

Statistical Inference Chapter 2

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1. (a) Let $g(x) = x^3$. Then g is monotonically increasing on $(0, 1)$. We get

$$g^{-1}(y) = y^{1/3} \implies \frac{d}{dy}g^{-1}(y) = \frac{1}{3y^{2/3}}.$$

Since $X \in (0, 1)$, $Y = X^3 \in (0, 1)$. Then by Theorem 2.1.5,

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy}g^{-1}(y) \right| \\ &= 42(y^{1/3})^5(1 - y^{1/3}) \cdot \frac{1}{3y^{2/3}} \\ &= 14y(1 - y^{1/3}), \quad y \in (0, 1). \end{aligned}$$

We also have

$$\begin{aligned} \int_0^1 14y(1 - y^{1/3}) \, dy &= 14 \int_0^1 y - y^{4/3} \, dy \\ &= 14 \left[\frac{1}{2}y^2 - \frac{3}{7}y^{7/3} \right]_0^1 \\ &= 14 \left(\frac{1}{2} - \frac{3}{7} \right) \\ &= 1. \end{aligned}$$

- (b) Let $g(x) = 4x + 3$. Then g is monotonically increasing on $(0, \infty)$. We get

$$g^{-1}(y) = \frac{y-3}{4} \implies \frac{d}{dy}g^{-1}(y) = \frac{1}{4}.$$

Since $X \in (0, \infty)$, $Y = 4X + 3 \in (3, \infty)$. Then by Theorem 2.1.5,

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy}g^{-1}(y) \right| \\ &= 7e^{-7 \cdot \frac{y-3}{4}} \cdot \frac{1}{4} \\ &= \frac{7}{4}e^{\frac{21}{4} - \frac{7}{4}y}, \quad y \in (3, \infty). \end{aligned}$$

We also have

$$\begin{aligned}
\int_3^\infty \frac{7}{4} e^{\frac{21}{4} - \frac{7}{4}y} dy &= \frac{7}{4} e^{\frac{21}{4}} \int_3^\infty e^{-\frac{7}{4}y} dy \\
&= \frac{7}{4} e^{\frac{21}{4}} \left[-\frac{4}{7} e^{-\frac{7}{4}y} \right]_3^\infty \\
&= \frac{7}{4} e^{\frac{21}{4}} \left(\frac{4}{7} e^{-\frac{21}{4}} \right) \\
&= 1.
\end{aligned}$$

(c) Let $g(x) = x^2$. Then g is monotonically increasing on $(0, 1)$. We get

$$g^{-1}(y) = \sqrt{y} \implies \frac{d}{dy} g^{-1}(y) = \frac{1}{2\sqrt{y}}.$$

Since $X \in (0, 1)$, $Y = X^2 \in (0, 1)$. Then by Theorem 2.1.5,

$$\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \\
&= 30y(1 - \sqrt{y})^2 \cdot \frac{1}{2\sqrt{y}} \\
&= 15\sqrt{y}(1 - \sqrt{y})^2, \quad y \in (0, 1).
\end{aligned}$$

We also have

$$\begin{aligned}
\int_0^1 15\sqrt{y}(1 - \sqrt{y})^2 dy &= 15 \int_0^1 \sqrt{y} - 2y + y^{3/2} dy \\
&= 15 \left[\frac{2}{3} y^{3/2} - y^2 + \frac{2}{5} y^{5/2} \right]_0^1 \\
&= 15 \left(\frac{2}{3} - 1 + \frac{2}{5} \right) \\
&= 1.
\end{aligned}$$

2. (a) Let $g(x) = x^2$. Then g is monotonically increasing on $(0, 1)$. We get

$$g^{-1}(y) = \sqrt{y} \implies \frac{d}{dy} g^{-1}(y) = \frac{1}{2\sqrt{y}}.$$

Since $X \in (0, 1)$, $Y = X^2 \in (0, 1)$. Then by Theorem 2.1.5,

$$\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \\
&= 1 \cdot \frac{1}{2\sqrt{y}} \\
&= \frac{1}{2\sqrt{y}}, \quad y \in (0, 1).
\end{aligned}$$

(b) Let $g(x) = -\log x$. Then g is monotonically decreasing on $(0, 1)$. We get

$$g^{-1}(y) = e^{-y} \implies \frac{d}{dy} g^{-1}(y) = -e^{-y}.$$

Since $X \in (0, 1)$, $Y = \log X \in (0, \infty)$. Then by Theorem 2.1.5,

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \\ &= \frac{(n+m+1)!}{n!m!} e^{-ny} (1 - e^{-y})^m \cdot |-e^{-y}| \\ &= \frac{(n+m+1)!}{n!m!} e^{-y(n+1)} (1 - e^{-y})^m, \quad y \in (0, \infty). \end{aligned}$$

(c) Let $g(x) = e^x$. Then g is monotonically increasing on $(0, \infty)$. We get

$$g^{-1}(y) = \ln y \implies \frac{d}{dy} g^{-1}(y) = \frac{1}{y}.$$

Since $X \in (0, \infty)$, $Y = e^X \in (0, \infty)$. Then by Theorem 2.1.5,

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \\ &= \frac{1}{\sigma^2} \ln y e^{-(\ln y / \sigma)^2 / 2} \cdot \frac{1}{y} \\ &= \frac{1}{\sigma^2} \frac{\ln y}{y} e^{-(\ln y / \sigma)^2 / 2}, \quad y \in (0, \infty). \end{aligned}$$

3. First of all,

$$X \in \{0, 1, 2, \dots\} \implies Y \in \left\{0, \frac{1}{2}, \frac{2}{3}, \dots\right\}.$$

