

Statistical Inference Chapter 2

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1. (a) Let $g(x) = x^3$. Then g is monotonically increasing on $(0, 1)$. We get

$$g^{-1}(y) = y^{1/3} \implies \frac{d}{dy}g^{-1}(y) = \frac{1}{3y^{2/3}}.$$

Since $X \in (0, 1)$, $Y = X^3 \in (0, 1)$. Then by Theorem 2.1.5,

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy}g^{-1}(y) \right| \\ &= 42(y^{1/3})^5(1 - y^{1/3}) \cdot \frac{1}{3y^{2/3}} \\ &= 14y(1 - y^{1/3}), \quad y \in (0, 1). \end{aligned}$$

We also have

$$\begin{aligned} \int_0^1 14y(1 - y^{1/3}) \, dy &= 14 \int_0^1 y - y^{4/3} \, dy \\ &= 14 \left[\frac{1}{2}y^2 - \frac{3}{7}y^{7/3} \right]_0^1 \\ &= 14 \left(\frac{1}{2} - \frac{3}{7} \right) \\ &= 1. \end{aligned}$$

- (b) Let $g(x) = 4x + 3$. Then g is monotonically increasing on $(0, \infty)$. We get

$$g^{-1}(y) = \frac{y-3}{4} \implies \frac{d}{dy}g^{-1}(y) = \frac{1}{4}.$$

Since $X \in (0, \infty)$, $Y = 4X + 3 \in (3, \infty)$. Then by Theorem 2.1.5,

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy}g^{-1}(y) \right| \\ &= 7e^{-7 \cdot \frac{y-3}{4}} \cdot \frac{1}{4} \\ &= \frac{7}{4}e^{\frac{21}{4} - \frac{7}{4}y}, \quad y \in (3, \infty). \end{aligned}$$

We also have

$$\begin{aligned}
\int_3^\infty \frac{7}{4} e^{\frac{21}{4} - \frac{7}{4}y} dy &= \frac{7}{4} e^{\frac{21}{4}} \int_3^\infty e^{-\frac{7}{4}y} dy \\
&= \frac{7}{4} e^{\frac{21}{4}} \left[-\frac{4}{7} e^{-\frac{7}{4}y} \right]_3^\infty \\
&= \frac{7}{4} e^{\frac{21}{4}} \left(\frac{4}{7} e^{-\frac{21}{4}} \right) \\
&= 1.
\end{aligned}$$

(c) Let $g(x) = x^2$. Then g is monotonically increasing on $(0, 1)$. We get

$$g^{-1}(y) = \sqrt{y} \implies \frac{d}{dy} g^{-1}(y) = \frac{1}{2\sqrt{y}}.$$

Since $X \in (0, 1)$, $Y = X^2 \in (0, 1)$. Then by Theorem 2.1.5,

$$\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \\
&= 30y(1 - \sqrt{y})^2 \cdot \frac{1}{2\sqrt{y}} \\
&= 15\sqrt{y}(1 - \sqrt{y})^2, \quad y \in (0, 1).
\end{aligned}$$

We also have

$$\begin{aligned}
\int_0^1 15\sqrt{y}(1 - \sqrt{y})^2 dy &= 15 \int_0^1 \sqrt{y} - 2y + y^{3/2} dy \\
&= 15 \left[\frac{2}{3} y^{3/2} - y^2 + \frac{2}{5} y^{5/2} \right]_0^1 \\
&= 15 \left(\frac{2}{3} - 1 + \frac{2}{5} \right) \\
&= 1.
\end{aligned}$$

2. (a) Let $g(x) = x^2$. Then g is monotonically increasing on $(0, 1)$. We get

$$g^{-1}(y) = \sqrt{y} \implies \frac{d}{dy} g^{-1}(y) = \frac{1}{2\sqrt{y}}.$$

Since $X \in (0, 1)$, $Y = X^2 \in (0, 1)$. Then by Theorem 2.1.5,

$$\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \\
&= 1 \cdot \frac{1}{2\sqrt{y}} \\
&= \frac{1}{2\sqrt{y}}, \quad y \in (0, 1).
\end{aligned}$$

- (b) Let $g(x) = -\log x$. Then g is monotonically decreasing on $(0, 1)$. We get

$$g^{-1}(y) = e^{-y} \implies \frac{d}{dy}g^{-1}(y) = -e^{-y}.$$

Since $X \in (0, 1)$, $Y = \log X \in (0, \infty)$. Then by Theorem 2.1.5,

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy}g^{-1}(y) \right| \\ &= \frac{(n+m+1)!}{n!m!} e^{-ny} (1 - e^{-y})^m \cdot | -e^{-y} | \\ &= \frac{(n+m+1)!}{n!m!} e^{-y(n+1)} (1 - e^{-y})^m, \quad y \in (0, \infty). \end{aligned}$$

- (c) Let $g(x) = e^x$. Then g is monotonically increasing on $(0, \infty)$. We get

$$g^{-1}(y) = \ln y \implies \frac{d}{dy}g^{-1}(y) = \frac{1}{y}.$$

Since $X \in (0, \infty)$, $Y = e^X \in (0, \infty)$. Then by Theorem 2.1.5,

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy}g^{-1}(y) \right| \\ &= \frac{1}{\sigma^2} \ln y e^{-(\ln y / \sigma)^2 / 2} \cdot \frac{1}{y} \\ &= \frac{1}{\sigma^2} \frac{\ln y}{y} e^{-(\ln y / \sigma)^2 / 2}, \quad y \in (0, \infty). \end{aligned}$$

