Notes for Chapter 0: Appetizers

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Definition. A <u>convex combination</u> of points $z_1, \dots, z_m \in \mathbb{R}^n$ is a linear combination with coefficients that are nonnegative and sum to 1, i.e. it is a sum of the form

$$\sum_{i=1}^m \lambda_i z_i, \quad \lambda_i \geq 0 \text{ and } \sum_{i=1}^m \lambda_i = 1.$$

Definition. The <u>convex hull</u> of a set $T \in \mathbb{R}^n$ is the set of all convex combinations of all finite collections of points in T, i.e.

$$\text{conv}(T) := \{\text{convex combinations of } z_1, \dots, z_m \in T \text{ for } m \in \mathbb{N}\}.$$

Theorem (Caratheodory Theorem). *Every point in the convex hull of a set $T \subseteq \mathbb{R}^n$ can be expressed as a convex combination of at most $n+1$ points from T.*

Proof. Denote the point as

$$p = a_1 x_1 + \cdots + a_m x_m, \; a_i \geq 0, \; \sum_{i=1}^m a_i = 0.$$

There are two cases that we can consider:
Case 1: $m \leq n+1$. Then p is already in the desired form and we don't need to worry about it.
Case 2: $m > n+1$. Then the set of $n-1$ points $\{x_2 - x_1, \dots, x_m - 1\}$ have to be linearly dependent because we have at least $n+1$ points in a subspace of $\mathbb{R}^n$. Let $b_2, \dots, b_m \in \mathbb{R}$ be not all zero such that

$$\sum_{i=2}^m b_i (x_i - x_1) = 0.$$

From the above, by adding an extra term when $i = 1$, there exists n numbers $c_1, \dots, c_n$ such that

$$\sum_{i=1}^m c_i x_i = 0 \text{ and } \sum_{i=1}^m c_i = 0.$$

Define $I = \{i \in \{1, 2, \dots, n\} : c_i > 0\}$. The set is nonempty by the results that we have above. Define

$$\alpha = \max_{i \in I} a_i / c_i.$$

Then we can rewrite our point p as

$$p = p - 0 = \sum_{i=1}^m a_i x_i - \alpha \sum_{i=1}^m c_i x_i = \sum_{i=1}^m (a_i - \alpha c_i) x_i,$$

which is a convex combination with at least one zero coefficient, meaning p can be written as a convex combination of $m-1$ points in T (we can check this!). By continuing to apply the above, we can eventually arrive at the case when p consists of a combination of exactly $n+1$ points, as desired. $\square$

Theorem (Approximate Caratheodory Theorem). *Consider a set $T \subseteq \mathbb{R}^n$ that is contained in the unit Euclidean ball. Then, for every point $x \in conv(T)$ and every $k \in \mathbb{N}$, one can find points $x_1, \dots, x_k \in T$ such that*

$$\left\| x - \frac{1}{k} \sum_{j=1}^k x_j \right\|_2 \leq \frac{1}{\sqrt{k}}.$$

Proof. We'll apply a technique called the *empirical method*. Fix $x \in \text{conv}(T)$ so

$$x = \lambda_1 z_1 + \cdots + \lambda_m z_m, \ \lambda_i \geq 0, \ \sum_{i=1}^{m} \lambda_i = 1.$$

From the above, we can consider the λ_i's as weights to a probability distribution. Define the random vector Z with its pmf being

$$P(Z = z_i) = \lambda_i, \ i = 1, 2, \ldots, m.$$

We can immediately get that the expected value of Z is

$$\mathbb{E}[Z] = \sum_{i=1}^{m} z_i P(Z = z_i) = \sum_{i=1}^{m} \lambda_i z_i = x.$$

Now consider $Z_1, \cdots, Z_k$ with the same distribution as Z. The strong law of large numbers tells us that

$$\frac{1}{k}\sum_{j=1}^{k} Z_j \to x \text{ almost surely as } k \to \infty.$$

For a more quantitative result, consider the mean-squared error:

$$\mathbb{E}\left[\left\| x - \frac{1}{k}\sum_{j=1}^{k} Z_j \right\|_2^2\right] = \frac{1}{k^2}\mathbb{E}\left[\left\| \sum_{j=1}^{k}(Z_j - x) \right\|_2^2\right] = \frac{1}{k^2}\sum_{j=1}^{k}\mathbb{E}[\|Z_j - x\|_2^2],$$

where the third equality is proved in exercise 3. For each term in the summation,

$$\begin{aligned}
\mathbb{E}[\|Z_j - x\|_2^2] &= \mathbb{E}[\|Z - \mathbb{E}[Z]\|_2^2] \\
&= \mathbb{E}[\|Z\|_2^2] - \|\mathbb{E}[Z]\|_2^2 \quad \text{(Exercise 1)} \\
&\leq \mathbb{E}[\|Z\|_2^2] \\
&\leq 1. \quad \text{(Since } Z \in T\text{)}.
\end{aligned}$$

Then, we get that

$$\mathbb{E}\left[\left\| x - \frac{1}{k}\sum_{j=1}^{k} Z_j \right\|_2^2\right] \leq \frac{1}{k}.$$

Therefore, there exists a realization $Z_1, \ldots, Z_k$ such that

$$\left\| x - \frac{1}{k}\sum_{j=1}^{k} Z_j \right\|_2^2 \leq \frac{1}{k}.$$

□

Covering Geometric Sets