

## Statistical Inference Chapter 2

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1. (a) Let  $g(x) = x^3$ . Then  $g$  is monotonically increasing on  $(0, 1)$ . We get

$$g^{-1}(y) = y^{1/3} \implies \frac{d}{dy}g^{-1}(y) = \frac{1}{3y^{2/3}}.$$

Since  $X \in (0, 1)$ ,  $Y = X^3 \in (0, 1)$ . Then by Theorem 2.1.5,

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy}g^{-1}(y) \right| \\ &= 42(y^{1/3})^5(1 - y^{1/3}) \cdot \frac{1}{3y^{2/3}} \\ &= 14y(1 - y^{1/3}), \quad y \in (0, 1). \end{aligned}$$

We also have

$$\begin{aligned} \int_0^1 14y(1 - y^{1/3}) \, dy &= 14 \int_0^1 y - y^{4/3} \, dy \\ &= 14 \left[ \frac{1}{2}y^2 - \frac{3}{7}y^{7/3} \right]_0^1 \\ &= 14 \left( \frac{1}{2} - \frac{3}{7} \right) \\ &= 1. \end{aligned}$$

- (b) Let  $g(x) = 4x + 3$ . Then  $g$  is monotonically increasing on  $(0, \infty)$ . We get

$$g^{-1}(y) = \frac{y-3}{4} \implies \frac{d}{dy}g^{-1}(y) = \frac{1}{4}.$$

Since  $X \in (0, \infty)$ ,  $Y = 4X + 3 \in (3, \infty)$ . Then by Theorem 2.1.5,

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy}g^{-1}(y) \right| \\ &= 7e^{-7 \cdot \frac{y-3}{4}} \cdot \frac{1}{4} \\ &= \frac{7}{4}e^{\frac{21}{4} - \frac{7}{4}y}, \quad y \in (3, \infty). \end{aligned}$$

We also have

$$\begin{aligned}
\int_3^\infty \frac{7}{4} e^{\frac{21}{4} - \frac{7}{4}y} dy &= \frac{7}{4} e^{\frac{21}{4}} \int_3^\infty e^{-\frac{7}{4}y} dy \\
&= \frac{7}{4} e^{\frac{21}{4}} \left[ -\frac{4}{7} e^{-\frac{7}{4}y} \right]_3^\infty \\
&= \frac{7}{4} e^{\frac{21}{4}} \left( \frac{4}{7} e^{-\frac{21}{4}} \right) \\
&= 1.
\end{aligned}$$

(c) Let  $g(x) = x^2$ . Then  $g$  is monotonically increasing on  $(0, 1)$ . We get

$$g^{-1}(y) = \sqrt{y} \implies \frac{d}{dy} g^{-1}(y) = \frac{1}{2\sqrt{y}}.$$

Since  $X \in (0, 1)$ ,  $Y = X^2 \in (0, 1)$ . Then by Theorem 2.1.5,

$$\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \\
&= 30y(1 - \sqrt{y})^2 \cdot \frac{1}{2\sqrt{y}} \\
&= 15\sqrt{y}(1 - \sqrt{y})^2, \quad y \in (0, 1).
\end{aligned}$$

We also have

$$\begin{aligned}
\int_0^1 15\sqrt{y}(1 - \sqrt{y})^2 dy &= 15 \int_0^1 \sqrt{y} - 2y + y^{3/2} dy \\
&= 15 \left[ \frac{2}{3} y^{3/2} - y^2 + \frac{2}{5} y^{5/2} \right]_0^1 \\
&= 15 \left( \frac{2}{3} - 1 + \frac{2}{5} \right) \\
&= 1.
\end{aligned}$$

2. (a) Let  $g(x) = x^2$ . Then  $g$  is monotonically increasing on  $(0, 1)$ . We get

$$g^{-1}(y) = \sqrt{y} \implies \frac{d}{dy} g^{-1}(y) = \frac{1}{2\sqrt{y}}.$$

Since  $X \in (0, 1)$ ,  $Y = X^2 \in (0, 1)$ . Then by Theorem 2.1.5,

$$\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \\
&= 1 \cdot \frac{1}{2\sqrt{y}} \\
&= \frac{1}{2\sqrt{y}}, \quad y \in (0, 1).
\end{aligned}$$

