F_{12}ECM

A PROGRAM FOR FINDING
THE FACTORS OF
THE TWELFTH FERMAT NUMBER

Elliptic Curve Method
and
Probabilities

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Chapter 1

Fermat numbers

Pierre de Fermat conjectured that every number of the form $F_n = 2^{2^n} + 1$, where n is a non-negative integer, is prime [6]. Today these positive integers are named Fermat numbers. The first five Fermat numbers are prime, but Leonhard Euler proved in 1732 that 641 divides F_5.

F_6 was completely factored by T Clausen, F. Landry and H. Le Lasseur in 1855. In 1970, M. A. Morrison and J. Brillhart cracked F_7 by the Continued Fraction method [10]. In 1980, R. P. Brent and J. M. Pollard used a modification of Pollard's rho method to factor F_8. R. P. Brent completely factored F_{11} in 1988 by ECM [3]. In 1990, A. K. Lenstra, H. W. Lenstra, M. S. Manasse and J. M. Pollard organized a distributed computation on approximately 700 workstations around the world and factored F_9 by the Number Field Sieve [8]. Finally R. P. Brent completely factored F_{10} in 1995 by ECM [3].

The smallest Fermat number which is not completely factored is F_{12}. Six prime factors are known, the 54-digit factor was found by Michael Vang in 2010 using GMP–ECM [12].

$$
\begin{aligned}
F_5 &= 641 \cdot 6700417 \\
F_6 &= 274177 \cdot 67280421310721 \\
F_7 &= 59649589127497217 \cdot 5704689200685129054721 \\
F_8 &= 1238926361552897 \cdot P_{62} \\
F_9 &= 2424833 \cdot 7455602825647884208337395736200454918783366342657 \cdot P_{99} \\
F_{10} &= 45592577 \cdot 6487031809 \cdot 4659775785220018543264560743076778192897 \cdot P_{252} \\
F_{11} &= 319489 \cdot 974849 \cdot 167988556341760475137 \cdot 3560841906445833920513 \cdot P_{564} \\
\\
F_{12} &= 114689 \cdot 26017793 \cdot 63766529 \cdot 190274191361 \cdot 1256132134125569 \cdot \\
&\quad 568630647535356955169033410940867804839360742060818433 \cdot C_{1133}
\end{aligned}
$$

Chapter 2

Elliptic curves modulo p

An elliptic curve over $\mathbb{Z}/p\mathbb{Z}$ is a set of points $P = (x, y) \in (\mathbb{Z}/p\mathbb{Z})^2$ such that

$$y^2 = x^3 + ax + b$$

with the condition $4a^3 + 27b^2 \neq 0$ and an additional point at infinity O.

The point $Q = (x, -y)$ is on the curve: $Q = -P$ is the point opposite of P. The curve has the property that if a non-vertical line intersects it at two points P and Q, then it will also have a third point R of intersection. The addition law is defined by $P + Q = -R$. If $P = Q$, the tangent of the curve at P is considered. If the line is vertical, we have $P + -P = O$. The points on an elliptic curve and the addition form an abelian group. O is the identity of the group: we have $P + O = O + P = P$.

Hasse's theorem on elliptic curves over finite fields provides an estimate of the number of points. If $\#E$ is the order of the group of an elliptic curve E over $\mathbb{Z}/p\mathbb{Z}$ then

$$|\#E - (p + 1)| \leq 2\sqrt{p}.$$

Given a prime $p > 3$ and any integer n such that $|n - (p + 1)| \leq 2\sqrt{p}$, there exists a and b such that $|\#E(a, b)| = n$. Furthermore, the numbers of points are uniformly distributed over the interval $[p + 1 - 2\sqrt{p}, p + 1 + 2\sqrt{p}]$.

To avoid the time-consuming inversion over $\mathbb{Z}/p\mathbb{Z}$, projective coordinates are preferred for computations. X, Y, Z are integers such that $x = X/Z$ and $y = Y/Z$. If $P \neq O$ then $Z \neq 0$ and the coordinates of the identity O are $(0, Y, 0)$.

$y^2 = x^3 + ax + b$ is called the Weierstrass form but there exist some alternative representations of elliptic curves. Some of them are faster for computations. F$_{12}$ECM representation is the Montgomery curve

$$By^2 = x^3 + Ax^2 + x.$$

The Montgomery curves are a subset of Weierstrass curves.

Montgomery coordinates are the projective coordinates (X, Z). The computation of Y is not needed for the Elliptic Curve Method.

Chapter 3

Elliptic Curve Method

3.1 Algorithm

The elliptic curve factorization method (ECM) is an extension of Pollard's $p-1$ algorithm [11].

Pollard's $p-1$ algorithm finds the prime factors p such that $p-1$ is B-smooth.
If $p > 2$ and e is a multiple of $p-1$ then by Fermat's little theorem we have $2^e \equiv 1 \pmod{p}$. For a fixed bound B, $M = \prod_{\substack{p \leq B \\ p \text{ prime}}} p^{\lfloor \log_p B \rfloor}$ is computed modulo n and finally $g = \gcd(2^M - 1, n)$.
If n is a composite integer, $p \mid n$, $p-1$ is B-powersmooth and $q \mid n$ but $q-1$ is not B-powersmooth then $1 < g < n$ and g is a multiple of p.
In practice, a two-stage variant of the algorithm is implemented: instead of requiring that $p-1$ has all its factors less than B, if all but one of them are less than B_1 and the remaining factor is less than B_2 then the range $]B_1; B_2]$ can be tested more quickly. M is computed for $B = B1$ and the second stage is $M' = \prod_{\substack{B_1 < p \leq B_2 \\ p \text{ prime}}} \left((2^M)^p - 1 \right)$. If p_n and p_{n+1} are two consecutive prime numbers then $A^{p_{n+1}} = A^{p_n} \cdot A^{d_n}$ where $d_n = p_{n+1} - p_n$. The d_n are relatively small then the values of $A^2, A^4, A^6, \ldots$ can be precomputed. Then $\left(2^M \right)^p$ is calculated with a single multiplication.

