable in polynomial time. Thus the equivalence query oracle could be replaced by a teacher with polynomially bounded computational resources.

11.3 The Algorithm

Before describing the algorithm, we need a few more definitions. We will use $\mathbf{T}$ to denote the logic constant "true" and $\mathbf{F}$ to denote the logic constant "false".

Definition 11.2 *Let x be an example; then* $\text{true}(x)$ *is the set of variables assigned the value* $\mathbf{T}$ *by x and* $\text{false}(x)$ *is the set of variables assigned the value* $\mathbf{F}$ *by x.*

By convention, $\mathbf{T} \in \text{true}(x)$ and $\mathbf{F} \in \text{false}(x)$.

Definition 11.3 *Let C be a Horn clause. Then* $\text{antecedent}(C)$ *is the set of variables that occur negated in C. If C contains an unnegated variable z, then* $\text{consequent}(C)$ *is just z. Otherwise, C contains only negated variables and* $\text{consequent}(C)$ *is* $\mathbf{F}$.

Definition 11.4 *An example x* covers *a Horn clause C if* $\text{antecedent}(C) \subseteq \text{true}(x)$*. Otherwise x does not cover C. The example x* violates *the Horn clause C if x covers C and* $\text{consequent}(C) \in \text{false}(x)$*.*

Observe that if x violates C then x must cover C, but the converse does not necessarily hold.

We are now ready to describe the algorithm. Let H_* be the target Horn sentence. The basic idea of the algorithm is that every negative example x violates some clause C of H_*. As in the algorithm presented for learning monotone DNF formulas, we would like to generate the target Horn sentence clause by forming the clause C from H_* that x violates and simply add it to our current hypothesis. However, here we cannot exactly determine C. We know that $\text{antecedent}(C) \subseteq \text{true}(x)$, and $\text{consequent}(C) \in \text{false}(x)$. Thus we will add to our current hypothesis all elements of the set

$$\text{clauses}(x) = \left\{ \left(\bigwedge_{v \in \text{true}(x)} v \right) \Rightarrow z : z \in \text{false}(x) \right\} \tag{11.1}$$

whenever a new negative counterexample x is obtained.

There are two problems which can occur. First of all, clauses with the "wrong" consequent can be added to the hypothesis. However, we can clean this up using the positive counterexamples. The second problem that occurs is the antecedent we select possible contains more variables than it should and thus it may not be false when the "real" antecedent is false. In order for the algorithm to work, it is important to ensure that we only have one set of clauses (as defined by Equation (11.1)) in the hypothesis corresponding to each clause in the target. Thus on a negative counterexample, we must first try to reduce the number

```
Learn-Horn-Sentence()
1    $S \leftarrow \emptyset$ ($s_i$ denotes $i$th element of $S$)
2    $H \leftarrow \emptyset$ (always true hypothesis)
3    UNTIL **equivalent**($H$) returns "yes" DO
4        BEGIN
5        Let $x$ be the counterexample returned
6        IF $x$ violates at least one clause of $H$
7            THEN ($x$ is a positive example)
8                remove from $H$ every clause that $x$ violates
9            ELSE ($x$ is a negative example)
10               BEGIN
11               FOR each $s_i \in S$ such that true($s_i \cap x$)
                     is properly contained in true($s_i$)
12                   BEGIN
13                   query **member**($s_i \cap x$)
14                   END
15               IF any of these queries is answered "no"
16                   THEN let $i$ be the least number such that
                         **member**($s_i \cap x$) was answered "no"
17                       refine $s_i \leftarrow s_i \cap x$
18                   ELSE add $x$ to end of $S$
19               ENDIF
20               $H \leftarrow \bigvee_{s \in S}$ clauses($s$)
21               END
22       ENDIF
23       END
24   Return $H$
```

Figure 11.1: Algorithm for learning Horn sentences.

of variables in the antecedent of a set of clauses currently in the hypothesis before using the formula in Equation (11.1) to create new clauses.

The algorithm maintains a sequence S of negative examples. Each new negative counterexample is used to either refine one element of S, or is added to the end of S. In order to learn all of the clauses of H_* we ensure that the clauses induced by the examples in S approximate *distinct* clauses of H_*. This will occur if the examples in S violate distinct clauses of H_*. To maintain this invariant, whenever a new negative counterexample could be used to refine several examples in the sequence S, only the earliest one in the set is refined. The algorithm is shown in Figure 11.1.

11.4 Example Run of Algorithm

Before analyzing this algorithm, we will examine an example run of it. Let $V = \{a, b, c, d\}$, and let $H_* = (ab \Rightarrow c) \wedge (d \Rightarrow b)$.

Initially $S = \emptyset$, $H = \emptyset$ (always true)

EQUIV(H)? 1101 (negative example)
$S = \{1101\}$
$H = (abd \Rightarrow c) \wedge (abd \Rightarrow \mathbf{F})$
Observe that the clause $(abd \Rightarrow c)$ is an approximation to $(ab \Rightarrow c)$. But note that for the example 1100 the clause $(ab \Rightarrow c)$ is not satisfied, but $(abd \Rightarrow c)$ is satisfied.

EQUIV(H)? 1100 (negative example)
MEMB(1100)? "no"
$S = \{1100\}$
$H = (ab \Rightarrow c) \wedge (ab \Rightarrow d) \wedge (ab \Rightarrow \mathbf{F})$

EQUIV(H)? 1001 (negative example)
MEMB(1000)? "yes"
$S = \{1100, 1001\}$
$H = (ab \Rightarrow c) \wedge (ab \Rightarrow d) \wedge (ab \Rightarrow \mathbf{F}) \wedge (ad \Rightarrow b) \wedge (ad \Rightarrow c) \wedge (ad \Rightarrow \mathbf{F})$

EQUIV(H)? 1110 (positive example)
$S = \{1100, 1001\}$
$H = (ab \Rightarrow c) \wedge (ad \Rightarrow b) \wedge (ad \Rightarrow c) \wedge (ad \Rightarrow \mathbf{F})$
Observe that the positive example only modify H and thus if there is every another negative example, it's effect is removed.

EQUIV(H)? 0011 (negative example)
MEMB(0000)? "yes"
MEMB(0001)? "no"
$S = \{1100, 0001\}$
$H = (ab \Rightarrow c) \wedge (ab \Rightarrow d) \wedge (ab \Rightarrow \mathbf{F}) \wedge (d \Rightarrow a) \wedge (d \Rightarrow b) \wedge (d \Rightarrow c) \wedge (d \Rightarrow \mathbf{F})$

EQUIV(H)? 0101 (positive example)
$S = \{1100, 1001\}$
$H = (ab \Rightarrow c) \wedge (d \Rightarrow b)$

EQUIV(H)? "yes"

11.5 Analysis of Algorithm

Clearly since the algorithm only halts when it receives "yes" from an equivalence query, the hypothesis output is correct. Thus what must be shown is that the number of membership and equivalence queries is polynomial in n and m (the number of clauses in H_*), and the computation time is polynomial in n and m. Here we shall just sketch the main ideas of the analysis, see the Angluin, Frazier, Pitt paper [5] for the details.

The key property which must be shown is that throughout the execution of the algorithm, at no time do two distinct elements of S violate the same clause of H_*. Once this has been shown, the remainder of the analysis is fairly straightforward. This key property is proven using the following two lemmas.

Lemma 11.1 *For each execution of the main loop of line 3, the following invariant holds. Suppose that in step 5 of the algorithm a negative example x is obtained such that for some clause C of H_* and for some $s_i \in S$, x violates C and s_i covers C. Then there is some $j \leq i$ such that in step 17 the algorithm will refine s_j by replacing it with $s_j \cap x$.*

This lemma is proven by induction on the number of iterations k of the main loop of line 3.

Lemma 11.2 *Let S be a sequence of elements constructed for target H_* by the algorithm. Then*

1. $\forall k \forall (i < k) \forall (C \in H_*)$ *if s_k violates C then s_i does not cover C*
2. $\forall k \forall (i \neq k) \forall (C \in H_*)$ *if s_k violates C, then s_i does not violate C.*

Here too, these two invariants are shown to hold by using an inductive proof.

An immediate corollary of the second property of the second lemma is that at no time do two distinct elements in S violate the same clause of H_*. And since each of the elements in S is a negative example if follows that every element of S violates at least one clause of H_*. Thus at no time doing the execution of the algorithm does S contain more than m elements. We are now ready to analyze the running time of the algorithm.

Theorem 11.1 *A Horn sentence consisting of m clauses over n variables can be learned exactly in time $\tilde{O}(m^3 n^4)$ using $O(m^2 n^2)$ equivalence queries and $O(m^2 n)$ membership queries*[5].

Proof Sketch: The sequence S is only changed by appending a new element to it, or refining an existing element. Thus $|S|$ never decreases. Since $|S| \leq m$ it follows that line 18 is executed at most m times. Observe that when an element of S is refined in line 17, it now contains strictly fewer variables assigned the value "true". Thus a given element can be refined at most n times, and so line 17 is executed at most mn times. Whenever the ELSE clause at line 9 is executed, either line 17 or 18 is executed. Thus lines 9–21 are executed at

[5]The "soft-oh" notation $\tilde{O}$ is like the standard "big-oh" notation except that logarithmic terms are ignored.

most $nm + m = (n+1)m$ times. So the total number of membership queries made are at most $(n+1)m^2$.

Observe that the cardinality of clauses(s) in any hypothesis H constructed in line 15 is at most $(n+1)m$. Each positive counterexample obtained in line 5 causes at least one clause to be removed from H, thus the equivalence queries can produce at most $(n+1)m$ positive counterexamples between modifications of S. Therefore, line 8 is executed at most $(n+1)^2m^2$ times. Thus the total number of equivalence queries is at most $(n+1)^2m^2 + (n+1)m + 1$. Finally, it can be shown that the time needed for each execution of the main loop is $\tilde{O}(n^2m)$.
■

CS 582T Computational Learning Theory **Spring Semester, 1991**

Topic 12: Learning With Abundant Irrelevant Attributes

Lecturer: Sally Goldman *Scribe: Marc Wallace*

12.1 Introduction

In these notes we first introduce the on-line (or mistake-bound) learning model. Then we consider when this model is applied to concept classes that contain a large number of irrelevant attributes. Most of this lecture comes from the paper, "Learning when Irrelevant Attributes Abound: A New Linear-threshold Algorithm," by Nick Littlestone [29].

12.2 On-line Learning Model

Observe that one property of the PAC-learning model is that it is a *batch model*—there is a separation between the *training phase* and the *performance phase*. Thus in the PAC model a learning session consists of two phases: the training phase and the performance phase. In the training phase the learner is presented with a set of instances labeled according to the target concept $c \in C_n$. At the end of this phase the learner must output a hypothesis h that classifies each $x \in X_n$ as either a positive or negative instance. Then in the performance phase, the learner uses h to predict the classification of new unlabeled instances. Since the learner never finds out the true classification for the unlabeled instances, all learning occurs in the training phase.

We now give an example from Goldman's thesis [18] to motivate the *on-line learning model*. This model is also known as the *mistake-bound learning model*. Suppose that when arriving at work (in Boston) you may either park in the street or park in a garage. In fact, between your office building and the garage there is a street on which you can always find a spot. On most days, street parking is preferable since you avoid paying the \$10 garage fee. Unfortunately, when parking on the street you risk being towed (\$50) due to street cleaning, snow emergency, special events, etc. When calling the city to find out when they tow, you are unable to get any reasonable guidance and decide the best thing to do is just learn from experience. There are many pieces of information that you might consider in making your prediction; e.g. the date, the day of the week, the weather. We make the following two assumptions: enough information is available to make good predictions if you know how to use it, and after you commit yourself to one choice or the other you learn of the right decision. In this example, the city has rules dictating when they tow; you just don't know them. If you park on the street at the end of the day you know if your car was towed; otherwise when walking to the garage you see if the street is clear (i.e. you learn if you would have been towed).

The on-line model is designed to study algorithms for learning to make accurate predictions in circumstances such as these. Formally, an on-line learning algorithm for C is an algorithm that runs under the following scenario. A *learning session* consists of a set of *trials*. In each trial, the learner is given an unlabeled instance $x \in X$. The learner uses its current hypothesis to predict if x is a positive or negative instance of the target concept $c \in C$ and then the learner is told the correct classification of x. If the prediction was incorrect, the learner has made a *mistake*. Note that in this model there is no training phase. Instead, the learner receives *unlabeled instances* throughout the entire learning session. However, after each prediction the learner "discovers" the correct classification. This feedback can then be used by the learner to improve its hypothesis. Observe that in this model it is beneficial for the learning algorithm to calculate hypothesis incrementally rather than starting from scratch each time.

In this learning model we shall evaluate the performance of a learning algorithm by the number of prediction mistakes made by the learning algorithm. We now describe the two most common ways in which this notion has been formalized.

Probabilistic Mistake Bound: Let D be some arbitrary and unknown probability distribution on the instance space. The *probabilistic mistake bound* is the probability that the learner's hypothesis disagrees with c on the t^{th} randomly drawn instance from D. Formally, given any $n \geq 1$ and any $c \in C_n$, the learner's goal is to output a hypothesis h such that the probability that h makes a mistake on the $t + 1^{st}$ trial is at most $p(n)t^{-\beta}$ for some polynomial $p(n)$ and $0 < \beta$.

Absolute Mistake Bound: The *absolute mistake bound* is the worst-case total number of mistakes made when the learner must make predictions for any, possibly infinite, sequence of instances. (Even if the instance space is finite, repetitions may occur.) Formally, given any $n \geq 1$ and any $c \in C_n$, the learner's goal is to make at most $p(n)$ mistakes for some polynomial $p(n)$.

Throughout the remainder of these notes we shall consider an on-line learning model where the absolute mistake bound criterion is used. In other words, we assume that an adversary selects the order in which the instances are presented to the learner and we evaluate the learner by the maximum number of mistakes made during the learning session. Our goal is to minimize the worst-case number of mistakes using an efficient learning algorithm (i.e. each prediction is made in polynomial time). Observe that such mistake bound are quite strong in that the order in which examples are presented does not matter; however, it is impossible to tell how early the mistakes will occur. See Haussler et al. [21] for a discussion of the relationship between the PAC model and on-line learning with the probabilistic mistake bound criterion.

After studying some general results about this on-line learning model we will focus on the situation in which the instances are drawn from the Boolean domain and proper classification for each instance can be determined by only a small fraction of the attribute space. Pattern recognition falls under this situation since a feature detector might extract a large number of features for the learner's consideration not knowing which few will prove useful. Another example is building new concepts as Boolean functions of previously learned concepts that are stored in a library. For this problem, the learner may need to sift through a large library of available concepts to find the suitable ones to use in expressing each new concept. (The concept class of k-DNF fits into this model where the terms come from the library.)

12.3 Definitions and Notation

In this section we describe the notation used by Littlestone [29]. A concept class C consists of concepts c which are Boolean functions $c : \{0,1\}^n \to \{0,1\}$.

- For any algorithm A and target concept $c \in C$, let $M_A(c)$ be the maximum over all possible sequences of examples of the number of mistakes that algorithm A makes while learning the concept c.
- Define $M_A(C) = \max_{c \in C} M_A(c)$. If C is the empty set then by definition $M_A(C) = -1$.
- Define $opt(C) = \min_A M_A(C)$, the minimum over all possible algorithms of the worst-case number of mistakes.
- An algorithm A is *optimal* for C if and only if $M_A(C) = opt(C)$.

12.4 Halving Algorithm

One algorithm which often yields a good mistake bound is the halving algorithm (as described in Section 10.2.2). We now review the halving algorithm using the notation of Littlestone. We shall then look at a variant of it that can be shown to perform optimally.

Let CONSIST be a subset of the concepts in C that are consistent with all previous examples. So initially, CONSIST $= C$. Given a target concept class C and an instance x, we define $\xi_i(C, x) = \{c \in C | c(x) = i\}$ for $i = 0$ or $i = 1$. Thus the sets $\xi_0(C, x)$ and $\xi_1(C, x)$ are the set of concepts that are 0 at x, and the set of concepts that are 1 at x, respectively.

Upon receiving an instance x, the halving algorithm computes the sets $\xi_0(\text{CONSIST}, x)$ and $\xi_1(\text{CONSIST}, x)$. If $|\xi_1(\text{CONSIST}, x)| > |\xi_0(\text{CONSIST}, x)|$ then the algorithm predicts 1, otherwise it predicts 0. After receiving feedback the learner updates CONSIST: if $c(x) = 0$ then set CONSIST $= \xi_0(\text{CONSIST}, x)$, and if $c(x) = 1$ then set CONSIST $= \xi_1(\text{CONSIST}, x)$.

Theorem 12.1 *For any nonempty target class C, $M_{HALVING}(C) \leq \log_2 |C|$.*

Proof: Since the halving algorithm predicts according to the majority, its response is consistent with at least half of CONSIST. Therefore the size of CONSIST drops by a factor of at least two at each mistake. And since there is a consistent function, $|\text{CONSIST}| \geq 1$ always, so the algorithm can make no more than $\log_2 |C|$ mistakes. ■

The halving algorithm tells us that we can always choose a concept class such that $\log_2 |C|$ mistakes are actually made. Hence we have

Theorem 12.2 *For C such that $|C|$ is finite, opt$(C) \leq \log_2 |C|$.*

Before considering an algorithm that often makes fewer mistakes than the halving algorithm, we briefly consider the relation between this on-line learning model and the model of learning with equivalence queries. If both an equivalence query algorithm and a mistake-bound algorithm are applied to learning the same concept (over the same representation class), we have:

$$\begin{aligned} \text{\# equiv. queries for exact id.} &\leq \text{\# mistakes} + 1 \\ \text{\# mistakes} &\leq \text{\# equiv. queries for exact id.} \end{aligned}$$

These inequalities follow from the simple observation that each mistake made in the on-line model serves as a counterexample to an equivalence query. Furthermore, an equivalence query with the correct hypothesis corresponds to a hypothesis for which no additional mistakes will occur.

12.5 Standard Optimal Algorithm

Along with the problem that the halving algorithm often requires an exponentially amount of time and space, it is not always an optimal algorithm. After giving some more definitions, we shall describe a modification of the halving algorithm that always performs optimally (although is still not computationally feasible in most cases).

A *mistake tree* for C over X is defined to be a binary tree each of whose nodes is a non-empty subset of C and each of whose internal nodes is labeled with an $x \in X$ which satisfies:

1. The root node is C (along with a label).
2. Given any internal node C' labeled with x, the left child (if present) is $\xi_0(C', x)$ and the right child (if present) is $\xi_1(C', x)$.

A *complete k-mistake tree* is a mistake tree that is a complete binary tree of height k. (The height of a binary tree is the number of edges in the longest path from the root to a leaf.) Finally, we define $K(C)$ as the largest integer k such that there exists a complete k-mistake tree for the concept class C. We shall use the convention that $K(\emptyset) = -1$.

In Figure 12.1 is an example of a complete 2-mistake tree. The concept class used is a simple one; $X = \{0,1\}^5$ and C consists of the five concepts, $f_i(x_1, \ldots, x_5) = x_i$ for $i = 1, \ldots, 5$.

As we shall show, these trees characterize the number of mistakes made by the optimal learning algorithm.

We now define the *standard optimal algorithm* (SOA):

```
Let CONSIST contain all $c \in C$ consistent with all past instances.
Predict 1 if $K(\xi_1(\text{CONSIST}, x)) > K(\xi_0(\text{CONSIST}, x))$.
Predict 0 otherwise.
```

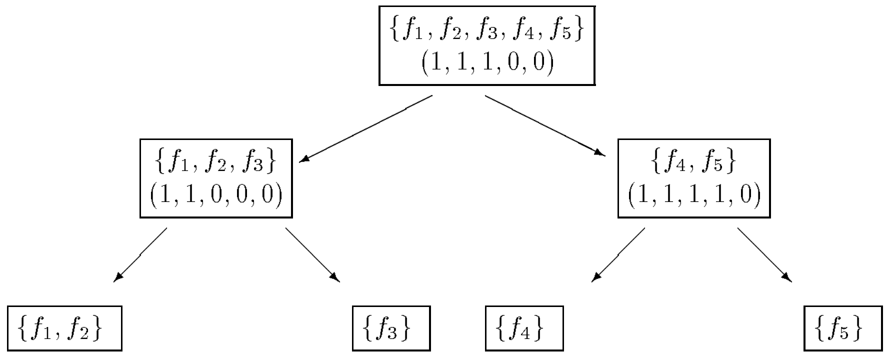


Figure 12.1: An example of a complete 2-mistake tree for a concept class C over instance space $X = \{0,1\}^5$ where C consists of the five concepts $f_i(x_1, \ldots, x_5) = x_i$ for $i = 1, \ldots, 5$.

So if a mistake is made the remaining consistent functions have the smaller maximal complete mistake tree. As we shall show this yields an optimal algorithm. However, we first prove the following two lemmas.

Lemma 12.1 *$opt(C) \geq K(C)$.*

Proof: If $C = \emptyset$ then by definition $K(C) = \Leftrightarrow 1$ and the lemma trivially follows. So assume that $C \neq \emptyset$ and $k = K(C)$. Given any algorithm A we show how the adversary can choose a target concept and a sequence of instances such that A makes at least k mistakes. If $k = 0$ the lemma is trivially satisfied. Otherwise, the adversary chooses the instance that is the root of a complete k-mistake tree for C. Regardless of A's prediction the adversary replies that the prediction is incorrect. The remaining consistent concepts form a complete $(k \Leftrightarrow 1)$-mistake tree, so by induction we are done. ■

Lemma 12.2 *Suppose that we run SOA in order to learn a concept in C where $x_1, \ldots, x_t$ is the sequences of instances given to SOA. Let* CONSIST$_i$ *be the value of the variable* CONSIST *at the beginning of the ith trial. Then for any $k \geq 0$ and $i \in \{1, \ldots, t\}$, if $K($*CONSIST$_i$*$) = k$, then SOA will make at most k mistakes during the trials $i, \ldots, t$.*

Proof: This will be a proof by induction on k. For the base case observe that by construction, the target function is always in CONSIST$_i$. If CONSIST$_i$ has two elements, we can always use an instance on which they differ as the root node of a 1-mistake tree, so $K($CONSIST$_i) = 0$ implies that CONSIST$_i$ has only the target function. Since $K(0) = \Leftrightarrow 1$ (by definition), SOA cannot make any mistakes when only the target function is left. Hence the base case $k = 0$ is proven.

We now prove the lemma for arbitrary $k > 0$, assuming it holds for $k \Leftrightarrow 1$. If SOA makes no mistakes during trials $i, \ldots, t \Leftrightarrow 1$ then the lemma is trivially true. So let j be the first trial among $i, \ldots, t \Leftrightarrow 1$ in which a mistake is made. We now prove by contradiction that $\xi_0(\text{CONSIST}_j, x_j)$ and $\xi_1(\text{CONSIST}_j, x_j)$ cannot both be complete k-mistake trees. Suppose there are k-mistake trees for both $\xi_0(\text{CONSIST}_j, x_j)$ and $\xi_1(\text{CONSIST}_j, x_j)$. Then we can combine these into $(k+1)$-mistake tree by using x_j as a root node. But by hypothesis, such a tree cannot exist (k is the largest size of a mistake tree). Therefore one of $K(\xi_0(\text{CONSIST}_j, x_j))$ or $K(\xi_1(\text{CONSIST}_j, x_j))$ is less than k. Since SOA always picks the larger of the two, and it made a mistake, then $K(\text{CONSIST}_{j+1})$ will be less than k. Using induction only $k \Leftrightarrow 1$ mistakes will be made from this point on. So only k mistakes can be made completing the inductive step.
■

Now we are ready to prove the main result of this section.

Theorem 12.3 $\mathit{opt}(C) = M_{SOA}(C) = K(C)$.

Proof: By setting $i = 1$ and $k = K(C)$ in the Lemma 12.2, we get $M_{SOA}(C) \leq K(C)$. Furthermore, by Lemma 12.1, $K(C) \leq \mathit{opt}(C)$. Combining these two inequalities it follows that $M_{SOA}(C) \leq \mathit{opt}(C)$. Finally, by definition, $\mathit{opt}(C) \leq M_{SOA}(C)$. ■

Let us now consider some lower bounds on $\mathit{opt}(C)$. The Vapnik-Chervonenkis dimension is useful in this respect.

Theorem 12.4 $\mathit{opt}(C) \geq VCD(C)$.

