A LAGRANGIAN SCHEME FOR THE INCOMPRESSIBLE EULER EQUATION USING OPTIMAL TRANSPORT

THOMAS O. GALLOUËT AND QUENTIN MÉRIGOT

Abstract. We approximate the regular solutions of the incompressible Euler equation by the solution of ODEs on finite-dimensional spaces. Our approach combines Arnold's interpretation of the solution of Euler's equation for incompressible and inviscid fluids as geodesics in the space of measure-preserving diffeomorphisms, and an extrinsic approximation of the equations of geodesics due to Brenier. Using recently developed semi-discrete optimal transport solvers, this approach yields numerical scheme able to handle problems of realistic size in 2D. Our purpose in this article is to establish the convergence of these scheme towards regular solutions of the incompressible Euler equation, and to provide numerical experiments on a few simple testcases in 2D.

Contents

1. Introduction

In this paper we investigate a discretization of Euler's equation for incompressible and inviscid fluids in a domain $\Omega \subseteq \mathbb{R}^d$ with Neumann boundary conditions:

$$\begin{cases} \partial_t v(t,x) + (v(t,x)\cdot\nabla)\, v(t,x) = -\nabla p(t,x), & \text{for } t\in[0,T],\ x\in\Omega\,,\\ \operatorname{div}(v(t,x)) = 0 & \text{for } t\in[0,T],\ x\in\Omega\,,\\ v(t,x)\cdot n = 0 & \text{for } t\in[0,T],\ x\in\partial\Omega\,,\\ v(0,x) = v_0. \end{cases} \tag{1.1}$$

As noticed by Arnold [2], in Lagrangian coordinates, Euler's equation can be interpreted as the equation of geodesics in the infinite-dimensional group of measure-preserving diffeomorphisms of Ω. To see this, we consider the flow map $\phi : [0,T]\times\Omega\to\Omega$ induced by the vector field v, that is:

$$\begin{cases} \frac{d}{dt}\phi(t,x) = v\,(t,\phi(t,x)) & \text{for } t\in[0,T],\ x\in\Omega\,,\\ \phi(0,\cdot) = \mathrm{id},\\ \partial_t\phi(0,\cdot) = v_0. \end{cases} \tag{1.2}$$

1991 *Mathematics Subject Classification.* 35Q31, 65M12, 65M50, 65Z05.
Key words and phrases. Incompressible Euler equation, Optimal Transport, Lagrangian numerical scheme, Hamiltonian.
The first author is supported by the ANR grant ISOTACE.

Using the incompressibility constraint $\operatorname{div}(v(t,x)) = 0$ and the initial condition $\phi(0) = \mathrm{id}$, one can check that $\phi(t,\cdot)$ belongs to the set of volume preserving maps $\mathbb{S}$, defined by

$$\mathbb{S} = \left\{ s \in \mathrm{L}^2(\Omega, \mathbb{R}^d) \mid s_\# \mathrm{Leb} = \mathrm{Leb} \right\},$$

where Leb is the restriction of the Lebesgue measure to the domain Ω and where the pushforward measure $s_\# \mathrm{Leb}$ is defined by the formula $s_\# \mathrm{Leb}(A) = \mathrm{Leb}(s^{-1}(A))$ for every measurable subset A of Ω. Euler's equation (1.1) can therefore be reformulated as

$$\begin{cases} \frac{d^2}{dt^2}\phi(t) = -\nabla p(t, \phi(t,x)) & \text{for } t \in [0,T],\ x \in \Omega, \\ \phi(t,\cdot) \in \mathbb{S} & \text{for } t \in [0,T], \\ \phi(0,\cdot) = \mathrm{id}, \\ \partial_t \phi(0,\cdot) = v_0. \end{cases} \tag{1.3}$$

This equation can be formally interpreted as the equation of geodesics in $\mathbb{S}$ as follows. First, note that the condition $\phi(t,\cdot) \in \mathbb{S}$ in (1.1) encodes the infinitesimal conditions $\operatorname{div} v(t,\cdot) = 0$ and $v(t,x) \cdot n(x) = 0$ in (1.3). This suggests that the tangent plane to $\mathbb{S}$ at a point $\phi \in \mathbb{S}$ should be the set $\{v \circ \phi \mid v \in \mathcal{H}_{\mathrm{div}}(\Omega)\}$, where $\mathcal{H}_{\mathrm{div}}(\Omega)$ denotes the set of divergence-free vector fields

$$\mathcal{H}_{\mathrm{div}}(\Omega) = \left\{ v \in \mathrm{L}^2(\Omega, \mathbb{R}^d) \mid \operatorname{div}(v) = 0,\ v \cdot n = 0 \right\}.$$

In addition, by the Helmoltz-Hodge decomposition, the orthogonal to $\mathcal{H}_{\mathrm{div}}(\Omega)$ in $\mathrm{L}^2(\Omega, \mathbb{R}^d)$ is the space of gradients of functions in $\mathcal{H}^1_0(\Omega)$. Therefore the evolution equation in (1.3) expresses that the acceleration of ϕ should be orthogonal to the tangent plane to $\mathbb{S}$ at ϕ, or in other words that $t \mapsto \phi(t,\cdot)$ should be a geodesic of $\mathbb{S}$. Note however that a solution to (1.3) does not need to be a *minimizing* geodesic between $\phi(0,\cdot)$ and $\phi(T,\cdot)$. The problem of finding a minimizing geodesic on $\mathbb{S}$ between two measure preserving maps amounts to solving equations (1.3), where the initial condition $\partial_t \phi(0,\cdot) = v_0$ is replaced by a prescribed coupling between the position of particles at initial and final times. It leads to generalized and non-deterministic solutions introduced Brenier [5], where particles are allowed to split and cross. Shnirel'man showed that this phenomena can happen even when the measure-preserving maps $\phi(0,\cdot)$ and $\phi(T,\cdot)$ are diffeomorphisms of Ω [17].

Our discretization of Euler's equations (1.1) relies on Arnold's interpretation as the equation of geodesics and exploit the extrinsic view given by the embedding of the set of measure preserving maps $\mathbb{S}$ in the Hilbert space $\mathbb{M} = \mathrm{L}^2(\Omega, \mathbb{R}^d)$. In our discretization the measure-preserving property is enforced through a penalization term involving the squared distance to the set of measure-preserving maps $\mathbb{S}$, as in [7]. The numerical implementation of this idea relies on Brenier's polar factorization theorem to compute the squared distance to $\mathbb{S}$ and on recently developed numerical solvers for optimal transport problems invoving a probability measure with density and a finitely-supported probability measure [3, 14, 8, 12]. This combination of ideas presented above has already been used to compute numerically minimizing geodesics between measure-preserving maps in [15], allowing the recovery of non-deterministic solutions predicted by Schnirel'man and Brenier. The object of this article is to determine whether this strategy can be used to construct a Lagrangian discretization for the more classical Cauchy problem for the Euler's equation (1.1), which is able to recover regular solutions to Euler's equation, both theoretically and experimentally.

Discretization in space: approximate geodesics. The construction of approximate geodesics presented here is strongly inspired by a particle scheme introduced by Brenier [7], in which the space of measure-preserving maps $\mathbb{S}$ was approximated by the space of permutations of a fixed tessellation of the domain Ω. To construct our numerical approximation we first approach the Hilbert space $\mathbb{M} = \mathrm{L}^2(\Omega, \mathbb{R}^d)$ with finite dimensional subspaces. Let

N be an integer and let P_N be a tessellation partition up to negligible set of Ω into N subsets $(\omega_i)_{1\leq i\leq N}$ satisfying

$$\begin{cases} \forall i \in \{1,\dots,N\},\ \mathrm{Leb}(\omega_i) = \dfrac{1}{N}\mathrm{Leb}(\Omega) \\ h_N := \max\limits_{1\leq i\leq N} \mathrm{diam}(\omega_i) \leq \dfrac{C}{N^{1/d}} \end{cases}$$

where C is independent of N. We consider $\mathbb{M}_N$ the space of functions from Ω to $\mathbb{R}^d$ which are constant on each of the subdomains (ω_i). To construct our approximate geodesics, we consider the squared distance to the set $\mathbb{S} \subseteq \mathbb{M}$ of measure-preserving maps:

$$d_{\mathbb{S}}^2 : m \in \mathbb{M}_n \mapsto \min_{s\in\mathbb{S}} \|m - s\|_{\mathbb{M}}^2.$$

The approximate geodesic model is described by the equations

$$\begin{cases} \ddot{m}(t) + \frac{\nabla \mathrm{d}_{\mathbb{S}}^2(m(t))}{2\epsilon^2} = 0, \quad \text{for } t\in[0,T]\,, \\ (m(0),\dot{m}(0)) \in \mathbb{M}_N^2 \end{cases} \tag{1.4}$$

which is the system associated to the Hamiltonian

$$H(m,v) = \frac{1}{2}\|v\|_{\mathbb{M}}^2 + \frac{d_{\mathbb{S}}^2(m)}{2\epsilon^2}. \tag{1.5}$$

Loosely speaking, equation (1.4) describes a physical system where the current point $m(t)$ moves by inertia in $\mathbb{M}_N$, but is deflected by a spring of strength $\frac{1}{\epsilon}$ attached to the nearest point $s(t)$ in $\mathbb{S}$. Note that the squared distance $\mathrm{d}_{\mathbb{S}}^2$ is semi-concave, and that its restriction to the finite-dimensional space $\mathbb{M}_N$ is differentiable at almost every point.

We now rewrite this systems of equations (1.4) in terms of projection on the sets $\mathbb{S}$ and $\mathbb{M}_N$. Since the space of measure-preserving maps $\mathbb{S}$ is closed but not convex, the orthogonal projection of $\mathbb{S}$ exists but is usually not uniquely defined. To simplify the exposition we will nonetheless associate to any point $m\in\mathbb{M}$ one of its projection $P_{\mathbb{S}}(m)$, i.e. any point in $\mathbb{S}$ such that $\|P_{\mathbb{S}}(m) - m\|_{\mathbb{M}} = \mathrm{d}_{\mathbb{S}}(m)$. We also denote $P_{\mathbb{M}_N} : \mathbb{M}\to\mathbb{M}_N$ the orthogonal projection on the linear subspace $\mathbb{M}_N \subseteq \mathbb{M}$. We can rewrite Eq. (1.4) in terms of these two projection operators:

$$\begin{cases} \ddot{m}(t) + \frac{m(t) - P_{\mathbb{M}_N}\circ P_{\mathbb{S}}(m(t))}{\epsilon^2} = 0, \quad \text{for } t > 0\,, \\ (m(0),\dot{m}(0)) \in \mathbb{M}_N^2 \end{cases} \tag{1.6}$$

From Proposition 5.2, the double projection $P_{\mathbb{M}_N}\circ P_{\mathbb{S}}(m)$ is uniquely defined for almost every $m\in\mathbb{M}_N$. We first prove that the system of equations (1.4) can be used to approximate regular solutions to Euler's equation (1.1). Our proof of convergence uses a modulated energy technic and requires a Lipschitz regularity assumption on the solution of Euler's equation. It also requires a technical condition on the computational domain.

Definition 1.1. An open subset Ω of $\mathbb{R}^d$ is called prox-regular with constant $r_\Omega > 0$ if every point within distance r_Ω from Ω has a unique projection on Ω.

Note that smooth and semi-convex domains are prox-regular for a constant r_Ω smaller than the minimal curvature radius of the boundary $\partial\Omega$. On the other hand, convex domains are prox-regular with constant $r_\Omega = +\infty$.

Theorem 1.2. *Let Ω be a connected prox-regular set. Let v, p be a strong solution of Euler's equations (1.1), let ϕ be the flow map induced by v given by (1.2) and assume that $v, p, \partial_t v, \partial_t p, \nabla v$ and ∇p are Lipschitz on Ω, uniformly on $[0,T]$. Suppose in addition that there exist a $\mathcal{C}^1$ curve $m : [0,T]\to\mathbb{R}$ satisfying the initial conditions*

$$m(0) = P_{\mathbb{M}_N}(\mathrm{id}), \quad \dot{m}(0) = P_{\mathbb{M}_N}(v(0,\cdot)),$$

which is twice differentiable and satisfies the second-order equation (1.4) *for all times in* $[0,T]$ *up to a (at most) countable number of exceptions. Then,*

$$\max_{t\in[0,T]} \|\dot m - v(t,\phi(t,\cdot))\|_{\mathbb{M}}^2 \leq C_1 \frac{h_N^2}{\varepsilon^2} + C_2\varepsilon^2 + C_3 h_N$$

where the constants C_1, C_2 *and* C_3 *only depend on the proximal constant of the domain, on the* L^∞ *norm (in space) of the velocity* $v(t,\cdot)$ *and on the Lipschitz norms (in space) of the velocity and its first derivatives* $v(t,\cdot), \nabla v(t,\cdot), \partial_t v(t,\cdot)$ *and of the pressure and its derivatives* $p(t,\cdot), \nabla p(t,\cdot), \partial_t p(t,\cdot)$.

The value of C_1, C_2 and C_3 is given more precisely at the end of Section 3. Note that the hypothesis on the solution m to the EDO is here for technical reasons. Removing it was not of our main concern in this paper since we also give a proof of convergence of the fully discrete numerical scheme without this assumption. It is likely that solutions to the EDO (1.4) satisfying this hypothesis can be constructed through di Perna-Lions or Bouchut-Ambrosio theory [1, 4, 13].

