Basics of Vectors and Matrices

This handout is meant to give a quick overview of the basic facts you need to know about vectors and matrices.

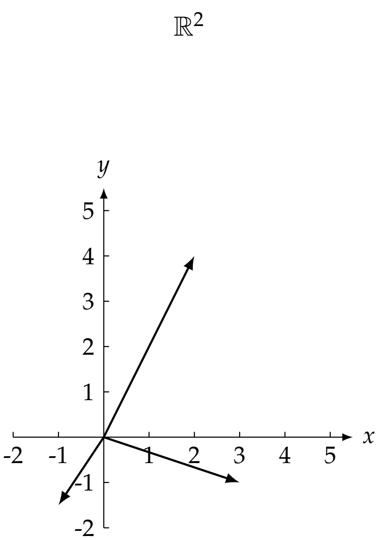
What is a vector?

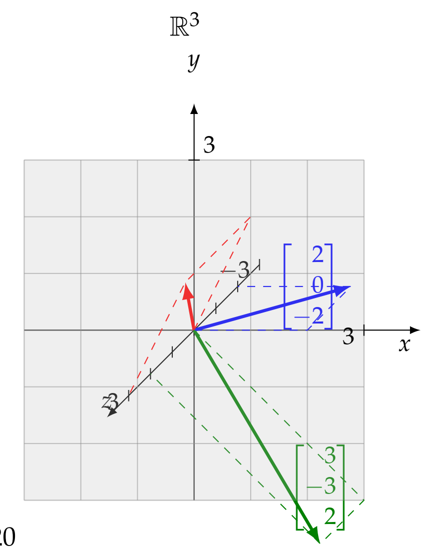
The simplest way to think about a vector is as a list of numbers, written in a column. For example, $\begin{bmatrix}1\\2\end{bmatrix}$, $\begin{bmatrix}1\\3\\-4\end{bmatrix}$, and $\begin{bmatrix}2.01\\1/3\\\sqrt{7}\\-\pi\\e\end{bmatrix}$ are all vectors. We refer to the individual numbers in a vector as the <u>components</u> or <u>entries</u> of the vector. We use $\mathbb{R}^n$ to refer to all vectors with n entries, where the entries are real numbers (the symbol $\mathbb{R}$ stands for "the real numbers").

When we want to give a vector a variable name, we typically use a letter with an arrow over it, like $\vec{v}$. So, for instance, if you see "$\vec{v}$ is a vector in $\mathbb{R}^3$," then you know that $\vec{v}$ is a vector with 3 components.

Visualizing vectors

We often visualize vectors as arrows, drawn *with their tails at the origin*. We think of vectors in $\mathbb{R}^2$ as arrows in the plane and vectors in $\mathbb{R}^3$ as arrows in space. For example, here are a few different vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$:





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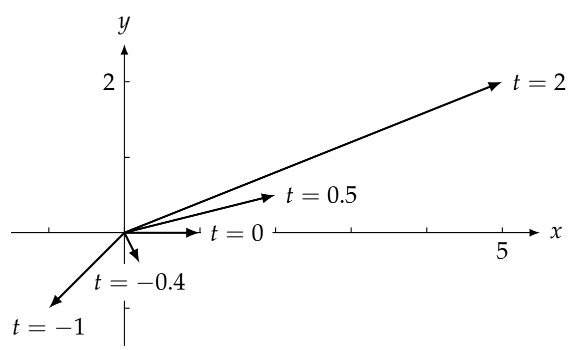
Although we can't visualize in more than 3 dimensions, we still think of vectors in $\mathbb{R}^5$ or $\mathbb{R}^{100}$ as arrows.

Visualizing infinite sets of vectors

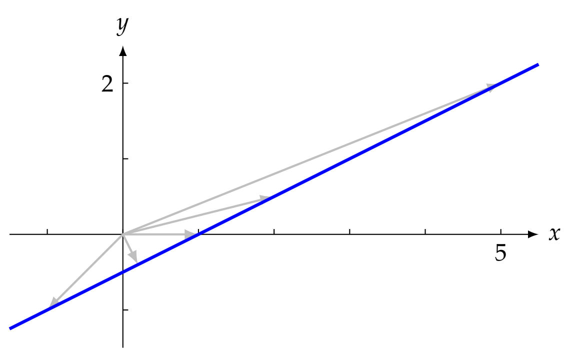
In linear algebra, we will often look at infinite sets of vectors. For example, we might consider the set of vectors of the form $\begin{bmatrix} 1+2t \\ t \end{bmatrix}$ where t is any real number. When visualizing an infinite set of vectors, we visualize all of the vectors (with their tails at the origin, as usual) and describe the shape formed by their heads.