Then

$$\begin{aligned} P(Y = y) &= P\left(\frac{X}{X+1} = y\right) \\ &= P\left(1 - \frac{1}{X+1} = y\right) \\ &= P\left(X = \frac{y}{1-y}\right) \\ &= \frac{1}{3} \left(\frac{2}{3}\right)^{y/(1-y)}, \quad y \in \left\{\frac{k}{k+1} : k \in \mathbb{N}_0\right\}. \end{aligned}$$

4. (a) It is not hard to see that $f(x) \geq 0 \forall x \in \mathcal{X}$ as both piecewise functions are exponentials. We also have

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) dx &= \int_{-\infty}^0 \frac{1}{2} \lambda e^{\lambda x} + \int_0^{\infty} \frac{1}{2} \lambda e^{-\lambda x} dx \\ &= \left[\frac{1}{2} e^{\lambda x} \right]_{-\infty}^0 + \left[-\frac{1}{2} e^{-\lambda x} \right]_0^{\infty} \\ &= \frac{1}{2} + \frac{1}{2} \\ &= 1. \end{aligned}$$

(b) For $t \leq 0$,

$$\begin{aligned} P(X < t) &= \int_{-\infty}^t \frac{1}{2} \lambda e^{\lambda x} dx \\ &= \left[\frac{1}{2} e^{\lambda x} \right]_{-\infty}^t \\ &= \frac{1}{2} e^{\lambda t}. \end{aligned}$$

For $t > 0$,

$$\begin{aligned} P(X < t) &= \frac{1}{2} + \int_0^t \frac{1}{2} \lambda e^{-\lambda x} dx \\ &= \frac{1}{2} + \left[-\frac{1}{2} e^{-\lambda x} \right]_0^t \\ &= \frac{1}{2} + \left(-\frac{1}{2} e^{-\lambda t} + \frac{1}{2} \right) \\ &= 1 - \frac{1}{2} e^{-\lambda t}. \end{aligned}$$

(c) For $t \leq 0$, $P(|X| < t) = 0$. For $t > 0$,

$$\begin{aligned} P(|X| < t) &= P(-t < X < t) \\ &= \int_{-t}^0 \frac{1}{2} \lambda e^{\lambda x} dx + \int_0^t \frac{1}{2} \lambda e^{-\lambda x} dx \\ &= \left[\frac{1}{2} e^{\lambda x} \right]_{-t}^0 + \left[-\frac{1}{2} e^{-\lambda x} \right]_0^t \\ &= \frac{1}{2} - \frac{1}{2} e^{-\lambda t} + \left(-\frac{1}{2} e^{-\lambda t} + \frac{1}{2} \right) \\ &= 1 - e^{-\lambda t}. \end{aligned}$$

5. Let $A_0 = \{\pi\}$, $A_1 = (0, \frac{\pi}{2})$, $A_2 = (\frac{\pi}{2}, \pi)$, $A_3 = (\pi, \frac{3\pi}{2})$, $A_4 = (\frac{3\pi}{2}, 2\pi)$, and let $g(x) = g_i(x) = \sin^2 x$. Then for each A_i ($i \neq 0$), $g_i(x) = g(x) \forall x \in A_i$, $g_i(x)$ is monotone on A_i . Moreover, $\mathcal{Y} = (0, 1)$ is the same for all i , and monotone on A_i , and

$$g^{-1}(y) = \arcsin(\sqrt{y}) \implies \frac{d}{dy} g^{-1}(y) = \frac{1}{2\sqrt{y(1-y)}}$$

is continuous on \mathcal{Y} for all i . Then by Theorem 2.1.8,

$$\begin{aligned} f_Y(y) &= \sum_{i=1}^4 f_X(g^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right| \\ &= 4 \cdot \frac{1}{2\pi} \cdot \left| \frac{1}{2\sqrt{y(1-y)}} \right| \\ &= \frac{1}{\pi\sqrt{y(1-y)}}, \quad y \in (0, 1). \end{aligned}$$

To use the cdf from (2.1.6), we first get that $x_1 = \arcsin(\sqrt{y})$, $x_2 = \pi - \arcsin(\sqrt{y})$. Note

$$P(Y \leq y) = 2P(X \leq x_1) + 2P(X \leq \pi) - 2P(X \leq x_2)$$

Then by differentiating the above we get

$$\begin{aligned} f_Y(y) &= 2f_X(x_1) \cdot \frac{d}{dy}(\sin^{-1} \sqrt{y}) - 2f_X(x_2) \cdot \frac{d}{dy}(\pi - \sin^{-1} \sqrt{y}) \\ &= 2 \cdot \frac{1}{2\pi} \cdot \frac{1}{2\sqrt{y(1-y)}} - 2 \cdot \frac{1}{2\pi} \cdot \left(-\frac{1}{2\sqrt{y(1-y)}}\right) \\ &= \frac{1}{\pi\sqrt{y(1-y)}}, \quad y \in (0, 1). \end{aligned}$$

6. (a) Let $g(x) = |x|^3$, $g_1(x) = -x^3$, $g_2(x) = x^3$. Let $A_0 = \{0\}$, $A_1 = (-\infty, 0)$, $A_2 = (0, \infty)$. Then we get $\mathcal{Y} = (0, \infty)$ so that all conditions for Theorem 2.1.8 are satisfied. Then

$$g_1^{-1}(y) = -y^{1/3} \implies \frac{d}{dy}g_1^{-1}(y) = -\frac{1}{3y^{2/3}}.$$

$$g_2^{-1}(y) = y^{1/3} \implies \frac{d}{dy}g_2^{-1}(y) = \frac{1}{3y^{2/3}}.$$

Then by Theorem 2.1.8,

$$\begin{aligned} f_Y(y) &= \sum_{i=1}^2 f_X(g_i^{-1}(y)) \left| \frac{d}{dy}g_i^{-1}(y) \right| \\ &= \frac{1}{2}e^{-y^{1/3}} \cdot \left| -\frac{1}{3y^{2/3}} \right| + \frac{1}{2}e^{-y^{1/3}} \cdot \left| \frac{1}{3y^{2/3}} \right| \\ &= \frac{1}{3}y^{-2/3}e^{-y^{1/3}}, \quad y \in (0, \infty). \end{aligned}$$

