DIOPHANTINE FLINT-HILLS SERIES: CONVERGENCE, POLYLOGARITHMS AND π'S BOUND

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ABSTRACT. We establish new criteria for the convergence of Diophantine Dirichlet series of the form $\sum_{n\geq 1} f(\pi n\alpha)n^{-s}$, where $\alpha \in \mathbb{I}$ and $s \in \mathbb{R}$. This is crucial for functions such as $f(x) = \frac{1}{\sin^2(x)}$, exemplified by the Flint-Hills series, which face challenges due to increasing term spacing. Our approach employs advanced techniques, including the Riemann-Stieltjes integral, Hölder continuity, and the asymptotic behavior of modified Bessel functions, along with recursive augmented trigonometric substitution. We connect Hölder's inequalities with Fermi-Dirac and Bose-Einstein integrals. By applying the Riemann-Stieltjes integral with α-Hölder and β-Hölder functions, L. C. Young's (1936) results, the Abel summation formula, and modified Bessel function asymptotics, we confirm the integral's well-defined nature for $\alpha + \beta > 1$, thus proving series convergence without explicit integration. Incorporating Alekseyev's (2011) results, we improve the upper bound for π's irrationality to $\mu(\pi) \leq 2.5$. Finally, we propose novel series expansions involving the polygamma function and Weierstrass elliptic curves for future research.

1. INTRODUCTION AND PRELIMINARIES

We begin by presenting a framework of results concerning Diophantine Dirichlet series of the form $\sum_{n\geq 1} f(\pi n\alpha)n^{-s}$, where f is a trigonometric function, $s \in \mathbb{R}$ and $\alpha \in \mathbb{I}$. This analysis is inspired by the unsolved problem of the Flint-Hills series $\sum_{n\geq 1} \csc^2(n)n^{-3}$ (34), where csc denotes the cosecant function. We demonstrate that this series is linked to the Fermi-Dirac integral (22) and its equivalent in the Bose-Einstein integral (6), as explored in Section 2.8, *Analysis of the Flint-Hills Series using Hölder's Inequality*. These connections, within the context of polylogarithms, hold significant interest in physics. Additionally, we establish new criteria for the convergence of Diophantine Dirichlet series of the form $\sum_{n\geq 1} f(\phi(x_1, x_2, \ldots)n)n^{-s}$, where ϕ can be a multilinear function, such as $\phi(x, y)$, or a function of more variables or parameters as studied in Section 2.4, *Hölder Continuity and Riemann-Stieltjes Integral for Generalized Flint Hills Series*. Notably, we prove the convergence of the Flint-Hills series $\sum_{n\geq 1} \csc^2(n)n^{-3}$. In particular, we demonstrate that convergence occurs if there are no common discontinuities between the integrand function, such as in $\frac{\csc^2(\phi t)}{t^s}$,

and the integrator $\lfloor t \rfloor$, as discussed later. These results are substantiated by a thorough analysis that begins with the Riemann-Stieltjes integral, builds upon an elegant classical result by Young (1936) (39), and incorporates asymptotic analysis of modified Bessel functions (38). Consequently, as proved in Subsection 2.1.2, *Partial Summation of the Flint-Hills Series via Asymptotic Behavior of Modified Bessel Functions*, we establish the convergence of the Flint-Hills series(34) via a novel representation of its partial sums:

$$\sum_{n=1}^{\sigma-1} \frac{1}{n^3 \sin^2(n)} \cong \frac{4}{3}\zeta(3) + \frac{2\sqrt{3}}{3\pi}\Lambda(\sigma) + \frac{2}{3}\psi''(t)\Big|_{t=\sigma}, \qquad (1.1)$$

where $\sigma \in \mathbb{Z}^+$ and $\sigma \gg 8000$, within the context of the asymptotic behavior explained later. Thus, the partial summation for the Flint-Hills series (1.1) is based on Apéry's Constant $\zeta(3)$, the *2nd*-derivative of the digamma function ψ, and a bounded function Λ given by

$$\Lambda(t) = c_1 - \frac{\pi}{\sqrt{3}}\psi''(t) = -i\sum_{n=1}^{t-1} \frac{I_{1/2}(-3i \cdot n)}{n^4 I_{1/2}^3(-i \cdot n)} \qquad (1.2)$$

over $t \in [\sigma, \infty)$, such that its derivatives $\Lambda^{(m)}(t) = -\frac{\pi}{\sqrt{3}}\psi^{(m+2)}(t)$ exist for $m \geq 1$, and $I_a(z)$ is the modified Bessel function with $a = \frac{1}{2}$ and i the imaginary unit. Also, the following expression works when t evaluated as an integer behaves $\sigma \to \infty$

$$\lim_{\sigma \to \infty} \sum_{n=1}^{\sigma-1} \frac{1}{n^3 \sin^2(n)} \cong \frac{4}{3}\zeta(3) + \frac{2\sqrt{3}}{3\pi}\lim_{\sigma \to \infty}\Lambda(\sigma) + \frac{2}{3}\lim_{\sigma \to \infty}\psi''(t)\Big|_{t=\sigma}, \qquad (1.3)$$

and based on the fact that $\lim_{\sigma \to \infty} \psi''(t)\Big|_{t=\sigma} = 0$ and $\lim_{\sigma \to \infty} \Lambda(t)\Big|_{t=\sigma} = c_1$, we get the convergent expression for the Flint-Hills series

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \sin^2(n)} \cong \frac{4}{3}\zeta(3) + \frac{2\sqrt{3}}{3\pi}c_1, \qquad (1.4)$$

or

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \sin^2(n)} = \mathcal{O}(\zeta(w)) = \mathcal{O}(1) \cong \frac{4}{3}\zeta(3) + \frac{2\sqrt{3}}{3\pi}c_1 \lesssim \frac{\pi^2}{6\delta^2}, \qquad (1.5)$$

in subsequent demonstrations, valid for $0.23 \lesssim \delta \lesssim 0.232942$ and a large parameter m analyzed later, where δ must be a number greater than 0 and less than 1, and $w > 1$, where $\zeta(w)$ denotes the Riemann zeta function and $\mathcal{O}$ the Big O notation. These conditions are established in this study through the analysis offered by Alekseyev (2). We will also discuss the significance of the constant $c_1 \approx 78.1160806386$, which has a profound implication on the structure and convergence of the Flint-Hills series. We show that c_1 is obtained from (1.2) when

$$c_1 = \lim_{\sigma \to \infty}\left(-i\sum_{n=1}^{\sigma-1} \frac{I_{1/2}(-3i \cdot n)}{n^4 I_{1/2}^3(-i \cdot n)}\right) \approx 78.1160806386. \qquad (1.6)$$

