Statistical Inference Chapter 2

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1. (a) Let $g(x) = x^3$. Then g is monotonically increasing on $(0, 1)$. We get

$$g^{-1}(y) = y^{1/3} \implies \frac{d}{dy} g^{-1}(y) = \frac{1}{3y^{2/3}}.$$

Since $X \in (0, 1),\ Y = X^3 \in (0, 1)$. Then by Theorem 2.1.5,

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \Big| \frac{d}{dy} g^{-1}(y) \Big| \\ &= 42(y^{1/3})^5(1 - y^{1/3}) \cdot \frac{1}{3y^{2/3}} \\ &= 14y(1 - y^{1/3}),\ y \in (0, 1). \end{aligned}$$

We also have

$$\begin{aligned} \int_0^1 14y(1 - y^{1/3})\ dy &= 14 \int_0^1 y - y^{4/3}\ dy \\ &= 14 \Big[\frac{1}{2} y^2 - \frac{3}{7} y^{7/3} \Big]_0^1 \\ &= 14 (\frac{1}{2} - \frac{3}{7}) \\ &= 1. \end{aligned}$$

(b) Let $g(x) = 4x + 3$. Then g is monotonically increasing on $(0, \infty)$. We get

$$g^{-1}(y) = \frac{y - 3}{4} \implies \frac{d}{dy} g^{-1}(y) = \frac{1}{4}.$$

Since $X \in (0, \infty),\ Y = 4X + 3 \in (3, \infty)$. Then by Theorem 2.1.5,

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \Big| \frac{d}{dy} g^{-1}(y) \Big| \\ &= 7e^{-7 \cdot \frac{y-3}{4}} \cdot \frac{1}{4} \\ &= \frac{7}{4} e^{\frac{21}{4} - \frac{7}{4} y},\ y \in (3, \infty). \end{aligned}$$

We also have

$$\begin{aligned}\int_3^\infty \frac{7}{4}e^{\frac{21}{4}-\frac{7}{4}y}\ dy &= \frac{7}{4}e^{\frac{21}{4}}\int_3^\infty e^{-\frac{7}{4}y}\ dy\\ &= \frac{7}{4}e^{\frac{21}{4}}\Big[-\frac{4}{7}e^{-\frac{7}{4}y}\Big]_3^\infty\\ &= \frac{7}{4}e^{\frac{21}{4}}(\frac{4}{7}e^{-\frac{21}{4}})\\ &= 1.\end{aligned}$$

(c) Let $g(x) = x^2$. Then g is monotonically increasing on $(0,1)$. We get

$$g^{-1}(y) = \sqrt{y} \implies \frac{d}{dy}g^{-1}(y) = \frac{1}{2\sqrt{y}}.$$

Since $X \in (0,1)$, $Y = X^2 \in (0,1)$. Then by Theorem 2.1.5,

$$\begin{aligned}f_Y(y) &= f_X(g^{-1}(y))\Big|\frac{d}{dy}g^{-1}(y)\Big|\\ &= 30y(1-\sqrt{y})^2 \cdot \frac{1}{2\sqrt{y}}\\ &= 15\sqrt{y}(1-\sqrt{y})^2,\ y \in (0,1).\end{aligned}$$

We also have

$$\begin{aligned}\int_0^1 15\sqrt{y}(1-\sqrt{y})^2\ dy &= 15\int_0^1 \sqrt{y} - 2y + y^{3/2}\ dy\\ &= 15\Big[\frac{2}{3}y^{3/2} - y^2 + \frac{2}{5}y^{5/2}\Big]_0^1\\ &= 15(\frac{2}{3} - 1 + \frac{2}{5})\\ &= 1.\end{aligned}$$

2. (a) Let $g(x) = x^2$. Then g is monotonically increasing on $(0,1)$. We get

$$g^{-1}(y) = \sqrt{y} \implies \frac{d}{dy}g^{-1}(y) = \frac{1}{2\sqrt{y}}.$$

Since $X \in (0,1)$, $Y = X^2 \in (0,1)$. Then by Theorem 2.1.5,

$$\begin{aligned}f_Y(y) &= f_X(g^{-1}(y))\Big|\frac{d}{dy}g^{-1}(y)\Big|\\ &= 1 \cdot \frac{1}{2\sqrt{y}}\\ &= \frac{1}{2\sqrt{y}},\ y \in (0,1).\end{aligned}$$

