Statistical Inference Chapter 4

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1. Since (X,Y) is distributed uniformly, the probabilities in this question are simply the ratio of the area satisfying the requirements to the area of the square, which is 2.

   (a) The circle $x^2+y^2<1$ has area π hence the answer is $\frac{\pi}{4}$.

   (b) The line $2x-y=0$ passes throught the origin, hence the answer is $\frac{1}{2}$.

   (c) For any (x,y) in the interior of the square, $|x+y|<2$ hence the probability is 1.

2. This is similar to the proof of Theorem 2.2.5.

   (a)

$$\begin{aligned}\mathbb{E}[ag_1(X,Y)+bg_2(X,Y)+c] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(ag_1(x,y)+bg_2(x,y)+c)f_{X,Y}(x,y)\ dydx \\ &= a\int_{-\infty}^{\infty}g_1(x,y)f_{X,Y}(x,y)\ dydx + b\int_{-\infty}^{\infty}g_2(x,y)f_{X,Y}(x,y)\ dydx \\ &\quad + c\int_{-\infty}^{\infty}f_{X,Y}(x,y)\ dydx \\ &= a\mathbb{E}[g_1(x,y)]+b\mathbb{E}[g_2(x,y)]+c.\end{aligned}$$

   (b)

3. For a fair die, each sample point in the sample space of size 36 has an equally likely chance of happening. Therefore we get that

$$\begin{aligned}P(X=0,Y=0) &= P(\{(1,1),(1,3),(1,5),(2,1),(2,3),(2,5)\}) \\ &= \frac{6}{36}=\frac{1}{6}.\end{aligned}$$

$$\begin{aligned}P(X=0,Y=1) &= P(\{(1,2),(1,4),(1,6),(2,2),(2,4),(2,6)\}) \\ &= \frac{6}{36}=\frac{1}{6}.\end{aligned}$$

$$\begin{aligned}P(X=1,Y=0) &= P(\{(3,1),(3,3),(3,5),(4,1),(4,3),(4,5),(5,1),(5,3),(5,5),(6,1),(6,3),(6,5)\}) \\ &= \frac{12}{36}=\frac{1}{3}.\end{aligned}$$

$$\begin{aligned}P(X=1,Y=1) &= P(\{(3,2),(3,4),(3,6),(4,2),(4,4),(4,6),(5,2),(5,4),(5,6),(6,2),(6,4),(6,6)\}) \\ &= \frac{12}{36}=\frac{1}{3}.\end{aligned}$$

   This matches the pmf of Example 4.1.5.

4. (a)

$$\int_0^1 \int_0^2 x + 2y \; dxdy = \int_0^1 \left[\frac{1}{2}x^2 + 2xy\right]_0^2 dy$$
$$= \int_0^1 2 + 4y \; dy$$
$$= [2y + 2y^2]_0^1$$
$$= 4,$$

which directly implies that $C = \frac{1}{4}$.

(b)

$$f_X(x) = \int_0^1 \frac{1}{4}(x + 2y) \; dy$$
$$= \frac{1}{4}\left[xy + y^2\right]_0^1$$
$$= \frac{1}{4}(x + 1), \; 0 < x < 2.$$

(c) This depends on the values of x and y. If either x or y is ≤ 0, the cdf is just 0. If $x \geq 2$ and $y \geq 1$, the cdf is 1. The remaining cases are:
- $x \in (0, 2)$ and $y \geq 1$.

$$F_{X,Y}(x, y) = \int_0^x \int_0^1 \frac{1}{4}(u + 2v) \; dvdu$$
$$= \frac{1}{4}\int_0^x [uv + v^2]_0^1 \; du$$
$$= \frac{1}{4}\int_0^x u + 1 \; du$$
$$= \frac{1}{4}\left[\frac{1}{2}u^2 + u\right]_0^x$$
$$= \frac{1}{8}x^2 + \frac{1}{4}x.$$

- $x \geq 1$ and $y \in (0, 1)$.

$$F_{X,Y}(x, y) = \int_0^y \int_0^2 \frac{1}{4}(u + 2v) \; dudv$$
$$= \frac{1}{4}\int_0^y \left[\frac{1}{2}u^2 + 2uv\right]_0^2 dv$$
$$= \frac{1}{4}\int_0^y 2 + 4v \; dv$$
$$= \frac{1}{4}[2v + 2v^2]_0^y$$
$$= \frac{1}{2}y^2 + \frac{1}{2}y.$$

- $x \in (0,2)$ and $y \in (0,1)$.

$$\begin{aligned} F_{X,Y}(x,y) &= \int_0^x \int_0^y \frac{1}{4}(u+2v)\ dvdu \\ &= \frac{1}{4}\int_0^x [uv+v^2]_0^y\ du \\ &= \frac{1}{4}\int_0^x uy+y^2\ du \\ &= \frac{1}{4}\left[\frac{1}{2}u^2y+uy^2\right]_0^x \\ &= \frac{1}{8}x^2y+\frac{1}{4}xy^2. \end{aligned}$$

All in all, the cdf for y is

$$F_{X,Y}(x,y) = \begin{cases} 0 & \text{if } x \le 0 \text{ or } y \le 0, \\ \frac{1}{8}x^2y+\frac{1}{4}xy^2 & \text{if } 0 < x < 2 \text{ and } 0 < y < 1, \\ \frac{1}{2}y^2+\frac{1}{2}y & \text{if } x \ge 1 \text{ and } 0 < y < 1, \\ \frac{1}{8}x^2+\frac{1}{4}x & \text{if } 0 < x < 2 \text{ and } y \ge 1. \\ 1 & \text{if } x, y \ge 1. \end{cases}$$

(d) Since $X \in (0,2)$, $Z = \frac{9}{(X+1)^2} \in (1,9)$.

5. (a)

$$\begin{aligned} P(X > \sqrt{Y}) &= \int_0^1 \int_0^{x^2} x+y\ dydx \\ &= \int_0^1 \left[xy+\frac{1}{2}y^2\right]_0^{x^2}\ dx \\ &= \int_0^1 x^3+\frac{1}{2}x^4\ dx \\ &= \left[\frac{1}{4}x^4+\frac{1}{10}x^5\right]_0^1 \\ &= \frac{7}{20}. \end{aligned}$$

(b)

$$\begin{aligned} P(X^2 < Y < X) &= \int_0^1 \int_{x^2}^{x} 2x\ dydx \\ &= \int_0^1 [2xy]_x^{x^2}\ dx \\ &= \int_0^1 2x^2-2x^3\ dx \\ &= \left[\frac{2}{3}x^3-\frac{1}{2}x^4\right]_0^1 \\ &= \frac{1}{6}. \end{aligned}$$

6. Let A, B be the time that A and B arrives respectively. Then $A, B \sim \text{Uniform}(1,2)$. Moreover, A and B are independent hence their joint distribution is the product of their marginals. That is,