Statistical Inference Chapter 2

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1. (a) Let $g(x) = x^3$. Then g is monotonically increasing on $(0,1)$. We get

$$g^{-1}(y) = y^{1/3} \implies \frac{d}{dy}g^{-1}(y) = \frac{1}{3y^{2/3}}.$$

Since $X \in (0,1),\ Y = X^3 \in (0,1)$. Then by Theorem 2.1.5,

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y))\Big|\frac{d}{dy}g^{-1}(y)\Big| \\ &= 42(y^{1/3})^5(1-y^{1/3}) \cdot \frac{1}{3y^{2/3}} \\ &= 14y(1-y^{1/3}),\ y \in (0,1). \end{aligned}$$

We also have

$$\begin{aligned} \int_0^1 14y(1-y^{1/3})\ dy &= 14\int_0^1 y - y^{4/3}\ dy \\ &= 14\Big[\frac{1}{2}y^2 - \frac{3}{7}y^{7/3}\Big]_0^1 \\ &= 14(\frac{1}{2} - \frac{3}{7}) \\ &= 1. \end{aligned}$$

(b) Let $g(x) = 4x+3$. Then g is monotonically increasing on $(0,\infty)$. We get

$$g^{-1}(y) = \frac{y-3}{4} \implies \frac{d}{dy}g^{-1}(y) = \frac{1}{4}.$$

Since $X \in (0,\infty),\ Y = 4X+3 \in (3,\infty)$. Then by Theorem 2.1.5,

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y))\Big|\frac{d}{dy}g^{-1}(y)\Big| \\ &= 7e^{-7\cdot\frac{y-3}{4}} \cdot \frac{1}{4} \\ &= \frac{7}{4}e^{\frac{21}{4}-\frac{7}{4}y},\ y \in (3,\infty). \end{aligned}$$

We also have

$$\begin{aligned}\int_3^\infty \frac{7}{4}e^{\frac{21}{4}-\frac{7}{4}y}\ dy &= \frac{7}{4}e^{\frac{21}{4}}\int_3^\infty e^{-\frac{7}{4}y}\ dy\\ &= \frac{7}{4}e^{\frac{21}{4}}\Big[-\frac{4}{7}e^{-\frac{7}{4}y}\Big]_3^\infty\\ &= \frac{7}{4}e^{\frac{21}{4}}(\frac{4}{7}e^{-\frac{21}{4}})\\ &= 1.\end{aligned}$$

(c) Let $g(x) = x^2$. Then g is monotonically increasing on $(0,1)$. We get

$$g^{-1}(y) = \sqrt{y} \implies \frac{d}{dy}g^{-1}(y) = \frac{1}{2\sqrt{y}}.$$

Since $X \in (0,1),\ Y = X^2 \in (0,1)$. Then by Theorem 2.1.5,

$$\begin{aligned}f_Y(y) &= f_X(g^{-1}(y))\Big|\frac{d}{dy}g^{-1}(y)\Big|\\ &= 30y(1-\sqrt{y})^2 \cdot \frac{1}{2\sqrt{y}}\\ &= 15\sqrt{y}(1-\sqrt{y})^2,\ y \in (0,1).\end{aligned}$$

We also have

$$\begin{aligned}\int_0^1 15\sqrt{y}(1-\sqrt{y})^2\ dy &= 15\int_0^1 \sqrt{y} - 2y + y^{3/2}\ dy\\ &= 15\Big[\frac{2}{3}y^{3/2} - y^2 + \frac{2}{5}y^{5/2}\Big]_0^1\\ &= 15(\frac{2}{3} - 1 + \frac{2}{5})\\ &= 1.\end{aligned}$$

2. (a) Let $g(x) = x^2$. Then g is monotonically increasing on $(0,1)$. We get

$$g^{-1}(y) = \sqrt{y} \implies \frac{d}{dy}g^{-1}(y) = \frac{1}{2\sqrt{y}}.$$

Since $X \in (0,1),\ Y = X^2 \in (0,1)$. Then by Theorem 2.1.5,

$$\begin{aligned}f_Y(y) &= f_X(g^{-1}(y))\Big|\frac{d}{dy}g^{-1}(y)\Big|\\ &= 1 \cdot \frac{1}{2\sqrt{y}}\\ &= \frac{1}{2\sqrt{y}},\ y \in (0,1).\end{aligned}$$

(b) Let $g(x) = -\log x$. Then g is monotonically decreasing on $(0,1)$. We get

$$g^{-1}(y) = e^{-y} \implies \frac{d}{dy}g^{-1}(y) = -e^{-y}.$$

Since $X \in (0,1),\ Y = \log X \in (0,\infty)$. Then by Theorem 2.1.5,

$$\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y))\Big|\frac{d}{dy}g^{-1}(y)\Big| \\
&= \frac{(n+m+1)!}{n!m!}e^{-ny}(1-e^{-y})^m \cdot |-e^{-y}| \\
&= \frac{(n+m+1)!}{n!m!}e^{-y(n+1)}(1-e^{-y})^m,\ y \in (0,\infty).
\end{aligned}$$

(c) Let $g(x) = e^x$. Then g is monotonically increasing on $(0,\infty)$. We get

$$g^{-1}(y) = \ln y \implies \frac{d}{dy}g^{-1}(y) = \frac{1}{y}.$$

Since $X \in (0,\infty),\ Y = e^X \in (0,\infty)$. Then by Theorem 2.1.5,

$$\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y))\Big|\frac{d}{dy}g^{-1}(y)\Big| \\
&= \frac{1}{\sigma^2}\ln y e^{-(\ln y/\sigma)^2/2} \cdot \frac{1}{y} \\
&= \frac{1}{\sigma^2}\frac{\ln y}{y}e^{-(\ln y/\sigma)^2/2},\ y \in (0,\infty).
\end{aligned}$$

