Definition. A <u>convex combination</u> of points $z_1, \ldots, z_m \in \mathbb{R}^n$ is a linear combination with coefficients that are nonnegative and sum to 1, i.e. it is a sum of the form

$$\sum_{i=1}^{m} \lambda_i z_i, \quad \lambda_i \geq 0 \text{ and } \sum_{i=1}^{m} \lambda_i = 1.$$

Definition. The <u>convex hull</u> of a set $T \in \mathbb{R}^n$ is the set of all convex combinations of all finite collections of points in T, i.e.

$$\operatorname{conv}(T) := \{\text{convex combinations of } z_1, \ldots, z_m \in T \text{ for } m \in \mathbb{N}\}.$$

Theorem (Caratheodory Theorem). Every point in the convex hull of a set $T \subseteq \mathbb{R}^n$ can be expressed as a convex combination of at most $n+1$ points from T.

Proof. Denote the point as

$$p = a_1 x_1 + \cdots + a_m x_m,\ a_i \geq 0,\ \sum_{i=1}^{m} a_i = 0.$$

There are two cases that we can consider:
Case 1: $m \leq n+1$. Then p is already in the desired form and we don't need to worry about it.
Case 2: $m > n+1$. Then the set of $n-1$ points $\{x_2 - x_1, \ldots, x_m - 1\}$ have to be linearly dependent because we have at least $n+1$ points in a subspace of $\mathbb{R}^n$. Let $b_2, \ldots, b_m \in \mathbb{R}$ be not all zero such that

$$\sum_{i=2}^{m} b_i (x_i - x_1) = 0.$$

From the above, by adding an extra term when $i = 1$, there exists n numbers $c_1, \ldots, c_n$ such that

$$\sum_{i=1}^{m} c_i x_i = 0 \text{ and } \sum_{i=1}^{m} c_i = 0.$$

Define $I = \{i \in \{1, 2, \ldots, n\} : c_i > 0\}$. The set is nonempty by the results that we have above. Define

$$\alpha = \max_{i \in I} a_i / c_i.$$

Then we can rewrite our point p as

$$p = p - 0 = \sum_{i=1}^{m} a_i x_i - \alpha \sum_{i=1}^{m} c_i x_i = \sum_{i=1}^{m} (a_i - \alpha c_i) x_i,$$

which is a convex combination with at least one zero coefficient, meaning p can be written as a convex combination of $m-1$ points in T (we can check this!). By continuing to apply the above, we can eventually arrive at the case when p consists of a combination of exactly $n+1$ points, as desired. $\square$

Theorem (Approximate Caratheodory Theorem). Consider a set $T \subseteq \mathbb{R}^n$ that is contained in the unit Euclidean ball. Then, for every point $x \in \operatorname{conv}(T)$ and every $k \in \mathbb{N}$, one can find points $x_1, \ldots, x_k \in T$ such that

$$\left\| x - \frac{1}{k} \sum_{j=1}^{k} x_j \right\|_2 \leq \frac{1}{\sqrt{k}}.$$

Proof. We'll apply a technique called the *empirical method*. Fix $x \in \text{conv}(T)$ so

$$x = \lambda_1 z_1 + \cdots + \lambda_m z_m,\ \ \lambda_i \geq 0,\ \ \sum_{i=1}^{m} \lambda_i = 1.$$

From the above, we can consider the λ_i's as weights to a probability distribution. Define the random vector Z with its pmf being

$$P(Z = z_i) = \lambda_i,\ \ i = 1, 2, \ldots, m.$$

We can immediately get that the expected value of Z is

$$\mathbb{E}[Z] = \sum_{i=1}^{m} z_i P(Z = z_i) = \sum_{i=1}^{m} \lambda_i z_i = x.$$

Now consider $Z_1, \cdots, Z_k$ with the same distribution as Z. The strong law of large numbers tells us that

$$\frac{1}{k} \sum_{j=1}^{k} Z_j \to x \text{ almost surely as } k \to \infty.$$

For a more quantitative result, consider the mean-squared error:

$$\mathbb{E}\left[\left\| x - \frac{1}{k} \sum_{j=1}^{k} Z_j \right\|_2^2\right] = \frac{1}{k^2} \mathbb{E}\left[\left\| \sum_{j=1}^{k} (Z_j - x) \right\|_2^2\right] = \frac{1}{k^2} \sum_{j=1}^{k} \mathbb{E}[\|Z_j - x\|_2^2],$$

where the third equality is proved in exercise 3. For each term in the summation,

$$\begin{aligned}
\mathbb{E}[\|Z_j - x\|_2^2] &= \mathbb{E}[\|Z - \mathbb{E}[Z]\|_2^2] \\
&= \mathbb{E}[\|Z\|_2^2] - \|\mathbb{E}[Z]\|_2^2 \quad \text{(Exercise 1)} \\
&\leq \mathbb{E}[\|Z\|_2^2] \\
&\leq 1. \quad \text{(Since } Z \in T\text{)}.
\end{aligned}$$

Then, we get that

$$\mathbb{E}\left[\left\| x - \frac{1}{k} \sum_{j=1}^{k} Z_j \right\|_2^2\right] \leq \frac{1}{k}.$$

