Distance & Similarity

Boston University CS 506 - Lance Galletti

at a high level
we have a
data set

Refund	Marital Status	Income	Age

Characteristics
for each data
point

Refund	Marital Status	Income	Age
1	Single	125k	25

Refund	Marital Status	Income	Age
1	Single	125k	25
0	Married	100k	27

Refund	Marital Status	Income	Age
1	Single	125k	25
0	Married	100k	27
0	Single	70k	22

Refund	Marital Status	Income	Age
1	Single	125k	25
0	Married	100k	27
0	Single	70k	22
1	Married	120k	30
0	Divorced	90k	28
0	Married	60k	37
1	Divorced	220k	24
0	Single	85k	23
0	Married	75k	23
0	Single	90k	26

Data

$$\text{\textbf{n} data points}\left\{\underbrace{\begin{pmatrix} x_{11} & \cdots & x_{1j} & \cdots & x_{1m} \\ \vdots & \ddots & \vdots & & \vdots \\ x_{i1} & \cdots & x_{ij} & \cdots & x_{im} \\ \vdots & & \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nj} & \cdots & x_{nm} \end{pmatrix}}_{\text{\textbf{m} features}}\right.$$

Data

$$\textbf{n}\text{ data points}\left\{\underbrace{\begin{pmatrix} x_{11} & \cdots & x_{1j} & \cdots & x_{1m} \\ \vdots & \ddots & \vdots & & \vdots \\ x_{i1} & \cdots & x_{ij} & \cdots & x_{im} \\ \vdots & & \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nj} & \cdots & x_{nm} \end{pmatrix}}_{\textbf{m}\text{ features}}\right.$$

Data point i

Data

Attribute / feature j

Data point i

$$\mathbf{n}\text{ data points}\left\{\underbrace{\begin{pmatrix} x_{11} & \cdots & x_{1j} & \cdots & x_{1m} \\ \vdots & \ddots & \vdots & & \vdots \\ x_{i1} & \cdots & x_{ij} & \cdots & x_{im} \\ \vdots & & \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nj} & \cdots & x_{nm} \end{pmatrix}}_{\mathbf{m}\text{ features}}\right.$$

Data

Attribute / feature j

Feature j of data point i

Data point i

$$\text{n data points}\left\{\underbrace{\begin{pmatrix} x_{11} & \cdots & x_{1j} & \cdots & x_{1m} \\ \vdots & \ddots & \vdots & & \vdots \\ x_{i1} & \cdots & \boxed{x_{ij}} & \cdots & x_{im} \\ \vdots & & \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nj} & \cdots & x_{nm} \end{pmatrix}}_{\text{m features}}\right.$$

data points rows

features are columns

Feature Space

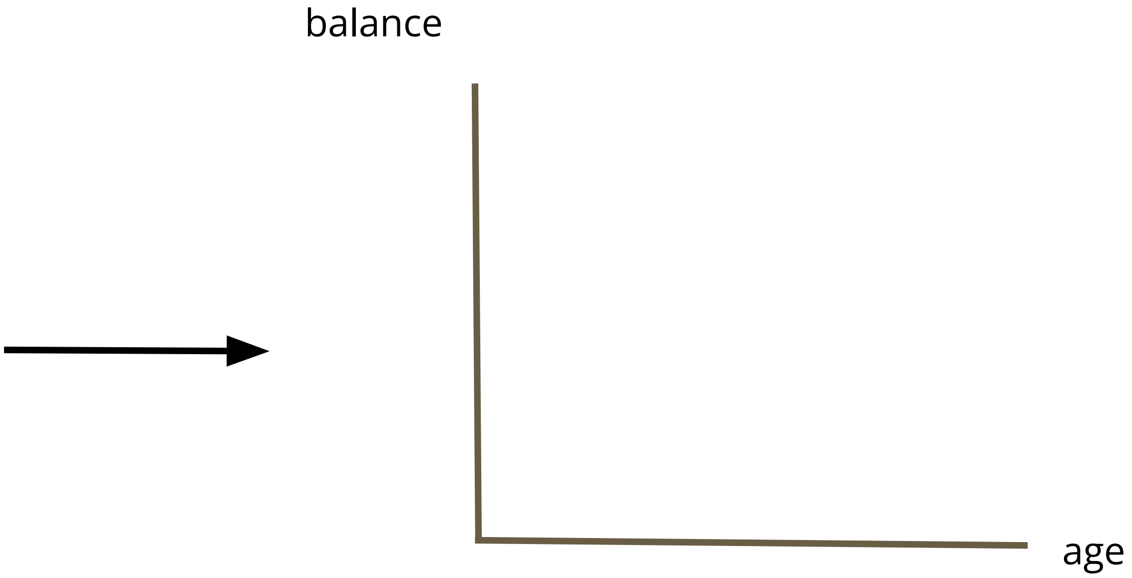
From our data we can generate a **feature space** of all possible values for the set of features in our data.

name	age	balance
Jane	25	150
John	30	100

Feature Space

From our data we can generate a **feature space** of all possible values for the set of features in our data.

