# Chapter 0 Exercises

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## Exercise 1

### (a)

$$\begin{aligned}
\mathbb{E}[\|Z - \mathbb{E}[Z]\|_2^2] &= \mathbb{E}[\|Z\|_2^2 - 2\langle Z, \mathbb{E}[Z]\rangle + \|\mathbb{E}[Z]\|_2^2] \\
&= \mathbb{E}[\|Z\|_2^2] - 2\mathbb{E}[Z]^T\mathbb{E}[Z] + \|\mathbb{E}[Z]\|_2^2 \\
&= \mathbb{E}[\|Z\|]_2^2 - \|\mathbb{E}[Z]\|_2^2.
\end{aligned}$$

### (b)

From part (a),

$$\begin{aligned}
\mathbb{E}[\|Z - \mathbb{E}[Z]\|_2^2] &= \mathbb{E}[\|Z\|]_2^2 - \|\mathbb{E}[Z]\|_2^2 \\
&= \frac{1}{2}\mathbb{E}[\|Z\|_2^2] - \mathbb{E}[Z]^T\mathbb{E}[Z] + \frac{1}{2}\mathbb{E}[\|Z\|_2^2] \\
&= \frac{1}{2}(\mathbb{E}[\|Z\|_2^2] - 2\mathbb{E}[Z^t]\mathbb{E}[Z'] + \frac{1}{2}\mathbb{E}[\|Z'\|_2^2]) \\
&= \frac{1}{2}(\mathbb{E}[\|Z\|_2^2] - 2\mathbb{E}[Z^TZ'] + \mathbb{E}[\|Z'\|_2^2]) \\
&= \frac{1}{2}\mathbb{E}[\|Z - Z'\|_2^2].
\end{aligned}$$

## Exercise 2

Let $\mu = \mathbb{E}[Z]$. First, notice that

$$\begin{aligned}
\mathbb{E}[\|Z-a\|_2^2] - \mathbb{E}[\|Z-\mu\|_2^2] &= \mathbb{E}[\|Z\|_2^2 - 2a^T Z + \|a\|_2^2 - \|Z\|_2^2 + 2\mu^T Z - \|\mu\|_2^2] \\
&= \|a\|_2^2 - 2(a^T - \mu^T)\mathbb{E}[Z] - \|\mu\|_2^2 \\
&= \|a\|_2^2 - 2a^T\mu + 2\|\mu\|_2^2 - \|\mu\|_2^2 \\
&= \|a-\mu\|_2^2.
\end{aligned}$$

From above, minimizing $\mathbb{E}[\|Z-a\|_2^2]$ in terms of $a$ is the same as minimizing the term we have above as the second term does not depend on $a$. The expression above is minimized exactly at $a^* = \mu = \mathbb{E}[Z]$ as the quantity is always greater than or equal to 0, and reaches the value 0 if and only if $a = \mu$.

## Exercise 3

$$
\begin{aligned}
\mathbb{E}\left[\left\|\sum_{j=1}^{k} Z_j\right\|_2^2\right] &= \mathbb{E}[(Z_1 + \cdots + Z_k)^T (Z_1 + \cdots + Z_k)] \\
&= \mathbb{E}\left[\sum_{j=1}^{k} \|Z_j\|_2^2 + \sum_{i \neq j} Z_i^T Z_j\right] \\
&= \mathbb{E}\left[\sum_{j=1}^{k} \|Z_j\|_2^2\right] + \sum_{i \neq j} \mathbb{E}[Z_i]^T \mathbb{E}[Z_j] \\
&= \mathbb{E}\left[\sum_{j=1}^{k} \|Z_j\|_2^2\right] + 0 \qquad (\mathbb{E}[Z_i] = 0) \\
&= \mathbb{E}\left[\sum_{j=1}^{k} \|Z_j\|_2^2\right].
\end{aligned}
$$

# Exercise 4

## (a)

We can consider these points as being chosen randomly at uniform from the unit ball in $n$ dimensions, i.e.

$$X_1, \dots, X_n \sim_{iid} \mathrm{Unif}(B_1^n) \implies \mathbb{E}[X_i] = 0.$$

Then by exercise 3,

$$\mathbb{E}\left[\left\|\sum_{i=1}^k X_i\right\|_2^2\right] = \sum_{i=1}^k \mathbb{E}[\|X_i\|_2^2] \le k.$$

Therefore there exists a realization $(x_1, \dots, x_n)$ for which

$$\left\|\sum_{i=1}^n x_i\right\|_2^2 \le k \implies \left\|\sum_{i=1}^n x_i\right\|_2 \le \sqrt{k}.$$

## (b)

We are bounding $\mathbb{E}[\|X_i\|_2^2]$ by 1, which is a tight bound.

## Exercise 5

# Exercise 6

The first inequality comes as follows: we can see that

$$\frac{n}{k} \leq \frac{n-i}{k-i}, \quad i = 1, 2, \ldots, k-1.$$

This is because by cross multiplication

$$n(k-i) = nk - ni \geq nk - ki = k(n-i).$$

Then

$$\left(\frac{n}{k}\right)^k = \frac{n}{k} \times \frac{n}{k} \times \cdots \times \frac{n}{k} \leq \frac{n}{k} \times \frac{n-1}{k-1} \times \cdots \times \frac{n-k+1}{1} = \binom{n}{k}.$$

The second inequality is trivial as $k \geq 1$. For the third inequality, we get

$$\begin{aligned}
\sum_{j=0}^{k} \binom{n}{j} \cdot \left(\frac{k}{n}\right)^k &\leq \sum_{j=0}^{k} \binom{n}{j} \cdot \left(\frac{k}{n}\right)^j \quad (k/n \leq 1) \\
&\leq \sum_{j=0}^{n} \binom{n}{j} \cdot \left(\frac{k}{n}\right)^j \quad (k/n \leq 1) \\
&= \left(1 + \frac{k}{n}\right)^n \quad \text{(Binomial Theorem)} \\
&< e^k.
\end{aligned}$$

## Exercise 7

Assume $n$ is large so that the $5/n$ radius near the surface is valid. The inner ball has radius $\frac{1}{2} - \frac{5}{n}$. Then the volume of the inner ball is $(\frac{1}{2} - \frac{5}{n})^n$ times the volums of the outer unit ball. In particular, as $n \to \infty$,

$$\left(\frac{1}{2} - \frac{5}{n}\right)^n = \left(\frac{1}{2}\right)^n \left(1 - \frac{10}{n}\right)^n \to 0.$$

This means that most of the points will be concentrated towards the surface of the $n$-dimensional ball.

# Exercise 8

Let $X \sim \mathrm{Unif}(B_1^n)$. Then the pdf of $X$ is

$$f_X(x) = \frac{1}{\mathrm{Vol}(B_1^n)},\ x \in B_1^n.$$

Now let's consider the random variable $\|X\|_2$, i.e. the 2-norm of the random vector. Since the random vector is distributed uniformly in the $n$-dimensional ball, we can define its CDF as a function of the radius $r$:

$$F_{\|X\|_2}(r) = P(\|X\|_2 \leq r) = r^n.$$

Correspondingly, we can find the PDF by just taking the derivative of the CDF:

$$f_{\|X\|_2}(r) = nr^{n-1},\ 0 \leq r \leq 1.$$

Then we can directly get that

$$\mathbb{E}[\|X\|_2] = \int_0^1 r \cdot nr^{n-1}\ dr = n \cdot \left[\frac{r^{n+1}}{n+1}\right]_0^1 = \frac{n}{n+1}.$$