Discretization in space and time. To obtain a numerical scheme we also need to discretize in time the Hamiltonian system (1.6). For simplicity of the analysis, we consider a symplectic Euler scheme. Let τ be the time step, for $m \in \mathbb{M}_N$ we denote by $P_{\mathbb{M}_N} P_{\mathbb{S}}(m)$ a random element in this set. The solution is the set of points M^n, V^n given by:

$$\begin{cases} (M^0, V^0) \in \mathbb{M}_N \\ V^{n+1} = V^n - \frac{\tau}{\epsilon^2}\left(M^n - P_{\mathbb{M}_N} \circ P_{\mathbb{S}}(M^n)\right) \\ M^{n+1} = M^n + \tau V^{n+1} \end{cases} \tag{1.7}$$

We also set $t^n = n\tau$. For the numerical scheme of our approximate geodesic flow we set a more precise theorem.

Theorem 1.3. *Let* Ω *be a connected prox-regular set,* ϵ *and* τ *be positive numbers. Let* v, p *be a strong solution of* (1.1)*, let* ϕ *be the flow map induced by* v *given by* (1.2) *and assume that* $v, p, \partial_t v, \partial_t p, \nabla v$ *and* ∇p *are Lipschitz on* Ω*, uniformly on* $[0,T]$*. Let* $(M^n, V^n)_{n\geq 0}$ *be a sequence generated by* (1.7) *with initial conditions*

$$M^0 = P_{\mathbb{M}_N}(\mathrm{id}),\ \ V^0 = P_{\mathbb{M}_N}(v(0,\cdot)).$$

Then,

$$\max_{n\in\mathbb{N}\cap[0,T/\tau]} \|V^n - v(t^n,\phi(t^n,\cdot))\|_{\mathbb{M}} \leq C_{(h_N\epsilon^{-1},\tau\epsilon^{-1})}\left[\epsilon^2 + h_N + \frac{h_N^2}{\epsilon^2} + \frac{\tau}{\epsilon^2}\right]$$

where the constant C *only depends on upper bounds of* $\tau\epsilon^{-1}$ *and* $h_N\epsilon^{-1}$*, on the proximal constant of the domain, on the* L^∞ *norm (in space) of the velocity* $v(t,\cdot)$ *and on the Lipschitz norms (in space) of the velocity and its first derivatives* $v(t,\cdot), \nabla v(t,\cdot), \partial_t v(t,\cdot)$ *and of the pressure and its derivatives* $p(t,\cdot), \nabla p(t,\cdot), \partial_t p(t,\cdot)$.

In order to use the numerical scheme (1.7), one needs to be able to compute the double projection operator $P_{\mathbb{M}_N} \circ P_{\mathbb{S}}$ or equivalently the gradient of the squared distance $\mathrm{d}_{\mathbb{S}}^2$ for (almost every) m in $\mathbb{M}_N$. Brenier's polar factorization problem [6] implies that the squared distance between a map $m : \Omega \to \mathbb{R}$ and the set $\mathbb{S}$ of measure-preserving maps equals the squared Wasserstein distance [18] between the restriction of the Lebesgue measure to Ω, denoted Leb, and its pushforward $m_\#\mathrm{Leb}$ under the map m:

$$\mathrm{d}_{\mathbb{S}}^2(m) = \min_{s\in\mathbb{S}} \|m - s\|^2 = \mathrm{W}_2^2(m_\#\mathrm{Leb}, \mathrm{Leb}).$$

Moreover, since m is piecewise-constant over the partition $(\omega_i)_{1\leq i\leq N}$, the push-forward measure $m_\#\,\mathrm{Leb}$ if finitely supported. Denoting $M_i \in \mathbb{R}^d$ the constant value of the map m on the subdomain ω_i we have,

$$m_\#\,\mathrm{Leb} = \sum_{1\leq i\leq N} \mathrm{Leb}(\omega_i)\delta_{M_i} = \frac{1}{N}\sum_{1\leq i\leq N} \delta_{M_i}.$$

Thus, computing the projection operator $P_{\mathbb{S}}$ amounts to the numerical resolution of an optimal transport problem between the Lebesgue measure on Ω and a finitely supported measure. Thanks to recent work [3, 14, 8, 12], this problem can be solved efficiently in dimensions $d = 2, 3$. We give more details in Section 5.

Remark 1.4. A scheme involving similar ideas, and in particular the use of optimal transport to impose incompressibility contraints, has recently been proposed for CFD simulations in computer graphics [9]. From the simulations presented in [9], the scheme seems to behave better numerically, and it also has the extra advantage of not depending on a penalization parameter ε. It would therefore be interesting to extend the convergence analysis presented in Theorem 1.3 to the scheme presented in [9]. This might however require new ideas, as our proof techniques rely heavily on the fact that the space-discretization is hamiltonian, which does not seem to be the case in [9].

Remark 1.5. Our discretization (1.4) resembles (and is inspired by) a space-discretization of Euler's equation (1.1) introduced by Brenier in [7]. The domain is also decomposed into subdomains $(\omega_i)_{1\leq i\leq N}$, and one considers the set $\mathbb{S}_N \subseteq \mathbb{S}$, which consists of measure-preserving maps $s : \Omega \to \Omega$ that are induced by a permutation of the subdomains. Equivalently, one requires that there exists $\sigma : \{1, \dots, N\} \to \{1, \dots, N\}$ such that $s(\omega_i) = \omega_{\sigma(j)}$. The space-discretization considered in [7] leads to an ODE similar to (1.4), but where the squared distance to $\mathbb{S}$ is replaced by the squared distance to $\mathbb{S}_N$. This choice of discretization imposes strong contraints on the relative size of the parameters τ, h_N and ϵ, namely that $h_N = \mathrm{O}(\varepsilon^8)$ and $\tau = \mathrm{O}(\varepsilon^4)$. Such constraints still exist with the discretization that we consider here, but they are milder. In Theorem 1.3 the condition $\tau = o(\epsilon^2)$ is due to the time discretization of (1.6) and can be improved using a scheme more accurate on the conservation of the Hamiltonian (1.5). However even with an exact time discretization of the Hamiltonian, the condition $\tau = o(\epsilon)$ remains mandatory, see section 4.

Acknowledgements. We would like to thank Yann Brenier for pointing us to [7], and for many interesting discussions at various stages of this work.

2. Preliminary discussion on geodesics

To illustrate the approached geodesic scheme we focus on the very simple example of $\mathbb{R}$ seen as $\mathbb{R} \times \{0\} \subset \mathbb{R}^2$. The geodesic is given by the function γ: $[0, T] \to \mathbb{R}^2$ with

$$\begin{cases} \gamma(t) = (t, 0),\ t \in [0, T], \\ \gamma(0) = (0, 0), \\ \dot\gamma(0) = (1, 0). \end{cases} \tag{2.1}$$

We suppose that we make an error of order (h_0, h_1) in the initial conditions. As in (1.4) we consider the solutions of the Hamiltonian system associated to:

$$H(z, v) = \frac{1}{2}||v||^2 + \frac{1}{2\epsilon^2} d^2_{\mathbb{R}\times\{0\}}(z). \tag{2.2}$$

That is

$$\begin{cases} \ddot z(t) = \frac{1}{\epsilon^2}\left(P_{\mathbb{R}}(z) - z\right) = \frac{1}{2\epsilon^2}\nabla d^2_{\mathbb{R}\times\{0\}}(z),\ t \in [0, T], \\ z(0) = (0, h_0), \\ \dot z(0) = (1, h_1). \end{cases} \tag{2.3}$$

where $P_{\mathbb{R}}(z)$ is the orthogonal projection from $\mathbb{R}^2$ onto $\mathbb{R}\times\{0\}$. Notice that we made a mistake of order h_0 on the initial position and h_1 on the initial velocity. In this case the solution is explicit and reads

$$z(t) = \left(t, h_0 \cos\frac{t}{\epsilon} + \epsilon h_1 \sin\frac{t}{\epsilon}\right). \tag{2.4}$$

A convenient way to quantify how far z is from being a geodesic is to use a modulated energy related to the Hamiltonian H and the solution γ. We define E_γ by

$$E_\gamma(z) = \frac{1}{2}||\dot z - \dot\gamma||^2 + \frac{1}{2\epsilon^2} d^2_{\mathbb{R}\times\{0\}}(z). \tag{2.5}$$

Notice that $E_\gamma(z)$ is symmetric since for all $t \in [0,T]$, $d^2_{\mathbb{R}\times\{0\}}(z) = 0$. A direct computation leads to

$$E_\gamma(z) = \frac{h_0^2}{\epsilon^2} + h_1^2. \tag{2.6}$$

This estimates shows that the velocity vector field $\dot z$ converges towards the geodesic velocity vector fields $\dot\gamma$ as soon as h_0 goes to 0 quicker then ϵ. Our construction of approached geodesics for the Euler equation follow this idea. Estimates (2.6) suggests that our convergence results for the incompressible Euler equation in Theorem 1.2 is sharp. A computation of the Hamiltonian (2.2) evaluated on the solution of the Euler symplectic scheme, with $h_1 = 0$ leads to

$$H(Z^n, V^n) \leq (1 - \frac{\tau^2}{\epsilon^2})^n H(Z^0, V^0).$$

It suggests again that the estimation $\tau = o(\epsilon^2)$ in Theorem 1.3 is sharp, even if one can hope for compensation to have in practice a much better convergence.

3. Convergence of the approximate geodesics model

3.1. **Preliminary lemma.** Before proving Theorem 1.2, we collect a few useful lemmas.

Lemma 3.1 (Projection onto the measure preserving maps $\mathbb{S}$)**.** *Let $m \in \mathbb{M} = \mathrm{L}^2(\Omega, \mathbb{R}^d)$. There exists a convex function $\varphi : \Omega \to \mathbb{R}$, which is unique up to an additive constant, such that s belongs to $\Pi_{\mathbb{S}}(m)$ if and only if $m = \nabla\varphi \circ s$ up to a negligible set. Moreover, $m - s$ is orthogonal to $\mathcal{H}_{\mathrm{div}}(\Omega) \circ s$:*

$$\forall v \in \mathcal{H}_{\mathrm{div}}(\Omega), \int_\Omega \langle m(x) - s(x) | v(s(x))\rangle \mathrm{d}x = 0. \tag{3.1}$$

Proof. The first part of the statement is Brenier's polar factorization theorem [6], and the uniqueness of ϕ follows from the connectedness of the domain. Using a regularization argument we deduce the orthogonality relation

$$\int_\Omega m(x)v(s(x)) = \int_\Omega \nabla\varphi \circ s(x) v(s(x)) = \int_\Omega \nabla\varphi v(x) = -\int_\Omega \varphi \nabla\cdot v(x) = 0. \qquad \square$$

Lemma 3.2 (Projection onto the piecewise constant set $\mathbb{M}_N$)**.** *The projection of a function $g \in L^2(\Omega, \mathbb{R}^d)$ on $\mathbb{M}_N$ is the following piecewise constant function :*

$$\Pi_{\mathbb{M}_N}(g) = \sum_{i=1}^N G_i \mathbf{1}_{\omega_i}, \ \textit{with } G_i := \frac{1}{\mathrm{Leb}(\omega_i)} \int_{\omega_i} g(x)\mathrm{d}x$$

and where $\mathbf{1}_{\omega_i}$ is the indicator function of the subdomain ω_i.

Proof. It suffices to remark that for any $m \in \mathbb{M}_N$, $m = \sum_{1\leq i\leq N} M_i \mathbf{1}_{\omega_i}$,

$$\langle g|m\rangle_{\mathbb{M}} = \int_\Omega \langle m(x)|g(x)\rangle \mathrm{d}x = \sum_{1\leq i\leq N} \langle M_i| \int_{\omega_i} g(x)\mathrm{d}x\rangle = \langle m| \sum_i G_i \mathbf{1}_{\omega_i}\rangle_{\mathbb{M}} \qquad \square$$

Lemma 3.3. *Let Ω be a prox-regular domain of $\mathbb{R}^d$ let $(V, \|.\|)$ be a normed vector space. Then, there exists a linear map $L : \mathcal{C}^0(\Omega, V) \to \mathcal{C}^0(\mathbb{R}^d, V)$ such that for any $f \in \mathcal{C}^0(\Omega, V)$,*

(i) $Lf|_\Omega = f$ *and* $\|Lf\|_{\mathrm{L}^\infty(\mathbb{R}^d,V)} \leq \|f\|_{\mathrm{L}^\infty(\Omega,V)}$

(ii) $\mathrm{Lip}(Lf) \leq \frac{6}{r_\Omega} \|f\|_{\mathrm{L}^\infty(\Omega,V)} + \mathrm{Lip}(f)$.