Fermat's little theorem can be extended to a group G such that each element of G is invertible. Pollard's $p-1$ is based on $(\mathbb{Z}/p\mathbb{Z})^\times$. A finite field of order q exists if and only if $q = p^k$, where p is a prime number and k is a positive integer. The order of $\mathbf{F}_q^\times$ is $p^k - 1$. A factor of $p^k - 1$ is sufficient for Pollard's method and the factors of $p^k - 1$ are the cyclotomic polynomials [1]. But in practice this algorithm is slower than ECM except $p + 1$.

Since the points on an elliptic curve over $\mathbb{Z}/p\mathbb{Z}$ forms an abelian group, Pollard's algorithm can be extended to this set. Because of Hasse's theorem, ECM is a "$p + 1 - a$" algorithm. a is unknown and $a \in [-2\sqrt{p}, 2\sqrt{p}]$. With different curves, we can expect that a is a random number in a large range.

If $\mathscr{P}$ is the probability that $p + 1 - a$ is B-smooth, with n curves the likelihood of success is $\mathscr{P}_n = 1 - (1 - \mathscr{P})^n$. If $\mathscr{P} \ll 1$ and $n \sim 1/\mathscr{P}$ then $\mathscr{P}_n = 1 - e^{-1} \approx 63.2\%$.
If $\mathscr{P}$ is 1% with Pollard's $p-1$, the likelihood of success with ECM is 63.2% with 100 curves and 99.99% with 1000 curves.

3.2 Largest prime factors

Dickman [5] proved that the probability that a large integer n has no prime factor exceeding n^α approaches a limit $F(\alpha)$ as $n \to \infty$, where

$$F(\alpha) = \begin{cases} 1 - \int_\alpha^1 F\left(\frac{t}{1-t}\right) \frac{dt}{t} & \text{if } 0 \le \alpha < 1, \\ 1 & \text{if } \alpha \ge 1. \end{cases}$$

Let $u = 1/\alpha$. $F(1/u) = 1 - \int_{1/u}^1 F\left(\frac{t}{1-t}\right) \frac{dt}{t}$. If $t' = 1/t$ we have $\frac{t}{1-t} = \frac{1}{t'-1}$ and $\frac{dt}{t} = -\frac{dt'}{t'}$ then $F(1/u) = 1 - \int_1^u F\left(\frac{1}{t'-1}\right) \frac{dt'}{t'}$. The relation becomes

$$F(1/u) = \rho(u) = \begin{cases} 1 - \int_1^u \frac{\rho(t-1)}{t} \, dt & \text{if } u > 1, \\ 1 & \text{otherwise.} \end{cases}$$

ρ is the Dickman function used to estimate the proportion of smooth numbers up to a given bound.

Differentiating both sides of the definition of $\rho(u)$ for $u > 1$ gives $t\rho'(t) = -\rho(t-1)$. Integration by parts of $\rho(t)$ and t is $\int_1^u \rho(t) \, dt = [t\rho(t)]_1^u - \int_1^u t\rho'(t) \, dt$. Hence $\int_1^u \rho(t) \, dt = u\rho(u) - 1 + \int_0^{u-1} \rho(t) \, dt$. Since $\int_0^1 \rho(t) \, dt = 1$ we have

$$\rho(u) = \frac{1}{u} \int_{u-1}^u \rho(t) \, dt.$$

This relation can be used for the numerical computation of ρ by approximating the integral with the trapezoidal formula [9]. If $1 \le u < 2$, $\rho(u) = 1 - \int_1^u \frac{dt}{t} = 1 - \log u$.

Knuth and Trabb Pardo [7] extended Dickman's theorem and shown that the probability that the kth largest prime factor of a number n is at most n^α tends to a limiting distribution $F_k(\alpha)$ as $n \to \infty$, where $F_0(\alpha) = 0$ for all α by convention and for $k \ge 1$

$$F_k(\alpha) = \begin{cases} 1 - \int_\alpha^1 \left(F_k\left(\frac{t}{1-t}\right) - F_{k-1}\left(\frac{t}{1-t}\right)\right) \frac{dt}{t} & \text{if } 0 \le \alpha < 1, \\ 1 & \text{if } \alpha \ge 1. \end{cases}$$

We can define the generalized Dickman function $\rho_k(u) = F_k(1/u)$ and we have

$$\rho_k(u) = \begin{cases} 1 - \int_1^u (\rho_k(t-1) - \rho_{k-1}(t-1)) \frac{dt}{t} & \text{if } u > 1 \text{ and } k \ge 1, \\ 1 & \text{if } 0 < u \le 1 \text{ and } k \ge 1, \\ 0 & \text{otherwise.} \end{cases}$$

The differential equation is $t\rho_k'(t) = -\rho_k(t-1) + \rho_{k-1}(t-1)$. Integrating by parts, we get

$$\rho_k(u) = \frac{1}{u}\left(\int_{u-1}^u \rho_k(t) \, dt + \int_0^{u-1} \rho_{k-1}(t) \, dt\right).$$