3. First of all,

$$X \in \{0,1,2,...\} \implies Y \in \Big\{0, \frac{1}{2}, \frac{2}{3}, ...\Big\}.$$

Then

$$\begin{aligned}
P(Y = y) &= P(\frac{X}{X+1} = y) \\
&= P(1 - \frac{1}{X+1} = y) \\
&= P(X = \frac{y}{1-y}) \\
&= \frac{1}{3}\Big(\frac{2}{3}\Big)^{y/(1-y)},\ y \in \Big\{\frac{k}{k+1} : k \in \mathbb{N}_0\Big\}.
\end{aligned}$$

4. (a) It is not hard to see that $f(x) \geq 0\ \forall x \in \mathcal{X}$ as both piecewise functions

are exponentials. We also have

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\ dx &= \int_{-\infty}^{0} \frac{1}{2}\lambda e^{\lambda x} + \int_{0}^{\infty} \frac{1}{2}\lambda e^{-\lambda x}\ dx \\
&= \Big[frac12e^{lambdax}\Big]_{-\infty}^{0} + \Big[-\frac{1}{2}e^{-\lambda x}\Big]_{0}^{\infty} \\
&= \frac{1}{2} + \frac{1}{2} \\
&= 1.
\end{aligned}$$

(b) For $t \leq 0$,

$$\begin{aligned}
P(X < t) &= \int_{-\infty}^{t} \frac{1}{2}\lambda e^{\lambda x}\ dx \\
&= \Big[\frac{1}{2}e^{\lambda x}\Big]_{\infty}^{t} \\
&= \frac{1}{2}e^{\lambda t}.
\end{aligned}$$

For $t > 0$,

$$\begin{aligned}
P(X < t) &= \frac{1}{2} + \int_{0}^{t} \frac{1}{2}\lambda e^{-\lambda x}\ dx \\
&= \frac{1}{2} + \Big[-\frac{1}{2}e^{-\lambda x}\Big]_{0}^{t} \\
&= \frac{1}{2} + (-\frac{1}{2}e^{-\lambda t} + \frac{1}{2}) \\
&= 1 - \frac{1}{2}e^{-\lambda t}.
\end{aligned}$$

(c) For $t \leq 0$, $P(|X| < t) = 0$. For $t > 0$,

$$\begin{aligned}
P(|X| < t) &= P(-t < X < t) \\
&= \int_{-t}^{0} \frac{1}{2}\lambda e^{\lambda x}\ dx + \int_{0}^{t} \frac{1}{2}\lambda e^{-\lambda x}\ dx \\
&= \Big[\frac{1}{2}e^{\lambda x}\Big]_{-t}^{0} + \Big[\frac{1}{2}e^{-\lambda x}\Big]_{0}^{t} \\
&= \frac{1}{2} - \frac{1}{2}e^{-\lambda t} + (-\frac{1}{2}e^{-\lambda t} + \frac{1}{2}) \\
&= 1 - e^{-\lambda t}.
\end{aligned}$$

5. Let $A_0 = \{\pi\}, A_1 = (0, \frac{\pi}{2}), A_2 = (\frac{\pi}{2}, \pi), A_3 = (\pi, \frac{3\pi}{2}), A_4 = (\frac{3\pi}{2}, 2\pi)$, and let $g(x) = g_i(x) = \sin^2 x$. Then for each $A_i (i \neq 0), g_i(x) = g(x) \forall x \in A_i,$, $g_i(x)$ is monotone on A_i. Moreover, $\mathcal{Y} = (0, 1)$ is the same for all i, and monotone on A_i, and

$$g^{-1}(y) = \arcsin(\sqrt{x}) \implies \frac{d}{dy}g^{-1}(y) = \frac{1}{2\sqrt{y(1-y)}}$$

is continuous on $\mathcal{Y}$ for all i. Then by Theorem 2.1.8,

$$
\begin{aligned}
f_Y(y) &= \sum_{i=1}^{4} f_X(g^{-1}(y))\Big|\frac{d}{dy}g_i^{-1}(y)\Big| \\
&= 4 \cdot \frac{1}{2\pi} \cdot \Big|\frac{1}{2\sqrt{y(1-y)}}\Big| \\
&= \frac{1}{\pi\sqrt{y(1-y)}},\ y \in (0,1).
\end{aligned}
$$

To use the cdf from (2.1.6), we first get that $x_1 = \arcsin(\sqrt{y}), x_2 = \pi - \arcsin(\sqrt{y})$. Note

$$
P(Y \leq y) = 2P(X \leq x_1) + 2P(X \leq \pi) - 2P(X \leq x_2)
$$

Then by differentiating the above we get

$$
\begin{aligned}
f_Y(y) &= 2f_X(x_1) \cdot \frac{d}{dy}(\sin^{-1}\sqrt{y}) - 2f_X(x_2) \cdot \frac{d}{dy}(\pi - \sin^{-1}\sqrt{y}) \\
&= 2 \cdot \frac{1}{2\pi} \cdot \frac{1}{2\sqrt{y(1-y)}} - 2 \cdot \frac{1}{2\pi} \cdot (-\frac{1}{2\sqrt{y(1-y)}}) \\
&= \frac{1}{\pi\sqrt{y(1-y)}},\ y \in (0,1).
\end{aligned}
$$