

F_{12} ECM

A PROGRAM FOR FINDING
THE FACTORS OF
THE TWELFTH FERMAT NUMBER

Elliptic Curve Method and Probabilities

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Chapter 1

Fermat numbers

Si je puis une fois tenir la raison fondamentale que 3, 5, 17, etc. sont nombres premiers, il me semble que je trouverai de très belles choses en cette matière,

*Fermat à Mersenne
25 décembre 1640*

Pierre de Fermat conjectured that every number of the form $F_n = 2^{2^n} + 1$, where n is a non-negative integer, is prime [6]. Today these positive integers are named Fermat numbers. The first five Fermat numbers are prime, but Leonhard Euler proved in 1732 that 641 divides F_5 .

F_6 was completely factored by T Clausen, F Landry and H. Le Lasseur in 1855. In 1970, M. A. Morrison and J. Brillhart cracked F_7 by the Continued Fraction method [10]. In 1980, R. P. Brent and J. M. Pollard used a modification of Pollard's rho method to factor F_8 . R. P. Brent completely factored F_{11} in 1988 by ECM [3]. In 1990, A. K. Lenstra, H. W. Lenstra, M. S. Manasse and J. M. Pollard organized a distributed computation on approximately 700 workstations around the world and factored F_9 by the Number Field Sieve [8]. Finally R. P. Brent completely factored F_{10} in 1995 by ECM [3].

The smallest Fermat number which is not completely factored is F_{12} . Six prime factors are known, the 54-digit factor was found by Michael Vang in 2010 using GMP-ECM [12].

$$\begin{aligned} F_5 &= 641 \cdot 6700417 \\ F_6 &= 274177 \cdot 67280421310721 \\ F_7 &= 59649589127497217 \cdot 5704689200685129054721 \\ F_8 &= 1238926361552897 \cdot P_{62} \\ F_9 &= 2424833 \cdot 7455602825647884208337395736200454918783366342657 \cdot P_{99} \\ F_{10} &= 45592577 \cdot 6487031809 \cdot 4659775785220018543264560743076778192897 \cdot P_{252} \\ F_{11} &= 319489 \cdot 974849 \cdot 167988556341760475137 \cdot 3560841906445833920513 \cdot P_{564} \\ F_{12} &= 114689 \cdot 26017793 \cdot 63766529 \cdot 190274191361 \cdot 1256132134125569 \cdot \\ &\quad 568630647535356955169033410940867804839360742060818433 \cdot C_{1133} \end{aligned}$$

Chapter 2

Elliptic curves modulo p

An elliptic curve over $\mathbb{Z}/p\mathbb{Z}$ is a set of points $P = (x, y) \in (\mathbb{Z}/p\mathbb{Z})^2$ such that

$$y^2 = x^3 + ax + b$$

with the condition $4a^3 + 27b^2 \neq 0$ and an additional point at infinity O .

The point $Q = (x, -y)$ is on the curve: $Q = -P$ is the point opposite of P . The curve has the property that if a non-vertical line intersects it at two points P and Q , then it will also have a third point R of intersection. The addition law is defined by $P + Q = -R$. If $P = Q$, the tangent of the curve at P is considered. If the line is vertical, we have $P + -P = O$. The points on an elliptic curve and the addition form an abelian group. O is the identity of the group: we have $P + O = O + P = P$.

Hasse's theorem on elliptic curves over finite fields provides an estimate of the number of points. If $\#E$ is the order of the group of an elliptic curve E over $\mathbb{Z}/p\mathbb{Z}$ then

$$|\#E - (p + 1)| \leq 2\sqrt{p}.$$

Given a prime $p > 3$ and any integer n such that $|n - (p + 1)| \leq 2\sqrt{p}$, there exists a and b such that $|\#E(a, b)| = n$. Furthermore, the numbers of points are uniformly distributed over the interval $[p + 1 - 2\sqrt{p}, p + 1 + 2\sqrt{p}]$.

To avoid the time-consuming inversion over $\mathbb{Z}/p\mathbb{Z}$, projective coordinates are preferred for computations. X, Y, Z are integers such that $x = X/Z$ and $y = Y/Z$. If $P \neq O$ then $Z \neq 0$ and the coordinates of the identity O are $(0, Y, 0)$.

$y^2 = x^3 + ax + b$ is called the Weierstrass form but there exist some alternative representations of elliptic curves. Some of them are faster for computations. $F_{12}ECM$ representation is the Montgomery curve

$$By^2 = x^3 + Ax^2 + x.$$

The Montgomery curves are a subset of Weierstrass curves.

Montgomery coordinates are the projective coordinates (X, Z) . The computation of Y is not needed for the Elliptic Curve Method.

Chapter 3

Elliptic Curve Method

3.1 Algorithm

The elliptic curve factorization method (ECM) is an extension of Pollard's $p - 1$ algorithm [11].

Pollard's $p - 1$ algorithm finds the prime factors p such that $p - 1$ is B -smooth.