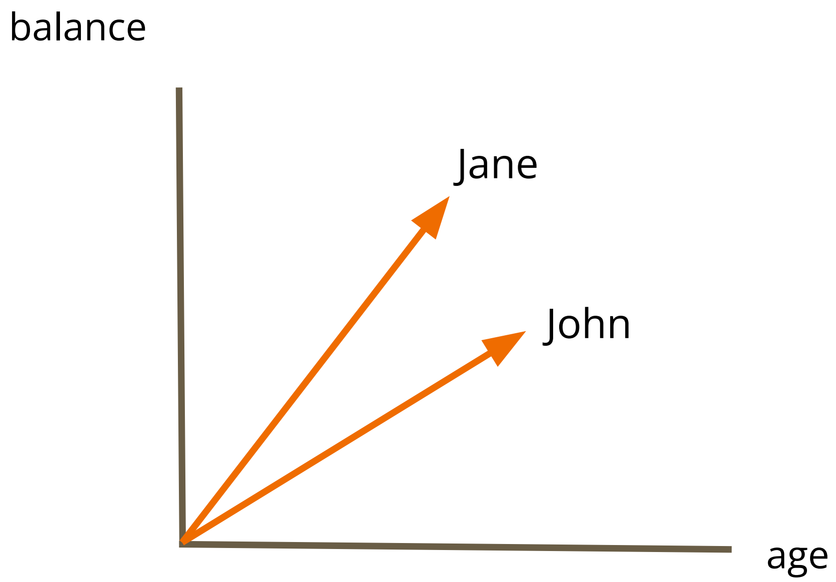
name	age	balance
Jane	25	150
John	30	100



Feature Space

From our data we can generate a **feature space** of all possible values for the set of features in our data.

name	age	balance
Jane	25	150
John	30	100



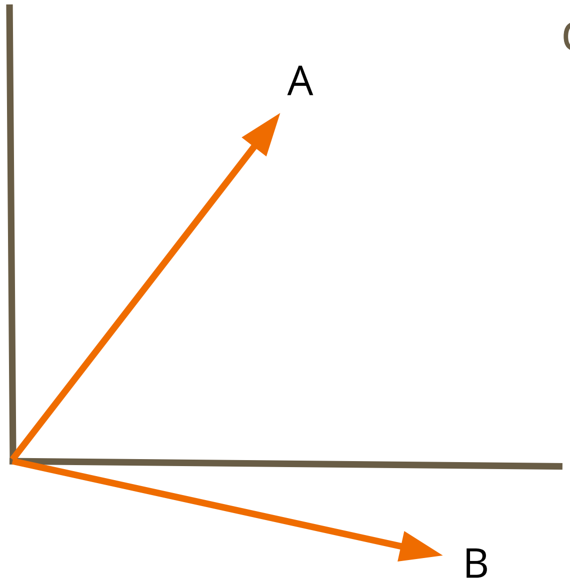
Our feature space is the Euclidean plane

Dissimilarity

In order to uncover interesting structure from our data, we need a way to **compare** data points.

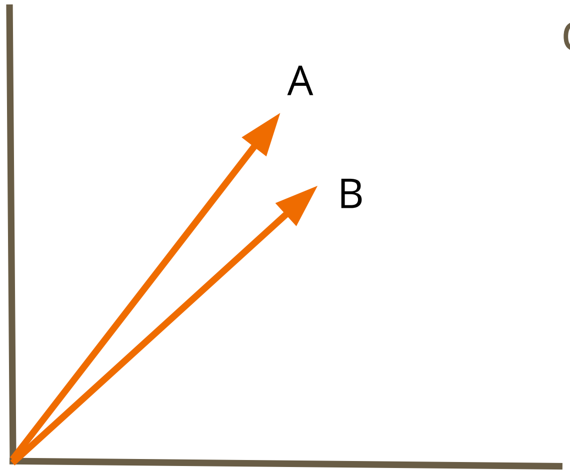
A **dissimilarity function** is a function that takes two objects (data points) and returns a **large value** if these objects are **dissimilar**.

Dissimilarity



dissim(A, B) is large

Dissimilarity



dissim(A, B) is small

Distance

A special type of dissimilarity function is a **distance** function

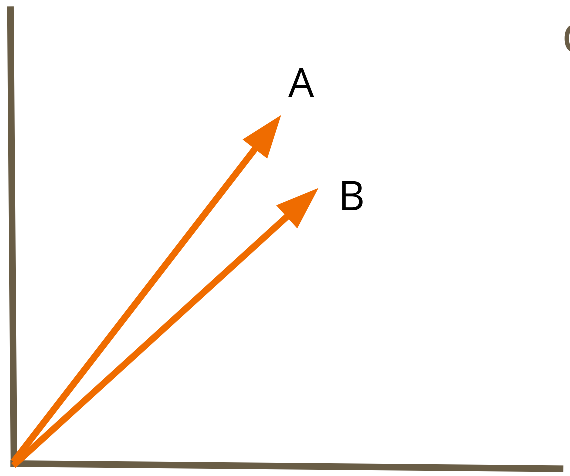
d is a distance function if and only if:

- $d(i, j) = 0$ if and only if $i = j$
- $d(i, j) = d(j, i)$
- $d(i, j) \leq d(i, k) + d(k, j)$

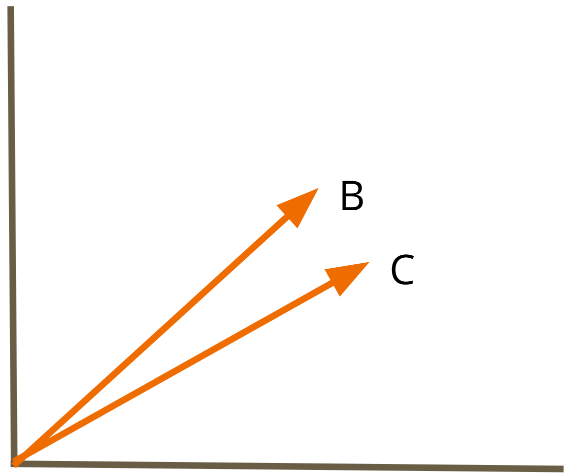
more intutive then dissimilarity

We don't **need** a distance function to compare data points, but why would we prefer using a distance function?