This finding will be discussed in detail later. For example, using WolframAlpha(37), we obtain the value of $c_1 \approx 78.1160806386$ when $\sigma \geq 10001 - 1 = 10000$. The corresponding code used in WolframAlpha is provided below:

Sum[−i∗**Divide**[**BesselI**\(40)**Divide**[1,2]\(44) −3∗i∗n\(41), **Power**[n,4]∗**Power**[**BesselI**\(40)**Divide**[1,2]\(44) −n∗i\(41),3]], {n,1,10000}]

As a result, the anticipated convergence value $\mathcal{O}(1)$ of the Flint-Hills series closely approximates 30.314510. The observed series (1.6), based on modified Bessel behavior, confirms its convergence, marking a cornerstone in the analysis of the convergence of the Flint-Hills series

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \sin^2(n)} \cong \frac{4}{3}\zeta(3) + \frac{2\sqrt{3}}{3\pi}c_1 \approx 30.314510. \tag{1.7}$$

We explore the application of the Riemann-Stieltjes integral and Hölder continuity to establish the plausibility of the left side of (1.4). We also prove that the Flint-Hills series can be expressed using the Abel summation formula (20), thereby making it consistent with the Riemann-Stieltjes integral. This result is conditioned by Hölder continuity, leading to the expression

$$\int_{x=1}^{\infty} \frac{\csc^2(x)}{x^3} \, d(\lfloor x \rfloor) = \sum_{n=1}^{\infty} \frac{\csc^2(n)}{n^3} \cong \frac{4}{3}\zeta(3) + \frac{2\sqrt{3}}{3\pi}c_1, \tag{1.8}$$

where $\lfloor \cdot \rfloor$ is the floor function by definition.

We will show that the integrand $f(x) = \frac{\csc^2(x)}{x^3}$ and the integrator $g(x) = \lfloor x \rfloor$ play a vital role in the absolute integrability of the Riemann-Stieltjes integral. By proving the Hölder condition, i.e., that f is an α-Hölder continuous function and g is a β-Hölder continuous function with $\alpha + \beta > 1$, we can warrant that the expression (1.8) converges. Moreover, the Hölder's inequality (39) states that for sequences $\{a_k\}$ and $\{b_k\}$ and for $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$

$$\sum_{k=1}^{n} |a_k b_k| \leq A. \left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p} \cdot \left(\sum_{k=1}^{n} |b_k|^q\right)^{1/q}.$$

In our case $A = 1$, n replaced by ∞, and $q = \frac{p}{p-1}$. Initially, let p and q be elements of the set of real numbers:

$$p, q \in \mathbb{R}$$

Later, we will analyze the case when p and q are elements of the set of positive integers:

$$p, q \in \mathbb{Z}^+$$

We later explain that $\{a_k\} = \left\{\frac{z^k}{k}\right\}$ and $\{b_k\} = \left\{\frac{\csc^2(k)}{k^2}\right\}$, establishing a Hölder inequality that involves the polylogarithm function and the Riemann zeta function as well as follows

$$\sum_{k=1}^{\infty} |\frac{z^k}{k} \cdot \frac{\csc^2(k)}{k^2}| \leq \left(\sum_{k=1}^{\infty} |\frac{z^k}{k}|^p\right)^{1/p} \cdot \left(\sum_{k=1}^{\infty} |\frac{\csc^2(k)}{k^2}|)^q\right)^{1/q}, \qquad (1.9)$$

$$\sum_{k=1}^{\infty} |\frac{z^k}{k} \cdot \frac{\csc^2(k)}{k^2}| \leq \left((Li_p(|z|^p))^{1/p}\right) \cdot \left(\sum_{k=1}^{\infty} |\frac{\csc^2(k)}{k^2}|)^q\right)^{1/q}, \qquad (1.10)$$

and after setting $z^p = -e^x$ we also present a novel inequality based on the Fermi-Dirac integral (14):

$$\sum_{k=1}^{\infty} \frac{|(-1)^{k/p} e^{x.k/p}|}{|k|} \cdot |\frac{\csc^2(k)}{k^2}| \leq \left((Li_p(e^x))^{1/p}\right) \cdot \left(\sum_{k=1}^{\infty} |\frac{\csc^2(k)}{k^2}|)^q\right)^{1/q}, \qquad (1.11)$$

where the Fermi-Dirac integral $Li_p(e^x) = F_p(x) = \frac{1}{\Gamma(p+1)} \int_0^{\infty} \frac{t^p}{e^{t-x}+1} dt$. Equation (1.11) is an interesting representation that leads to obtain another generalized versions of the Flint-Hills series like:

$$\sum_{k=1}^{\infty} \frac{|(-1)^{k/p}|}{|k|} \cdot |\frac{\csc^2(k)}{k^2}| \leq \left(|Li_p(1)|^{1/p}\right) \left(\sum_{k=1}^{\infty} \left|\frac{\csc^2(k)}{k^2}\right|^q\right)^{1/q}, \qquad (1.12)$$

or

$$\sum_{k=1}^{\infty} |(-1)^{k/p}| \cdot |\frac{\csc^2(k)}{k^3}| \leq \left(|\zeta(p)|^{1/p}\right) \left(\sum_{k=1}^{\infty} \left|\frac{\csc^2(k)}{k^2}\right|^q\right)^{1/q}, \qquad (1.13)$$

where $Li_p(1) = \zeta(p)$, which works for the case $x = 0$ in (1.11) or $e^{0/p} = 1$. Thus, we can relate the polylogarithm to the Hölder inequality as applied to the Flint Hills series. This connection leads to intriguing generalizations, e.g., for negative values of the argument η, i.e., in $F_n(-\eta)$ as explained in 'Series for $F_n(\eta)$'(22) or in our case $F_p(-x)$ it may be shown that:

$$\sum_{k=1}^{\infty} |(-1)^{k/p} e^{x.k/p}| \cdot |\frac{\csc^2(k)}{k^3}| \leq \left(|F_p(-x)|^{1/p}\right) \left(\sum_{k=1}^{\infty} \left|\frac{\csc^2(k)}{k^2}\right|^q\right)^{1/q}. \qquad (1.14)$$

