Statistical Inference Chapter 3

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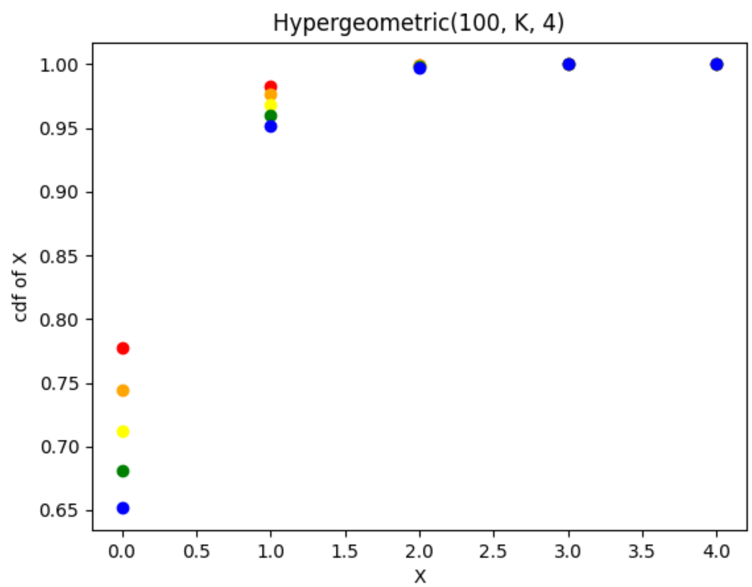
1. We first note that the pmf of X is
$$p_X(x) = \frac{1}{N_1 - N_0 + 1},\ x \in \{N_0, N_0 + 1, ..., N_1\}.$$
Then we get the expectation to be
$$\begin{aligned}\mathbb{E}[X] &= \sum_{x=N_0}^{N_1} x \frac{1}{N_1 - N_0 + 1} \\ &= \frac{1}{N_1 - N_0 + 1} \cdot \frac{N_1 - N_0 + 1}{2}(2N_0 + (N_1 - N_0 + 1 - 1)) \\ &= \frac{N_1 + N_0}{2}.\end{aligned}$$
As for the variance, we get
$$\begin{aligned}\mathbb{E}[X^2] &= \sum_{x=N_0}^{N_1} x^2 \frac{1}{N_1 - N_0 + 1} \\ &= \frac{1}{N_1 - N_0 + 1}\Big(\sum_{x=1}^{N_1} x^2 - \sum_{x=1}^{N_0 - 1} x^2\Big) \\ &= \frac{1}{N_1 - N_0 + 1}\Big(\frac{N_1(N_1 + 1)(N_1 + 2) - (N_0 - 1)(N_0)(2N_0 - 1)}{6}\Big)\end{aligned}$$
So that
$$\begin{aligned}Var(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= 1\end{aligned}$$

2. Let X = number of defective parts in the sample. Then
$X \sim$ Hypergeometric$(100, n, K)$.

   (a) Firstly, we need $n = 6$ because for the same K, increasing n decreases the value of the Hypergeometric pmf (image shown at end of answer). Then with $n = 6$,
$$\begin{aligned}P(X = 0) &= \frac{\binom{6}{0}\binom{94}{K}}{\binom{100}{K}} \\ &= \frac{(100 - k) \cdots (100 - K - 5)}{100 \cdots 95}.\end{aligned}$$

After some trial and error with the calculations, we have that when $K = 31$, $P(X = 0) = 0.10056$, but when $K = 32$, $P(X = 0) = 0.09182$. Therefore, the sample size must be at least 32.



(b) By the same reasoning above, we need $n = 6$. Then with this n,

$$P(X = 0 \text{ or } 1) = \frac{\binom{6}{0}\binom{94}{K}}{\binom{100}{K}} + \frac{\binom{6}{1}\binom{94}{K-1}}{\binom{100}{K}}.$$

Again, by trial and error, when $K = 50$, $P(X = 0 \text{ or } 1) = 0.10220$, but when $K = 51$, $P(X = 0 \text{ or } 1) = 0.09331$ hence the sample size must be at least 51.

3. In this case a car has to pass through for the first second, no car passes in the next three seconds (three second wait), and no car passes through for another three seconds (person is crossing the road).