3. First of all,

$$X \in \{0, 1, 2, \dots\} \implies Y \in \left\{0, \frac{1}{2}, \frac{2}{3}, \dots\right\}.$$

Then

$$\begin{aligned} P(Y = y) &= P\left(\frac{X}{X+1} = y\right) \\ &= P\left(1 - \frac{1}{X+1} = y\right) \\ &= P\left(X = \frac{y}{1-y}\right) \\ &= \frac{1}{3} \left(\frac{2}{3}\right)^{y/(1-y)}, \quad y \in \left\{\frac{k}{k+1} : k \in \mathbb{N}_0\right\}. \end{aligned}$$

4. (a) It is not hard to see that $f(x) \geq 0 \forall x \in \mathcal{X}$ as both piecewise functions

are exponentials. We also have

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x) dx &= \int_{-\infty}^0 \frac{1}{2} \lambda e^{\lambda x} + \int_0^{\infty} \frac{1}{2} \lambda e^{-\lambda x} dx \\
&= \left[\frac{1}{2} e^{\lambda x} \right]_{-\infty}^0 + \left[-\frac{1}{2} e^{-\lambda x} \right]_0^{\infty} \\
&= \frac{1}{2} + \frac{1}{2} \\
&= 1.
\end{aligned}$$

(b) For $t \leq 0$,

$$\begin{aligned}
P(X < t) &= \int_{-\infty}^t \frac{1}{2} \lambda e^{\lambda x} dx \\
&= \left[\frac{1}{2} e^{\lambda x} \right]_{-\infty}^t \\
&= \frac{1}{2} e^{\lambda t}.
\end{aligned}$$

For $t > 0$,

$$\begin{aligned}
P(X < t) &= \frac{1}{2} + \int_0^t \frac{1}{2} \lambda e^{-\lambda x} dx \\
&= \frac{1}{2} + \left[-\frac{1}{2} e^{-\lambda x} \right]_0^t \\
&= \frac{1}{2} + \left(-\frac{1}{2} e^{-\lambda t} + \frac{1}{2} \right) \\
&= 1 - \frac{1}{2} e^{-\lambda t}.
\end{aligned}$$

(c) For $t \leq 0$, $P(|X| < t) = 0$. For $t > 0$,

$$\begin{aligned}
P(|X| < t) &= P(-t < X < t) \\
&= \int_{-t}^0 \frac{1}{2} \lambda e^{\lambda x} dx + \int_0^t \frac{1}{2} \lambda e^{-\lambda x} dx \\
&= \left[\frac{1}{2} e^{\lambda x} \right]_{-t}^0 + \left[-\frac{1}{2} e^{-\lambda x} \right]_0^t \\
&= \frac{1}{2} - \frac{1}{2} e^{-\lambda t} + \left(-\frac{1}{2} e^{-\lambda t} + \frac{1}{2} \right) \\
&= 1 - e^{-\lambda t}.
\end{aligned}$$

5. Let $A_0 = \{\pi\}$, $A_1 = (0, \frac{\pi}{2})$, $A_2 = (\frac{\pi}{2}, \pi)$, $A_3 = (\pi, \frac{3\pi}{2})$, $A_4 = (\frac{3\pi}{2}, 2\pi)$, and let $g(x) = g_i(x) = \sin^2 x$. Then for each $A_i (i \neq 0)$, $g_i(x) = g(x) \forall x \in A_i$, $g_i(x)$ is monotone on A_i . Moreover, $\mathcal{Y} = (0, 1)$ is the same for all i , and monotone on A_i , and

$$g^{-1}(y) = \arcsin(\sqrt{y}) \implies \frac{d}{dy} g^{-1}(y) = \frac{1}{2\sqrt{y(1-y)}}$$

is continuous on \mathcal{Y} for all i . Then by Theorem 2.1.8,

$$\begin{aligned} f_Y(y) &= \sum_{i=1}^4 f_X(g^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right| \\ &= 4 \cdot \frac{1}{2\pi} \cdot \left| \frac{1}{2\sqrt{y(1-y)}} \right| \\ &= \frac{1}{\pi\sqrt{y(1-y)}}, \quad y \in (0, 1). \end{aligned}$$

To use the cdf from (2.1.6), we first get that $x_1 = \arcsin(\sqrt{y})$, $x_2 = \pi - \arcsin(\sqrt{y})$. Note

$$P(Y \leq y) = 2P(X \leq x_1) + 2P(X \leq \pi) - 2P(X \leq x_2)$$

Then by differentiating the above we get

$$\begin{aligned} f_Y(y) &= 2f_X(x_1) \cdot \frac{d}{dy}(\sin^{-1} \sqrt{y}) - 2f_X(x_2) \cdot \frac{d}{dy}(\pi - \sin^{-1} \sqrt{y}) \\ &= 2 \cdot \frac{1}{2\pi} \cdot \frac{1}{2\sqrt{y(1-y)}} - 2 \cdot \frac{1}{2\pi} \cdot \left(-\frac{1}{2\sqrt{y(1-y)}} \right) \\ &= \frac{1}{\pi\sqrt{y(1-y)}}, \quad y \in (0, 1). \end{aligned}$$

6. (a) For $g(x) = x^2$, $x \in [-1, 2]$, there is no partition $\{A_i\}$ of the interval which could produce the same \mathcal{Y} for all i . Therefore, we cannot use Theorem 2.1.8 in this case. To solve directly, we get