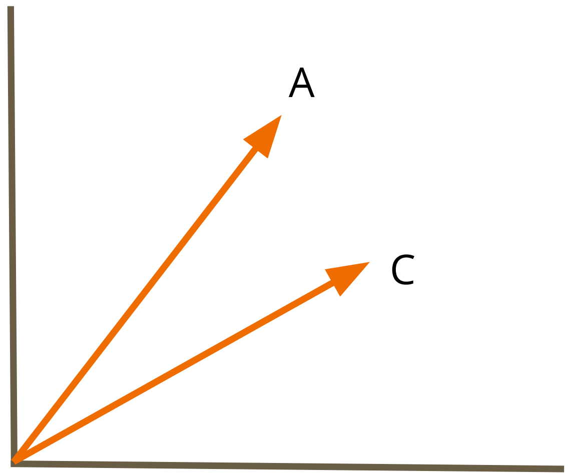
- completely costemizable



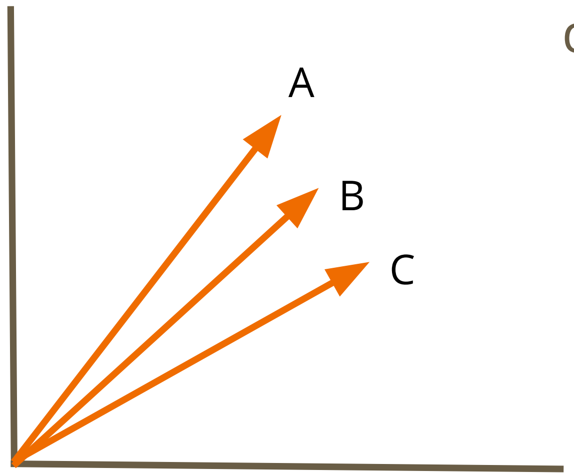
dissim(A, B) is small



dissim(B, C) is small



dissim(A, C) not necessarily small



d(A, B) is small

d(B, C) is small

Triangle inequality guarantees d(A, C) small

Minkowski Distance

For **x**, **y** points in **d**-dimensional real space

I.e. **x = [x_1 , … , x_d]** and **y = [y_1 , … , y_d]**

p ≥ 1

$$L_p(x, y) = \left(\sum_{i=1}^{d} |x_i - y_i|^p \right)^{\frac{1}{p}}$$

When **p** = 2 -> Euclidean Distance

When **p** = 1 -> Manhattan Distance

looking at differences
between feature i

$$\sqrt[p]{\sum_{i=1}^{2} |x_i - y_i|^p}$$

$$\left(|x_1 - y_1|^p + |x_2 - y_2|^p\right)^{\frac{1}{p}}$$

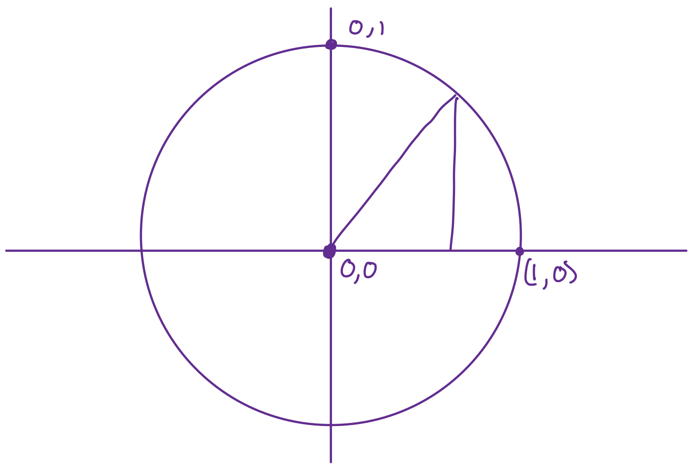
parameter p is up to you to customize

d = deminsion (characteristics / attributes)

$p \geq 1$

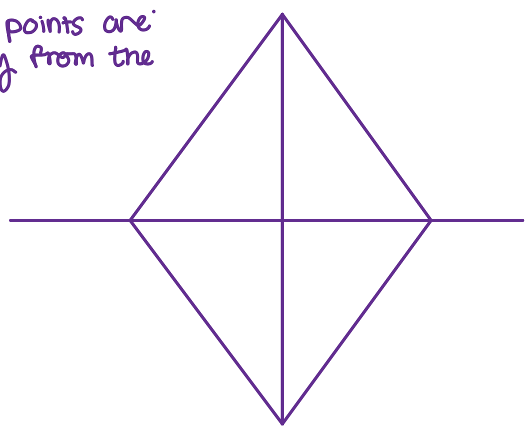
↳ look @ dataset and see how points interact and adapt p based on this

