Probability that two Integers are Relatively Prime

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I. Lemmas and setup

Definition: Two lattice points P and Q are said to be **mutually visible** if the line segment which joins them contains no lattice points other than the endpoints P and Q.

Definition: The Mobius function μ is a function on the intergers defined as $\mu(1) = 1$, and for n>1 s.t. $n = p_1^{a_1} ... p_k^{a_k}$:

$\mu(n) = (-1)^k, \quad$ if $a_1 = ... = a_k = 1$
$\mu(n) = 0, \quad$ otherwise

Note: A fun exercise to get familiar with the Mobius function is to prove that every square has Mobius 0.

Lemma 1:

$$\sum_{n \leq x} \frac{\mu(n)}{n^2} = \frac{6}{\pi^2} + O\left(\frac{1}{x}\right)$$

where $\mu(n)$ is the Mobius function

Proof: Notice that $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}$ converges because it is dominated by $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}$ (which converges to $\frac{\pi^2}{6}$). In fact, because of properties of the distribution of the prime numbers, $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$. By spliiting the sum into a difference of two known sums we get:

$$\begin{aligned}\sum_{n \leq x} \frac{\mu(n)}{n^2} &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} - \sum_{n \geq x} \frac{\mu(n)}{n^2} \\ &= \frac{6}{\pi^2} + O\left(\sum_{n \geq x} \frac{1}{n^2}\right) \\ &= \frac{6}{\pi^2} + O\left(\frac{1}{x}\right)\end{aligned}$$

Note: The fact that $O\left(\sum_{n \geq x} \frac{1}{n^2}\right) = O\left(\frac{1}{x}\right)$ comes from the fact that $\sum_{n \geq x} \frac{1}{n^s} = O\left(x^{1-s}\right)$ for $s > 1$ which we will not prove here.

Definition: The totient function $\varphi(n)$ is the sum of positive integers less than n that are relatively prime to n.

Note: As an exercise, try to show the interesting relation between the Mobius and Totient functions: $\varphi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$ where the sum is over the divisors d of n

Lemma 2:

$$\sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2} x^2 + O(x \log x)$$

Proof: Because $\varphi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$, we have:

$$\begin{aligned}
\sum_{n \leq x} \varphi(n) &= \sum_{n \leq x} \sum_{d|n} \mu(d) \frac{n}{d} \\
&= \sum_{q,d \text{ s.t. } qd \leq x} \mu(d) q \\
&= \sum_{d \leq x} \mu(d) \sum_{q \leq x/d} q \\
&= \sum_{d \leq x} \mu(d) \left[\frac{1}{2} \left(\frac{x}{d} \right)^2 + O\left(\frac{x}{d} \right) \right] \\
&= \frac{1}{2} x^2 \sum_{d \leq x} \frac{\mu(d)}{d^2} + O\left(x \sum_{d \leq x} \frac{1}{d} \right) \\
&= \frac{1}{2} x^2 \left[\frac{6}{\pi^2} + O\left(\frac{1}{x} \right) \right] + O\left(x \log x \right) \qquad \text{from Lemma 1} \\
&= \frac{3}{\pi^2} x^2 + O(x \log x)
\end{aligned}$$

Note: The fact that $\sum_{q \leq x/d} q = \frac{1}{2} \left(\frac{x}{d} \right)^2 + O\left(\frac{x}{d} \right)$ comes from the fact that $\sum_{n \leq x} n^\alpha = \frac{x^{\alpha+1}}{\alpha+1} + O(x^\alpha)$.

Lemma 3: Two lattice points (a, b) and (m, n) are mutually visible if and only if a-m and b-n are relatively prime.

Proof: Since (a-m, b-n) and (0,0) are mutually visible if and only if (a, b) and (m, n) are mutually visible, suffices to prove the above lemma for (m,n) = (0,0).

($\Rightarrow$) Assume (a, b) is visible from the origin and let d = $\gcd(a, b)$. If d > 1 then a = da′, b = db′ and the lattice point (a′, b′) is on the line segment joining (0, 0) to (a, b). But (a, b) is assumed to be visible from the origin. This contradiction implies that d=1.

($\Leftarrow$) Assume $\gcd(a, b) = 1$. If a lattice point (a′, b′) is on the line segment joining (0,0) to (a,b) then a′ = ta and b′ = tb where $0 < t < 1$. This implies t is rational because b and b′ are integers just like a and a′. Hence we can write t = r/s where r, s are integers with $\gcd(r, s) = 1$. Thus, sa′ = ar and sb′ = br. So, s | ar and s | br. But $\gcd(s, r) = 1$ so s | a, s | b. Meaning s=1 since $\gcd(a, b) = 1$. This contradicts the inequality $0 < t < 1$ thus (a, b) is visible from the origin.

II. Theorem

Theorem: The set of lattice points visible from the origin has density $\frac{6}{\pi^2}$. This set of points is the same as the proportion of pairs of integers that are coprime.

Proof: Consider a large square region in the xy-plane defined by $|x| \leq r$, $|y| \leq r$. Let N(r) denote the number of lattice points in this square, and let N′(r) denote the number of lattice points which are visible from the origin. We want to find:

$$\lim_{r\to\infty} \frac{N'(r)}{N(r)}$$

Notice that by symmetry we can split the plane into eight identical regions so that N′(r) is equal to 8 (the 8 closest points to the origin) plus 8 times the number of visible points in each region. Let us find the number of visible points in just one of these regions:

$$\{(x,y)|2 \leq x \leq r, \quad 1 \leq y \leq x\}$$

Notice that:

$$\begin{aligned} N'(r) &= 8 \sum_{1\leq n\leq r} \varphi(n) \qquad \text{by lemma 3} \\ &= \frac{24}{\pi^2} + O(r \log r) \qquad \text{by lemma 2} \end{aligned}$$

And, $N(r) = (2\lfloor r \rfloor + 1)^2 = 4r^2 + O(r)$

Hence,

$$\frac{N'(r)}{N(r)} = \frac{\frac{24}{\pi^2} + O(r\log r)}{4r^2 + O(r)} = \frac{\frac{6}{\pi^2} + O(\frac{\log r}{r})}{1 + O(\frac{1}{r})}$$

Hence,

$$\lim_{r\to\infty} \frac{N'(r)}{N(r)} = \frac{6}{\pi^2}$$