Let's look at the set S of all vectors of the form $\begin{bmatrix} 1+2t \\ t \end{bmatrix}$ where t is a real number.

As a start, we can try some different values of t just to get an idea of what vectors are in this set. For example, when $t = 0$, $\begin{bmatrix} 1+2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; when $t = 0.5$, $\begin{bmatrix} 1+2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0.5 \end{bmatrix}$. Let's draw several of these:



However, what we really want to visualize is what the heads of all of these arrows, together with the infinitely many that we didn't draw, form. In this case, it's a line (we'll see in Example how we can be sure of this):

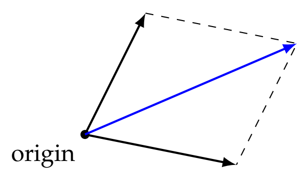


So, the infinite set S is really a line (more specifically, it's the line $y = \dfrac{x-1}{2}$).

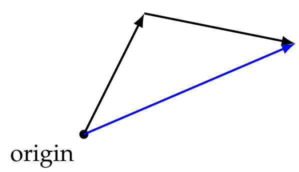
Vector arithmetic: adding and multiplying by scalars

To add vectors, simply add the corresponding entries. For example, $\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}\pi\\\sqrt{3}\\-7\end{bmatrix} = \begin{bmatrix}1+\pi\\2+\sqrt{3}\\-4\end{bmatrix}$. Note that you can only add vectors with the same number of components; $\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}3\\4\\5\end{bmatrix}$ is not defined.

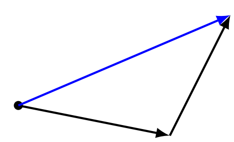
Vector addition has a nice visualization; if you have two vectors $\vec{v}$ and $\vec{w}$, you can make a parallelogram with $\vec{v}$ and $\vec{w}$ as two of the sides; then, $\vec{v}+\vec{w}$ is the vector that goes from the origin to the opposite vertex of the parallelogram:



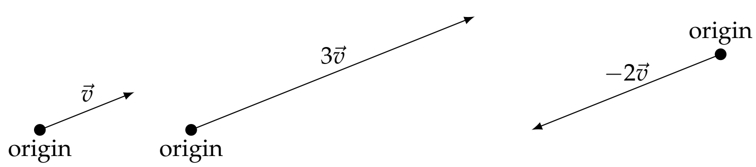
It is often convenient to draw only a triangle instead of the entire parallelogram when we're trying to visualize the sum of two vectors, as in the following two pictures:



OR

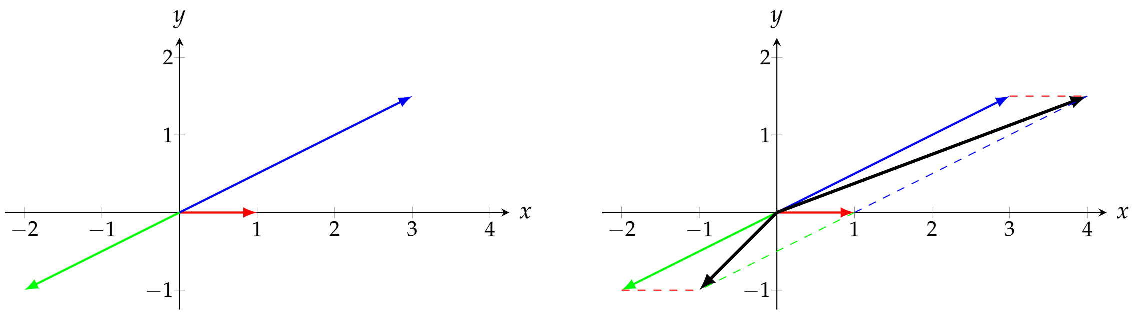


A <u>scalar</u> is simply another word for a number. To multiply a vector by a scalar, multiply each component of the vector by the scalar. For example, $2\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}2\\4\\6\end{bmatrix}$. Graphically, $k\vec{v}$ can be interpreted like this: if $k \geq 0$, then $k\vec{v}$ is the vector that goes in the same direction as $\vec{v}$ but is k times as long as $\vec{v}$; if $k < 0$, then $k\vec{v}$ is the vector that goes in exactly the opposite direction from $\vec{v}$ and is $|k|$ times as long:



These two operations, addition and scalar multiplication, will be key to our study of linear algebra.