Proof. Let r_Ω be the prox-regularity constant of Ω, and let Ω' be a tubular neighborhood of radius $r_\Omega/2$ around Ω, i.e. $\Omega' = \{x \in \mathbb{R}^d \mid \mathrm{d}(x, \Omega) \leq r_\Omega/2\}$. Denote $p : \Omega' \to \overline{\Omega}$ the function which maps a point of Ω' to the closest point in $\overline{\Omega}$. From Theorem 4.8.(8) in [10], the map p is 2-Lipschitz. We now define the function Lf by

$$Lf(x) = \begin{cases} \chi(\|x - p(x)\|) f(p(x)) & \text{if } x \in \Omega' \\ 0 & \text{if not,} \end{cases}$$

where $\chi(r) = \max(1 - 2r/r_\Omega, 0)$. Remark that $\|\chi\|_{L^\infty}$ is bounded by one, implying that $\|Lf\|_{L^\infty(\mathbb{R}^d)} \leq \|f\|_{L^\infty(\Omega)}$. For the Lipschitz continuity estimates we distinguish three cases. First, if x, y both belong to $\Omega' \times \Omega'$, we have

$$\begin{aligned} \|Lf(x) - Lf(y)\| &= \|\chi(\|x - p(x)\|) f(p(x)) - \chi(\|y - p(y)\|) f(p(y))\| \\ &\leq |\chi(\|x - p(x)\|)| \cdot \|f(p(x)) - f(p(y))\| \\ &\qquad + |\chi(\|y - p(y)\|) - \chi(\|x - p(x)\|)| \cdot \|f(p(y))\| \\ &\leq \|\chi\|_{\mathrm{L}^\infty(\mathbb{R})} \mathrm{Lip}(f) \cdot \|x - y\| + \mathrm{Lip}(\chi \circ (\mathrm{Id} \, - p)) \|f\|_{L^\infty(\Omega)} \|x - y\| \\ &\leq \left(\frac{6}{r_\Omega} \|f\|_{L^\infty(\Omega)} + \mathrm{Lip}(f) \right) \|x - y\| . \end{aligned}$$

If x belongs to Ω' and y belongs to $\mathbb{R}^d \setminus \Omega'$ one has $Lf(y) = 0$ so that

$$\begin{aligned} \|Lf(x) - Lf(y)\| &\leq \|\chi(\|x - p(x)\|) f(p(x))\| \\ &\leq \frac{2}{r_\Omega} \|f\|_{L^\infty(\Omega)} |r_\Omega/2 - \|x - p(x)\|| \\ &\leq \frac{2}{r_\Omega} |\|f\|_{L^\infty(\Omega)} \|y - x\| . \end{aligned}$$

Finally, if x, y are outside of Ω', $Lf(x) = Lf(y) = 0$ and there is nothing to prove. ☐

We are now ready to prove Theorem 1.2. In the following the dot refer to the time derivative and $\langle .|. \rangle$ to the Hilbert scalar on $\mathbb{M}$. By abuse of notation we denote by the same variable a Lipschitz function defined on Ω and its (also Lipschitz) extension defined on the whole space $\mathbb{R}^d$ thanks to Lemma 3.3. The space $\mathbb{R}^d$ is equipped with the Euclidian norm, and the space of $d \times d$ matrices are equiped with the dual norm. The Lipschitz constants that we consider are with respect to these two norms. Finally for a curve $X : t \in [0, T] \mapsto X(t, \cdot)$ we denote $\mathrm{Lip}_{[0,\mathrm{T}]}(X) = \sup_{t \in [0,T]} \mathrm{Lip}(X(t, \cdot))$.

3.2. **Proof of Theorem 1.2.** Let v be a solution of (1.1) and m a solution of (1.4) and for any $t \in [0, T]$, denote $s(t) = P_{\mathbb{S}}(m(t))$. In other words, $s(t)$ is an arbitrary choice of a projection of $m(t)$ on $\mathbb{S}$. Equation (1.4) is the ODE associated to the Hamiltonian

$$H(m, v) = \frac{1}{2} \|v\|_{\mathbb{M}}^2 + \frac{d_{\mathbb{S}}^2(m)}{2\epsilon^2}.$$

We therefore consider a energy involving this Hamiltonian, modulated with the exact solution v:

$$E_v(t) = \frac{1}{2} \|\dot{m}(t) - v(t, m(t))\|_{\mathbb{M}}^2 + \frac{d_{\mathbb{S}}^2(m)}{2\epsilon^2}. \tag{3.2}$$

We will control E_v using a Gronwall estimate.

Remark 3.4. Note that we need to use Lemma 3.3 to define the modulated energy E_v since the maps $m(t,\cdot) \in \mathbb{M}_N$ can send points outside of Ω when Ω is not convex.

3.2.1. *Time derivative.* We compute $\frac{d}{dt}E_v(t)$ and modify the expression in order to identify terms of quadratic order. Since the Hamiltonian $H(\dot{m}(t), m(t))$ is preserved, we find

$$\frac{d}{dt}E_v(t) = \underbrace{-\langle \ddot{m}(t), v(t,m(t))\rangle}_{I_1} \underbrace{-\langle \dot{m}(t) - v(t,m(t)), \partial_t v(t,m(t)) + (\dot{m}(t)\cdot\nabla)\, v(t,m(t))\rangle}_{I_2}. \tag{3.3}$$

Using the EDO (1.4), I_1 can be rewritten as

$$\begin{aligned}\epsilon^2 I_1 &= \langle m(t) - P_{\mathbb{M}_N}(s(t)), v(t,m(t))\rangle \\ &= \langle m(t) - s(t), v(t,m(t))\rangle + \langle s(t) - P_{\mathbb{M}_N}(s(t)), v(t,m(t))\rangle \\ &= \underbrace{\langle m(t) - s(t), v(t,m(t)) - v(t,s(t))\rangle}_{\epsilon^2 I_3},\end{aligned}$$

where we have used that $s(t) - P_{\mathbb{M}_N}(s(t))$ is orthogonal to $\mathbb{M}_N$ and that $m(t) - s(t)$ is orthogonal to $\mathcal{H}_{\mathrm{div}}(\Omega)\circ s$, see Lemmas 3.2 and 3.1. To handle the term I_2 we define for $X \in \mathbb{M}$ the two following operators, often called material derivatives:

$$\begin{cases} D_t v(t,X) &= \partial_t v(t,X) + (v(t,X)\cdot\nabla)\, v(t,X), \\ D_t p(t,X) &= \partial_t p(t,X) + \langle v(t,X), \nabla p(t,X)\rangle . \end{cases} \tag{3.4}$$

Remark that Euler's equation (1.1) implies that $D_t v(t,s(t)) = -\nabla p(t,s(t))$. This leads to

$$\begin{aligned} I_2 &= -\langle \dot{m}(t) - v(t,m(t)), \partial_t v(t,m(t)) + (v(t,m(t))\cdot\nabla)\, v(t,m(t))\rangle \\ &\quad \underbrace{-\langle \dot{m}(t) - v(t,m(t)), (\dot{m}(t) - v(t,m(t))\cdot\nabla)\, v(t,m(t))\rangle}_{I_4} \\ &= I_4 \underbrace{-\langle \dot{m}(t) - v(t,m(t)), D_t v(t,m(t)) - D_t v(t,s(t))\rangle}_{I_5} + \underbrace{\langle \dot{m}(t) - v(t,m(t)), \nabla p(t,s(t))\rangle}_{I_6} \end{aligned}$$

We rewrite I_6 as

$$\begin{aligned} I_6 &= \underbrace{-\langle \dot{m}(t) - v(t,m(t)), \nabla p(t,m(t)) - \nabla p(t,s(t))\rangle}_{I_7} + \langle \dot{m}(t) - v(t,m(t)), \nabla p(t,m(t))\rangle \\ &= I_7 + \frac{d}{dt}\underbrace{\int_\Omega p(t,m(t,x)))dx}_{-J(t)} - \int_\Omega \partial_t p(t,m(t,x)) - \langle v(t,m(t,x)), \nabla p(t,m(t,x))\rangle\, dx \\ &= -\frac{d}{dt}J(t) + I_7 - \underbrace{\int_\Omega D_t p(t,m(t,x))dx}_{I_8}. \end{aligned}$$

Remark 3.5. The quantity $I_5 + I_7$ control the fact that the extension of (v,p) constructed by Lemma 3.3 is not a solution of the Euler equation on $\mathbb{R}^d$ (in particular, $I_5 + I_7$ vanishes if (v,p) solves Euler's equation on $\mathbb{R}^d$).

3.2.2. *Estimates.* Many of the integrals I_i can be easily bounded using the energy E_v and Cauchy-Schwarz and Young's inequalities. First,

$$\begin{aligned} I_3 &\leq \left| \frac{\langle m(t) - s(t), v(t,m(t)) - v(t,s(t))\rangle}{\epsilon^2} \right| \\ &\leq \mathrm{Lip}(v(t))\frac{\|m(t) - s(t)\|^2_{\mathbb{M}}}{\epsilon^2} \leq \mathrm{Lip}_{[0,\mathrm{T}]}(v)E_v(t). \end{aligned} \tag{3.5}$$

Furthermore

$$I_4 \le \sup_{x\in\mathbb{R}^d} ||\nabla v(t,x)|| \|\dot m(t) - v(t,m(t))\|^2_{\mathbb{M}} \le \mathrm{Lip}_{[0,\mathrm{T}]}(v) E_v(t), \tag{3.6}$$

Where C depends only on the dimension d. To estimate I_5 and later I_8 we first remark that $D_t v$ and $D_t p$ are Lipschitz operators with

$$\begin{aligned}\mathrm{Lip}_{[0,\mathrm{T}]}(D_t v) &\le \mathrm{Lip}_{[0,\mathrm{T}]}(\partial_t v) + \mathrm{Lip}_{[0,\mathrm{T}]}(v)\|\nabla v\|_{L^\infty} + \mathrm{Lip}_{[0,\mathrm{T}]}(\nabla v)\|v\|_{L^\infty} \\ &\le \mathrm{Lip}_{[0,\mathrm{T}]}(\partial_t v) + \mathrm{Lip}_{[0,\mathrm{T}]}(v)\,\mathrm{Lip}_{[0,\mathrm{T}]}(v) + \mathrm{Lip}_{[0,\mathrm{T}]}(\nabla v)\|v\|_{L^\infty}\end{aligned} \tag{3.7}$$

$$\begin{aligned}\mathrm{Lip}_{[0,\mathrm{T}]}(p_t v) &\le \mathrm{Lip}_{[0,\mathrm{T}]}(\partial_t p) + \mathrm{Lip}_{[0,\mathrm{T}]}(v)\|\nabla p\|_{L^\infty} + \mathrm{Lip}_{[0,\mathrm{T}]}(\nabla p)\|v\|_{L^\infty} \\ &\le \mathrm{Lip}_{[0,\mathrm{T}]}(\partial_t p) + \mathrm{Lip}_{[0,\mathrm{T}]}(v)\,\mathrm{Lip}_{[0,\mathrm{T}]}(p) + \mathrm{Lip}_{[0,\mathrm{T}]}(\nabla p)\|v\|_{L^\infty}.\end{aligned} \tag{3.8}$$

For I_5 we obtain — using $\mathrm{d}_{\mathbb{S}}(m(t)) = \|m(t) - s(t)\|_{\mathbb{M}} \le \epsilon\sqrt{E_v(t)}$ and $\|\dot m(t) - v(t,m(t))\|_{\mathbb{M}} \le \sqrt{E_v(t)}$ to get from the second to the third line —,

$$\begin{aligned} I_5 &\le |\langle \dot m(t) - v(t,m(t)), D_t v(t,m(t)) - D_t v(t,s(t))\rangle| \\ &\le \mathrm{Lip}_{[0,\mathrm{T}]}(D_t v)\|\dot m(t) - v(t,m(t))\|_{\mathbb{M}}\|m(t) - s(t)\|_{\mathbb{M}} \\ &\le \epsilon\,\mathrm{Lip}_{[0,\mathrm{T}]}(D_t v) E_v(t)\end{aligned} \tag{3.9}$$

The quantity I_7 can be bounded using the same arguments,

$$\begin{aligned} I_7 &\le |\langle \dot m(t) - v(t,m(t)), \nabla p(t,m(t)) - \nabla p(t,s(t))\rangle| \\ &\le \mathrm{Lip}_{[0,\mathrm{T}]}(\nabla p)\|\dot m(t) - v(t,m(t))\|_{\mathbb{M}}\|m(t) - s(t)\|_{\mathbb{M}} \\ &\le \epsilon\,\mathrm{Lip}_{[0,\mathrm{T}]}(\nabla p) E_v(t).\end{aligned} \tag{3.10}$$

Finally to estimate I_8 and J we can assume that $\int_\Omega p(t,x)\mathrm{d}x = 0$ since the pressure is defined up to a constant. Using that $s(t)$ is measure-preserving, this gives

$$\begin{aligned}\int_\Omega D_t p(t,s(t,x))dx &= \int_\Omega \partial_t p(t,s(t,x)) + \langle v(t,s(t,x)), \nabla p(t,s(t,x)))\rangle\, dx \\ &= \int_\Omega \partial_t p(t,x))dx + \int_\Omega \langle v(t,x), \nabla p(t,x))\rangle\, dx = 0,\end{aligned}$$

Therefore, using Young's inequality,

$$\begin{aligned} I_8 &\le \left|\int_\Omega D_t p(t,m(t,x))dx - \int_\Omega D_t p(t,s(t,x))dx\right| \le \mathrm{Lip}_{[0,\mathrm{T}]}(D_t p)\|m(t) - s(t)\|_{L^1(\Omega)} \\ &\le \frac{1}{2}\frac{||m(t) - s(t)||^2_{L^2(\Omega)}}{2\epsilon^2} + C\,\mathrm{Lip}_{[0,\mathrm{T}]}(D_t p)\epsilon^2 \\ &\le \frac{1}{2}E_v(t) + \mathrm{cst}(\Omega)\,\mathrm{Lip}_{[0,\mathrm{T}]}(D_t p)\epsilon^2,\end{aligned} \tag{3.11}$$

where in this estimates and in the following estimates $\mathrm{cst}(\Omega)$ is a constant depending only on $\mathrm{Leb}(\Omega)$. Similarly

$$\begin{aligned}|J(t)| &\le \left|\int_\Omega p(t,m(t,x))) - p(t,s(t,x)))dx\right| \le \mathrm{Lip}_{[0,\mathrm{T}]}(p)||m(t) - s(t)||_{L^1(\Omega)} \\ &\le \frac{1}{2}E_v(t) + \mathrm{cst}(\Omega)\,\mathrm{Lip}_{[0,\mathrm{T}]}(p)\epsilon^2.\end{aligned} \tag{3.12}$$

Remark also that

$$|J(0)| \le \mathrm{Lip}_{[0,\mathrm{T}]}(p) h_N. \tag{3.13}$$

Remark 3.6. The two last estimates show that we can add $\frac{\mathrm{d}}{\mathrm{d}t}J$ into the Gronwall argument. It is a general fact that the derivative of a controlled quantity can be added. This is a classical way of controlling the term of order one in the energy.