unit circle: only looks this way under euclidean distance



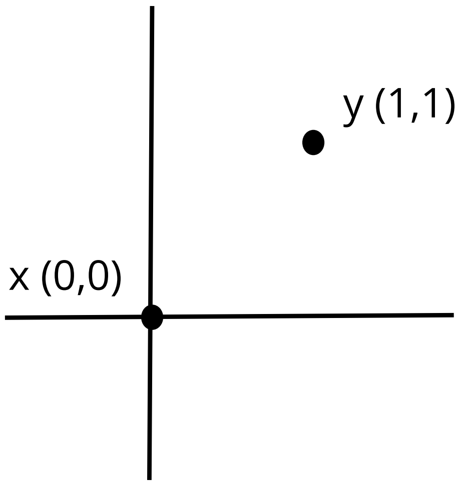
under manhattan distance:

where all points are one away from the origin



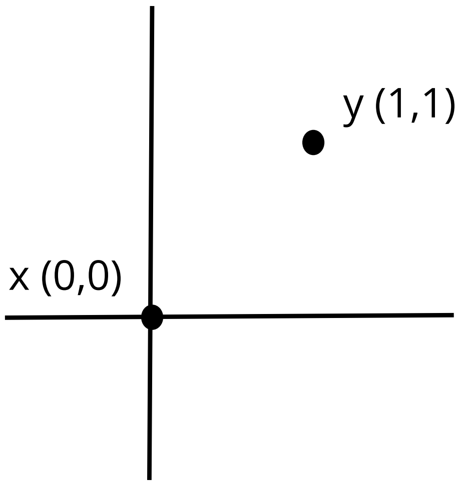
Example

d = 2



Example

d = 2

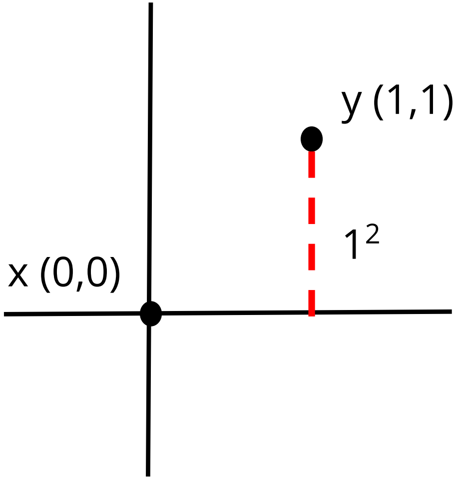


p = 2

$$L_p(x, y) = \left(\sum_{i=1}^{d} |x_i - y_i|^p \right)^{\frac{1}{p}}$$

Example

d = 2

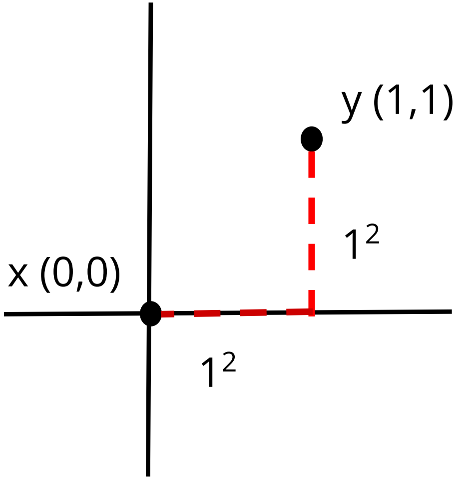


p = 2

$$L_p(x,y) = \left(\sum_{i=1}^{d} |x_i - y_i|^p \right)^{\frac{1}{p}} = 1^2 + 1^2 = \sqrt[2]{2} = 1$$