- (b) Let $g(x) = g_1(x) = g_2(x) = 1 - x^2$. Let $A_0 = \{0\}$, $A_1 = (-1, 0)$, $A_2 = (0, 1)$. Then we get

$$g_1^{-1}(y) = -\sqrt{1-y} \implies \frac{d}{dy}g_1^{-1}(y) = \frac{1}{2\sqrt{1-y}},$$

$$g_2^{-1}(y) = \sqrt{1-y} \implies \frac{d}{dy}g_2^{-1}(y) = -\frac{1}{2\sqrt{1-y}}.$$

Then we get $\mathcal{Y} = (0, 1)$ so that all conditions for Theorem 2.1.8 are satisfied.

Then by Theorem 2.1.8,

$$\begin{aligned}
f_Y(y) &= \sum_{i=1}^2 f_X(g^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right| \\
&= \frac{3}{8} (-\sqrt{1-y} + 1)^2 \cdot \left| \frac{1}{2\sqrt{1-y}} \right| \\
&\quad + \frac{3}{8} (\sqrt{1-y} + 1)^2 \cdot \left| -\frac{1}{2\sqrt{1-y}} \right| \\
&= \frac{3}{8} (1-y-2\sqrt{1-y}+1) \cdot \frac{1}{2\sqrt{1-y}} \\
&\quad + \frac{3}{8} (1-y+2\sqrt{1-y}+1) \cdot \frac{1}{2\sqrt{1-y}} \\
&= \frac{3}{8} (1-y)^{1/2} + \frac{3}{8} (1-y)^{-1/2}, \quad y \in (0, 1).
\end{aligned}$$

(Note for g_1 we chose the negative root because $x < 0$).

- (c) Let $g_1(x) = 1 - x^2$, $g_2(x) = 1 - x$. Let $A_0 = \{0\}$, $A_1 = (-1, 0)$, $A_2 = (0, 1)$. Then we get

$$\begin{aligned}
g_1^{-1}(y) &= -\sqrt{1-y} \implies \frac{d}{dy} g_1^{-1}(y) = \frac{1}{2\sqrt{1-y}}. \\
g_2^{-1}(y) &= 1-y \implies \frac{d}{dy} g_2^{-1}(y) = -1.
\end{aligned}$$

Then we get $\mathcal{Y} = (0, 1)$ so that all conditions for Theorem 2.1.8 are satisfied. Then by Theorem 2.1.8,

$$\begin{aligned}
f_Y(y) &= \sum_{i=1}^2 f_X(g^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right| \\
&= \frac{3}{8} (-\sqrt{1-y} + 1)^2 \cdot \left| \frac{1}{2\sqrt{1-y}} \right| \\
&\quad + \frac{3}{8} (1-y+1)^2 \cdot |-1| \\
&= \frac{3}{16\sqrt{1-y}} (1-\sqrt{1-y})^2 + \frac{3}{8} (2-y)^2, \quad y \in (0, 1).
\end{aligned}$$

7. (a) For $g(x) = x^2$, $x \in [-1, 2]$, there is no partition $\{A_i\}$ of the interval which could produce the same \mathcal{Y} for all i . Therefore, we cannot use Theorem 2.1.8 in this case. To solve directly, we get

$$\begin{aligned}
f_Y(y) &= \sum_{i=1}^4 f_X(g^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right| \\
&=
\end{aligned}$$

8. (a) It is easy to see that

$$\lim_{x \rightarrow -\infty} F_X(x) = 0, \quad \lim_{x \rightarrow +\infty} F_X(x) = 1.$$

Moreover, both 0 and $1 - e^{-x}$ are non-decreasing on their respective intervals, and

$$\lim_{x \rightarrow 0^+} F_X(x) = 0$$

so that F_X is right continuous and therefore is a valid cdf. Its inverse is

$$F_X^{-1}(y) = -\ln(1 - y)$$

(b) Again, we can see that

$$\lim_{x \rightarrow -\infty} F_X(x) = 0, \quad \lim_{x \rightarrow +\infty} F_X(x) = 1.$$

$e^x/2, 1 - (e^{-x}/2)$ are increasing, and $1/2$ is noncreasing on their respective intervals, and

$$\lim_{x \rightarrow 0} F_X(x) = \frac{1}{2}, \quad \lim_{x \rightarrow 1} F_X(x) = \frac{1}{2}$$

so that F_X is continuous hence right continuous so is a valid cdf. Its inverse is

$$F_X^{-1}(y) = \begin{cases} \ln(2x) & 0 < y < \frac{1}{2} \\ -\ln(2 - 2x) & \frac{1}{2} \leq y < 1. \end{cases}$$

(c) Again, we can see that

$$\lim_{x \rightarrow -\infty} F_X(x) = 0, \quad \lim_{x \rightarrow +\infty} F_X(x) = 1.$$