Proof: Let $\{v_1, \ldots, v_k\} \subseteq X$ be any set shattered by C. Then clearly a complete k-mistake tree can be constructed for C, where all internal nodes at a depth i are labeled with v_{i+1}. Apply this procedure to $k = VCD(C)$. ■

We note however that this is not a tight lower bound. Let C be the a concept class of the instance space $X = \{1, \ldots, 2^n \Leftrightarrow 1\}$ where $C = \{c_i : X \to \{0,1\} | c_i(x_j) = 1$ if and only if j < i$\}$. Thus, these are concepts of "all things less than i". Clearly $VCD(C) = 1$ since for two concepts c_i and c_j, the greater of $\{i, j\}$ cannot be covered by one concept without covering the other. Yet, we can show that $\mathit{opt}(C) \geq n$ by constructing a complete n-mistake tree as follows: label the root node 2^{n-1}. Each branch of the tree is the same type but half as big, so by inductively constructing the binary tree, we have a complete n-mistake tree.

12.6 The Linear Threshold Algorithm: WINNOW1

In this section we describe an algorithm that efficiently deals with a large number of irrelevant attributes when learning in a Boolean domain. If desired this algorithm can be implemented within a neural network framework as a simple linear threshold function. This algorithm is similar to the classical perceptron algorithm, except that it uses a multiplicative weight-update scheme that permits it to perform much better than classical perceptron training

prediction	correct	name	update scheme
1	0	elimination	if $x_i = 1$ then set $w_i = 0$.
0	1	promotion	if $x_i = 1$ then set $w_i = \alpha \cdot w_i$.

Figure 12.2: The update scheme used by WINNOW1.

algorithms when many attributes are irrelevant. From empirical evidence it appears that for the perceptron algorithm the number of mistakes grows linearly with the number of irrelevant attributes. Here we shall describe an algorithm for which the number of mistakes only grows logarithmically with the number of irrelevant attributes.

A *linearly separable Boolean function* is a map $f : \{0,1\}^n \to \{0,1\}$ such that there exists a hyperplane in $\Re^n$ that separates the inverse images $f^{-1}(0)$ and $f^{-1}(1)$ (i.e. the hyperplane separates the point on which the function is 1 from those on which it is 0). An example of a linearly separable function is any monotone disjunction: if $f(x_1,\ldots,x_n) = x_{i_1} \vee \ldots \vee x_{i_k}$, then the hyperplane $x_{i_1} + \ldots + x_{i_k} = 1/2$ is a separating hyperplane.

We present a limited form, WINNOW1, and will later generalize it. WINNOW1 is a linear threshold algorithm over the Boolean space $X = \{0,1\}^n$ that is designed for learning monotone disjunctions. There are n real valued weights $w_1,\ldots,w_n$ maintained by the algorithm. Initially each weight is set to 1. Also, a real number θ, called the *threshold*, is utilized. When WINNOW1 receives an instance $x = (x_1,\ldots,x_n)$, it predicts as follows:

- if $\sum_{i=1}^n w_i x_i > \theta$ then it predicts 1.
- if $\sum_{i=1}^n w_i x_i \leq \theta$ then it predicts 0.

When a mistake is made, the weights with non-zero x_i are updated as shown in Figure 12.2. Note that the threshold θ is never altered. Good values for the parameters θ and α are $\theta = n/2$ and $\alpha = 2$.

We now present three lemmas which will be used to prove a later theorem. All three have as preconditions that WINNOW1 is run with $\alpha > 1$ and $\theta \geq 1/\alpha$ for the learning of k-literal monotone disjunctions.

Lemma 12.3 *Let u be the number of promotion steps that have occurred in some sequence of trials, and v the number of elimination steps in the same trials. Then $v \leq (n/\theta) + (\alpha \Leftrightarrow 1)u$.*

Proof: Consider the sum $\sum_{i=1}^n w_i$. Initially this sum is n since all of the weights are initially 1. At each promotion, the sum can increase by no more than $(\alpha \Leftrightarrow 1)\theta$, since the sum (over all x_i that are on) must be less than θ for a promotion to occur. Similarly, at each elimination step, the sum must be decreased by at least θ. Hence

$$0 \leq \sum_{i=1}^{n} w_i \leq n + \theta(\alpha \Leftrightarrow 1)u \Leftrightarrow \theta v.$$

Thus

$$\theta v \leq n + \theta(\alpha \Leftrightarrow 1)u$$

giving the desired result. ■

Lemma 12.4 *For all i, $w_i \leq \alpha\theta$ (after any number of trials).*

Proof: Since $\theta \geq 1/\alpha$, for all i, initially $w_i \leq \alpha\theta$. We now proceed by induction on the number of steps. Note that w_i is only increased by a promotion step when $x_i = 1$ and $\sum_{i=1}^{n} w_i x_i \leq \theta$. These conditions can only occur together if $w_i \leq \theta$ prior to promotion. Thus $w_i \leq \alpha\theta$ after the promotion step. So after any step the claim is true, hence it is always true. ■

Lemma 12.5 *After u promotion steps and any number of eliminations, there exists an i such that $\log_\alpha w_i \geq u/k$.*

Proof: Let the k-literal disjunction we are learning be of the form

$$f(x_1, \ldots, x_n) = x_{i_1} \vee \ldots \vee x_{i_k}$$

Let the set $R = \{i_1, \ldots, i_k\}$, and consider the product $P = \prod_{j \in R} w_j$. Since $f = 0$ if and only if $x_j = 0$ for all $j \in R$, and elimination occurs when $f = 0$, elimination cannot affect the product P at all. Also, at each promotion step, P is increased by at least a factor of α, since at least one of the x_is was on for $i \in R$. At first we have $P = 1$. After u promotions (and any number of eliminations), $P \geq \alpha^u$. Taking logs of both sides, we have $\sum_{i \in R} \log_\alpha w_i \geq u$. Since $|R| = k$, for some $i \in R$ we must have that $\log_\alpha w_i \geq u/k$. ■

We are finally ready to prove an upperbound on the number of mistakes made by WINNOW1 when learning a k-literal monotone disjunction.

Theorem 12.5 *For the learning of k-literal monotone disjunctions, if WINNOW1 is run with $\alpha > 1$ and $\theta \geq 1/\alpha$, then the total number of mistakes is at most $\alpha k(\log_\alpha \theta + 1) + n/\theta$.*

Proof: The total number of mistakes is clearly the number of promotion steps plus the number of elimination steps $(u + v)$. Lemmas 12.4 and 12.4 yield the following bound on v:

$$u/k \leq \log_\alpha w_i \leq \log_\alpha \alpha\theta = \log_\alpha \theta + 1$$

So

$$u \leq k(\log_\alpha \theta + 1)$$

Plugging in this value of u into the inequality of Lemma 12.3, we obtain the following bound on v:

$$v \leq (n/\theta) + (\alpha \Leftrightarrow 1)k(\log_\alpha \theta + 1)$$

Adding the two bounds together gives the desired result. ■

Observe that WINNOW1 need not have prior knowledge of k although the number of mistakes will depend on k. Littlestone's paper also discusses how to optimally choose θ and α if an upperbound on k is known a priori.

Noticing that k-literal monotone disjunctions are 1-term monotone k-DNF formulas, we state (without proof) the following lower bound on the VC dimension and thus on the number of mistakes made in the on-line model. See Littlestone's paper for the proof.

Lemma 12.6 *For $1 \leq k \leq n$ and $1 \leq l \leq \binom{n}{k}$, let C be the class of l-term monotone k-DNF formulas. Let m be any integer with $k \leq m \leq n$ with $\binom{m}{k} \geq l$. Then $VCD(C) \geq kl\lfloor \log_2(n/m) \rfloor$.* ■

We now apply this theorem to two special cases. If $l = 1$ and $m = k$ then we get that $VCD(C) \geq k\lfloor \log_2 n/k \rfloor$ where C contains all conjunctions of at most k variables chosen from n variables. Likewise, if $k = 1$ and $m = l$ then $VCD(C) \geq l\lfloor \log_2 nl \rfloor$ where C contains all disjunctions of at most l variables chosen from n variables.

12.7 Extensions: WINNOW2

Let $X = \{0,1\}^n$ be the instance space, and $0 < \delta \leq 1$. We define $F(X, \delta)$ to be the set of all functions $X \to \{0,1\}$ with the following property: for each $f \in F(X, \delta)$ there exist $\mu_1, \ldots, \mu_n \geq 0$ such that for all instances $x = (x_1, \ldots, x_n) \in X$,

$$\sum_{i=1}^{n} \mu_i x_i \geq 1 \mathrm{iff(x)} = 1$$

and

$$\sum_{i=1}^{n} \mu_i x_i < 1 \Leftrightarrow \delta \mathrm{iff(x)} = 0$$

That is, the inverse images of 0 and 1 are separated by at least δ.

An example of such a class of functions would be the r-of-k threshold functions. These functions are defined by selecting k variables which are "important". Then, $f(x) = 1$ whenever r or more of these k significant variables are on. Let the selected variables be $x_{i_1}, \ldots, x_{i_k}$.

$$f(x) = 1 \Longleftrightarrow x_{i_1} + \ldots + x_{i_k} \geq r$$

Thus,

$$f(x) = 1 \Longleftrightarrow (1/r)x_{i_1} + \ldots + (1/r)x_{i_k} \geq 1$$

Also,

$$f(x) = 0 \Longleftrightarrow x_{i_1} + \ldots + x_{i_k} \leq r \Leftrightarrow 1$$

Thus,

$$f(x) = 0 \Longleftrightarrow (1/r)x_{i_1} + \ldots + (1/r)x_{i_k} \leq 1 \Leftrightarrow (1/r)$$

Hence r-of-k threshold functions are contained in $F(\{0,1\}^n, 1/r)$.

We describe the algorithm WINNOW2 here. It is basically the same as WINNOW1, but updates made in response to mistakes is slightly different: instead of eliminating weights entirely, they are divided by α. One could call this altered step a demotion. See Figure 12.3 for the update scheme used by WINNOW2.

A theorem similar to the one for WINNOW1 exists to give an upper bound on the number of mistakes that WINNOW2 will make. We state without proof the theorem here. For a proof see Littlestone's paper.

prediction	correct	name	update scheme
1	0	demotion	if $x_i = 1$ then set $w_i = w_i/\alpha$.
0	1	promotion	if $x_i = 1$ then set $w_i = w_i * \alpha$.

Figure 12.3: The update scheme used by WINNOW2.

Theorem 12.6 *Let $0 < \delta \le 1$, and the target function be in $F(X,\delta)$ for $X = \{0,1\}^n$. If the appropriate $\mu_1, \ldots, \mu_n$ are chosen so that the target function satisfies the definition for $F(X,\delta)$, and WINNOW2 is run with values $\alpha = 1 + \delta/2$ and $\theta \ge 1$ and the algorithm gets its instances from X, then the number of mistakes is bounded above by*

$$\frac{8}{\delta^2}\frac{n}{\theta} + \left(\frac{5}{\delta} + \frac{14\ln\theta}{\delta^2}\right)\sum_{i=1}^{n}\mu_i$$

For r-of-k threshold functions we have $\delta = 1/r$ and $\sum \mu_i = k/r$. So setting $\theta = n$ yields a mistake bound of $8r^2 + 5k + 14kr\ln n$.

12.8 Transformations to Other Concept Classes

Suppose we had a learnable concept class, and wanted to learn a second class. If morphisms existed which map instances between the two concept spaces, and mapped from the answers of one to the answers of the second, then it would not seem to be difficult to learn the second class by simply pretending to be learning the first. This is the basic idea behind these transformations.

Let the instance space of the derived algorithm be X_1, and that of the original algorithm be X_2. The transformations will take the form of $T_i : X_1 \to X_2$ and $T_p : \{0,1\} \to \{0,1\}$. The mapping T_i maps between the instance spaces, and T_p, the map between predictions, is either the identity function or the function which interchanges 0 and 1. Let A_1 be the algorithm desired to learn concepts over X_1, and let A_2 be the algorithm provided that learns concepts over X_2. When A_1 gets an instance $x \in X_1$, it sends the instance $T_i(x)$ to A_2. Then A_2 generates a prediction y. Next, A_2 sends this prediction to A_1 which outputs $T_p(y)$. Finally, any reinforcements are passed directly to A_2.

So suppose we have an algorithm A_2, and we want to derive an algorithm to learn C_1. We need not only a C_2 that A_2 can learn, but also two mappings T_i, T_p such that for all $g \in C_1$, there exists $f \in C_2$ such that $T_p \circ f \circ T_i = g$.

Theorem 12.7 *Suppose we are given maps $T_i : X_1 \to X_2$ and $T_p : \{0,1\} \to \{0,1\}$, an original algorithm A that takes instances from X_2, and a derived algorithm B (defined as above). Suppose also that we have a function $g : X_1 \to \{0,1\}$ that we want B to learn. If $f : X_2 \to \{0,1\}$ is a function that A can learn with some bounded number of mistakes such that $T_p \circ f \circ T_i = g$, then B will learn g with at most $M_A(f)$ mistakes.*

Proof: This proof is trivial. Basically, B can only make a mistake when A has. ■

We now go through several example transformations.

Example 12.1 *Arbitrary Disjunctions*

We first consider learning arbitrary disjunctions, using WINNOW1 as A_1. We can determine the correct signs (negated variable or not) for all variables by finding a single negative example. Predict positive until we get a negative example (one mistake at most). Let $(z_1, \ldots, z_n)$ be the negative example. All future instances are sent to WINNOW1 using

$$T_i(x_1, \ldots, x_n) = (x_1 \oplus z_1, \ldots, x_n \oplus z_n).$$

Such a mapping makes sense, since if z_i is off in the negative example, it cannot be a negated variable, and vice versa, so only the negated variables are negated by the mapping. (What is done to irrelevant variables does not matter.) Finally, let T_p be the identity function. It is easily shown that the conditions of Theorem 12.7 are satisfied. So the mistake bound for learning non-monotone disjunctions with this technique is just one more than the corresponding mistake bound for learning monotone disjunctions.

Example 12.2 *Arbitrary Conjunctions*

The example of arbitrary disjunctions can be extended to learning k-literal monotone conjunctions. Let A_2 be the algorithm described in the above example, let $T_i = (1 \Leftrightarrow x_1, \ldots, 1 \Leftrightarrow x_n)$, and let $T_p(r) = 1 \Leftrightarrow r$. Again, using DeMorgan's law it is easily verified that the conditions of Theorem 12.7 hold. Thus, the number of mistakes will be bounded by $2k \log_2 n + 2$.

Example 12.3 *k-DNF for fixed k*

As another example, consider learning k-DNF for a fixed k. We notice that this class is more difficult, since it is not a linearly separable class. Let the original algorithm be WINNOW1 over k-literal disjunctions. Let n_1 be the number of variables in the derived concept class, and let n_2, the number of variables over which the original algorithm is run, be set to $n_2 = \sum_{i=0}^{k} 2^i \binom{n}{i}$. Then we have that

$$T_i(x) = (c_1(x), \ldots, c_{n_2}(x))$$

where $x = (x_1, \ldots, x_{n_1})$ and the c_i's range over all possible conjunctions which could be terms in a k-DNF formula. Notice that given any k-DNF we can represent it as such. If such a g is given we can construct an f in the derived space which is a disjunction of all the variables whose corresponding terms are in the expansion of g.

By expanding the summation for n_2, we can see that $n_2 \leq (2n)^k + 1$. Since the original algorithm will be learning an l-literal monotone disjunction, we will make $O(l \log n^k) =$

$O(lk \log n)$ mistakes. Compare this to the algorithm Valiant presented which can be forced to make $\binom{n}{k} \Leftrightarrow l$ mistakes in the worst case.

By using the VC dimension, we can find a lower bound on the mistake bound. Taking $m = \lceil kl^{1/k} \rceil$, we have

$$\binom{m}{k} \geq \frac{m^k}{k^k} \geq 1$$

Thus a lower bound would be given, when $kl^{1/k} \leq n$, by:

$$kl \left\lfloor \log_2 \frac{n}{\lceil kl^{1/k} \rceil} \right\rfloor$$

If, however, l is known, an even better bound of $4l + 2kl \log_2 \left(\frac{2n}{l^{1/k}} \right)$ can be achieved.

CS 582T Computational Learning Theory **Spring Semester, 1991**

Topic 13: Learning Regular Sets

Lecturer: Sally Goldman *Scribe: Ellen Witte*

13.1 Introduction

In this lecture we consider the problem of learning an unknown regular set using membership and equivalence queries. The material presented in this lecture comes from the paper, "Learning Regular Sets from Queries and Counterexamples," by Dana Angluin [2]. Since deterministic finite-state acceptors (dfa's) recognize the same languages as regular sets, this is equivalent to the problem of learning an unknown dfa. We will express our hypothesis in the form of a dfa by giving the initial state, final (accepting) states and transition function. Our goal will be to find a dfa with the *minimum* number of states that recognizes the unknown regular set.

Gold [15] has shown that finding a minimum dfa consistent with an arbitrary set of positive and negative examples is NP-hard. The learning algorithm has the advantage of being able to select the examples for the membership queries, thus the set of examples used to help construct the dfa is not arbitrary. To be certain we are not asking for too much with membership and equivalence queries, we would like both types of queries to be computable in time polynomial in the number of states in a minimum dfa recognizing the regular set and polynomial in the length of the hypothesis dfa. The teacher has access to the minimum dfa, thus a membership query can be answered by simply tracing the dfa using the given string. In addition, there is a polynomial time algorithm for testing the equivalence of two dfa's which returns a counterexample if the dfa's are not equivalent.

13.2 The Learning Algorithm

In the following discussion we use the notation L^* to denote the learning algorithm, U to denote the unknown regular set to be learned, and A to denote the alphabet of the regular language. (The alphabet is known to the learner.)

The learning algorithm develops a hypothesis by using membership queries to determine whether or not certain strings are in the set U. The results of these queries are stored in an *observation table* maintained by the learner. Periodically the learner will construct a hypothesis dfa from the observation table and perform an equivalence query to see if the hypothesis is correct. If the hypothesis is not correct then the counterexample will be used to modify the observation table. Next we describe the structure of the observation table. Later we will see how the hypothesis dfa is constructed from the table.

T	λ
λ	1
a	0
b	0

Table 13.1: Initial observation table.

The observation table represents a mapping T from a set of finite strings to {0,1}. The function T is such that $T(u) = 1$ if and only if $u \in U$. The strings in the table are formed by concatenating an element from the set $S \cup S \cdot A$ with an element from the set E, where S is a nonempty finite prefix-closed set of strings and E is a nonempty finite suffix-closed set of strings. $S \cdot A = \{s \cdot a : \ s \in S, a \in A\}$, where $\cdot$ is the concatenation operator. The table can be represented by a two-dimensional array with rows labeled by elements of $S \cup S \cdot A$ and columns labeled by elements of E. The entry in row $s \in S \cup S \cdot A$ and column $e \in E$ contains $T(s \cdot e)$. The observation table will be denoted (S, E, T).

In the initial observation table, $S = E = \{\lambda\}$, where λ represents the empty string. This table has one column and $1 + |A|$ rows. For example, if $A = \{a, b\}$ and the regular language $U = \{u : \ u$ contains an even number of both a's and b's$\}$ then Table 13.1 is the initial table. The double horizontal line separates the rows labeled with elements of S from the rows labeled with elements of $S \cdot A$. We will denote the row of the table labeled by $s \in S \cup S \cdot A$ by $\text{row}(s)$. In the example, $\text{row}(a) = (0)$.

We define two properties of an observation table. An observation table is *closed* if for all $t \in S \cdot A$, there exists an $s \in S$ such that $\text{row}(t) = \text{row}(s)$. An observation table is *consistent* if whenever $s_1, s_2 \in S$ satisfy $\text{row}(s_1) = \text{row}(s_2)$, then for all $a \in A$, $\text{row}(s_1 \cdot a) = \text{row}(s_2 \cdot a)$. Table 13.1 is not closed since $a \in S \cdot A$ and $\text{row}(a) = (0)$, but there is no $s \in S$ such that $\text{row}(s) = (0)$. The table is consistent.

We now define the dfa, denoted $M(S, E, T)$, corresponding to the closed, consistent observation table (S, E, T). (The observation table must be closed and consistent otherwise the dfa is undefined.) $M(S, E, T)$ is the acceptor over alphabet A, with state set Q, initial state q_0, accepting state set F and transition function δ where:

$$\begin{aligned}
&Q = \{\text{row}(s) : \ s \in S\} \\
&q_0 = \text{row}(\lambda) \\
&F = \{\text{row}(s) : \ s \in S, T(s) = 1\} \\
&\delta(\text{row}(s), a) = \text{row}(s \cdot a)
\end{aligned}$$

We can show that $M(S, E, T)$ is a well defined acceptor. First, it is always true that $\lambda \in S$ since S is nonempty and prefix-closed. Thus $q_0 = \text{row}(\lambda)$ is well defined. Second, $F = \{\text{row}(s) : s \in S, T(s) = 1\}$ is well defined. For F to be ill defined there must exist $s_1, s_2 \in S$ such that $\text{row}(s_1) = \text{row}(s_2)$ but $T(s_1) \neq T(s_2)$. Since E is nonempty and suffix-closed, $\lambda \in E$. Thus $\text{row}(s_1)$ contains $T(s_1 \cdot \lambda) = T(s_1)$, and $\text{row}(s_2)$ contains $T(s_2 \cdot \lambda) = T(s_2)$. Since $\text{row}(s_1) = \text{row}(s_2)$, it must be true that $T(s_1) = T(s_2)$. Thus, F is well defined. Third, $\delta(\text{row}(s), a) = \text{row}(s \cdot a)$ is well defined. There are two ways for δ to be ill defined. First, there could exist $s_1, s_2 \in S$ such that $\text{row}(s_1) = \text{row}(s_2)$ but $\text{row}(s_1 \cdot a) \neq \text{row}(s_2 \cdot a)$. Second, it could be that $\text{row}(s \cdot a)$ is not in Q. Because the table is consistent, the first condition cannot occur. Because the table is closed, the second condition cannot occur. Thus δ is well defined.

We now give the learning algorithm L^*, which maintains an observation table (S, E, T). The table is modified to reflect responses to membership and equivalence queries. The hypothesis dfa posed by each equivalence query is the dfa $M(S, E, T)$ corresponding to the current observation table.

Algorithm L^*

```
Initialize $S$ and $E$ to $\{\lambda\}$.
Ask membership queries for $\lambda$ and $a$, $\forall a \in A$.
Construct the initial observation table $(S, E, T)$.
Repeat:
    While $(S, E, T)$ is not closed or not consistent:
        If $(S,E,T)$ is not consistent,
            find $s_1, s_2 \in S, a \in A, e \in E$ such that $\text{row}(s_1) = \text{row}(s_2)$
                and $T(s_1 \cdot a \cdot e) \neq T(s_2 \cdot a \cdot e)$,
            add $a \cdot e$ to $E$,
            extend $T$ to $(S \cup S \cdot A) \cdot E$ using membership queries.
        If $(S,E,T)$ is not closed,
            find $s_1 \in S, a \in A$ such that $\text{row}(s_1 \cdot a) \neq \text{row}(s) \forall s \in S$,
            add $s_1 \cdot a$ to $S$,
            extend $T$ to $(S \cup S \cdot A) \cdot E$ using membership queries.
    Perform an equivalence query with $M = M(S, E, T)$.
    If answer is "no" with counterexample $t$,
        add $t$ and its prefixes to $S$,
        extend $T$ to $(S \cup S \cdot A) \cdot E$ using membership queries.
Until answer is "yes" from equivalence query.
```

We note that the runtime of the algorithm depends upon the length of the longest counterexample, since the table must include each counterexample t and all its prefixes. The algorithm will never remove a row or column from the table. The only adjustments to the table are the addition of rows and columns. Also, the algorithm must test the closure and consistency of the current observation table. The closure is easy to check; the consistency is a bit more time consuming, but not too difficult. Before analyzing the correctness of the algorithm we consider an example execution of L^*.

13.3 An Example

T_1	λ
λ	1
a	0
b	0

Table 13.2: Initial observation table.

T_2	λ
λ	1
a	0
b	0
aa	1
ab	0

Table 13.3: Second observation table.

In this section we trace the execution of L^* as it learns the regular set U defined in the previous section. That is, $U = \{u : u$ contains an even number of both a's and b's$\}$. The alphabet $A = \{a, b\}$. The initial table was given in the previous section and is repeated as Table 13.2. To distinguish the versions of the observation table as the algorithm progresses, we label the i^{th} table in the upper left corner with T_i.

