Chapter 0 Exercises

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Exercise 1

(a)

$$\begin{aligned}
\mathbb{E}[\|Z - \mathbb{E}[Z]\|_2^2] &= \mathbb{E}[\|Z\|_2^2 - 2\langle Z, \mathbb{E}[Z]\rangle + \|\mathbb{E}[Z]\|_2^2] \\
&= \mathbb{E}[\|Z\|_2^2] - 2\mathbb{E}[Z]^T\mathbb{E}[Z] + \|\mathbb{E}[Z]\|_2^2 \\
&= \mathbb{E}[\|Z\|]_2^2 - \|\mathbb{E}[Z]\|_2^2.
\end{aligned}$$

(b)

From part (a),

$$\begin{aligned}
\mathbb{E}[\|Z - \mathbb{E}[Z]\|_2^2] &= \mathbb{E}[\|Z\|]_2^2 - \|\mathbb{E}[Z]\|_2^2 \\
&= \frac{1}{2}\mathbb{E}[\|Z\|_2^2] - \mathbb{E}[Z]^T\mathbb{E}[Z] + \frac{1}{2}\mathbb{E}[\|Z\|_2^2] \\
&= \frac{1}{2}(\mathbb{E}[\|Z\|_2^2] - 2\mathbb{E}[Z^t]\mathbb{E}[Z'] + \frac{1}{2}\mathbb{E}[\|Z'\|_2^2]) \\
&= \frac{1}{2}(\mathbb{E}[\|Z\|_2^2] - 2\mathbb{E}[Z^TZ'] + \mathbb{E}[\|Z'\|_2^2]) \\
&= \frac{1}{2}\mathbb{E}[\|Z - Z'\|_2^2].
\end{aligned}$$

Exercise 2

Let $\mu = \mathbb{E}[Z]$. First, notice that

$$\begin{aligned}
\mathbb{E}[\|Z-a\|_2^2] - \mathbb{E}[\|Z-\mu\|_2^2] &= \mathbb{E}[\|Z\|_2^2 - 2a^T Z + \|a\|_2^2 - \|Z\|_2^2 + 2\mu^T Z - \|\mu\|_2^2] \\
&= \|a\|_2^2 - 2(a^T - \mu^T)\mathbb{E}[Z] - \|\mu\|_2^2 \\
&= \|a\|_2^2 - 2a^T\mu + 2\|\mu\|_2^2 - \|\mu\|_2^2 \\
&= \|a-\mu\|_2^2.
\end{aligned}$$

From above, minimizing $\mathbb{E}[\|Z-a\|_2^2]$ in terms of a is the same as minimizing the term we have above as the second term does not depend on a. The expression above is minimized exactly at $a^* = \mu = \mathbb{E}[Z]$ as the quantity is always greater than or equal to 0, and reaches the value 0 if and only if $a = \mu$.