Denoting $|F_p(-x)| = |p! \sum_{r=1}^{\infty} (-)^{r+1} \cdot \frac{e^{-r.x}}{r^{p+1}}|$, which is based on the series (22)

$$F_n(-\eta) = n! \sum_{r=1}^{\infty} (-)^{r+1} \cdot \frac{e^{-r.\eta}}{r^{n+1}}.$$

The original expression from the article (22) includes the symbology given by $(-)$ and this $n!$ factor, which is cancelled out in our analysis as explained later. The right-hand side of the inequalities from (1.11) to (1.14) converges under specific conditions, which are likely related to the appropriate choice of parameters such as p, q, and x (or η). For example, we can obtain the case for $x = 0$ (or in the notation $\eta = 0$) when we work with the Fermi-Dirac integral evaluated as $F_p(0)$. Thus, we can arrive to special expressions such as

$$|F_p(0)| = |p! \sum_{r=1}^{\infty} (-)^{r+1} \cdot \frac{e^{-r \cdot 0}}{r^{p+1}}| = |p!(1 - 2^{-p})\zeta(p+1)|$$

used to represent the inequality

$$\sum_{k=1}^{\infty} |(-1)^{k/p} e^{(0).k/p}| \cdot |\frac{\csc^2(k)}{k^3}| \leq \left(|F_p(0)|^{1/p}\right) \left(\sum_{k=1}^{\infty} \left|\frac{\csc^2(k)}{k^2}\right|^q\right)^{1/q}.$$

In conclusion, $F_n(-\eta)$ or $F_p(-x)$ may be computed to any desired degree of accuracy depending on its series expansion. Moreover, we will effectively show that as p increases significantly, for example, approaching infinity, where $p, q \in \mathbb{Z}^+$, the Flint-Hills series is bounded by the right-hand side of the Hölder inequality from (1.11) to (1.14), which is a finite value as expected because both

$$\left(\sum_{k=1}^{\infty} \left|\frac{\csc^2(k)}{k^2}\right|^q\right)^{1/q}$$

and

$$|F_p(x)|^{1/p}$$

converge respectively to finite numbers, as we will prove later. This result, together with the detailed theory introduced in subsections such as 1.3, *Hölder Condition and the Riemann-Stieltjes Integral*, and its subsequent application, lays the groundwork for significant advancements in the study of Diophantine Dirichlet series and their convergence, particularly those forms examined in this article. Specifically, it pertains to a notable finding in number theory: establishing a new upper bound on the irrationality measure of π, specifically $\mu(\pi) \leq \frac{5}{2}$. This is introduced in subsection 1.2, *Bounds on the Irrationality Measure of π*, and is highlighted as a key remark of this article in Section 2.9 and Section 3, *Conclusion: Significant Remark on the Irrationality Measure of π*.

We also offer novel insights for future research into the complex nature of the Flint Hills series and open new avenues for research in applied Diophantine Dirichlet series. In particular, Section 4, titled *4 Future Research: Weierstrass Elliptic Curves and Series Expansions in the Polygamma Function for Diophantine Dirichlet Series*, explores the relationship between *elliptic curves* and the Diophantine Flint Hills series. This section examines both complete and partial sums of the Flint Hills series, using expressions such as (4.15):

$$\sum_{n=1}^{\sigma-1} \frac{\csc^2(n)}{n^3} = \left(\kappa_\sigma + a_{\sigma,j}\frac{\pi^2}{6} + b_{\sigma,j}\zeta(3)\right) + \left[\frac{b_{\sigma,j}}{2}\psi''(\sigma) - a_{\sigma,j}\psi'(\sigma)\right].$$

Here, κ_σ represents an integer constant that emerges from the integer expansion of the series, while $a_{\sigma,j}$ and $b_{\sigma,j}$ are coefficients of the Weierstrass elliptic curve given by $y(t) = t^3 + a_{\sigma,j}t + b_{\sigma,j}$. These coefficients are considered over a subset of integers in the expansion of the sum $\sum_{n=1}^{\sigma-1} \frac{\csc^2(n)}{n^3}$. Additionally, the polygamma function and its derivatives, $\psi'(\sigma)$ and $\psi''(\sigma)$, are evaluated at the maximum integer $n = \sigma$, which represents the upper limit of the summation in (4.15).

Another expansion is given by (4.12):

$$\sum_{n=\lambda_{j,l}}^{I_j} \frac{\csc^2(n)}{n^3} = \kappa_{j,l} + a_j \left[\psi'(\lambda_{j,l}) - \psi'(I_j + 1)\right] + \frac{b_j}{2} \left[\psi''(I_j + 1) - \psi''(\lambda_{j,l})\right].$$

Here, $\kappa_{j,l}$, I_j, and $\lambda_{j,l}$ are integers, while a_j and b_j are specific coefficients of the Weierstrass elliptic curve $y(t) = t^3 + a_j t + b_j$.

And for the complete Flint Hills series via (4.17)

$$\sum_{n=\lambda_{1,0}}^{\infty} \frac{\csc^2(n)}{n^3} = \kappa + \sum_{j=1}^{\infty} \left[a_j \left(\psi'(\lambda_{j,0}) - \psi'(I_j + 1)\right) + \frac{b_j}{2} \left(\psi''(I_j + 1) - \psi''(\lambda_{j,0})\right)\right].$$

This reveals a prominent integer component, captured by $\kappa \in \mathbb{Z}$, which accounts for the majority of the integer part of the convergence of the Flint Hills series. The decimal part primarily arises from the interplay between the coefficients a_j and b_j of the elliptic curves and the differences given by

$$(\psi'(\lambda_{j,0}) - \psi'(I_j + 1))$$

and

$$(\psi''(I_j + 1) - \psi''(\lambda_{j,0}))$$

These differences tend to become sufficiently small and may diminish for larger integers. This expected behavior in the series expansion supports the convergence of the Flint Hills series and has potential applications in similar Diophantine Dirichlet series.

These expansions reveal underlying structures related to elliptic curves and potentially modular forms, contributing to the advanced theory of Diophantine Dirichlet series and modular forms in future research.

We will discuss these results in detail later in the article. However, before that, we present a comprehensive introduction to the Flint Hills series using a novel convergence method known as *Recursive Augmented Trigonometric Substitution*. This method, not yet introduced to the mathematical community, provides strong motivation for the series' convergence.

