Statistical Inference Chapter 1

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1. (a) $\Omega = \{(x_1, x_2, x_3, x_4) : x_i \in \{H, T\}\}$.
   (b) If there are N leaves on the plant, $\Omega = [N]$.
   (c) $\Omega = \{t : t \in \mathbb{R},\ t \geq 0\}$.
   (d) $\Omega = \{w : w \in \mathbb{R}_+\}$.
   (e) If there are n components, $\Omega = \{i/n : i \in \{0, 1, ..., n\}\}$.
2. (a)

$$\begin{aligned} x \in A \setminus B &\iff x \in A \text{ and } x \notin B \\ &\iff x \in A \text{ and } x \notin A \cap B \\ &\iff x \in A \setminus (A \cap B). \end{aligned}$$

   Also,

$$\begin{aligned} x \in A \setminus B &\iff x \in A \text{ and } x \notin B \\ &\iff x \in A \text{ and } x \in B^c \\ &\iff x \in A \cap B^c. \end{aligned}$$

   Therefore $A \setminus B = A \setminus (A \cap B) = A \cap B^c$.

   (b) By the distributive law,

$$\begin{aligned} (B \cap A) \cup (B \cap A^c) &= B \cap (A \cup A^c) \\ &= B. \end{aligned}$$

   (c)

$$\begin{aligned} x \in B \setminus A &\iff x \in B \text{ and } x \notin A \\ &\iff x \in B \text{ and } x \in A^c \\ &\iff x \in B \cap A^c. \end{aligned}$$

   (d) From part b), we have

$$\begin{aligned} A \cup B &= A \cup ((B \cap A) \cup (B \cap A^c)) \\ &= A \cup (B \cap A) \cup A \cup (B \cap A^c) \\ &= A \cup A \cup (B \cap A^c) \\ &= A \cup (B \cap A^c). \end{aligned}$$

3. (a)

$$\begin{aligned} x \in A \cup B &\iff x \in A \text{ or } x \in B \\ &\iff x \in B \cup A. \\ x \in A \cap B &\iff x \in A \text{ and } x \in B \\ &\iff xin B \cap A. \end{aligned}$$

(b)

$$\begin{aligned} x \in A \cup (B \cup C) &= x \in A \text{ or } x \in B \cup C \\ &= x \in A \cup B \text{ or } x \in C \\ &= x \in (A \cup B) \cup C. \end{aligned}$$

(c)

$$\begin{aligned} x \in (A \cup B)^c &\iff x \notin A \cup B \\ &\iff x \in A^c \text{ and } x \in B^c \\ &\iff x \in A^c \cap B^c. \\ x \in (A \cap B)^c &\iff x \notin A \cap B \\ &\iff x \in A^c \text{ or } x \in B^c \\ &\iff x \in A^c \cup B^c. \end{aligned}$$

4. (a) This is $P(A \cup B)$, so we get $P(A) + P(B) - P(A \cap B)$.
   (b) This is $P(A \Delta B)$, so we get $P(A) + P(B) - 2P(A \cap B)$.
   (c) This is again $P(A \cup B)$, so we get $P(A) + P(B) - P(A \cap B)$.
   (d) This is $P((A \cap B)^c)$, so we get $1 - P(A \cap B)$.

5. (a) $A \cap B \cap C = \{$a U.S. birth resulting in identical twin females$\}$.
   (b) $P(A \cap B \cap C) = \frac{1}{90} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{540}$.

6. $p_0 = (1-u)(1-w), p_1 = u(1-w) + w(1-u), p_2 = uw$. For them to be equal,

$$\begin{aligned} p_0 = p_2 &\implies 1 - u - w + uw = uw \\ &\implies u + w = 1, \\ p_1 = p_2 &\implies u + w - 2uw = uw \\ &\implies uw = \frac{1}{3}. \end{aligned}$$

The above two equations imply $u(1-u) = \frac{1}{3}$, which has no real solutions in $\mathbb{R}$. Therefore we can't choose such u, w satisfying $p_0 = p_1 = p_2$.

7. (a) This is just having an extra case of hitting outside of the dart board. So

$$P(\text{scoring } i \text{ points}) = \begin{cases} 1 - \frac{\pi r^2}{A} & i = 0 \\ \frac{\pi r^2}{A} \cdot \frac{1}{5^2}((6-i)^2 - (5-i)^2) & i = 1, ..., 5 \end{cases}$$

(b)

$$P(\text{scoring } i \text{ points}|\text{board is hit}) = \frac{P(\text{scoring } i \text{ points, board is hit})}{P(\text{board is hit})}$$
$$= \frac{\pi r^2}{A} \cdot \frac{1}{5^2}((6-i)^2 - (5-i)^2)/\frac{\pi r^2}{A}$$
$$= \frac{1}{5^2}((6-i)^2 - (5-i)^2),\ i = 1, ..., 5$$

For $i = 0$, we will definitely score given that we hit the board so $P(\text{scoring } 0 \text{ points}|\text{board is hit}) = 0$, which is consistent with the probability distribution in Example 1.2.7 as well.