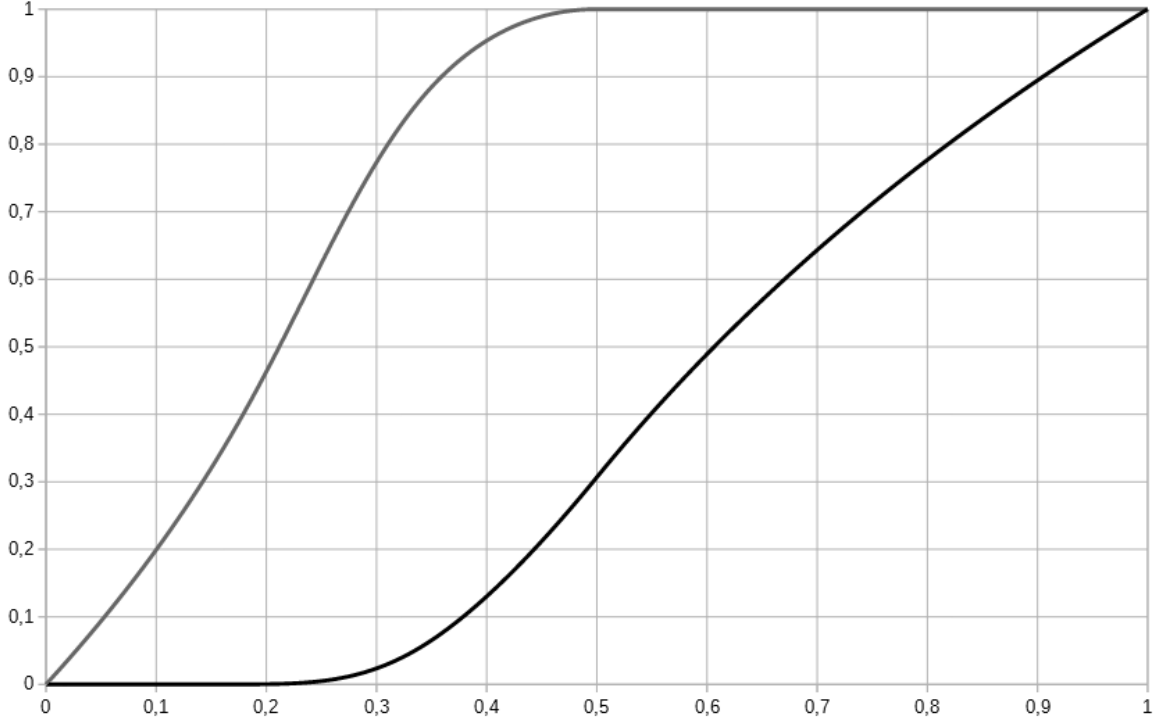This relation can be used for the numerical computation of ρ_2.

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Figure 3.1: $F(\alpha)$ and $F_2(\alpha)$.

The sizes of the two largest prime factors are correlated: let $n \approx 10^{100}$ and $p \approx 10^{30}$ be the largest prime factor of n. We have $\alpha = 0.3$ and $F(\alpha) \approx 0.02365$. However $F_2(0.0131) \approx 0.02365$ and $n^{0.0131} \approx 20.4$. The probability that the largest prime factor of n is at most 10^{30} is equal to the probability that its second largest prime factor is at most 20.4. But this result cannot be used to set B_1 and B_2. If the largest prime factor of n is a 30-digit prime, the second largest prime factor is certainly larger than 20.4 (see subsection 3.3.1).

3.3   Probabilities

Let B_1 and B_2 be the bounds of a two-stage ECM, n be the number of curves and p be a prime factor of F_{12}. $\mathscr{P}(B_1, B_2, n, p)$ is the chance for ECM to find p.

If ECM finds p, because of Hasse's theorem we have $\#E \approx p$. $F(\alpha)$ is the probability that the largest prime factor of $\#E$ is at most p^α. One must have $B_2 \gtrsim p^\alpha$. Let $P(B_2) = F(\log B_2 \,/\, \log p)$. If the second largest prime factor is larger than B_1 then the likelihood of success for ECM is $\mathscr{P}(B_2, B_2, n, p) = 1 - (1 - P(B_2))^n$.

Note that $\lim_{n \to \infty} (1 - x/n)^n = e^{-x}$. Hence, if $P(B_2) \ll 1$ we have $\mathscr{P}(B_2, B_2, n, p) \approx 1 - e^{-P(B_2)n}$. If $\mathscr{P}^*(B_2, B_2, n, p) = 1 - e^{-1} \approx 63.2\%$ is chosen as an acceptable likelihood of success, we have the condition $P(B_2)n = 1$.

The number of operations per curve is about $\log(B_1\#) \sim B_1$ for stage 1 and about $B_2/\log B_2$ for stage 2 if $B_1 \ll B_2$. Hence, computation time is proportional to $(B_1 + B_2/\log B_2)\, n$. If $B_1 = K \cdot B_2/\log B_2$

then $(B_2/\log B_2)\, n$ must be minimal.