$e^x/4, 1 - (e^{-x}/4)$ are both increasing on their respective intervals, and

$$\lim_{x \rightarrow 0^+} F_X(x) = \frac{3}{4} = F_X(0)$$

so that F_X is right continuous and therefore is a valid cdf. Its inverse is

$$F_X^{-1}(y) = \begin{cases} \ln(4x) & 0 < y < \frac{1}{4} \\ -\ln(4 - 4x) & \frac{3}{4} \leq y < 1 \end{cases}$$

9. We first find the cdf of X :

$$F_X(x) = \begin{cases} 0 & x \leq 1 \\ \frac{1}{4}(x - 1)^2 & 1 < x < 3 \\ 1 & x \geq 3 \end{cases}$$

Then we have

$$\lim_{x \rightarrow 1} F_X(x) = 0, \quad \lim_{x \rightarrow 3} F_X(x) = 1.$$

hence X has a continuous cdf. Let $u(x) = F_X(x)$. Then $u(x)$ is nondecreasing and by Theorem 2.1.10, $Y = u(X)$ has a uniform distribution.

10. (a)

11. (a)

$$\begin{aligned}
\mathbb{E}[X^2] &= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}} \left([-xe^{-\frac{x^2}{2}}]_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} dx \right) \\
&= \frac{1}{\sqrt{2\pi}} \cdot \sqrt{2\pi} \\
&= 1.
\end{aligned}$$

From Example 2.1.7,

$$\begin{aligned}
f_Y(y) &= \frac{1}{2\sqrt{y}} (f_X(\sqrt{y}) + f_X(-\sqrt{y})) \\
&= \frac{1}{2\sqrt{y}} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \right) \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{y}} e^{-\frac{y}{2}}, \quad y > 0.
\end{aligned}$$

Using integration by parts,

$$\begin{aligned}
\mathbb{E}[Y] &= \int_0^{\infty} y \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{y}} e^{-\frac{y}{2}} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \sqrt{y} e^{-\frac{y}{2}} dy \\
&= \frac{1}{\sqrt{2\pi}} \left([-2\sqrt{y} e^{-\frac{y}{2}}]_0^{\infty} + \int_0^{\infty} \frac{1}{\sqrt{y}} e^{-\frac{y}{2}} dy \right) \\
&= \frac{1}{\sqrt{2\pi}} \cdot \sqrt{2\pi} \\
&= 1.
\end{aligned}$$

(Note that the term on the right is the kernel of the Chi-squared distribution defined in Example 2.1.9 earlier.)

(b) We first find the cdf of Y .

$$F_Y(y) = P(|X| \leq y) = P(-y \leq X \leq y) = F_X(y) - F_X(-y).$$

Therefore the pdf of Y is just

$$\begin{aligned}
f_Y(y) &= f_X(y) + f_X(-y) \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \\
&= \sqrt{\frac{2}{\pi}} e^{-\frac{y^2}{2}}, \quad y \in [0, \infty).
\end{aligned}$$

Therefore we can find the mean and variance of Y :

$$\begin{aligned}\mathbb{E}[Y] &= \int_0^\infty y \sqrt{\frac{2}{\pi}} e^{-\frac{y^2}{2}} dy \\ &= \sqrt{\frac{2}{\pi}} [-e^{-\frac{y^2}{2}}]_0^\infty \\ &= \sqrt{\frac{2}{\pi}}.\end{aligned}$$

From part (a),

$$\mathbb{E}[Y^2] = \mathbb{E}[|X|^2] = \mathbb{E}[X^2] = 1.$$

Therefore,

$$\text{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = 1 - \frac{2}{\pi}.$$

12. We have that $X \sim \text{Uniform}(0, \frac{\pi}{2})$ and $Y = d \tan X$. Let $g(x) = d \tan x$. Then g is increasing on $(0, \frac{\pi}{2})$. For $X \in (0, \frac{\pi}{2})$, $Y \in (0, \infty)$. We have that $g^{-1}(y) = \arctan y/d$ has a continuous derivative on $(0, \infty)$. Then by Theorem 2.1.5,

$$\begin{aligned}f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \\ &= \frac{2}{\pi} \frac{1}{1 + (y/d)^2} \cdot \frac{1}{d} \\ &= \frac{2}{\pi d} \frac{1}{1 + (y/d)^2}, \quad y \in (0, \infty).\end{aligned}$$

Then $Y \sim \text{Cauchy}(0, d)$ so therefore $\mathbb{E}[Y] = \infty$.

13. We have that For $X = k$, we can either have k tails followed by a head or k heads followed by a tail. Then

$$P(X = k) = (1-p)^k p + p^k (1-p), \quad k = 1, 2, \dots$$

Then

$$\begin{aligned}\mathbb{E}[X] &= \sum_{k=1}^{\infty} k[(1-p)^k p + p^k (1-p)] \\ &= p(1-p) \sum_{k=1}^{\infty} k(1-p)^{k-1} p + kp^{k-1}(1-p) \\ &= p(1-p) \left(\frac{1}{p^2} + \frac{1}{(1-p)^2} \right) \\ &= \frac{1-2p+2p^2}{p(1-p)}.\end{aligned}$$

14. (a)

$$\begin{aligned}\mathbb{E}[X] &= \int_0^\infty x f_X(x) dx \\ &= [x F_X(x)]_0^\infty - \int_0^\infty F_X(x) dx\end{aligned}$$

15. We can assume without loss of generality that $X \leq Y$ as the other case is similar. Then $X \wedge Y = X$, $X \vee Y = Y$. Taking expectations on both sides gives the result.