Example

d = 2

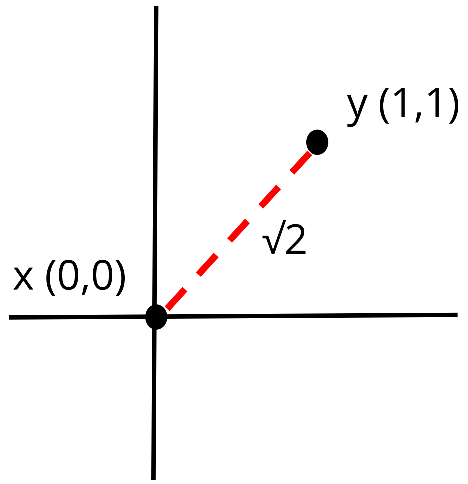


p = 2

$$L_p(x, y) = \left(\sum_{i=1}^{d} |x_i - y_i|^p \right)^{\frac{1}{p}}$$

Example

d = 2

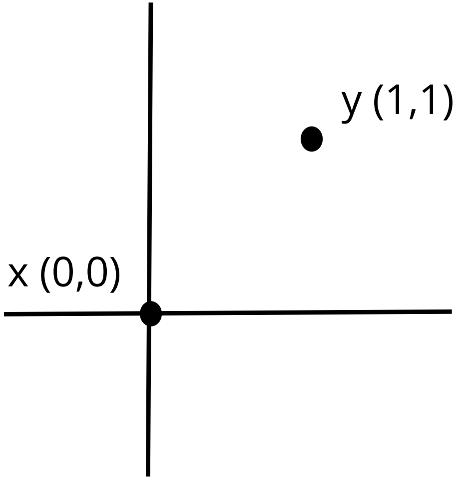


p = 2

$$L_p(x, y) = \left(\sum_{i=1}^{d} |x_i - y_i|^p \right)^{\frac{1}{p}}$$

Example

d = 2

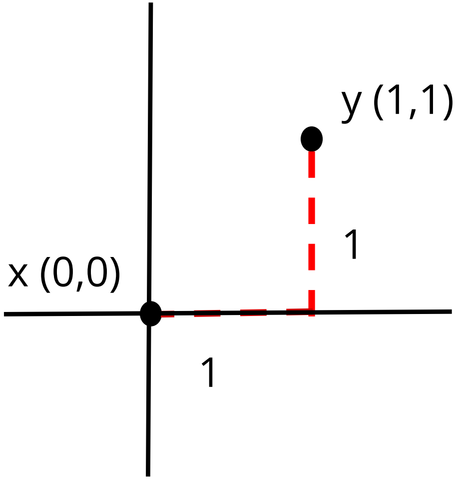


p = 1

$$L_p(x,y) = \left(\sum_{i=1}^{d} |x_i - y_i|^p \right)^{\frac{1}{p}}$$

Example

d = 2

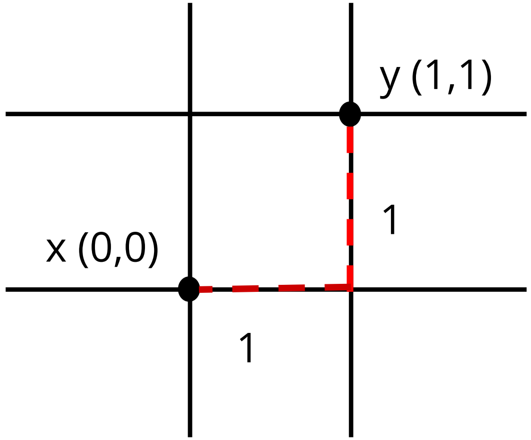


p = 1

$$L_p(x,y) = \left(\sum_{i=1}^{d} |x_i - y_i|^p \right)^{\frac{1}{p}}$$

Example

d = 2



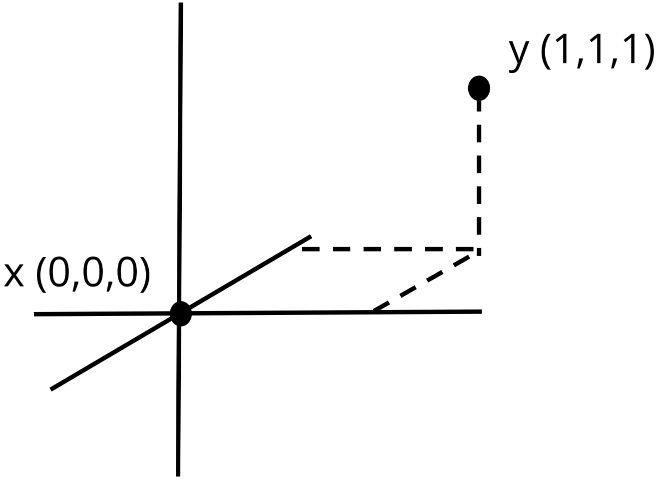
manhattan distance

p = 1

$$L_p(x,y) = \left(\sum_{i=1}^{d} |x_i - y_i|^p \right)^{\frac{1}{p}}$$

Example

d = 3

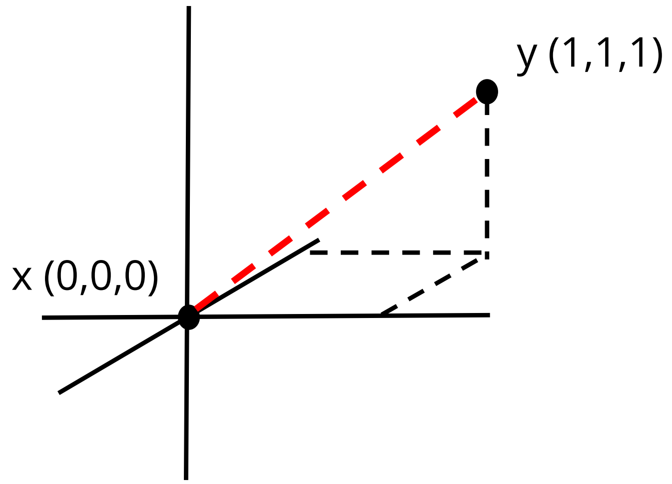


p = 2

$$L_p(x, y) = \left(\sum_{i=1}^{d} |x_i - y_i|^p \right)^{\frac{1}{p}}$$

Example

d = 3

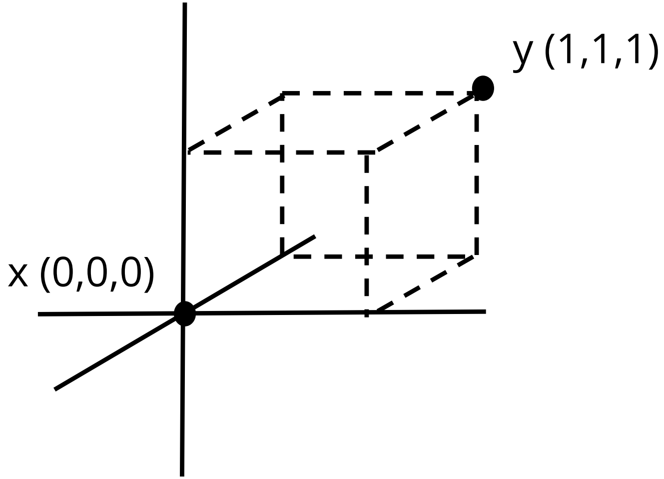


p = 2

$$L_p(x,y) = \left(\sum_{i=1}^{d} |x_i - y_i|^p\right)^{\frac{1}{p}}$$

Example

d = 3

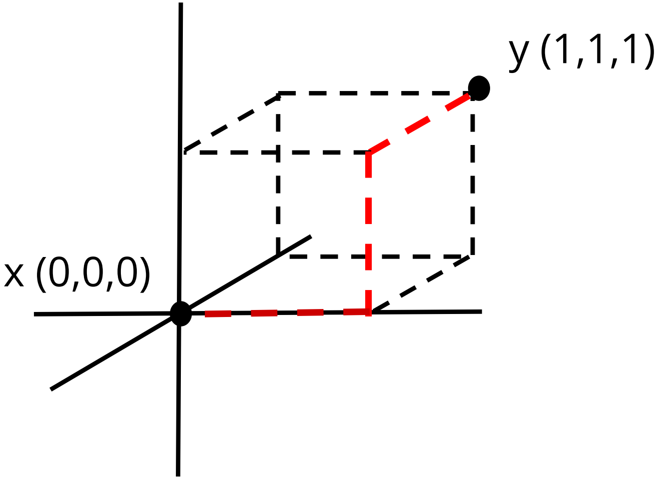