(b) Let $g(x) = -\log x$. Then g is monotonically decreasing on $(0,1)$. We get

$$g^{-1}(y) = e^{-y} \implies \frac{d}{dy}g^{-1}(y) = -e^{-y}.$$

Since $X \in (0,1),\ Y = \log X \in (0,\infty)$. Then by Theorem 2.1.5,

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y))\Big|\frac{d}{dy}g^{-1}(y)\Big| \\ &= \frac{(n+m+1)!}{n!m!}e^{-ny}(1-e^{-y})^m \cdot |-e^{-y}| \\ &= \frac{(n+m+1)!}{n!m!}e^{-y(n+1)}(1-e^{-y})^m,\ y \in (0,\infty). \end{aligned}$$

(c) Let $g(x) = e^x$. Then g is monotonically increasing on $(0,\infty)$. We get

$$g^{-1}(y) = \ln y \implies \frac{d}{dy}g^{-1}(y) = \frac{1}{y}.$$

Since $X \in (0,\infty),\ Y = e^X \in (0,\infty)$. Then by Theorem 2.1.5,

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y))\Big|\frac{d}{dy}g^{-1}(y)\Big| \\ &= \frac{1}{\sigma^2}\ln y e^{-(\ln y/\sigma)^2/2} \cdot \frac{1}{y} \\ &= \frac{1}{\sigma^2}\frac{\ln y}{y}e^{-(\ln y/\sigma)^2/2},\ y \in (0,\infty). \end{aligned}$$

3. First of all,

$$X \in \{0,1,2,...\} \implies Y \in \Big\{0, \frac{1}{2}, \frac{2}{3}, ...\Big\}.$$

Then

$$\begin{aligned} P(Y=y) &= P(\frac{X}{X+1} = y) \\ &= P(1 - \frac{1}{X+1} = y) \\ &= P(X = \frac{y}{1-y}) \\ &= \frac{1}{3}\Big(\frac{2}{3}\Big)^{y/(1-y)},\ y \in \Big\{\frac{k}{k+1} : k \in \mathbb{N}_0\Big\}. \end{aligned}$$

4. (a) It is not hard to see that $f(x) \geq 0\ \forall x \in \mathcal{X}$ as both piecewise functions are exponentials. We also have

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\ dx &= \int_{-\infty}^{0} \frac{1}{2}\lambda e^{\lambda x} + \int_{0}^{\infty} \frac{1}{2}\lambda e^{-\lambda x}\ dx \\ &= \Big[\frac{1}{2}e^{\lambda x}\Big]_{-\infty}^{0} + \Big[-\frac{1}{2}e^{-\lambda x}\Big]_{0}^{\infty} \\ &= \frac{1}{2} + \frac{1}{2} \\ &= 1. \end{aligned}$$

(b) For $t \leq 0$,

$$\begin{aligned} P(X < t) &= \int_{-\infty}^{t} \frac{1}{2}\lambda e^{\lambda x} \, dx \\ &= \left[\frac{1}{2}e^{\lambda x}\right]_{\infty}^{t} \\ &= \frac{1}{2}e^{\lambda t}. \end{aligned}$$

For $t > 0$,

$$\begin{aligned} P(X < t) &= \frac{1}{2} + \int_{0}^{t} \frac{1}{2}\lambda e^{-\lambda x} \, dx \\ &= \frac{1}{2} + \left[-\frac{1}{2}e^{-\lambda x}\right]_{0}^{t} \\ &= \frac{1}{2} + (-\frac{1}{2}e^{-\lambda t} + \frac{1}{2}) \\ &= 1 - \frac{1}{2}e^{-\lambda t}. \end{aligned}$$

(c) For $t \leq 0$, $P(|X| < t) = 0$. For $t > 0$,

$$\begin{aligned} P(|X| < t) &= P(-t < X < t) \\ &= \int_{-t}^{0} \frac{1}{2}\lambda e^{\lambda x} \, dx + \int_{0}^{t} \frac{1}{2}\lambda e^{-\lambda x} \, dx \\ &= \left[\frac{1}{2}e^{\lambda x}\right]_{-t}^{0} + \left[\frac{1}{2}e^{-\lambda x}\right]_{0}^{t} \\ &= \frac{1}{2} - \frac{1}{2}e^{-\lambda t} + (-\frac{1}{2}e^{-\lambda t} + \frac{1}{2}) \\ &= 1 - e^{-\lambda t}. \end{aligned}$$