16. From Exercise 2.14,

$$\begin{aligned}\mathbb{E}[T] &= \int_0^\infty ae^{-\lambda t} + (1-a)e^{-\mu t} dt \\ &= \left[-\frac{a}{\lambda}e^{-\lambda t} + \frac{a-1}{\mu}e^{-\mu t} \right]_0^\infty \\ &= \frac{a}{\lambda} + \frac{1-a}{\mu}.\end{aligned}$$

17. (a)

$$\int_0^m 3x^2 = [x^3]_0^m = m^3 = \frac{1}{2} \implies m = \frac{1}{\sqrt[3]{2}}.$$

(b) This is the pdf of the standard Cauchy distribution, which has median 0.

18.

$$\begin{aligned}\mathbb{E}[|X - a|] &= \int_{-\infty}^\infty |x - a|f_X(x) dx \\ &= \int_{-\infty}^a -(x - a)f_X(x) dx + \int_a^\infty (x - a)f_X(x) dx.\end{aligned}$$

Taking the derivative with respect to a ,

$$\frac{d}{da}\mathbb{E}[|X - a|] = \int_{-\infty}^a f_X(x) dx - \int_a^\infty f_X(x) dx.$$

Setting the above to 0 yields that a is the median. By the second derivative test,

$$\frac{d^2}{da^2}\mathbb{E}[|X - a|] = 2f(a) > 0$$

so that we have a minimum.

19.

20. Let X be the number of children until the first daughter. Then $X \sim \text{Geom}(p)$. Then $\mathbb{E}[X] = \frac{1}{p}$.

21. Since $y = g(x)$ and $g(x)$ is monotone, $x = g^{-1}(y) \implies dx = \frac{d}{dy}g^{-1}(y)dy$.

$$\begin{aligned}\mathbb{E}[g(X)] &= \int_{-\infty}^\infty g(x)f_X(x) dx \\ &= \int_{-\infty}^\infty g(g^{-1}(y))f_X(g^{-1}(y))\frac{d}{dy}g^{-1}(y) dy \\ &= \int_{-\infty}^\infty yf_Y(y) dy \\ &= \mathbb{E}[Y].\end{aligned}$$

22. (a) It is clear that $f(x) > 0$ when $0 < x < \infty$. In here we will just calculate the kernel and show that it is the reciprocal of $\frac{4}{\beta^3\sqrt{\pi}}$.

$$\begin{aligned}\int_0^\infty x^2 e^{-x^2/\beta^2} dx &= \left[-\frac{\beta^2}{2} x e^{-x^2/\beta^2} \right]_0^\infty + \int_0^\infty \frac{\beta^2}{2} e^{-x^2/\beta^2} dx \\ &= 0 + \int_0^\infty \frac{\beta^3}{4} e^{-u^2} du \quad (u = \frac{x}{\beta}) \\ &= \frac{\beta^3\sqrt{\pi}}{4},\end{aligned}$$

which is correct.

- (b) Using integration by parts,

$$\begin{aligned}\mathbb{E}[X] &= \frac{4}{\beta^3\sqrt{\pi}} \int_0^\infty x^3 e^{-x^2/\beta^2} dx \\ &= \frac{4}{\beta^3\sqrt{\pi}} \left[\left[-\frac{\beta^2}{2} x^2 e^{-x^2/\beta^2} \right]_0^\infty + \beta^2 \int_0^\infty x e^{-x^2/\beta^2} dx \right] \\ &= \frac{4}{\beta^3\sqrt{\pi}} \left[0 + \beta^2 \left[-\frac{\beta^2}{2} e^{-x^2/\beta^2} \right]_0^\infty \right] \\ &= \frac{2\beta}{\sqrt{\pi}}. \\ \mathbb{E}[X^2] &= \frac{4}{\beta^3\sqrt{\pi}} \int_0^\infty x^4 e^{-x^2/\beta^2} dx \\ &= \frac{4}{\beta^3\sqrt{\pi}} \left[\left[-\frac{\beta^2}{2} x^3 e^{-x^2/\beta^2} \right]_0^\infty + \frac{3\beta^2}{2} \int_0^\infty x^2 e^{-x^2/\beta^2} dx \right] \\ &= \frac{4}{\beta^3\sqrt{\pi}} \left(0 + \frac{3\beta^2}{2} \cdot \frac{\beta^3\sqrt{\pi}}{4} \right) \\ &= \frac{3\beta^2}{2}. \\ \text{Var } X &= \frac{3\beta^2}{2} - \left(\frac{2\beta}{\sqrt{\pi}} \right)^2 \\ &= \beta^2 \left(\frac{3}{2} - \frac{4}{\pi} \right).\end{aligned}$$

23. (a) First of all, $X \in (-1, 1)$ hence $Y = X^2 \in [0, 1)$.

$$\begin{aligned}F_Y(y) &= P(X^2 \leq y) \\ &= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}).\end{aligned}$$

Taking derivatives,

$$\begin{aligned}
 f_Y(y) &= f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} - f_X(-\sqrt{y}) \cdot \left(-\frac{1}{2\sqrt{y}}\right) \\
 &= \frac{1}{2}(1 + \sqrt{y}) \cdot \frac{1}{2\sqrt{y}} + \frac{1}{2}(1 - \sqrt{y}) \cdot \left(\frac{1}{2\sqrt{y}}\right) \\
 &= \frac{1}{2}y^{-1/2}, \quad y \in (0, 1).
 \end{aligned}$$

(b)

$$\begin{aligned}
 \mathbb{E}[Y] &= \int_0^1 \frac{1}{2}\sqrt{y} \, dy \\
 &= \left[\frac{1}{3}y^{3/2}\right]_0^1 \\
 &= \frac{1}{3}. \\
 \mathbb{E}[Y^2] &= \int_0^1 \frac{1}{2}y^{3/2} \, dy \\
 &= \left[\frac{1}{5}y^{5/2}\right]_0^1 \\
 &= \frac{1}{5}. \\
 \text{Var } Y &= \frac{1}{5} - \left(\frac{1}{3}\right)^2 \\
 &= \frac{4}{45}.
 \end{aligned}$$