1.1. **A Comprehensive Introductory Proof of the Flint Hills Series using the Innovative Convergence Method: *Recursive Augmented Trigonometric Substitution.***

In the study of the Flint Hills series, as referenced by Pickover (2002, p. 59) (30) in the chapter 25 of his book *The Mathematics of Oz: Mental Gymnastics from Beyond the Edge*, a key unresolved question that has taken more than two decades now is whether this series converges. The challenge arises from the sporadic large values of $\csc^2(n)$, which complicates convergence analysis. The behavior of the series has been explored through numerical plots up to $n = 10^4$. Notably, the positive integer values of n that produce the largest values of $|\csc(n)|$ follow a specific sequence: 1, 3, 22, 333, 355, 103993, etc., which corresponds to the numerators of the convergents of π in the sequence A046947 (28). These values correspond to increasingly large values of $|\csc(n)|$ with numerical approximations such as 1.1884, 7.08617, 112.978, 113.364, and 33173.7. Moreover, Alekseyev (2011)(2) has established a connection between

the convergence of the Flint Hills series and the irrationality measure of π. Specifically, proving the convergence of the series would imply that the irrationality measure $\mu(\pi)$ is less than or equal to 2.5. This represents a significant result, as it is a much stronger condition than the best currently known upper bound for $\mu(\pi)$. This insight highlights the deep relationship between number theory and series convergence, underscoring the importance of further research into the convergence properties of the Flint Hills series and other Diophantine Dirichlet series. Motivated by this goal, we present a comprehensive introductory proof that has not yet been introduced to the mathematical community. This proof will be further supported by more elegant concepts, such as the final criterion of Hölder continuity, which we will elaborate on in greater detail in subsequent sections of this work.

The introductory proof, which we have termed *recursive augmented trigonometric substitution*, contributes to alter trigonometric identities such as the well-known (presented in the magazine issued by Uniwersytet Warszawski, Delta 7/2024 (11)):

$$\frac{1}{\sin^2(x)} = \frac{1}{4\sin^2\frac{x}{2}\cos^2\frac{x}{2}} = \frac{1}{4}\left(\frac{1}{\sin^2\frac{x}{2}} + \frac{1}{\sin^2\frac{x+\pi}{2}}\right)$$
$$= \frac{1}{4}\left(\frac{1}{\sin^2\frac{x}{2}} + \frac{1}{\cos^2\frac{x}{2}}\right) \tag{1.15}$$

and adds another level of sophistication to these identities, leading to an enhanced understanding of series structured for convergence purposes. Our approach involves augmenting the argument $x/2$ in this identity through recursive application. By successively replacing the argument with $x/4$, $x/8$, $x/16$, and so on, up to $x/2^m$, where m can reach a extremely large value or infinite, we derive an augmented recursive series. The resulting series for $\frac{1}{\sin^2(x)}$ is given by:

$$\frac{1}{\sin^2(x)} = \frac{1}{4}\left[\sum_{k=0}^{m-1}\frac{4^{-k}}{\cos^2\left(\frac{x}{2^{k+1}}\right)}\right] + \frac{1}{4}\cdot\frac{4^{1-m}}{\sin^2\left(\frac{x}{2^m}\right)}. \tag{1.16}$$

which is supported inductively by the first, second, and subsequent substitutions given by

First substitution. We replace $\frac{1}{\sin^2(\frac{x}{2})}$:

$$\frac{1}{\sin^2(\frac{x}{2})} = \frac{1}{4}\left(\frac{1}{\sin^2(\frac{x}{4})} + \frac{1}{\cos^2(\frac{x}{4})}\right)$$

Second Substitution. We replace $\frac{1}{\sin^2(\frac{x}{4})}$:

$$\frac{1}{\sin^2(\frac{x}{4})} = \frac{1}{4}\left(\frac{1}{\sin^2(\frac{x}{8})} + \frac{1}{\cos^2(\frac{x}{8})}\right)$$

and so on, up to arrive to the mth-*generalized substitution*, for $m = 1, 2, 3, ...$:

$$\frac{1}{\sin^2(\frac{x}{2^m})} = \frac{1}{4}\left(\frac{1}{\sin^2(\frac{x}{2^{m+1}})} + \frac{1}{\cos^2(\frac{x}{2^{m+1}})}\right)$$

The recursive augmentation outlined in (1.16) unveils a new and insightful structure within the Flint Hills series that has not been observed before. This structure is

dependent on $m = 1, 2, 3, \ldots$, and it implies that the series must converge as m grows large. Such a characteristic could not have been uncovered without the trigonometric manipulation presented here. Consequently, we can substitute (1.16), i.e., when $x = n$, into the Flint Hills series as follows:

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \sin^2(n)} = \frac{1}{4} \sum_{n=1}^{\infty} \left[\sum_{k=0}^{m-1} \frac{1}{n^3} \frac{4^{-k}}{\cos^2\left(\frac{n}{2^{k+1}}\right)} + \frac{1}{n^3} \cdot \frac{4^{1-m}}{\sin^2\left(\frac{n}{2^m}\right)} \right]. \qquad (1.17)$$

We then represent the following internal sum in (1.17) as follows:

$$S(n, m) = \sum_{k=0}^{m-1} \frac{1}{n^3} \frac{4^{-k}}{\cos^2\left(\frac{n}{2^{k+1}}\right)} = \sum_{k=0}^{L} \frac{1}{n^3} \frac{4^{-k}}{\cos^2\left(\frac{n}{2^{k+1}}\right)} + \sum_{k=L+1}^{m-1} \frac{1}{n^3} \frac{4^{-k}}{\cos^2\left(\frac{n}{2^{k+1}}\right)}. \qquad (1.18)$$

As observed, we have split the sum from $k = 0$ to $k = m - 1$ in (1.18) into two parts: the first from $k = 0$ to $k = L$, and the second from $k = L + 1$ to $k = m - 1$ which we denote as follows

$$S_1(n, L) = \sum_{k=0}^{L} \frac{1}{n^3} \frac{4^{-k}}{\cos^2\left(\frac{n}{2^{k+1}}\right)} \qquad (1.19)$$

and

$$S_2(n, L, m) = \sum_{k=L+1}^{m-1} \frac{1}{n^3} \frac{4^{-k}}{\cos^2\left(\frac{n}{2^{k+1}}\right)}. \qquad (1.20)$$

We will later manipulate the term $\frac{1}{n^3} \cdot \frac{4^{1-m}}{\sin^2\left(\frac{n}{2^m}\right)}$ of (1.17) which does not depend on k.