8. (a) From the example given,

$$P(\text{scoring } i \text{ points}) = \frac{(6-i)^2 - (5-i)^2}{5^2}, i = 1, ..., 5.$$

(b) Expanding the above,

$$\frac{(6-i)^2 - (5-i)^2}{5^2} = \frac{11 - 2i}{r^2},$$

which is a decreasing function of i.

(c)

$$\frac{11 - 2i}{5^2} > 0 \text{for } i = 1, ..., 5$$

hence the first axiom is satisfied.

$$P(S) = P(\text{hitting the board}) = 1,$$

hence the second axiom is satisfied. For $i \neq j$,

$$P(i \cup j) = \text{Area of ring } i + \text{Area of ring } j = P(i) + P(j),$$

hence the third axiom is satisfied so $P(\text{scoring } i \text{ points})$ is a probability function.

9. (a) Suppose $x \in (\cup_\alpha A_\alpha)^c$. Then $x \notin A_\alpha$ for all $\alpha \in \Gamma$ so $x \in A_\alpha^c$ for all $\alpha \in \Gamma$. Therefore $x \in \cap_\alpha A_\alpha$.

Now suppose $x \in \cap_\alpha A_\alpha^c$. Then for all $\alpha \in \Gamma$, $x \in A_\alpha^c$ hence $x \notin A_\alpha$, then $x \notin \cup_\alpha A_\alpha$ so $x \in (\cup_\alpha A_\alpha)^c$.

(b) Suppose $x \in (\cap_\alpha A_\alpha)^c$. Then $x \notin \cap_\alpha A_\alpha$ so $x \notin A_\alpha$ for some $\alpha \in \Gamma$. Then $x \in A_\alpha^c$ for some $\alpha \in \Gamma$. Therefore $x \in \cup_\alpha A_\alpha^c$.

Now suppose $x \in \cup_\alpha A_\alpha^c$. Then $x \in A_\alpha^c$ for some $\alpha \in \Gamma$ so $x \notin A_\alpha$ for some $\alpha \in \Gamma$. Then $x \notin \cap_\alpha$ thus $x \in (\cap_\alpha)^c$.

10. We have

$$\left(\bigcup_{i=1}^n A_i\right)^c = \bigcap_{i=1}^n A_i^c,\ \left(\bigcap_{i=1}^n A_i\right)^c = \bigcup_{i=1}^n A_i^c$$

Proof of first equality:
Suppose $x \in (\cup_{i=1}^n A_i)^c$. Then $x \notin \cup_{i=1}^n A_i$ so $x \notin A_i$ for all i, meaning $x \in A_i^c$ for all i. Therefore $x \in \cap_{i=1}^n A_i^c$. Now suppose $x \in \cap_{i=1}^n A_i^c$. Then $x \notin A_i$ for all i, hence $x \notin \cup_{i=1}^n A_i$, hence $x \in (\cup_{i=1}^n A_i)^c$.

Proof of second equality:
Suppose $x \in (\cap_{i=1}^n A_i)^c$. Then $x \notin \cap_{i=1}^n A_i$ so $x \notin A_i$ and so $x \in A_i^c$ for some i, meaning $x \in \cup_{i=1}^n A_i^c$. Now suppose $x \in \cup_{i=1}^n A_i^c$. Then $x \notin A_i$ for some i hence $x \in (\cap_{i=1}^n A_i)^c$.

11. (a) $\emptyset \in \mathcal{B}$ hence property 1 is satisfied. $\emptyset^c = S \in \mathcal{B}$, $S^c = \emptyset \in \mathcal{B}$ hence property 2 is satisfied. $\emptyset \cup S = S \in \mathcal{B}$ hence property 3 is satisfied so $\mathcal{B}$ is a sigma algebra.

    (b) $\emptyset$ is a subset of S hence $\emptyset \in \mathcal{B}$ hence property 1 is satisfied. For any set $A \in \mathcal{B}$, $A^c = S \setminus A \in B$ hence property 2 is satisfied. Any finite union of elements in $\mathcal{B}$ will be a subset of S, which will be in $\mathcal{B}$ so $\mathcal{B}$ is a sigma algebra.