5. Let $A_0 = \{\pi\}, A_1 = (0, \frac{\pi}{2}), A_2 = (\frac{\pi}{2}, \pi), A_3 = (\pi, \frac{3\pi}{2}), A_4 = (\frac{3\pi}{2}, 2\pi)$, and let $g(x) = g_i(x) = \sin^2 x$. Then for each $A_i (i \neq 0), g_i(x) = g(x) \forall x \in A_i,$, $g_i(x)$ is monotone on A_i. Moreover, $\mathcal{Y} = (0, 1)$ is the same for all i, and monotone on A_i, and

$$g^{-1}(y) = \arcsin(\sqrt{x}) \implies \frac{d}{dy}g^{-1}(y) = \frac{1}{2\sqrt{y(1-y)}}$$

is continuous on $\mathcal{Y}$ for all i. Then by Theorem 2.1.8,

$$\begin{aligned} f_Y(y) &= \sum_{i=1}^{4} f_X(g^{-1}(y)) \Big| \frac{d}{dy} g_i^{-1}(y) \Big| \\ &= 4 \cdot \frac{1}{2\pi} \cdot \Big| \frac{1}{2\sqrt{y(1-y)}} \Big| \\ &= \frac{1}{\pi\sqrt{y(1-y)}}, \ y \in (0, 1). \end{aligned}$$

To use the cdf from (2.1.6), we first get that $x_1 = \arcsin(\sqrt{y}), x_2 = \pi - \arcsin(\sqrt{y})$. Note

$$P(Y \le y) = 2P(X \le x_1) + 2P(X \le \pi) - 2P(X \le x_2)$$

Then by differentiating the above we get

$$\begin{aligned}
f_Y(y) &= 2f_X(x_1) \cdot \frac{d}{dy}(\sin^{-1}\sqrt{y}) - 2f_X(x_2) \cdot \frac{d}{dy}(\pi - \sin^{-1}\sqrt{y}) \\
&= 2 \cdot \frac{1}{2\pi} \cdot \frac{1}{2\sqrt{y(1-y)}} - 2 \cdot \frac{1}{2\pi} \cdot (-\frac{1}{2\sqrt{y(1-y)}}) \\
&= \frac{1}{\pi\sqrt{y(1-y)}}, \ y \in (0,1).
\end{aligned}$$

6. (a) Let $g(x) = |x|^3, g_1(x) = -x^3, g_2(x) = x^3$. Let $A_0 = \{0\}, A_1 = (-\infty, 0), A_2 = (0, \infty)$. Then we get $\mathcal{Y} = (0, \infty)$ so that all conditions for Theorem 2.1.8 are satisfied. Then

$$g_1^{-1}(y) = -y^{1/3} \implies \frac{d}{dy}g_1^{-1}(y) = -\frac{1}{3y^{2/3}}.$$

$$g_2^{-1}(y) = y^{1/3} \implies \frac{d}{dy}g_2^{-1}(y) = \frac{1}{3y^{2/3}}.$$

Then by Theorem 2.1.8,

$$\begin{aligned}
f_Y(y) &= \sum_{i=1}^{2} f_X(g^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right| \\
&= \frac{1}{2}e^{-y^{1/3}} \cdot \left| -\frac{1}{3y^{2/3}} \right| + \frac{1}{2}e^{-y^{1/3}} \cdot \left| \frac{1}{3y^{2/3}} \right| \\
&= \frac{1}{3}y^{-2/3}e^{-y^{1/3}}, \ y \in (0, \infty).
\end{aligned}$$

(b) Let $g(x) = g_1(x) = g_2(x) = 1 - x^2$. Let $A_0 = \{0\}, A_1 = (-1, 0), A_2 = (0, 1)$. Then we get

$$g_1^{-1}(y) = -\sqrt{1-y} \implies \frac{d}{dy}g_1^{-1}(y) = \frac{1}{2\sqrt{1-y}},$$

$$g_2^{-1}(y) = \sqrt{1-y} \implies \frac{d}{dy}g_2^{-1}(y) = -\frac{1}{2\sqrt{1-y}}.$$

Then we get $\mathcal{Y} = (0, 1)$ so that all conditions for Theorem 2.1.8 are satisfied.