3.3. **Gronwall argument.** Collecting estimates (3.5), (3.6), (3.9), (3.10), (3.11) and (3.12) we get

$$\begin{aligned}&\frac{d}{dt}\left(E_v(t)+J(t)\right) \leq I_3+I_4+I_5+I_7+I_8+J(t)-J(t)\\&\leq \left[2\,\mathrm{Lip}_{[0,\mathrm{T}]}(v)+\epsilon\,\mathrm{Lip}_{[0,\mathrm{T}]}(D_t v)+\epsilon\,\mathrm{Lip}_{[0,\mathrm{T}]}(\nabla p)+1\right]\left(E_v(t)+J(t)\right)\\&+\mathrm{cst}(\Omega)\left(\mathrm{Lip}_{[0,\mathrm{T}]}(D_t p)+\mathrm{Lip}_{[0,\mathrm{T}]}(p)\right)\epsilon^2\end{aligned}$$

Setting

$$\begin{cases}\widetilde{C}_1 &= 2\,\mathrm{Lip}_{[0,\mathrm{T}]}(v)+\epsilon\,\mathrm{Lip}_{[0,\mathrm{T}]}(D_t v)+\epsilon\,\mathrm{Lip}_{[0,\mathrm{T}]}(\nabla p)+1,\\ \widetilde{C}_2 &= \left(\mathrm{Lip}_{[0,\mathrm{T}]}(D_t p)+\mathrm{Lip}_{[0,\mathrm{T}]}(p)\right),\end{cases}$$

we obtain

$$\frac{d}{dt}\left(E_v(t)+J(t)\right) \leq \mathrm{cst}(\Omega)\widetilde{C}_1(E_v(t)+J(t))+\mathrm{cst}(\Omega)\widetilde{C}_2\epsilon^2.$$

We deduce that for any $t\in[0,T]$:

$$E_v(t) \leq \left((E_v(0)+J(0))+\mathrm{cst}(\Omega)\widetilde{C}_2 T\epsilon^2\right)e^{\widetilde{C}_1 T}-J(t)$$

Using one more time (3.12) we obtain

$$E_v(t)\leq 2\left(E_v(0)+\mathrm{Lip}_{[0,\mathrm{T}]}(p)h_N+\mathrm{cst}(\Omega)\widetilde{C}_2 T\epsilon^2\right)e^{\widetilde{C}_1 T}+\mathrm{cst}(\Omega)\,\mathrm{Lip}_{[0,\mathrm{T}]}(p)\epsilon^2.$$

Finally using that

$$E_v(0)=\frac{1}{2}\|P_{\mathbb{M}}(v_0)-v_0\|^2_{\mathbb{M}}+\frac{\mathrm{d}^2_{\mathbb{S}}(\mathrm{Id})}{2\epsilon^2}\leq\frac{h_N^2}{2}+\frac{h_N^2}{2\epsilon^2}$$

and

$$\begin{aligned}\|\dot m(t)-v(t,\phi(t))\|^2_{\mathbb{M}} &\leq 2\,\|\dot m(t)-v(t,m(t))\|^2_{\mathbb{M}}+\|v(t,m(t))-v(t,\phi(t))\|^2_{\mathbb{M}}\\&\leq 2E_v(t)+2(\mathrm{Lip}_{[0,\mathrm{T}]}(v))^2\,\|m(t)-\phi(t)\|^2_{\mathbb{M}}\\&\leq 2E_v(t)+2(\mathrm{Lip}_{[0,\mathrm{T}]}(v))^2\mathrm{d}^2_{\mathbb{S}}(m(t))\\&\leq 2(1+(\mathrm{Lip}_{[0,\mathrm{T}]}(v))^2\epsilon^2)E_v(t).\end{aligned}\tag{3.14}$$

we conclude

$$\begin{aligned}\|\dot m(t)-v(t,\phi(t))\|^2_{\mathbb{M}} &\leq (2+(\mathrm{Lip}_{[0,\mathrm{T}]}(v))^2\epsilon^2)E_v(t)\\&\leq 2(1+(\mathrm{Lip}_{[0,\mathrm{T}]}(v))^2\epsilon^2)\left[2\left(\frac{h_N^2}{2}+\frac{h_N^2}{2\epsilon^2}+\mathrm{Lip}_{[0,\mathrm{T}]}(p)h_N+\mathrm{cst}(\Omega)\widetilde{C}_2T\epsilon^2\right)e^{\widetilde{C}_1T}\right.\\&\left.+\ \mathrm{cst}(\Omega)\,\mathrm{Lip}_{[0,\mathrm{T}]}(p)\epsilon^2\right] &(3.15)\\&\leq C_1\frac{h_N^2}{\epsilon^2}+C_2\epsilon^2+C_3h_N &(3.16)\end{aligned}$$

where

$$\begin{cases}C_1 &= 2(1+(\mathrm{Lip}_{[0,\mathrm{T}]}(v))^2\epsilon^2)e^{\widetilde{C}_1T}\\ C_2 &= \mathrm{cst}(\Omega)(1+(\mathrm{Lip}_{[0,\mathrm{T}]}(v))^2)\left(\widetilde{C}_2Te^{\widetilde{C}_1T}+\mathrm{Lip}_{[0,\mathrm{T}]}(p)\right)\\ C_3 &= \mathrm{cst}(\Omega)(1+(\mathrm{Lip}_{[0,\mathrm{T}]}(v))^2\epsilon^2)\,\mathrm{Lip}_{[0,\mathrm{T}]}(p)\end{cases}$$

where we used that ϵ and h_N are smaller than $\mathrm{cst}(\Omega)$. Observe that the RHS of (3.16) goes to zero as $\frac{h_N}{\epsilon}$ and ϵ goes to zero. It finishes the proof of Theorem 1.2. In order to track down the regularity assumptions, we give the value of $\widetilde{C}_1$, $\widetilde{C}_2$ in term of the data:

$$\begin{aligned}
\widetilde{C}_1 &= 1 + 2\,\mathrm{Lip}_{[0,\mathrm{T}]}(v) + \epsilon\,\mathrm{Lip}_{[0,\mathrm{T}]}(\nabla p) \\
&\quad + \epsilon\left(\mathrm{Lip}_{[0,\mathrm{T}]}(\partial_t v) + (\mathrm{Lip}_{[0,\mathrm{T}]}(v))^2 + \mathrm{Lip}_{[0,\mathrm{T}]}(\nabla v)\|v\|_{L^\infty}\right), \\
\widetilde{C}_2 &= \left(\mathrm{Lip}_{[0,\mathrm{T}]}(D_t p) + \mathrm{Lip}_{[0,\mathrm{T}]}(p)\right) \\
&= \mathrm{Lip}_{[0,\mathrm{T}]}(p) + \mathrm{Lip}_{[0,\mathrm{T}]}(\partial_t p) + \mathrm{Lip}_{[0,\mathrm{T}]}(v)\,\mathrm{Lip}_{[0,\mathrm{T}]}(p) + \mathrm{Lip}_{[0,\mathrm{T}]}(\nabla p)\|v\|_{L^\infty}.
\end{aligned}$$

Remark 3.7. A close look to the explicit value of $\widetilde{C}_1$, $\widetilde{C}_2$ and estimation (3.15), together with a diagonal argument shows that our scheme approximate solutions less regular than supposed in Theorem 1.2. For example we can set the following theorem: Let v, p be a solution of Euler's equation (1.1). Suppose that v is Lipschitz in space. Suppose although that there exists $(v_k, p_k)_{k\in\mathbb{N}}$ a sequence of regular (in the sense of Theorem 1.2) solutions of (1.1) such that $v_k(0,\cdot) \longrightarrow v(0,\cdot)$ in $\mathbb{M}$ and $\mathrm{Lip}_T(v_k) \longrightarrow \mathrm{Lip}_T(v)$. Then there exists $h_N(k)$ and $\epsilon(k)$, polynomials in the data, such that $\|\dot{m}(t)[v_k(0),\epsilon(k)] - v\,(t, m(t)[v_k(0),\epsilon(k)])\|^2_{\mathbb{M}}$ goes to zero as k goes to infinity.

With an exponential dependance on the data we can approach the L^2 adherence of all the regular velocity fields solutions to Euler equation.

4. Convergence of the Euler symplectic numerical scheme

In this section we prove a more general version of Theorem 1.3 allowing a kind of a posteriori estimates. The proof follows the one given in 3.1 for Theorem 1.2 with some additional terms. It combined two Gronwall estimates. The first one is a continuos Gronwall argument on the segment $[n\tau, (n+1)\tau]$, the second one is a discrete Gronwall argument. For both steps we use the modulated energy.

Theorem 4.1. *Let Ω be a connected prox-regular set, ϵ and τ be positive numbers. Let v, p be a strong solution of (1.1), let ϕ be the flow map induced by v given by (1.2) and assume that $v, p, \partial_t v, \partial_t p, \nabla v$ and ∇p are Lipschitz on Ω, uniformly on $[0,T]$. Let $(M^n, V^n)_{n\geq 0}$ be a sequence generated by (1.7) with initial conditions*

$$M^0 = P_{\mathbb{M}_N}(\mathrm{id}),\ \ V^0 = P_{\mathbb{M}_N}(v(0,\cdot)).$$

Finally let

$$H^n = H(M^n, V^n) = \frac{1}{2}\|V^n\|^2_{\mathbb{M}} + \frac{d^2_{\mathbb{S}}(M^n)}{2\epsilon^2},$$

and

$$\kappa = \max_{n\in\mathbb{N}\cap[0,T/\tau]}\left(H^n - H^0\right).$$

Then,

$$\max_{n\in\mathbb{N}\cap[0,T/\tau]}\|V^n - v(t^n, \phi(t^n,\cdot))\|_{\mathbb{M}} \leq C_{(h_N\epsilon^{-1},\tau\epsilon^{-1})}\left[\epsilon^2 + h_N + \frac{h_N^2}{\epsilon^2} + \frac{\tau}{\epsilon} + \kappa\right]$$

where the constant C only depends on upper bounds of $\tau\epsilon^{-1}$ and $h_N\epsilon^{-1}$, on the proximal constant of the domain, the measure of the domain $\mathrm{Leb}(\Omega)$, the dimension d, on the L^∞ norm (in space) of the velocity $v(t,\cdot)$ and on the Lipschitz norms (in space) of the velocity and its first derivatives $v(t,\cdot), \nabla v(t,\cdot), \partial_t v(t,\cdot)$ and of the pressure and its derivatives $p(t,\cdot), \nabla p(t,\cdot), \partial_t p(t,\cdot)$.

We now prove Theorem 4.1. The proof follows the one given in 3.1 for Theorem 1.2 with some additional terms. It combined two Gronwall estimates. The first one is a continuos Gronwall argument on the segment $[n\tau, (n+1)\tau]$, the second one is a discrete Gronwall argument. For both steps we use the modulated energy.

For a solution of (1.7) and $s \in [0,1]$ we denote

$$\begin{cases} V^{n+s} & = V^n - s\tau \frac{M^n - P_{\mathbb{M}} \circ P_{\mathbb{S}}(M^n)}{\epsilon^2} \\ M^{n+s} & = M^n + s\tau V^{n+1}, \end{cases} \tag{4.1}$$

the linear interpolation between (M^n, V^n) and (M^{n+1}, V^{n+1}).

4.1. **Preliminary lemma.** We start with a lemma quantifying the conservation of the Hamiltonian.

