Distance & Similarity

Boston University CS 506 - Lance Galletti

Data

$$\left. \begin{array}{c} \\ \\ \\ n \text{ data} \\ \text{points} \\ \\ \\ \end{array} \right\{ \begin{pmatrix} x_{11} & \dots & x_{1j} & \dots & x_{1m} \\ \vdots & \ddots & \vdots & & \vdots \\ x_{i1} & \dots & x_{ij} & \dots & x_{im} \\ \vdots & & \vdots & \ddots & \vdots \\ x_{n1} & \dots & x_{nj} & \dots & x_{nm} \end{pmatrix}}_{m \text{ features}}$$

Feature Space

From our data we can generate a **feature space** of all possible values for the set of features in our data.

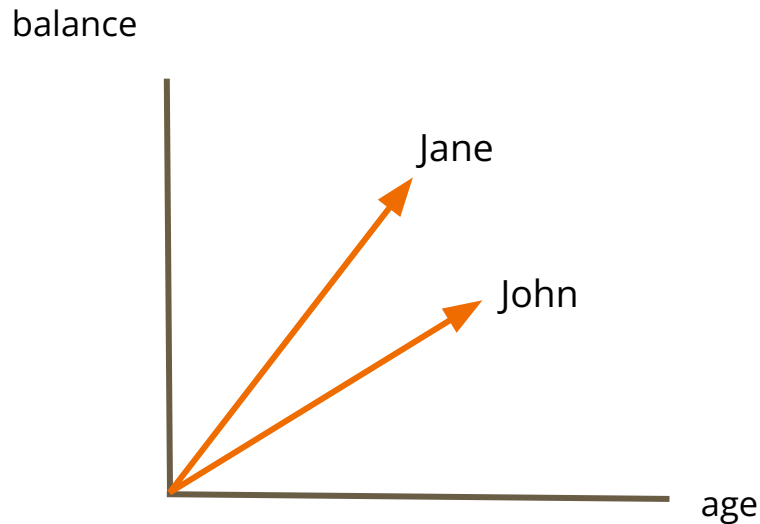| name	age	balance
Jane	25	150
John	30	100

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name	age	balance
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balance

age

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Our feature space is the Euclidean plane

Dissimilarity

In order to uncover interesting structure from our data, we need a way to **compare** data points.

A **dissimilarity function** is a function that takes two objects (data points) and returns a **large value** if these objects are **dissimilar**.

A special type of dissimilarity function is a **distance** function

Distance

d is a distance function if and only if:

- $d(i, j) = 0$ if and only if $i = j$
- $d(i, j) = d(j, i)$
- $d(i, j) \leq d(i, k) + d(k, j)$

We don't **need** a distance function to compare data points, but why would we prefer using a distance function?

Minkowski Distance

For **x**, **y** points in **d**-dimensional real space

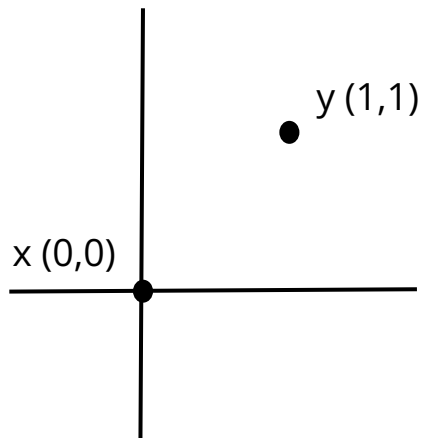I.e. **x = [x$_1$, … , x$_d$]** and **y = [y$_1$, … , y$_d$]**

p ≥ 1

$$L_p(x, y) = \left(\sum_{i=1}^{d} |x_i - y_i|^p \right)^{\frac{1}{p}}$$

When **p** = 2 -> Euclidean Distance

When **p** = 1 -> Manhattan Distance

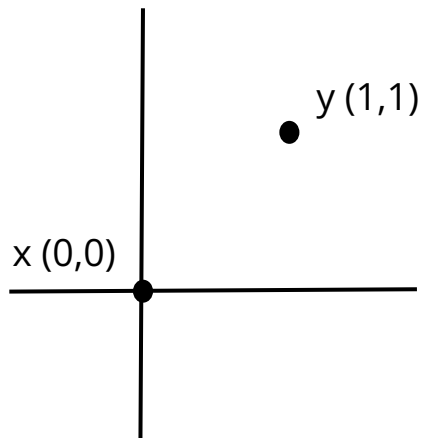# Example

d = 2

Example

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x (0,0)