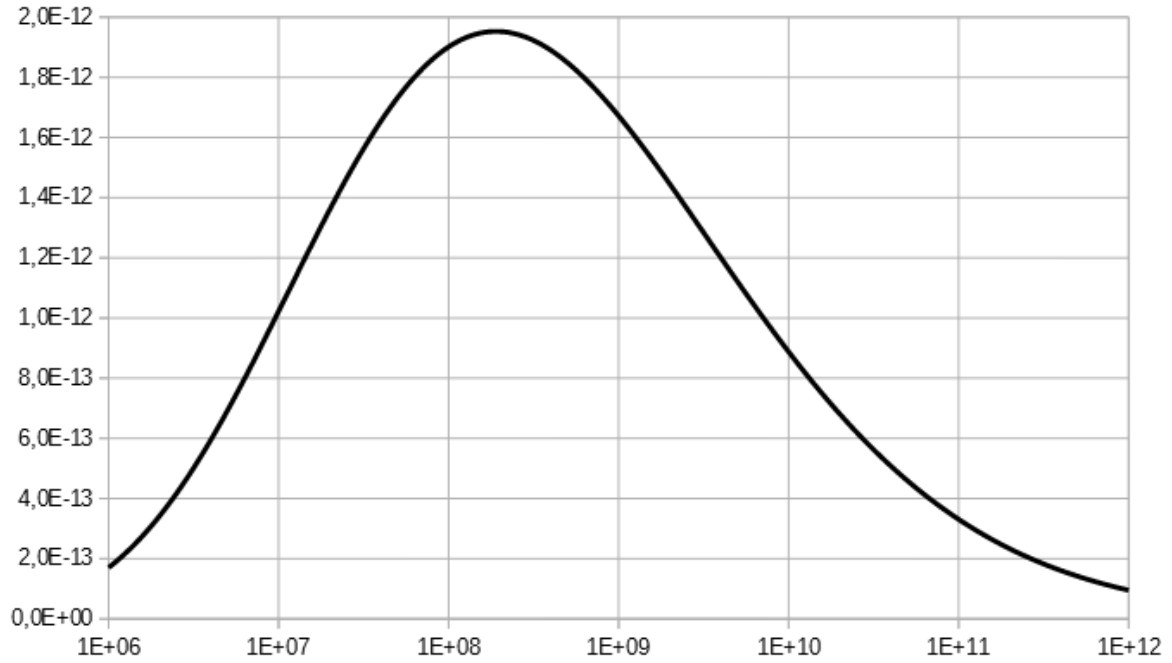Finally the conditions are B_2 is the maximum point of $P(B_2)\log B_2/B_2$ and $n = 1/P(B_2)$.

Figure 3.2: $P(B_2)\log B_2/B_2$ for $p = 5 \cdot 10^{49}$.

Let $p = 5 \cdot 10^{49}$ be an average 50-digit prime. $P(B_2)\log B_2/B_2$ is maximum at $B_2 \approx 1.9 \cdot 10^8$. We have $P(B_2) \approx 1.95 \cdot 10^{-5}$ and $n \approx 51400$.
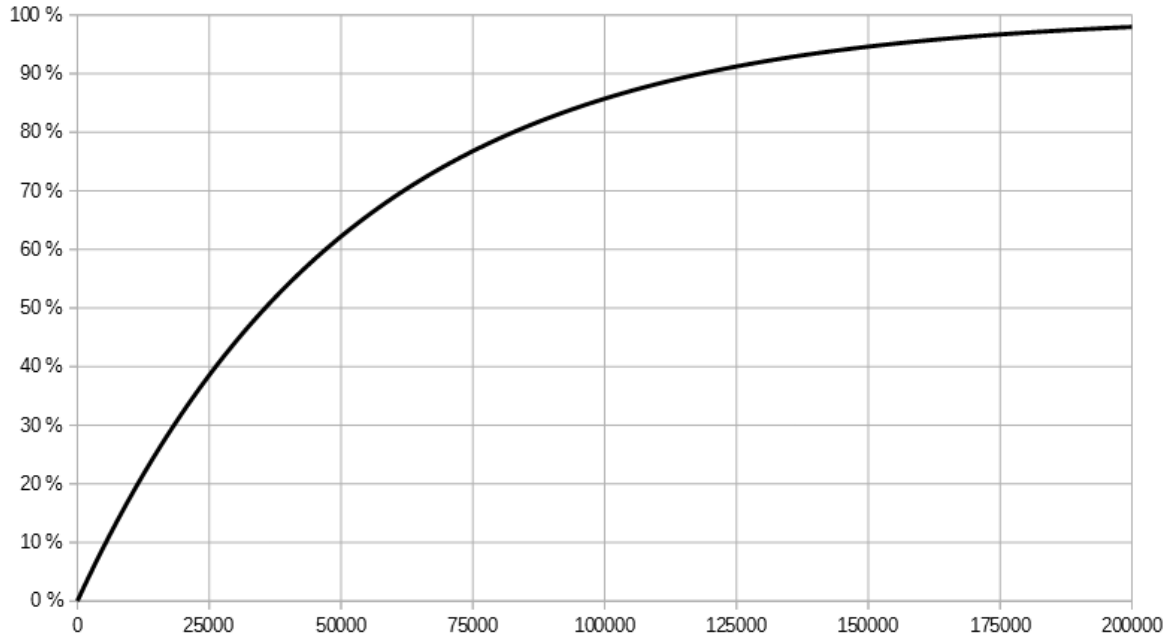

