Quad Sieve for twin and Sophie Germain primes

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If we search for twin and Sophie Germain prime numbers, we first test the primality of $k\,2^n - 1$. If it is prime, then we check the primality of $k\,2^n + 1$, $k\,2^{n-1} - 1$ and $k\,2^{n+1} - 1$. A quad sieve ensures that the four forms don't have any small prime divisor.

Based on Bateman and Horn conjecture, the expected numbers of remaining candidates, primes of the form $k\,2^n - 1$, twin primes and of Sophie Germain primes in a fixed sieved range are calculated in this note.

Conjecture (Bateman and Horn).
Let $f_1, f_2, \cdots, f_m$ be a set of m distinct irreducible polynomials with integral coefficients and a positive leading coefficient. Let f be their product and suppose that f does not vanish identically modulo any prime. Let $w(p)$ be the number of solutions to the congruence $f(x) \equiv 0 \pmod{p}$. An integer n is prime-generating if every polynomial $f_i(n)$ produces a prime number.
π_f is the number of prime-generating integers for $n_1 \leq n \leq n_2$. If $n_2 - n_1 \to \infty$ then

$$\begin{aligned}\pi_f &\sim C_f \sum_{n=n_1}^{n_2} \frac{1}{\log f_1(n) \cdot \log f_2(n) \cdots \log f_m(n)} \\ &\sim \frac{C_f}{D_f} \int_{n_1}^{n_2} \frac{dt}{(\log t)^m} \\ &\sim \frac{C_f}{D_f} \left(\frac{n_2}{(\log n_2)^m} - \frac{n_1}{(\log n_1)^m} \right),\end{aligned}$$

where D_f is the product of the degrees of the polynomials $f_1, f_2, \cdots, f_m$ and

$$C_f = \prod_{p\ prime} \frac{1 - w(p)/p}{(1 - 1/p)^m}.$$

C_f is a sieving constant. Let S be a set containing n m-tuples of random integers. S is sieved for $p \leq p_{max}$: an element is removed from the set if at least one integer of the m-tuple is divisible by p. The expected number of remaining elements is $\prod_{p \leq p_{max}} (1 - 1/p)^m\, n$. If the m-tuples of the set are generated with the polynomials f_i then the expected number of remaining elements is $\mathrm{cand}_{p_{max}}(n) = \prod_{p \leq p_{max}} (1 - w(p)/p)\, n$.

$k\,2^n - 1$ is odd and $C_{k\,2^n-1} = 2$.

For twin primes, we have $f_1(x) = 2x - 1$ and $f_2(x) = 2x + 1$. $w(2) = 0$ and $w(p) = 2$ for $p > 2$. $C_{(k\,2^n-1,\,k\,2^n+1)} = 4\,C_2$, where C_2 is the twin prime constant.
$C_{(k\,2^n-1,\,k\,2^n+1)} \approx 2.6406473$.

$f_1(x) = 2x - 1$ and $f_2(x) = 4x - 1$ for Sophie Germain primes. We still have $w(2) = 0$ and $w(p) = 2$ for $p > 2$ then $C_{(k\,2^n-1,\,k\,2^{n+1}-1)} = C_{(k\,2^n-1,\,k\,2^n+1)}$.

The functions of the quadruplet are $f_1(x) = 2x - 1$, $f_2(x) = 4x - 1$, $f_3(x) = 8x - 1$ and $f_4(x) = 4x + 1$. $w(2) = 0$, $w(3) = 2$ and $w(p) = 4$ for $p > 3$.
$C_{(k\,2^{n-1}-1,\,k\,2^n-1,\,k\,2^{n+1}-1,\,k\,2^n+1)} \approx 8.3023617$.

In the following table, the formula with the summation is applied.

Number of prime patterns: k is odd, $1 \le k \le 10^8$

	Form	Actual	Expected.
single	$8k-1$	5143331	5143482.8
twin	$8k-1, 8k+1$	350779	350497.4
Sophie Germain I	$4k-1, 8k-1$	363795	363558.9
Sophie Germain II	$8k-1, 16k-1$	338858	338346.4
quadruplet	$4k-1, 8k-1, 8k+1, 16k-1$	3042	2978.2

Mertens' third theorem is $\lim_{n\to\infty} \log n \prod_{p\le n}(1-1/p) = e^{-\gamma}$, where $\gamma \approx 0.57721566$ is the Euler's constant.
We have $\mathrm{cand}_{p_{max}}(n) = \prod_{p\le p_{max}}(1 - w(p)/p)\, n \approx C_f \prod_{p\le p_{max}}(1-1/p)^m\, n$, hence

$$\mathrm{cand}_{p_{max}}(n) \approx C_f \left(\frac{e^{-\gamma}}{\log p_{max}}\right)^m n.$$

For quad sieve, we have

$$\mathrm{cand}_{p_{max}}(n) \approx \frac{0.82504063\,n}{(\log p_{max})^4}.$$

If N has no prime factor $p \le p_{max}$ then the likelihood that N is prime is

$$\mathscr{P}(N) = \prod_{p\le p_{max}}(1-1/p)^{-1}\,\frac{1}{\log N} \approx \frac{\log p_{max}}{e^{-\gamma}\log N}.$$