If $p > 2$ and e is a multiple of $p - 1$ then by Fermat's little theorem we have $2^e \equiv 1 \pmod{p}$. For a fixed bound B , $M = \prod_{\substack{p \leq B \\ p \text{ prime}}} p^{\lfloor \log_p B \rfloor}$ is computed modulo n and finally $g = \gcd(2^M - 1, n)$.

If n is a composite integer, $p \mid n$, $p - 1$ is B -powersmooth and $q \mid n$ but $q - 1$ is not B -powersmooth then $1 < g < n$ and g is a multiple of p .

In practice, a two-stage variant of the algorithm is implemented: instead of requiring that $p - 1$ has all its factors less than B , if all but one of them are less than B_1 and the remaining factor is less than B_2 then the range $]B_1; B_2]$ can be tested more quickly. M is computed for $B = B_1$ and the second stage is $M' = \prod_{\substack{B_1 < p \leq B_2 \\ p \text{ prime}}} ((2^M)^p - 1)$. If p_n and p_{n+1} are two consecutive prime numbers then

$A^{p_{n+1}} = A^{p_n} \cdot A^{d_n}$ where $d_n = p_{n+1} - p_n$. The d_n are relatively small then the values of A^2, A^4, A^6, \dots can be precomputed. Then $(2^M)^{p_n}$ is calculated with a single multiplication.

Fermat's little theorem can be extended to a group G such that each element of G is invertible. Pollard's $p - 1$ is based on $(\mathbb{Z}/p\mathbb{Z})^\times$. A finite field of order q exists if and only if $q = p^k$, where p is a prime number and k is a positive integer. The order of \mathbb{F}_q^\times is $p^k - 1$. A factor of $p^k - 1$ is sufficient for Pollard's method and the factors of $p^k - 1$ are the cyclotomic polynomials [1]. But in practice this algorithm is slower than ECM except $p + 1$.

Since the points on an elliptic curve over $\mathbb{Z}/p\mathbb{Z}$ forms an abelian group, Pollard's algorithm can be extended to this set. Because of Hasse's theorem, ECM is a " $p + 1 - a$ " algorithm. a is unknown and $a \in [-2\sqrt{p}, 2\sqrt{p}]$. With different curves, we can expect that a is a random number in a large range.

If \mathcal{P} is the probability that $p + 1 - a$ is B -smooth, with n curves the likelihood of success is $\mathcal{P}_n = 1 - (1 - \mathcal{P})^n$. If $\mathcal{P} \ll 1$ and $n \sim 1/\mathcal{P}$ then $\mathcal{P}_n = 1 - e^{-1} \approx 63.2\%$.

If \mathcal{P} is 1% with Pollard's $p - 1$, the likelihood of success with ECM is 63.2% with 100 curves and 99.99% with 1000 curves.

3.2 Largest prime factors

Dickman [5] proved that the probability that a large integer n has no prime factor exceeding n^α approaches a limit $F(\alpha)$ as $n \rightarrow \infty$, where

$$F(\alpha) = \begin{cases} 1 - \int_{\alpha}^1 F\left(\frac{t}{1-t}\right) \frac{dt}{t} & \text{if } 0 \leq \alpha < 1, \\ 1 & \text{if } \alpha \geq 1. \end{cases}$$

Let $u = 1/\alpha$. $F(1/u) = 1 - \int_{1/u}^1 F\left(\frac{t}{1-t}\right) \frac{dt}{t}$. If $t' = 1/t$ we have $\frac{t}{1-t} = \frac{1}{t'-1}$ and $\frac{dt}{t} = -\frac{dt'}{t'}$ then $F(1/u) = 1 - \int_1^u F\left(\frac{1}{t'-1}\right) \frac{dt'}{t'}$. The relation becomes

$$F(1/u) = \rho(u) = \begin{cases} 1 - \int_1^u \frac{\rho(t-1)}{t} dt & \text{if } u > 1, \\ 1 & \text{otherwise.} \end{cases}$$

ρ is the Dickman function used to estimate the proportion of smooth numbers up to a given bound.

Differentiating both sides of the definition of $\rho(u)$ for $u > 1$ gives $t \rho'(t) = -\rho(t-1)$. Integration by parts of $\rho(t)$ and t is $\int_1^u \rho(t) dt = [t \rho(t)]_1^u - \int_1^u t \rho'(t) dt$. Hence $\int_1^u \rho(t) dt = u \rho(u) - 1 + \int_0^{u-1} \rho(t) dt$. Since $\int_0^1 \rho(t) dt = 1$ we have

$$\rho(u) = \frac{1}{u} \int_{u-1}^u \rho(t) dt.$$

This relation can be used for the numerical computation of ρ by approximating the integral with the trapezoidal formula [9]. If $1 \leq u < 2$, $\rho(u) = 1 - \int_1^u \frac{dt}{t} = 1 - \log u$.