24. (a)

$$\begin{aligned}
 \mathbb{E}[X] &= \int_0^1 ax^a \, dx = \left[\frac{a}{a+1}x^{a+1}\right]_0^1 = \frac{a}{a+1}. \\
 \mathbb{E}[X^2] &= \int_0^1 ax^{a+1} \, dx = \left[\frac{a}{a+2}x^{a+2}\right]_0^1 = \frac{a}{a+2}. \\
 \text{Var } X &= \frac{a}{a+2} - \left(\frac{a}{a+1}\right)^2 = \frac{a}{(a+2)(a+1)^2}.
 \end{aligned}$$

(b)

$$\begin{aligned}
 \mathbb{E}[X] &= \sum_{k=1}^n \frac{k}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}. \\
 \mathbb{E}[X^2] &= \sum_{k=1}^n \frac{k^2}{n} = \frac{n(n+1)(2n+1)}{6n} = \frac{(n+1)(2n+1)}{6}. \\
 \text{Var } X &= \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 = \frac{n^2+1}{12}.
 \end{aligned}$$

(c)

$$\begin{aligned}\mathbb{E}[X] &= \frac{3}{2} \int_0^2 x^3 - 2x^2 + x \, dx = \frac{3}{2} \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2 \right]_0^2 = 1. \\ \mathbb{E}[X^2] &= \frac{3}{2} \int_0^2 x^4 - 2x^3 + x^2 \, dx = \frac{3}{2} \left[\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3 \right]_0^2 = \frac{8}{5}. \\ \text{Var } X &= \frac{8}{5} - 1^2 = \frac{3}{5}.\end{aligned}$$

25. (a) Let $Y = -X$. Then $g(x) = g^{-1}(x) = -x$. We have that

$$f_{-X}(x) = f_X(-x) \cdot |-1| = f_X(x) \forall x$$

so that X and $-X$ are identically distributed.

(b) Let $\varepsilon > 0$ be given. Then

$$\begin{aligned}M_X(0 + \varepsilon) &= \int_{-\infty}^{\infty} e^{\varepsilon x} f_X(x) \, dx \\ &= - \int_{\infty}^{-\infty} e^{(-\varepsilon u)} f_X(u) \, du \quad (u = -x) \\ &= \int_{-\infty}^{\infty} e^{(0-\varepsilon)u} f_X(u) \, du \\ &= M_X(0 - \varepsilon).\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we are done.

26. (a) $N(\mu, \sigma^2)$ is symmetric about μ , DoubleExp(μ, b) is symmetric about μ , and t_n is symmetric about 0.

(b)

$$\begin{aligned}\int_a^{\infty} f(x) \, dx &= \int_0^{\infty} f(a + \varepsilon) \, d\varepsilon \quad (\varepsilon = x - a) \\ &= \int_0^{\infty} f(a - \varepsilon) \, d\varepsilon \\ &= \int_a^{\infty} f(x) \, dx \quad (x = a + \varepsilon)\end{aligned}$$

Since f is a valid pdf, a has to be the median.

(c)

$$\begin{aligned}
\mathbb{E}[X] - a &= \mathbb{E}[X - a] \\
&= \int_{-\infty}^{\infty} (x - a)f(x) dx \\
&= \int_{-\infty}^a (x - a)f(x) dx + \int_a^{\infty} (x - a)f(x) dx \\
&= \int_0^{\infty} -\varepsilon f(a - \varepsilon) d\varepsilon + \int_0^{\infty} \varepsilon f(a + \varepsilon) d\varepsilon \\
&= - \int_0^{\infty} \varepsilon f(a + \varepsilon) d\varepsilon + \int_0^{\infty} \varepsilon f(a + \varepsilon) d\varepsilon \\
&= 0.
\end{aligned}$$

Here, we substituted $\varepsilon = a - x$ for the first integral and $\varepsilon = x - a$ for the second integral (sorry for the confusing notation).

- (d) If $a < 0$, for $\varepsilon > a$, $f(a - \varepsilon) = 0$ but $f(a + \varepsilon) > 0$. If $a \geq 0$, the same is true, hence $f(x)$ is not a symmetric pdf.
- (e) For the mean,

$$\begin{aligned}
\mathbb{E}[X] &= \int_0^{\infty} x e^{-x} dx \\
&= [-x e^{-x} - e^{-x}]_0^{\infty} \\
&= 1.
\end{aligned}$$

For the median,

$$\int_0^a e^{-x} dx = \frac{1}{2} \implies a = \log 2.$$

Since $\log 2 < 1$, the median is less than the mean.

27. (a) The standard normal has a unique mode at $x = 0$.
- (b) The Uniform(0, 1) does not have a unique mode as all $x \in (0, 1)$ is a mode.
- (c) First suppose that the mode is unique. Let a be the mean and b be the mode suppose that $a \neq b$. We can assume without loss of generality that $a = b + \varepsilon$. Since $f(x)$ is unimodal, $f(b) > f(b + \varepsilon) \geq f(b + 2\varepsilon)$, and $f(b - 2\varepsilon) \geq f(b - \varepsilon) > f(b)$, contradicting to our assumption that f is symmetric about b .
- Now suppose that the mode is not unique. Then it is the same case except that there is a region (x_1, x_2) such that b is a mode for all $b \in (x_1, x_2)$.
- (d) f is monotonically decreasing on $[0, \infty)$ hence it is unimodal with mode 0.