We noted earlier that this table is consistent, but not closed since $a \in S \cup A$ but $\text{row}(a) = 0 \neq \text{row}(s)$ for any $s \in S$. Accordingly, L^* moves a to S, extends $S \cup A$ to include extensions of a and fills in the new entries in the first column of the table. Table 13.3 is the second observation table.

This table is closed and consistent, so the algorithm poses an equivalence query with the dfa $M(S, E, T)$. This dfa has two states, $q_0 = \text{row}(\lambda)$ and $q_1 = \text{row}(a)$. The starting state is q_0, the accepting state is q_0 and the transition function is as shown in Table 13.4. To distinguish the various hypothesis dfa's, we label the transition function corresponding to observation table i with δ_i.

This dfa does not recognize U thus the equivalence query returns a counterexample t. Let us assume the counterexample is $t = bb$. It is easy to see that $t \in U$, but t is not accepted by the hypothesis dfa. Accordingly, L^* adds bb and all its prefixes to S and adds the extensions of bb to $S \cdot A$. L^* then fills in the new entries in the first column. Table 13.5 is the resulting observation table.

The third observation table is closed, but not consistent, since $\text{row}(a) = \text{row}(b)$ but

δ_2	a	b
q_0	q_1	q_1
q_1	q_0	q_1

Table 13.4: Dfa for second observation table.

T_3	λ
λ	1
a	0
b	0
bb	1
aa	1
ab	0
ba	0
bba	0
bbb	0

Table 13.5: Third observation table.

T_4	λ	a
λ	1	0
a	0	1
b	0	0
bb	1	0
aa	1	0
ab	0	0
ba	0	0
bba	0	1
bbb	0	0

Table 13.6: Fourth observation table.

δ_4	a	b
q_0	q_1	q_2
q_1	q_0	q_2
q_2	q_2	q_0

Table 13.7: Dfa for fourth observation table.

T_5	λ	a
λ	1	0
a	0	1
b	0	0
bb	1	0
ab	0	0
abb	0	1
aa	1	0
ba	0	0
bba	0	1
bbb	0	0
aba	0	0
$abba$	1	0
$abbb$	0	0

Table 13.8: Fifth observation table.

row(aa) $\neq$ row(ba). The algorithm adds a to E and fills in the values of T for the new column. Table 13.6 is the result.

Table 13.6 is a closed, consistent observation table. The algorithm constructs the dfa $M(S,E,T)$, which has three states, q_0 = row(λ), q_1 = row(a) and q_2 = row(b). The starting state is q_0, the accepting state is q_0 and the transition function is as shown in Table 13.7.

This dfa is not quite correct, thus the equivalence query returns a counterexample t. Let us assume the counterexample is $t = abb$, which is not in U but is accepted by the dfa. The algorithm adds abb and all its prefixes and extensions to the observation table and fills in the values of T. Table 13.8 shows the result.

Table 13.8 is closed, but not consistent since row(b) = row(ab) but row(bb) $\neq$ row(abb). The algorithm adds b to E and fills in the values of T for this new column. The result is Table 13.9.

This table is closed and consistent, so the algorithm constructs $M(S,E,T)$. This dfa has four states, q_0 = row(λ), q_1 = row(a), q_2 = row(b) and q_3 = row(ab). The starting state is q_0, the accepting state is q_0 and the transition function is as shown in Table 13.10.

This dfa accepts exactly the language U, thus the equivalence query returns "yes" and the algorithm halts. This is the minimum size dfa to recognize the language consisting of all strings with an even number of a's and an even number of b's.

T_6	λ	a	b
λ	1	0	0
a	0	1	0
b	0	0	1
bb	1	0	0
ab	0	0	0
abb	0	1	0
aa	1	0	0
ba	0	0	0
bba	0	1	0
bbb	0	0	1
aba	0	0	1
$abba$	1	0	0
$abbb$	0	0	0

Table 13.9: Sixth observation table.

δ_6	a	b
q_0	q_1	q_2
q_1	q_0	q_3
q_2	q_3	q_0
q_3	q_2	q_1

Table 13.10: Dfa for sixth observation table.

13.4 Algorithm Analysis

There are three requirements of the algorithm which we address in this section. First, we show that the algorithm is correct, that is, L^* finds a *minimum* dfa to recognize U. In addressing this issue we assume the algorithm terminates and show that, assuming termination, the algorithm finds a minimum dfa to recognize U. This leads to the second issue, showing that the algorithm terminates. Third, we show that the time complexity of the algorithm is polynomial in the number of states in the minimum dfa and polynomial in the length of the longest counterexample. We consider these requirements in the next three subsections.

13.4.1 Correctness of L^*

Assuming L^* terminates, the claim that L^* produces a dfa that recognizes U is trivial. The condition for termination is that the equivalence query returns "yes", indicating that the dfa recognizes U. The claim that L^* produces a *minimum* dfa to recognize U is more complicated. The key to this claim is the following theorem about the acceptor $M(S, E, T)$ constructed from a closed, consistent observation table (S, E, T).

Theorem 13.1 *If (S,E,T) is a closed, consistent observation table, then the acceptor M(S,E,T) is consistent with the finite function T. Any other acceptor consistent with T but inequivalent to M(S,E,T) must have more states.*

Proof: This theorem is proven by several lemmas.

Lemma 13.1 *Assume that (S,E,T) is a closed, consistent observation table. For the acceptor M(S,E,T) and every $s \in (S \cup S \cdot A)$, $\delta(q_0, s) = \text{row}(s)$.*

Proof: The proof is by induction on the length of s.

The base case is $|s| = 0$ implying that $s = \lambda$. By definition, $q_0 = \text{row}(\lambda)$, thus

$$\delta(q_0, s) = \delta(\text{row}(\lambda), \lambda) = \text{row}(\lambda) = \text{row}(s).$$

For the induction step we assume the lemma holds for $|s| \leq k$ and show it holds for strings of length $k+1$. Let $t \in (S \cup S \cdot A)$ with $|t| = k+1$. Clearly, $t = s \cdot a$ for some string s of length k and some $a \in A$. By the construction of the observation table, $s \in S$. Thus,

$$\begin{aligned} \delta(q_0, t) &= \delta(q_0, s \cdot a), \\ &= \delta(\delta(q_0, s), a), \\ &= \delta(\text{row}(s), a), && \text{by induction hypothesis,} \\ &= \text{row}(s \cdot a), && \text{by definition of } \delta, \\ &= \text{row}(t). \end{aligned}$$

■

Lemma 13.2 *Assume that (S, E, T) is a closed, consistent observation table. The acceptor $M(S, E, T)$ is consistent with the finite function T. That is, for every $s \in (S \cup S \cdot A)$ and $e \in E$, $\delta(q_0, s \cdot e) \in F$ if and only if $T(s \cdot e) = 1$.*

Proof: The proof is by induction on the length of e.

In the base case, $|e| = 0$, thus $e = \lambda$. By the preceding lemma, $\delta(q_0, s \cdot e) = \text{row}(s)$. If $s \in S$ then by the definition of F, $\text{row}(s) \in F$ if and only if $T(s) = 1$. If $s \in S \cup A$ then since the table is closed, $\text{row}(s) = \text{row}(s_1)$ for some $s_1 \in S$. Now $\text{row}(s_1) \in F$ if and only if $T(s_1) = 1$, which is true if and only if $T(s) = 1$, since $\text{row}(s) = \text{row}(s_1)$.

In the induction step we assume the lemma holds for all e with $|e| \leq k$. Let $e \in E$ with $|e| = k + 1$. Since E is suffix closed, $e = a \cdot e_1$ for some $a \in A$ and some $e_1 \in E$ of length k. Let s be any element of $(S \cup S \cdot A)$. Because the observation table is closed, there exists a string $s_1 \in S$ such that $\text{row}(s) = \text{row}(s_1)$. Then,

$$\begin{aligned}
\delta(q_0, s \cdot e) &= \delta(\delta(q_0, s), a \cdot e_1), && \\
&= \delta(\text{row}(s), a \cdot e_1), && \text{by preceding lemma,} \\
&= \delta(\text{row}(s_1), a \cdot e_1), && \text{since row}(s) = \text{row}(s_1), \\
&= \delta(\delta(\text{row}(s_1), a), e_1), && \\
&= \delta(\text{row}(s_1 \cdot a), e_1), && \text{by definition of } \delta, \\
&= \delta(\delta(q_0, s_1 \cdot a), e_1), && \text{by preceding lemma,} \\
&= \delta(q_0, s_1 \cdot a \cdot e_1). &&
\end{aligned}$$

By the induction hypothesis on e_1, $\delta(q_0, s_1 \cdot a \cdot e)$ is in F if and only if $T(s_1 \cdot a \cdot e) = 1$. Since $\text{row}(s) = \text{row}(s_1)$ and $a \cdot e_1 = e$ is in E, $T(s_1 \cdot a \cdot e) = T(s \cdot a \cdot e) = T(s \cdot e)$. Therefore, $\delta(q_0, s \cdot e) \in F$ if and only if $T(s \cdot e) = 1$, as claimed by the lemma.

■

Lemma 13.3 *Assume that (S,E,T) is a closed, consistent observation table. Suppose M(S,E,T) has n states. If $M' = (Q', q_0', F', \delta')$ is any dfa consistent with T that has n or fewer states, then M' is isomorphic to $M(S, E, T)$.*

Proof: The proof consists of exhibiting an isomorphism.

For each $q' \in Q'$, define $\text{row}(q')$ to be the function f from E to {0,1} such that $f(e) = 1$ if and only if $\delta'(q', e) \in F'$.

Since M' is consistent with T, for each $s \in (S \cup S \cdot A)$ and each $e \in E$, $\delta'(q_0', s \cdot e) \in F'$ if and only if $T(s \cdot e) = 1$. Also, since $\delta'(q_0', s \cdot e) = \delta'(\delta'(q_0', s), e)$ we know that $\delta'(\delta'(q_0', s), e) \in F'$ if and only if $T(s \cdot e) = 1$. Therefore, $\text{row}(\delta'(q_0', s)) = \text{row}(s)$ in $M(S, E, T)$. As s ranges over all of S, $\text{row}(\delta'(q_0', s))$ ranges over all elements of Q, implying that M' must have at least n states. Since the statement of the lemma presumed that M' had n or fewer states we can conclude that M' has exactly n states.

Thus, for each $s \in S$ there is a a unique $q' \in Q'$ for which $\text{row}(s) = \text{row}(q')$, namely, $\delta'(q_0', s)$. We can define a one-to-one and onto mapping between the states of $M(S, E, T)$ and the states Q' of M'. Specifically, for each $s \in S$, we define $\phi(\text{row}(s)) = \delta'(q_0', s)$. It remains to be shown that this mapping takes q_0 to q_0', that it preserves the transition function and that it takes F to F'. Each of these can be verified in a straightforward way using the definitions above. The details can be found in the paper by Angluin [2].

■

Lemma 2 and Lemma 3 together prove the two parts of Theorem 1. Namely, Lemma 2 shows that $M(S,E,T)$ is consistent with T and Lemma 3 shows that any dfa consistent with T is either isomorphic to $M(S,E,T)$ or has more states. Thus $M(S,E,T)$ is a smallest dfa consistent with T.

13.4.2 Termination of L^*

In the previous subsection we ignored the question of whether or not L^* will terminate and concentrated on showing that if it terminates, the output will be correct. In this subsection we address the issue of termination.

To see that the algorithm will terminate we need the following lemma. At first glance the lemma appears similar to the last lemma of the previous section. The key difference is that the following lemma does not assume that the observation table is closed and consistent. It is important to consider this more general case because as the algorithm proceeds, there will be times when the observation table is not closed and consistent.

Lemma 13.4 *Let (S,E,T) be an observation table. Let n denote the number of different values of* row(s) *for $s \in S$. (Note that if (S,E,T) is closed and consistent, then this will be the number of states in $M(S,E,T)$.) Any dfa consistent with T must have at least n states.*

Proof:

Let $M = (Q, \delta, q_0, F)$ be a dfa consistent with T. Define $f(s) = \delta(q_0, s)$ for every $s \in S$. In other words, $f(s)$ is the final state of M when run with input s. Suppose s_1 and s_2 are elements of S such that $\text{row}(s_1) \neq \text{row}(s_2)$. Then there exists an $e \in E$ such that $T(s_1 \cdot e) \neq T(s_2 \cdot e)$. Since M is consistent with T, exactly one of $\delta(q_0, s_1 \cdot e)$ and $\delta(q_0, s_2 \cdot e)$ is in F. Thus, $\delta(q_0, s_1 \cdot e)$ and $\delta(q_0, s_2 \cdot e)$ must be distinct states, implying that $f(s_1 \cdot e) \neq f(s_2 \cdot e)$. Since there are n different values of row(s), $f(s)$ must take on at least n distinct values. Thus M has at least n states.

■

Let n be the number of states in a minimum dfa M_U for the unknown language U. To prove termination we show that the number of distinct values of row(s) for $s \in S$ increases monotonically up to n as L^* runs.

First, consider what happens when a string is added to E because the table is not consistent. The number of distinct values of row(s) must increase by at least one. Two previously equal values, row(s_1) and row(s_2), are no longer equal after E is augmented. Any two unequal values will remain unequal.

Second, consider what happens when a string $s_1 \cdot a$ is added to S because the table is not closed. By definition, row$(s_1 \cdot a)$ differs from row(s) for all $s \in S$. Thus the number of distinct values of row(s) is increased by at least one.

From these two situations we can conclude that the total number of operations of either type over the entire run of L^* is at most $n \Leftrightarrow 1$. (There is initially one value of row(s) and there cannot be more than n.) Thus the algorithm will enter the while loop at most $n \Leftrightarrow 1$ times. This means that L^* always eventually finds a closed, consistent observation table (S,E,T) and makes an equivalence query with $M(S,E,T)$.

We must show that the number of equivalence queries is limited (i.e., that the algorithm does not get stuck in the repeat loop). If an equivalence query $M(S, E, T)$ is incorrect with counterexample t, then by Theorem 1, M_U must have at least one more state than $M(S, E, T)$. Furthermore, L^* must eventually make another equivalence query $M(S', E', T')$ which is consistent with T (since T' extends T) and also classifies t the same as M_U (since $t \in S'$ and $\lambda \in E$). This implies that $M(S', E', T')$ is inequivalent to $M(S, E, T)$ and thus has at least one more state that $M(S, E, T)$.

From this we conclude that L^* can make at most $n \Leftrightarrow 1$ incorrect equivalence queries, since the number of states in the successive queries is monotonically increasing from one and cannot exceed $n \Leftrightarrow 1$. Since L^* will eventually make another equivalence query, it will terminate with a correct query.

13.4.3 Runtime complexity of L^*

The runtime of the algorithm depends in part of the length of the longest counterexample. We let m be the length of the longest counterexample and analyze the runtime in terms of m and n, the number of states in a minimum dfa for the unknown language. In addition, we let k denote the cardinality of the alphabet A.

First we determine the space needed by the observation table. Initially, $|S| = |E| = 1$. Each time (S, E, T) is discovered to be not closed, $|S| \to |S| + 1$. Each time (S, E, T) is discovered to be not consistent, $|E| \to |E| + 1$. For each counterexample of length at most m, at most m strings are added to S.

From this and the analysis of the termination of L^* we see that $|E| \leq n$ and for all $e \in E$, $|e| \leq n \Leftrightarrow 1$. Also, $|S| \leq n + m(n \Leftrightarrow 1)$. The first term results because the observation table may be discovered to be not closed at most $n \Leftrightarrow 1$ times. The second term results because there may be at most $n \Leftrightarrow 1$ counterexamples, each of which adds at most m strings to S. The maximum length of any string in S is increased by one each time the table is found not be to closed. Thus for all $s \in S$, $|s| \leq m + n \Leftrightarrow 1$.

Thus, the table size, $|(S \cup S \cdot A) \cdot E|$ is at most

$$(k + 1)(n + m(n \Leftrightarrow 1))n = O(mn^2).$$

The maximum length of any string in $(S \cup S \cdot A) \cdot E$ is at most

$$(m + n \Leftrightarrow 1) + 1 + n \Leftrightarrow 1 = m + 2n \Leftrightarrow 1 = O(m + n).$$

Thus the observation table takes space $O(m^2n^2 + mn^3)$.

Now we determine the time needed for the computation performed by L^*. Checking if the observation table is closed and consistent can be done in time polynomial in the size of the table. This is done at most $n \Leftrightarrow 1$ times. Adding a string to S or E requires at most $O(mn)$ membership queries of strings of length at most $O(m + n)$. The total number of membership queries is $O(mn^2)$. Given a closed and consistent table, $M(S, E, T)$ can be constructed in time polynomial in the size of the table. This must be done at most $n \Leftrightarrow 1$ times. Thus the computation time is polynomial.

Finally, note that if the counterexamples are always of the minimum possible length then $m \leq n$ and all results are polynomial in n. The algorithm which tests for the equivalence of two dfa's can identify a counterexample of minimum length. Also, the bound on the number of membership queries was improved to $O(kn^2 + n \log m)$ by Rivest and Schapire [36].

CS 582T Computational Learning Theory **Spring Semester, 1991**

Topic 14: Results on Learnability and the VC Dimension

Lecturer: Sally Goldman *Scribe: Weilan Wu*

14.1 Introduction

In these notes we more carefully study the relationship between the Vapnik-Chervonenkis Dimension and the sample complexity of PAC learning. The material presented here comes from the paper "Results on Learnability and the Vapnik-Chervonenkis Dimension," by Nathan Linial, Yishay Mansour, and Ronald Rivest [28].

Recall in Topic 7 that we studied a result of Blumer, et al. [12] stating that a concept class C which is non-trivial and well-behaved is PAC learnable if and only if the VC dimension of C is finite. Here we demonstrate that by using *dynamic sampling* versus *static sampling* that one can PAC learn some concept classes with infinite VC dimension. In other words, the Blumer, Ehrenfeucht, Haussler and Warmuth result assumes that the learner asks for a single sample of a given size and must then processes these examples. If instead the learner can divide the learning session into stages where in each stage the learner asks for some examples and then performs some processing, then it is possible to learn any concept class that can be written as the countable union

$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots$$

where each concept class $\mathcal{C}_d$ has VC-dimension at most d.

In these notes we shall say that a concept class C is *PAC learnable* if there exists a learning algorithm that outputs a hypothesis meeting the PAC criterion. We say a concept class C is *polynomially PAC learnable* if it uses time and sample complexity that are polynomial in $\frac{1}{\epsilon}$ and $\frac{1}{\delta}$.

All previous PAC algorithms that we have studied assume a *static sampling* model in which the learning algorithm must draw all examples before any computation is performed. In addition to using a static sampling model, typically PAC learning algorithms are *consistent* in that the concept C they return agrees with the classification of each example of the sample.

Using this notation, we will restate the (Blumer et al. [12]) result as below.

Theorem 14.1 *A concept class C is PAC learnable with static sampling if and only if C has finite VC-dimension. Furthermore the bounds of the sample complexity are:*

$$m(\epsilon, \delta) = O\left(\frac{1}{\epsilon}\ln\frac{1}{\delta} + \frac{d}{\epsilon}\ln\frac{1}{\epsilon}\right)$$

and

$$m(\epsilon, \delta) = \Omega\left(\frac{1}{\epsilon}\ln\frac{1}{\delta} + \frac{d}{\epsilon}\right)$$

Based on these bounds, Blumer et. al. also showed many concept classes to be polynomially PAC learnable. More generally, if C is a class with finite VC dimension and there exists a polynomial time algorithm to find a concept in C that is consistent with a given sample, then C is polynomially PAC learnable.

14.2 Dynamic Sampling

Now we consider the notion of *dynamic sampling*, in which the number of examples examined increases with the complexity of target concept. It turns out that *dynamic sampling* does indeed enrich the class of PAC learnable concepts, compared to static sampling. By this method, one can establish the learnability of various concept class with an *infinite* Vapnik-Chervonenkis dimension.

When using dynamic sampling, the PAC learning algorithm alternates between drawing examples and doing computations. A *stage* of the PAC learning algorithm of consists of drawing a set examples and performing the subsequent computations. Thus a static sampling algorithm is equivalent to a one-stage dynamic sampling algorithm. We note that PAC learning a class with infinite VC dimension may require an unbounded number of stages.

14.3 Learning Enumerable Concept Classes

We first define an enumerable concept class and then prove that any such class is PAC learnable using dynamic sampling. This illustrates the power of dynamic sampling relative to static sampling, since an enumerable class of concepts may have infinite VC dimension.

Let $C = \{C_1, C_2, \ldots, \}$ be a recursively enumerable concept class, such that for each C_i, membership in C_i is decidable. Note that C may have infinite VC-dimension, for example, let N be the set of natural numbers, and C be the set of all finite subsets of N.

Now we give the algorithm for PAC learning C.

Algorithm Enumerable-Learner

1. Let i=1.
2. Draw enough examples so that the total number m_i of examples drawn so far is at least $\frac{1}{\epsilon} \ln \frac{2i^2}{\delta}$.
3. If C_i is consistent with all examples seen so far then output C_i. Otherwise increase i by 1 and return to step 2.

Theorem 14.2 *Enumerable-Learner can PAC learn any enumerable concept class C.*

Proof: Let T be the target concept. Recall the concept C_i is "ϵ -bad" if $P(T \oplus C_i) > \epsilon$. The probability that an ϵ- bad concept C_i is output is at most $(1 \Leftrightarrow \epsilon)^{m_i}$. For $m_i \geq \frac{1}{\epsilon} \ln \frac{2i^2}{\delta}$,

$(1 \Leftrightarrow \epsilon)^{m_i} < \frac{\delta}{i^2}\frac{6}{\pi^2}$ holds.

To show this, just note that

$$m_i \ln(1 \Leftrightarrow \epsilon) < \Leftrightarrow \epsilon m_i$$

Now substitute the lower bounds for m_i, we get

$$m_i \ln(1 \Leftrightarrow \epsilon) < \ln \frac{\delta}{2i^2} < \ln \frac{\delta}{2i^2} + \ln \frac{12}{\pi^2} < \ln(\frac{\delta}{i^2}\frac{6}{\pi^2})$$

therefore

$$(1 \Leftrightarrow \epsilon)^{m_i} < \frac{\delta}{i^2}\frac{6}{\pi^2}$$

Since

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$$

the probability that Enumerable-Learner outputs an ϵ-bad concept is at most

$$\frac{6}{\pi^2} \sum_{i=1}^{\infty} \frac{\delta}{i^2} = \delta.$$

■

14.4 Learning Decomposable Concept Classes

The result just described does not handle the uncountable concept classes. To learn such classes we introduce the notion of a *decomposable concept class*.

Definition 14.1 *A concept class C is decomposable if it can be written as a countable union, $C = C_1 \vee C_2 \vee \ldots$, where each concept class C_d has VC dimension at most d.*

In many cases, this decomposition can be done in such a way that $C_i \subseteq C_{i+1}$ for all i, and those concepts in $C_d \Leftrightarrow C_{d-1}$ can naturally be said to have *size* d. For example, if each concept is represented by a binary string, we might let C_d be the set of concepts whose binary encoding has at most $d \Leftrightarrow 1$ bits. If $X = [0,1]$ and C is finite unions of subintervals of X, then C_d is the set of concepts which are the union of at most $d/2$ subintervals of [0,1]. In other cases, the "natural" size measure might be polynomially related to d_i; the results can be easily extended to these cases.

Before describing the PAC learning algorithm for decomposable classes we need the following definitions.

Definition 14.2 *For target concept $T \in C$, the size of the target concept T is defined as $size(T) = min\{d \mid T \in C_d\}$.*

Now we want the complexity of learning algorithm to be polynomial in $size(T)$, as well as $\frac{1}{\epsilon}$, $\frac{1}{\delta}$. Furthermore, we wish to have the PAC learning algorithm determine $size(T)$ itself.

Definition 14.3 *A concept is uniformly decomposable if it is decomposable and there exists an algorithm A, which given d and a sample can produce a concept $c \in C_d$ consistent with the sample or else output "none" if no such concept exists. If A runs in time polynomial in d and the number of examples we say that C is polynomially uniformly decomposable.*

Theorem 14.3 *Any uniformly decomposable concept class is PAC learnable using dynamic sampling. If C is polynomially uniformly decomposable, then the time and sample complexity are polynomial in the size of target concept. (The sample complexity polynomial in either case.)*

Proof: First of all, we give the algorithm Uniformly-Decomposable-Learner, which can PAC learn any uniformly decomposable concept class.