Lemma 4.2 (Conservation of the Hamiltonian)**.** *For $s \in [0,1]$ and $n \in \mathbb{N} \cap [0, T/\tau]$ there holds*

$$\left(1 - \frac{\tau^2}{\epsilon^2}\right) H^{n+1} \leq H^n, \tag{4.2}$$

$$H^n \leq e^{T\tau\epsilon^{-2}} \left(\frac{1}{2}\|V^0\|^2_{\mathbb{M}} + \frac{h_N^2}{2\epsilon^2}\right), \tag{4.3}$$

and

$$H^{n+s} \leq H^n + \frac{\tau^2}{\epsilon^2}\left(\frac{1}{2}\|V^0\|^2_{\mathbb{M}} + \frac{h_N^2}{2\epsilon^2}\right) e^{T\tau\epsilon^{-2}}, \tag{4.4}$$

Proof. The proof is based on the 1-semiconcavity of $\frac{1}{2}d^2_{\mathbb{S}}$, see Proposition 5.2 for details. On the one hand the 1-semiconcavity of $\frac{1}{2}d^2_{\mathbb{S}}$ reads

$$\frac{d^2_{\mathbb{S}}(M^{n+s})}{2\epsilon^2} \leq \frac{d^2_{\mathbb{S}}(M^n)}{2\epsilon^2} + s\tau\left\langle V^{n+1}, \frac{M^n - P_{\mathbb{M}} \circ P_{\mathbb{S}}(M^n)}{\epsilon^2}\right\rangle + \frac{s^2\tau^2}{2\epsilon^2}\|V^{n+1}\|^2_{\mathbb{M}},$$

where we used that $[M^n - P_{\mathbb{M}} \circ P_{\mathbb{S}}(M^n)] \in \nabla^- \mathrm{d}^2_{\mathbb{S}}(M^n)$ and (4.1). On the other hand, (4.1) again, leads to

$$\frac{\|V^{n+s}\|^2_{\mathbb{M}}}{2} = \frac{\|V^n\|^2_{\mathbb{M}}}{2} - s\tau\left\langle V^n, \frac{M^n - P_{\mathbb{M}} \circ P_{\mathbb{S}}(M^n)}{\epsilon^2}\right\rangle + s^2\tau^2\left\|\frac{M^n - P_{\mathbb{M}} \circ P_{\mathbb{S}}(M^n)}{\epsilon^2}\right\|^2_{\mathbb{M}}$$

Summing both equations and using (4.1) gives

$$H^{n+s} \leq H^n + \frac{\tau^2 s(s-1)}{\epsilon^2}\frac{\|M^n - P_{\mathbb{M}} \circ P_{\mathbb{S}}(M^n)\|^2_{\mathbb{M}}}{\epsilon^2} + s^2\frac{\tau^2}{\epsilon^2}\frac{\|V^{n+1}\|^2_{\mathbb{M}}}{2} \tag{4.5}$$

Taking $s = 1$ in (4.5) proves (4.2). The inequality (4.3) is a direct consequence of (4.2). Plugging this information in (4.5) leads

$$\begin{aligned} H^{n+s} &\leq H^n + \frac{\tau^2}{\epsilon^2}H^{n+1} && (4.6)\\ &\leq H^n + \frac{\tau^2}{\epsilon^2}\left(\frac{1}{2}\|V^0\|^2_{\mathbb{M}} + \frac{h_N^2}{2\epsilon^2}\right) e^{T\tau\epsilon^{-2}} && (4.7) \end{aligned}$$

□

4.2. **The modulated energy.** The Hamiltonian at a step n is

$$H^n = H(M^n, V^n) = \frac{1}{2}\|V^n\|^2_{\mathbb{M}} + \frac{d^2_{\mathbb{S}}(M^n)}{2\epsilon^2}.$$

The modulated energy at time $n\tau$ is

$$E^n = \frac{1}{2}\|V^n - v(n\tau, M^n)\|^2_{\mathbb{M}} + \frac{d^2_{\mathbb{S}}(M^n)}{2\epsilon^2}. \tag{4.8}$$

For $s \in [0,1]$ we consider

$$\begin{cases} H^{n+s} & = \frac{1}{2} \left\| V^{n+s} \right\|_{\mathbb{M}}^2 + \frac{\mathrm{d}_{\mathbb{S}}^2(M^{n+s})}{2\epsilon^2}, \\ E^{n+s} & = \frac{1}{2} \left\| V^{n+s} - v\left(n\tau + s\tau, M^{n+s}\right) \right\|_{\mathbb{M}}^2 + \frac{\mathrm{d}_{\mathbb{S}}^2(M^{n+s})}{2\epsilon^2}. \end{cases} \tag{4.9}$$

Remark that

$$E^{n+s} = H^{n+s} - \left\langle V^{n+s}, v\left(n\tau + s\tau, M^{n+s}\right)\right\rangle + \frac{1}{2} \left\| v\left(n\tau + s\tau, M^{n+s}\right)\right\|_{\mathbb{M}}^2, \tag{4.10}$$

therefore for any $s \in [0,1]$ and $n \in \mathbb{N} \cap [0, T/\tau]$

$$E^{n+s} = E^n + H^{n+s} - H^n + \int_0^s d^{n+\theta} d\theta, \tag{4.11}$$

where

$$d^{n+s} = \frac{d}{ds}\left[- \left\langle V^{n+s}, v\left(n\tau + s\tau, M^{n+s}\right)\right\rangle + \frac{1}{2} \left\| v\left(n\tau + s\tau, M^{n+s}\right)\right\|_{\mathbb{M}}^2 \right].$$

We compute d^{n+s} using (4.1) and the notations $v_p^{n+s} = v(n\tau + s\tau, M^p)$, $s^{n+s} = P_{\mathbb{S}}(M^{n+s})$ and $v_{s^{n+s}}^{n+s} = v(n\tau + s\tau, s^{n+s})$.

Remark 4.3. The leading idea of the following computation is to make appear terms of quadratic order. To control the linear term we have to rewrite it as a derivative of a small quantity and add it in the Gronwall argument.

$$\begin{aligned} d^{n+s} &= - \left\langle \frac{d}{ds} V^{n+s}, v_{n+s}^{n+s} \right\rangle - \left\langle V^{n+s}, \frac{d}{ds} v_{n+s}^{n+s} \right\rangle + \left\langle v_{n+s}^{n+s}, \frac{d}{ds} v_{n+s}^{n+s} \right\rangle \\ &= \underbrace{\tau\epsilon^{-2} \left\langle M^n - P_{\mathbb{M}} \circ P_{\mathbb{S}}(M^n), v_{n+s}^{n+s} \right\rangle}_{I_1} \underbrace{- \left\langle V^{n+s} - v_{n+s}^{n+s}, \tau \partial_t v_{n+s}^{n+s} + \frac{d}{ds} M^{n+s} \cdot \nabla v_{n+s}^{n+s} \right\rangle}_{I_2} \end{aligned}$$

The term I_1 rewrites

$$\begin{aligned} \tau\epsilon^2 I_1 &= \left\langle M^n - P_{\mathbb{M}} \circ P_{\mathbb{S}}(M^n), v_{n+s}^{n+s} \right\rangle \\ &= \left\langle M^n - s^n, v_{n+s}^{n+s} \right\rangle + \left\langle s^n - P_{\mathbb{M}} \circ P_{\mathbb{S}}(M^n), v_{n+s}^{n+s} \right\rangle \\ &= \underbrace{\left\langle M^n - s^n, v_{n+s}^{n+s} - v_{s^n}^{n+s} \right\rangle}_{\tau\epsilon^2 I_3} \end{aligned}$$

Here we had to control the fact that, due to the double projection, the norm of the acceleration $\| M^n - P_{\mathbb{M}} \circ P_{\mathbb{S}}(M^n) \|_{\mathbb{M}}^2$ is not equal to $d_{\mathbb{S}}^2(M^n)$. We used the orthogonality property of the double projection to control this problem. On the one hand $s^n - P_{\mathbb{M}} \circ P_{\mathbb{S}}(M^n)$ is orthogonal to $\mathbb{M}$ since it is a linear subspace. On the other hand $M^n - s^n$ is orthogonal to the tangent space of $\mathbb{S}$ at s^n, see Lemma 3.1.

To handle I_2 we used the material derivatives defined by (3.4),

$$
\begin{aligned}
I_2 &= -\left\langle V^{n+s} - v^{n+s}_{n+s}, \tau\partial_t v^{n+s}_{n+s} + \frac{d}{ds}M^{n+s}\cdot\nabla v^{n+s}_{n+s}\right\rangle \\
I_2 &= -\left\langle V^{n+s} - v^{n+s}_{n+s}, \tau\partial_t v^{n+s}_{n+s} + \tau v^{n+s}_{n+s}\cdot\nabla v^{n+s}_{n+s}\right\rangle \\
&\underbrace{-\left\langle V^{n+s} - v^{n+s}_{n+s}, \left(\frac{d}{ds}M^{n+s} - \tau v^{n+s}_{n+s}\right)\cdot\nabla v^{n+s}_{n+s}\right\rangle}_{I_4} \\
&= I_4 \underbrace{-\tau\left\langle V^{n+s} - v^{n+s}_{n+s}, D_t v^{n+s}_{n+s} - D_t v^{n+s}_{s^{n+s}}\right\rangle}_{I_5} + \underbrace{\tau\left\langle V^{n+s} - v^{n+s}_{n+s}, \nabla p^{n+s}_{s^{n+s}}\right\rangle}_{I_6}.
\end{aligned}
$$

We used that $D_t v^{n+s}_{s^{n+s}} = -\nabla p^{n+s}_{s^{n+s}}$. We rewrite I_6 using $\frac{d}{ds}M^{n+s} = \tau V^{n+1}$.

$$
\begin{aligned}
I_6 &= \underbrace{\tau\left\langle V^{n+s} - v^{n+s}_{n+s}, \nabla p^{n+s}_{s^{n+s}} - \nabla p^{n+s}_{n+s}\right\rangle}_{I_7} + \tau\left\langle V^{n+s} - v^{n+s}_{n+s}, \nabla p^{n+s}_{n+s}\right\rangle \\
&= I_7 + \left\langle \frac{d}{ds}M^{n+s}, \nabla p^{n+s}_{n+s}\right\rangle + \tau\left\langle V^{n+s} - V^{n+1}, \nabla p^{n+s}_{n+s}\right\rangle - \tau\left\langle v^{n+s}_{n+s}, \nabla p^{n+s}_{n+s}\right\rangle \\
&= I_7 + \frac{d}{ds}\underbrace{\int_\Omega p^{n+s}_{n+s}dx}_{-J^{n+s}} - \tau\int_\Omega \partial_t p^{n+s}_{n+s} - \tau\left\langle v^{n+s}_{n+s}, \nabla p^{n+s}_{n+s}\right\rangle dx \\
&+ \underbrace{(1-s)\tau^2\epsilon^{-2}\left\langle M^n - P_{\mathbb{M}}\circ P_{\mathbb{S}}(M^n), \nabla p^{n+s}_{n+s}\right\rangle}_{I_8} \\
&= I_7 + I_8 - \frac{d}{ds}J^{n+s} \underbrace{- \tau\int_\Omega D_t p^{n+s}_{n+s}dx}_{I_9},
\end{aligned}
$$

We need to estimate all the terms in the following formula.

$$
d^{n+s} = I_3 + I_4 + I_5 + I_7 + I_8 + I_9 - \frac{d}{ds}J^{n+s} \tag{4.12}
$$

4.3. **Gronwall estimates on $[n\tau,(n+1)\tau]$.** Using 4.1 and Young's inequality we obtain for I_3:

$$
\begin{aligned}
I_3 &= \tau\epsilon^{-2}\left\langle M^n - s^n, v^{n+s}_{n+s} - v^{n+s}_{s^n}\right\rangle \\
&\le \tau\,\mathrm{Lip}_{[0,\mathrm{T}]}(v)\frac{\|M^n - s^n\|_{\mathbb{M}}\|M^{n+s} - s^n\|_{\mathbb{M}}}{\epsilon^2} \\
&\le \tau\,\mathrm{Lip}_{[0,\mathrm{T}]}(v)\frac{\|M^n - s^n\|_{\mathbb{M}}\|M^{n+s} - M^n\|_{\mathbb{M}}}{\epsilon^2} + \tau\,\mathrm{Lip}_{[0,\mathrm{T}]}(v)\frac{\|M^n - s^n\|_{\mathbb{M}}\|M^n - s^n\|_{\mathbb{M}}}{\epsilon^2} \\
&\le \tau\,\mathrm{Lip}_{[0,\mathrm{T}]}(v)\left(\frac{\|M^n - s^n\|^2_{\mathbb{M}}}{\epsilon^2} + \tau\epsilon^{-1}\frac{\|M^n - s^n\|_{\mathbb{M}}}{\epsilon}\|V^{n+1}\|_{\mathbb{M}}\right) \\
&\le 2\tau\,\mathrm{Lip}_{[0,\mathrm{T}]}(v)E^n + \mathrm{Lip}_{[0,\mathrm{T}]}(v)\tau^3\epsilon^{-2}H^n \\
&\le 2\tau\,\mathrm{Lip}_{[0,\mathrm{T}]}(v)E^n + \mathrm{Lip}_{[0,\mathrm{T}]}(v)\tau^3\epsilon^{-2}(H^0+\kappa).
\end{aligned} \tag{4.13}
$$

Since $\frac{d}{ds}M^{n+s} = \tau V^{n+1}$, I_4 rewrites

$$
\begin{aligned}
\tau^{-1} I_4 &= -\left\langle V^{n+s} - v^{n+s}_{n+s}, \left(V^{n+1} - v^{n+s}_{n+s}\right) \cdot \nabla v^{n+s}_{n+s}\right\rangle \\
&= -\left\langle V^{n+s} - v^{n+s}_{n+s}, \left(V^{n+s} - v^{n+s}_{n+s}\right) \cdot \nabla v^{n+s}_{n+s}\right\rangle \\
&- \left\langle V^{n+s} - v^{n+s}_{n+s}, \left(V^{n+1} - V^{n+s}\right) \cdot \nabla v^{n+s}_{n+s}\right\rangle \\
&\leq ||\nabla v(n\tau + s\tau)||_{L^\infty(\Omega)} \left\|V^{n+s} - v^{n+s}_{n+s}\right\|^2_{\mathbb{M}} \\
&+ \tau(1-s)\epsilon^{-2}\left\langle V^{n+s} - v^{n+s}_{n+s}, \left(M^n - P_{\mathbb{M}} \circ P_{\mathbb{S}}(M^n)\right) \cdot \nabla v^{n+s}_{n+s}\right\rangle \\
&\leq \mathrm{Lip}_{[0,\mathrm{T}]}(v) E^{n+s} + \tau(1-s)\epsilon^{-2}\left\langle V^{n+s} - v^{n+s}_{n+s}, \left(M^n - s^n\right) \cdot \nabla v^{n+s}_{n+s}\right\rangle \\
&\leq \mathrm{Lip}_{[0,\mathrm{T}]}(v) \left(E^{n+s} + \tau\epsilon^{-1}\left\|V^{n+s} - v^{n+s}_{n+s}\right\|_{\mathbb{M}} \frac{\|M^n - s^n\|_{\mathbb{M}}}{\epsilon}\right) \\
&\leq \mathrm{Lip}_{[0,\mathrm{T}]}(v) \left((1+\tau\epsilon^{-1})E^{n+s} + \tau\epsilon^{-1}E^n\right) \\
&\leq \mathrm{Lip}_{[0,\mathrm{T}]}(v)(1+\tau\epsilon^{-1})E^{n+s} + \mathrm{Lip}_{[0,\mathrm{T}]}(v)\tau\epsilon^{-1}E^n
\end{aligned} \tag{4.14}
$$