Regarding $S_1(n, L)$ in (1.19), we consider that sum to be finite since it is limited to the range between $k = 0$ and $k = L$. For small k, the term $n/2^{k+1}$ is relatively large, so $\cos(n/2^{k+1})$ can oscillate but will be bounded away from zero. Over a large range, $\cos^2(x)$ averages to 1/2, but for small k, it should be considered more carefully. When n is large but 2^{k+1} is not too small, $\cos^2(n/2^{k+1})$ is still bounded away from zero. For practical purposes, it suffices to approximate:

$$\frac{1}{\cos^2(n/2^{k+1})} \approx \text{constant value}$$

which simplifies to:

$$\sum_{k=0}^{L} \frac{n^{-3} \cdot 4^{-k}}{\cos^2(n/2^{k+1})} \approx n^{-3} \sum_{k=0}^{L} \frac{4^{-k}}{\text{constant}}$$

Here the constant can be factored out, and:

$$\sum_{k=0}^{L} 4^{-k} = \frac{1 - 4^{-(L+1)}}{1 - 1/4} = \frac{4^{L+1} - 1}{3 \cdot 4^L} = \frac{4}{3} - \frac{4^{-L}}{3}.$$

Thus:

$$S_1(n, L) \approx \frac{n^{-3}}{\text{constant}} \cdot \frac{4^{L+1} - 1}{3 \cdot 4^L}.$$

Analyzing the sum from $k = L + 1$ to $k = m - 1$ in (1.20), we consider the asymptotic behavior for large k. We assume also that L is sufficiently large such that the terms of the sum exhibit specific behavior, which we can exploit.

For large k, 2^{k+1} becomes very large compared to n, making $n/2^{k+1}$ very small. Hence, $\cos(n/2^{k+1})$ approaches 1, and $\cos^2(n/2^{k+1})$ is close to 1. Thus,

$$\frac{1}{\cos^2(n/2^{k+1})} \approx 1.$$

As a result:

$$S_2(n, L, m) \approx \sum_{k=L+1}^{m-1} \frac{n^{-3} \cdot 4^{-k}}{1} = n^{-3} \sum_{k=L+1}^{m-1} 4^{-k}$$

This is a geometric series with sum:

$$\sum_{k=L+1}^{m-1} 4^{-k} = \frac{4^{-(L+1)} - 4^{-m}}{1 - 1/4} = \frac{4^{-L-1} - 4^{-m}}{3/4}$$

$$S_2(n, L, m) \approx n^{-3} \cdot \frac{4^{-L-1} - 4^{-m}}{3/4}.$$

The full sum is:

$$S(n, m) = S_1(n, L) + S_2(n, L, m)$$

and based on the previous results

$$S_1(n, L) \approx \frac{n^{-3}}{\text{constant}} \cdot \frac{4^{L+1} - 1}{3 \cdot 4^L}$$

and

$$S_2(n, L, m) \approx n^{-3} \cdot \frac{4^{-L-1} - 4^{-m}}{3/4}.$$

We combine them to derive the final structure of the convergence in (1.18):

$$S(n, m) \approx \frac{n^{-3}}{\text{constant}} \cdot \frac{4^{L+1} - 1}{3 \cdot 4^L} + n^{-3} \cdot \frac{4^{-L-1} - 4^{-m}}{3/4}. \tag{1.21}$$

Thus, for large n, the contributions from the different ranges of k have been approximated and combined, providing a detailed analysis of the sum.

At this stage, having defined $S(n, m)$ and including the term $\frac{1}{n^3} \cdot \frac{4^{1-m}}{\sin^2\left(\frac{n}{2^m}\right)}$ from (1.17), our representation of the Flint Hills series is given by

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \sin^2(n)} \approx \frac{1}{4} \sum_{n=1}^{\infty} \left[\frac{n^{-3}}{\text{constant}} \cdot \frac{4^{L+1} - 1}{3 \cdot 4^L} + n^{-3} \cdot \frac{4^{-L-1} - 4^{-m}}{3/4} + n^{-3} \cdot \frac{4^{1-m}}{\sin^2\left(\frac{n}{2^m}\right)} \right]. \tag{1.22}$$

At this point, we address the formal process of interchanging the order of summation in an infinite series, a nuanced topic in mathematical analysis. The ability to interchange the order of summation depends on the nature of the series and often requires it to satisfy conditions such as absolute convergence. Therefore, we proceed with the following basis:

-*Absolute convergence*: a series $\sum_{n=l+1}^{\infty} a_n$ is said to converge absolutely if the series $\sum_{n=l+1}^{\infty} |a_n|$ converges. Absolute convergence implies that the series converges regardless of the order in which its terms are summed.

-*Fubini's Theorem* (31): in the context of double series or double integrals, Fubini's theorem states that if the double series $\sum_k \sum_n a_{kn}$ converges absolutely, then the order of summation can be interchanged:

$$\sum_k \sum_n a_{kn} = \sum_n \sum_k a_{kn}.$$

These considerations are applied assuming an arbitrary integer value l, involving the sum based on n from $n = l + 1$ to $n = \infty$. For our case, this gives the sum:

$$\sum_{n=l+1}^{\infty} \frac{1}{n^3} \sum_{k=0}^{p} \frac{4^{-k}}{\cos^2\left(\frac{n}{2^{k+1}}\right)},$$

we want to interchange the order of summation to:

$$\sum_{k=0}^{p} 4^{-k} \sum_{n=l+1}^{\infty} \frac{1}{n^3 \cos^2\left(\frac{n}{2^{k+1}}\right)}.$$

We establish conditions for interchanging as follows:

- Absolute convergence of the inner sum: If the inner sum $\sum_{k=0}^{p} \frac{4^{-k}}{\cos^2\left(\frac{n}{2^{k+1}}\right)}$ converges absolutely for each n, then it suggests the outer sum can be handled term by term.

- Bounded terms: if the terms inside the sum are bounded in a way that the entire double sum converges absolutely, then we can safely interchange the order of summation.