(b) Let  $g(x) = -\log x$ . Then  $g$  is monotonically decreasing on  $(0, 1)$ . We get

$$g^{-1}(y) = e^{-y} \implies \frac{d}{dy} g^{-1}(y) = -e^{-y}.$$

Since  $X \in (0, 1)$ ,  $Y = \log X \in (0, \infty)$ . Then by Theorem 2.1.5,

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \\ &= \frac{(n+m+1)!}{n!m!} e^{-ny} (1 - e^{-y})^m \cdot |-e^{-y}| \\ &= \frac{(n+m+1)!}{n!m!} e^{-y(n+1)} (1 - e^{-y})^m, \quad y \in (0, \infty). \end{aligned}$$

(c) Let  $g(x) = e^x$ . Then  $g$  is monotonically increasing on  $(0, \infty)$ . We get

$$g^{-1}(y) = \ln y \implies \frac{d}{dy} g^{-1}(y) = \frac{1}{y}.$$

Since  $X \in (0, \infty)$ ,  $Y = e^X \in (0, \infty)$ . Then by Theorem 2.1.5,

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \\ &= \frac{1}{\sigma^2} \ln y e^{-(\ln y / \sigma)^2 / 2} \cdot \frac{1}{y} \\ &= \frac{1}{\sigma^2} \frac{\ln y}{y} e^{-(\ln y / \sigma)^2 / 2}, \quad y \in (0, \infty). \end{aligned}$$

3. First of all,

$$X \in \{0, 1, 2, \dots\} \implies Y \in \left\{0, \frac{1}{2}, \frac{2}{3}, \dots\right\}.$$

Then

$$\begin{aligned} P(Y = y) &= P\left(\frac{X}{X+1} = y\right) \\ &= P\left(1 - \frac{1}{X+1} = y\right) \\ &= P\left(X = \frac{y}{1-y}\right) \\ &= \frac{1}{3} \left(\frac{2}{3}\right)^{y/(1-y)}, \quad y \in \left\{\frac{k}{k+1} : k \in \mathbb{N}_0\right\}. \end{aligned}$$

4. (a) It is not hard to see that  $f(x) \geq 0 \forall x \in \mathcal{X}$  as both piecewise functions are exponentials. We also have

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) dx &= \int_{-\infty}^0 \frac{1}{2} \lambda e^{\lambda x} + \int_0^{\infty} \frac{1}{2} \lambda e^{-\lambda x} dx \\ &= \left[ \frac{1}{2} e^{\lambda x} \right]_{-\infty}^0 + \left[ -\frac{1}{2} e^{-\lambda x} \right]_0^{\infty} \\ &= \frac{1}{2} + \frac{1}{2} \\ &= 1. \end{aligned}$$

(b) For  $t \leq 0$ ,

$$\begin{aligned} P(X < t) &= \int_{-\infty}^t \frac{1}{2} \lambda e^{\lambda x} dx \\ &= \left[ \frac{1}{2} e^{\lambda x} \right]_{-\infty}^t \\ &= \frac{1}{2} e^{\lambda t}. \end{aligned}$$

For  $t > 0$ ,

$$\begin{aligned} P(X < t) &= \frac{1}{2} + \int_0^t \frac{1}{2} \lambda e^{-\lambda x} dx \\ &= \frac{1}{2} + \left[ -\frac{1}{2} e^{-\lambda x} \right]_0^t \\ &= \frac{1}{2} + \left( -\frac{1}{2} e^{-\lambda t} + \frac{1}{2} \right) \\ &= 1 - \frac{1}{2} e^{-\lambda t}. \end{aligned}$$

(c) For  $t \leq 0$ ,  $P(|X| < t) = 0$ . For  $t > 0$ ,

$$\begin{aligned} P(|X| < t) &= P(-t < X < t) \\ &= \int_{-t}^0 \frac{1}{2} \lambda e^{\lambda x} dx + \int_0^t \frac{1}{2} \lambda e^{-\lambda x} dx \\ &= \left[ \frac{1}{2} e^{\lambda x} \right]_{-t}^0 + \left[ -\frac{1}{2} e^{-\lambda x} \right]_0^t \\ &= \frac{1}{2} - \frac{1}{2} e^{-\lambda t} + \left( -\frac{1}{2} e^{-\lambda t} + \frac{1}{2} \right) \\ &= 1 - e^{-\lambda t}. \end{aligned}$$