Then by Theorem 2.1.8,

$$
\begin{aligned}
f_Y(y) &= \sum_{i=1}^{2} f_X(g^{-1}(y))\Big|\frac{d}{dy}g_i^{-1}(y)\Big| \\
&= \frac{3}{8}(-\sqrt{1-y}+1)^2 \cdot \Big|\frac{1}{2\sqrt{1-y}}\Big| \\
&\qquad + \frac{3}{8}(\sqrt{1-y}+1)^2 \cdot \Big| -\frac{1}{2\sqrt{1-y}}\Big| \\
&= \frac{3}{8}(1-y-2\sqrt{1-y}+1)\cdot\frac{1}{2\sqrt{1-y}} \\
&\qquad + \frac{3}{8}(1-y+2\sqrt{1-y}+1)\cdot\frac{1}{2\sqrt{1-y}} \\
&= \frac{3}{8}(1-y)^{1/2} + \frac{3}{8}(1-y)^{-1/2},\ y\in(0,1).
\end{aligned}
$$

(Note for g_1 we chose the negative root because $x < 0$).

(c) Let $g_1(x) = 1-x^2, g_2(x) = 1-x$. Let $A_0 = \{0\}, A_1 = (-1,0), A_2 = (0,1)$. Then we get

$$
g_1^{-1}(y) = -\sqrt{1-y} \implies \frac{d}{dy}g_1^{-1}(y) = \frac{1}{2\sqrt{1-y}}.
$$

$$
g_2^{-1}(y) = 1-y \implies \frac{d}{dy}g_2^{-1}(y) = -1.
$$

Then we get $\mathcal{Y} = (0,1)$ so that all conditions for Theorem 2.1.8 are satisfied. Then by Theorem 2.1.8,

$$
\begin{aligned}
f_Y(y) &= \sum_{i=1}^{2} f_X(g^{-1}(y))\Big|\frac{d}{dy}g_i^{-1}(y)\Big| \\
&= \frac{3}{8}(-\sqrt{1-y}+1)^2 \cdot \Big|\frac{1}{2\sqrt{1-y}}\Big| \\
&\qquad + \frac{3}{8}(1-y+1)^2 \cdot |-1| \\
&= \frac{3}{16\sqrt{1-y}}(1-\sqrt{1-y})^2 + \frac{3}{8}(2-y)^2,\ y\in(0,1).
\end{aligned}
$$

7. (a) For $g(x) = x^2,\ x\in[-1,2]$, there is no partition $\{A_i\}$ of the interval which could produce the same $\mathcal{Y}$ for all i. Therefore, we cannot use Theorem 2.1.8 in this case. To solve directly, we get

$$
\begin{aligned}
f_Y(y) &= \sum_{i=1}^{4} f_X(g^{-1}(y))\Big|\frac{d}{dy}g_i^{-1}(y)\Big| \\
&=
\end{aligned}
$$

8. (a) It is easy to see that

$$
\lim_{x\to-\infty} F_X(x) = 0,\ \lim_{x\to+\infty} F_X(x) = 1.
$$

Moreover, both 0 and $1 - e^{-x}$ are non-decreasing on their respective intervals, and

$$\lim_{x \to 0^+} F_X(x) = 0$$

so that F_X is right continuous and therefore is a valud cdf. Its inverse is

$$F_X^{-1}(y) = -\ln(1-y)$$

(b) Again, we can see that

$$\lim_{x \to -\infty} F_X(x) = 0, \quad \lim_{x \to +\infty} F_X(x) = 1.$$

$e^x/2, 1-(e^{-x}/2)$ are increasing, and $1/2$ is noncreasing on their respective intervals, and

$$\lim_{x \to 0} F_X(x) = \frac{1}{2}, \ \lim_{x \to 1} F_X(x) = \frac{1}{2}$$

so that F_X is continuous hence right continuous so is a valid cdf. Its inverse is

$$F_X^s{-1}(y) = \begin{cases} \ln(2x) & 0 < y < \frac{1}{2} \\ -\ln(2-2x) & \frac{1}{2} \le y < 1. \end{cases}$$

(c) Again, we can see that

$$\lim_{x \to -\infty} F_X(x) = 0, \quad \lim_{x \to +\infty} F_X(x) = 1.$$

$e^x/4, 1-(e^{-x}/4)$ are both increasing on their respective intervals, and

$$\lim_{x \to 0^+} F_X(x) = \frac{3}{4} = F_X(0)$$

so that F_X is right continuous and therefore is a valid cdf. Its inverse is

$$F_X^{-1}(y) = \begin{cases} \ln(4x) & 0 < y < \frac{1}{4} \\ -\ln(4-4x) & \frac{3}{4} \le y < 1 \end{cases}$$

9. We first find the cdf of X:

$$F_X(x) = \begin{cases} 0 & x \le 1 \\ \frac{1}{4}(x-1)^2 & 1 < x < 3 \\ 1 & x \ge 3 \end{cases}$$

Then we have

$$\lim_{x \to 1} F_X(x) = 0, \ \lim_{x \to 3} F_X(x) = 1.$$

hence X has a continuous cdf. Let $u(x) = F_X(x)$. Then $u(x)$ is nondecreasing and by Theorem 2.1.10, $Y = u(X)$ has a uniform distribution.