In Example , we looked at the set S of all vectors of the form $\begin{bmatrix}1+2t\\t\end{bmatrix}$. Now that we know some vector algebra, we can rewrite this: $\begin{bmatrix}1+2t\\t\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}2t\\t\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} + t\begin{bmatrix}2\\1\end{bmatrix}$. Since we know how to interpret both addition and scalar multiplication graphically, this gives us an easier way to think about the set S. To get any vector of the form $\begin{bmatrix}1\\0\end{bmatrix} + t\begin{bmatrix}2\\1\end{bmatrix}$, we add $\begin{bmatrix}1\\0\end{bmatrix}$ to a scalar multiple of $\begin{bmatrix}2\\1\end{bmatrix}$. In the left picture below, two such scalar multiples are shown in green and blue. If we add each to $\begin{bmatrix}1\\0\end{bmatrix}$ (which is shown in red), we get the two black vectors in the right picture.



As you should see, if we look at all such sums, we get exactly the line we drew in Example . If you're not convinced yet, take a look at the interactive demo at https://www.geogebra.org/m/jjWBx4N3.

Dot product, angles, and lengths

Although we can multiply a vector by a scalar, we cannot multiply two vectors together. However, there is an operation called the dot product, which is quite useful in describing the geometry of vectors.

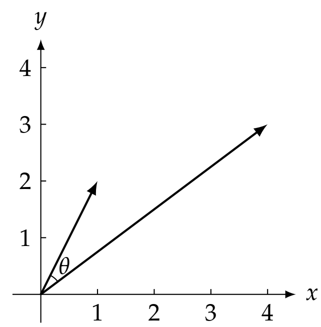
Definition 1. *The dot product of two vectors in $\mathbb{R}^n$, denoted using the symbol $\cdot$, is defined by* $\begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix} \cdot \begin{bmatrix}y_1\\ \vdots \\ y_n\end{bmatrix} = x_1y_1 + \cdots + x_ny_n$.

$$\begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}4\\5\\6\end{bmatrix} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32.$$

There are two things to note: the dot product of two vectors is a scalar, and we can only dot two vectors that have the same number of entries. The dot product has two main applications:

- The length of a vector $\vec{v}$, denoted $\|\vec{v}\|$, is $\sqrt{\vec{v} \cdot \vec{v}}$. A vector whose length is 1 is called a unit vector.
- The angle θ between two vectors $\vec{v}, \vec{w}$ is defined by the formula $\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos\theta$.

Suppose we'd like to find the angle θ between the vectors $\vec{v} = \begin{bmatrix}1\\2\end{bmatrix}$ and $\vec{w} = \begin{bmatrix}4\\3\end{bmatrix}$ in $\mathbb{R}^2$.



Since $\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos\theta$, we start by calculating $\vec{v} \cdot \vec{w}$, $\|\vec{v}\|$, and $\|\vec{w}\|$:

$$\begin{aligned} \vec{v} \cdot \vec{w} &= (1)(4) + (2)(3) = 10 \\ \|\vec{v}\| &= \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{(1)(1) + (2)(2)} = \sqrt{5} \\ \|\vec{w}\| &= \sqrt{\vec{w} \cdot \vec{w}} = \sqrt{(4)(4) + (3)(3)} = 5 \end{aligned}$$

Therefore, $10 = (\sqrt{5})(5)\cos\theta$, so $\cos\theta = \dfrac{10}{5\sqrt{5}} = \dfrac{2}{\sqrt{5}}$, and $\theta = \arccos\left(\dfrac{2}{\sqrt{5}}\right) \approx 26.6°$.

We say that two vectors in $\mathbb{R}^n$ are perpendicular or orthogonal if the angle between them is 90°.

Matrices

In linear algebra, one of the fundamental objects we study is the matrix. For now, you can think of a matrix simply as a rectangular grid of numbers. For example, $\begin{bmatrix} 1 & 7 & 2 \\ 0 & 3 & 5 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$ are both examples of matrices. The former has 2 rows and 3 columns, so we say it is a 2×3 matrix; the latter is a 4×2 matrix.

There are a few matrix operations that we will use frequently. First, we can **add** two matrices of the same size by adding the corresponding entries. For example, $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & -2 & \pi \\ \sqrt{2} & 1 & -3 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 3+\pi \\ 4+\sqrt{2} & 6 & 3 \end{bmatrix}$.