Therefore, there exists a realization $Z_1, \ldots, Z_k$ such that

$$\left\| x - \frac{1}{k} \sum_{j=1}^{k} Z_j \right\|_2^2 \leq \frac{1}{k}.$$

$\square$

Covering Geometric Sets

Convex Sets and Functions

Definition. A subset $K \subseteq \mathbb{R}^n$ is a convex set if, for any pair of points in K, the line segment connecting these two points is also contained in K, i.e.

$$\lambda x + (1-\lambda)y \in K \quad \forall x, y \in K, \lambda \in [0,1].$$

Let $K \in \mathbb{R}^n$ be a convex subset. A function $f : K \to \mathbb{R}$ is a convex function if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \quad \forall x, y \in K, \lambda \in [0,1].$$

f is concave if the inequality above is reversed, or equivalently, if $-f$ is convex.

Norms and Inner Products

Why Concentration Inequalities?

From previous chapters, the simplest concentration inequality is Chebyshev's Inequality, which is quite general but the bounds can often can be too weak. We can look at the following example:

Example. Toss a fair coin N times. What is the probability that we get at least $\frac{3}{4}$ heads? Let S_N denote the number of heads, then $S_N \sim \text{Binom}(N, \frac{1}{2})$. We get

$$\mathbb{E}[S_N] = \frac{N}{2}, \text{Var}(S_n) = \frac{N}{4}.$$

Using Chebyshev's Inequality, we get

$$P(S_N \geq \frac{3}{4}N) \leq P(\left|S_N - \frac{N}{2}\right| \geq \frac{N}{4}) \leq \frac{4}{N}.$$

This means probabilistic bound from above converges linearly in N.
However, by using the Central Limit Theorem, we get a very different result: If we let S_N be a sum of independent $Be(\frac{1}{2})$ random variables. Then by the De Moivre-Laplace CLT, the random variable

$$Z_N = \frac{S_N - N/2}{\sqrt{N/4}}$$

converges to the standard normal distribution $N(0, 1)$. Then for a large N,

$$P(S_{\geq}\frac{3}{4}N) = P(Z_N \geq \sqrt{N/4}) \approx P(Z \geq \sqrt{N/4})$$

where $Z \sim N(0, 1)$. We will use the following proposition:

Proposition (Gaussian tails)**.** *Let $Z \sim N(0, 1)$. Then for all $t > 0$,*

$$\frac{t}{t^2+1} \cdot \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \leq P(Z \geq t) \leq \frac{1}{t} \cdot \frac{1}{\sqrt{2\pi}} e^{-t^2/2}.$$

Proof. The first inequality is proved in exercise 2.2. For the second inequality, by making the change of variables $x = t + y$,

$$\begin{aligned} P(Z \geq t) &= \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-x^2/2}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t^2/2} e^{-ty} e^{-y^2/2}\, dy \\ &\leq \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \int_0^\infty e^{-ty}\, dy \quad (e^{-y^2/2} \leq 1) \\ &= \frac{1}{t} \cdot \frac{1}{\sqrt{2\pi}} e^{-t^2/2}. \end{aligned}$$

$\square$

Therefore the probability of having at least $\frac{3}{4}N$ heads is bounded by

$$\frac{1}{\sqrt{2\pi}} e^{-N/8},$$

which is much better than the linear convergence we had above. However, this reasoning is not rigorous, as the approximation error decays slowly, which can be shown via the CLT below:

Theorem (Berry-Esseen CLT)**.** Let $X_1, X_2, \dots$ be a sequence of i.i.d. random variables with mean μ and variance σ^2, and let $S_N = X_1 + \cdots + X_N$, and let

$$Z_N = \frac{S_N - \mathbb{E}[S_N]}{\sqrt{\mathrm{Var}(S_N)}}.$$

Then for every $N \in \mathbb{N}$ and $t \in \mathbb{R}$ we have

$$|P(Z_N \geq t) - P(Z \geq t)| \leq \frac{\rho}{\sqrt{N}},$$

where $Z \sim N(0,1)$ and $\rho = \mathbb{E}[|X_1 - \mu|^3]/\sigma^3$.

Therefore the approximation error decays at a rate of $1/\sqrt{N}$. Moreover, this bound cannot be improved, as for even N, the probability of exactly half the flips being heads is

$$P(S_N = \frac{N}{2}) = 2^{-N}\binom{N}{N/2} \approx \sqrt{\frac{2}{\pi N}}.$$

where the last approximation uses Stirling approximation.
All in all, we need theory for concentration which bypasses the Central Limit Theorem.

Hoeffding Inequality

Definition. A random variable X has the <u>Rademacher Distribution</u> if it takes values -1 and 1 with probability $1/2$ each, i.e.

$$P(X = -1) = P(X = 1) = \frac{1}{2}.$$

Theorem (Hoeffding Inequality)**.** Let $X_1, , X_N$ be independent Rademacher random variables, and let $a = (a_1, \cdots, a_n) \in \mathbb{R}^n$ be fixed. Then for any $t \geq 0$,

$$P\left(\sum_{i=1}^{N} a_i X_i \geq t\right) \leq \exp\left(-\frac{t^2}{2\|a\|_2^2}\right).$$

Proof. See how this looks with no box $\square$