y (1,1)

p = 2

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p = 1
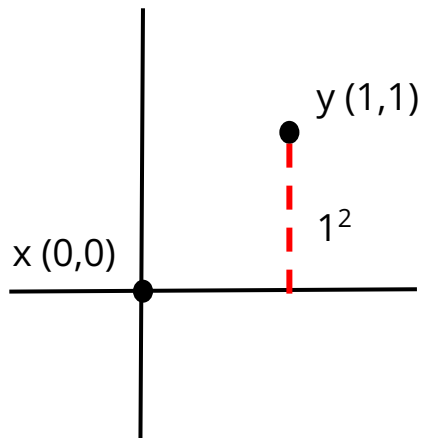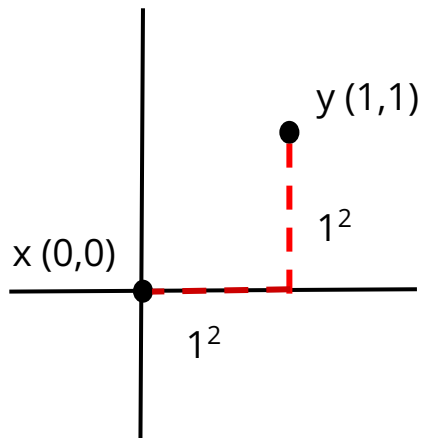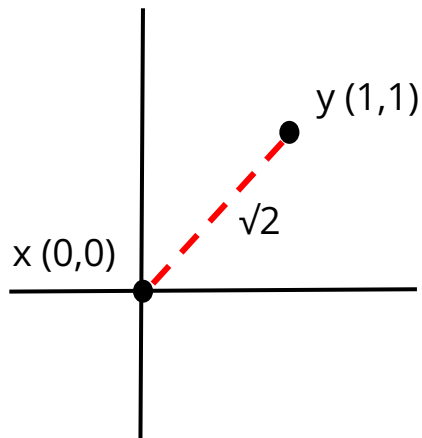
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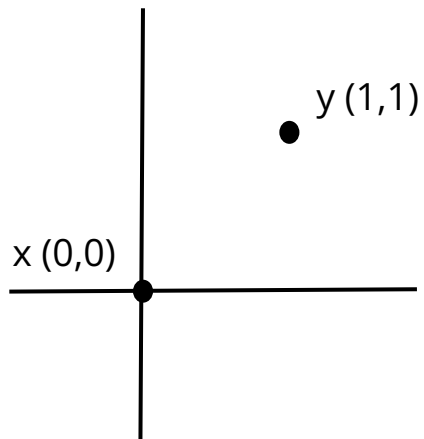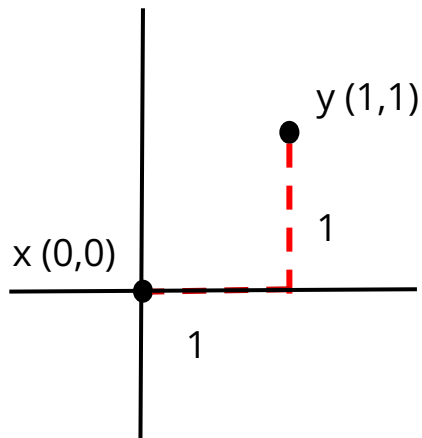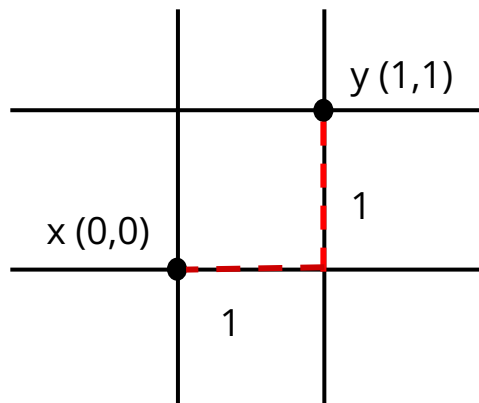
Example