If two matrices have different sizes, then their sum is not defined; for example, $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 5 \\ 4 & 2 & 0 \\ 7 & 5 & 6 \end{bmatrix}$ is not defined.

We can also **multiply a matrix by a scalar**, simply by multiplying each entry of the matrix by the scalar. For example, $3\begin{bmatrix} 1 & 7 \\ \pi & e \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 21 \\ 3\pi & 3e \\ 6 & -3 \end{bmatrix}$. Order is not important here: $\begin{bmatrix} 1 & 7 \\ \pi & e \\ 2 & -1 \end{bmatrix} 3$ is also equal to $\begin{bmatrix} 3 & 21 \\ 3\pi & 3e \\ 6 & -3 \end{bmatrix}$.

Finally, we can **multiply a matrix by a vector**, which is a slightly more subtle operation. Here is the definition:

Definition 2. *Let A be an $n \times m$ matrix and $\vec{x}$ be a vector in $\mathbb{R}^m$. We can think of each column of A as a vector; if* *we call these vectors* $\vec{v}_1, \ldots, \vec{v}_m$ *(in the order they appear in the matrix), and if* $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$*, then the product* $A\vec{x}$ *is defined to be* $x_1\vec{v}_1 + \cdots + x_m\vec{v}_m$.

Let's evaluate the product $\begin{bmatrix} 1 & 3 & -1 \\ 2 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$. According to Definition 2, this is equal to $1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 3 \\ -2 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 4 \end{bmatrix}$, which simplifies to $\boxed{\begin{bmatrix} -1 \\ 10 \end{bmatrix}}$.

⚠ There are some subtleties about multiplying a matrix by a vector, and it's important to get very comfortable with these:

- You can only multiply an $n \times m$ matrix by a vector with m components, and the result will have n components.
- Order *is* important: for example, if A is an 5×3 matrix and $\vec{x}$ is a vector in $\mathbb{R}^3$, $A\vec{x}$ is defined, but $\vec{x}A$ is not.

Optional – Cross product and scalar triple product ($\mathbb{R}^3$ only)

There is one last vector operation, the cross product, that is handy to know about; however, you don't need to memorize its definition as we will never rely on it in 21b. It applies *only* to vectors in $\mathbb{R}^3$.

Definition 3. *If* $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ *and* $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$*, their cross product* $\vec{v} \times \vec{w}$ *is* $\begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix}$.

Notice that the cross product of two vectors in $\mathbb{R}^3$ is another vector in $\mathbb{R}^3$. The cross product is used primarily to generate a vector perpendicular to two given vectors in $\mathbb{R}^3$: $\vec{v} \times \vec{w}$ is perpendicular to both $\vec{v}$ and $\vec{w}$.

$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ -1 \end{bmatrix}$. You can compute some dot products to verify that $\begin{bmatrix} 7 \\ -3 \\ -1 \end{bmatrix}$ is orthogonal to both $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$.

Functions

Many branches of mathematics are concerned with studying *functions* with particular properties. For example, single variable calculus is largely concerned with studying functions of one variable that are differentiable. In linear algebra, the sort of function that we study is called a linear transformation, and the goal of this handout is to explain what a linear transformation is.

If you've studied calculus, you've studied functions with different types of inputs and outputs. For example, you've studied functions like $f(x) = x^2 + 2$, where the input is a real number (x) and the output is another number. Other examples of functions are $f(x) = \sin(2x)$, $f(x, y) = x^2 + 2y$, etc.

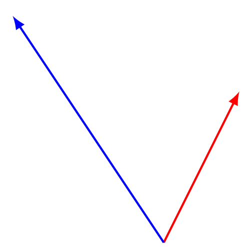
In linear algebra, we'll study functions where the input and output are both vectors; such functions are often also called transformations. (A linear transformation is a special kind of transformation, as we'll explain soon.) Our functions will often be named T (for "transformation"), and we will write $T : \mathbb{R}^m \to \mathbb{R}^n$ to mean that the inputs of T are vectors in $\mathbb{R}^m$ and the outputs of T are vectors in $\mathbb{R}^n$. We call $\mathbb{R}^m$ the domain of the function and $\mathbb{R}^n$ the codomain. For example, we could define a function $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \\ x_1 x_2 \end{bmatrix}$. This function has domain $\mathbb{R}^2$ (that is, the inputs are vectors in $\mathbb{R}^2$) and codomain $\mathbb{R}^3$ (the outputs are vectors in $\mathbb{R}^3$).