We used that $\left\langle V^{n+s} - v^{n+s}_{n+s}, \left(s^n - P_{\mathbb{M}} \circ P_{\mathbb{S}}(M^n)\right) \cdot \nabla v^{n+s}_{n+s}\right\rangle = 0$ since $s^n - P_{\mathbb{M}} \circ P_{\mathbb{S}}(M^n)$ is orthogonal to $\mathbb{M}_N$ and the quantity ∇v^{n+s}_{n+s} is a symmetric operator from $\mathbb{M}_N$ to $\mathbb{M}_N$. At the antepenultimate line we used Young's inequality. The estimates of I_5 and I_7 are similar to the semi-discrete case.

$$
\begin{aligned}
\tau^{-1} I_5 &\leq \left|\left\langle V^{n+s} - v^{n+s}_{n+s}, D_t v^{n+s}_{n+s} - D_t v^{n+s}_{s^{n+s}}\right\rangle\right| \\
&\leq \mathrm{Lip}_{[0,\mathrm{T}]}(D_t v)\left\|V^{n+s} - v^{n+s}_{n+s}\right\|_{\mathbb{M}} \left\|M^{n+s} - s^{n+s}\right\|_{\mathbb{M}} \\
&\leq \mathrm{Lip}_{[0,\mathrm{T}]}(D_t v)\left\|V^{n+s} - v^{n+s}_{n+s}\right\|_{\mathbb{M}} \left\|M^{n+s} - s^{n+s}\right\|_{\mathbb{M}} \\
&\leq \epsilon\, \mathrm{Lip}_{[0,\mathrm{T}]}(D_t v) E^{n+s}
\end{aligned} \tag{4.15}
$$

The quantity I_7 is of the same kind.

$$
\begin{aligned}
\tau^{-1} I_7 &\leq \left|\left\langle V^{n+s} - v^{n+s}_{n+s}, \nabla p^{n+s}_{s^{n+s}} - \nabla p^{n+s}_{n+s}\right\rangle\right| \\
&\leq \epsilon\, \mathrm{Lip}_{[0,\mathrm{T}]}(\nabla p) E^{n+s}
\end{aligned} \tag{4.16}
$$

For the estimation of I_8 we use $\left\langle s^n - P_{\mathbb{M}} \circ P_{\mathbb{S}}(M^n), \nabla p^{n+s}_{n+s}\right\rangle = 0$ to get

$$
\begin{aligned}
I_8 &= (1-s)\tau^2\epsilon^{-2}\left\langle M^n - P_{\mathbb{M}} \circ P_{\mathbb{S}}(M^n), \nabla p^{n+s}_{n+s}\right\rangle \\
&\leq \tau^2\epsilon^{-2}||\nabla p(n\tau + s\tau)||_{L^\infty(\Omega)}||M^n - s^n||_{\mathbb{M}} \\
&\leq \tau^2\epsilon^{-1}\,\mathrm{Lip}_{[0,\mathrm{T}]}(v)\frac{||M^n - s^n||_{\mathbb{M}}}{\epsilon} \\
&\leq \tau^2\epsilon^{-1}E^n + \frac{1}{2}\tau^2\epsilon^{-1}\left(\mathrm{Lip}_{[0,\mathrm{T}]}(v)\right)^2.
\end{aligned} \tag{4.17}
$$

To estimate J and I_9 recall that $\int_\Omega D_t p(t, s^n(t,x))dx = 0$ and we set $\int_\Omega p(t,x)dx = 0$.

$$
\begin{aligned}
\tau^{-1} I_9 &\leq \mathrm{Lip}_{[0,\mathrm{T}]}(D_t p)||M^{n+s} - s^{n+s}||_{L^1(\Omega)} \\
&\leq \frac{1}{2}E^{n+s} + \left(\mathrm{Lip}_{[0,\mathrm{T}]}(D_t p)\right)^2\epsilon^2
\end{aligned} \tag{4.18}
$$

Similarly

$$
\begin{aligned}
|J^{n+s}| = |J(n\tau + \tau s)| &\leq \left|\int_\Omega p^{n+s}_{n+s} - p^{n+s}_{s^{n+s}}dx\right| \leq \mathrm{Lip}_{[0,\mathrm{T}]}(p)||M^{n+s} - s^{n+s}||_{L^1(\Omega)} \\
&\leq \frac{1}{2}E^{n+s} + \left(\mathrm{Lip}_{[0,\mathrm{T}]}(p)\right)^2\epsilon^2.
\end{aligned} \tag{4.19}
$$

Note also that $J^0 \leq \mathrm{Lip}_{[0,\mathrm{T}]}(p)h_N$ still holds see (3.13).

4.4. **Gronwall argument on** $[n\tau,(n+1)\tau]$**.** From now and for clarity we do not track the constants anymore, C will be a constant depending only on T, Ω, $\mathrm{Lip}_{[0,\mathrm{T}]}(v)$, $\mathrm{Lip}_{[0,\mathrm{T}]}(p)$, $\mathrm{Lip}_{[0,\mathrm{T}]}(\nabla p)$, $\mathrm{Lip}_{[0,\mathrm{T}]}(D_t v)$ and $\mathrm{Lip}_{[0,\mathrm{T}]}(D_t p)$. The constant C can change between estimates. Collecting estimates (4.13), (4.14), (4.15), (4.16), (4.17), (4.18) and (4.19) and intergreting equation (4.12) from 0 to s we obtain

$$J^{n+s}+\int_0^s d^{n+\theta}d\theta \leq J^n + 2\tau\,\mathrm{Lip}_{[0,\mathrm{T}]}(v)E^n + \mathrm{Lip}_{[0,\mathrm{T}]}(v)\tau^3\epsilon^{-2}(H^0+\kappa) \tag{4.20}$$

$$+\tau\,\mathrm{Lip}_{[0,\mathrm{T}]}(v)\int_0^s(1+\tau\epsilon^{-1})E^{n+\theta}d\theta + \mathrm{Lip}_{[0,\mathrm{T}]}(v)\tau^2\epsilon^{-1}E^n \tag{4.21}$$

$$+\tau\epsilon\,\mathrm{Lip}_{[0,\mathrm{T}]}(D_t v)\int_0^s E^{n+\theta}d\theta + \tau\epsilon\,\mathrm{Lip}_{[0,\mathrm{T}]}(\nabla p)\int_0^s E^{n+\theta}d\theta \tag{4.22}$$

$$+\tau^2\epsilon^{-1}E^n + \frac{1}{2}\tau^2\epsilon^{-1}\left(\mathrm{Lip}_{[0,\mathrm{T}]}(v)\right)^2$$

$$+\frac{\tau}{2}\int_0^s E^{n+\theta}d\theta + \tau\epsilon^2\left(\mathrm{Lip}_{[0,\mathrm{T}]}(D_t p)\right)^2$$

$$+\tau\int_0^s J^{n+\theta} - J^{n+\theta}ds$$

$$\leq J^n + C\tau(1+\tau\epsilon^{-1})E^n + C\tau\epsilon^2 + C\tau^2\epsilon^{-1} + (C+\kappa)\tau^3\epsilon^{-2} \tag{4.23}$$

$$+\tau C\int_0^s(1+\tau\epsilon^{-1})\left(E^{n+\theta}+J^{n+\theta}\right)d\theta.$$

Remark that we only kept the first order terms using $\epsilon \leq C$. Plugging (4.23) into (4.11) we obtain

$$\begin{aligned} E^{n+s}+J^{n+s} &\leq E^n + J^n + H^{n+s} - H^n \\ &+ C\tau(1+\tau\epsilon^{-1})E^n + C\tau\epsilon^2 + C\tau^2\epsilon^{-1} + (H^0+\kappa)\tau^3\epsilon^{-2} \\ &+ \tau C\int_0^s(1+\tau\epsilon^{-1})\left(E^{n+\theta}+J^{n+\theta}\right)d\theta. \end{aligned} \tag{4.24}$$

Let The Gronwall Lemma on $[0,1]$ implies

$$\begin{aligned} E^{n+s}+J^{n+s} \leq\ & \left[E^n + J^n + +C\tau(1+\tau\epsilon^{-1})E^n + C\tau\epsilon^2 + C\tau^2\epsilon^{-1}\right. \\ & \left.+(H^0+\kappa)\tau^3\epsilon^{-2}\right]e^{C\tau(1+\tau\epsilon^{-1})} \\ & + H^{n+s} - H^n + \underbrace{\int_0^s\left(H^{n+s}-H^n\right)C\tau(1+\tau\epsilon^{-1})e^{\int_\theta^1 C\tau(1+\tau\epsilon^{-1})\zeta d\zeta}}_{R} \end{aligned}$$

Using Lemma 4.2 and in particular (4.6) we find

$$\begin{aligned} R &\leq \frac{\tau^2}{\epsilon^2}H^{n+1}\int_0^s C\tau(1+\tau\epsilon^{-1})e^{C\tau(1+\tau\epsilon^{-1})(1-\theta)} \\ &\leq \frac{\tau^2}{\epsilon^2}(H^0+\kappa)\left[e^{C\tau(1+\tau\epsilon^{-1})}-1\right] \end{aligned}$$

Going back to (4.24) yields

$$\begin{aligned} E^{n+s}+J^{n+s} \leq\ & \left[\left(1+C\tau(1+\tau\epsilon^{-1})\right)\left(E^n+J^n\right)\right. \\ & \left.+C\tau\epsilon^2 + C\tau^2\epsilon^{-1} + (H^0+\kappa)\tau^3\epsilon^{-2}\right]e^{C\tau(1+\tau\epsilon^{-1})} \\ & + H^{n+s} - H^n + \tau^2\epsilon^{-2}(H^0+\kappa)\left[e^{C\tau(1+\tau\epsilon^{-1})}-1\right], \end{aligned}$$

and in particular

$$
\begin{aligned}
E^{n+1}+J^{n+1} \leq \Big[& \left(1+C\tau(1+\tau\epsilon^{-1})\right)(E^n+J^n) \\
& + C\tau\epsilon^2 + C\tau^2\epsilon^{-1} + (H^0+\kappa)\tau^3\epsilon^{-2} \\
& + H^{n+s} - H^n + \tau^2\epsilon^{-2}(H^0+\kappa)\left[e^{C\tau(1+\tau\epsilon^{-1})}-1\right]\Big] e^{C\tau(1+\tau\epsilon^{-1})},
\end{aligned}
\tag{4.25}
$$

4.5. **Discrete Gronwall step.** From (4.25) and the descrete Gronwall inequality we deduce, for any $n \in \mathbb{N} \cap [0, T/\tau]$:

$$
\begin{aligned}
E^n + J^n \leq \Big[& E^0 + J^0 + TC\epsilon^2 + CT\tau\epsilon^{-1} + (H^0+\kappa)\tau^2\epsilon^{-2} + H^n - H^0 \\
& + \tau^2\epsilon^{-2}(H^0+\kappa)\frac{T}{\tau}\left[e^{C\tau(1+\tau\epsilon^{-1})}-1\right]\Big] \left(1+C\tau(1+\tau\epsilon^{-1})\right)^n e^{CT(1+\tau\epsilon^{-1})} \\
\leq C\Big[& E^0 + J^0 + \epsilon^2 + \tau\epsilon^{-1} + \kappa \\
& + (H^0+\kappa)\tau^2\epsilon^{-2}(1+1+\tau\epsilon^{-1})e^{CT(1+\tau\epsilon^{-1})}\Big] e^{CT(1+\tau\epsilon^{-1})}.
\end{aligned}
$$

We used the mean value theorem to obtain the last line. Let $C_{h_N\epsilon^{-1},\tau\epsilon^{-1}}$ be a constant depending on the previous quantity but also an upper bound of $h_N\epsilon^{-1}$ and $\tau\epsilon^{-1}$. Using (4.19) one last time leads

$$
E^n \leq C_{(h_N\epsilon^{-1},\tau\epsilon^{-1})}\left[E^0 + J^0 + \epsilon^2 + \tau\epsilon^{-1} + \kappa + H^0\tau^2\epsilon^{-2}\right] + C\epsilon^2.
$$

Including the initial errors and rearranging the terms yields

$$
E^n \leq C_{(h_N\epsilon^{-1},\tau\epsilon^{-1})}\left[\epsilon^2 + h_N + \frac{h_N^2}{\epsilon^2} + \kappa + \frac{\tau}{\epsilon}\right].
$$

Using (3.14) we conclude

$$
\begin{aligned}
& \max_{n\in\mathbb{N}\cap[0,T/\tau]} \|V^n - v(t^n,\phi(t^n,\cdot))\|_{\mathbb{M}} \\
& \qquad \leq C \max_{n\in\mathbb{N}\cap[0,T/\tau]} E^n \leq C_{(h_N\epsilon^{-1},\tau\epsilon^{-1})}\left[\epsilon^2 + h_N + \frac{h_N^2}{\epsilon^2} + \kappa + \frac{\tau}{\epsilon}\right].
\end{aligned}
\tag{4.26}
$$

It finishes the proof of Theorem 4.1.