Algorithm Uniformly-Decomposable-Learner

1. Let d=1.
2. Draw enough examples so that the total number m_d of examples drawn so far is at least $max(\frac{4}{\epsilon}\ln\frac{8d^2}{\delta}, \frac{8d}{\epsilon}\ln\frac{13}{\epsilon})$.
3. If there is a $C \in C_d$ which is consistent with all examples seen so far,then output C. Otherwise increment d by 1 and return to step 2.

Now we prove the above algorithm will PAC learn all the uniformly decomposable concept classes.

The number of examples at each stage, m_d, is chosen by the requirement from Blumer, et al. [12] with the probability that an ϵ-bad concept is output is at most $\frac{\delta}{4d^2}$. Summing up $\frac{\delta}{4d^2}$ over all d, we get that the probability of an ϵ-bad concept to be output is at most δ.

Now we show that the algorithm will halt. Since at each stage d is incremented by 1, when d reaches $size(T)$ (after $size(T)$ steps), there is a concept in C_d (namely T), which is consistent with all the examples. At this point, algorithm will halt. Further more, the number of examples seen by the algorithm is polynomial in $size(T)$. For the case that the concept classes that polynomially uniformly decomposable, the running of the algorithm is polynomial in $size(T)$ as well. ■

As an illustration of the power of these techniques, the following classes are PAC learnable, even though they are uncountable and have infinite VC dimension.

Theorem 14.4 *The concept class C_{FI}, defined as set of finite union of subintervals of [0,1] is PAC learnable.*

Proof: Decompose the concept class such that each C_i includes concepts with at most $\frac{i}{2}$ subintervals (only defined for i even). To show C_{FI} is polynomially uniformly decomposable, we need to exhibit an algorithm, A_{FI} that given a sample and index d, finds a concept in C_d consistent with the samples (if one exists) or replies "none" (if such a concept does not exist).

The algorithm is based upon the observation that number of alternations (switches from "+" to "⇔" or "⇔" to "+") in the sample is at most twice number of subintervals in target. (Technically to make this work, we have to assume that there is a "⇔" at the points 0 and 1. Another alternative is to look at the number of "blocks" of positive examples, where any two blocks are separated by at least 1 negative example.)

So if number of alternatives is greater than d, A_{FI} outputs "none", else it outputs a concept with minimal number of subintervals that is consistent with the sample

Clearly concept with minimum number of subintervals that is consistent with an example is easy to find. Therefore we know that C_{FI} is polynomially PAC learnable.

We also note that the concept class C_{PR}, the set of regions in $\Re^2$, defined by an inequality of the form $y \leq f(x)$, where f is any polynomial of finite degree with real coefficients, is PAC learnable.

14.5 Reducing the Number of Stages

In this section we briefly explore ways to reduce the number of stages in a PAC algorithm which uses dynamic sampling. Observe that in Uniformly-Decomposable-Learner the number of stages may be as large as $n = size(T)$. We now show that this can be improved for the concept class C_{FI}.

Theorem 14.5 *To PAC learn the concept class C_{FI} the number of stages required is $O(\lg \lg n)$.*

Proof: In each stage update the value of d to d^2 rather than $d + 1$. (Note that the decomposition of C_{FI} is such that if $i < j$ then $C_i \subset C_j$.) ■

Furthermore, it can be shown that this bound for C_{FI} is tight.

Theorem 14.6 *Any algorithm that PAC learns C_{FI}, with respect to the uniform distribution using a number of examples that is bounded by a polynomial in the number n of subintervals of the target concept requires at least $\Omega(\log \log n)$ stages.*

See the Linial, Mansour, Rivest paper for the proof of this result. Finally, we show that not every concept class of infinite VC dimension requires an unbounded number of stages.

Theorem 14.7 *Let C_N denote the concept class of all subsets of the natural numbers. Then C_N can be PAC learned with a two-stage learning algorithm.*

Proof: In the first stage we draw a sample of size $(2/\epsilon)\lg(2/\delta)$. Let M denote the largest integer appearing in this sample. With probability at least $1 \Leftrightarrow \delta/2$ the probability associated with integers greater than M is at most $\epsilon/2$. In the second stage we consider the induced problem of learning the restriction of the target concept to the natural numbers at most M. This reduces the problem to one having a finite VC-dimension (i.e., M), which can be solved with a static sampling algorithm with parameters $\epsilon/2$ and $\delta/2$. ■

A simple generalization of this argument applies in a straightforward manner whenever the instance space is countable.

CS 582T Computational Learning Theory **Spring Semester, 1991**

Topic 15: Learning in the Presence of Malicious Noise

Lecturer: Sally Goldman *Scribe: Nilesh Jain*

15.1 Introduction

Earlier in the course, we saw the different types of noises (errors) that can affect the training instances. The worst case is when these errors are generated by an adversary with unbounded computational resources, and knowledge of the function being learned, the probability distribution on the examples, and the internal state of the learning algorithm. The goal of the adversary is to foil the learning algorithm. This type of noise is known as *malicious noise*. With a fixed probability β $(0 \leq \beta < 1)$, the adversary gets to pick the instance and label.

After defining the required terms and notions, we shall obtain the upper and lower bounds on the noise rate that can be tolerated by any learning algorithm for a specified concept class $\mathcal{C}$. Then, we will show the relationship between learning with errors and traditional complexity theory. This will be done by giving approximation-preserving reductions from standard optimization problems to natural learning problems, and vice-versa. The results obtained here also apply to learning under random misclassification noise covered earlier. The material presented in the lecture comes from the paper "Learning in the Presence of Malicious Errors," by Michael Kearns and Ming Li [25]. Portions of the notes are taken from this paper.

15.2 Definitions and Notation

We shall be considering the 2-button model. For a fixed target concept $c \in \mathcal{C}$, target distributions D^+ and D^-, and fixed $0 \leq \beta < 1$, we define two *oracles with malicious errors* as follows: in oracle POS_M^β, with probability $1 \Leftrightarrow \beta$, a point $x \in \mathrm{pos}(c)$ is drawn according to D^+ and returned, and with probability β, an arbitrary point is returned and in oracle NEG_M^β, with probability $1 \Leftrightarrow \beta$, a point $x \in \mathrm{neg}(c)$ is drawn according to D^- and returned, and with probability β, an arbitrary point is returned.

$E_M(\mathcal{C})$, the *optimal malicious error rate* for a class of functions $\mathcal{C}$, is the largest value of β that can be tolerated by any learning algorithm (not necessarily polynomial time) for $\mathcal{C}$. We will obtain upper bounds on this. $E_M^p(\mathcal{C})$ is the largest error rate tolerated by a polynomial time learning algorithm for $\mathcal{C}$. We will obtain lower bounds on this by giving efficient learning algorithms.

A concept class $\mathcal{C}$ over X is *distinct* if there exist concepts $c_1, c_2 \in \mathcal{C}$ and points $u, v, w, x \in X$ satisfying $u \in c_1, u \notin c_2, v \in c_1, v \in c_2, w \notin c_1, w \in c_2$, and $x \notin c_1, x \notin c_2$.

15.3 Upper Bounds on $E_M(\mathcal{C})$

In this section, we will give a theorem bounding $E_M(\mathcal{C})$.

Theorem 15.1 *For a distinct concept class $\mathcal{C}$, $E_M(\mathcal{C}) < \frac{\epsilon}{1+\epsilon}$.*

Proof: We will prove the theorem by using a technique called as the *method of induced distributions*: two or more concepts $\{c_i\} \subseteq \mathcal{C}$ are chosen, such that if c_i is the target concept, then for all $i \neq j$, c_j is ϵ-bad. Then adversaries are given for generating errors such that regardless of which c_i is the target concept, the behavior of the oracle POS_M^β is identical, and similarly for the oracle NEG_M^β, thus making it impossible for the learning algorithm to distinguish the true target concept.

Coming to our theorem, we know that $\mathcal{C}$ is distinct. Let $c_1, c_2 \in \mathcal{C}$ and $u, v, w, x \in X$ as per the definition of distinct concept classes. Now consider the following table which gives the actual distribution D_i, the adversary's choice in the case of noise and the induced distribution I_i for target concept c_i.

Actual distribution	Adversary's choice if noise	Induced distribution
$D_1^+(u) = \epsilon$		$I_1^+(u) = (1 \Leftrightarrow \beta)\epsilon$
$D_1^+(v) = 1 \Leftrightarrow \epsilon$	w	$I_1^+(v) = (1 \Leftrightarrow \beta)(1 \Leftrightarrow \epsilon)$
		$I_1^+(w) = \beta$
$D_1^-(w) = \epsilon$		$I_1^-(u) = \beta$
$D_1^-(x) = 1 \Leftrightarrow \epsilon$	u	$I_1^-(w) = (1 \Leftrightarrow \beta)\epsilon$
		$I_1^-(x) = (1 \Leftrightarrow \beta)(1 \Leftrightarrow \epsilon)$
$D_2^+(w) = \epsilon$		$I_2^+(u) = \beta$
$D_2^+(v) = 1 \Leftrightarrow \epsilon$	u	$I_2^+(v) = (1 \Leftrightarrow \beta)(1 \Leftrightarrow \epsilon)$
		$I_2^+(w) = (1 \Leftrightarrow \beta)\epsilon$
$D_2^-(u) = \epsilon$		$I_2^-(u) = (1 \Leftrightarrow \beta)\epsilon$
$D_2^-(x) = 1 \Leftrightarrow \epsilon$	w	$I_2^-(w) = \beta$
		$I_2^-(x) = (1 \Leftrightarrow \beta)(1 \Leftrightarrow \epsilon)$

From the table, we see that if $\beta = (1 \Leftrightarrow \beta)\epsilon$, then the induced distributions are identical. Also, $\beta = (1 \Leftrightarrow \beta)\epsilon \Rightarrow \beta = \frac{\epsilon}{1+\epsilon}$. Note that if $\beta > \frac{\epsilon}{1+\epsilon}$, then the adversary can choose to draw from the actual distribution some of the time, so that the effective error rate is $\frac{\epsilon}{1+\epsilon}$.

Thus we see that the learning algorithm can only handle $\beta < \frac{\epsilon}{1+\epsilon}$. Note that this result holds regardless of the time or sample complexity of the learning algorithms for $\mathcal{C}$. ∎

Since the bound for Theorem 15.1 holds even for algorithms with unbounded computational resources, the best we can hope for distinct $\mathcal{C}$ is $E_M(\mathcal{C}) = \Theta(\epsilon)$. Monomials, k-DNF and symmetric functions are all distinct concept classes.

We will now consider the hardness results for algorithms that learn from only positive or only negative examples. First, we need a definition.

A concept class $\mathcal{C}$ is *positive t-splittable* if there exist concepts $c_1, \ldots, c_t \in \mathcal{C}$ and points $u_1, \ldots, u_t, v \in X$ such that c_j includes v and all the u_i's except u_j. In other words, $\forall i, j\ u_i \in c_j$ if $i \neq j$ and $\forall i\ v \in c_i$.

As an example, monomials on n variables are positive n-splittable. Let $c_1 = x_1, \ldots, c_n = x_n$ and let $u_1 = 011\ldots 11, u_2 = 101\ldots 11, \ldots, u_n = 111\ldots 10, v = 111\ldots 11$. Also, if $\mathcal{C}$ is a concept class with $\text{VCD}(\mathcal{C}) = d$, then $\mathcal{C}$ is both positive and negative d-splittable.

Theorem 15.2 *For positive t-splittable $\mathcal{C}$, any algorithm calling on POS_M^β can tolerate an error rate $\beta \leq \frac{\epsilon}{t-1}$ regardless of time or sample complexity.*

Proof: The theorem is proved using the method of induced distributions. Since $\mathcal{C}$ is positive t-splittable, let $c_1, \ldots, c_t \in \mathcal{C}$ and $u_1, \ldots, u_t, v \in X$ be as in the definition of positive t-splittable. Let c_j be the target concept. Consider the following actual distribution:

$$\begin{array}{rcll} D_j^+(v) & = & 1 \Leftrightarrow \epsilon & \\ D_j^+(u_i) & = & \frac{\epsilon}{t-1} & \forall i \neq j \\ D_j^-(u_j) & = & 1 & \end{array}$$

In case of noise, the adversary chooses u_j. Thus, the induced distribution is

$$\begin{array}{rcll} I_j^+(v) & = & (1 \Leftrightarrow \beta)(1 \Leftrightarrow \epsilon) & \\ I_j^+(u_i) & = & (1 \Leftrightarrow \beta)\left(\frac{\epsilon}{t-1}\right) & \forall i \neq j \\ I_j^+(u_j) & = & \beta & \end{array}$$

If $\beta = (1 \Leftrightarrow \beta)\left(\frac{\epsilon}{t-1}\right)$, then the induced distributions I_j^+ are identical for all u_i. Solving for β, we get

$$\begin{array}{rcl} \beta & = & \dfrac{\frac{\epsilon}{t-1}}{1 + \frac{\epsilon}{t-1}} \\ & = & \dfrac{\epsilon}{t \Leftrightarrow 1 + \epsilon} \\ & \leq & \dfrac{\epsilon}{t \Leftrightarrow 1} \end{array}$$

Thus we see that the learning algorithm can only tolerate an error rate of $\beta < \frac{\epsilon}{t-1}$. ■

The above theorem leads to the following corollaries.

Corollary 15.1 *A learning algorithm for monomials (using only positive or only negative examples) can tolerate an error rate of $\beta < \frac{\epsilon}{n-1}$ where n is the number of variables in the monomial.*

Corollary 15.2 *A learning algorithm for k-DNF (using only positive or only negative examples) can tolerate an error rate of $\beta < \frac{c_0 \epsilon}{n^k}$ where c_0 is a constant > 0.*

Corollary 15.3 *A learning algorithm for symmetric functions (using only positive or only negative examples) can tolerate an error rate of $\beta < \frac{\epsilon}{n-1}$ where n is the number of variables in the symmetric functions.*

The above corollaries agree with some earlier work by Valiant [44]. Valiant proved the following two theorems.

Theorem 15.3 *Monomials are learnable using calls to POS_M^β for $\beta \le \frac{c_0 \epsilon}{n}$ for constant c_0.*

The learning algorithm for the above is slightly different from the algorithm for learning monomials in a noise-free environment. In this case, variable x_i is deleted from the current hypothesis only if $x_i = 0$ in at least $\frac{2c_0\epsilon}{n}$ of the examples seen so far.

Theorem 15.4 *k-DNF is learnable using calls to NEG_M^β for $\beta \le \frac{c_0\epsilon}{n^k}$ for constant c_0.*

15.4 Generalization of Occam's Razor

In the previous section, we saw the limitations on the error rate that a learning algorithm can tolerate, even if it had unbounded computational resources. In this section, we will concentrate on the largest error rate that a polynomial time learning algorithm can tolerate. We will give a technique for building robust algorithms which is a generalization of Occam's Razor.

Let A be an algorithm accessing POS_M^β and NEG_M^β, and taking inputs $\epsilon > 0, \delta < 1$. Suppose that for target concept $c \in \mathcal{C}$ and $0 \le \beta < \frac{\epsilon}{4}$, A makes m calls to POS_M^β and receives examples $u_1, \ldots, u_m$, and m calls to NEG_M^β and receives examples $v_1, \ldots, v_m$, and outputs $h_A \in H$ satisfying with probability at least $1 \Leftrightarrow \delta$:

$$(i)\ \frac{\{u_i : u_i \notin h_A\}}{m} \le \frac{\epsilon}{2}$$

$$(ii)\ \frac{\{v_i : v_i \in h_A\}}{m} \le \frac{\epsilon}{2}$$

Thus, with high probability, h_A agrees with a fraction of at least $1 \Leftrightarrow \frac{\epsilon}{2}$ of the sample. Such an algorithm A will be called a *β-robust Occam Algorithm for $\mathcal{C}$.*

Theorem 15.5 *Let $\beta < \frac{\epsilon}{4}$, and let A be a β-robust Occam Algorithm for $\mathcal{C}$ outputting $h_A \in H$. Then A is a PAC-learning algorithm for $\mathcal{C}$ for $m = O\left(\frac{1}{\epsilon}\ln\frac{1}{\delta} + \frac{1}{\epsilon}\ln|H|\right)$.*

Proof: We will show that if condition (i) holds for A, then $e^+(h_A) < \epsilon$, and similarly, of condition (ii) holds, then $e^-(h_A) < \epsilon$.

Suppose $h \in H$ is ϵ-bad, i.e. $e^+(h) \ge \epsilon$. Then, the probability that h agrees with an example x from POS_M^β is at most

$$(1 \Leftrightarrow \beta)(1 \Leftrightarrow \epsilon) + \beta \le 1 \Leftrightarrow \frac{3\epsilon}{4}$$

for $\beta < \frac{\epsilon}{4}$. Thus, the probability that h agrees with at least a fraction $1 \Leftrightarrow \frac{\epsilon}{2}$ of the m positive examples is

$$LE\left(\frac{3\epsilon}{4}, m, \frac{\epsilon}{2}m\right) \le e^{-m\epsilon/24}$$

using Chernoff bounds. Thus, the probability that some ϵ-bad h agrees with a fraction $1 \Leftrightarrow \frac{\epsilon}{2}$ of the m positive examples is at most

$$|H|e^{-m\epsilon/24}.$$

We want this to be at most $\frac{\delta}{2}$. Thus, by solving

$$|H|e^{-m\epsilon/24} \leq \frac{\delta}{2}$$

for m, we get

$$m \geq \frac{24}{\epsilon}\left(\ln|H| + \ln\frac{2}{\delta}\right)$$

On the other hand, if c is the target concept, then the probability that c does not agree with a fraction $1 \Leftrightarrow \frac{\epsilon}{2}$ of the m positive examples is

$$GE\left(\frac{\epsilon}{4}, m, \frac{\epsilon}{2}m\right) \leq \frac{\delta}{2}$$

for $\beta \leq \frac{\epsilon}{4}$ and m from above. Observe that $m = O\left(\frac{1}{\epsilon}\ln\frac{1}{\delta} + \frac{1}{\epsilon}\ln|H|\right)$. Also, for such an m, any hypothesis agreeing with a fraction $1 \Leftrightarrow \frac{\epsilon}{2}$ of a sample of size m from POS_M^β must have $e^+(h) < \epsilon$ with high probability. ■

The above theorem can be used to prove the correctness of the following learning algorithms by Valiant [44].

Theorem 15.6 *There is a polynomial time learning algorithm A for monomials which can learn from positive examples and tolerate an error rate of $\Theta\left(\frac{\epsilon}{n}\right)$ where n is the number of variables in the monomial.*

Theorem 15.7 *There is a polynomial time learning algorithm A for k-DNF which can learn from negative examples and tolerate an error rate of $\Theta\left(\frac{\epsilon}{n^k}\right)$ where n is the number of variables in the k-DNF formula.*

15.5 Using Positive and Negative Examples to Improve Learning Algorithms

In this section, we show that by using both positive and negative examples, we can improve the noise rate that can be handled by the learning algorithm.

Theorem 15.8 *Let $\mathcal{C}$ be a polynomial time learnable concept class in the error-free model by algorithm A with sample complexity $s_A(\epsilon, \delta, n)$ and let $s = s_A(\frac{\epsilon}{8}, \frac{1}{2}, n)$. Then, we can learn $\mathcal{C}$ in polynomial time with an error rate of $\beta = \Omega\left(\min\left(\frac{\epsilon}{8}, \frac{\ln s}{s}\right)\right)$.*

Proof: Assume $\beta < \frac{\ln s}{s}$. Run algorithm A using oracles POS_M^β and NEG_M^β with accuracy set to $\frac{\epsilon}{8}$ and confidence set to $\frac{1}{2}$. The probability that no errors occur during this run of A is

$$\begin{aligned}(1 \Leftrightarrow \beta)^s &\geq \left(1 \Leftrightarrow \frac{\ln s}{s}\right)^s \\ &= \left(\left(1 \Leftrightarrow \frac{\ln s}{s}\right)^{\frac{s}{\ln s}}\right)^{\ln s}\end{aligned}$$

Recall that

$$e^x = O(1 + x)$$

Putting $x = \Leftrightarrow \frac{\ln s}{s}$, we get

$$e^{-\frac{\ln s}{s}} = O\left(1 \Leftrightarrow \frac{\ln s}{s}\right)$$

which implies

$$e^{-1} = \left(e^{-\frac{\ln s}{s}}\right)^{\frac{s}{\ln s}} = O\left(\left(1 \Leftrightarrow \frac{\ln s}{s}\right)^{\frac{s}{\ln s}}\right)$$

Thus, we get

$$\begin{aligned}(1 \Leftrightarrow \beta)^s &\geq O\left(e^{-\ln s}\right) \\ &= O\left(\frac{1}{s}\right)\end{aligned}$$

Let a successful run be one that has no errors. Using Chernoff bounds, it can be shown that for $r = O\left(s \ln \frac{1}{\delta}\right)$, the probability of no successful runs in r tries is at most $\frac{\delta}{2}$.

Thus, by running algorithm A r times, we get hypotheses $h_1, \ldots, h_r$, one of which is $\frac{\epsilon}{8}$-good. By doing our standard hypothesis testing, we can find this hypothesis. Note that if a hypothesis is $\frac{\epsilon}{8}$-good, then regardless of what the adversary does, we get an error rate of at most $(1 \Leftrightarrow \beta)\frac{\epsilon}{8} + \beta \leq \frac{\epsilon}{8}$ for the range of β we are assuming. Thus, we can use the standard hypothesis testing. ∎

Earlier, we saw that polynomial time algorithms for learning monomials using only positive examples can tolerate an error rate of $\Theta\left(\frac{\epsilon}{n}\right)$. Applying the above theorem improves this bound by a log factor as can be seen in the following corollary.

Corollary 15.4 *There is a polynomial time learning algorithm A for monomials which can learn from both positive and negative examples and tolerate an error rate of $\Omega\left(\frac{\epsilon}{n} \ln \frac{n}{\epsilon}\right)$ where n is the number of variables in the monomial.*

Also, in an earlier topic, we saw that any learning algorithm A for a concept class $\mathcal{C}$ must have sample complexity

$$s_A(\epsilon, \delta) = \Omega\left(\frac{1}{\epsilon} \ln \frac{1}{\delta} + \frac{d}{\epsilon}\right)$$

where $d = \text{VCD}(\mathcal{C})$. Applying Theorem 15.8, we get an algorithm for $\mathcal{C}$ that can tolerate a malicious error rate of $c_0 \left(\ln \frac{d}{\epsilon}\right) \frac{\epsilon}{d}$ for a constant c_0. Note that Corollary 15.4 follows from this more general result since the VC dimension of monomials is $\Theta(n)$.

15.6 Relationship between Learning Monomials with Errors and Set Cover

In this section, we will generalize the Set Cover problem to Partial Cover and show its relationship to learning monomials with errors. We first define the partial cover problem.

Problem: PC (Partial Cover)
Instance: Finite sets $S_1, \ldots, S_n$, (without loss of generality, assume $\cup_{i=1}^{n} S_i = \{1, \ldots, m\}$), positive real costs $c_1, \ldots, c_n$, and a positive fraction $0 < p \leq 1$.
Output: $J' \subseteq \{1, \ldots, n\}$ such that $|\cup_{j \in J'} S_j| \geq pm$ and $\text{PCcost}(J') = \sum_{j \in J'} c_j$ is minimized.

The Partial Cover Problem is clearly NP-complete, since it contains set cover as a special case ($p = 1$). We now give the greedy approximation algorithm for Partial Cover.

Let $J = \{1, \ldots, n\}$ and $T = \{1, \ldots, m\}$.

Algorithm G for the Partial Cover Problem:

0. Set $J^* = 0, m = |\cup_{j \in J'} S_j|$.
1. Let $q = pm - |\cup_{j \in J^*} S_j|$. $\forall j \notin J^*$, if $|S_j| > q$, delete any $|S_j| - q$ elements from S_j.
2. If $|\cup_{j \in J'} S_j| \geq pm$ then stop: J^* is a partial cover. Otherwise find a k maximizing the ratio $\frac{|S_k|}{c_k}$.
3. Add k to J^*, and replace each S_j by $S_j - S_k$. Return to Step 1.

It can be shown that G gives a cover which is within $\Theta(\log m)$ of the optimal. The following theorem demonstrates that an approximation algorithm for a covering problem can be used to obtain a robust algorithm for monomials. See the Kearns and Li paper for both of these proofs.