10. (a)

11. (a)

$$\begin{aligned}
\mathbb{E}[X^2] &= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}} \Big([-xe^{-\frac{x^2}{2}}]_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} \ dx \Big) \\
&= \frac{1}{\sqrt{2\pi}} \cdot \sqrt{2\pi} \\
&= 1.
\end{aligned}$$

From Example 2.1.7,

$$\begin{aligned}
f_Y(y) &= \frac{1}{2\sqrt{y}} (f_x(\sqrt{y}) + f_X(-\sqrt{y})) \\
&= \frac{1}{2\sqrt{y}} \Big(\frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \Big) \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{y}} e^{-\frac{y}{2}}, \quad y > 0.
\end{aligned}$$

Using integration by parts,

$$\begin{aligned}
\mathbb{E}[Y] &= \int_0^{\infty} y \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{y}} e^{-\frac{y}{2}} \ dy \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \sqrt{y} e^{-\frac{y}{2}} \ dy \\
&= \frac{1}{\sqrt{2\pi}} \Big([-2\sqrt{y} e^{-\frac{y}{2}}]_0^{\infty} + \int_0^{\infty} \frac{1}{\sqrt{y}} e^{-\frac{y}{2}}) \ dy \\
&= \frac{1}{\sqrt{2\pi}} \cdot \sqrt{2\pi} \\
&= 1.
\end{aligned}$$

(Note that the term on the right is the kernel of the Chi-squared distribution defined in Example 2.1.9 earlier.)

(b) We first find the cdf of Y.

$$F_Y(y) = P(|X| \leq y) = P(-y \leq X \leq y) = F_X(y) - F_X(-y).$$

Therefore the pdf of Y is just

$$\begin{aligned}
&f_Y(y) \\
&\quad = f_X(y) + f_X(-y) \\
&\quad = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \\
&\quad = \sqrt{\frac{2}{\pi}} e^{-\frac{y^2}{2}}, \quad y \in [0, \infty).
\end{aligned}$$

Therefore we can find the mean and variance of Y:

$$\begin{aligned}\mathbb{E}[Y] &= \int_0^\infty y\sqrt{\frac{2}{\pi}}e^{-\frac{y^2}{2}} \\ &= \sqrt{\frac{2}{\pi}}[-e^{-\frac{y^2}{2}}]_0^\infty \\ &= \sqrt{\frac{2}{\pi}}.\end{aligned}$$

From part (a),

$$\mathbb{E}[Y^2] = \mathbb{E}[|X|^2] = \mathbb{E}[X^2] = 1.$$

Therefore,

$$\mathrm{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = 1 - \frac{2}{\pi}.$$

12. We have that $X \sim$ Uniform$(0, \frac{pi}{2})$" and $Y = d\tan X$. Ket $g(x) = d\tan x$. Then g is increasing on $(0, \frac{\pi}{2})$. For $X \in (0, \frac{\pi}{2}), \quad Y \in (0, \infty)$. We have that $g-1(y) = \arctan y/d$ has a continuous derivative on $(0, \infty)$. Then by Theorem 2.1.5,

$$\begin{aligned}f_Y(y) &= f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| \\ &= \frac{2}{\pi}\frac{1}{1+(y/d)^2} \cdot \frac{1}{d} \\ &= \frac{2}{\pi d}\frac{1}{1+(y/d)^2}, \quad y \in (0, \infty).\end{aligned}$$

Then $Y \sim$ Cauchy$(0, d)$ so therefore $\mathbb{E}[Y] = \infty$.

