Long Range Dependence

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Abstract

This paper will motivate and introduce finite variance long-range dependent series. We first give an overview of time series, then attempt to justify the existence of each non-equivalent definition of LRD by looking at various examples such as FARIMA(p, d, q) series and Fractional Brownian Motion.

I. Introduction

A stochastic process $\{X(t)\}_{t\in \mathrm{T}}$ is stationary if

$$\{X(t)\}_{t\in \mathrm{T}} \stackrel{d}{=} \{X(t+h)\}_{t\in \mathrm{T}}$$

But this assumption is often too strong in practice. A way to relax this assumption is to limit the constraints to mean and covariance. We define a stochastic process $\{X(t)\}_{t\in \mathrm{T}}$ to be weak stationary (or second-order stationary) if for any $s < t \in \mathrm{T}$ the two following hold:

$$\mathbb{E}X(t) = \mathbb{E}X(0)$$

and

$$\begin{aligned} \mathrm{Cov}(X(t), X(s)) &= \mathrm{Cov}(X(t-s), X(0)) \\ &= \gamma_X(t-s) \end{aligned}$$

Let $h = t - s$, we call $\gamma_X(h)$ the autocovariance function and $\rho(h) = \gamma_X(h)/\gamma_X(0)$ the autocorrelation function of the time series $\{X(t)\}_{t\in\mathbb{Z}}$. This is enough to fully characterize the dependence structure and therefore the process itself. Sample counterparts of the ACF and ACVF are good measures of the dependence structure of the time series. They can also allow us to uncover simpler processes that together model our process.

MA Processes

A stochastic process $\{X(n)\}_{n\in\mathbb{Z}}$ is called a Moving Average process of order q if

$$X(n) = X_n = Z_n + \sum_{k=1}^{q} \theta_k Z_{n-k} \qquad (1)$$

where $Z_n \sim WN(0, \sigma_Z^2)$.

Example: MA(1)

Using (1), let X_n be MA(1):

$$X_n = Z_n + \theta Z_{n-1}$$

Hence we can find the ACVF and ACF of X_n:

$$\gamma_X(h) = \mathbb{E}X_h X_0 = \begin{cases} \sigma_Z^2(1+\theta^2),\ h = 0 \\ \sigma_Z^2\theta,\ h = 1 \\ 0,\ h \geq 2 \end{cases}$$

So,

$$\rho_X(h) = \begin{cases} 1,\ h = 0 \\ \frac{\theta}{1+\theta^2},\ h = 1 \\ 0,\ h \geq 2 \end{cases}$$

Remark 1

After lag q, The ACF (and ACVF) of an MA(q) are 0. The PACF tails off gradually.

AR processes

A weak stationary time series $\{X(n)\}_{n\in\mathbb{Z}}$ is called autoregressive of order p if

$$X_n = \sum_{k=1}^{p} \varphi_k X_{n-k} + Z_n \tag{2}$$

with $Z_n \sim WN(0, \sigma_Z^2)$.

Let $p = 1$ and notice that because (2) is equivalent to

$$X_n = \varphi(\varphi X_{n-2} + Z_{n-1}) + Z_n$$

We have that

$$X_n = \sum_{k=0}^{\infty} \varphi^k Z_{n-k} \tag{3}$$

Since WLOG $X(0) = 0$. Which by geometric series properties only converges in L^2 for $|\varphi| < 1$
When this is the case, it is easy to show that $\rho_X(h) = \varphi^{|h|}$. That is, (3) is well-defined when

$$\sum_{h=0}^{\infty} |\varphi^h| = \sum_{h=0}^{\infty} |\rho_X(h)| < \infty \tag{4}$$

Remark In contrast with the MA(q), the ACF and ACVF of an AR(p) tail off gradually. While the PACF is 0 after lag p.