Theorem 15.9 *There is a polynomial time learning algorithm A for monomials of length at most l which can tolerate an error rate of $\frac{c_0 \epsilon}{l} \left(\frac{1}{\log \frac{l \ln n}{\epsilon}} \right)$ where n is the number of variables in the monomial and c_0 is a positive constant.*

Now we shall show that by approaching the best known upper bound of ϵ on $E_M(M_n)$, where M_n is the class of monomials over n variables, we are actually improving the best known approximation algorithms for set cover.

Theorem 15.10 *Suppose A is a polynomial time learning algorithm for learning monomials using monomials, and can tolerate a malicious error rate of $\beta = \frac{\epsilon}{r(n)}$, then there is a polynomial time algorithm A' for set cover that finds a weighted cover with cost at most $4r(n)$ times the optimal cost, where n is the number of sets.*

Proof: We will give a reduction showing how algorithm A can be used as a subroutine to obtain the desired set cover approximation algorithm by associating the weighted set covers with monomials. Let $J_{\text{opt}} \in \{1, \ldots, n\}$ be an optimal cover of $T = \{1, \ldots, m\}$, and let $\text{PCcost}(J_{\text{opt}}) = C_{\text{PCopt}}$. Each cover $\{i_1, \ldots, i_l\}$ is associated with monomial $x_{i_1} \wedge \ldots \wedge x_{i_l}$. Let M_{opt} be the monomial corresponding to the optimal cover J_{opt}. The basic idea is for A' to run A with M_{opt} as the target concept. We then force the output of algorithm A to be a monomial with error close to M_{opt}. This monomial will correspond to a cover that is close to optimal.

We will use NEG_M^{β} to force the output of A to correspond to a cover. To achieve this goal we put a uniform distribution over the following set of m negative examples. For each $i \in T$, define $\vec{e_i}$ to be an n-bit vector whose j^{th} bit is 0 if and only if $i \in S_j$. For example, let $m = 5$ and $S_1 = \{1, 3, 5\}$, $S_2 = \{1, 2\}$, $S_3 = \{1, 4, 5\}$, and $S_4 = \{1, 3\}$. Then $\vec{e_1} = 0000$, $\vec{e_2} = 1011$, $\vec{e_3} = 0110$, $\vec{e_4} = 1101$, and $\vec{e_5} = 0101$. Let $E = \cup_{i \in T}\, \vec{e_i}$. Observe that $J' = \{i_1, \ldots, i_l\}$ is a cover of T if and only if every vector in E is a negative example of the corresponding monomial. Thus since J_{opt} is a cover, M_{opt} is consistent with all the negative examples. Finally, we let $\epsilon < \frac{1}{m}$. Note that $\frac{1}{\epsilon}$ is polynomial in the size of the set cover instance. Since we force A to output a monomial consistent with the sample, A' outputs a cover. Note that at this point we have used the assumption that A learns monomials by monomials because we do not have a natural mapping between non-monomials and possible covers.

Next we use POS_M^{β} to force the cost of the cover to be low. First, without loss of generality, we assume the costs are scaled so that $\sum_{i=1}^{n} c_i \leq \frac{\epsilon}{r(n)} < 1$. Next we define the induced distribution I^+ on the positive examples. We will then show that this distribution can be obtained. For each $1 \leq i \leq n$, let $\vec{p_i}$ be the n-bit vector that is 1 everywhere except in the i^{th} position. Now I^+ places a probability of c_i on all $\vec{p_i}$, and probability $1 \Leftrightarrow \sum_{i=1}^{n} c_i$ on $\vec{1}^n$. The probability that a monomial $x_{i_1} \wedge \ldots \wedge x_{i_l}$ disagrees with a positive example is $\sum_{k=1}^{l} c_{i_k} = \text{COST}(J)$ where $J = \{i_1, \ldots, i_l\}$ is the corresponding cover.

Thus, the smallest possible rate of disagreements with I^+ is C_{PCopt} and is achieved by M_{opt}. Since $C_{\text{PCopt}} \leq \sum_{i=1}^{n} c_i \leq \frac{\epsilon}{r(n)} = \beta$, this distribution can be induced by a malicious oracle.

A good hypothesis output by algorithm A must have a rate of disagreement of at most 2ϵ with M_{opt} on I^+. One factor of ϵ comes from the disagreement with D^+, and the other factor comes from the error $\beta < \epsilon$ induced by the malicious oracle. Finally, we do not know the actual value of C_{PCopt}. So, we repeatedly divide ϵ by 2 and run A with the new value of ϵ, but without changing I^+. This forces A to output better and better monomials, until it reaches a point where $C_{\text{PCopt}} < \frac{\epsilon}{r(n)} \leq 2C_{\text{PCopt}}$. At that point, the hypothesis output will have error $< 2\epsilon \leq 4r(n)\epsilon$.

The only problem with this occurs when C_{PCopt} is extremely small causing exponential number of iterations of A. This can be avoided by first using the greedy approximation algorithm G and discarding all the sets whose cost is higher than the greedy cover. This ensures that in the new (smaller) instance, all the costs are polynomially related (within a multiplicative $\log m$ factor), and thus we can maintain the polynomial running time. ■

If in the above theorem, $r(n) = o(\log n)$, then A would imply a significant improvement in the existing approximation for set cover. As for our learning problem, the only other alternative (besides improving on the best known set covering result) is to use a hypothesis space larger than just monomials.

CS 582T Computational Learning Theory **Spring Semester, 1991**

Topic 16: Inferring Graphs from Walks

Lecturer: Sally Goldman *Scribe: Andy Fingerhut*

16.1 Introduction

In this lecture we consider a problem that is different from the other topics we have seen—we study the problem of inferring an edge-colored graph from observing the sequence of edge colors output from a walk of the graph. The material presented here comes from the paper, "Inferring Graphs From Walks," by Javed Aslam and Ronald Rivest [9].

Consider an undirected, edge-colored multigraph $G = (V, E, c)$ with vertex set V, edge set E, and edge coloring $c : E \rightarrow \Sigma$. A *walk* on G is an alternating sequence of vertices and edges $v_1 e_1 v_2 e_2 \ldots v_n e_n v_{n+1}$ which starts with a vertex, ends with a vertex, and $(\forall i)(e_i$ is adjacent to v_i and $v_{i+1})$. The *output* of a walk is the sequence of colors seen during the walk. For the above walk, this sequence is $c(e_1)c(e_2)\ldots c(e_n)$. We say that a sequence of colors y is *consistent* with graph G if y is the output of some walk on G. We say a graph G has *degree bound* D if every vertex in G has degree at most D.

For example, given the graph in Figure 16.1 (b), a walk is `AaBbCcAcCbBbCcAdCeBfAfBeC`, where the edges are here (uniquely) represented by their colors. The output of this walk is `abccbbcdeffe`.

The key question we shall consider here is the following:

Problem: Minimum Consistent Inference (MCI)
Instance: Finite set Σ of colors. A string $s \in \Sigma^+$ (where Σ^+ is the set of all finite, non-empty strings over alphabet Σ), and degree bound $D \geq 2$.
Objective: Find a *smallest* (i.e. fewest number of vertices) undirected, edge-colored multigraph $G = (V, E, c)$ such that s is consistent with G and $\Delta(G) \leq D$. (The maximum degree of G, $\Delta(G)$, is the largest degree of any vertex in G).

Observe that the set of possible solutions (i.e. all graphs that are consistent with s) to an instance to MCI can be very large. For example, given the walk output `abccbbcdeffe`, and $D = 6$, any of the graphs in Figure 16.1 are candidate solutions. Graph (a) is a minimum consistent degree-bound 2 graph, graph (b) is a minimum consistent degree-bound 4 graph, and graph (d) is a minimum consistent degree-bound 6 (or higher) graph.

Note that if the number of distinct symbols in s is at most D, then the optimal solution is always similar to that in Figure 16.1 (d), with exactly 2 vertices and all edges appearing between them. Furthermore, this is also the optimal solution if there is no degree bound—it is this degree bound that makes this problem interesting.

We shall describe a polynomial time algorithm to solve MCI for the special case in which $D = 2$. The complexity for the problem when $D > 2$ remains unresolved. That is, for this

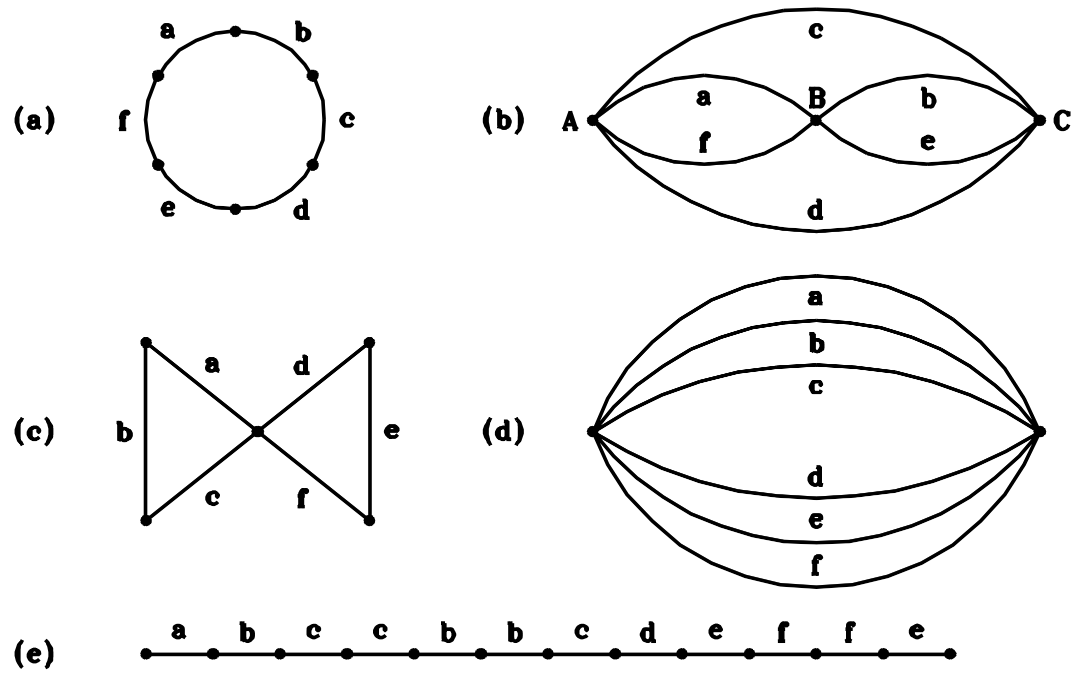


Figure 16.1: Graphs consistent with the walk `abccbbcdeffe`.

general case there is no known polynomial time algorithm, yet it also has not been shown to be NP-hard.

16.2 Intuition

When D is fixed at 2, the only possible solutions to MCI are cycles and linear chains (cycles with a single edge removed). Disconnected graphs which are possible solutions can always be reduced to a smaller connected graph by removing every component except the one containing the desired walk.

A linear chain can be identified with a sequence of edge colors seen in an end-to-end traversal of the chain (in either direction). A cycle can be identified with any of the color sequences obtained by traversing the cycle once around, starting and ending at the same vertex. For the remainder of this topic, strings of symbols in Σ^+ will be used to denote both walk outputs and degree bound 2 graphs.

Given any sequence y, with length denoted by $|y|$, it is clear that there is a candidate solution which is a cycle with $|y|$ vertices and edges (the cycle represented by y). Therefore the optimum solution has at most $|y|$ vertices. We may also get a lower bound on the value of any candidate solution. Any such solution must have at least one edge of each color that appears in y. Unfortunately, exhaustively trying all cycles satisfying the above two conditions would take exponential time. To solve this problem in polynomial time, a more clever approach is necessary.

Although the smallest degree 2 graph consistent with a walk is often a cycle (and certainly can always be made into a cycle), it is useful to first consider the problem of finding the smallest chain consistent with a given walk. We can then use the ideas developed for this simpler problem to solve the more general problem.

Consider the sequence $y =$baabcddccde. The smallest chain which is consistent with it is $z =$abcde. In general, how can we get from y to z? Looking at the substring cddccd, we see that it is of the form xx^Rx where $x =$ cd (x^R denotes the reverse of string x). Such a substring can be thought of as a portion of the walk which traverses consecutive edges in x, then goes back over them in reverse, and then forward again. If xx^Rx is replaced by x, then that portion of the walk still starts and ends at the same vertices. Therefore, the shorter string represents a smaller chain that is still consistent with the original walk. Furthermore, observe that a prefix of the string is baab. This is of the form x^Rx where $x =$ ab. By replacing this with x, the result is a shorter sequence representing a smaller chain. The original sequence is still a walk on this graph; it merely starts at a different vertex. Similar transformations can be performed on a suffix.

This suggests the following approach to solving the problem of inferring a minimum length linear chain consistent with a walk. Starting with the walk output given, apply reductions of the above forms until it is no longer possible. Now, one may (and should) ask themselves a few questions about the usefulness of this approach. Is it possible for more than one reduction to be applicable to a sequence? If so, are all resulting strings consistent with the original walk? Furthermore, which order of application leads to the optimal solution (i.e. the shortest linear chain)? In other words, for a given order of application what guarantees that the resulting string is both *consistent* and *optimal*?

To answer the first question, it is possible for more than one reduction to be applicable, as exhibited by the sequence abbbccbbbbc. Figure 16.2 shows all possible reductions of the form $xx^Rx \rightarrow x$ that can be performed on this sequence. Since there may be more than one reduction possible on a sequence, which one should be chosen? In the example of Figure 16.2, it does not seem to matter which reductions are performed first. The eventual result of continuing reductions until no more are possible is always abc. So in this case, the order of reductions does not matter, and thus any applicable reduction can be applied. In general, does performing any applicable reduction lead to a consistent and optimal solution? We shall prove that for this problem the order of reductions does not matter. Such questions have been asked in other contexts for other kinds of reductions. See the Aslam, Rivest paper [9] for references.

16.3 Preliminaries

Before proving that any order of reductions produces a consistent and optimal linear chain, we first need some definitions.

Let $\mathcal{S}$ be a set. A *binary relation* on $\mathcal{S}$ is a subset of $\mathcal{S} \times \mathcal{S}$. Let $\rightarrow$ be a binary relation on $\mathcal{S}$. If $(x, y) \in \rightarrow$, we write $x \rightarrow y$.

Let ι be the *identity relation* on $\mathcal{S}$, $\iota = \{(x, x) | x \in \mathcal{S}\}$. Given two binary relations $\rightarrow_{\text{A}}$ and

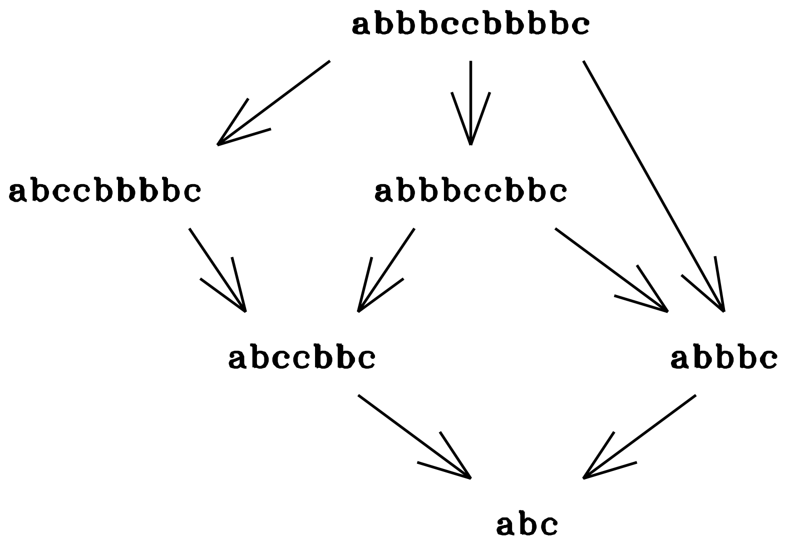


Figure 16.2: Possible reductions on `abbbccbbbbc`

$\rightarrow_{\text{B}}$, we define their *composition* as $\rightarrow_{\text{A}} \cdot \rightarrow_{\text{B}} = \{(x,y) | (\exists z)(x \rightarrow_{\text{A}} z \wedge z \rightarrow_{\text{B}} y)\}$. An element $x \in \mathcal{S}$ is *$\rightarrow$-reducible* if $(\exists y \in \mathcal{S})(x \rightarrow y)$, otherwise x is *$\rightarrow$-irreducible*. The particular relation involved is often omitted when it should be clear from the context. Define the following inductively:

$\overset{0}{\rightarrow}$	$=$	ι	0 steps in relation $\rightarrow$
$\overset{i}{\rightarrow}$	$=$	$\rightarrow \cdot \overset{i-1}{\rightarrow}, \forall i > 0$	exactly i steps in relation $\rightarrow$
$\overset{*}{\rightarrow}$	$=$	$\bigcup_{0 \leq i} \overset{i}{\rightarrow}$	0 or more steps
$\overset{\wedge}{\rightarrow}$	$=$	$\Big\{(x,y) \| x \overset{*}{\rightarrow} y \wedge y \text{ is irreducible}\Big\}$	0 or more steps ending at a dead end

Definition 16.1 *If $x \overset{*}{\rightarrow} y$ and y is irreducible, then y is called a* $\rightarrow$-normal form *of x.*

Note that for an arbitrary relation $\rightarrow$ and element x, there may be many normal forms of x—it is not necessarily unique.

Definition 16.2 *A binary relation $\rightarrow$ is* noetherian *if and only if there is no infinite sequence $x_1 \rightarrow x_2 \rightarrow \ldots$ (i.e. there are no "arrow-sequences" which are infinite to the right).*

Note that any element x of a noetherian relation must have at least one normal form. (If it did not, then there must be a sequence which is infinite to the right).

Definition 16.3 *A binary relation* $\to$ *is* confluent *if and only if* $(\forall w, x, y)(w \overset{*}{\to} x \wedge w \overset{*}{\to} y \Rightarrow (\exists z)(x \overset{*}{\to} z \wedge y \overset{*}{\to} z))$ *(i.e., for any element* w*, if both* x *and* y *can be reached from* w*, then there is an element* z *which can be reached from both* x *and* y*).*

We note that the confluence property is equivalent to the well known Church-Rosser property.

Definition 16.4 *A binary relation* $\to$ *is* locally confluent *if and only if* $(\forall w, x, y)(w \to x \wedge w \to y \Rightarrow (\exists z)(x \overset{*}{\to} z \wedge y \overset{*}{\to} z))$ *(i.e., for any element* w*, if both* x *and* y *are immediate successors of* w*, then there is an element* z *which can be reached from both* x *and* y*).*

The following well known results (they can be found in Huet [23]) will be very useful here.

Theorem 16.1 *A noetherian relation is confluent if and only if it is locally confluent.*

Theorem 16.2 *If a binary relation is confluent, then the normal form of any element, if it exists, is unique.*

So, for a confluent noetherian relation, every element x has exactly one normal form. Denote this normal form by $\hat{x}$.

Definition 16.5 *A binary relation* $\to$ *is* strictly decreasing *if* $(\forall x, y)(x \overset{*}{\to} y \Rightarrow |y| < |x|)$.

It should be clear that every strictly decreasing relation is noetherian.

The way that all of these definitions and theorems are used is as follows. For a particular problem which involves finding a minimum consistent chain from a given type of walk, find a relation $\to$ on strings such that $y \overset{*}{\to} z$ if and only if y is a walk on chain z. Prove that $\to$ is locally confluent and strictly decreasing. The rewards for this work are (1) every string y has a unique normal form $\hat{y}$; (2) the smallest chain consistent with any walk y is $\hat{y}$ (to be proven later); (3) an algorithm to find $\hat{y}$ can choose to perform $\to$-reductions in any order desired (by confluence).

16.4 Inferring Linear Chains from End-to-end Walks

In this section, we examine the problem of inferring chains from walks. We will further restrict the problem to inferring chains from *end-to-end* walks, in which it is known that the walk begins at one end of the chain and stops at the other end. We later consider the problem of inferring chains from general walks, and then the problem of inferring any degree 2 graph from a general walk.

Following the outline at the end of the previous section, define the relation $\to_{\mathrm{B}}$ on the set Σ^+ as

$$\to_{\mathrm{B}} = \left\{ (pxx^R xq, pxq) | p, q \in \Sigma^* \wedge x \in \Sigma^+ \right\}.$$

It should be clear that if $y \to_{\mathrm{B}} z$, then y is an end-to-end walk on z with a single "zig-zag" in the middle. We now prove that regardless of the order in which $\to_{\mathrm{B}}$ is applied, the resulting walk is consistent and that all linear chains consistent with the walk can be obtained in this manner.

Lemma 16.1 *For all $y, z \in \Sigma^+$, y is an end-to-end walk on chain $z \Leftrightarrow y \overset{*}{\to}_{\mathrm{B}} z$.*

Proof: ($\Rightarrow$) Since y is an end-to-end walk on the chain z, there is a sequence of vertices in the chain which can represent that walk. Such an embedding partitions y into segments that are the portions of y seen between reversals or "folds" in the walk. For example, the walk `abccbbcd` on chain `abcd` can be partitioned into the segments `abc`, `cb`, and `bcd`.

Note that the sequence of segments are alternating "right" and "left" segments, starting and ending with a "right" segment. Therefore there must be an odd number of segments. Also, the second segment is no longer than the first, and the second to last segment is no longer than the last.

By contradiction, assume that the implication $\Rightarrow$ is false. Then there is a shortest counterexample y such that y is an end-to-end walk on some z, but $y \overset{*}{\to}_{\mathrm{B}} z$ does not hold. Let s be a shortest segment of an embedding of y into z. There are several cases.

1. The segment s is the only segment. Then $y = z$ and $y \overset{*}{\to}_{\mathrm{B}} z$ holds trivially giving the desired contradiction.

2. There are several segments, and s is the first or last segment. As noted above, we can then select s to be either the second or second to last segment, respectively, since they are guaranteed to be no longer than s. Thus this case can be handled using case 3 below.

3. There are several segments, and s is a "middle" segment. Since s is a shortest segment, the previous segment ends with s^R and the next segment begins with s^R. Thus $y = ps^Rss^Rq \to_{\mathrm{B}} ps^Rq$. There are two subcases.

   (a) $ps^Rq \overset{*}{\to}_{\mathrm{B}} z$. Then $y \overset{*}{\to}_{\mathrm{B}} z$ giving the desired contradiction.

   (b) $ps^Rq \overset{*}{\to}_{\mathrm{B}} z$ does not hold. Then ps^Rq, which is clearly also a walk on z, is a shorter counterexample than y giving a contradiction.

($\Leftarrow$) Easily proven by induction on the number of "steps" in the relation $\to_{\mathrm{B}}$ between y and z, given the observation immediately preceding the lemma. ∎

It should be clear that $\to_{\mathrm{B}}$ is strictly decreasing. Every $\to_{\mathrm{B}}$-reduction removes at least two symbols from a string. Thus to prove that $\to_{\mathrm{B}}$ is confluent we need just show that it is locally confluent.

Lemma 16.2 *The relation $\to_{\mathrm{B}}$ is locally confluent.*

```
Input:
  $y$ - string of symbols
  $n$ - length of $y$
Output:
  $z$ - string of symbols corresponding to $\hat{y}$
  $m$ - length of $z$
Procedure:
  $m := 0$
  for $i := 1$ to $n$
    $m := m + 1$
    $z[m] := y[i]$
    if $z$ has a suffix of the form $xx^Rx$ then
      $l :=$ length of the $xx^Rx$ suffix
      $m := m \Leftrightarrow \frac{2}{3}l$
    endif
  end for
```

Figure 16.3: Algorithm for obtaining $\rightarrow_{\mathrm{B}}$-normal form of y.

Proof Sketch: Here is the basic idea behind this proof. Suppose that $w \rightarrow_{\mathrm{B}} u$ (so $w = p_1xx^Rxq_1$ and $u = p_1xq_1$ for some p_1, x, q_1). Also suppose that $w \rightarrow_{\mathrm{B}} v$ (so $w = p_2yy^Ryq_2$ and $v = p_2yq_2$ for some p_2, y, q_2). We must show that there exists a z such that $u \overset{*}{\rightarrow}_{\mathrm{B}} z$ and $v \overset{*}{\rightarrow}_{\mathrm{B}} z$. Notice that w must contain both an xx^Rx and yy^Ry substring. If these substrings do not overlap, then z is trivially found. The case in which they do overlap is a bit more complicated and handled by induction on the number of steps to get to z. ■

Finally, we apply this confluence result to prove that regardless of the order in which $\rightarrow_{\mathrm{B}}$-reductions are performed the same resulting string is obtained.