13. We have that For $X = k$, we can either have k tails followed by a head or k heads followed by a tail. Then

$$P(X = k) = (1-p)^k p + p^k(1-p), \quad k = 1, 2, ...$$

Then

$$\begin{aligned}\mathbb{E}[X] &= \sum_{k=1}^\infty k[(1-p)^k p + p^k(1-p)] \\ &= p(1-p)\sum_{k=1}^\infty k(1-p)^{k-1}p + kp^{k-1}(1-p) \\ &= p(1-p)\Big(\frac{1}{p^2} + \frac{1}{(1-p)^2}) \\ &= \frac{1-2p+2p^2}{p(1-p)}.\end{aligned}$$

14. (a)

$$\begin{aligned}\mathbb{E}[X] &= \int_0^\infty x f_X(x)\ dx \\ &= [xF_X(x)]_0^\infty - \int_0^\infty F_X(x)\ dx\end{aligned}$$

15. We can assume without loss of generality that $X \leq Y$ as the other case is similar. Then $X \wedge Y = X, X \vee Y = Y$. Taking expectations on both sides gives the result.

16. From Exercise 2.14,

$$\begin{aligned}\mathbb{E}[T] &= \int_0^\infty ae^{-\lambda t} + (1-a)e^{-\mu t}\ dt \\ &= \left[-\frac{a}{\lambda}e^{-\lambda t} + \frac{a-1}{\mu}e^{-\mu t}\right]_0^\infty \\ &= \frac{a}{\lambda} + \frac{1-a}{\mu}.\end{aligned}$$

17. (a)

$$\int_0^m 3x^2 = [x^3]_0^m = m^3 = \frac{1}{2} \implies m = \frac{1}{\sqrt[3]{2}}.$$

(b) This is the pdf of the standard Cauchy distribution, which has median 0.

18.

$$\begin{aligned}\mathbb{E}[|X-a|] &= \int_{-\infty}^\infty |x-a| f_X(x)\ dx \\ &= \int_{-\infty}^a -(x-a)f_X(x)\ dx + \int_a^\infty (x-a)f_X(x)\ dx.\end{aligned}$$

Taking the derivative with respect to a,

$$\frac{d}{da}\mathbb{E}[|X-a|] = \int_{-\infty}^a f_X(x)\ dx - \int_a^\infty f_X(x)\ dx.$$

Setting the above to 0 yields that a is the median. By the second derivative test,

$$\frac{d^2}{da^2}\mathbb{E}[|X-a|] = 2f(a) > 0$$

so that we have a minimum.

19.

20. Let X be the number of children until the first daughter. Then $X \sim \text{Geom}(p)$. Then $\mathbb{E}[X] = \frac{1}{p}$.

21. Since $y = g(x)$ and $g(x)$ is monotone, $x = g^{-1}(y) \implies dx = \frac{d}{dy}g^{-1}(y)dy$.

$$\begin{aligned}\mathbb{E}[g(X)] &= \int_\infty^\infty g(x)f_X(x)\ dx \\ &= \int_{-\infty}^\infty g(g^{-1}(y))f_X(g^{-1}(y))\frac{d}{dy}g^{-1}(y)\ dy \\ &= \int_{-\infty}^\infty yf_Y(y)\ dy \\ &= \mathbb{E}[Y].\end{aligned}$$

22. (a) It is clear that $f(x) > 0$ when $0 < x < \infty$. In here we will just calculate the kernel and show that it is the reciprocal of $\frac{4}{\beta^3\sqrt{\pi}}$.

$$\begin{aligned}
\int_0^\infty x^2 e^{-x^2/\beta^2}\ dx &= \Big[-\frac{\beta^2}{2} x e^{-x^2/\beta^2}\Big]_0^\infty + \int_0^\infty \frac{\beta^2}{2} e^{-x^2/\beta^2}\ dx \\
&= 0 + \int_0^\infty \frac{\beta^3}{4} e^{-u^2}\ du \quad (u = \frac{x}{\beta}) \\
&= \frac{\beta^3\sqrt{\pi}}{4},
\end{aligned}$$

which is correct.