5. Let  $A_0 = \{\pi\}$ ,  $A_1 = (0, \frac{\pi}{2})$ ,  $A_2 = (\frac{\pi}{2}, \pi)$ ,  $A_3 = (\pi, \frac{3\pi}{2})$ ,  $A_4 = (\frac{3\pi}{2}, 2\pi)$ , and let  $g(x) = g_i(x) = \sin^2 x$ . Then for each  $A_i$  ( $i \neq 0$ ),  $g_i(x) = g(x) \forall x \in A_i$ ,  $g_i(x)$  is monotone on  $A_i$ . Moreover,  $\mathcal{Y} = (0, 1)$  is the same for all  $i$ , and monotone on  $A_i$ , and

$$g^{-1}(y) = \arcsin(\sqrt{y}) \implies \frac{d}{dy} g^{-1}(y) = \frac{1}{2\sqrt{y(1-y)}}$$

is continuous on  $\mathcal{Y}$  for all  $i$ . Then by Theorem 2.1.8,

$$\begin{aligned} f_Y(y) &= \sum_{i=1}^4 f_X(g^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right| \\ &= 4 \cdot \frac{1}{2\pi} \cdot \left| \frac{1}{2\sqrt{y(1-y)}} \right| \\ &= \frac{1}{\pi\sqrt{y(1-y)}}, \quad y \in (0, 1). \end{aligned}$$

To use the cdf from (2.1.6), we first get that  $x_1 = \arcsin(\sqrt{y})$ ,  $x_2 = \pi - \arcsin(\sqrt{y})$ . Note

$$P(Y \leq y) = 2P(X \leq x_1) + 2P(X \leq \pi) - 2P(X \leq x_2)$$

Then by differentiating the above we get

$$\begin{aligned} f_Y(y) &= 2f_X(x_1) \cdot \frac{d}{dy}(\sin^{-1} \sqrt{y}) - 2f_X(x_2) \cdot \frac{d}{dy}(\pi - \sin^{-1} \sqrt{y}) \\ &= 2 \cdot \frac{1}{2\pi} \cdot \frac{1}{2\sqrt{y(1-y)}} - 2 \cdot \frac{1}{2\pi} \cdot \left(-\frac{1}{2\sqrt{y(1-y)}}\right) \\ &= \frac{1}{\pi\sqrt{y(1-y)}}, \quad y \in (0, 1). \end{aligned}$$

6. (a) Let  $g(x) = |x|^3$ ,  $g_1(x) = -x^3$ ,  $g_2(x) = x^3$ . Let  $A_0 = \{0\}$ ,  $A_1 = (-\infty, 0)$ ,  $A_2 = (0, \infty)$ . Then we get  $\mathcal{Y} = (0, \infty)$  so that all conditions for Theorem 2.1.8 are satisfied. Then

$$g_1^{-1}(y) = -y^{1/3} \implies \frac{d}{dy}g_1^{-1}(y) = -\frac{1}{3y^{2/3}}.$$

$$g_2^{-1}(y) = y^{1/3} \implies \frac{d}{dy}g_2^{-1}(y) = \frac{1}{3y^{2/3}}.$$

Then by Theorem 2.1.8,

$$\begin{aligned} f_Y(y) &= \sum_{i=1}^2 f_X(g_i^{-1}(y)) \left| \frac{d}{dy}g_i^{-1}(y) \right| \\ &= \frac{1}{2}e^{-y^{1/3}} \cdot \left| -\frac{1}{3y^{2/3}} \right| + \frac{1}{2}e^{-y^{1/3}} \cdot \left| \frac{1}{3y^{2/3}} \right| \\ &= \frac{1}{3}y^{-2/3}e^{-y^{1/3}}, \quad y \in (0, \infty). \end{aligned}$$

- (b) Let  $g(x) = g_1(x) = g_2(x) = 1 - x^2$ . Let  $A_0 = \{0\}$ ,  $A_1 = (-1, 0)$ ,  $A_2 = (0, 1)$ . Then we get

$$g_1^{-1}(y) = -\sqrt{1-y} \implies \frac{d}{dy}g_1^{-1}(y) = \frac{1}{2\sqrt{1-y}},$$

$$g_2^{-1}(y) = \sqrt{1-y} \implies \frac{d}{dy}g_2^{-1}(y) = -\frac{1}{2\sqrt{1-y}}.$$

Then we get  $\mathcal{Y} = (0, 1)$  so that all conditions for Theorem 2.1.8 are satisfied.