Knuth and Trabb Pardo [7] extended Dickman's theorem and shown that the probability that the k^{th} largest prime factor of a number n is at most n^α tends to a limiting distribution $F_k(\alpha)$ as $n \rightarrow \infty$, where $F_0(\alpha) = 0$ for all α by convention and for $k \geq 1$

$$F_k(\alpha) = \begin{cases} 1 - \int_{\alpha}^1 (F_k\left(\frac{t}{1-t}\right) - F_{k-1}\left(\frac{t}{1-t}\right)) \frac{dt}{t} & \text{if } 0 \leq \alpha < 1, \\ 1 & \text{if } \alpha \geq 1. \end{cases}$$

We can define the generalized Dickman function $\rho_k(u) = F_k(1/u)$ and we have

$$\rho_k(u) = \begin{cases} 1 - \int_1^u (\rho_k(t-1) - \rho_{k-1}(t-1)) \frac{dt}{t} & \text{if } u > 1 \text{ and } k \geq 1, \\ 1 & \text{if } 0 < u \leq 1 \text{ and } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

The differential equation is $t \rho'_k(t) = -\rho_k(t-1) + \rho_{k-1}(t-1)$. Integrating by parts, we get

$$\rho_k(u) = \frac{1}{u} \left(\int_{u-1}^u \rho_k(t) dt + \int_0^{u-1} \rho_{k-1}(t) dt \right).$$

This relation can be used for the numerical computation of ρ_2 .

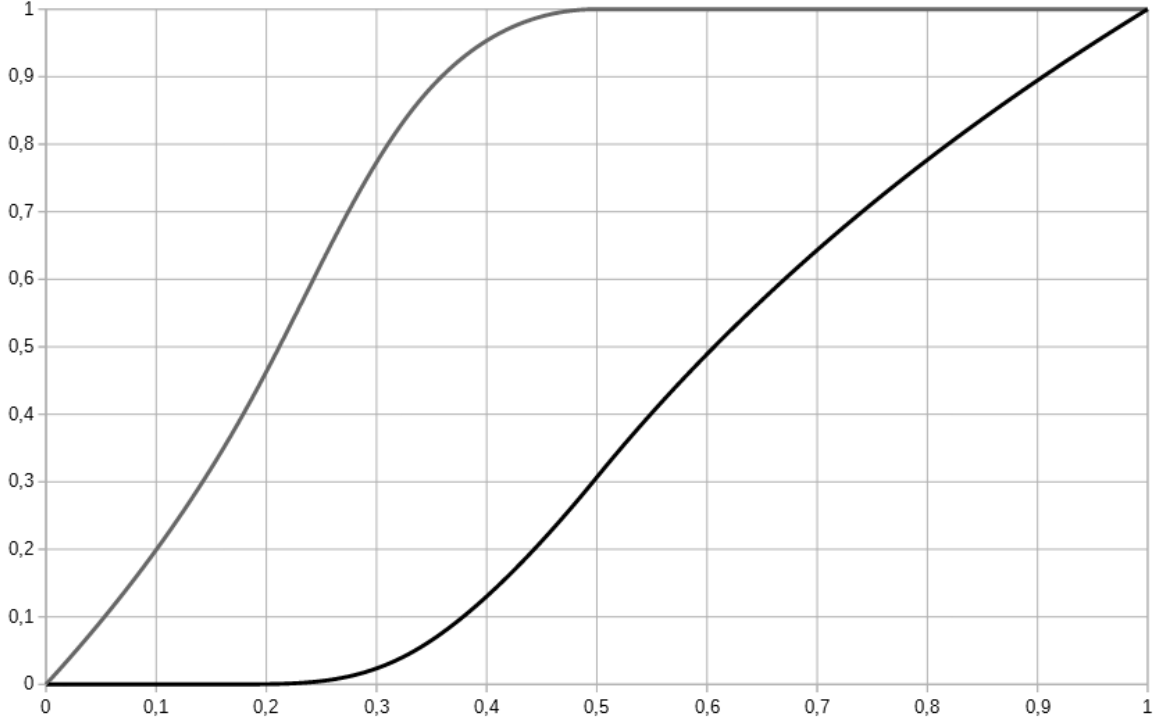


Figure 3.1: $F(\alpha)$ and $F_2(\alpha)$.

The sizes of the two largest prime factors are correlated: let $n \approx 10^{100}$ and $p \approx 10^{30}$ be the largest prime factor of n . We have $\alpha = 0.3$ and $F(\alpha) \approx 0.02365$. However $F_2(0.0131) \approx 0.02365$ and $n^{0.0131} \approx 20.4$. The probability that the largest prime factor of n is at most 10^{30} is equal to the probability that its second largest prime factor is at most 20.4. But this result cannot be used to set B_1 and B_2 . If the largest prime factor of n is a 30-digit prime, the second largest prime factor is certainly larger than 20.4 (see subsection 3.3.1).