(b) Using integration by parts,

$$\begin{aligned}
\mathbb{E}[X] &= \frac{4}{\beta^3\sqrt{\pi}} \int_0^\infty x^3 e^{-x^2/\beta^2}\ dx \\
&= \frac{4}{\beta^3\sqrt{\pi}} \left[\Big[-\frac{\beta^2}{2} x^2 e^{-x^2/\beta^2}\Big]_0^\infty + \beta^2 \int_0^\infty x e^{-x^2/\beta^2}\ dx\right] \\
&= \frac{4}{\beta^3\sqrt{\pi}} \Big[0 + \beta^2 [-\frac{\beta^2}{2} e^{-x^2/\beta^2}]_0^\infty\Big] \\
&= \frac{2\beta}{\sqrt{\pi}}. \\
\mathbb{E}[X^2] &= \frac{4}{\beta^3\sqrt{\pi}} \int_0^\infty x^4 e^{-x^2/\beta^2}\ dx \\
&= \frac{4}{\beta^3\sqrt{\pi}} \left[\Big[-\frac{\beta^2}{2} x^3 e^{-x^2/\beta^2}\Big]_0^\infty + \frac{3\beta^2}{2} \int_0^\infty x^2 e^{-x^2/\beta^2}\ dx\right] \\
&= \frac{4}{\beta^3\sqrt{\pi}} \Big(0 + \frac{3\beta^2}{2} \cdot \frac{\beta^3\sqrt{\pi}}{4}\Big) \\
&= \frac{3\beta^2}{2}. \\
\operatorname{Var} X &= \frac{3\beta^2}{2} - \Big(\frac{2\beta}{\sqrt{\pi}}\Big)^2 \\
&= \beta^2\Big(\frac{3}{2} - \frac{4}{\pi}\Big).
\end{aligned}$$

23. (a) First of all, $X \in (-1, 1)$ hence $Y = X^2 \in [0, 1)$.

$$\begin{aligned}
F_Y(y) &= P(X^2 \le y) \\
&= P(-\sqrt{y} \le X \le \sqrt{y}) \\
&= F_X(\sqrt{y}) - F_X(-\sqrt{y}).
\end{aligned}$$

Taking derivatives,

$$\begin{aligned}
f_Y(y) &= f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - f_X(-\sqrt{y}) \cdot \Big(-\frac{1}{2\sqrt{y}}\Big) \\
&= \frac{1}{2}(1+\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} + \frac{1}{2}(1-\sqrt{y}) \cdot \Big(\frac{1}{2\sqrt{y}}\Big) \\
&= \frac{1}{2}y^{-1/2},\ y \in (0,1).
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbb{E}[Y] &= \int_0^1 \frac{1}{2}\sqrt{y}\ dy \\
&= \Big[\frac{1}{3}y^{3/2}\Big]_0^1 \\
&= \frac{1}{3}. \\
\mathbb{E}[Y^2] &= \int_0^1 \frac{1}{2}y^{3/2}\ dy \\
&= \Big[\frac{1}{5}y^{5/2}\Big]_0^1 \\
&= \frac{1}{5}. \\
\operatorname{Var} Y &= \frac{1}{5} - \Big(\frac{1}{3}\Big)^2 \\
&= \frac{4}{45}.
\end{aligned}$$

24. (a)

$$\begin{aligned}
\mathbb{E}[X] &= \int_0^1 ax^a\ dx = \Big[\frac{a}{a+1}x^{a+1}\Big]_0^1 = \frac{a}{a+1}. \\
\mathbb{E}[X^2] &= \int_0^1 ax^{a+1}\ dx = \Big[\frac{a}{a+2}x^{a+2}\Big]_0^1 = \frac{a}{a+2}. \\
\operatorname{Var} X &= \frac{a}{a+2} - \Big(\frac{a}{a+1}\Big)^2 = \frac{a}{(a+2)(a+1)^2}.
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{k=1}^n \frac{k}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}. \\
\mathbb{E}[X^2] &= \sum_{k=1}^n \frac{k^2}{n} = \frac{n(n+1)(2n+1)}{6n} = \frac{(n+1)(2n+1)}{6}. \\
\operatorname{Var} X &= \frac{(n+1)(2n+1)}{6} - \Big(\frac{n+1}{2}\Big)^2 = \frac{n^2+1}{12}.
\end{aligned}$$

(c)

$$\mathbb{E}[X] = \frac{3}{2}\int_0^2 x^3 - 2x^2 + x \; dx = \frac{3}{2}\left[\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2\right]_0^2 = 1.$$

$$\mathbb{E}[X^2] = \frac{3}{2}\int_0^2 x^4 - 2x^3 + x^2 \; dx = \frac{3}{2}\left[\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3\right]_0^2 = \frac{8}{5}.$$

$$\operatorname{Var} X = \frac{8}{5} - 1^2 = \frac{3}{5}.$$

25. (a) Let $Y = -X$. Then $g(x) = g^{-1}(x) = -x$. We have that

$$f_{-X}(x) = f_X(-x) \cdot |-1| = f_X(x) \forall x$$

so that X and $-X$ are identically distributed.