Integrated processes

Let $d \in \mathbb{Z}$ be positive. In an AR process of order d, if the first $d-1$ φ_j's are zero and $\varphi_d = 1$ then X_n becomes

$$X_n - X_{n-d} = Z_n \tag{5}$$

Such a process is called Integrated of order d.

Examples of mixtures

By mixing models we can model more general time series. This may be more useful in practice.

ARMA(p,q):

Combining (1) and (2) we get the ARMA(p,q):

$$X_n - \sum_{k=1}^{p} \varphi_k X_{n-k} = Z_n + \sum_{k=1}^{q} \theta_k Z_{n-k} \tag{6}$$

which exists if the characteristic polynomial $1 - \varphi_1 z - ...\varphi_p z^p = 0$ does not have roots on the unit circle. This is consistent with (3).

At this point it becomes more convenient to express processes in terms of an operator B, called the backshift operator, defined as:

$$B^k X_n = X_{n-k},\ B^0 = I,\ k \in \mathbb{Z}$$

We write:

$$\varphi(B)X_n = \theta(B)Z_n \tag{7}$$

where $\varphi(z) = 1 - \varphi_1 z - ... - \varphi_p z^p$
and $\theta(z) = 1 + \theta_1 z + ... + \theta_p z^p$

ARIMA(p, d, q)

Adding an integrated component to the ARMA(p,q) process, we get the following process we call ARIMA(p,d,q)

$$\varphi(B)(1-B)^d X_n = \theta(B)Z_n \tag{8}$$

II. Definition of LRD in Terms of ACF

Motivating Example for LRD

FARIMA(0, d, 0)

Attempting to extend the definition of ARIMA where d is only integer valued, we define a FARIMA(0, d, 0) process X_n as

$$X_n = (I-B)^{-d} Z_n$$

For $d \in I$ where I is some interval to be determined later. Notice that we can also write this as

$$X_n = \sum_{j=0}^{\infty} b_j^{(-d)} B^j Z_n = \sum_{j=0}^{\infty} b_j^{(-d)} Z_{n-j} \tag{9}$$

where

$$b_j^{(-d)} = \frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(d)} \tag{10}$$

are the coefficients of the Taylor expansion of the above $(I-B)^{-d}$

As we have seen before in I.2, (9) is well defined if

$$\sum_{j=0}^{\infty} b_j^2 < \infty \tag{11}$$

Using Stirling's approximation formula, we find that (9) is well defined for $d < \frac{1}{2}$. So $I = (-\infty, 1/2)$

The F in FARIMA stands for Fractional.

Values of d

If $d < 0$ but not a negative integer,

$$\sum_{h=-\infty}^{\infty} |\gamma_X(h)| \leq \sigma^2 \left(\sum_j |b_j| \right)^2 < \infty$$

If $d = -1$ then $X_n = Z_n - Z_{n-1} = \Delta Z_n$ and hence for $d = -1, -2, ...$, $X_n = \Delta^{-d} Z_n$. Hence in these cases, $\gamma_X(h) = 0$ for $|h| > |d|$ by Remark 1.

Hence, for $d < 0$ the autocovariances of the times series are absolutely summable.

If $d = 0$, then $X_n = Z_n$ is just white noise, so the autocovariances are absolutely summable.

If $0 < d < 1/2$ then the autocovariances are not absolutely summable.

Definitions

Short Range Dependence

A time series X_n is short range dependent (SRD) if

$$\sum_{h=-\infty}^{\infty} |\gamma_X(h)| < \infty$$

That is, the so called long-run variance of the times series is constant. This corresponds to $d \leq 0$ above.

A special case of SRD time series is when the long-run variance is 0. Such a process is called antipersistent.