3.3 Probabilities

Let B_1 and B_2 be the bounds of a two-stage ECM, n be the number of curves and p be a prime factor of F_{12} . $\mathcal{P}(B_1, B_2, n, p)$ is the chance for ECM to find p .

If ECM finds p , because of Hasse's theorem we have $\#E \approx p$. $F(\alpha)$ is the probability that the largest prime factor of $\#E$ is at most p^α . One must have $B_2 \gtrsim p^\alpha$. Let $P(B_2) = F(\log B_2 / \log p)$. If the second largest prime factor is larger than B_1 then the likelihood of success for ECM is $\mathcal{P}(B_2, B_2, n, p) = 1 - (1 - P(B_2))^n$.

Note that $\lim_{n \rightarrow \infty} (1 - x/n)^n = e^{-x}$. Hence, if $P(B_2) \ll 1$ we have $\mathcal{P}(B_2, B_2, n, p) \approx 1 - e^{-P(B_2)n}$. If $\mathcal{P}^*(B_2, B_2, n, p) = 1 - e^{-1} \approx 63.2\%$ is chosen as an acceptable likelihood of success, we have the condition $P(B_2)n = 1$.

The number of operations per curve is about $\log(B_1 \#) \sim B_1$ for stage 1 and about $B_2 / \log B_2$ for stage 2 if $B_1 \ll B_2$. Hence, computation time is proportional to $(B_1 + B_2 / \log B_2) n$. If $B_1 = K \cdot B_2 / \log B_2$

then $(B_2/\log B_2) n$ must be minimal.

Finally the conditions are B_2 is the maximum point of $P(B_2) \log B_2 / B_2$ and $n = 1/P(B_2)$.

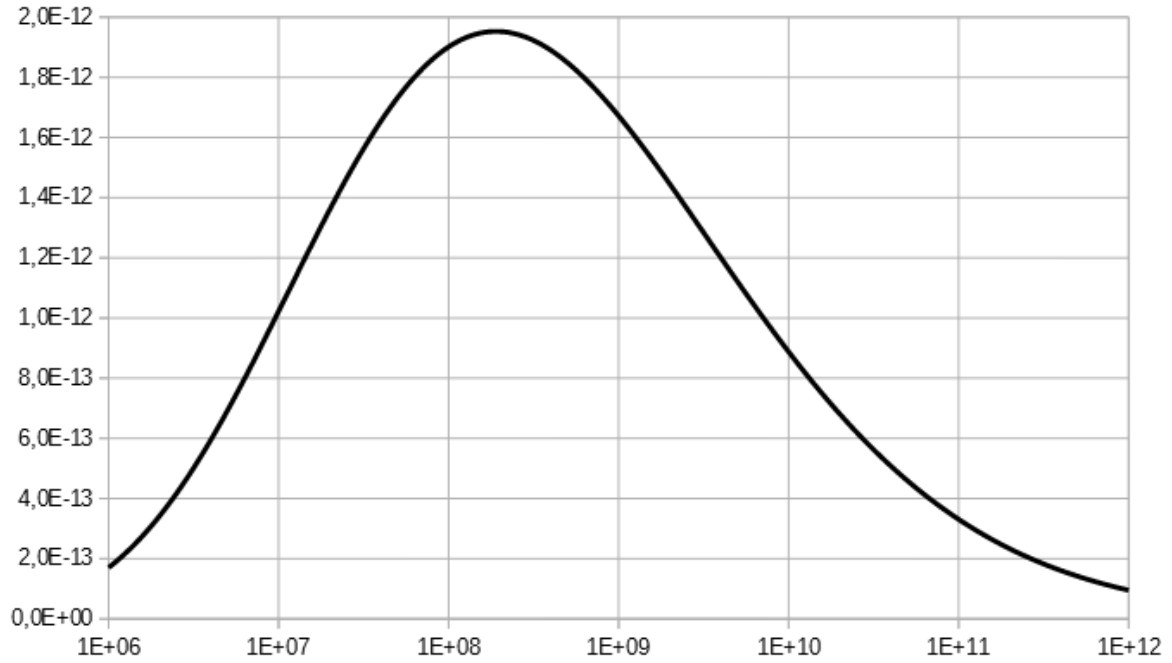


Figure 3.2: $P(B_2) \log B_2 / B_2$ for $p = 5 \cdot 10^{49}$.

Let $p = 5 \cdot 10^{49}$ be an average 50-digit prime. $P(B_2) \log B_2 / B_2$ is maximum at $B_2 \approx 1,9 \cdot 10^8$. We have $P(B_2) \approx 1,95 \cdot 10^{-5}$ and $n \approx 51400$.