Lemma 16.3 *If y is an end-to-end walk, then the shortest linear chain that is consistent with y is $\hat{y}$, the normal form of y with respect to the relation $\rightarrow_{\mathrm{B}}$.*

Proof: Observe that $y \overset{*}{\rightarrow}_{\mathrm{B}} \hat{y}$ by definition, so y is an end-to-end walk on $\hat{y}$ by Lemma 16.1. Suppose that there is some other chain z such that y is an end-to-end walk on z and $|z| < |\hat{y}|$. Then $y \overset{*}{\rightarrow}_{\mathrm{B}} z$ by Lemma 16.1. By the confluence of $\rightarrow_{\mathrm{B}}$, there must be a string w such that $\hat{y} \overset{*}{\rightarrow}_{\mathrm{B}} w$ and $z \overset{*}{\rightarrow}_{\mathrm{B}} w$. By definition of $\hat{y}$, the only possibility is $w = \hat{y}$. Therefore $z \overset{*}{\rightarrow}_{\mathrm{B}} \hat{y}$. Since $\rightarrow_{\mathrm{B}}$ is strictly decreasing, this implies that $|z| \geq |\hat{y}|$, a contradiction. ■

We now build on this theoretical framework to obtain an efficient algorithm for inferring the shortest linear chain consistent with an end-to-end walk. The algorithm is shown in Figure 16.3.

Lemma 16.4 *Given input y, the algorithm shown in Figure 16.3 produces $\hat{y}$, the normal form of y with respect to $\rightarrow_{\mathrm{B}}$.*

Proof: Let z_i be the value of z after the ith iteration of the `for` loop has completed. Clearly $y \overset{*}{\rightarrow}_{\text{B}} z$ holds. Now we show that z_i is $\rightarrow_{\text{B}}$-irreducible for all i by induction. The basis that z_1 is irreducible is obvious; it consists of only a single symbol. Now for all $i > 1$, we need to show that z_{i-1} is irreducible implies that z_i is irreducible. There are two cases.

1. A suffix of the form xx^Rx was found. In this case, z_i is a proper prefix of z_{i-1}. Therefore, if z_{i-1} is irreducible (i.e., contains no substrings of the form xx^Rx), then z_i is irreducible.

2. No suffix of the form xx^Rx was found. In this case, z_i is z_{i-1} plus a single symbol appended to the end. Suppose z_i is reducible. Then it must contain a substring of the form xx^Rx. There are two subcases.

   (a) The substring ends at the last symbol of z_i. This cannot be the case, however, since such a suffix was not found by the algorithm.

   (b) The substring ends before the last symbol of z_i. This is also impossible, for then z_{i-1} would be reducible.

We conclude that z_i is irreducible. ■

If the search for the suffix is done in an obvious, exhaustive manner, then it requires $O(i^2)$ time for a string of length i. The worst case behavior of the algorithm occurs when the input string y is irreducible. It then requires $O(n^3)$ time.

16.5 Inferring Linear Chains from General Walks

Before returning to the problem of inferring a linear chain from a walk, we first consider the problem of inferring a linear chain from a general walk (as opposed to an end-to-end walk). A *walk* an a linear chain z is a sequence of colors corresponding to the edges traversed in some walk on z that begins a some vertex v_i and at some point passes through both the leftmost and rightmost vertices of the chain.

Observe that embedded in the general walk of the chain there must be an end-to-end walk. The approach used here is just an extension of the approach seen in the last section. However, now we must have special reductions to handle the head and the tail of the walk. Following the definition of $\rightarrow_{\text{B}}$, we define the following relation $\rightarrow_{\text{H}}$ on the set Σ^+ as

$$\rightarrow_{\text{H}} = \left\{ (x^Rxq, xq) | q \in \Sigma^* \wedge x \in \Sigma^+ \right\}.$$

Likewise we define the relation $\rightarrow_{\text{T}}$ on the set Σ^+ as

$$\rightarrow_{\text{T}} = \left\{ (pxx^R, px) | p \in \Sigma^* \wedge x \in \Sigma^+ \right\}.$$

Finally, we define

$$\rightarrow_{\text{HBT}} = \rightarrow_{\text{B}} \bigcup \rightarrow_{\text{H}} \bigcup \rightarrow_{\text{T}} .$$

Then proceeding in a very similar fashion as in the last section it can be shown that the binary relation $\rightarrow_{\text{HBT}}$ is confluent. This property can be used to prove the following key theorem.

Theorem 16.3 *If y is a walk, then the shortest linear chain that can produce y is $\hat{y}$, the normal form of y with respect to the relation $\rightarrow_{\text{HBT}}$.*

See the Aslam, Rivest paper [9] for the details of these proofs.

To further simplify the algorithm it would be nice if we could perform the three type of reductions one at a time. We now prove that this approach can be applied.

Lemma 16.5 *If $w \overset{\wedge}{\rightarrow}_{\text{B}} x \overset{\wedge}{\rightarrow}_{\text{H}} y \overset{\wedge}{\rightarrow}_{\text{T}} z$, then z is the normal form of w with respect to the relation $\rightarrow_{\text{HBT}}$.*

Proof: Clearly, we have that $w \overset{*}{\rightarrow}_{\text{B}} x \overset{*}{\rightarrow}_{\text{H}} y \overset{*}{\rightarrow}_{\text{T}} z \Rightarrow w \overset{*}{\rightarrow}_{\text{HBT}} z$. Thus we need just show that z is irreducible under $\rightarrow_{\text{HBT}}$. Clearly z is irreducible under $\rightarrow_{\text{T}}$. Also z is a prefix of y by the definition of $\rightarrow_{\text{T}}$. Suppose that z is not irreducible under $\rightarrow_{\text{H}}$, but then y is not irreducible under $\rightarrow_{\text{H}}$ which contradicts the definition of $\overset{\wedge}{\rightarrow}_{\text{H}}$. Furthermore, note that z is a substring of x by the definitions of $\rightarrow_{\text{H}}$ and $\rightarrow_{\text{T}}$. Suppose z is not irreducible under $\rightarrow_{\text{B}}$. Then x is not irreducible under $\rightarrow_{\text{B}}$ contradicting the definition of $\overset{\wedge}{\rightarrow}_{\text{B}}$. Thus z is irreducible under $\rightarrow_{\text{B}}$, $\rightarrow_{\text{H}}$, and $\rightarrow_{\text{T}}$, and hence irreducible under $\rightarrow_{\text{HBT}}$. ■

So putting these ideas all together, to obtain an algorithm for inferring a linear chain from a general walk w we can just perform the following steps:

1. $x \leftarrow$ normal-form of w with respect to $\rightarrow_{\text{B}}$
2. $y \leftarrow$ normal-form of x with respect to $\rightarrow_{\text{H}}$
3. $z \leftarrow$ normal-form of y with respect to $\rightarrow_{\text{T}}$

It then follows from the above results that z is the normal form of w with respect to $\rightarrow_{\text{HBT}}$. All that remains is to give an algorithm to find the normal form of a string with respect to $\rightarrow_{\text{H}}$ and $\rightarrow_{\text{T}}$. See Figure 16.4 for an algorithm to find the normal form of a string with respect to $\rightarrow_{\text{T}}$. Clearly a similar algorithm can be given to find the normal form of a string with respect to $\rightarrow_{\text{H}}$. The proof of correctness for the algorithm given in Figure 16.4 is similar (but easier) than the proof of Lemma 16.4 and is thus omitted here.

Finally we analyze the running time of this algorithm for inferring a linear chain from a general walk. We have already seen that the first step can be performed in $O(n^3)$ time. Now we just need to analyze the algorithm given in Figure 16.4. If the input is irreducible then $O(n^2)$ time is spend trying to find a suffix of the form xx^R. On the other hand, if the input is not irreducible $O(i^2)$ time is spent finding each xx^R suffix of length $2i$ or verifying one does not exist. Thus we get the recurrence

$$T(n) = \max_i \{T(n \Leftrightarrow i) + O(i^2)\} = O(n^2).$$

Thus the overall running time is $O(n^3)$.

```
*In*put:
  $y$ - string of symbols
  $n$ - length of $y$
*Output:*
  $z$ - string of symbols corresponding to $\hat{y}$
  $m$ - length of $z$
*Procedure:*
  $l := 0$
  while $2l \leq n$
      if $y$ has a length $2l$ suffix of the form $xx^R$ then
        $n := n \Leftrightarrow l$
        $l := 1$
      else
        $l := l + 1$
      endif
  end for
```

Figure 16.4: Algorithm for obtaining $\rightarrow_{\mathrm{T}}$-normal form of y

16.6 Inferring Cycles from Walks

In this section we consider the original problem MCI (for the special case of $D = 2$). To find the minimum cycle consistent with a given walk output, it is *not* sufficient to find the minimum consistent chain and then join the end vertices of the chain to form a cycle. That procedure certainly produces a cycle consistent with the walk, but not necessarily a smallest one. For example, the walk `abcdabcd` has as a minimum consistent chain the chain represented by the same string. However, a minimum consistent cycle is the one represented by the string `abcd`. To find a correct method, some more properties of walks on cycles must be examined.

Consider a cycle C. For any vertex k in C, define C_k to be the chain obtained by "breaking" the cycle at v_k. For example for the cycle shown in Figure 16.5, $C_1 =$ `abcd`. Consider a walk output w consistent with C. If every possible embedding of w in C uses every edge of C, then we call it a *completing* walk on C. Some example completing walks on the cycle shown in Figure 16.5 are `bcdabbbc` and `cdaadcba`. Note that if w is not a completing walk on C, then C cannot be a minimum consistent cycle for w.

Suppose we are given a walk output w and a minimum consistent cycle C for w. Then we know that w is a completing walk for C. Consider a particular embedding for w in C. At some point in the embedding, there is a first time when every edge of C has been used at least once. Call the vertex visited at this time k. Let x be the prefix of w corresponding to all edges traversed until vertex k is reached for the first time (there must be such a point, or

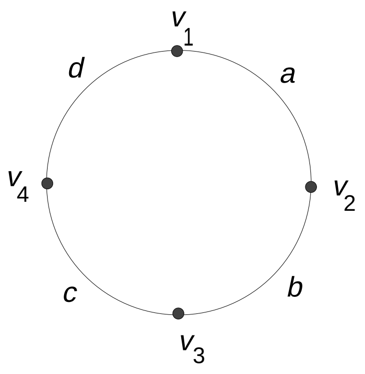


Figure 16.5: An example cycle.

else the walk would not have completed first at vertex k, but at some other vertex). Let y be the next portion of w corresponding to all edges traversed until vertex k is again reached. Let z be the remainder of w. So for the walk `cdaadcba` on the cycle of Figure 16.5, $k = v_2$, $x =$ `cda`, $y =$ `adcb`, and $z =$ `a`. Observe that $w = xyz$. Furthermore, the three properties clearly must hold.

1. y is an end-to-end walk on the chain C_k. Thus by Lemma 16.1, $y \overset{*}{\rightarrow}_{\mathrm{B}} C_k$.
2. x^R embeds into C_k starting at vertex k.
3. z embeds into C (not necessarily C_k) starting at vertex k.

Before giving the algorithm, we need one last lemma.

Lemma 16.6 *C_k is irreducible.*

Proof: Suppose it is reducible. Then $y \overset{*}{\rightarrow}_{\mathrm{B}} C_k = ptt^Rtq \rightarrow_{\mathrm{B}} ptq$. Then by Lemma 16.1, y is an end-to-end walk on ptq. It is also easy to show that x^R embeds on the chain ptq starting at the "left end", and z embeds on the cycle represented by ptq starting at the break. So we have found a cycle smaller than C which is consistent with the walk w. This is a contradiction, since C was defined to be a minimum consistent cycle. ■

Now, given a walk output w, there is no known simple way to discover the strings x, y, and z. Instead we must try all possible non-empty substrings y of w and eliminate those not meeting the three conditions given above. This basic algorithm is shown in Figure 16.6. Clearly w is a walk on each cycle that is a success. Furthermore, since the minimum consistent cycle is among the successes, this algorithm is correct. We now analyze the running time of this basic algorithm. For a given walk output w with $|w| = n$, there are $O(n^2)$ substrings to be tested. Each iteration requires $O(n^3)$ time to calculate y and $O(n^2)$ time to check if x^R and z embed in K. Thus the algorithm clearly runs in polynomial time.

```
Input:
  $w$ - walk output
Output:
  $C$ - string representing a smallest cycle consistent with $w$
Procedure:
  for all $x, z \in \Sigma^*$, $y \in \Sigma^+$
    let $w = xyz$
    $\hat{y}$ := $B$-normal-form of $y$
    $K$ := cycle such that $K_1 = \hat{y}$
    if $x^R$ and $z$ embed in $K$ starting at vertex 1 then
      call $K$ a success
    endif
  end for
  $C$ := smallest success
```

Figure 16.6: Basic algorithm for finding smallest cycle consistent with a general walk.

The algorithm shown in Figure 16.6 can be made to run more efficiently. Observe that the algorithm given early in Figure 16.3 actually computes the normal forms of *all* prefixes of y. Thus we can modify this algorithm so that it also checks for embeddings. The resulting algorithm is shown in Figure 16.7. For a given walk output w, *any* undirected edge-colored multigraph G with n vertices and m edges, and a starting vertex $v \in V$, it can be determined whether w can be embedded in G starting at v in $O(|w|(n+m))$ time. Simply keep track of the reachable vertex set at each step of the walk output. Since the length of the walks and the number of vertices and edges in the graphs are always at most n, this check can be done in $O(n^2)$ time. Thus each iteration of the inner loop on variable j can be performed in $O(n^2)$ time resulting in an overall running time of $O(n^4)$ for the algorithm.

```
Input:
    $w$ - walk output
    $n$ - length of $w$
Output:
    $C$ - string representing a smallest cycle consistent with $w$
    $L$ - length of $C$
Procedure:
    $C := w$
    $L := n$
    for $i := 1$ to $n$
        $x := w[1 \ldots (i \Leftrightarrow 1)]$
        $m := 0$
        for $j := i$ to $n$
            $m := m + 1$
            $y[m] := w[j]$
            if $y$ has a suffix of the form $tt^Rt$ then
                $l :=$ length of the $tt^Rt$ suffix
                $m := m \Leftrightarrow \frac{2}{3}l$
            endif
            if $m < L$ and
              both $x^R$ and $z = w[(j + 1) \ldots n]$ embed
              in cycle represented by $y$ starting at 1 then
              $C := y$
              $L := m$
            endif
        end for
    end for
```

Figure 16.7: More efficient algorithm for finding smallest cycle consistent with a general walk.

CS 582T Computational Learning Theory **Spring Semester, 1991**

Topic 17: Learning in an Infinite Attribute Space

Lecturer: Sally Goldman *Scribe: Kevin Ruland*

17.1 Introduction and Description of Model

In the real world there is a potentially infinite number of adjectives one can use to describe objects. The number of attributes is limited only by the structure of the language and precision of measurements. However, even though the possible number of attributes is very large the number of attributes to distinguish a particular object is usually relatively small. We will now consider learning in such an environment while requiring little time and space as long as each individual object possess only a finite number of attributes. The standard way of representing attributes as a bit vector will not work in this setting because the vector representing any object must have infinite length. Instead we will represent an object by the list of attributes the object has leaving out of the list the attributes the object does not have. One can think of attributes not in a particular object's representation as either attributes the object does not possess or as unknown but irrelevant attributes. The material presented here comes from the paper "Learning Boolean Functions in an Infinite Attribute Space," by Avrim Blum [10].

Using the notation in Blum:

- $\mathcal{A}$ is the set of all attributes. For simplicity enumerate the attributes, a_1, a_2, a_3,...
- $x \subseteq \mathcal{A}$ is an instance if $|x| < \infty$. An instance is represented as the list of attributes it has.
- $X = \{\, x \subseteq \mathcal{A} \mid |x| < +\infty \,\}$ is the instance space.
- $f\colon X \to \{0,1\}$ is a concept if for some finite set $R_f \subseteq \mathcal{A}$ of relevant attributes, $f(x) = f(x \cap R_f)$, $\forall x$. F is the target concept.
- $k = |R_F|$ is the number of relevant attributes in the target.
- n is the maximum number of attributes possessed by any example.

For simplicity of analysis, assume k and n are known a priori. We can remove this assumption by applying the standard doubling technique.

17.2 Learning Monotone Disjunctions

Valiant's algorithm for learning monotone disjunctions first built the disjunction of all attributes. Then, when it made a mistake on a negative example, removed all the attributes in the example. It is clear that this algorithm will not work in this model because one cannot build the disjunction of all attributes. Instead, the positive examples are used to indicate which attributes might be significant.

Learn-Monotone Disjunctions

1. Predict FALSE until the first positive example is encountered.
   Initialize f to the disjunction of all attributes in this example.
   Use f for all future predictions.
2. If a mistake is made on a positive example then add onto f all attributes in the example.
3. If a mistake is made on a negative example then remove from f all attributes in the example.

Claim 17.1 *This algorithm makes at most kn mistakes on a target of k attributes.*

Proof: Each mistake made on a positive example will add at least one relevant attribute because the example satisfies the target F but not f. Since only one mistake is made in step 1 and at least 1 attribute is added, at most $k \Leftrightarrow 1$ mistakes can occur in step 2. Now, for every mistake made on a positive example, at most $n \Leftrightarrow 1$ irrelevant attributes are added to f so the most irrelevant attributes in f is $k(n \Leftrightarrow 1)$. In step 3, all irrelevant attributes are removed. This can require up to $k(n \Leftrightarrow 1)$ mistakes because each mistake on a negative example removes at least one irrelevant attributes. Therefore fewer than kn mistakes are made in total. ■

17.3 Learning Monotone k-CNF

In the standard learning model, one can build an algorithm to learn k-CNF simply by building the conjunction of all disjunctions up to size k over the attribute space. Then each time a positive example is seen, remove those clauses inconsistent with the example. This approach will not work in this model because there is an infinite number of possible clauses and possibly an infinite number of clauses consistent with any particular positive example. So we need a procedure that "grows" each of the clauses from "seeds" of size one and discard those seeds that become too large.

A seed for a monotone clause is a disjunction of a subset of attributes in the clause. For example, a_4 is a seed for both $a_1 \vee a_4$ and $a_2 \vee a_3 \vee a_4$.

Property (*) of seeds If a clause c is a *seed* of a clause c_F then every example that satisfies c_F but not c has at least one attribute in c_F that is not in c.

Learn-Monotone-k-CNF

```
1. Predict FALSE until the first positive example is received. Initialize $f$
   to the conjunction of all attributes in $x$. Use $f$ to make all future predictions
2. If a mistake is made on positive example $x$, consider each clause $c$ in $f$
   not satisfied by $x$. If $c$ contains $k$ attributes remove it from $f$.
   Otherwise, replace it with all clauses made by adding any one attribute in $x$ to $c$.
```

Note that f is always consistent with all negative examples because a negative example must falsify some clause c_F of F and, by the converse of property (*), it must also falsify all seeds of c_F.

Claim 17.2 *This algorithm makes at most $(n+1)^k$ mistakes.*

Proof: Step 1 creates the invariant that every clause in the target F has some seed in f because any positive example has at least one attribute from every clause in the target. Step 2 maintains this invariant because, if c is replaced then by property (*) at least one of the new clauses created will also be a seed. If c is removed it was not satisfied by a positive example and it already contained k attributes so it cannot be a seed for any clause in F.

We complete the proof with a simple potential argument. For each clause in f of m attributes place a cost of $(n+1)^{k-m}$. The total cost for f is the sum of all clauses in f. The initial cost is at most $n(n+1)^{k-1} < (n+1)^k$. For each iteration of step 2, if for a particular clause, $m = k$ then the total cost is decreased by 1. However if $m < k$, then the total cost is decreased by $n(n+1)^{k-(m+1)} \Leftrightarrow (n+1)^{k-m} \geq 1$. Thus the total cost is decreased by at least 1 with each mistake made in step 2. Since the total cost for any nontrivial target is at least one, the total number of mistakes is at most $(n+1)^k$. ■

17.4 Learning Non-Monotone k-CNF

In some cases the target concept could depend on the absence of some attributes. For example, the concept of "penguin" depends on the birds inability to fly. We cannot, as we have done in previous models, simply add a new attribute for the negative of every attribute, because then each object will depend on an infinite list of attributes. If a concept depends on the absence of an attribute, positive examples will not contain this attribute. Some negative examples, however, will. We need to build seeds for the non-monotonic clauses from the negative examples.

Subsets of attributes in a non-monotone clause will not, in general, possess property (*). For example, a_1 is a subset of $a_1 \vee \overline{a_2}$ but does not satisfy property (*) for the instance $x = \{a_2\}$. So a revised definition of seed is needed. A seed for a non-monotone clause is a disjunction of all negated attributes and any subset of the non-negated attributes in the clause. For example, the seeds for $a_1 \vee a_2 \vee \overline{a_3} \vee \overline{a_4}$ must contain $\overline{a_3}$ and $\overline{a_4}$. Note that if the clause contains no negated attributes then the seeds are exactly those described in the previous section.

Learn-Non-Monotone-k-CNF

1. Predict FALSE until the first positive example is received. Initialize f to the conjunction of all attributes in x. Use f to make all future predictions.
2. If a mistake made on positive example x, consider each clause c in f not satisfied by x. If c contains k attributes remove it from f. Otherwise, replace it with all clauses made by adding any one attribute in x to c.
3. If a mistake is made on a negative example x, then for each subset $\{a_{i_1}, a_{i_2}, \ldots, a_{i_r}\} \subseteq x$, $(r \leq k)$ add to f the clause $(\overline{a_{i_1}} \vee \overline{a_{i_2}} \vee \cdots \vee \overline{a_{i_r}})$.

Claim 17.3 *This procedure makes at most $(tk+1)(n+1)^k$ mistakes on a k-CNF formula of t clauses.*

Proof Sketch: Similar to the previous proof, step 2 maintains the invariant that once a clause in F has a seed in f it will continue to. Step 3 adds to f a seed for some clause that previously had no seed. So the algorithm makes at most t mistakes on negative examples.

Each mistake on a negative example adds to f at most $\binom{n}{m}$ clauses of m literals for all $m \leq k$. Using the same potential as before, each mistake on a negative example adds a total cost of at most

$$\sum_{m=1}^{k} \binom{n}{m} (n+1)^{k-m} \leq \sum_{m=1}^{k} (n+1)^k \leq k(n+1)^k.$$

Since at most t mistakes are made on negative examples the total cost of f will increase by at most $tk(n+1)^k$ for all negative examples. Finally since the initial cost is $(n+1)^k$ and each positive example reduces the cost by at least one, the total number of mistakes made is at most $(tk+1)(n+1)^k$. ■

17.5 Generalized Halving Algorithm

Blum generalizes the halving algorithm of Littlestone to the infinite attribute model. He concludes that any concept class in the infinite-attribute model is learnable in polynomial number of mistakes if k is finite. This result ignores possible run time constraints.

Definition 17.1 *For some $f \in \mathcal{C}$, let $f_S(x) = f(x \cap S)$.*

Definition 17.2 *Given $S = \{a_{i_1}, a_{i_2}, \ldots, a_{i_m}\} \subset \mathcal{A}$ let $C(k,S) = \{\, f_S \mid f \in \mathcal{C}, |R_f| \leq k \,\}$.*

So $|C(k,S)|$ is the number of functions in $\mathcal{C}$ on at most k relative attributes that differ over examples whose attributes come from S.

Sequential Halving Algorithm($\mathcal{C}$,k)

```
1. $S \leftarrow \{\}$
2. $H \leftarrow C(k,S)$, $T \leftarrow \{\}$
3. On input $x$ predict according to majority vote on functions in $H$. If the
   prediction is incorrect, remove from $H$ those functions that are incorrect
   and let $T \leftarrow T \cup x$. Continue until $H$ is empty.
4. If $H$ is empty, Let $S \leftarrow S \cup T$ and go to step 2.
```

Theorem 17.1 *This algorithm makes at most $O(k^2 \lg(kn|C(k)|))$ mistakes.*

Previous knowledge of k can be removed by the standard doubling technique.