Long-Range Dependence

Condition I. The times series X_n has a linear representation

$$X_n = \mu_X + \sum_{k=0}^{\infty} \psi_k Z_{n-k} = \mu_X + \sum_{k=-\infty}^{\infty} \psi_{n-j} Z_j$$

with $\{\psi_k\}_{k \geq 0}$ satisfying

$$\psi_k = L_1(k) k^{d-1}$$

where $d \in (0, 1/2)$ and $L_1(k)$ is positive on $[c, \infty]$ for some $c \geq 0$ satisfies, for any $a > 0$

$$\lim_{k \to \infty} \frac{L(ak)}{L(k)} = 1$$

we call such a function slowly varying at infinity.

Condition II. The autocovariance function of X_n satisfies

$$\gamma_X(k) = L_2(k) k^{2d-1},\ k = 0, 1, 2, ...$$

with d as before and L_2 slowly varying at infinity.

Condition III. The autocovariances of the time series of X_n are not absolutely summable, that is,

$$\sum_{h=-\infty}^{\infty} |\gamma_X(h)| = \infty$$

Condition V. The time series X_n satisfies

$$\text{Var}(X_1 + ... + X_N) = L_5(N) N^{2d+1}$$

Definition A weak stationary time series X_n is called long-range dependent (LRD) if one of the non-equivalent conditions above are satisfied. The parameter $d \in (0, 1/2)$ is called the long-range dependence parameter.

Note: It can be shown that

$$\text{I} \implies \text{II} \implies \text{III, V}$$

Examples

ARMA(p,q)

Can show that the autocovariances of an ARMA(p,q) are not absolutely summable. Hence ARMA(p,q) processes are SRD.

Markov Chains with finite state space

Let $X = X_n$ be a Markov Chain with finite state space. Let P be the transition matrix. Assume X is is irreducible (all states are connected) and aperiodic. Then, let $\pi_i > 0,\ i = 1, ..., m$ be the invariant probabilities.

A Markov Chain X with initial distribution $\mathbb{P}(X_0 = i) = \pi_i$ is strictly stationary

It can be shown that if P has distinct eigenvalues, then

$$|\gamma_{g(X)}(k)| \leq Cs^k, \text{ some } 0 < s < 1$$

for some function g and some s that depends on the eigenvalues of P. That is, $g(X_n)$ has at least exponentially decreasing autovariances and hence is SRD.

FARIMA(p,d,q)

Definition The series X_n is called FARIMA(p, d, q) if it satisfies

$$X_n = \varphi^{-1}(B)\theta(B)(I - B)^{-d} Z_n$$

with $d < 1/2$

Remark The operator $(1 - B)^{-d}$ is responsible for LRD features of X_n while $\varphi^{-1}(B)\theta(B)$ is responsible for SRD behavior.

If the $\varphi(z)$ has roots outside of the unit circle, then the FARIMA(p, d, q) series with $0 < d < 1/2$ is LRD in the sense of conditions I, II, III, and V.

III. Definition of LRD in Terms of Self-Similarty

Self-Similar Stationary Increment

Definitions

Definition A process $\{X(t)\}_{t\in\mathbb{R}}$ is self-similar if there is an $H > 0$, called the self-similarity parameter such that for all $c > 0$,

$$\{X(ct)\}_{t\in\mathbb{R}} \overset{d}{=} \{c^H X(t)\}_{t\in\mathbb{R}} \tag{12}$$

Theorem If $X = X(t)$ is H-SS, then $Y(t) = e^{-tH}X(e^t)$ is stationary. Conversely, If $Y = Y(t)$ is stationary, then $X(t) = t^H X(\log(t)))$ is H-SS.

Definition A process $\{X(t)\}_{t\in\mathbb{R}}$ has stationary increments if, for any $h \in \mathbb{R}$

$$\{X(t+h) - X(t)\}_{t\in\mathbb{R}} \overset{d}{=} \{X(t) - X(0)\}_{t\in\mathbb{R}}$$

Proposition Let $\{X(t)\}_{t\in\mathbb{R}}$ be an H-SSSI process. Then, if $\mathbb{E}|X(1)|^2 < \infty$, then for $s, t \in \mathbb{R}$

$$\mathbb{E}X(t)X(s) = \frac{\mathbb{E}X(1)^2}{2}\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\}$$

Definition A Gaussian H-SSSI process $\{B_H(t)\}_{t\in\mathbb{R}}$ with $0 < H \leq 1$ is called Fractional Brownian Motion.