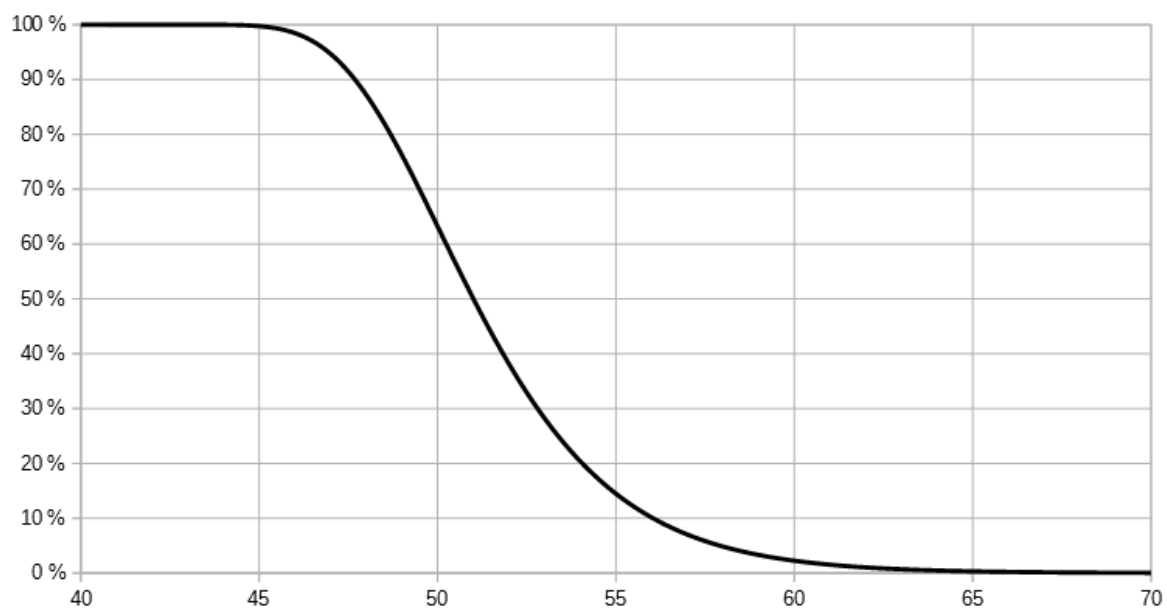Figure 3.3: Probability of success for n curves, $p = 5 \cdot 10^{49}$ and $B_2 = 1.9 \cdot 10^8$.

Figure 3.4: Chance to find a x-digit factor, $B_2 = 190 \cdot 10^6$ and $n = 51400$.
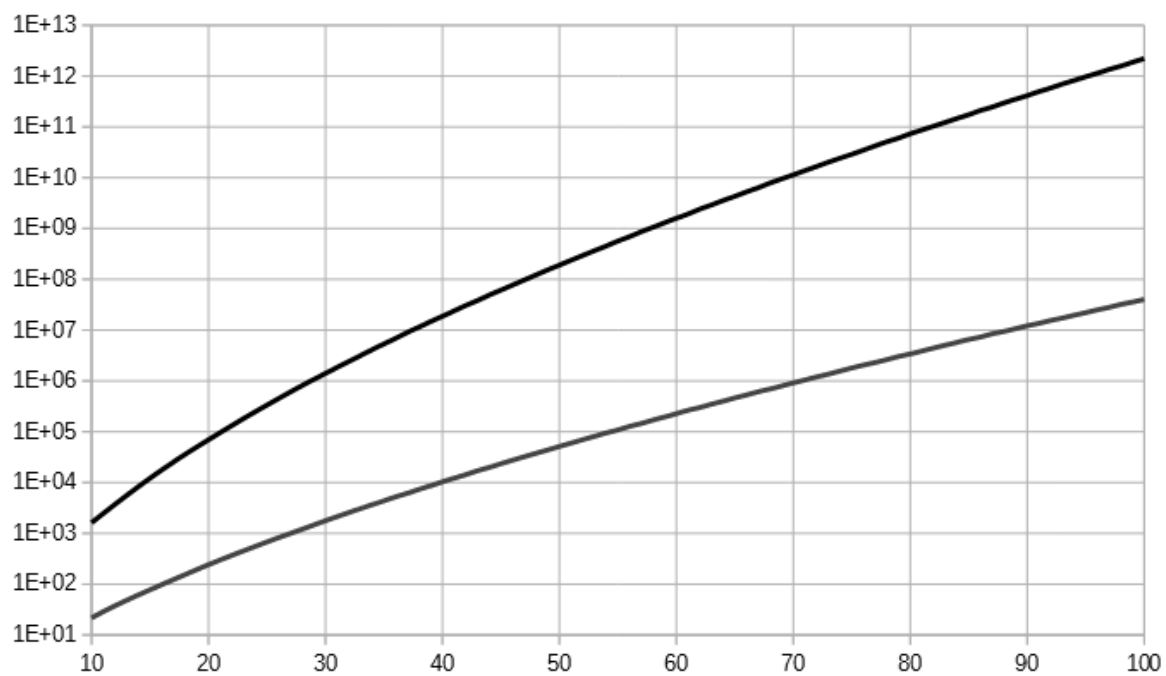

Figure 3.5: Optimal B_2 and number of curves as a function of the number of digits.

3.3.1 The ratio B_2/B_1

The generalized Dickman function F_2 cannot be used to compute B_1 as a function of B_2 (see section 3.2). Let $G(\alpha, \beta)$ be the probability that the largest prime factor of a number n is at most n^α and that the second largest prime factor of n is at most n^β. Following Knuth and Trabb Pardo [7] heuristic derivation, we have:

$$G(\alpha, \beta) = \begin{cases} F(\beta) + \int_\beta^\alpha F\left(\frac{\beta}{1-t}\right) \frac{dt}{t} & \text{if } 0 \le \beta < \alpha, \\ F(\alpha) & \text{if } \beta \ge \alpha. \end{cases}$$

If $\beta \le \alpha$ we have $G(\alpha, \beta) = F(\beta) + \int_\beta^\alpha \rho\left(\frac{1-t}{\beta}\right) \frac{dt}{t}$ and $G(1/u, 1/v) = \rho(v) + \int_{1/v}^{1/u} \rho\left((1-t)v\right) \frac{dt}{t}$.