Visualizing a function $T : \mathbb{R}^m \to \mathbb{R}^n$

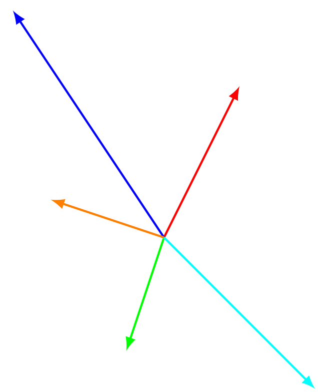
When you studied calculus, you probably visualized functions by drawing their graphs. This is too hard to do for functions $T : \mathbb{R}^m \to \mathbb{R}^n$, so we use a different type of picture to visualize such functions.

Consider the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$.

From the definition of this function, we see that, for example, $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $T\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$. To visualize this, we draw a diagram showing both the domain and the codomain, with the domain on the left. We can then draw some input vectors in the domain and the corresponding output vectors in the codomain:

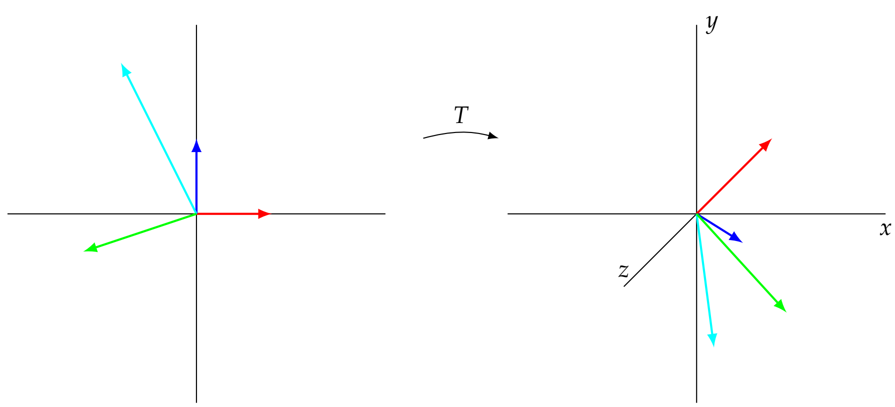


For a better picture, we can draw more inputs and the corresponding outputs:



From this picture, you can probably see that T actually has a simple description in words: $T(\vec{v})$ is the reflection of $\vec{v}$ over the x-axis in $\mathbb{R}^2$. (Most functions $T : \mathbb{R}^m \to \mathbb{R}^n$ do not have a simple geometric interpretation like this, but the ones that do are valuable examples.)

Consider the function $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix}$. Again, we can visualize this by drawing what T does to several vectors in the domain. For example, $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$; of course, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is a vector in $\mathbb{R}^3$, so we must visualize it in 3-dimensional space (which is hard to do on a 2-dimensional sheet of paper!).



Here's an interactive version of this picture (where you can rotate the codomain to see it more easily).

As you can imagine, this visualization is only useful in relatively simple cases. After all, it's impossible to visualize $\mathbb{R}^7$, so we have no hope of visualizing a function $T : \mathbb{R}^7 \to \mathbb{R}^2$. Nonetheless, visualizing in simple cases will help you build up your intuition so that you can understand what's happening when we study functions from $\mathbb{R}^{10}$ to $\mathbb{R}^{17}$, even if you can't visualize them directly!

Linear Transformations

Linear transformations are a special class of functions from $\mathbb{R}^m$ to $\mathbb{R}^n$; here is the definition.

Definition 4. *A linear transformation is a function $T : \mathbb{R}^m \to \mathbb{R}^n$ satisfying the following two properties:*

1. *$T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ for all $\vec{x}, \vec{y}$ in $\mathbb{R}^m$. (In words, we say that "T preserves addition".)*
2. *$T(k\vec{x}) = kT(\vec{x})$ for all $\vec{x}$ in $\mathbb{R}^m$ and all scalars (real numbers) k. (In words, we say that "T preserves scalar multiplication".)*

This definition probably seems quite abstract at the moment; an excellent habit when you read an abstract definition is to think about some examples, so let's do that now.

Consider the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 3x_2 \\ 2x_1 - x_2 \end{bmatrix}$. We'd like to decide whether this is a linear transformation. Reading Definition 4, we see that we need to decide whether T preserves addition and scalar multiplication.