p = 1

$$L_p(x,y) = \left(\sum_{i=1}^{d} |x_i - y_i|^p\right)^{\frac{1}{p}}$$

Example

d = 3



p = 1

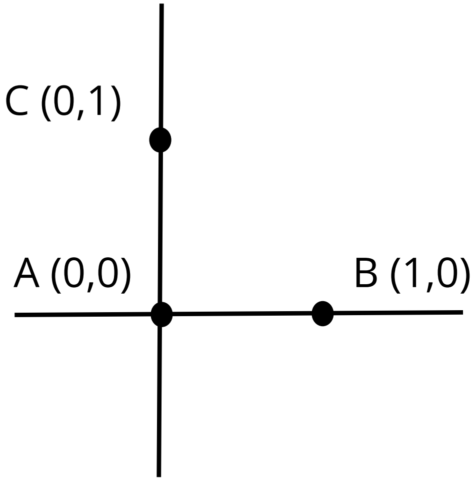
$$L_p(x, y) = \left(\sum_{i=1}^{d} |x_i - y_i|^p \right)^{\frac{1}{p}}$$

Minkowski Distance

Is $\mathbf{L_p}$ a distance function when $\mathbf{0 < p < 1}$?

Minkowski Distance

Is $\mathbf{L_p}$ a distance function when $\mathbf{0 < p < 1}$?

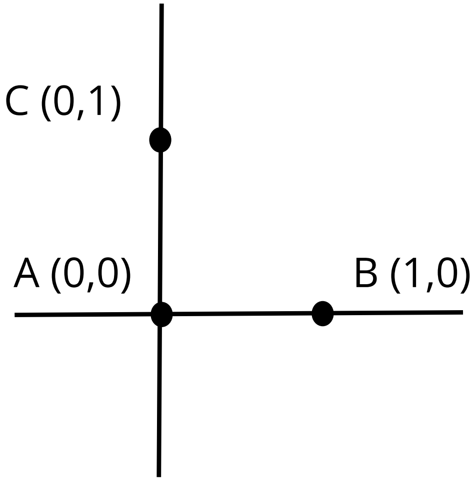


Minkowski Distance

Is $\mathbf{L_p}$ a distance function when $\mathbf{0 < p < 1}$?

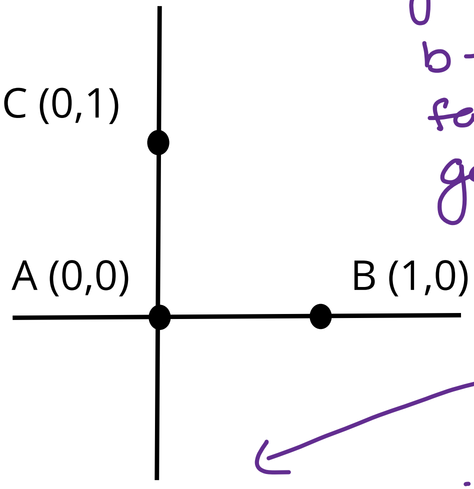
D(B,A) = D(A, C) = 1

D(B, C) = $\mathbf{2^{1/p}}$



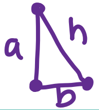
Minkowski Distance

Is L_p a distance function when $0 < p < 1$?



going from b → c is faster then going B → A → C showing its not a distance function

triangle inequality doesnt hold

↳ 

h > a+b

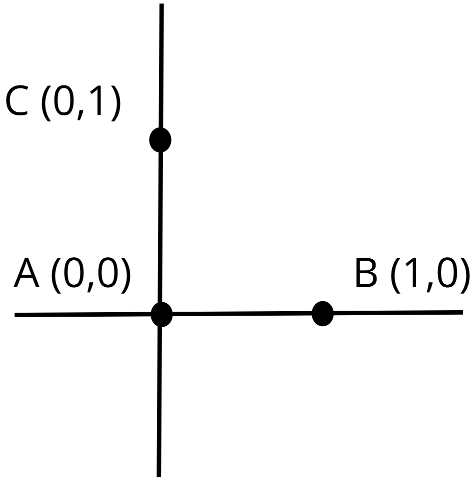
D(B,A) + D(A, C) = 2

D(B, C) = $2^{1/p}$

But… if $p < 1$ then $1/p > 1$

Minkowski Distance

Is $\mathbf{L_p}$ a distance function when $\mathbf{0 < p < 1}$?