Let $t' = (1-t)v + 1$. We have $t = (v+1-t')/v$, $dt = -dt'/v$ and $\frac{dt}{t} = -\frac{dt'}{v+1-t'}$. Hence,

$$G(1/u, 1/v) = \sigma(u, v) = \rho(v) + \int_{v+1-v/u}^v \frac{\rho(t-1)\, dt}{v+1-t}.$$

This relation can be used for the numerical computation of σ.

$P(B_2) = F(\log B_2 / \log p)$ can be replaced with $P(B_2, B_1) = G(\log B_2 / \log p, \log B_1 / \log p)$. Now the likelihood of success for ECM is $\mathscr{P}(B_2, B_1, n, p) = 1 - (1 - P(B_2, B_1))^n$.

The computation time is proportional to $(K \cdot B_1 + B_2/\log B_2)\, n$, where $K \approx 5.6$ depends on the implementation. Hence, $(B_2; B_1)$ must be the maximum point of the function

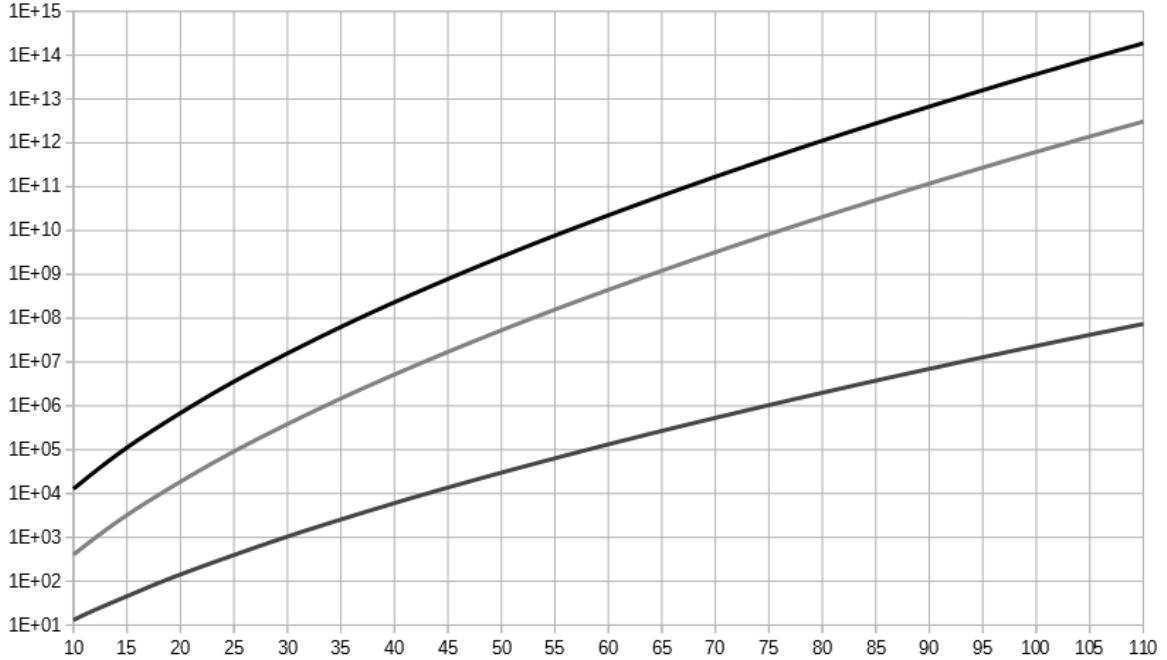$$\frac{P(B_2, B_1)}{K \cdot B_1 + B_2/\log B_2}.$$

Figure 3.6: Optimal B_2, B_1 and number of curves as a function of the number of digits.

3.3.2 ECM and F_{12}

The present status is

$$F_{12} = P_6 \cdot P_8 \cdot P_8 \cdot P_{12} \cdot P_{16} \cdot P_{54} \cdot C_{1133}.$$

The size of the next prime factor is unknown and we can't set B_1 and B_2 as a function of its number of digits. The number of curves needed to find a new factor is unknown but it is a variable that will increase over time. We can set the searched prime factor as a function of the index of the curve.

The number of digits, B_1 and B_2 can be computed as a function of number of curves with subsection 3.3.1. But now we don't check n curves with a fixed set of parameters but the ith curve is tested with different settings $B_1(i)$ and $B_2(i)$.

However subsection 3.3.1 can be the starting point. For integer values of $\log p$, $n(\lfloor \log p \rfloor)$ is calculated. If n curves are tested for each value of $\lfloor \log p \rfloor$, the probability of success is larger than $1 - e^{-1}$ because $n(\lfloor \log p \rfloor - 1), n(\lfloor \log p \rfloor - 2), \ldots$ have already been tested. If $n(\lfloor \log p \rfloor) - n(\lfloor \log p \rfloor - 1)$ curves are tested then the probability is smaller than $1 - e^{-1}$ because $B_1(\lfloor \log p \rfloor - 1) < B_1(\lfloor \log p \rfloor)$ and $B_2(\lfloor \log p \rfloor - 1) < B_2(\lfloor \log p \rfloor)$.