Corollary If $\{X(t)\}_{t\in\mathbb{R}}$ satisfies the following conditions:

(i) It is a Gaussian process with zero mean and X(0) = 0

(ii) $\mathbb{E}(X(t))^2 = \sigma^2|t|^{2H}$ for all $t \in \mathbb{R}$, $\sigma > 0$, and $0 < H \leq 1$

(iii) $\mathbb{E}(X(t) - X(s))^2 = \mathbb{E}(X(t-s))^2$ for all $s, t \in \mathbb{R}$

Then, $\{X(t)\}_{t\in\mathbb{R}}$ is fractional Brownian Motion.

Time Domain Representation of FBM

Proposition For $H \in (0, 1)$, standard FBM admits the following integral representation

$$\{B_H(t)\}_{t\in\mathbb{R}} =$$

$$\{\frac{1}{c_1(H)}\int_{\mathbb{R}}\left((t-u)_+^{H-1/2} - (-u)_-^{H-1/2}\right)B(du)\}_{t\in\mathbb{R}} \tag{13}$$

Proof Must show that RHS is well defined (i.e. the integrand is square integrable), and show that the three conditions of the Corollary are satisfied.

Values of H If $1/2 < H < 1$, then we can express (13) as

$$\{\frac{H-1/2}{c_1(H)}\int_{\mathbb{R}}\left(\int_0^t (s-u)_+^{H-3/2} ds\right)B(du)\}_{t\in\mathbb{R}}$$

If $H = 1/2$ then FBM reduces to Brownian Motion.

Bifractional Brownian Motion

Definition Bifractional Brownian Motion is a natural extension to FBM and is defined as a zero mean Gaussian process with covariance function

$$\mathbb{E}B_{H,K}(t)B_{H,K}(s) = \frac{1}{2^K}\left((t^{2H} + s^{2H})^K - |t-s|^{2HK}\right)$$

where $H, K \in (0, 1)$

Note When $K = 1$, or $H = 1/2$, biFBM reduces to FBM.

Proposition Let $B_{H,K}$ be biFBM, then:
(i) It is (HK)-self-similar
(ii) It has stationary increments iff $K = 1$
(iii) For all $s, t \geq 0$,

$$\frac{|t-s|^{2HK}}{2^K} \leq \mathbb{E}(B_{H,K}(t) - B_{H,K}(s))^2 \leq \frac{|t-s|^{2HK}}{2^{1-K}}$$

LRD from SSSI

H-SSSI differencing

Let $Y = Y(t)$ be an H-SSSI process with $0 < H < 1$ and consider the stationary process

$$X_n = Y(n) - Y(n-1),\ n \in \mathbb{Z}$$

Notice that X_n has zero mean, $\mathbb{E}X_n^2 = \mathbb{E}Y(1)^2$, and

$$\gamma_X(k) = \frac{\mathbb{E}X(1)^2}{2}\{|k+1|^{2H} + 2|k|^{2H} - |k-1|^{2H}\}$$

$$\sim \mathbb{E}Y(1)^2 H(2H-1)k^{2H-2}$$

Consequence

By the above observation, if $1/2 < H < 1$ then the series X is LRD in the sense of condition II (and hence III) with

$$d = H - \frac{1}{2} \in \left(0, \frac{1}{2}\right) \tag{14}$$

So d and H are closely related.
Note When Y is FBM, X is called fractional Gaussian Noise.

IV. Bibliography

Long-Range Dependence and Self-Similarity (Cambridge Series in Statistical and Probabilistic Mathematics) **1st Edition, 2017**, Murad Taqqu, Vladas Pipiras