d = 3

x (0,0,0)

y (1,1,1)

p = 2

$$L_p(x, y) = \left(\sum_{i=1}^{d} |x_i - y_i|^p \right)^{\frac{1}{p}}$$

Example

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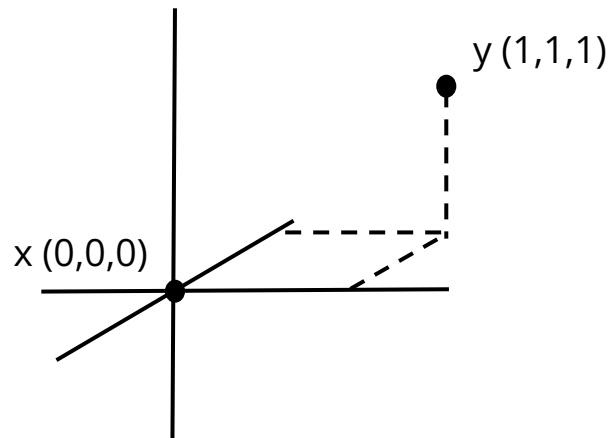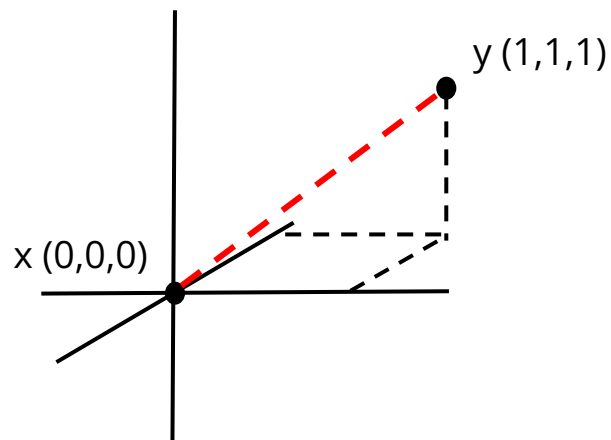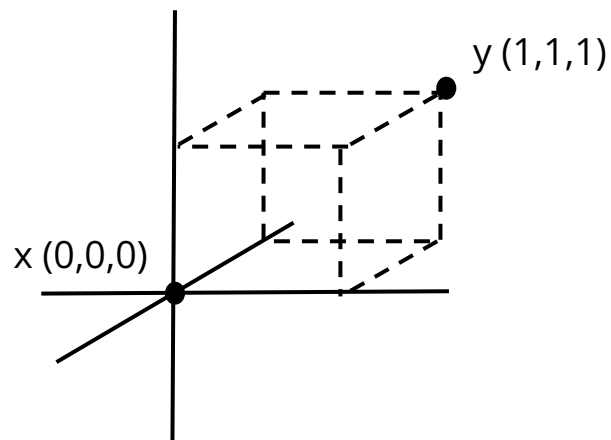
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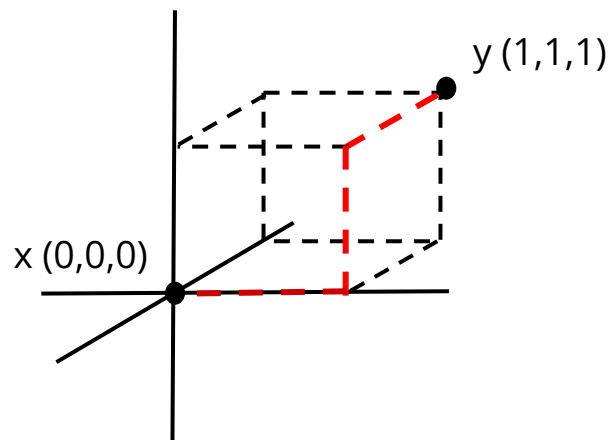
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Minkowski Distance

Is L_p a distance function when $0 < p < 1$?