D(B,A) + D(A, C) = 2

D(B, C) = $\mathbf{2^{1/p}}$

So **D(B, C) > D(B, A) + D(A, C)** which violates the triangle inequality

Jaccard Similarity

How similar are the following documents?

	w_1	w_2	…	w_d
x	1	0	…	1
y	1	1	…	0

Jaccard Similarity

One way is to use the Manhattan distance which will return the size of the set difference

	w_1	w_2	...	w_d
x	1	0	...	1
y	1	1	...	0

$$L_1(x, y) = \sum_{i=1}^{d} |x_i - y_i|$$

Jaccard Similarity

One way is to use the Manhattan distance which will return the size of the set difference

	w_1	w_2	...	w_d
x	1	0	...	1
y	1	1	...	0

$$L_1(x, y) = \sum_{i=1}^{d} |x_i - y_i|$$

Will only be 1 when $x_i \neq y_i$

Jaccard Similarity

But how can we distinguish between these two cases?

	w_1	w_2	...	w_{d-1}	w_d
x	1	1	1	0	1
y	1	1	1	1	0

Only differ on the last two words

	w_1	w_2
x	0	1
y	1	0

Completely different

Jaccard Similarity

But how can we distinguish between these two cases?

	w_1	w_2	...	w_{d-1}	w_d
x	1	1	1	0	1
y	1	1	1	1	0

Only differ on the last two words

	w_1	w_2
x	0	1
y	1	0

Completely different

Both have Manhattan distance of 2

Jaccard Similarity

↳ gives context to similarity

We need to account for the size of the intersection!

Given two documents x and y:

$$JSim(x, y) = \frac{|x \cap y|}{|x \cup y|}$$

Jaccard Similarity

We need to account for the size of the intersection!

Given two documents x and y:

$$JSim(x, y) = \frac{|x \cap y|}{|x \cup y|}$$

$$JDist(x, y) = 1 - \frac{|x \cap y|}{|x \cup y|}$$

Here, x is the set of words (not the binary vector representation)

Jaccard Similarity

$$JDist(x, y) = 1 - \frac{|x \cap y|}{|x \cup y|}$$

	w_1	w_2	...	w_{d-1}	w_d
x	1	1	1	0	1
y	1	1	1	1	0

Only differ on the last two words

	w_1	w_2
x	0	1
y	1	0

Completely different

What is the jaccard distance in each?

Jaccard Similarity

$$JDist(x,y) = 1 - \frac{|x \cap y|}{|x \cup y|}$$

Jaccard distance

Here, x is the set of words (not the binary vector representation)

in manhattan distance we use 0,1

Cosine Similarity

A **similarity** function is a function that takes two objects (data points) and returns a **large value** if these objects are **similar**.

$$\mathbf{s(x, y) = cos(\theta)}$$

where $\boldsymbol{\theta}$ is the angle between **x** and **y**

referenced as a si

Cosine Similarity

A **similarity** function is a function that takes two objects (data points) and returns a **large value** if these objects are **similar**.

$$\mathbf{s(x, y) = cos(\theta)}$$

where $\boldsymbol{\theta}$ is the angle between **x** and **y**

two proportional vectors have a cosine similarity of:

Cosine Similarity

A **similarity** function is a function that takes two objects (data points) and returns a **large value** if these objects are **similar**.

$$\mathbf{s(x, y) = cos(\boldsymbol{\theta})}$$

where $\boldsymbol{\theta}$ is the angle between **x** and **y**

two proportional vectors have a cosine similarity of: 1

Cosine Similarity

A **similarity** function is a function that takes two objects (data points) and returns a **large value** if these objects are **similar**.

$$\mathbf{s(x, y) = cos(\boldsymbol{\theta})}$$

where $\boldsymbol{\theta}$ is the angle between **x** and **y**

two proportional vectors have a cosine similarity of: 1

two orthogonal vectors have a similarity of:

Cosine Similarity

A **similarity** function is a function that takes two objects (data points) and returns a **large value** if these objects are **similar**.

$$\mathbf{s(x, y) = cos(\theta)}$$

where $\boldsymbol{\theta}$ is the angle between **x** and **y**

two proportional vectors have a cosine similarity of: 1

two orthogonal vectors have a similarity of: 0

Cosine Similarity

A **similarity** function is a function that takes two objects (data points) and returns a **large value** if these objects are **similar**.

$$\mathbf{s(x, y) = cos(\theta)}$$

where $\boldsymbol{\theta}$ is the angle between **x** and **y**

two proportional vectors have a cosine similarity of: 1

two orthogonal vectors have a similarity of: 0

two opposite vectors have a similarity of:

Cosine Similarity

A **similarity** function is a function that takes two objects (data points) and returns a **large value** if these objects are **similar**.