Then by Theorem 2.1.8,

$$\begin{aligned}
f_Y(y) &= \sum_{i=1}^2 f_X(g^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right| \\
&= \frac{3}{8} (-\sqrt{1-y} + 1)^2 \cdot \left| \frac{1}{2\sqrt{1-y}} \right| \\
&\quad + \frac{3}{8} (\sqrt{1-y} + 1)^2 \cdot \left| -\frac{1}{2\sqrt{1-y}} \right| \\
&= \frac{3}{8} (1-y - 2\sqrt{1-y} + 1) \cdot \frac{1}{2\sqrt{1-y}} \\
&\quad + \frac{3}{8} (1-y + 2\sqrt{1-y} + 1) \cdot \frac{1}{2\sqrt{1-y}} \\
&= \frac{3}{8} (1-y)^{1/2} + \frac{3}{8} (1-y)^{-1/2}, \quad y \in (0, 1).
\end{aligned}$$

(Note for  $g_1$  we chose the negative root because  $x < 0$ ).

- (c) Let  $g_1(x) = 1 - x^2$ ,  $g_2(x) = 1 - x$ . Let  $A_0 = \{0\}$ ,  $A_1 = (-1, 0)$ ,  $A_2 = (0, 1)$ . Then we get

$$\begin{aligned}
g_1^{-1}(y) &= -\sqrt{1-y} \implies \frac{d}{dy} g_1^{-1}(y) = \frac{1}{2\sqrt{1-y}}. \\
g_2^{-1}(y) &= 1 - y \implies \frac{d}{dy} g_2^{-1}(y) = -1.
\end{aligned}$$

Then we get  $\mathcal{Y} = (0, 1)$  so that all conditions for Theorem 2.1.8 are satisfied. Then by Theorem 2.1.8,

$$\begin{aligned}
f_Y(y) &= \sum_{i=1}^2 f_X(g^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right| \\
&= \frac{3}{8} (-\sqrt{1-y} + 1)^2 \cdot \left| \frac{1}{2\sqrt{1-y}} \right| \\
&\quad + \frac{3}{8} (1-y + 1)^2 \cdot |-1| \\
&= \frac{3}{16\sqrt{1-y}} (1 - \sqrt{1-y})^2 + \frac{3}{8} (2-y)^2, \quad y \in (0, 1).
\end{aligned}$$

7. (a) For  $g(x) = x^2$ ,  $x \in [-1, 2]$ , there is no partition  $\{A_i\}$  of the interval which could produce the same  $\mathcal{Y}$  for all  $i$ . Therefore, we cannot use Theorem 2.1.8 in this case. To solve directly, we get

$$\begin{aligned}
f_Y(y) &= \sum_{i=1}^4 f_X(g^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right| \\
&=
\end{aligned}$$

8. (a) It is easy to see that

$$\lim_{x \rightarrow -\infty} F_X(x) = 0, \quad \lim_{x \rightarrow +\infty} F_X(x) = 1.$$

Moreover, both  $0$  and  $1 - e^{-x}$  are non-decreasing on their respective intervals, and

$$\lim_{x \rightarrow 0^+} F_X(x) = 0$$

so that  $F_X$  is right continuous and therefore is a valid cdf. Its inverse is

$$F_X^{-1}(y) = -\ln(1 - y)$$

(b) Again, we can see that

$$\lim_{x \rightarrow -\infty} F_X(x) = 0, \quad \lim_{x \rightarrow +\infty} F_X(x) = 1.$$