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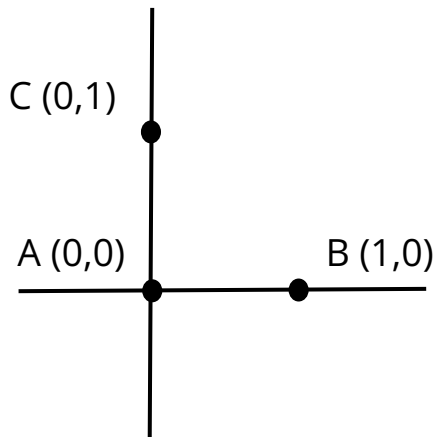
Minkowski Distance

Is L_p a distance function when **0 < p < 1** ?

D(B,A) = D(A, C) = 1

D(B, C) = $2^{1/p}$

Minkowski Distance

Is L_p a distance function when **0 < p < 1** ?

C (0,1)

A (0,0)   B (1,0)

D(B,A) + D(A, C) = 2

D(B, C) = 2$^{1/p}$

But... if **p < 1** then **1/p > 1**

Minkowski Distance

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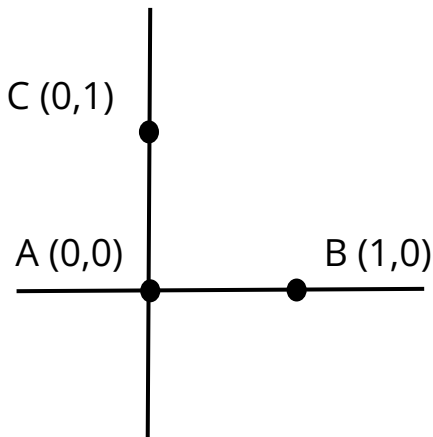
So **D(B, C) > D(B, A) + D(A, C)** which violates the triangle inequality

Cosine Similarity

A **similarity** function is a function that takes two objects (data points) and returns a **large value** if these objects are **similar**.

$$s(x, y) = \cos(\theta)$$

where θ is the angle between **x** and **y**

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Cosine Similarity

To get a corresponding **dissimilarity** function, we can usually try

$$d(x, y) = 1 / s(x, y)$$

or

$$d(x, y) = k - s(x, y) \text{ for some } k$$

Here, we can use

$$d(x, y) = 1 - s(x, y)$$
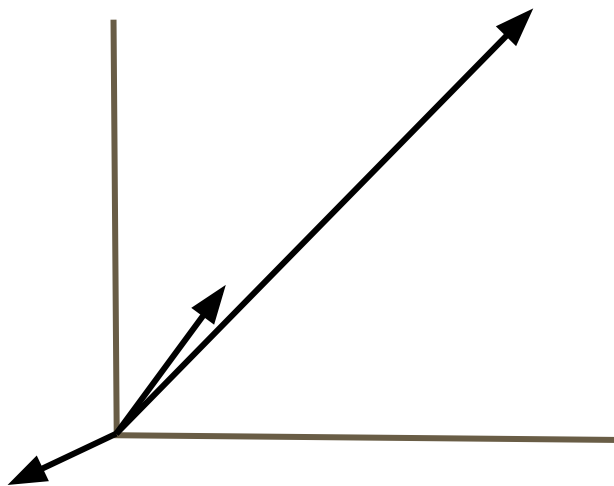
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When should you use **cosine (dis)similarity** over **euclidean distance**?