We search for λ such that if $n'(\lfloor \log p \rfloor) = n(\lfloor \log p \rfloor) - \lambda \cdot n(\lfloor \log p \rfloor - 1)$ then the probability remains constant and equal to $1 - e^{-1}$. $\lambda \approx 0.92$ is suitable. By inverting this function, for each index an estimate of $\log p$ is computed and then $B_1(\log p)$ and $B_2(\log p)$.
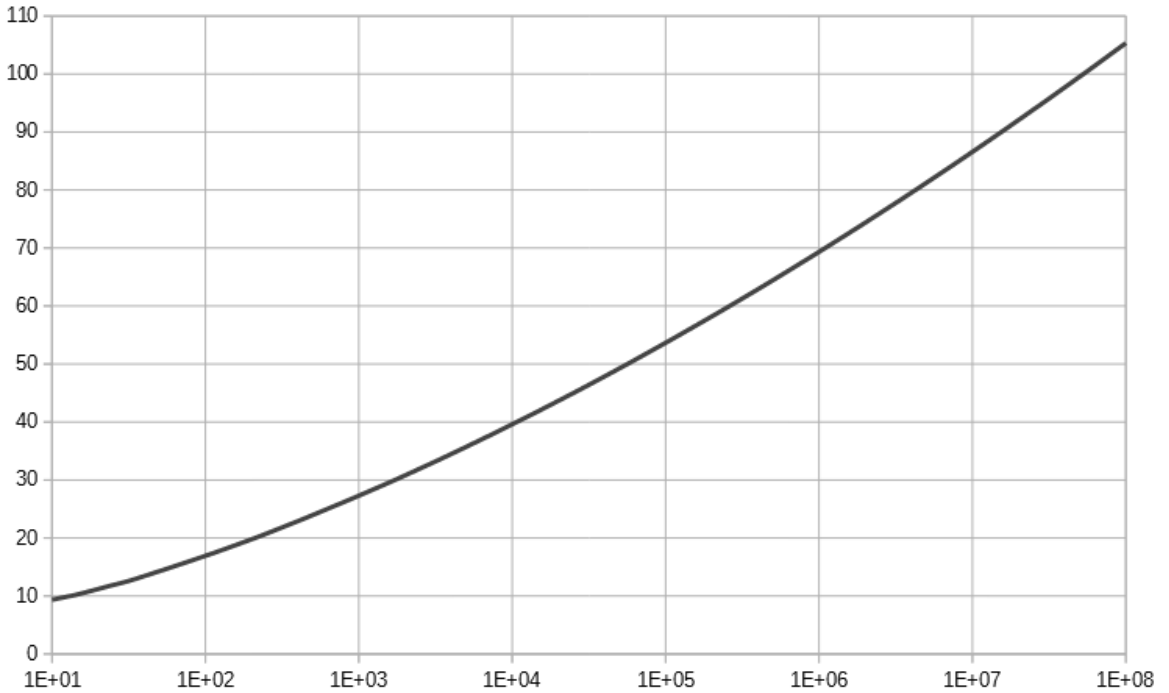

Figure 3.7: Expected number of digits as a function of the index of the curve.
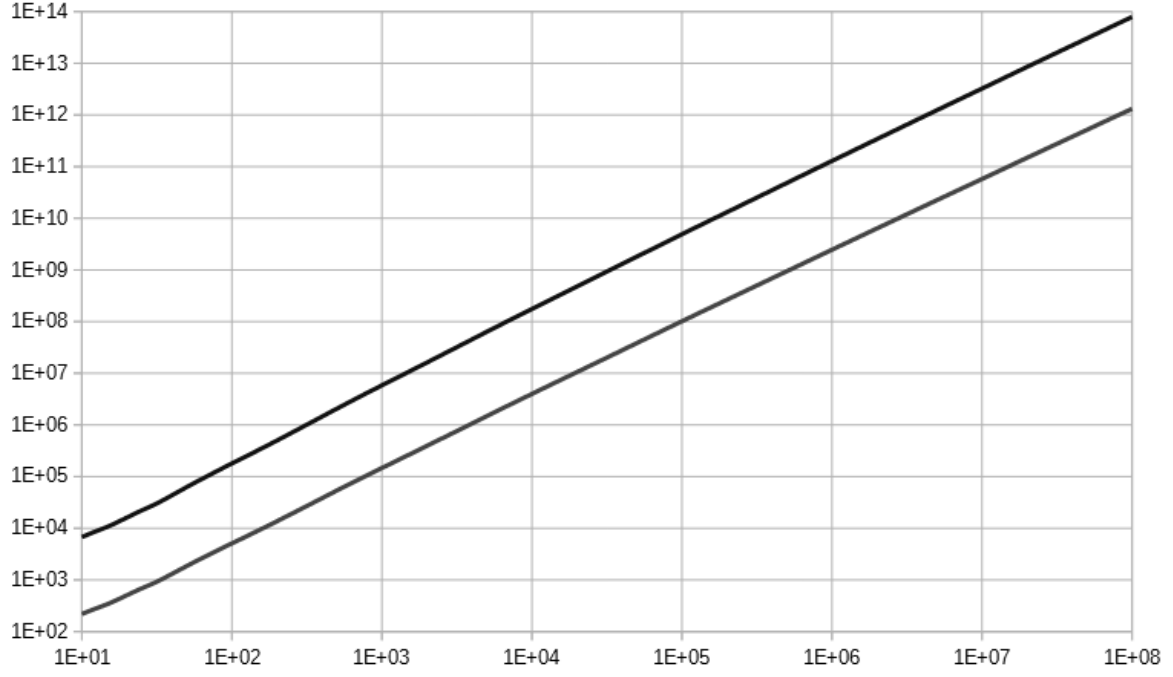
Figure 3.8: B_2 and B_1 as a function of the index of the curve.