Remark 4.4. Lemma 4.2 gives an upper bound for κ in Theorem 4.1 namely

$$
\kappa \leq \sum_{i=0}^{T/\tau-1} \left|H^{i+1} - H^i\right| \leq \frac{\tau}{\epsilon^2} T e^{T\tau\epsilon^{-2}} \left(\frac{1}{2}\|V^0\|_{\mathbb{M}}^2 + \frac{h_N^2}{2\epsilon^2}\right).
$$

Using this upper bound Theorem 4.1 becomes Theorem 1.3 and the condition $\kappa = o(1)$ becomes $\tau = o(\epsilon^2)$. However numerically one can expect some compensation in H^n and thus obtain a better "a posteriori bound" for κ in order to get rid of the assumption $\tau = o(\epsilon^2)$, see Figure 5.4 for instance. Notice that we do not have real a posteriori estimate since the constants in Theorem 4.1 do not depend only on the numerical solution but also on the, unknown, limit solution. The condition $\tau = o(\epsilon)$ seems mandatory for this method to work.

Remark 4.5. A close look to the constant leads to a similar result as the one given in Remark 3.7: namely the convergence of the numerical scheme towards less regular solutions of the Euler's equations.

Remark 4.6. The method of the proof is robust and can be easily adapted to other numerical scheme. Better is the preservation of the Hamiltonian better will be the scheme, in particular it will improve Lemma 4.2.

5. Numerical implementation and experiments

5.1. **Numerical implementation.** We discuss here the implementation of the numerical scheme (1.7) and in particular the computation of the double projection $P_{\mathbb{M}_N} \circ P_{\mathbb{S}}(m)$ for a piecewise constant function $m \in \mathbb{M}_N$. Using Brenier's polar factorisation theorem, the projection of m on $\mathbb{S}$ amounts to the resolution of an optimal transport problem between Leb and the finitely supported measure $m_\#\,\mathrm{Leb}$. Such optimal transport problems can be solved numerically using the notion of Laguerre diagram from computational geometry.

Definition 5.1 (Laguerre diagram)**.** Let $M = (M_1, \dots, M_N) \in (\mathbb{R}^d)^N$ and let $\psi_1, \dots, \psi_N \in \mathbb{R}$. The Laguerre diagram is a decomposition of $\mathbb{R}^d$ into convex polyhedra defined by

$$\mathrm{Lag}_i(M, \psi) = \left\{ x \in \mathbb{R}^d \mid \forall j \in \{1, \dots, N\},\ \|x - M_i\|^2 + \psi_i \le \|x - M_i\|^2 + \psi_j \right\}.$$

In the following proposition, we denote $\Pi_{\mathbb{S}}(m) = \{s \in \mathbb{S} \mid \|m - s\| = \mathrm{d}_{\mathbb{S}}(m)\}$.

Proposition 5.2. *Let $m \in \mathbb{M}_N \setminus \mathbb{D}_N$ and define $M_i = m(\omega_i) \in \mathbb{R}^d$. There exist scalars $(\psi_i)_{1 \le i \le N}$, which are unique up to an additive constant, such that*

$$\forall i \in \{1, \dots, N\}, \qquad \mathrm{Leb}(\mathrm{Lag}_i(M, \psi)) = \frac{1}{N} \mathrm{Leb}(\Omega) \tag{5.1}$$

We denote $L_i := \mathrm{Lag}_i(M, \psi)$.Then, a function $s \in \mathbb{S}$ is a projection of m on $\mathbb{S}$ if and only if it maps the subdomain ω_i to the Laguerre cell L_i up to a negligible set, that is:

$$\Pi_{\mathbb{S}}(m) = \{s \in \mathbb{S} \mid \forall i \in \{1, \dots, N\},\ \mathrm{Leb}(s(\omega_i) \Delta L_i) = 0\} \tag{5.2}$$

where Δ denotes the symmetric difference. Moreover, $\mathrm{d}^2_{\mathbb{S}}(m)$ is differentiable at m and, setting $B_i = \frac{1}{\mathrm{Leb}(L_i)} \int_{L_i} x \mathrm{d}x$,

$$\begin{aligned} &\mathrm{d}^2_{\mathbb{S}}(m) = \sum_{1 \le i \le N} \int_{L_i} \|x - M_i\|^2 \, \mathrm{d}x \\ &\nabla \mathrm{d}^2_{\mathbb{S}}(m) = 2(m - P_{\mathbb{M}_N} \circ P_{\mathbb{S}}(m)) \text{ with } P_{\mathbb{M}_N} \circ P_{\mathbb{S}}(m) = \sum_{1 \le i \le N} B_i \mathbf{1}_{L_i}. \end{aligned} \tag{5.3}$$

Proof. The existence of a vector $(\psi_i)_{1 \le i \le N}$ satisfying Equation (5.1) follows from optimal transport theory (see Section 5 in [3] for a short proof), and its uniqueness follows from the connectedness of the domain Ω. In addition, the map $T : \Omega \to \{M_1, \dots, M_N\}$ defined by $T(L_i) = M_i$ (up to a negligible set) is the gradient of a convex function and therefore a quadratic optimal transport between Leb and the measure $\frac{\mathrm{Leb}(\Omega)}{N} \sum_i \delta_{M_i}$. By Brenier's polar factorization theorem, summarized in Lemma 3.1,

$$\begin{aligned} s \in \Pi_{\mathbb{S}}(m) \iff m = T \circ s \text{ a.e.} &\iff \forall i \in \{1, \dots, N\},\ \mathrm{Leb}(\omega_i \Delta (T \circ s)^{-1}(\{M_i\})) = 0 \\ &\iff \forall i \in \{1, \dots, N\},\ \mathrm{Leb}(s(\omega_i) \Delta L_i) = 0, \end{aligned}$$

where the last equality holds because s is measure preserving. To prove the statement on the differentiability of $\mathrm{d}^2_{\mathbb{S}}$, we first note that the function $\mathrm{d}^2_{\mathbb{S}}$ is 1-semi-concave, since

$$D(m) := \|m\|^2 - \mathrm{d}^2_{\mathbb{S}}(m) = \|m\|^2 - \min_{s \in \mathbb{S}} \|m - s\|^2 = \max_{s \in \mathbb{S}} 2\langle m|s\rangle - \|s\|^2$$

is convex. The subdifferential of D at m is given by $\partial D(m) = \{P_{\mathbb{M}_N}(s) \mid s \in \Pi_{\mathbb{S}}(m)\}$, so that D (and hence $\mathrm{d}^2_{\mathbb{S}}$) is differentiable at m if and only if $P_{\mathbb{M}_N}(\Pi_{\mathbb{S}}(m))$ is a singleton. Now, note from Lemma 3.2 that for $s \in \Pi_{\mathbb{S}}(m)$

$$P_{\mathbb{M}_N}(s) = \sum_{1 \le i \le N} \mathrm{bary}(s(\omega_i)) \mathbf{1}_{\omega_i} = \sum_{1 \le i \le N} \mathrm{bary}(L_i) \mathbf{1}_{\omega_i}.$$

This shows that $P_{\mathbb{M}_N}(\Pi_{\mathbb{S}}(m))$ is a singleton, and therefore establishes the differentiability of $\mathrm{d}^2_{\mathbb{S}}$ at m, together with the desired formula for the gradient. □

The difficulty to implement the numerical scheme (1.7) is the resolution of the discrete optimal transport problem (5.1), a non-linear system of equations which must be solved at every iteration. We resort to the damped Newton's algorithm presented in [11] (see also [16]) and more precisely on its implementation in the open-source PyMongeAmpere library[1].

5.1.1. *Construction of the tessellation of the domain.* The fixed tessellation $(\omega_i)_{1\leq i\leq N}$ of the domain Ω is a collection of Laguerre cells that are computed through a simple fixed-point algorithm similar to the one presented in [8]. We start from a random sampling $(C_i^0)_{1\leq i\leq N}$ of Ω. At a given step $k\geq 0$, we compute $(\psi_i)_{1\leq i\leq N}\in\mathbb{R}^N$ such that

$$\forall i\in\{1,\dots,N\},\ \mathrm{Leb}(\mathrm{Lag}_i(C,\psi)) = \frac{1}{N}\mathrm{Leb}(\Omega),$$

and we then update the new position of the centers (C_i^{k+1}) by setting $C_i^{k+1} := \mathrm{bary}(\mathrm{Lag}_i(C^k,\psi))$. After a few iterations, a fixed-point is reached and we set $\omega_i := \mathrm{Lag}_i(C^k,\psi)$.

5.1.2. *Iterations.* To implement the symplectic Euler scheme for (1.6), we start with $M_i^0 := \mathrm{bary}(\omega_i)$ and $V_i^0 := v_0(M_i^0)$. Then, at every iteration $k\geq 0$, we use Algorithm 1 in [11] to compute a solution $(\psi_i^k)_{1\leq i\leq N}\in\mathbb{R}^N$ to Equation (5.1) with $M = M^k$, i.e. such that

$$\forall i\in\{1,\dots,N\},\ \mathrm{Leb}(\mathrm{Lag}_i(M^k,\psi^k)) = \frac{1}{N}\mathrm{Leb}(\Omega).$$

Finally, we update the positions $(M_i^{k+1})_{1\leq i\leq N}$ and the speeds $(V_i^{k+1})_{1\leq i\leq N}$ by setting

$$\begin{cases} V_i^{k+1} = V_i^k + \frac{\tau}{\varepsilon^2}(\mathrm{bary}(\mathrm{Lag}_i(M^k,\psi^k) - M_i^k) \\ M_i^{k+1} = M_i^k + \tau V_i^{k+1} \end{cases} \tag{5.4}$$

5.2. **Beltrami flow in the square.** Our first testcase is constructed from a stationary solution to Euler's equation in 2D. On the unit square $\Omega = [-\frac{1}{2},\frac{1}{2}]^2$, we consider the Beltrami flow constructed from the time-independent pressure and speed:

$$\begin{cases} p_0(x_1,x_2) = \frac{1}{2}(\sin(\pi x_1)^2 + \sin(\pi x_2)^2) \\ v_0(x_1,x_2) = (-\cos(\pi x_1)\sin(\pi x_2), \sin(\pi x_1)\cos(\pi x_2)) \end{cases}$$

In Figure 1, we display the computed numerical solution using a low number of particles ($N = 900$) in order to show the shape of the Laguerre cells associated to the solution.

5.3. **Kelvin-Helmoltz instability.** For this second testcase, the domain is the rectangle $\Omega = [0,2]\times[-.5,.5]$ periodized in the first coordinate by making the identification identification $(4,x_2)\sim(0,x_2)$ for $x_2\in[-.5,.5]$. The initial speed v_0 is discontinuous at $x_2 = 0$: the upper part of the domain has zero speed, and the bottom part has unit speed:

$$v_0(x_1,x_2) = \begin{cases} 0 \text{ if } x_2\geq 0 \\ 1 \text{ if } x_2 < 0 \end{cases}$$

This speed profile corresponds to a stationnary but unstable solution to Euler's equation. If the subdomains $(\omega_i)_{1\leq i\leq N}$ are computed following §5.1.1, the perfect symmetry under horizontal translations is lost, and in Figure 2 we observe the formation of vortices whose radius increases with time. This experiment involves $N = 300\,000$ particles, with parameters $\tau = 0.005$ and $\varepsilon = 0.0025$, and $2\,000$ timesteps.

[1] https://github.com/mrgt/PyMongeAmpere

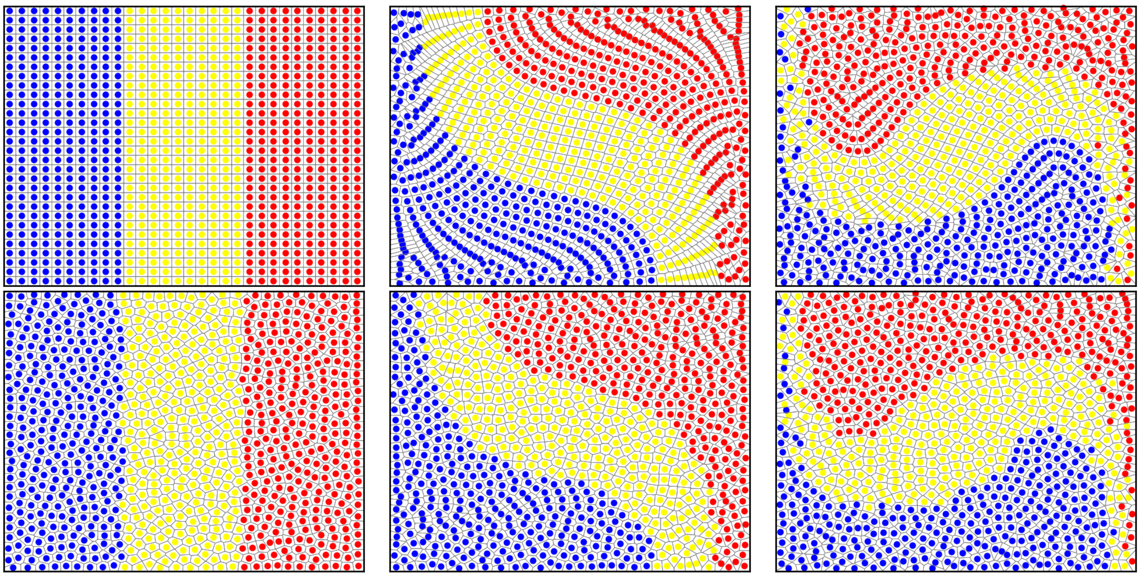

FIGURE 1. (Top row) Beltrami flow in the square, with $N = 900$ particles, $\tau = 1/50$ and $\varepsilon = .1$. The particles are colored depending on their initial position in the square. From left to right, we display the Laguerre cells and their barycenters at timesteps $k = 0, 24$ and 49. The partition $(\omega_i)_{1\leq i\leq N}$ is induced by a regular grid. (Bottom row) Same experiment, but where the partition $(\omega_i)_{1\leq i\leq N}$ is optimized using the algorithm described in §5.1.1.