When **direction** matters more than **magnitude**

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Euclidean distance
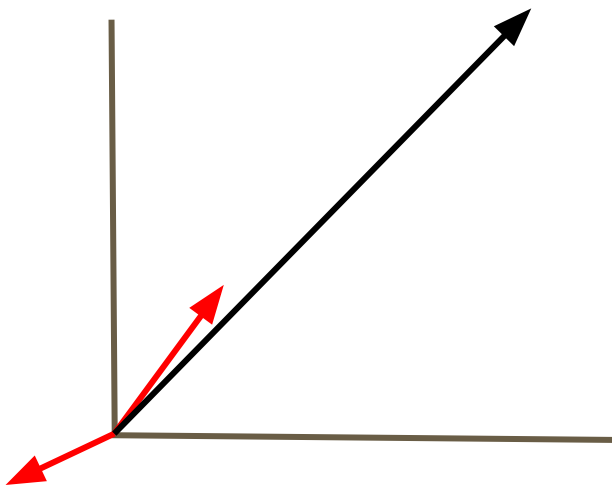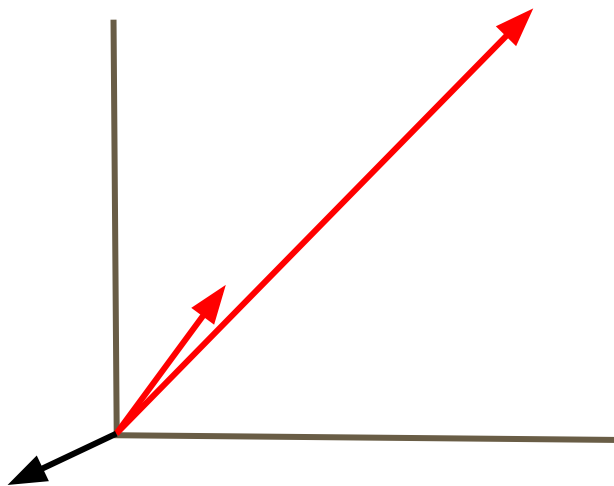
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Close under Cosine
Similarity

Jaccard Similarity

How similar are the following documents?

	w_1	w_2	...	w_d
x	1	0	...	1
y	1	1	...	0

Jaccard Similarity

One way is to use the Manhattan distance which will return the size of the set difference

	w_1	w_2	...	w_d
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y	1	1	...	0

$$L_1(x, y) = \sum_{i=1}^{d} |x_i - y_i|$$

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Will only be 1 when $x_i \neq y_i$

Jaccard Similarity

But how can we distinguish between these two cases?

	w_1	w_2	...	w_{d-1}	w_d
x	1	1	1	0	1
y	1	1	1	1	0

Only differ on the last two words

	w_1	w_2
x	0	1
y	1	0

Completely different

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Completely different

Both have Manhattan distance of 2

Jaccard Similarity

We need to account for the size of the intersection!
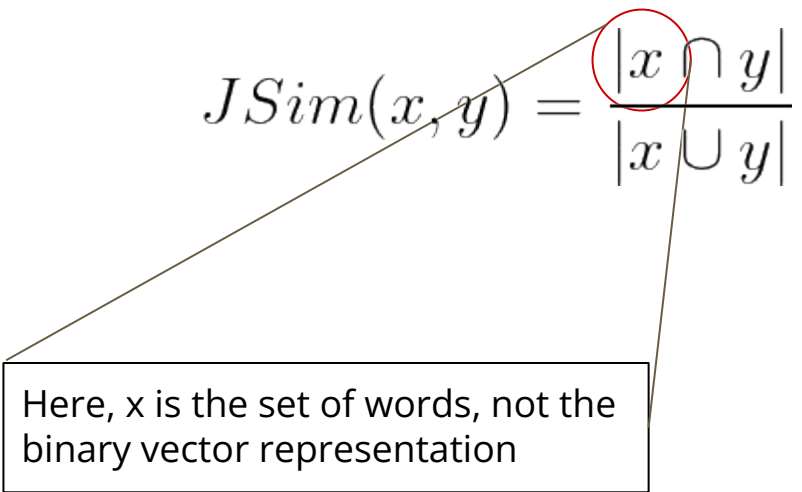
Given two documents x and y:

$$JSim(x, y) = \frac{|x \cap y|}{|x \cup y|}$$

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$$JDist(x, y) = 1 - \frac{|x \cap y|}{|x \cup y|}$$

Here, x is the set of words, not the binary vector representation

Jaccard Similarity

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	w_1	w_2
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What is the jaccard distance in each?

Jaccard Similarity

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A quick note on Norms

- Distance from the origin
  - Minkowski Distance <=> Lp Norm
  - Not all distances can create a norm

- Notion of size
- Has the following properties:
  - $p(x + y) \leq p(x) + p(y)$
  - $p(ax) = |a|p(x)$
  - $p(x) = 0$ iff $x = 0$
  - $p(x) \geq 0$ for all x