$\log B_2 = f(\log i)$ where i is the ith curve is close to a linear function. The error of the estimate $B_2 = 400\,i^{\sqrt{2}}$ is less than 5% for $10^4 \le i \le 10^8$. The ratio of $B_2 / B_1 = 26 + 1.8 \log i$ is always slightly larger than the optimal value. Finally we have $\log p \approx (3 + 1.3 \log i)^{5/3}$.

The following parameters are applied: let i be the ith curve and d be the expected number of digits of the prime factor, then

$$B_2 = 400\,i^{\sqrt{2}},$$
$$B_1 = B_2 / (26 + 1.8 \log i),$$
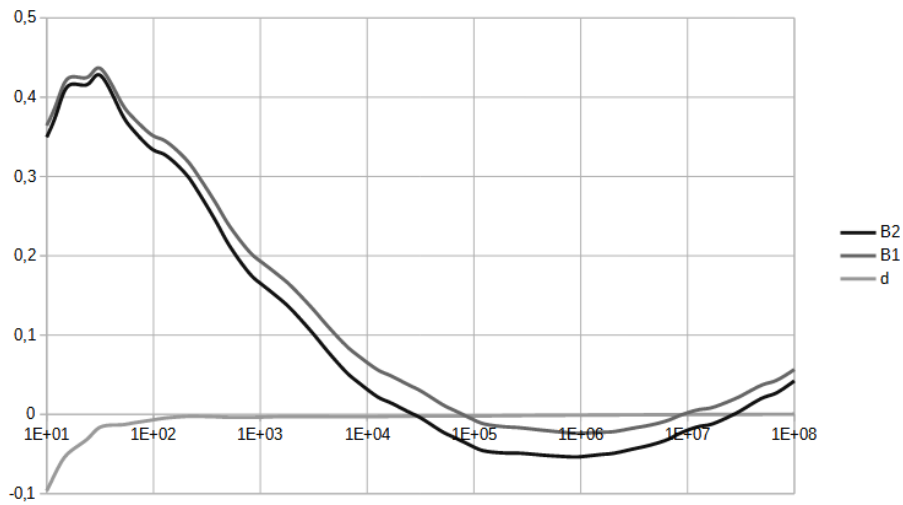$$d = (1.8 + 0.79 \log i)^{5/3}.$$

Figure 3.9: Relative error with applied parameters as a function of the index of the curve.

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3.3.3 Test of the probabilistic model

Chapter 4

Implementation

Bibliography

[1] Eric Bach and Jeffrey Shallit, *Factoring with cyclotomic polynomials*, Math. Comp. **52** (1989), 201–219, DOI: `https://doi.org/10.1090/S0025-5718-1989-0947467-1`.

[2] Richard P. Brent, *Factorization of the tenth Fermat number*, Math. Comp. **68** (1999), 429–451, DOI: `https://doi.org/10.1090/S0025-5718-99-00992-8`.

[3] Richard P. Brent, *Factorization of the tenth and eleventh Fermat numbers*, Report TR-CS-96-02, Computer Sciences Laboratory, Australian National Univ., Canberra, Feb. 1996, `https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.70.6415&rep=rep1&type=pdf`.

[4] Richard P. Brent and John M. Pollard, *Factorization of the eighth Fermat number*, Math. Comp. **36** (1981), 627–630, DOI: `https://doi.org/10.1090/S0025-5718-1981-0606520-5`.

[5] Karl Dickman, *On the Frequency of Numbers Containing Prime Factors of a Certain Relative Magnitude*, Arkiv för Mat., Astron. och Fys. 22A, 1-14, 1930.

[6] Pierre de Fermat, *Lettre à Marin Mersenne*, `https://www.archive.org/stream/oeuvresdefermat942ferm#page/212/mode/2up`.

[7] Donald E. Knuth and Luis Trabb Pardo, *Analysis of a simple factorization algorithm*, Theoretical Computer Science, Volume 3, Issue 3, December 1976, Pages 321–348, DOI: `https://doi.org/10.1016/0304-3975(76)90050-5`.

[8] A. K. Lenstra, H. W. Lenstra, M. S. Manasse and J. M. Pollard *The factorization of the ninth Fermat number*, Math. Comp. **61** (1993), 319–349, DOI: `https://doi.org/10.1090/S0025-5718-1993-1182953-4`.

[9] J. van de Lune and E. Wattel, *On the numerical solution of a differential-difference equation arising in analytic number theory*, Math. Comp. **23** (1969), 417–421, DOI: `https://doi.org/10.1090/S0025-5718-1969-0247789-3`.

[10] Michael A. Morrison and John Brillhart, *A method of factoring and the factorization of F_7*, Math. Comp. **29** (1975), 183–205, DOI: `https://doi.org/10.1090/S0025-5718-1975-0371800-5`.

[11] J. M. Pollard, *Theorems on factorization and primality testing*, Mathematical Proceedings of the Cambridge Philosophical Society, Volume **76**, Issue 3, November 1974, pp. 521 - 528, DOI: `https://doi.org/10.1017/S0305004100049252`.

[12] `https://caramel.loria.fr/f12.txt`.