Statistical Inference Chapter 1

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1. (a) $\Omega = \{(x_1, x_2, x_3, x_4) : x_i \in \{H, T\}\}$.
   (b) If there are N leaves on the plant, $\Omega = [N]$.
   (c) $\Omega = \{t : t \in \mathbb{R},\ t \geq 0\}$.
   (d) $\Omega = \{w : w \in \mathbb{R}_+\}$.
   (e) If there are n components, $\Omega = \{i/n : i \in \{0, 1, ..., n\}\}$.
2. (a)

$$\begin{aligned} x \in A \setminus B &\iff x \in A \text{ and } x \notin B \\ &\iff x \in A \text{ and } x \notin A \cap B \\ &\iff x \in A \setminus (A \cap B). \end{aligned}$$

   Also,

$$\begin{aligned} x \in A \setminus B &\iff x \in A \text{ and } x \notin B \\ &\iff x \in A \text{ and } x \in B^c \\ &\iff x \in A \cap B^c. \end{aligned}$$

   Therefore $A \setminus B = A \setminus (A \cap B) = A \cap B^c$.

   (b) By the distributive law,

$$\begin{aligned} (B \cap A) \cup (B \cap A^c) &= B \cap (A \cup A^c) \\ &= B. \end{aligned}$$

   (c)

$$\begin{aligned} x \in B \setminus A &\iff x \in B \text{ and } x \notin A \\ &\iff x \in B \text{ and } x \in A^c \\ &\iff x \in B \cap A^c. \end{aligned}$$

   (d) From part b), we have

$$\begin{aligned} A \cup B &= A \cup ((B \cap A) \cup (B \cap A^c)) \\ &= A \cup (B \cap A) \cup A \cup (B \cap A^c) \\ &= A \cup A \cup (B \cap A^c) \\ &= A \cup (B \cap A^c). \end{aligned}$$

3. (a)

$$\begin{aligned} x \in A \cup B &\iff x \in A \text{ or } x \in B \\ &\iff x \in B \cup A. \\ x \in A \cap B &\iff x \in A \text{ and } x \in B \\ &\iff x in B \cap A. \end{aligned}$$

(b)

$$\begin{aligned} x \in A \cup (B \cup C) &= x \in A \text{ or } x \in B \cup C \\ &= x \in A \cup B \text{ or } x \in C \\ &= x \in (A \cup B) \cup C. \end{aligned}$$

(c)

$$\begin{aligned} x \in (A \cup B)^c &\iff x \notin A \cup B \\ &\iff x \in A^c \text{ and } x \in B^c \\ &\iff x \in A^c \cap B^c. \\ x \in (A \cap B)^c &\iff x \notin A \cap B \\ &\iff x \in A^c \text{ or } x \in B^c \\ &\iff x \in A^c \cup B^c. \end{aligned}$$

4. (a) This is $P(A \cup B)$, so we get $P(A) + P(B) - P(A \cap B)$.
   (b) This is $P(A \Delta B)$, so we get $P(A) + P(B) - 2P(A \cap B)$.
   (c) This is again $P(A \cup B)$, so we get $P(A) + P(B) - P(A \cap B)$.
   (d) This is $P((A \cap B)^c)$, so we get $1 - P(A \cap B)$.

5. (a) $A \cap B \cap C = \{$a U.S. birth resulting in identtical twin females$\}$.
   (b) $P(A \cap B \cap C) = \frac{1}{90} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{540}$.

6. $p_0 = (1-u)(1-w), p_1 = u(1-w) + w(1-u), p_2 = uw$. For them to be equal,

$$\begin{aligned} p_0 = p_2 &\implies 1 - u - w + uw = uw \\ &\implies u + w = 1, \\ p_1 = p_2 &\implies u + w - 2uw = uw \\ &\implies uw = \frac{1}{3}. \end{aligned}$$

The above two equations imply $u(1-u) = \frac{1}{3}$, which has no real solutions in $\mathbb{R}$. Therefore we can't choose such u, w satisfying $p_0 = p_1 = p_2$.

7. (a) This is just having an extra case of hitting outside of the dart board. So

$$P(\text{scoring } i \text{ points}) = \begin{cases} 1 - \frac{\pi r^2}{A} & i = 0 \\ \frac{\pi r^2}{A} \cdot \frac{1}{5^2}((6-i)^2 - (5-i)^2) & i = 1, ..., 5 \end{cases}$$

(b)

$$\begin{aligned}
P(\text{scoring } i \text{ points}|\text{board is hit}) &= \frac{P(\text{scoring } i \text{ points, board is hit})}{P(\text{board is hit})} \\
&= \frac{\pi r^2}{A} \cdot \frac{1}{5^2}((6-i)^2 - (5-i)^2) / \frac{\pi r^2}{A} \\
&= \frac{1}{5^2}((6-i)^2 - (5-i)^2), \ i = 1, ..., 5
\end{aligned}$$

For $i = 0$, we will definitely score given that we hit the board so $P(\text{scoring 0 points}|\text{board is hit}) = 0$, which is consistent with the probability distribution in Example 1.2.7 as well.

8. (a) From the example given,

$$P(\text{scoring } i \text{ points}) = \frac{(6-i)^2 - (5-i)^2}{5^2}, i = 1, ..., 5.$$

(b) Expanding the above,

$$\frac{(6-i)^2 - (5-i)^2}{5^2} = \frac{11 - 2i}{r^2},$$

which is a decreasing function of i.

(c)

$$\frac{11 - 2i}{5^2} > 0 \text{for } i = 1, ..., 5$$

hence the first axiom is satisfied.

$$P(S) = P(\text{hitting the board}) = 1,$$

hence the second axiom is satisfied. For $i \neq j$,

$$P(i \cup j) = \text{Area of ring } i + \text{Area of ring } j = P(i) + P(j),$$

hence the third axiom is satisfied so $P(\text{scoring } i \text{ points})$ is a probability function.

(a) Suppose $x \in (\cup_\alpha A_\alpha)^c$. Then $x \notin A_\alpha$ for all $\alpha \in \Gamma$ so $x \in A_\alpha^c$ for all $\alpha \in \Gamma$. Therefore $x \in \cap_\alpha A_\alpha$.
Now suppose $x \in \cap_\alpha A_\alpha^c$. Then for all $\alpha \in \Gamma$, $x \in A_\alpha^c$ hence $x \notin A_\alpha$, then $x \notin \cup_\alpha A_\alpha$ so $x \in (\cup_\alpha A_\alpha)^c$.

(b) Suppose $x \in (\cap_\alpha A_\alpha)^c$. Then $x \notin \cap_\alpha A_\alpha$ so $x \notin A_\alpha$ for some $\alpha \in \Gamma$. Then $x \in A_\alpha^c$ for some $\alpha \in \Gamma$. Therefore $x \in \cup_\alpha A_\alpha^c$.
Now suppose $x \in \cup_\alpha A_\alpha^c$. Then $x \in A_\alpha^c$ for some $\alpha \in \Gamma$ so $x \notin A_\alpha$ for some $\alpha \in \Gamma$. Then $x \notin \cap_\alpha$ thus $x \in (\cap_\alpha)^c$.