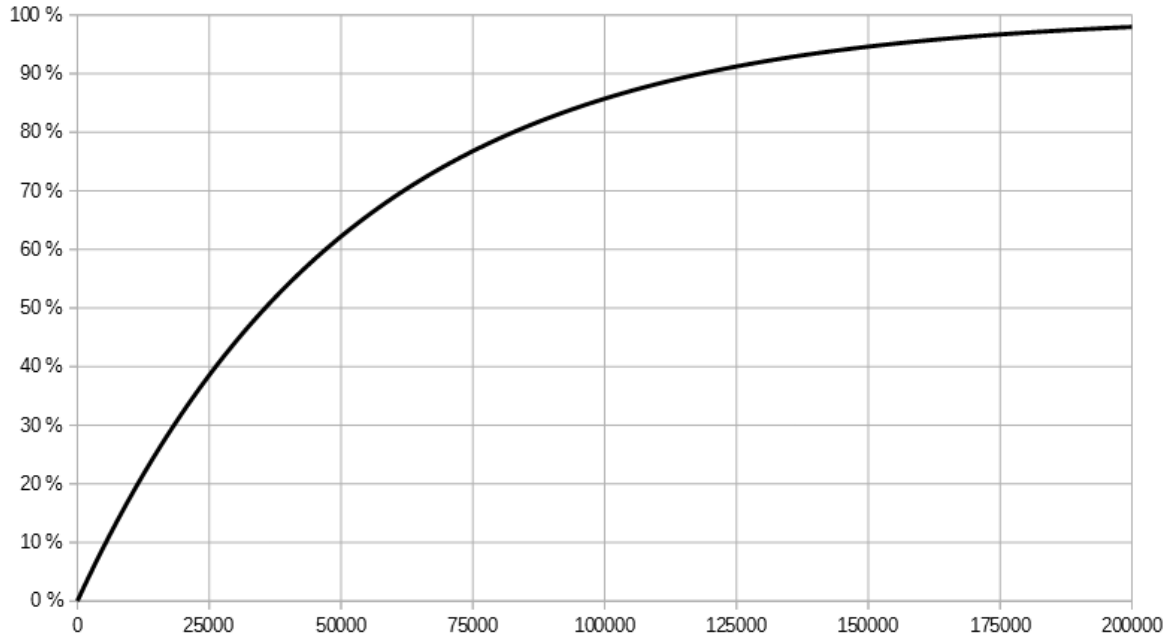


Figure 3.3: Probability of success for n curves, $p = 5 \cdot 10^{49}$ and $B_2 = 1,9 \cdot 10^8$.

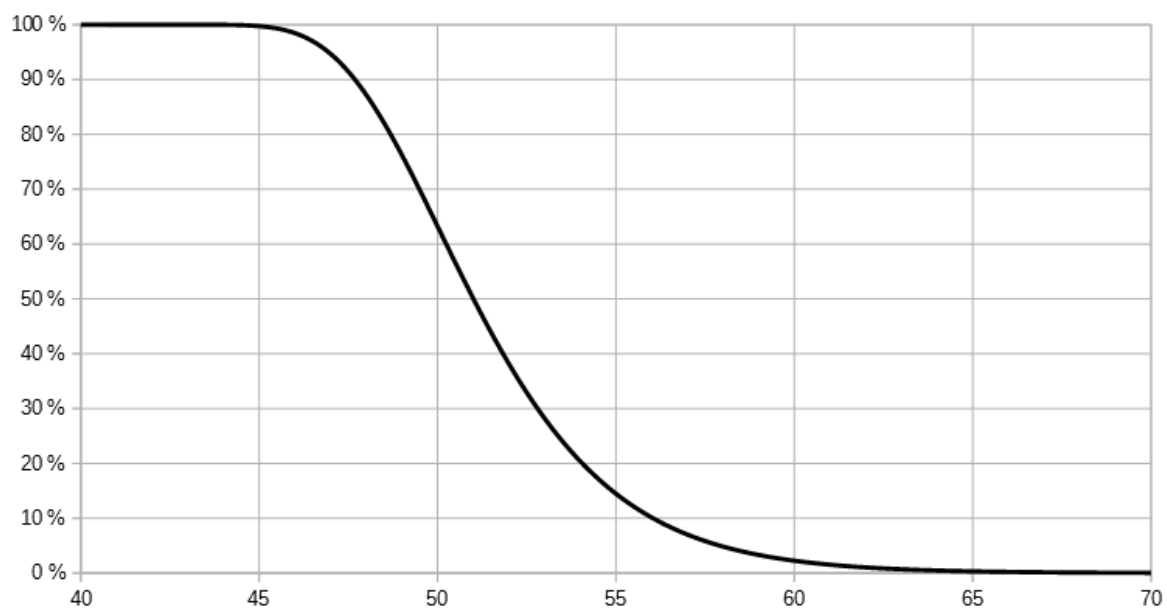


Figure 3.4: Chance to find a x -digit factor, $B_2 = 190 \cdot 10^6$ and $n = 51400$.