Let S be a set of pairs (k, n). A quad sieve is applied to this set. The expected number of primes is

$$\pi_{k\,2^n-1} \approx \frac{1.4694571}{(\log p_{max})^3}\sum_S \frac{1}{\log(k\,2^n-1)}.$$

If n is fixed and k is odd, $k_1 \le k \le k_2$ and $1 \ll k\,2^n$, we have:

$$\pi_s \approx \frac{0.73472856}{(\log p_{max})^3}\int_{k_1}^{k_2}\frac{dt}{\log(t\,2^n)}.$$

Applying $\mathscr{P}(N)$ twice for twin primes, we obtain

$$\pi_t \approx \frac{1.30860477}{(\log p_{max})^2}\int_{k_1}^{k_2}\frac{dt}{\log(t\,2^n)^2},$$

and $\pi_{SG} \approx 2\,\pi_t$.

Quad sieve prime occurrences:

$p_{max} = 10^4$, $n = 100$, k is odd, $1 \le k \le 10^9$

	Actual	Expected.
candidates	57069	57324.8
primes	10461	10562.8
twin	1950	1946.6
Sophie Germain	3825	3893.2

$p_{max} = 10^5$, $n = 100$, k is odd, $1 \le k \le 10^9$

	Actual	Expected.
candidates	23660	23480.2
primes	5419	5408.2
twin	1280	1245.8
Sophie Germain	2481	2491.6

$p_{max} = 10^5$, $n = 1000$, k is odd, $1 \le k \le 10^{10}$

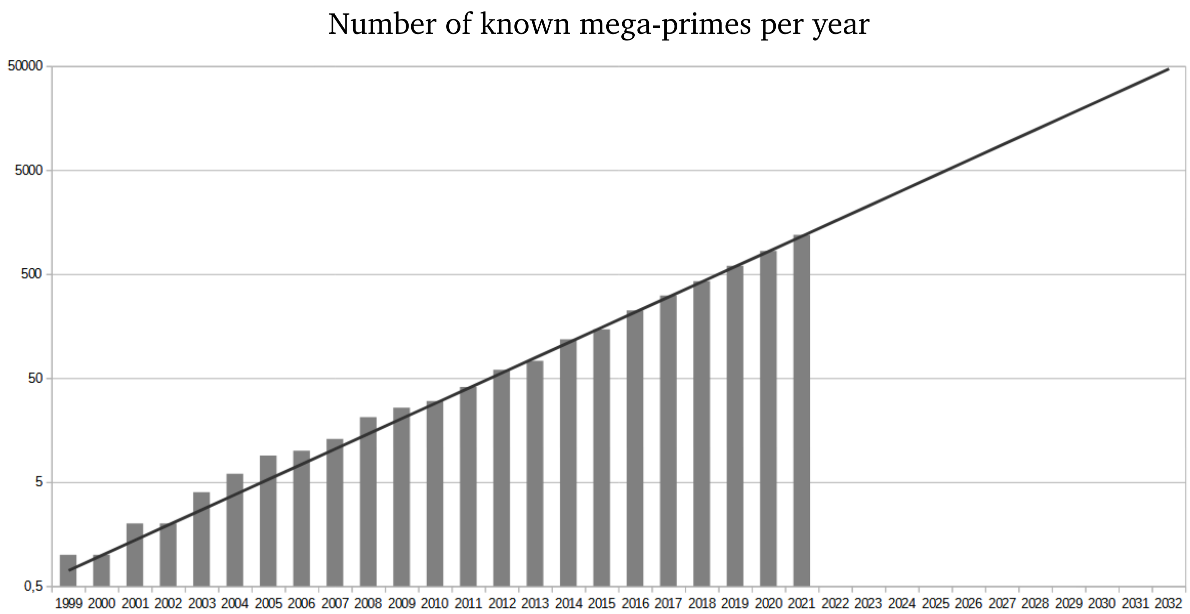
	Actual	Expected.
candidates	234241	234802.4
primes	6784	6732.2
twin	175	193.0
Sophie Germain	392	386.1

If $n = 3321925$ then $k\,2^n - 1$ has more than one million digits. For $1 \le k \le 2^{52}$ the expected values are

p_{max}	candidates	primes	twin	SG
10^{12}	3,187,000,000	68,120	1.46	2.91
10^{13}	2,314,000,000	53,578	1.24	2.48
10^{14}	1,720,000,000	42,898	1.07	2.14
10^{15}	1,306,000,000	34,877	0.93	1.86

In practice, we can set $3321925 \le n \le 3321925 + 63$ and $1 \le k \le 2^{46}$. The subset $1 \le k \le 2^{32}$ was sieved for $p_{max} < 10^{12}$. 194535 candidates were expected and 195115 remain.

If the range is sieved to 10^{15} on GPU and that the number of mega-primes found each year is still growing at the same rate then a mega twin prime could be discovered within the next ten years.


Number of known mega-primes per year
50000
5000
500
50
5
0,5
1999 2000 2001 2002 2003 2004 2005 2006 2007 2008 2009 2010 2011 2012 2013 2014 2015 2016 2017 2018 2019 2020 2021 2022 2023 2024 2025 2026 2027 2028 2029 2030 2031 2032