    (c) Suppose $\mathcal{F}_1, \mathcal{F}_2$ are sigma algebras on the sample space S. Since $\emptyset \in \mathcal{F}_1$ and $\emptyset \in \mathcal{F}_2$, $\emptyset \in \mathcal{F}_1 \cap \mathcal{F}_2$ so property 1 is satisfied. Suppose $A \subseteq \mathcal{F}_1 \cap \mathcal{F}_2$. Then $A \subseteq \mathcal{F}_1$ and $A \subseteq \mathcal{F}_2$. Since $\mathcal{F}_1, \mathcal{F}_2$ are both sigma algebras, $A^c \in \mathcal{F}_1$ and $A^c \in \mathcal{F}_2$. Therefore $A^c \in \mathcal{F}_1 \cap \mathcal{F}_2$ so property 2 is satisfied. Suppose $A_1, A_2, \dots \in \mathcal{F}_1 \cap \mathcal{F}_2$. Then $A_i \in \mathcal{F}_1$ and $A_i \in \mathcal{F}_2$. Since $\mathcal{F}_1, \mathcal{F}_2$ are both sigma algebras, $\cup_i A_i \in \mathcal{F}_1$ and $\cup_i A_i \in \mathcal{F}_2$ hence $\cup_i A_i \in \mathcal{F}_1 \cap \mathcal{F}_2$ hence property 3 is satisfied so $\mathcal{F}_1 \cap \mathcal{F}_2$ is a sigma algebra.

12. (a) 12.1

13. A, B cannot be disjoint. If they are,

$$P(A \cup B) = P(A) + P(B) = \frac{1}{3} + \frac{1}{4} = \frac{13}{12} > 1,$$

    which is not possible.

14. For each element, we can choose to include it or exclude it in the subset. Since there are n elements, the number of subsets that can be formed is 2^n. A more formal proof can be done using bijections.

15. Now that the base case of $k = 2$ has been done, assume that this is true for k seperate tasks. Then for each of the $n_1 \times n_2 \times \cdots \times n_k$ ways, we have n_{k+1} choices for the $(k+1)$th task. Therefore the entire job can be done in

$$\underbrace{1 \times n_{k+1} + 1 \times n_{k+1} + \cdots + 1 \times n_{k+1}}_{n_1 \times \ldots \times n_k \text{ terms}} = n_1 n_2 \cdots n_{k+1}.$$

16. (a) 26^3

    (b) $26^3 + 26^2$

    (c) $26^4 + 26^3 + 26^2$

17. This is just choosing 2 numbers out of n of them, which is $\binom{n}{2} = \frac{n(n+1)}{2}$.

18. There are a total of n^n ways of putting n balls into n cells. For exactly one cell to be empty, there will also be another cell which has exactly 2 balls in it. Therefore there are $\binom{n}{2}$ ways of picking these special buckets. Since the order of putting in the balls matters, the answer is $\binom{n}{2} n!/n^n$.

19. (a) By part (b), this is $\binom{6}{4} = 15$.

    (b) We can consider the n variables as bins, and the r partial derivatives as balls. Then we are putting r unlabeled balls into n unlabeled bins. There are a total of $\binom{n+r-1}{n-1} = \binom{n+r-1}{r}$ ways of doing this.

20. First of all, there are many different ways such that there is at least one call per day. Staying consistent with Casella's answers, if there is 6 calls on 1 day and 1 call on the other six days, we will denote this configuration as 6111111. All possible configs and the number of ways to form them are shown in the table below:

Config	Number of Ways	Answer
6111111	$7\binom{12}{7} \cdot 6!$	4656960
5211111	$7\binom{12}{5} \cdot 6\binom{7}{2} \cdot 5!$	82825280
4221111	$7\binom{12}{4} \cdot \binom{6}{2}\binom{8}{2}\binom{6}{2} \cdot 4!$	523908000
4311111	$7\binom{12}{4} \cdot 6\binom{8}{3} \cdot 5!$	139708800
3321111	$\binom{7}{2}\binom{12}{3}\binom{9}{3} \cdot 5\binom{6}{2} \cdot 4!$	698544000
3222111	$7\binom{12}{3} \cdot \binom{6}{3}\binom{9}{2}\binom{7}{2}\binom{5}{2} \cdot 3!$	1397088000
2222211	$\binom{7}{5}\binom{12}{2}\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2} \cdot 2!$	314344800
Total		3162075840

For example, for the config 6111111, there are $\binom{12}{6}$ ways for picking the calls for the day with 6 calls, 7 ways for the 6-call day to be in, and 6! ways for rearranging the rest of the 1-call days. A similar reasoning follows for the rest of the configs as well. All in all, the answer is about

$$\frac{3162075840}{7^{12}} \approx 0.2285.$$

21. There are $\binom{2n}{2r}$ ways of choosing the shoes. For there to be no matching pair, there are $\binom{n}{2r}$ ways of choosing, and for each choice within the $2r$ shoes, it can be either a left or right foot so there is a factor of 2^{2r}. Therefore out final answer is $\binom{n}{2r}2^{2r}/\binom{2n}{2r}$.