28. (a) From part (c) of Exercise 2.26, $\mathbb{E}[X] = a$. Then

$$\begin{aligned}
 \mu_3 &= \int_{-\infty}^{\infty} (x-a)^3 f(x) dx \\
 &= \int_{-\infty}^a (x-a)^3 f(x) dx + \int_a^{\infty} (x-a)^3 f(x) dx \\
 &= \int_{-\infty}^0 u^3 f(a+u) du + \int_0^{\infty} u^3 f(a+u) du \quad (u = x-a) \\
 &= \int_0^{\infty} (-v)^3 f(a-v) dv + \int_0^{\infty} u^3 f(a+u) du \quad (v = -u) \\
 &= - \int_0^{\infty} v^3 f(a+v) dv + \int_0^{\infty} u^3 f(a+u) du \quad (f(a-v) = f(a+v)) \\
 &= 0.
 \end{aligned}$$

(b) First of all,

$$E[X] = \int_0^{\infty} x e^{-x} dx = [-x e^{-x} - e^{-x}]_0^{\infty} = 1.$$

Then

$$\begin{aligned}
 \mu_2 &= \int_0^{\infty} (x-1)^2 e^{-x} dx \\
 &= [- (x-1)^2 e^{-x} - 2(x-1)e^{-x} - 2e^{-x}]_0^{\infty} \\
 &= 0 - (-1 + 2 - 2) \\
 &= 1, \\
 \mu_3 &= \int_0^{\infty} (x-1)^2 e^{-x} dx \\
 &= [- (x-1)^3 e^{-x} - 3(x-1)^2 e^{-x} - 6(x-1)e^{-x} - 6e^{-x}]_0^{\infty} \\
 &= 0 - (1 - 3 + 6 - 6) \\
 &= 2.
 \end{aligned}$$

Therefore $\alpha_3 = \frac{2}{1^{3/2}} = 2$.

(c) The first pdf is the standard normal, so for any even number $n = 2k, k \in \mathbb{N}$, $\mathbb{E}[X^n] = (n-1)!!$ so $\alpha_4 = \frac{3}{1^2} = 3$.

For the second pdf,

$$\begin{aligned}
 \mathbb{E}[X^2] &= \int_{-1}^1 \frac{1}{2} x^2 dx = \left[\frac{1}{6} x^3 \right]_{-1}^1 = \frac{1}{3}. \\
 \mathbb{E}[X^2] &= \int_{-1}^1 \frac{1}{2} x^4 dx = \left[\frac{1}{10} x^5 \right]_{-1}^1 = \frac{1}{5}. \\
 \mu_4 &= \frac{1}{5} / \left(\frac{1}{3} \right)^2 = \frac{9}{5}.
 \end{aligned}$$

For the third pdf, since it is symmetric and unimodal, $\mathbb{E}[X] = 0$. Then

$$\begin{aligned}\mathbb{E}[X^2] &= \int_{-\infty}^0 \frac{1}{2} x^2 e^x dx + \int_0^{\infty} \frac{1}{2} x^2 e^{-x} dx = 2. \\ \mathbb{E}[X^4] &= \int_{-\infty}^0 \frac{1}{2} x^4 e^x dx + \int_0^{\infty} \frac{1}{2} x^4 e^{-x} dx = 24. \\ \mu_4 &= \frac{24}{2^2} = 6.\end{aligned}$$

We can see that the larger the kurtosis, the more peaked the pdf is.

29. (a) For the Binomial(n, p) distribution,

$$\begin{aligned}\mathbb{E}[X(X-1)] &= \sum_{k=0}^n k(k-1) \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=2}^n n(n-1) \binom{n-2}{k-2} p^k (1-p)^{n-k} \\ &= n(n-1)p^2 \sum_{l=0}^{n-2} \binom{n-2}{l} p^l (1-p)^{n-2-l} \\ &= n(n-1)p^2,\end{aligned}$$

where we used the substitution $l = k - 2$.

For the Poisson(λ) distribution,

$$\begin{aligned}\mathbb{E}[X(X-1)] &= \sum_{k=0}^{\infty} k(k-1) \frac{\lambda^k e^{-\lambda}}{k!} \\ &= \sum_{k=2}^{\infty} k(k-1) \frac{\lambda^k e^{-\lambda}}{k!} \\ &= \lambda^2 \sum_{k=2}^{\infty} \frac{\lambda^{k-2} e^{-\lambda}}{(k-2)!} \\ &= \lambda^2.\end{aligned}$$

(b) Since $\text{Var } X = \mathbb{E}[X(X-1)] + \mathbb{E}[X] - (\mathbb{E}[X])^2$, for the binomial,

$$\text{Var } X = n(n-1)p^2 + np - (np)^2 = np(1-p).$$

For the Poisson,

$$\text{Var } X = \lambda^2 + \lambda - \lambda^2 = \lambda.$$

(c)

30. (a)

$$M(t) = \int_0^c e^{tx} \cdot \frac{1}{c} dx = \frac{1}{c} \left[\frac{1}{t} e^{tx} \right]_0^c = \frac{1}{ct} (e^{ct} - 1).$$

(b)

$$\begin{aligned}
 M(t) &= \int_0^c e^{tx} \cdot \frac{2x}{c^2} dx \\
 &= \frac{2}{c^2} \left[\frac{x}{t} e^{tx} - \frac{1}{t^2} e^{tx} \right]_0^c \\
 &= \frac{2}{c^2} \left(\frac{c}{t} e^{ct} - \frac{1}{t^2} e^{ct} + \frac{1}{t^2} \right) \\
 &= \frac{2}{ct} e^{ct} - \frac{2}{c^2 t^2} e^{ct} + \frac{2}{c^2 t^2}.
 \end{aligned}$$

(c)

31. No such distribution exists. First note that $M_X(0) = \mathbb{E}[e^0] = 1$. If the mgf were to be that stated in the question, $M_X(0) = 0$, which is incorrect.