17.6 Other Topics

Blum also describes how to combine the ideas of Littlestone's WINNOW1 with this new model. This new algorithm makes at most $O(k \lg n)$ mistakes on any disjunction of k literals. Although we refer to reader to Blum's paper for the algorithm achieving the $O(k \lg n)$ mistake bound, we briefly describe a simple idea to obtain an $O(k^2 \log(kn))$ mistake bound. Begin with a small set of attributes S, say those appearing in the first positive example. Next run WINNOW1 using only the attributes in S. (Note that WINNOW1 must be modified to handle the concepts "true" and "false".) If more than $k \log(|S|)$ mistakes are made then some example on which a mistake was made has some relevant attribute that was ignored. Finally, just add the at most $nk \log(|S|)$ attributes ignored and repeat. This result is particularly useful if the target depends on a small number of relevant attributes, each example is fairly large, and the attribute set is enormous. He extends this idea to learning any k-DNF formula on t terms with at most $O(tk^2 \log(t+n))$ mistakes.

Finally, Blum also describes how any algorithm that learns in a finite attribute space using membership queries can be modified to learn in the new model with few additional mistakes and a small time penalty.

CS 582T Computational Learning Theory **Spring Semester, 1991**

Topic 18: The Weighted Majority Algorithm

Lecturer: Sally Goldman *Scribe: Marc Wallace*

18.1 Introduction

In these notes we study the construction of a prediction algorithm in a situation in which a learner faces a sequence of trials, with a prediction to be made in each, and the goal of the learner is to make few mistakes. In particular, consider the case in which the learner has reason to believe that one of some pool of known algorithms will perform well, but the learner does not know which one. A simple and effective method, based on weighted voting, is introduced for constructing a compound algorithm in such a circumstance. This algorithm is shown to be robust with respect to errors in the data. We then discuss other situations in which such a compound algorithm can be applied. The material presented here comes from the paper "The Weighted Majority Algorithm," by Nick Littlestone and Manfred Warmuth [30].

Before describing the weighted majority algorithm in detail we briefly discuss some learning problems in which the above scenario applies. The first case is one that we have seen before. Suppose one knows that the correct prediction comes from some target concept selected from some known concept class, then one can apply the weighted majority algorithm where each concept in the class is one of the algorithms in the pool. As we shall see for such situations, the weighted majority algorithm is just the natural generalization of the halving algorithm. Another situation in which the weighted majority algorithm can be used in the situation in which one has a set of algorithms for making predictions, one of which will work well on a given data set. However, it may be that the best prediction algorithm to use depends on the data set and thus it is not known a priori which algorithm to use. Finally, as we shall discuss in more detail below the weighted majority algorithm can often be applied to help in situations in which the prediction algorithm has a parameter that must be selected and the best choice for the parameter depends on the target. In such cases one can build the pool of algorithms by choosing various values for the parameter.

The basic idea behind the weighted majority algorithm is to give each algorithm in the pool the input instance, get all the predictions, combine these into one prediction, and then if wrong pass that data along to the algorithms in the pool. Observe that the halving algorithm could be considered as a weak master algorithm by simply taking the majority prediction and throwing out all bad predictors at each mistake. Unfortunately this algorithm is not robust against noise. If one algorithm is perfect, but that source is noisy, then the main algorithm will reject it and may never converge to one solution (it will run out of algorithms).

We define the number of *anomalies* in a sequence of trials as the least number of inconsistent trials of any algorithm in the pool. Notice if there are any anomalies, then the halving algorithm will fail to converge by eliminating all functions.

18.2 Weighted Majority Algorithm

In this section we introduce the weighted majority algorithm.

Weighted Majority Algorithm (WM)

```
Initially assign non-negative weights (say 1) to each algorithm in the pool
To make a prediction for an instance:
    Let $q_0$ be the total weight of algs that predict 0
    Let $q_1$ be the total weight of algs that predict 1
    Predict 0 iff $q_0 \geq q_1$
If a mistake is made, multiply the weights of the algorithms agreeing
with the incorrect prediction by some fixed non-negative $\beta < 1$.
```

Notice that using $\beta = 0$ would yield the halving algorithm. However, it is more interesting to consider what happens as $\beta \to 1$. Suppose the pool is called F, and a given sequence of trials has m anomalies with respect to F. Then

$$\#\text{ mistakes } \leq c(\log |F| + m)$$

where c is some constant dependent on β.

In general the maximum number of mistakes will be:

- $O(\log |F| + m)$ if one algorithm in F makes at most m mistakes.
- $O(\log(|F|/k) + m)$ if each of a set of k algorithms in F makes at most m mistakes.
- $O(\log(|F|/k) + (m/k))$ if the total number of mistakes of a set of k algorithms in F is at most m mistakes.

As an application of WM (weighted majority), we return to the use of WINNOW1 to learn r-of-k threshold functions. Normally the mistake bound is $O(kn \log n)$. However, if the learner has prior knowledge of a close upper bound for r, the mistake bound can be reduced to $b_r = O(kr \log n)$ by carefully selecting the parameters for WINNOW1. Let A_r denote the algorithm WINNOW1 with a guess of r. Now any sequence that cannot be represented by an r'-of-k function for $r' \leq r$ will fail phenomenally, and make far too many mistakes. However, we can apply WM to the set of algorithms $\{A_{2^i}\}$ for all i satisfying $2^i \leq n$. Then the number of mistakes will be $O(\log \log n + b_r) = O(kr \log n)$.

Now clearly if the size of our algorithm pool is not polynomial the weighted majority algorithm is not computationally feasible. Still, many reasonable pools of algorithms will be of polynomial size, or can be reduced to such without a dramatic increase in the average number of mistakes.

18.3 Analysis of Weighted Majority Algorithm

Let $A = \{A_1, \ldots, A_{|A|}\}$ be the pool of algorithms, and w_i the weight of A_i. The basic structure of the proofs that follow will be: First, show that after each trial, if a mistake occurs then the sum of the weights is decreases by at least a factor of u for some $u < 1$. Second, let W_{init} be the total initial weight and W_{fin} be a lower bound for the total final weights; then $W_{init}u^m \geq W_{fin}$, where m is the number of mistakes. This implies that

$$m \leq \frac{\log \frac{W_{init}}{W_{fin}}}{\log \frac{1}{u}}.$$

We now prove a general theorem bounding the number of mistakes made by the weighted majority algorithm. We can then apply to obtain the three results given above.

Theorem 18.1 *Let S be a sequence of instances and m_i the number of mistakes made by A_i on S. Let m be the number of mistakes made by WM on S when applied to the pool A. Then*

$$m \leq \frac{\log \frac{W_{init}}{W_{fin}}}{\log \frac{2}{1+\beta}}$$

where $W_{fin} = \sum_{i=1}^{n} w_i \beta^{m_i}$.

Notice that if the initial weights are all 1, we have

$$m \leq \frac{\log \frac{|A|}{\sum_{i=1}^{n} \beta^{m_i}}}{\log \frac{2}{1+\beta}}$$

Proof: We notice that the weights are updated only when their algorithm makes a mistake; and since A_i can make only m_i mistakes, we must have $w_{i_{final}} \geq w_{i_{init}} \beta^{m_i}$. Furthermore, at the end the total current weight must be greater than or equal to W_{fin}.

Now, during each trial, let $q_0 + q_1$ be the current total weight, where q_0 and q_1 are as defined in the definition of WM. Suppose we have a trial in which a mistake was made. Without loss of generality we assume the prediction was zero. Then the total weight after the trial will be:

$$\beta q_0 + q_1 \leq \beta q_0 + q_1 + \frac{1 \Leftrightarrow \beta}{2}(q_0 \Leftrightarrow q_1) = \frac{1+\beta}{2}(q_0 + q_1)$$

If no mistake is made then the weights remain the same, of course. So we have $u = \frac{1+\beta}{2} < 1$.

Finally, since we know $W_{init}u^m \geq W_{fin}$ we take logs of both sides and get

$$m \leq \frac{\log \frac{W_{init}}{W_{fin}}}{\log \frac{2}{1+\beta}}$$

■

Now we provide some (more useful) corollaries. Assume $\beta > 0$ and all initial weights are one. Let $|A| = n$.

Corollary 18.1 *Assume there is a subpool of A (of size k) such that each algorithm in it makes no more than m mistakes on a set of instances S. Then WM applied to A makes no more than*

$$\frac{\log \frac{n}{k} + m \log \frac{1}{\beta}}{\log \frac{2}{1+\beta}}$$

mistakes on S.

Proof: $W_{init} = n$ and $W_{fin} \geq k\beta^m$. Apply the theorem. ■

The theorem and the corollary give us the first two relations stated above concerning the order of the mistake bounds: $O(\log |F| + m)$ if one algorithm in F makes at most m mistakes, and $O(\log(|F|/k) + m)$ if each of a set of k algorithms in F makes at most m mistakes. The next corollary will yield the third relation. However, before proving this corollary we review some basic definitions and lemmas.

Definition 18.1 *A function $f : \mathbb{R} \to \mathbb{R}$ is concave (respectively convex) over an interval D of $\mathbb{R}$ if for all $x \in D$, $f''(x) \geq 0$ ($f''(x) \leq 0$).*

The following to standard lemmas [20, 32] are often useful.

Lemma 18.1 *Let f be a function from $\mathbb{R}$ to $\mathbb{R}$ that is concave over some interval D of $\mathbb{R}$. Let q be a natural number, and let $x_1, x_2, \ldots, x_q \in D$. Then*

$$\sum_{i=1}^{q} x_i \leq S \;\Rightarrow\; \sum_{i=1}^{q} f(x_i) \geq q f(S/q).$$

Lemma 18.2 *Let f be a function from $\mathbb{R}$ to $\mathbb{R}$ that is convex over some interval D of $\mathbb{R}$. Let q be a natural number, and let $x_1, x_2, \ldots, x_q \in D$. Then*

$$\sum_{i=1}^{q} x_i \leq S \;\Rightarrow\; \sum_{i=1}^{q} f(x_i) \leq q f(S/q).$$

We are now ready to prove the corollary.

Corollary 18.2 *Assume there is a subpool of A (of size k) such that all algorithms (together) make no more than m mistakes on a set of instances S. Then WM applied to A makes no more than*

$$\frac{\log \frac{n}{k} + \frac{m}{k} \log \frac{1}{\beta}}{\log \frac{2}{1+\beta}}$$

mistakes on S.

Proof: Without loss of generality we can assume the first k algorithms of A are the subpool in question. Then $\sum_{i=1}^{k} m_i \leq m$. Since the sum of the final weights is greater than the sum of k of the final weights (or equal), we have

$$w_{fin} \geq \sum_{i=1}^{k} w_{i_{fin}} \geq \sum_{i=1}^{k} \beta^{m_i}$$

Finally, from Lemma 18.1 it follows that

$$\sum_{i=1}^{k} \beta^{m_i} \geq k\beta^{\frac{m}{k}}$$

■

As a final note, if $\beta = 0$ and $m = 0$ the number of mistakes made will be no more than $\log_2 \frac{n}{k}$.

18.4 Variations on Weighted Majority

In this section we demonstrate the generality of the weighted majority algorithm by discussing some useful learning situation in which it can be applied.

18.4.1 Shifting Target

We modify WM so that it can recover more easily if the target is changed. For example, suppose a particular subset of algorithms predicts well in the beginning, but after some period of time some other set of algorithms has the best performance (and the initial set may yield terribly wrong predictions). We want the overall performance to remain good.

For simplicity of notation and mathematics, assume that the subpools which do well (either initially or after a while) are all of size one: it is not too difficult to generalize but the basic ideas can easily be lost in the dredge of notation.

Our new algorithm will be called WML. WML has two basic parameters, β and γ, with $0 < \beta < 1$ and $0 \leq \gamma < 1/2$. Each algorithm in A has a non-negative weight associated with it. Everything is basically the same as WM except for the updating: whenever WM would multiply a weight by β, WML will multiply by β if and only if the weight is larger than $\gamma/|A|$ times the total weight of all algorithms in A.

The idea, therefore, is to follow the same updating scheme, unless a weight is already extremely low. If it is then it is probably negligible, so do not bother updating it. This allows us to take a previously poorly performing algorithm (which has just started predicting extremely well) and put lots of weight on it, since its weight has not been allowed dropped too low.

Now suppose that some algorithm makes m_1 mistakes in the first segment of some sequence of instances, another makes m_2 mistakes in the next segment, and so on. Of course, WML does not know when the segments start and finish. Still, the number of mistakes will be no greater than

$$c(S \log |A| + \sum_{i=1}^{s} m_i)$$

where S is the number of segments there are and c is some constant.

This mistake bound makes sense, since the $S \log |A|$ part is really $\sum_{i=1}^{S} \log |A|$. When there is a change between segments, the WML will make some constant times $\log |A|$ mistakes while the weights are updated so that the (currently) accurate algorithm receives the majority of the weight. The other m_i is from the accurate algorithms themselves, as according to the bounds for WM.

18.4.2 Selection from an Infinite Pool

Suppose we have a (countably) infinite pool of algorithms from which to choose. Clearly we cannot ask for input from all of them. But by using ever increasing sets of the algorithms one can approximate the standard WM model, with only an additional term of order $\log i$ where it is the ith algorithm which predicts well.

Previously (in homework 3, problem 1) we saw that one could apply the halving algorithm to such an infinite set, and achieve a mistake bound of order $\log i$. Since the WM is merely an extension of the halving algorithm, it is possible to construct a variant of the WM algorithm which increases its pool over time, in which the number of mistakes is bounded by $c(\log i + m_i)$, where the constant c depends not only on the initial parameters, but on how big the initial set of algorithms grows.

18.4.3 Pool of Functions

We now consider the application of WM to a pool of Boolean functions on some domain. This is a true generalization of the halving algorithm. Now, suppose there is a function consistent with the sequence of trials. We wish to obtain better bounds for the number of mistakes on this function, at the possible expense of greater numbers of mistakes on the other functions. Hence, the following theorem:

Theorem 18.2 *Let $f_1, \ldots, f_n$ be a pool of functions, and $m_1, \ldots, m_n$ be integers such that $\sum_{i=1}^{n} 2^{-m_i} < 2$. If WM is applied to the pool with initial weights $w_j = 2^{m_i}$ and $\beta = 0$, and if the sequence of trials is consistent with some f_i, then WM will make no more than m_i mistakes.*

Proof: Recall from the discussion of WM that for m the number of mistakes, we have

$$m \leq \frac{\log \frac{W_{init}}{W_{fin}}}{\log \frac{2}{1+\beta}}$$

where $W_{fin} = \sum_{i=1}^{n} w_i \beta^{m_i}$, and m_i is the number of mistakes made by the ith algorithm on a given sequence.

For $\beta = 0$ the denominator drops out. Now, $W_{init} = \sum_{i=1}^{n} 2^{-m_i} < 2$. And since f_i makes no mistakes at all, $W_{fin} \geq$ initial weight of $f_i = 2^{-m_i}$. Thus the number of mistakes m is no more than:

$$m \leq \log W_{init} \Leftrightarrow \log W_{fin} \leq \log 2 \Leftrightarrow \log 2^{-m_i} = 1 + m_i$$

which yields the desired result. ∎

18.4.4 Randomized Responses

Notice that we can consider WM to be a deterministic algorithm which predicts 1 whenever $\frac{q_1}{q_0+q_1} \geq \frac{1}{2}$.

Consider a randomized version in which 1 is predicted with probability $\frac{q_1}{q_0+q_1}$. This algorithm may make more initial mistakes when the probability is near 1/2, but it can be shown that we can update the weights so that the rate of mistakes (in the long run) can be made arbitrarily close to the rate of mistakes of the best prediction algorithm in the pool. This would yield an improvement of a factor of two over the limiting bounds for the learning rate of the deterministic version of WM.

CS 582T Computational Learning Theory **Spring Semester, 1991**

Topic 19: Implementing the Halving Algorithm

Lecturer: Sally Goldman

In this lecture we will demonstrate how approximate counting schemes can be used to implement a randomized version of the halving algorithm. We will apply this technique to the problem of learning a total order in the mistake bound learning model. The material presented in this lecture comes from the paper, "Learning Binary Relations and Total Orders," by Sally Goldman, Ronald Rivest, and Robert Schapire [16]. These scribe notes are an abbreviated version of Chapter 4 of Goldman's thesis [18].

We formalize the problem of learning a total order as follows. The instance space $X_n = \{1, \ldots, n\} \times \{1, \ldots, n\}$. An instance (i, j) in X_n is in the target concept if and only if object i precedes object j in the target total order.

If computation time is not a concern, then the halving algorithm makes only $O(n \lg n)$ mistakes, since there are $n!$ possible target concepts. However, we are interested in efficient algorithms and thus our goal is to design an efficient version of the halving algorithm. In the next section we discuss the relation between the halving algorithm and approximate counting. Then we show how to use an approximate counting scheme to efficiently implement a randomized version of the approximate halving algorithm, and apply this result to the problem of learning a total order on a set of n elements.

19.1 The Halving Algorithm and Approximate Counting

In this section we first review the halving algorithm (mostly to give the notation we shall use) and then discuss approximate counting schemes. Let $\mathcal{V}$ denote the set of concepts in C_n that are consistent with the feedback from *all previous queries*. Given an instance x in X_n, for each concept in $\mathcal{V}$ the halving algorithm computes the prediction of that concept for x and predicts according to the majority. Finally, all concepts in $\mathcal{V}$ that are inconsistent with the correct classification are deleted. We now define an *approximate halving algorithm* to be the following generalization of the halving algorithm. Given instance x in X_n an approximate halving algorithm predicts in agreement with at least $\varphi|\mathcal{V}|$ of the concepts in $\mathcal{V}$ for some constant $0 < \varphi \leq 1/2$.

Theorem 19.1 *The approximate halving algorithm makes at most* $\log_{(1-\varphi)^{-1}} |C_n|$ *mistakes for learning* C_n.

Proof: Each time a mistake is made, the number of concepts that remain in $\mathcal{V}$ are reduced by a factor of at least $1 \Leftrightarrow \varphi$. Thus after at most $\log_{(1-\varphi)^{-1}} |C_n|$ mistakes there is only one consistent concept left in C_n. ■

When given an instance $x \in X_n$, one way to predict as dictated by the halving algorithm is to count the number of concepts in $\mathcal{V}$ for which x is negative and for which x is positive and then predict with the majority. As we shall see, by extending this idea we can implement the approximate halving algorithm using an approximate counting scheme.

We now introduce the notion of an approximate counting scheme for counting the number of elements in a finite set $\mathcal{S}$. Let x be a description of a set $\mathcal{S}_x$ in some natural encoding. An *exact counting scheme* on input x outputs $|\mathcal{S}_x|$ with probability 1. Such a scheme is polynomial if it runs in time polynomial in $|x|$. Sometimes exact counting can be done in polynomial time; however, many counting problems are $\#\mathcal{P}$-complete and thus assumed to be intractable. (For a discussion of the class $\#\mathcal{P}$ see Valiant [42].) For many $\#\mathcal{P}$-complete problems good approximations are possible. A *randomized approximation scheme*, R, for a counting problem satisfies the following condition for all $\epsilon, \delta > 0$:

$$\Pr\left[\frac{|\mathcal{S}_x|}{(1+\epsilon)} \leq R(x,\epsilon,\delta) \leq |\mathcal{S}_x|(1+\epsilon)\right] \geq 1 \Leftrightarrow \delta$$

where $R(x,\epsilon,\delta)$ is R's estimate on input x, ϵ, and δ. In other words, with high probability, R estimates $|\mathcal{S}_x|$ within a factor of $1+\epsilon$. Such a scheme is *fully polynomial* if it runs in time polynomial in $|x|$, $\frac{1}{\epsilon}$, and $\lg \frac{1}{\delta}$. For further discussion see Sinclair [39].

We now review work on counting the number of linear extensions of a partial order. That is, given a partial order on a set of n elements, the goal is to compute the number of total orders that are linear extensions of the given partial order. We discuss the relationship between this problem and that of computing the volume of a convex polyhedron. Given a convex set S and an element a of $\Re^n$, a *weak separation oracle*

1. Asserts that $a \in S$, or

2. Asserts that $a \notin S$ and supplies a reason why. In particular for closed convex sets in $\Re^n$, if $a \notin S$ then there exists a hyperplane separating a from S. So if $a \notin S$, the oracle responds with such a separating hyperplane as the reason why $a \notin S$.

We now discuss how to reduce the problem of counting the number of extensions of a partial order on n elements to that of computing the volume of a convex n-dimensional polyhedron given by a separation oracle. The polyhedron built in the reduction will be a subset of the $[0,1]^n$ (i.e. the unit hypercube in $\Re^n$) where each dimension corresponds to one of the n elements. Observe that any inequality $x_i > x_j$ defines a halfspace in $[0,1]^n$. Let $\Delta(t)$ denote the polyhedron obtained by taking the *intersection* of the halfspaces given by the inequalities of the partial order t. (See Figure 19.1 for an example with $n = 3$.) For any pair of total orders t_1 and t_2, the polyhedra $\Delta(t_1)$ and $\Delta(t_2)$ are simplices that only intersect in a face (zero volume): a pair of elements say x_i and x_j that are ordered differently in t_1 and t_2 (such a pair must exist) define a hyperplane $x_i = x_j$ that separates $\Delta(t_1)$ and $\Delta(t_2)$. Let T_n be the set of all $n!$ total orders on n elements. Then

$$[0,1]^n = \bigcup_{t \in T_n} \Delta(t). \qquad (19.1)$$

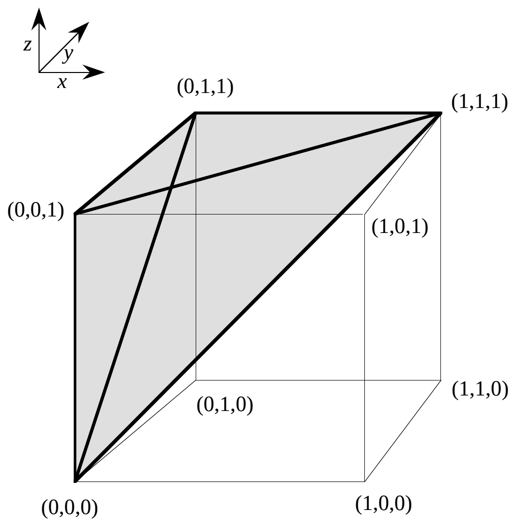


Figure 19.1: The polyhedron formed by the total order $z > y > x$.

From equation (19.1) and the observation that the volumes of the polyhedra formed by each total order is equal, it follows that the volume of the polyhedron defined by any total order is $1/n!$. Thus it follows that for any partial order P

$$\frac{\text{number of extensions of } P}{n!} = \text{ volume of } \Delta(P). \tag{19.2}$$

Rewriting equation (19.2), we obtain that

$$\text{number of extensions of } P = n! \cdot (\text{volume of } \Delta(P)) \,. \tag{19.3}$$

Finally, we note that the weak separation oracle is easy to implement for any partial order. Given inputs a and S, it just checks each inequality of the partial order to see if a is in the convex polyhedron S. If a does not satisfy some inequality then reply that $a \notin S$ and return that inequality as the separating hyperplane. Otherwise, if a satisfies all inequalities, reply that $a \in S$.

Dyer, Frieze and Kannan [13] give a fully-polynomial randomized approximation scheme (*fpras*) to approximate the volume of a polyhedron given a separation oracle. From equation (19.3) we see that this fpras for estimating the volume of a polyhedron can be easily applied to estimate the number of extensions of a partial order. (A similar result was given independently by Matthews [31].)

19.2 Learning a Total Order

In this section we show how to use a fpras to implement a randomized version of the approximate halving algorithm, and apply this result for the problem of learning a total order on a set of n elements.

We first prove an $\Omega(n \lg n)$ lower bound on the number of mistakes made by any learning algorithm. We use the following result of Kahn and Saks [24]: Given any partial order P that is not a total order there exists an incomparable pair of elements x_i,x_j such that

$$\frac{3}{11} \leq \frac{\text{number of extensions of } P \text{ with } x_i \leq x_j}{\text{number of extensions of } P} \leq \frac{8}{11}.$$

So the adversary can always pick a pair of elements so that regardless of the learner's prediction, the adversary can force a mistake while only eliminating a constant fraction of the remaining total orders.

Finally, we present a polynomial prediction algorithm making $n \lg n + (\lg e) \lg n$ mistakes with very high probability. We first show how to use an exact counting algorithm R, for counting the number of concepts in C_n consistent with a given set of examples, to implement the halving algorithm.