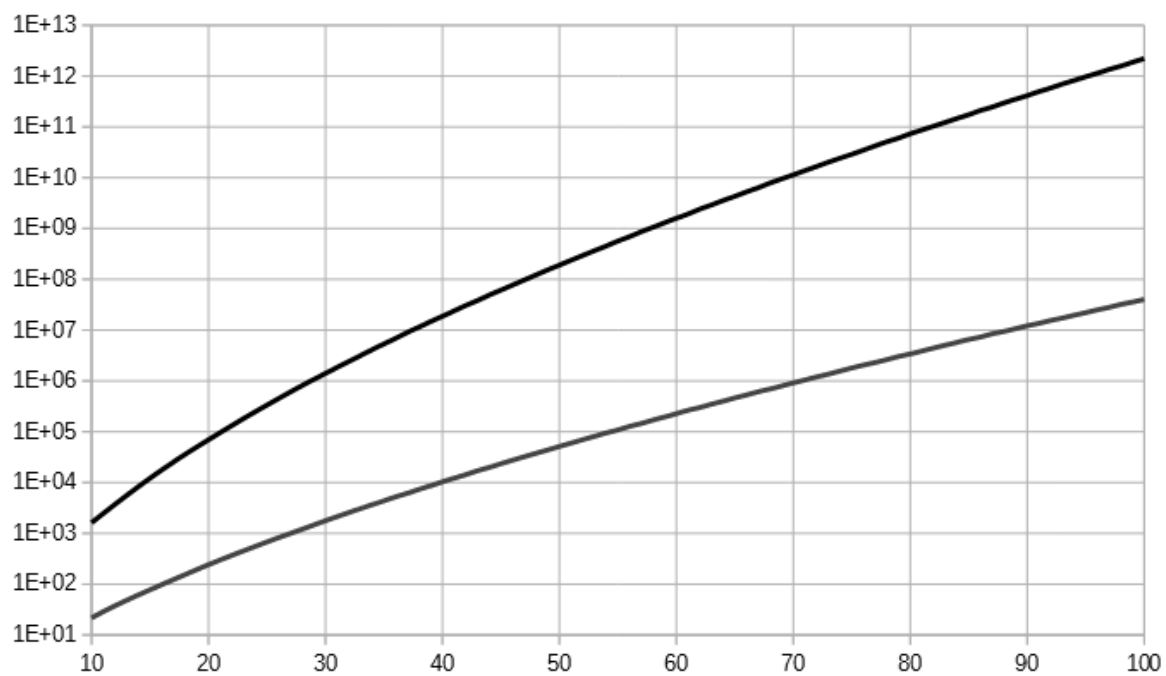


Figure 3.5: Optimal B_2 and number of curves as a function of the number of digits.

3.3.1 The ratio B_2/B_1

The generalized Dickman function F_2 cannot be used to compute B_1 as a function of B_2 (see section 3.2). Let $G(\alpha, \beta)$ be the probability that the largest prime factor of a number n is at most n^α and that the second largest prime factor of n is at most n^β . Following Knuth and Trabb Pardo [7] heuristic derivation, we have:

$$G(\alpha, \beta) = \begin{cases} F(\beta) + \int_{\beta}^{\alpha} F\left(\frac{\beta}{1-t}\right) \frac{dt}{t} & \text{if } 0 \leq \beta < \alpha, \\ F(\alpha) & \text{if } \beta \geq \alpha. \end{cases}$$

If $\beta \leq \alpha$ we have $G(\alpha, \beta) = F(\beta) + \int_{\beta}^{\alpha} \rho\left(\frac{1-t}{\beta}\right) \frac{dt}{t}$ and $G(1/u, 1/v) = \rho(v) + \int_{1/v}^{1/u} \rho((1-t)v) \frac{dt}{t}$.

Let $t' = (1-t)v + 1$. We have $t = (v+1-t')/v$, $dt = -dt'/v$ and $\frac{dt}{t} = -\frac{dt'}{v+1-t'}$. Hence,

$$G(1/u, 1/v) = \sigma(u, v) = \rho(v) + \int_{v+1-v/u}^v \frac{\rho(t-1) dt}{v+1-t}.$$

This relation can be used for the numerical computation of σ .

$P(B_2) = F(\log B_2 / \log p)$ can be replaced with $P(B_2, B_1) = G(\log B_2 / \log p, \log B_1 / \log p)$. Now the likelihood of success for ECM is $\mathcal{P}(B_2, B_1, n, p) = 1 - (1 - P(B_2, B_1))^n$.