22. (a) We need 15 days from each month, hence our answer is

$$\frac{\binom{31}{15}\binom{30}{15}\cdots\binom{31}{15}}{\binom{366}{180}} \approx 0.167 \times 10^{-8}.$$

    (b) We can just exclude the days from September so our answer is $\binom{336}{30}/\binom{366}{30}$.

23. There can be 0 to n heads for both players, which are disjoint events. Therefore

$$\begin{aligned}
P(\text{Same number of heads}) &= \Big[\sum_{x=0}^{n}\binom{n}{x}\Big(\frac{1}{2}\Big)^x\Big(\frac{1}{2}\Big)^{n-x}\Big]^2 \\
&= \Big(\frac{1}{4}\Big)^n \sum_{x=0}^{n}\binom{n}{x}^2 \\
&= \binom{2n}{n}\Big(\frac{1}{4}\Big)^n.
\end{aligned}$$

(Note that the summation ends up in $\binom{2n}{n}$ as one can think about this being equivalent to choosing n people from $2n$ people: We divide the $2n$ people into two groups of n people. We can pick k people from the first group and pick $n-k$ from the second group. A more formal proof uses generating functions.)

24. (a) Player A can win on the 1st, 3rd, ..., toss. We have

$$\begin{aligned} P(\text{A wins}) &= \sum_{k=1}^{\infty} P(\text{A wins on } k\text{th toss}) \\ &= \sum_{k=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{2k-2} \\ &= \frac{2}{3}. \end{aligned}$$

(b) With the same idea as above,

$$\begin{aligned} P(\text{A wins}) &= \sum_{k=1}^{\infty} P(\text{A wins on } k\text{th toss}) \\ &= \sum_{k=1}^{\infty} p(1-p)^{2k-2} \\ &= \frac{p}{1-(1-p)^2}. \end{aligned}$$

(c) Taking the derivative with respect to p,

$$\frac{d}{dp}\frac{p}{1-(1-p)^2} = \frac{p^2}{(1-(1-p)^2)^2} > 0.$$

Therefore this function is an increasing function in p, and its minimum occurs at $p = 0$. By L'Hopital's rule we have

$$\lim_{p\to 0^+} \frac{p}{1-(1-p)^2} = \frac{1}{2},$$

hence for $p \in (0,1)$, $P(\text{A wins}) > \frac{1}{2}$.

25. Suppose that the order matters for the two children. Then

$$\begin{aligned} &P(\text{Both children are boys — at least one is a boy}) \\ =\ &\frac{P(\text{Both children are boys, at least one is a boy})}{P(\text{At least one is a boy})} \\ =\ &\frac{1}{3}. \end{aligned}$$

26. Let X be the number of tosses until a 6 appears. Then $X \sim \text{Geom}(\frac{1}{6})$.

$$\begin{aligned} P(X > 5) &= 1 - P(X \leq 4) \\ &= 1 - \sum_{k=0}^{4} \frac{1}{6}\left(\frac{5}{6}\right)^k \\ &= ... \end{aligned}$$

27. (a) If n is odd, each k term cancels out with the $n-k$ term so the statement is correct. If n is even, by Pascal's identity,

(b) By the Binomial Theorem, we have

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Taking derivatives with respect to x both sides gives

$$n(1+x)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k} x^{k-1}.$$

Plugging in $x=1$ gives the result.

(c)

$$\begin{aligned}\sum_{k=1}^{n}(-1)^{k+1}k\binom{n}{k} &= \sum_{k=1}^{n}(-1)^{k+1}n\binom{n-1}{k-1} \\ &= n\sum_{j=0}^{n}(-1)^{j}\binom{n-1}{j} \\ &= 0 \quad \text{(From part a.)}\end{aligned}$$

Here we used the formula $k\binom{n}{k} = n\binom{n-1}{k-1}, k > 0$.