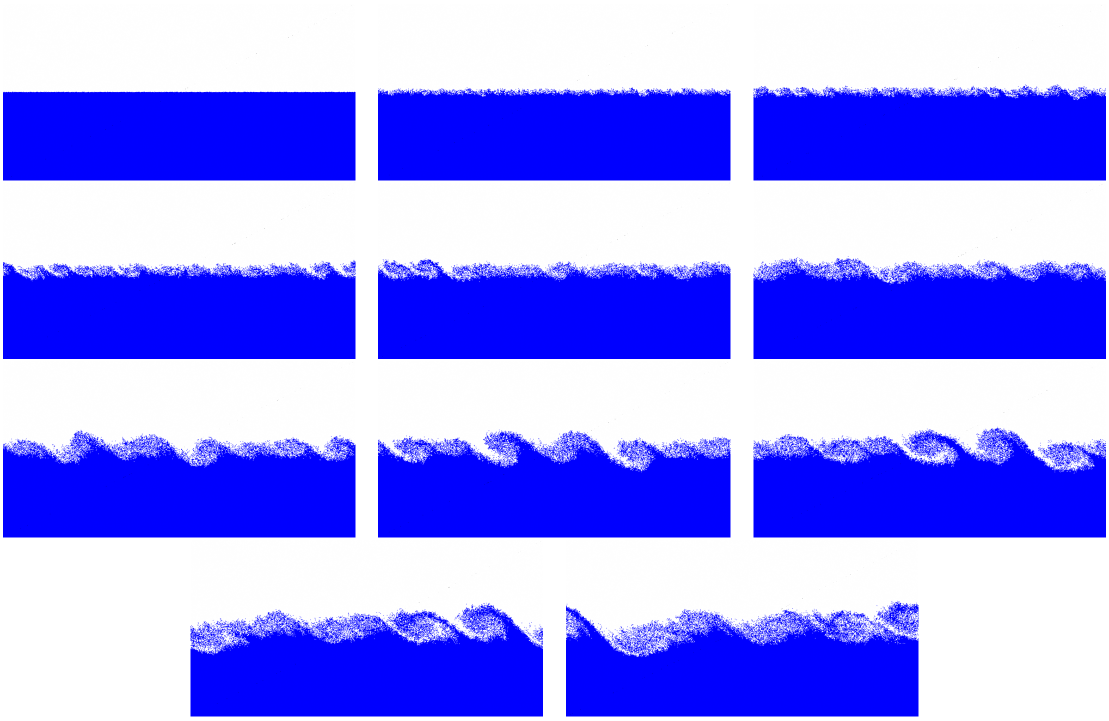

FIGURE 2. Numerical illustration of the Kelvin-Helmotz instability on a rectangle with periodic conditions (in the horizontal coordinate) involving a discontinuous initial speed. The parameters chosen for this experiment are given in §5.4.

5.4. **Rayleigh-Taylor instability.** For this last testcase, the particles are assigned a density ρ_i, and are subject to the force of the gravity $\rho_i G$, where $G = (0, -10)$. This changes the numerical scheme to

$$\begin{cases} V_i^{k+1} = V_i^k + \tau \left(\frac{1}{\varepsilon^2} (\mathrm{bary}(\mathrm{Lag}_i(M^k, \psi^k) - M_i^k) + \rho_i G \right) \\ M_i^{k+1} = M_i^k + \tau V_i^{k+1} \end{cases} \tag{5.5}$$

The computational domain is the rectangle $\Omega = [-1, 1] \times [-3, 3]$, and the initial distribution of particles is given by $C_i = \mathrm{bary}(\omega_i)$, where the partition $(\omega_i)_{1 \leq i \leq N}$ is constructed according to §5.1.1. The fluid is composed of two phases, the heavy phase being on top of the light phase:

$$\rho_i = \begin{cases} 3 & \text{if } C_{i2} > \eta \cos(\pi C_{i1}) \\ 1 & \text{if } C_{i2} \leq \eta \cos(\pi C_{i1}) \end{cases},$$

where $\eta = 0.2$ in the experiment and where we denoted C_{i1} and C_{i2} the first and second coordinates of the point C_i. Finally, we have set $N = 50\,000$, $\varepsilon = 0.002$ and $\tau = 0.001$ and we have run 2000 timesteps. The computation takes less than six hours on a single core of a regular laptop. Note that it does not seem straighforward to adapt the techniques used in the proofs of convergence presented here to this setting, where the force depends on the density of the particle. Our purpose with this testcase is only to show that the numerical scheme behaves reasonably well in more complex situations.

xs

References

[1] L. Ambrosio. Transport equation and cauchy problem for BV vector fields. *Inventiones mathematicae*, 158(2):227–260, 2004.

[2] V. Arnold. Sur la géométrie différentielle des groupes de lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. *Annales de l'institut Fourier*, 16(1):319–361, 1966.

[3] F. Aurenhammer, F. Hoffmann, and B. Aronov. Minkowski-type theorems and least-squares clustering. *Algorithmica*, 20(1):61–76, 1998.

[4] F. Bouchut. Renormalized solutions to the vlasov equation with coefficients of bounded variation. *Archive for rational mechanics and analysis*, 157(1):75–90, 2001.

[5] Y. Brenier. The least action principle and the related concept of generalized flows for incompressible perfect fluids. *Journal of the American Mathematical Society*, 1989.

[6] Y. Brenier. Polar factorization and monotone rearrangement of vector-valued functions. *Communications on pure and applied mathematics*, 44(4):375–417, 1991.

[7] Y. Brenier. Derivation of the Euler equations from a caricature of Coulomb interaction. *Communications in Mathematical Physics*, 212(1):93–104, 2000.

[8] F. de Goes, K. Breeden, V. Ostromoukhov, and M. Desbrun. Blue noise through optimal transport. *ACM Transactions on Graphics (TOG)*, 31(6):171, 2012.

[9] F. de Goes, C. Wallez, J. Huang, D. Pavlov, and M. Desbrun. Power particles: an incompressible fluid solver based on power diagrams. *ACM Transactions on Graphics (TOG)*, 34(4):50, 2015.

[10] H. Federer. Curvature measures. *Transactions of the American Mathematical Society*, 93(3):418–491, 1959.

[11] J. Kitagawa, Q. Mérigot, and B. Thibert. Convergence of a newton algorithm for semi-discrete optimal transport. *arXiv preprint arXiv:1603.05579*, 2016.

[12] B. Lévy. A numerical algorithm for L^2 semi-discrete optimal transport in 3d. *ESAIM M2AN*, 49(6), 2015.

[13] P.-L. Lions. Sur les équations différentielles ordinaires et les équations de transport. *Comptes Rendus de l'Académie des Sciences-Series I-Mathematics*, 326(7):833–838, 1998.

[14] Q. Mérigot. A multiscale approach to optimal transport. *Computer Graphics Forum*, 30(5):1583–1592, 2011.

[15] Q. Mérigot and J.-M. Mirebeau. Minimal geodesics along volume preserving maps, through semi-discrete optimal transport. *arXiv preprint arXiv:1505.03306*, 2015.

[16] J.-M. Mirebeau. Discretization of the 3d monge-ampere operator, between wide stencils and power diagrams. *arXiv preprint arXiv:1503.00947*, 2015.

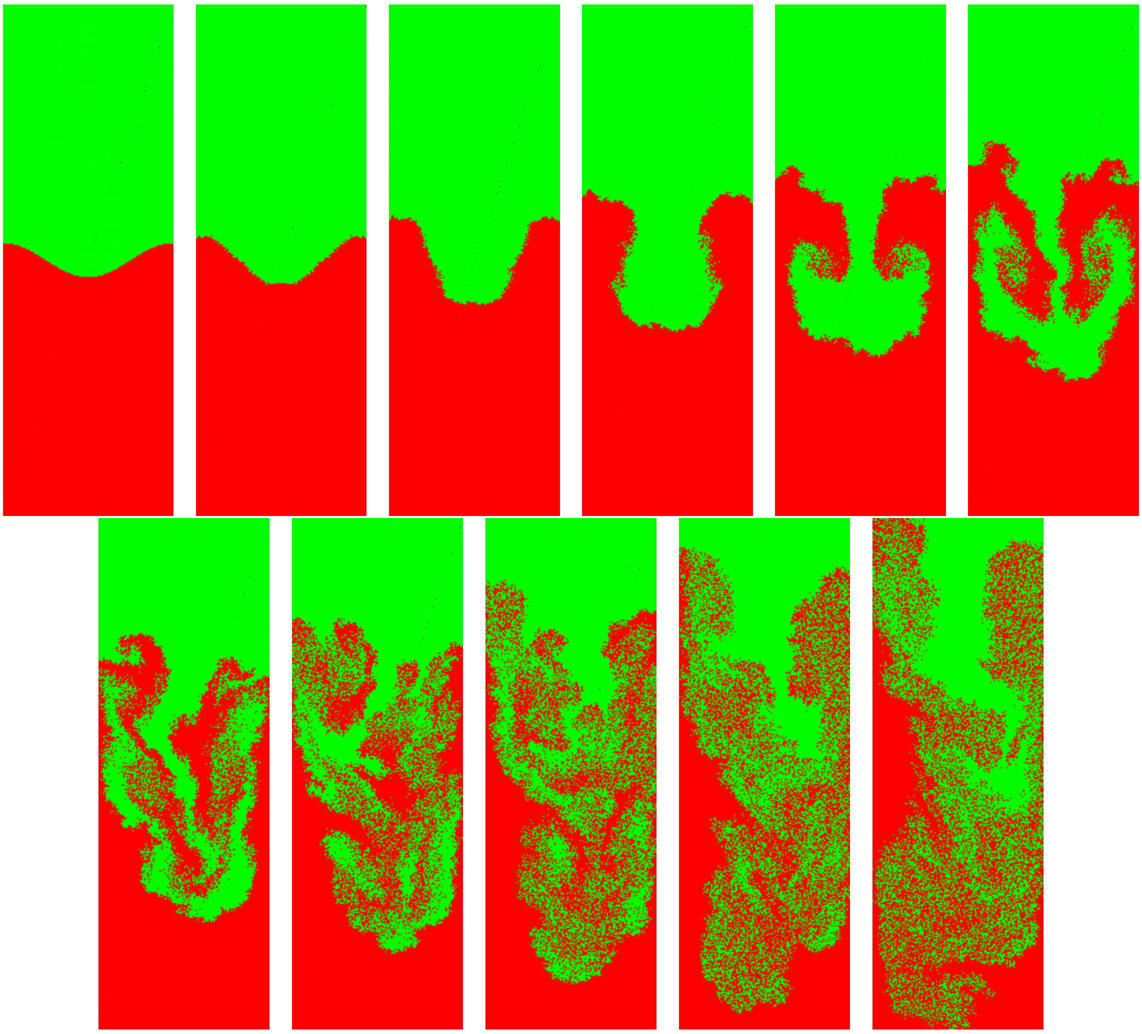

FIGURE 3. Numerical illustration of the Rayleigh-Taylor instability occuring when a heavy fluid (in green) is placed over a lighter fluid (in red) at timesteps $n = 0, 200, 400, \ldots, 2000$. The parameters chosen for this experiment are given in §5.4.

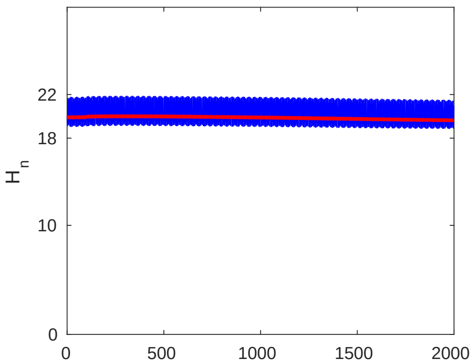

FIGURE 4. Preservation of the Hamiltonian by the discrete scheme.

[17] A. I. Shnirelman. Generalized fluid flows, their approximation and applications. *Geometric and Functional Analysis*, 1994.
[18] C. Villani. *Optimal transport: old and new.* Springer Verlag, 2009.

CMLS, École polytechnique, CNRS, Université Paris-Saclay, 91128 Palaiseau Cedex, France. E-mail: thomas.gallouet@polytechnique.edu

CNRS / Université Paris-Dauphine, Paris, France. E-mail: quentin.merigot@univ-dauphine.fr