The computation time is proportional to $(K \cdot B_1 + B_2 / \log B_2) n$, where $K \approx 5.6$ depends on the implementation. Hence, $(B_2; B_1)$ must be the maximum point of the function

$$\frac{P(B_2, B_1)}{K \cdot B_1 + B_2 / \log B_2}.$$

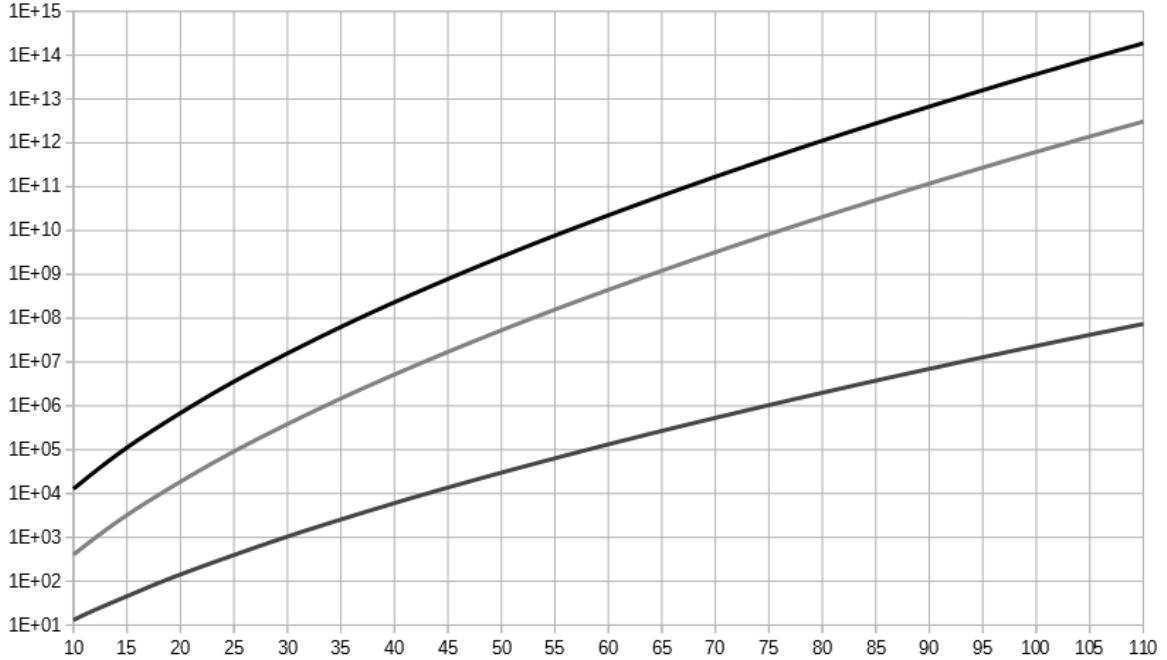


Figure 3.6: Optimal B_2, B_1 and number of curves as a function of the number of digits.

3.3.2 ECM and F_{12}

The present status is

$$F_{12} = P_6 \cdot P_8 \cdot P_8 \cdot P_{12} \cdot P_{16} \cdot P_{54} \cdot C_{1133}.$$

The size of the next prime factor is unknown and we can't set B_1 and B_2 as a function of its number of digits. The number of curves needed to find a new factor is unknown but it is a variable that will increase over time. We can set the searched prime factor as a function of the index of the curve.

The number of digits, B_1 and B_2 can be computed as a function of number of curves with subsection 3.3.1. But now we don't check n curves with a fixed set of parameters but the i^{th} curve is tested with different settings $B_1(i)$ and $B_2(i)$.

However subsection 3.3.1 can be the starting point. For integer values of $\log p$, $n(\lfloor \log p \rfloor)$ is calculated. If n curves are tested for each value of $\lfloor \log p \rfloor$, the probability of success is larger than $1 - e^{-1}$ because $n(\lfloor \log p \rfloor - 1), n(\lfloor \log p \rfloor - 2), \dots$ have already been tested. If $n(\lfloor \log p \rfloor) - n(\lfloor \log p \rfloor - 1)$ curves are tested then the probability is smaller than $1 - e^{-1}$ because $B_1(\lfloor \log p \rfloor - 1) < B_1(\lfloor \log p \rfloor)$ and $B_2(\lfloor \log p \rfloor - 1) < B_2(\lfloor \log p \rfloor)$.

We search for λ such that if $n'(\lfloor \log p \rfloor) = n(\lfloor \log p \rfloor) - \lambda \cdot n(\lfloor \log p \rfloor - 1)$ then the probability remains constant and equal to $1 - e^{-1}$. $\lambda \approx 0.92$ is suitable. By inverting this function, for each index an estimate of $\log p$ is computed and then $B_1(\log p)$ and $B_2(\log p)$.

[TODO] Find $B_1 = f(\log p), B_2$.