28. We have that

$$\int_0^n \log x \; dx = [x\log x - x]_0^n = n\log n - n.$$

$$\begin{aligned}\int_1^{n+1} \log x \; dx &= [x\log x - x]_1^{n+1} \\ &= (n+1)\log(n+1) - (n+1) - (\log 1 - 1) \\ &= (n+1)\log(n+1) - n.\end{aligned}$$

Then we get the average of the two integrals to be

$$\begin{aligned}\frac{1}{2}\Big(\int_0^n \log x \; dx + \int_1^{n+1} \log x \; dx\Big) &= \frac{1}{2}(n\log n - n + (n+1)\log(n+1) - n) \\ &\approx (n+\frac{1}{2})\log n - n\end{aligned}$$

Define the sequence $a_n = \log(n!) - (n+\frac{1}{2})\log n - n$. Then for the problem, it is enough to show that $\lim_{n\to\infty} a_n = c$ for some nonzero constant c. To avoid the factorial, consider

$$a_n - a_{n+1} = \Big(n+\frac{1}{2}\Big)\log\Big(1+\frac{1}{n}\Big) - 1.$$

By the comparison test, the series above converges hence has a limit. Hence we get

$$\lim_{N\to\infty}\sum_{n=1}^{N} a_n - a_{n+1} = \lim_{N\to\infty} a_1 - a_{N+1} = c \implies \lim_{n\to\infty} a_n = a_1 - c,$$

which is a constant hence the proof is complete.

29. (a) Ordered samples of 4, 4, 12, 12:
$(4,4,12,12),(4,12,4,12),(4,12,12,4),(12,4,4,12),(12,4,12,4),(12,12,4,4)$.
Ordered Samples of 2, 9, 9, 12:
$(2,9,9,12),(2,9,12,9),(2,12,9,9),(9,2,9,12),(9,2,12,9),(12,2,9,9)$,
$(9,9,2,12),(9,12,2,9),(12,9,2,9),(9,9,12,2),(9,12,9,2),(12,9,9,2)$.

(b) Same as part a.

(c) There are a total of 6^6 ways of drawing an ordered sample with replacement from $1,2,7,8,14,20$. There are $\frac{6!}{2!2!} = 180$ ways of forming the ordered sample $2,7,7,8,14,14$. Therefore the probability of getting the specific unordered sample is just $\frac{180}{6^6}$.

(d) There are $k!$ ways of ordering the sample. For each number, the order with a different number is considered the same sample. Therefore the answer is

$$\frac{k!}{k_1!k_2!\cdots k_m!}.$$

(e) We can think of the m distinct numbers as m bins, and creating a sample of size k with replacement as putting k balls in the m bins. From before, we already know that there are a total of $\binom{k+m-1}{k}$ ways of doing this.

30.

31. (a) There are $n!$ ways of generating the ordered set $\{x_1,...,x_n\}$ from the set, and there are n^n ways of generating size n ordered samples from the set. Therefore the probability with the average being $(x_1+\cdots+x_n)/n$ is just $\frac{n!}{n^n}$. Now consider any other set having a different sample average. Then the outcome will have m numbers repeated $k_1,\ldots,k_m$ times respectively, and at least one of the k_i's will satisfy $2 \le k_i \le n$. Hence the probability of getting this sample is then

$$\frac{n!}{k_1!k_2!\cdots k_m!n^n} < \frac{n!}{n^n}, \text{ since } k_1\cdots k_m > 1$$

. Hence the sample with average $(x_1+\cdots+x_n)/n$ is the most likely one.

32.

33. Let M/F denote the event that a person is male/female, and let C denote the event that a person is color-blind. Using Bayes' Rule,

$$\begin{aligned} P(M|C) &= \frac{P(C|M)P(M)}{P(C|F)P(F)+P(C|M)P(M)} \\ &= \frac{0.05\cdot 0.5}{0.0025\cdot 0.5+0.05\cdot 0.5} \\ &\approx 0.9524. \end{aligned}$$

34. (a) Let L_i be the event that the rodent is from litter i, B be the event that the rodent has brown hair, and G be the event that the rodent has grey hair. By the Law

of Total Probability,

$$\begin{aligned} P(B) &= P(B|L_1)P(L_1) + P(B|L_2)P(L_2) \\ &= \frac{2}{3} \cdot \frac{1}{2} + \frac{3}{5} \cdot \frac{1}{2} \\ &= \frac{19}{30}. \end{aligned}$$

(b) With the same notation as above,

$$\begin{aligned} P(L_1|B) &= \frac{P(B|L_1)P(L_1)}{P(B)} \\ &= \frac{1/3}{19/30} \\ &= \frac{10}{19}. \end{aligned}$$