(b) Let $\varepsilon > 0$ be given. Then

$$\begin{aligned} M_X(0+\varepsilon) &= \int_{-\infty}^{\infty} e^{\varepsilon x} f_X(x) \; dx \\ &= -\int_{\infty}^{-\infty} e^{(-\varepsilon u)} f_X(u) \; du \quad (u = -x) \\ &= \int_{-\infty}^{\infty} e^{(0-\varepsilon)u} f_X(u) \; du \\ &= M_X(0-\varepsilon). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we are done.

26. (a) $N(\mu, \sigma^2)$ is symmetric about μ, DoubleExp(μ, b) is symmetric about μ, and t_n is symmetric about 0.

(b)

$$\begin{aligned} \int_a^{\infty} f(x) \; dx &= \int_0^{\infty} f(a+\varepsilon) \; d\varepsilon \quad (\varepsilon = x - a) \\ &= \int_0^{\infty} f(a-\varepsilon) \; d\varepsilon \\ &= \int_a^{\infty} f(x) \; dx \quad (x = a + \varepsilon) \end{aligned}$$

Since f is a valid pdf, a has to be the median.

(c)

$$\begin{aligned}
\mathbb{E}[X] - a &= \mathbb{E}[X = a] \\
&= \int_{-\infty}^{\infty} (x-a)f(x)\ dx \\
&= \int_{-\infty}^{a} (x-a)f(x)\ dx + \int_{a}^{\infty} (x-a)f(x)\ dx \\
&= \int_{0}^{\infty} -\varepsilon f(a-\varepsilon)\ d\varepsilon + \int_{0}^{\infty} \varepsilon f(a+\varepsilon)\ d\varepsilon \\
&= -\int_{0}^{\infty} \varepsilon f(a+\varepsilon)\ d\varepsilon + \int_{0}^{\infty} \varepsilon f(a+\varepsilon)\ d\varepsilon \\
&= 0.
\end{aligned}$$

Here, we substituted $\varepsilon = a - x$ for the first integral and $\varepsilon = x - a$ for the second integral (sorry for the confusing notation).

(d) If $a < 0$, for $\varepsilon > a$, $f(a-\varepsilon) = 0$ but $f(a+\varepsilon) > 0$. If $a \geq 0$, the same is true, hence $f(x)$ is not a symmetric pdf.

(e) For the mean,

$$\begin{aligned}
\mathbb{E}[X] &= \int_{0}^{\infty} xe^{-x}\ dx \\
&= [-xe^{-x} - e^{-x}]_0^{\infty} \\
&= 1.
\end{aligned}$$

For the median,

$$\int_0^a e^{-x} = \frac{1}{2} \implies a = \log 2.$$

Since $\log 2 < 1$, the median is less than the mean.

27. (a) The standard normal has a unique mode at $x = 0$.

(b) The Uniform$(0, 1)$ does not have a unique mode as all $x \in (0, 1)$ is a mode.

(c) First suppose that the mode is unique. Let a be the mean and b be the mode suppose that $a \neq b$. We can assume without loss of generality that $a = b + \varepsilon$. Since $f(x)$ is unimodal, $f(b) > f(b+\varepsilon) \geq f(b+2\varepsilon)$, and $f(b-2\varepsilon) \geq f(b-\varepsilon) > f(b)$, contradicting to our assumption that f is symmetric about b.

Now suppose that the mode is not unique. Then it is the same case except that there is a region (x_1, x_2) such that b is a mode for all $b \in (x_1, x_2)$.

(d) f is monotonically decreasing on $[0, \infty)$ hence it is unimodal with mode 0.

28. (a) From part (c) of Exercise 2.26, $\mathbb{E}[X] = a$. Then

$$\begin{aligned}
\mu_3 &= \int_{-\infty}^{\infty} (x-a)^3 f(x)\ dx \\
&= \int_{-\infty}^{a} (x-a)^3 f(x)\ dx + \int_{a}^{\infty} (x-a)^3 f(x)\ dx \\
&= \int_{-\infty}^{0} u^3 f(a+u)\ du + \int_{0}^{\infty} u^3 f(a+u)\ du \quad (u = x-a) \\
&= \int_{0}^{\infty} (-v)^3 f(a-v)\ dv + \int_{0}^{\infty} u^3 f(a+u)\ du \quad (v = -u) \\
&= -\int_{0}^{\infty} v^3 f(a+v)\ dv + \int_{0}^{\infty} u^3 f(a+u)\ du \quad (f(a-v) = f(a+v)) \\
&= 0.
\end{aligned}$$

(b)