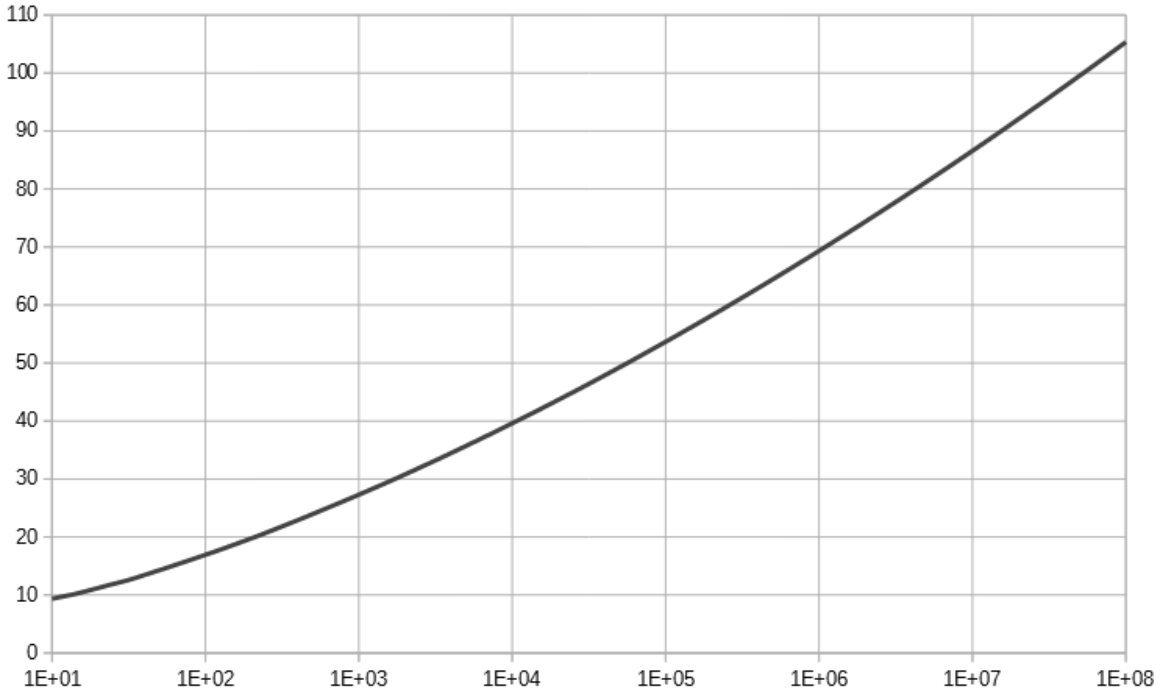


Figure 3.7: Expected number of digits as a function of the index of the curve.

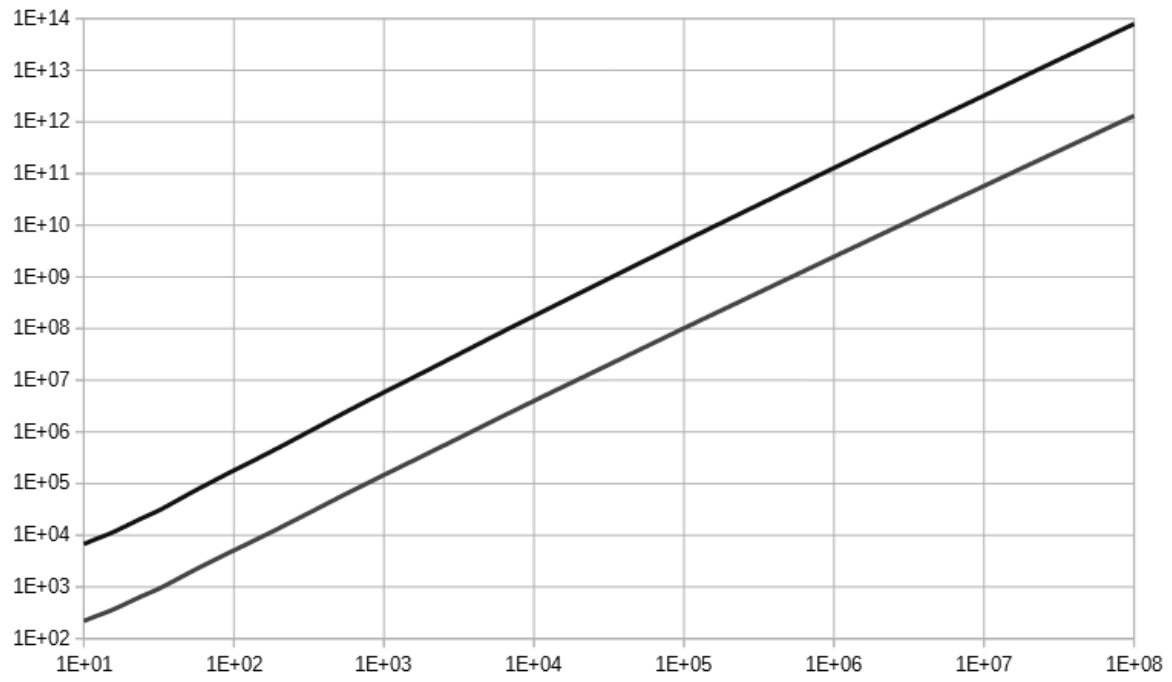


Figure 3.8: B_2 and B_1 as a function of the index of the curve.

3.3.3 Test of the probabilistic model

Chapter 4

Implementation

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