32.

$$\frac{d}{dt} S(t) \Big|_{t=0} = \frac{1}{M_X(t)} \cdot M'_X(t) \Big|_{t=0} = \frac{1}{1} \cdot M'_X(0) = \mathbb{E}[X].$$

$$\begin{aligned}
 \frac{d^2}{dt^2} S(t) \Big|_{t=0} &= -\frac{1}{M_X^2(t)} \cdot (M'_X(t))^2 + \frac{1}{M_X(t)} M''_X(t) \Big|_{t=0} \\
 &= -\frac{1}{1^2} (M'_X(0))^2 + \frac{1}{1} M''_X(0) \\
 &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\
 &= \text{Var } X.
 \end{aligned}$$

33. (a)

$$\begin{aligned}
 M_X(t) &= \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x e^{-\lambda}}{x!} \\
 &= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} \\
 &= e^{-\lambda} \cdot e^{\lambda e^t} \\
 &= e^{\lambda(e^t - 1)}.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \mathbb{E}[X] &= M'_X(0) = \lambda e^t \cdot e^{\lambda(e^t - 1)} \Big|_{t=0} = \lambda, \\
 \mathbb{E}[X^2] &= M''_X(0) = \lambda e^t \cdot e^{\lambda(e^t - 1)} + (\lambda e^t)^2 \cdot e^{\lambda(e^t - 1)} \Big|_{t=0} = \lambda^2 + \lambda, \\
 \text{Var } X &= \lambda^2 + \lambda - \lambda^2 = \lambda.
 \end{aligned}$$

(b)

$$\begin{aligned}
M_X(t) &= \sum_{x=0}^{\infty} e^{tx} \cdot p(1-p)^x \\
&= p \sum_{x=0}^{\infty} ((1-p)e^t)^x \\
&= \frac{p}{1 - (1-p)e^t}, \quad t < -\log(1-p).
\end{aligned}$$

Therefore

$$\mathbb{E}[X] = M'_X(0) = \frac{-p(-(1-p)e^t)}{(1 - (1-p)e^t)^2} \Big|_{t=0} = \frac{p(1-p)e^t}{(1 - (1-p)e^t)^2} \Big|_{t=0} = \frac{1-p}{p}.$$

$$\begin{aligned}
\mathbb{E}[X^2] &= M''_X(0) \\
&= \frac{p(1-p)e^t(1 - (1-p)e^t)^2 - p(1-p)e^t 2(1 - (1-p)e^t)(-(1-p)e^t)}{(1 - (1-p)e^t)^4} \Big|_{t=0} \\
&= \frac{p^3(1-p + 2p^2(1-p)^2)}{p^4} \\
&= \frac{p(1-p) + 2(1-p)^2}{p^2}.
\end{aligned}$$

$$\text{Var } X = \frac{p(1-p) + 2(1-p)^2}{p^2} - \left(\frac{1-p}{p}\right)^2 = \frac{1-p}{p^2}.$$

(c)

$$\begin{aligned}
M_X(t) &= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x^2 - 2\mu x - 2t\sigma^2 x + \mu^2)/2\sigma^2} dx \\
&= e^{(2\mu t\sigma^2 + t^2\sigma^4)/2\sigma^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x - (\mu + t\sigma^2))^2} dx \\
&= e^{\mu t + \sigma^2 t^2/2}.
\end{aligned}$$

Where we have completed the square on the exponential by doing

$$\begin{aligned}
x^2 - 2\mu x - 2t\sigma^2 x + \mu^2 &= x^2 - 2(\mu + t\sigma^2)x + (\mu + t\sigma^2)^2 - (\mu + t\sigma^2)^2 + \mu^2 \\
&= (x - (\mu + t\sigma^2))^2 - (2\mu t\sigma^2 + t^2\sigma^4).
\end{aligned}$$

Therefore,

$$\mathbb{E}[X] = M'_X(0) = (\mu + \sigma^2 t) e^{\mu t + \sigma^2 t^2/2} \Big|_{t=0} = \mu.$$

$$\begin{aligned}
\mathbb{E}[X^2] &= M''_X(0) \\
&= \sigma^2 e^{\mu t + \sigma^2 t^2/2} + (\mu + \sigma^2 t)^2 e^{\mu t + \sigma^2 t^2/2} \Big|_{t=0} \\
&= \sigma^2 + \mu^2.
\end{aligned}$$

$$\text{Var } X = \sigma^2 + \mu^2 - \mu^2 = \sigma^2.$$

34. Since $X \sim N(0, 1)$,

$$\mathbb{E}[X^n] = \begin{cases} 0, & \text{if } n \text{ odd,} \\ (n-1)!!, & \text{if } n \text{ even.} \end{cases}$$

Therefore $\mathbb{E}[X] = \mathbb{E}[X^3] = \mathbb{E}[X^5] = 0$, $\mathbb{E}[X^2] = 1$, $\mathbb{E}[X^4] = 3$.

On the other hand,

$$E[Y^n] = \frac{1}{6}(-\sqrt{3})^n + \frac{1}{6}(\sqrt{3})^n,$$

so $\mathbb{E}[Y] = \mathbb{E}[Y^3] = \mathbb{E}[Y^5] = 0$, $\mathbb{E}[Y^2] = 1$, $\mathbb{E}[Y^4] = 3$, so

$$\mathbb{E}[X^n] = \mathbb{E}[Y^n], \quad n = 1, 2, 3, 4, 5.$$