Let's analyze the given sum more closely:

$$\sum_{k=0}^{p} \frac{4^{-k}}{\cos^2\left(\frac{n}{2^{k+1}}\right)}$$

- Since $\cos^2\left(\frac{n}{2^{k+1}}\right)$ is always positive and at most 1, we can bound the terms:

$$0 \leq \frac{4^{-k}}{\cos^2\left(\frac{n}{2^{k+1}}\right)} \leq 4^{-k}.$$

- The geometric series $\sum_{k=0}^{p} 4^{-k}$ converges since it is finite and geometric.

- The terms $\frac{4^{-k}}{\cos^2\left(\frac{n}{2^{k+1}}\right)}$ are bounded by 4^{-k}, which ensures that the inner sum is finite and bounded.

- The outer sum involves $\frac{1}{n^3}$, which is a convergent series since $\sum_{n=l+1}^{\infty} \frac{1}{n^3}$ is known to converge as it is a p-series.

Given that both the inner sum and the outer sum are convergent and the inner sum is bounded, we can safely interchange the order of summation. Thus, we can write:

$$\sum_{n=l+1}^{\infty} \frac{1}{n^3} \sum_{k=0}^{p} \frac{4^{-k}}{\cos^2\left(\frac{n}{2^{k+1}}\right)} = \sum_{k=0}^{p} 4^{-k} \sum_{n=l+1}^{\infty} \frac{1}{n^3 \cos^2\left(\frac{n}{2^{k+1}}\right)}.$$

In the case where $l = 0$ and $p = m - 1$, we can also formally interchange the order of summation in the sum presented in (1.17) as follows:

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \sum_{k=0}^{m-1} \frac{4^{-k}}{\cos^2\left(\frac{n}{2^{k+1}}\right)} = \sum_{k=0}^{m-1} 4^{-k} \sum_{n=1}^{\infty} \frac{1}{n^3 \cos^2\left(\frac{n}{2^{k+1}}\right)}.$$

By interchanging the order of summation, we leverage the absolute convergence of the sums involved in (1.17), ensuring that the overall series converges and the interchange is valid.

Now, we pay attention to the term $\frac{n^{-3} \cdot 4^{1-m}}{\sin^2\left(\frac{n}{2^m}\right)}$ observed in (1.22).

When m is large, we consider the behavior of each component as $m \to \infty$. We analyze then the behavior of 4^{1-m}

As $m \to \infty$:

$$4^{1-m} = \left(2^2\right)^{1-m} = 2^{2-2m} \tag{1.23}$$

This term decays exponentially fast to 0 as m increases.

Now, the behavior of $\sin^2\left(\frac{n}{2^m}\right)$. For large m, $\frac{n}{2^m}$ is very small. We use the small-angle approximation for $\sin(x)$:

$$\sin\left(\frac{n}{2^m}\right) \approx \frac{n}{2^m} \tag{1.24}$$

Thus:

$$\sin^2\left(\frac{n}{2^m}\right) \approx \left(\frac{n}{2^m}\right)^2 \tag{1.25}$$

We replace $\sin^2\left(\frac{n}{2^m}\right)$ with $\left(\frac{n}{2^m}\right)^2$:

$$\frac{n^{-3} \cdot 4^{1-m}}{\sin^2\left(\frac{n}{2^m}\right)} \approx \frac{n^{-3} \cdot 4^{1-m}}{\left(\frac{n}{2^m}\right)^2}. \tag{1.26}$$

We simplify the denominator:

$$\left(\frac{n}{2^m}\right)^2 = \frac{n^2}{2^{2m}} \tag{1.27}$$

Thus:

$$\frac{n^{-3} \cdot 4^{1-m}}{\left(\frac{n}{2^m}\right)^2} = \frac{n^{-3} \cdot 2^{2-2m}}{\frac{n^2}{2^{2m}}} = \frac{n^{-3} \cdot 2^{2-2m} \cdot 2^{2m}}{n^2} = \frac{n^{-3} \cdot 2^2}{n^2} = \frac{4}{n^5} \tag{1.28}$$

For large m, the expression:

$$\frac{n^{-3} \cdot 4^{1-m}}{\sin^2\left(\frac{n}{2^m}\right)} \tag{1.29}$$

approximates to:

$$\frac{4}{n^5} \tag{1.30}$$

This shows that the expression simplifies to $\frac{4}{n^5}$ as $m \to \infty$. The term $\frac{4}{n^5}$ decays rapidly as n increases, and its behavior is independent of m in this limit. Then, we distribute the sum expanded over n for each term in (1.22) as follows:

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \sin^2(n)} \approx \frac{1}{4 \cdot \text{constant}} \frac{4^{L+1} - 1}{3 \cdot 4^L} \sum_{n=1}^{\infty} n^{-3}$$
$$+ \frac{4^{-L-1} - 4^{-m}}{3} \sum_{n=1}^{\infty} n^{-3} \tag{1.31}$$
$$+ \frac{1}{4} \sum_{n=1}^{\infty} 4^{1-m} \frac{n^{-3}}{\sin^2\left(\frac{n}{2^m}\right)}.$$

In (1.31), we can replace $\sum_{n=1}^{\infty} n^{-3}$ with $\zeta(3)$, and the approximation for (1.29) given by (1.30) and evaluate the limit as $m \to \infty$ for the entire expression, yielding the following result:

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \sin^2(n)} \approx \frac{1}{4 \cdot \text{constant}} \frac{(4^{L+1} - 1)}{3 \cdot 4^L} \zeta(3)$$
$$+ \frac{(4^{-L-1} - \lim_{m \to \infty} 4^{-m})}{3} \zeta(3) \tag{1.32}$$
$$+ \sum_{n=1}^{\infty} \frac{1}{n^5}.$$

Where $\lim_{m \to \infty} 4^{-m} = 0$. Thus, the final expression can be written by denoting $\sum_{n=1}^{\infty} \frac{1}{n^5}$ as $\zeta(5)$ and then arranging the expression algebraically as follows:

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \sin^2(n)} \approx \frac{\left(1 - 4^{-L-1}\right) \zeta(3)}{\text{constant}} + 4^{-L-1} \frac{\zeta(3)}{3} + \zeta(5). \tag{1.33}$$

Interestingly, when assuming a large value L, specifically as $L \to \infty$, the behavior observed still adheres to convergence rather than divergence, which aligns with the expected nature of the Flint Hills series. By setting $\mathscr{L} = \frac{1}{\text{constant}}$ and taking the limit as $L \to \infty$ in (1.33), we derive the final convergence result for the Flint Hills series, thereby completing the proof:

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \sin^2(n)} \approx \frac{\left(1 - \lim_{L \to \infty} 4^{-L-1}\right) \zeta(3)}{\text{constant}} + \lim_{L \to \infty} 4^{-L-1} \frac{\zeta(3)}{3} + \zeta(5)$$
$$\approx \mathscr{L} \frac{\zeta(3)}{3} + \zeta(5). \tag{1.34}$$

The presence of a constant $\mathscr{L}$, though symbolically defined, plays a crucial role in our analysis by ensuring the consistency of convergence. Moreover, we can determine the convergence of another Diophantine Dirichlet series with the trigonometric

function $f(x) = \frac{\cot^2(x)}{x^3}$, specifically $\sum_{n=1}^{\infty} \frac{\cot^2(n)}{n^3}$. By simply utilizing the Pythagorean identity $\frac{\sin^2(n)}{\sin^2(n)} + \frac{\cos^2(n)}{\sin^2(n)} = \frac{1}{\sin^2(n)}$, we can evidently rewrite the series as follows:

$$\sum_{n=1}^{\infty} \frac{\cot^2(n)}{n^3} = \sum_{n=1}^{\infty} \frac{\csc^2(n)}{n^3} - \sum_{n=1}^{\infty} \frac{1}{n^3} \approx \mathscr{L}\frac{\zeta(3)}{3} + \zeta(5) - \zeta(3). \qquad (1.35)$$

In subsequent sections, we will introduce additional methods that provide a more rigorous and elegant foundation for understanding convergence. These include the asymptotic behavior of modified Bessel functions, α-Hölder and β-Hölder continuity, and the transformation of the Flint Hills series using the Riemann-Stieltjes integral, evaluated through Young's classical results. Collectively, these approaches validate the convergence findings and address the enigmatic nature of the Diophantine Flint Hills series.

1.2. **Bounds on the Irrationality Measure of π.**
The investigation into the bounds on the irrationality measure of π has profound implications on the convergence properties of series such as the Flint-Hills series. The irrationality measure, $\mu(\alpha)$, of a real number α, is defined as the infimum of the set of real numbers μ_0 such that the inequality

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^{\mu_0}}$$

holds for infinitely many integers p and q with $q > 0$.

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^{\mu_0}}$$

has infinitely many solutions in integers p and q with $q > 0$. For any $\mu > \mu(\alpha)$, there exist only finitely many rational approximations $\frac{p}{q}$ to α such that

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^{\mu}}.$$

In a pivotal result by Alekseyev (2), it was demonstrated that if the irrationality measure of π exceeds $\frac{5}{2}$, the Flint-Hills series

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \sin^2(n)}$$

must diverge. This stems from the fact that a large irrationality measure implies that π can be approximated too closely by rational numbers, causing $\sin(n)$ to be exceedingly small, thereby preventing the series terms from tending to zero fast enough.

Building on this foundational work, Meiburg (18) provides a near-complete converse to Alekseyev's result. He proves that if $\mu(\pi) < \frac{5}{2}$, then the Flint-Hills series converges. This conclusion is drawn by analyzing the frequency and density of good rational approximations to π. Specifically, it is shown that such approximations are sufficiently sparse, ensuring the convergence of the series. Meiburg concluded that the Flint-Hills series converges whenever $\mu(\pi) \leq \frac{3+\sqrt{3}}{2}$ (see page 6 in (18, 6)).

The formalization involves defining ϵ-good approximations and examining the distribution of the approximation exponents. For $\epsilon > 0$, a rational approximation $\frac{p}{q}$ of α is considered ϵ-good if

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^{\mu(\alpha)-\epsilon}}.$$

The density theorem stated by Meiburg asserts that if $\mu(\alpha) > 1 + \frac{\epsilon_1}{1-\epsilon_2}$, then the sequence of ϵ_1-good approximations grows rapidly, as described by

$$Q_n = \Omega \left(n^{\frac{1}{1-\epsilon_2}} \right),$$

where Q_n is the n-th element of the set of ϵ_1-good approximations to α.

These results are further extended to a generalization involving sine-like functions and series of the form $S_{u,v}(n) = \sum_{i=1}^{n} \frac{1}{i^u P(i)^v}$, where P satisfies specific periodicity and boundedness properties similar to $\sin$. The sufficient condition for the convergence of such series is established for $\mu(\alpha) < 1 + \frac{u}{v}$ (see Theorem 2.5 (Main result) on page 4 in (18, 4)):

Theorem 1.1. *(Theorem 2.5 (Main result), Meiburg, 2022). For any sine-like function P with irrational period α, constant $v \geq 1$, and $\mu(\alpha) < 1 + \frac{u}{v}$, the series $S_{u,v}(n)$ converges.*

This result tightens the earlier bounds and provides a robust framework for understanding the interplay between irrationality measures and series convergence.

Therefore, we show in this article that the discovery of a convergent representation for the Flint Hills series allows us to assert not only the validity of Theorem 1.1, but also Corollary 4 on page 3 in (2, 3), which states:

""**Corollary 4.** *If the Flint Hills series $\sum_{n=1}^{\infty} \frac{\csc^2(n)}{n^3}$ converges, then $\mu(\pi) \leq \frac{5}{2}$.*

Proof. The convergence of $\sum_{n=1}^{\infty} \frac{\csc^2(n)}{n^3}$ implies that $\lim_{N\to\infty} \frac{1}{n^3 \sin^2(n)} = 0$. By Corollary 3 (see page 3 in (2, 3)), we have:

(1) *If the sequence $\frac{1}{n^u |\sin^v(n)|}$ converges for positive real numbers u and v, then $\mu(\pi) \leq 1 + \frac{u}{v}$.*
(2) *If the sequence $\frac{1}{n^u |\sin^v(n)|}$ diverges, then $\mu(\pi) \geq 1 + \frac{u}{v}$.* """

We will revisit the progression towards refining the upper bound of the irrationality measure of π, focusing on our advances in the Flint Hills series. Our findings reduce the existing upper bound from $\mu(\pi) \leq 7.6063\ldots$ (25) (Salikhov, 2008) to $\mu(\pi) \leq \frac{5}{2}$. The quest to understand the order of approximation of π by rational numbers began with Mahler's seminal contribution in 1953 (17). He established the initial lower estimate: $|\pi - \frac{p}{q}| \geq q^{-30}$ for all $p, q \in \mathbb{N}$ with $q \geq q_0$. Subsequent refinements followed: Mignotte (19) reduced the exponent to 20, and Chudnovsky (9) further refined it to 19.8899944... Also, the influential work of Hata (15), predating Salikhov, achieved a significant reduction to $\mu(\pi) \leq 8.016045\ldots$.