$$\mathbf{s(x, y) = cos(\theta)}$$

where $\boldsymbol{\theta}$ is the angle between **x** and **y**

two proportional vectors have a cosine similarity of: 1

two orthogonal vectors have a similarity of: 0

two opposite vectors have a similarity of: - 1

Cosine Similarity

To get a corresponding **dissimilarity** function, we can usually try

$$d(x, y) = 1 / s(x, y)$$

or

$$d(x, y) = k - s(x, y) \text{ for some } k$$

Here, we can use

$$d(x, y) = 1 - s(x, y)$$

Cosine Similarity

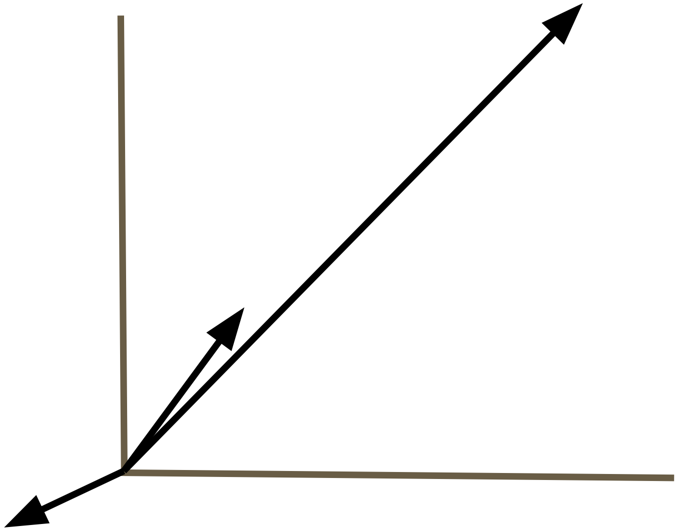
When should you use **cosine (dis)similarity** over **euclidean distance**?

When **direction** matters more than **magnitude**

Cosine Similarity

When should you use **cosine (dis)similarity** over **euclidean distance**?

When **direction** matters more than **magnitude**

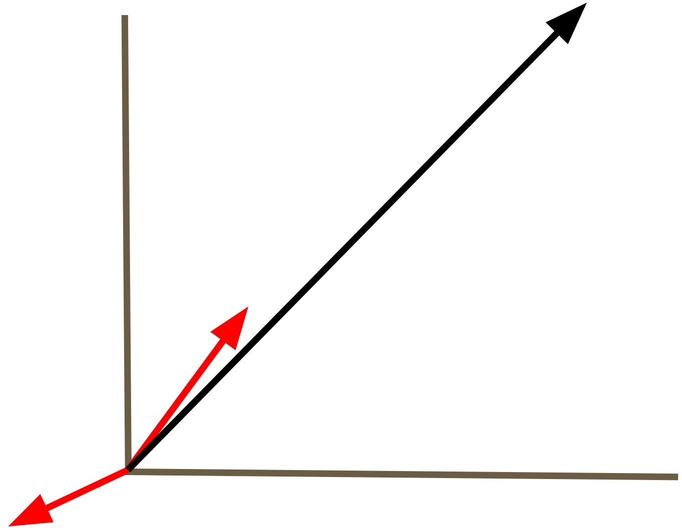


Cosine Similarity

When should you use **cosine (dis)similarity** over **euclidean distance**?

When **direction** matters more than **magnitude**

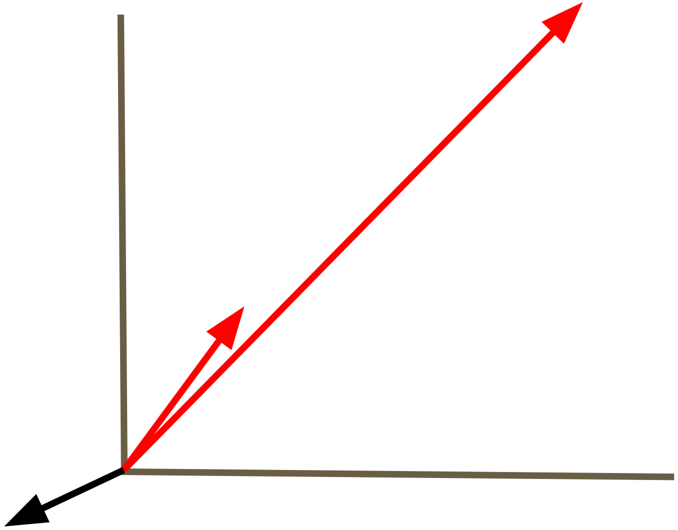
Close under
Euclidean distance



Cosine Similarity

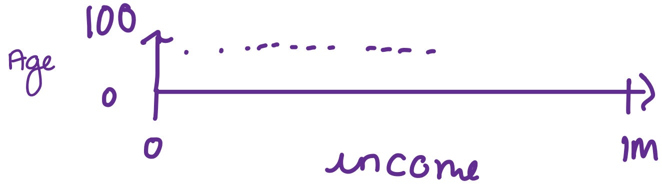
When should you use **cosine (dis)similarity** over **euclidean distance**?

When **direction** matters more than **magnitude**



Close under Cosine Similarity

Scale of income obliterated scale of age



you can't tell what the variability is here

A quick Note on Norms

$$d(A, B) = \|A - B\|$$

distance btwn A & B = $\|A-B\|$

norm X = distance(0, X)

Size = Distance from the origin $\qquad d(0, X) = \|X\|$

- Minkowski Distance <=> Lp Norm
- Not all distances can create a Norm