- *Does T preserve addition? That is, is it true that $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ for all $\vec{x}, \vec{y}$ in $\mathbb{R}^2$?*

  To decide this, let's write everything out. If we write $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, then

  $$\begin{aligned} T(\vec{x} + \vec{y}) &= T\left(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} (x_1 + y_1) + 3(x_2 + y_2) \\ 2(x_1 + y_1) - (x_2 + y_2) \end{bmatrix} \quad \text{by definition of } T \end{aligned}$$

  On the other hand,

  $$\begin{aligned} T(\vec{x}) + T(\vec{y}) &= T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + T\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} x_1 + 3x_2 \\ 2x_1 - x_2 \end{bmatrix} + \begin{bmatrix} y_1 + 3y_2 \\ 2y_1 - y_2 \end{bmatrix} \end{aligned}$$

  If we compare the two previous equations, we can see that $T(\vec{x} + \vec{y})$ is indeed equal to $T(\vec{x}) + T(\vec{y})$. So, T preserves addition.

- *Does T preserve scalar multiplication? That is, is it true that $T(k\vec{x}) = kT(\vec{x})$ for all $\vec{x}$ in $\mathbb{R}^2$ and all scalars (real numbers) k?*

  Again, we simply write everything out. If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then

  $$T(k\vec{x}) = T\left(\begin{bmatrix} kx_1 \\ kx_2 \end{bmatrix}\right) = \begin{bmatrix} kx_1 + 3kx_2 \\ 2kx_1 - kx_2 \end{bmatrix}$$

  while

  $$kT(\vec{x}) = k\begin{bmatrix} x_1 + 3x_2 \\ 2x_1 - x_2 \end{bmatrix}$$

  So, $T(k\vec{x}) = kT(\vec{x})$, and T preserves scalar multiplication.

Since T preserves both addition and scalar multiplication, $\boxed{T \text{ is a linear transformation}}$.

Note: See the Mathematica page of the course website for applets that will help you visualize linear transformations such as this one.

Consider the function from Example , $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix}$. Let's decide whether this is a linear transformation. Again, this means deciding whether T preserves addition and scalar multiplication.

- *Does T preserve addition? That is, is it true that $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ for all $\vec{x}, \vec{y}$ in $\mathbb{R}^2$?*

  To decide this, let's write everything out. If we write $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, then

  $$T(\vec{x} + \vec{y}) = T\left(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} \text{ by definition of } T$$

  On the other hand,

  $$\begin{aligned} T(\vec{x}) + T(\vec{y}) &= T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + T\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ y_1 \\ y_2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} \end{aligned}$$

  If we compare the two previous equations, we can see that $T(\vec{x} + \vec{y})$ is **not** equal to $T(\vec{x}) + T(\vec{y})$. So, T does not preserve addition.

Since T does not preserve addition, T $\boxed{\text{is not a linear transformation}}$.

Note: Here, we showed that T does not preserve addition in general; you could also show this by giving a specific example. For example, $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, so we see that $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \neq T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$. This single example is enough to ascertain that T does not preserve addition, so T cannot be a linear transformation.

In math, we often gain insight by looking at special cases of a general idea. We know that every linear transformation preserves scalar multiplication; what does this say when the scalar is 0? If $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, then $T(0\vec{x}) = 0T(\vec{x})$ for any vector $\vec{x}$ in $\mathbb{R}^m$. This can be written more simply as $T(\vec{0}) = \vec{0}$. That is, we have shown:

Fact 5. *If $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, then $T(\vec{0}) = \vec{0}$.*

This fact sometimes gives us a quick way to show that a given function is *not* a linear transformation. For example, in Example , $T\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\\0\end{bmatrix} \neq \vec{0}$. Therefore, by Fact 5, we can immediately be sure that the function T in Example is **not** a linear transformation.

Be careful not to confuse Fact 5 with its converse(1). It's **not** true that, if $T : \mathbb{R}^m \to \mathbb{R}^n$ is a function satisfying $T(\vec{0}) = \vec{0}$, then T is a linear transformation.

$T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}x_1x_2\\x_1+x_2\end{bmatrix}$ satisfies $T(\vec{0}) = \vec{0}$, but it is not a linear transformation (can you explain why not?).

(1)If you have a statement of the form, "If A, then B", then the *converse* statement is "If B, then A". For example, the converse of the statement "If $a > 0$, then $a+1 > 0$" is "If $a+1 > 0$, then $a > 0$."