$e^x/2, 1 - (e^{-x}/2)$  are increasing, and  $1/2$  is noncreasing on their respective intervals, and

$$\lim_{x \rightarrow 0} F_X(x) = \frac{1}{2}, \quad \lim_{x \rightarrow 1} F_X(x) = \frac{1}{2}$$

so that  $F_X$  is continuous hence right continuous so is a valid cdf. Its inverse is

$$F_X^{-1}(y) = \begin{cases} \ln(2x) & 0 < y < \frac{1}{2} \\ -\ln(2 - 2x) & \frac{1}{2} \leq y < 1. \end{cases}$$

(c) Again, we can see that

$$\lim_{x \rightarrow -\infty} F_X(x) = 0, \quad \lim_{x \rightarrow +\infty} F_X(x) = 1.$$

$e^x/4, 1 - (e^{-x}/4)$  are both increasing on their respective intervals, and

$$\lim_{x \rightarrow 0^+} F_X(x) = \frac{3}{4} = F_X(0)$$

so that  $F_X$  is right continuous and therefore is a valid cdf. Its inverse is

$$F_X^{-1}(y) = \begin{cases} \ln(4x) & 0 < y < \frac{1}{4} \\ -\ln(4 - 4x) & \frac{3}{4} \leq y < 1 \end{cases}$$

9. We first find the cdf of  $X$ :

$$F_X(x) = \begin{cases} 0 & x \leq 1 \\ \frac{1}{4}(x - 1)^2 & 1 < x < 3 \\ 1 & x \geq 3 \end{cases}$$

Then we have

$$\lim_{x \rightarrow 1} F_X(x) = 0, \quad \lim_{x \rightarrow 3} F_X(x) = 1.$$

hence  $X$  has a continuous cdf. Let  $u(x) = F_X(x)$ . Then  $u(x)$  is nondecreasing and by Theorem 2.1.10,  $Y = u(X)$  has a uniform distribution.

10. (a)

11. (a)

$$\begin{aligned}
\mathbb{E}[X^2] &= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}} \left( [-xe^{-\frac{x^2}{2}}]_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} dx \right) \\
&= \frac{1}{\sqrt{2\pi}} \cdot \sqrt{2\pi} \\
&= 1.
\end{aligned}$$

From Example 2.1.7,

$$\begin{aligned}
f_Y(y) &= \frac{1}{2\sqrt{y}} (f_X(\sqrt{y}) + f_X(-\sqrt{y})) \\
&= \frac{1}{2\sqrt{y}} \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \right) \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{y}} e^{-\frac{y}{2}}, \quad y > 0.
\end{aligned}$$

Using integration by parts,

$$\begin{aligned}
\mathbb{E}[Y] &= \int_0^{\infty} y \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{y}} e^{-\frac{y}{2}} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \sqrt{y} e^{-\frac{y}{2}} dy \\
&= \frac{1}{\sqrt{2\pi}} \left( [-2\sqrt{y} e^{-\frac{y}{2}}]_0^{\infty} + \int_0^{\infty} \frac{1}{\sqrt{y}} e^{-\frac{y}{2}} dy \right) \\
&= \frac{1}{\sqrt{2\pi}} \cdot \sqrt{2\pi} \\
&= 1.
\end{aligned}$$

(Note that the term on the right is the kernel of the Chi-squared distribution defined in Example 2.1.9 earlier.)

(b) We first find the cdf of  $Y$ .

$$F_Y(y) = P(|X| \leq y) = P(-y \leq X \leq y) = F_X(y) - F_X(-y).$$

Therefore the pdf of  $Y$  is just

$$\begin{aligned}
f_Y(y) &= f_X(y) + f_X(-y) \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \\
&= \sqrt{\frac{2}{\pi}} e^{-\frac{y^2}{2}}, \quad y \in [0, \infty).
\end{aligned}$$



Therefore we can find the mean and variance of  $Y$ :

$$\begin{aligned}\mathbb{E}[Y] &= \int_0^\infty y \sqrt{\frac{2}{\pi}} e^{-\frac{y^2}{2}} dy \\ &= \sqrt{\frac{2}{\pi}} [-e^{-\frac{y^2}{2}}]_0^\infty \\ &= \sqrt{\frac{2}{\pi}}.\end{aligned}$$

From part (a),

$$\mathbb{E}[Y^2] = \mathbb{E}[|X|^2] = \mathbb{E}[X^2] = 1.$$

Therefore,

$$Var(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = 1 - \frac{2}{\pi}.$$