Lemma 19.1 *Given a polynomial-time algorithm R to exactly count the number of concepts in C consistent with a given set E of examples, one can efficiently implement the halving algorithm for C.*

Proof: We show how to use R to efficiently make the predictions required by the halving algorithm. To make a prediction for an instance x in X_n the following procedure is used: Construct E^- from E by appending x as a negative example to E. Use the counting algorithm R to count the number of concepts $C^- \in \mathcal{V}$ that are consistent with E^-. Next construct E^+ from E by appending x as a positive example to E. As before, use R to count the number of concepts $C^+ \in \mathcal{V}$ that are consistent with E^+. Finally if $|C^-| \geq |C^+|$ then predict that x is a negative example; otherwise predict that x is a positive example.

Clearly a prediction is made in polynomial time, since it just requires calling R twice. It is also clear that each prediction is made according to the majority of concepts in $\mathcal{V}$. ■

We modify this basic technique to use a fpras instead of the exact counting algorithm to obtain an efficient implementation of a randomized version of the approximate halving algorithm.

Theorem 19.2 *Let R be a* fpras *for counting the number of concepts in C_n consistent with a given set E of examples. If $|X_n|$ is polynomial in n, one can produce a prediction algorithm that for any $\delta > 0$ runs in time polynomial in n and* $\lg \frac{1}{\delta}$ *and makes at most* $\lg |C_n| \left(1 + \frac{\lg e}{n}\right)$ *mistakes with probability at least* $1 - \delta$.

Proof: The prediction algorithm implements the procedure described in Lemma 19.1 with the exact counting algorithm replaced by the fpras $R(n, \frac{1}{n}, \frac{\delta}{2|X_n|})$. Consider the prediction for an instance $x \in X_n$. Let $\mathcal{V}$ be the set of concepts that are consistent with all previous instances. Let r^+ (respectively r^-) be the number of concepts in $\mathcal{V}$ for which x is a positive (negative) instance. Let $\hat{r}^+$ (respectively $\hat{r}^-$) be the estimate output by R for r^+ (r^-). Since R is a fpras, with probability at least $1 \Leftrightarrow \frac{\delta}{|X_n|}$

$$\frac{r^-}{1+\epsilon} \leq \hat{r}^- \leq (1+\epsilon)r^- \quad \text{and} \quad \frac{r^+}{1+\epsilon} \leq \hat{r}^+ \leq (1+\epsilon)r^+$$

where $\epsilon = 1/n$. Without loss of generality, assume that the algorithm predicts that x is a negative instance, and thus $\hat{r}^- \geq \hat{r}^+$. Combining the above inequalities and the observation that $r^- + r^+ = |\mathcal{V}|$, we obtain that $r^- \geq \frac{|\mathcal{V}|}{1+(1+\epsilon)^2}$.

We define an *appropriate* prediction to be a prediction that agrees with *at least* $\frac{|\mathcal{V}|}{1+(1+\epsilon)^2}$ of the concepts in $\mathcal{V}$. To analyze the mistake bound for this algorithm, suppose that each prediction is appropriate. For a single prediction to be appropriate, both calls to the fpras R must output a count that is within a factor of $1+\epsilon$ of the true count. So any given prediction is appropriate with probability at least $1 \Leftrightarrow \frac{\delta}{|X_n|}$, and thus the probability that all predictions are appropriate is at least

$$1 \Leftrightarrow |X_n| \left(\frac{\delta}{|X_n|} \right) = 1 \Leftrightarrow \delta.$$

Clearly if all predictions are appropriate then the above procedure is in fact an implementation of the approximate halving algorithm with $\varphi = \frac{1}{1+(1+\epsilon)^2}$ and thus by Theorem 19.1 at most $\log_{(1-\varphi)^{-1}} |C_n|$ mistakes are made. Substituting ϵ with its value of $\frac{1}{n}$ and simplifying the expression we obtain that with probability at least $1 \Leftrightarrow \delta$,

$$\# \text{ mistakes} \leq \frac{\lg |C_n|}{\lg \frac{1}{1-\varphi}} = \frac{\lg |C_n|}{\lg \left(1 + \frac{n^2}{n^2+2n+1}\right)}. \tag{19.4}$$

Since $\frac{n^2}{n^2+2n+1} \geq 1 \Leftrightarrow \frac{2}{n}$,

$$\begin{aligned}
\frac{1}{\lg \left(1 + \frac{n^2}{n^2+2n+1}\right)} &\leq \frac{1}{\lg \left(1 + 1 \Leftrightarrow \frac{2}{n}\right)} \\
&= \frac{1}{1 + \lg \left(1 \Leftrightarrow \frac{1}{n}\right)} \\
&= 1 \Leftrightarrow \frac{\lg \left(1 \Leftrightarrow \frac{1}{n}\right)}{1 + \lg \left(1 \Leftrightarrow \frac{1}{n}\right)}
\end{aligned}$$

We now derive two inequalities which we use to finish up the analysis. Since for all x, $e^x \geq 1 + x$, by letting $x = \Leftrightarrow 1/n$, taking logarithms of both sides and then adding 1 to each side, we obtain that

$$1 + \lg \left(1 \Leftrightarrow \frac{1}{n}\right) \leq 1 \Leftrightarrow \frac{\lg e}{n}.$$

Next be letting $x = 1/(n \Leftrightarrow 1)$ in $e^x \geq 1 + x$ and raising both sides to the power of $n \Leftrightarrow 1$ we get that:

$$e \geq \left(1 + \frac{1}{n \Leftrightarrow 1}\right)^{n-1} = \left(\frac{n}{n \Leftrightarrow 1}\right)^{n-1}.$$

Taking the logarithm of both sides and simplifying yields that

$$\lg\left(1 \Leftrightarrow \frac{1}{n}\right) \geq \frac{\Leftrightarrow \lg e}{n \Leftrightarrow 1}.$$

We now apply the inequalities $\lg\left(1 \Leftrightarrow \frac{1}{n}\right) \geq \frac{-\lg e}{n-1}$ and $1 + \lg\left(1 \Leftrightarrow \frac{1}{n}\right) \leq 1 \Leftrightarrow \frac{\lg e}{n}$ to get that

$$\begin{aligned}
\frac{\lg\left(1 \Leftrightarrow \frac{1}{n}\right)}{1 + \lg\left(1 \Leftrightarrow \frac{1}{n}\right)} &\geq \frac{\frac{-\lg e}{n-1}}{1 \Leftrightarrow \frac{\lg e}{n}} \\
&= \frac{\Leftrightarrow \lg e}{n \Leftrightarrow 1 \Leftrightarrow \frac{n-1}{n} \lg e} \\
&\geq \frac{\Leftrightarrow \lg e}{n}
\end{aligned}$$

Finally, applying these inequalities to equation (19.4) yields that

$$\#\text{ mistakes} \leq \frac{\lg |C_n|}{\lg\left(1 + \frac{n^2}{n^2+2n+1}\right)} \leq \lg |C_n| \left(1 + \frac{\lg e}{n}\right).$$

■

Note that we could modify the above proof by not requiring that all predictions be appropriate. In particular if we allow γ predictions not to be appropriate then we get a mistake bound of $\lg |C_n| \left(1 + \frac{\lg e}{n}\right) + \gamma$.

We now apply this result to obtain the main result of this section. Namely, we describe a randomized polynomial prediction algorithm for learning a total order in the case that the adversary selects the query sequence.

Theorem 19.3 *There exists a prediction algorithm A for learning total orders such that on input δ (for all $\delta > 0$), and for any query sequence provided by the adversary, A runs in time polynomial in n and $\lg \frac{1}{\delta}$ and makes at most $n \lg n + (\lg e) \lg n$ mistakes with probability at least $1 \Leftrightarrow \delta$.*

Proof Sketch: We apply the results of Theorem 19.2 using the fpras for counting the number of extensions of a partial order given independently by Dyer, Frieze and Kannan [13], and by Matthews [31]. We know that with probability at least $1 \Leftrightarrow \delta$, the number of mistakes is at most $\lg |C_n| \left(1 + \frac{\lg e}{n}\right)$. Since $|C_n| = n!$ the desired result is obtained. ■

We note that the probability that A makes more than $n \lg n + (\lg e) \lg n$ mistakes does not depend on the query sequence selected by the adversary. The probability is taken over the coin flips of the randomized approximation scheme.

CS 582T Computational Learning Theory **Fall Semester, 1990**

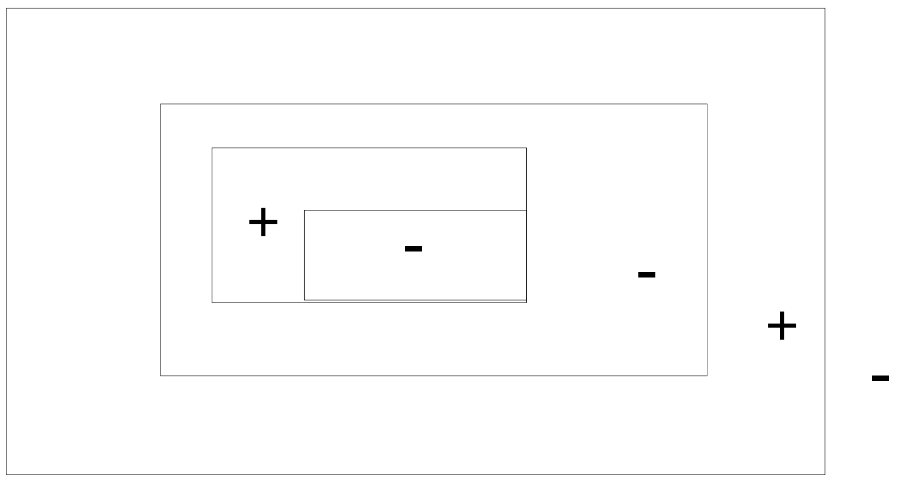
Choices of Topics

March 18, 1991

Below are a list of possible topics to cover during the remainder of this course. Please mark your first and second choices. (If you are interested in some topic not listed here please come and talk to me about it.)

1. **Learning in the Presence of Malicious Noise**: Study both upper and lower bounds on the noise rate that can be tolerated when the examples are corrupted by malicious noise (i.e. with probability ν the adversary selects the instance and label).

2. **Learning Nested Differences of Intersection-Closed Concept Classes:** Present technique for converting an algorithm for learning an intersection-closed concept class into an algorithm that can learn any concept class expressed as nested differences of concepts from an intersection-closed class. Here is an example of the nested difference of the intersection-closed class of axis-parallel rectangles in the plane:



3. **Learnability and the VC Dimension**: Demonstrate that by using *dynamic sampling* versus *static sampling* one can learn concept classes with infinite VC dimension for decomposable concept class. In other words, the Blumer, Ehrenfeucht, Haussler and Warmuth result assumes that the learner asks for a single sample of a given size and must then process these examples. If instead the learner can divide the learning session into stages where in each stage the learner asks for some examples and then

performs some processing, then it is possible to learn any concept class that can be written as the countable union

$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots$$

where each concept class $\mathcal{C}_d$ has VC dimension at most d.

4. **On the Sample Complexity of Weak Learning**: Study the sample complexity needed to weakly learn. Although Schapire has shown that any concept weakly learnable is also strongly learnable, his work does not address the number of examples needed to weakly learn. It is shown to be important to distinguish between those learning algorithms that output *deterministic hypothesis* and those that output *randomized* hypothesis.

5. **Learning DFA With Membership and Equivalence Queries**: Present an algorithm that can learn any DFA with membership and equivalence queries in time polynomial in the number of states in the minimum DFA and the length of the longest counterexample.

6. **Learning Read-Once Formulas with Membership and Equivalence Queries**: Present an algorithm that can learn any read-once formula in time polynomial in the number of variables and the length of the formula. Also an algorithm to learn any monotone read-once formula using only membership queries is presented.

7. **Separating the PAC and Mistake-Bound Model over the Boolean Domain**: Over the domain $\{0,1\}^n$ it is known that if the learner is allowed unlimited computational resources then any concept class learning in one of these two models is learnable in the other. In addition, any polynomial-time learning algorithm for a concept class in the mistake-bound model can be transformed into one that learns the class in the PAC model. Here it is shown that if one-way functions exists then the converse does not hold. Namely, a concept class over $\{0,1\}^n$ is given that is not learnable in the mistake-bound model but is learnable in the PAC model. Furthermore, the concept class remains hard to learn in the mistake-bound model even if the learner is allowed a polynomial number of membership queries.

8. **Using Approximate Counting Schemes to Efficiently Implement a Randomized Version of the Halving Algorithm**: Study a technique to efficiently implement the halving algorithm given an appropriate approximate counting scheme. In particular, it is shown how a scheme to approximate the number of extensions of a partial order can be used to learn the concept class of total orders under the mistake bound model (with adversary presentation) making only $O(n \lg n)$ mistakes.

9. **The Weighted Majority Algorithm**: Study construction of a prediction algorithm in a situation in which a learner faces a sequence of trials, with a prediction to be made in each, and the goal of the learner is to make few mistakes. In particular, consider

the case in which the learner has reason to believe that one of some pool of known algorithms will perform well, but the learner does not know which one. A simple and effective method, based on weighted voting, is introduced for constructing a compound algorithm in such a circumstance. This algorithm is shown to be robust with respect to errors in the data.

10. **Learning Boolean Functions in an Infinite Attribute Space**: Consider a model for learning Boolean functions in domains described by a potentially infinite number of attributes. (The learner is just shown the attributes which are true and must run in time polynomial in the number of attributes revealed.) It is shown that many of the basic Boolean functions learnable in the standard model such as k-CNF and k-DNF are still learnable in the new model, though by more complicated algorithms.

11. **Training a 3-Node Neural Network is NP-Complete**: It is shown that given a simple 2-layer 3-node, n-input neural network whose nodes compute linear threshold functions of their inputs, that it is NP-complete to decide whether there exist linear threshold functions for the three nodes of this network so that it will produce output consistent with a given set of training examples. The proof involves translating the learning problem into a geometrical setting. An equivalent statement of the result is that it is NP-complete to decide whether two sets of Boolean vectors in n-dimensional space can be separated by two hyperplanes so that some quadrant contains all the elements of one set and none of the other.

12. **Inferring Graphs from Walks**: Consider the problem of inferring an undirected, degree-bounded, edge-colored graph from the sequence of edge colors seen in a walk of that graph. This problem can be viewed as reconstructing the *structure* of a Markov chain from its output. A polynomial-time algorithm for the inference of underlying graphs of degree-bound 2 is presented.

CS 582T Computational Learning Theory **January 25, 1991**

Homework Assignment 1

Solutions Due: February 8

Please write up all solutions clearly, concisely, and legibly.

1. Consider Boolean functions f that can by defined on $\{0,1\}^n$ by a nested **if–then–else** statement of the form:

$$f(x_1, x_2, \ldots, x_n) = \textbf{if } l_1 \textbf{ then } c_1 \textbf{ elseif } l_2 \textbf{ then } c_2 \cdots \textbf{ elseif } l_k \textbf{ then } c_k \textbf{ else } c_{k+1}$$

where the l_j's are literals (either one of the variables or their negations), and the c_j's are either **T** (true) or **F** (false). (Such a function is said to be computed by a *simple decision list.*)

   (a) Suppose you have a set of labeled examples that are consistent with a function computed by a simple decision list. Show how to find in polynomial time some simple decision list that is consistent with the examples.

   (b) Show how to generalize your algorithm to handle k-decision lists, where the condition in each **if** statement may be the conjunction of up to k literals. (Here k is a fixed constant.)

   (c) Argue the k-decision lists are PAC learnable.

2. We have seen in class that an algorithm that is capable of finding a hypothesis consistent with a given set of labeled examples can be turned into a PAC learning algorithm. Argue that the converse is true: a PAC learning algorithm can be used (with high probability) to find a hypothesis consistent with a given set of labeled examples. (Hint: Show how to define an appropriate probability distribution on the given set of examples, and how to set ϵ and δ, so that the PAC learning algorithm returns a hypothesis consistent with the given set of examples, with high probability.)

3. Describe a PAC learning algorithm for the class of concepts k-RS(n) (the RS is for "ring-sum"); this class contains all boolean formula over the n boolean variables $\{x_1, \ldots, x_n\}$, where each such formula is the exclusive-or of a set of terms, and each term is the conjunction of at most k literals. (Once again, think of k as a fixed constant.) For example, the formula

$$x_1\overline{x_3}x_5 \oplus x_2 \oplus \overline{x_4}$$

is a member of 3-RS(5). Be sure to describe how many examples are required as a function of n, k, ϵ, and δ, and how the learning algorithm computes its hypothesis. (Hints: For the former, base your computation on the number of formula in k-RS(n). For the later, define a variable for each possible term, and solve linear equations modulo 2.)

4. Consider the class of concepts defined on the interval on the interval $(0, 1)$ where each concept is of the form $(0, r]$ for some real $r \in (0, 1)$. For the concept $(0, r]$, all real numbers x, $0 < x \leq r$, are *positive* instances while all real numbers y, $r < y < 1$, are *negative* instances. There is one such concept for each real number $r \in \{0, 1\}$.

   (a) Show that this concept class is PAC learnable under the assumption that the examples are drawn *uniformly* from the range $(0, 1)$. Specify the number of examples needed as a function of δ and ϵ, and explain how you construct your hypothesis from the given data.

   (b) Show that this concept class is PAC learnable when the examples are drawn according to some *unknown* distribution. Specify the number of examples needed as a function of δ and ϵ, and explain how you construct your hypothesis from the given data.

5. Let y be a Bernoulli random variable that is 1 with probability p. Our goal is to compute a good estimate for p from observing a set of independent draws of y. Let $\hat{p}$ be an estimate of p obtained from m independent trials. That is $\hat{p}$ is just:

   $$\text{(number of trials in which } y = 1\text{) / (total number of trials)}$$

   How large must m be so that $\Pr[|\hat{p} \Leftrightarrow p| \geq \gamma] \leq \delta$. That is we want our estimate for p to be within γ of the true value for p with probability at least $1 \Leftrightarrow \delta$.

CS 582T Computational Learning Theory **February 20, 1991**

Homework Assignment 2

Solutions Due: March 6

Please write up all solutions clearly, concisely, and legibly.

1. We return to the class of concepts defined on the interval $(0, 1)$ where each concept is of the form $(0, r]$ for some real $r \in (0, 1)$. Suppose that the source of labeled examples is subject to random misclassification noise of a rate $\beta < 1/4$. Give a PAC learning algorithm for this problem. Be sure to specify how many examples are needed as a function of ϵ and δ (and β if you wish).

2. If C_1 and C_2 are concept classes, and $c_1 \in C_1$ and $c_2 \in C_2$, then $c_1 \vee c_2$ is the concept whose positive examples are exactly the set $c_1 \cup c_2$. Note that $c_1 \vee c_2$ may not be an element of either C_1 or C_2. We can then define the concept class $C_1 \vee C_2 = \{c_1 \vee c_2 : c_1 \in C_1, c_2 \in C_2\}$. The definition for the class $C_1 \wedge C_2$ is analogous.

   (a) Prove that if C_1 is PAC learnable, and C_2 is PAC learnable from negative examples, then $C_1 \vee C_2$ is PAC learnable.

   (b) Prove that if C_1 is PAC learnable from positive examples and C_2 is PAC learnable from positive examples then $C_1 \wedge C_2$ is PAC learnable from positive examples.

3. Let C, C_1, C_2 be concept classes of finite VC-dimension over instance space X.

   (a) Define $\overline{C} = \{X \Leftrightarrow c : c \in C\}$. What is $VCD(\overline{C})$ in terms of $VCD(C)$?

   (b) For concept class C, let $VCD(C) = d$ and let S be any set of m distinct instances. Prove that $|\Pi_C(S)| \leq \Phi_d(m) = \sum_{i=0}^{d} \binom{m}{i}$. (Recall that $\Pi_C(S)$ includes all distinct ways that C can classify the S instances.)

   Hint: Use induction on m and d. (Think about $m = d$ as the base case for any given d.) Also, it may be useful to show that $\Phi_d(m) = \Phi_d(m \Leftrightarrow 1) + \Phi_{d-1}(m \Leftrightarrow 1)$.

   (c) For concept classes C_1, C_2, prove that $VCD(C_1 \cup C_2) \leq VCD(C_1) + VCD(C_2) + 1$. Also provide an example to demonstrate that this result is the best possible such result.

4. Show how to exactly identify any unknown read-once-DNF formula using a polynomial number of equivalence and membership queries. In a read-once formula each variable occurs at most once. (The one instance of a variable may be either negated or unnegated—so a read-once formula need not be a monotone formula.) The number of queries should be polynomial in the number of terms and variables in the unknown formula.

Big Hint: Prove as a lemma that if f is the unknown read-once-DNF formula over the variables $X_1, \ldots, X_n$, and $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are two assignments to these variables such that x and y agree in all but one variable assignment, yet $f(x) = 1$ and $f(y) = 0$, then $S(x)$ is a term of f, where $S(x)$ is the conjunction of the literals in the set

$$\{X_i : f(x \oplus i) = 0 \text{ and } x_i = 1\} \cup \left\{\overline{X_i} : f(x \oplus i) = 0 \text{ and } x_i = 0\right\}$$

where $x \oplus i$ is the assignment x with the ith variable flipped. (For example $00110 \oplus 3 = 00010$.)

EXTRA CREDIT:

Let C be a concept class of finite VC-dimension over instance space X. Let

$$C^{(k)} = \{\cup_{i=1}^{k} c_i : c_i \in C\}.$$

Prove that $VCD(C^{(k)}) = O(k \ln(k) \cdot VCD(C))$. (Note the same result holds for k-way intersection.)

CS 582T Computational Learning Theory **March 27, 1991**

Homework Assignment 3

Solutions Due: April 9

Please write up all solutions clearly, concisely, and legibly.

1. Suppose that $f_1, f_2, \ldots$ is an enumeration of some set of computable total 0-1-valued functions. More precisely, this enumeration is computed by some program M; the output of M on input i is a program P_i that computes f_i.

   Show how to learn an unknown function f, which is guaranteed to be one of the f_i's, in the mistake-bound model described in the Littlestone paper in such a way that you make only $O(\lg t)$ mistakes, where t is the least index such that $f = f_t$.

2. In this problem we considered a simple case of learning with queries where the feedback can be erroneous. The learner and adversary agree on a number n, and then the adversary thinks of a number between 1 and n, inclusive. The learner must find out which number the adversary has selected by asking questions of the form, "Is your number less than t?" for various t. A binary-search approach allows you to ask at most $\lg n$ questions before finding the number.

   To make this an interesting problem, suppose that the adversary is allowed to incorrectly respond to at most *one* question. How many questions must the learner now ask? (A bound of $3 \lg n$ is easy: the learner can just ask each question three times and take the majority vote of the adversary's responses.) Give a learning algorithm that uses a number of queries of the form $\lg n + f(n)$ where $f(n) = o(\lg n)$, that is, f grows asymptotically more slowly than the logarithm function.

3. Argue that functions representable as 1-decision lists are threshold functions. That is, given any 1-decision list $f(x_1, x_2, \ldots, x_n) =$ **if** l_1 **then** c_1 **elseif** l_2 **then** $c_2 \cdots$ **elseif** l_k **then** c_k **else** c_{k+1}, show how to construct a threshold function equivalent to this decision list. One way to achieve this goal is to do the following:

   (a) Show that, without loss of generality, you can assume that $c_k = 1$ and $c_{k+1} = 0$.

   (b) Suppose that the threshold circuit not only has access to the x_i for all i, but also has access to the negated variables $\overline{x_i}$ for all i. (We will remove this additional information in the next step.) Given this assumption, describe a threshold function equivalent to f.

   (c) Now modify the previous part so that the threshold function does not have access to the negated variables.

   Observe that now we can use Littlestone's Winnow2 combined with the transformation described in Example 6 from Littlestone's paper (pg. 312) to learn 1-decision lists in the mistake-bound model.

4. In this problem we consider r-of-k threshold functions. (Recall that for this concept class, $X = \{0,1\}^n$. For a chosen set of k ($k \leq n$) relevant variables and a given number r ($1 \leq r \leq k$), an r-of-k threshold function is 1 if and only if at least r of the k relevant variables are 1.

   Assuming that both r and k are unknown to the learner, show that the class of r-of-k threshold functions can be learned in the mistake-bound model using the halving algorithm. What mistake bound do you obtain? How does this compare to the mistake bound obtained by Winnow2? (That is, does one algorithm always perform better than the other algorithm? If not, state under what conditions each algorithm is best.)

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