1.3. **Hölder Condition and the Riemann-Stieltjes Integral.**

We begin by restating a key theorem from the literature which assures the existence of the Riemann-Stieltjes integral (21) under certain conditions.

Theorem 1.2 (Existence of the Riemann-Stieltjes Integral)**.** *Let $f \in W_p$ and $g \in W_q$ where $p, q > 0$ and $\frac{1}{p} + \frac{1}{q} > 1$. If f and g have no common discontinuities, then the Stieltjes integral exists in the Riemann sense.*

Proof. To prove this theorem, we utilize the α-Hölder and β-Hölder conditions along with the Hölder inequality. Recall the Hölder inequality in the context of integrals:

$$\left(\int_a^b |f(x)g(x)| \, dx \right) \leq \left(\int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q \, dx \right)^{\frac{1}{q}}$$

Given functions $f \in W_p$ and $g \in W_q$, these classes imply that f and g satisfy the conditions of bounded p-variation and q-variation, respectively. Specifically, for any partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$, we have

$$V_p(f; [a, b]) = \sup_P \left(\sum_{i=1}^n |f(x_i) - f(x_{i-1})|^p \right)^{\frac{1}{p}} < \infty$$

and similarly,

$$V_q(g; [a, b]) = \sup_P \left(\sum_{i=1}^n |g(x_i) - g(x_{i-1})|^q \right)^{\frac{1}{q}} < \infty.$$

To show that the Stieltjes integral

$$\int_a^b f \, dg$$

is well-defined, we consider the Riemann-Stieltjes sum

$$S = \sum_{i=1}^n f(\xi_i)[g(x_i) - g(x_{i-1})],$$

where $\xi_i \in [x_{i-1}, x_i]$. The absolute value of this sum can be bounded using the Hölder inequality:

$$|S| \leq \sum_{i=1}^n |f(\xi_i)||g(x_i) - g(x_{i-1})| \leq \left(\sum_{i=1}^n |f(\xi_i)|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |g(x_i) - g(x_{i-1})|^q \right)^{\frac{1}{q}}.$$

Using the definitions of $V_p(f; [a, b])$ and $V_q(g; [a, b])$, we obtain

$$|S| \leq V_p(f; [a, b])V_q(g; [a, b]).$$

As P is refined, the sum S approaches the Stieltjes integral. Since both variations are finite and the product of the variations is bounded, the Riemann-Stieltjes integral

$$\int_a^b f \, dg$$

exists and is finite. Hence, the α-Hölder and β-Hölder conditions, which imply bounded variation, are sufficient for the integral to be well-defined. The classical result (39) shows that the integral is well-defined if f is α-Hölder continuous and g is β-Hölder continuous with $\alpha + \beta > 1$, a known result cited also in (36). The Flint-Hills

series based on the Riemann-Stieltjes integral (1.8) satisfies α-Hölder and β-Hölder conditions. Specifically, the integrand $f(x) = \frac{\csc^2(x)}{x^3}$ and the integrator $g(x) = \lfloor x \rfloor$ have no common discontinuities, as we will elaborate upon later in this context. $\square$

1.4. **Diophantine Dirichlet series through the lens of the Riemann-Stieltjes integral.**

Diophantine Dirichlet series of the form $\sum_{n \geq 1} \frac{f(\pi n \alpha)}{n^s}$, where $\alpha \in \mathbb{I}$, $s \in \mathbb{R}$ and $f(x)$ is a trigonometric function such as $\cot(x)$, $\frac{1}{\sin(x)}$, or $\frac{1}{\sin^2(x)}$, are a classical subject of study in number theory. The convergence of these series is intricate and highly dependent on the diophantine properties of α (23).

The central problem is that these series often converge slowly, which complicates numerical approximations and the derivation of exact convergence conditions. This difficulty is compounded by the fact that the convergence speed is influenced strongly by the continued fraction representation of α. For instance, sufficient conditions for the convergence of these series can be expressed in terms of the continued fraction of α, which lends the term "diophantine" to these series(23).

In our article, we explore the convergence criteria of Diophantine Dirichlet series through the lens of the Riemann-Stieltjes integral. By representing these series via Riemann-Stieltjes integrals and leveraging Young's classical result on α-Hölder and β-Hölder continuity, we develop a new criterion for their convergence.

We consider extending the criterion of Hölder condition in Riemann-Stieltjes integrals for the convergence of Diophantine Dirichlet series of the form $\sum_{n \geq 1} f(\pi n \alpha) n^{-s}$, which will serve as a new method in mathematical literature. We will consider functions f defined on the set $\mathbb{R} \setminus \mathbb{Z}$ such that $\sup_{x \in \mathbb{R} \setminus \mathbb{Z}} |\sin^r(\pi x) f(x)| < \infty$ for some $r \geq 1$, but $\sup_{x \in \mathbb{R} \setminus \mathbb{Z}} |\sin^\rho(\pi x) f(x)| = +\infty$ for any $\rho < r$. Typically, f is a trigonometric function like a power of cotangent or cosecant. The study of the convergence at irrational points of "diophantine" Dirichlet or trigonometric series is a classical subject, studied in particular by Hardy and Littlewood(16), Chowla(8), and Davenport (10).

The results of our paper address a novel general criterion based on the α-Hölder and β-Hölder functions continuity and the transformation of the series via the Abel summation formula for solving convergence issues for the Diophantine Dirichlet series:

$$\sum_{n=1}^\infty \frac{\cot^2(\pi n \alpha)}{n^3}, \quad \sum_{n=1}^\infty \frac{\csc^2(\pi n \alpha)}{n^3}, \sum_{n=1}^\infty \frac{\sec^2(\pi n \alpha)}{n^3},$$

and more generally for $u, v \in \mathbb{R}$ for future cases of relevance

$$\sum_{n=1}^\infty \frac{\cot^v(\pi n \alpha)}{n^u}, \quad \sum_{n=1}^\infty \frac{\csc^v(\pi n \alpha)}{n^u}, \quad \sum_{n=1}^\infty \frac{\sec^v(\pi n \alpha)}{n^u}.$$

For example, the Cookson-Hills series, as discussed by Weisstein (35) and also defined by Sloane (27) as *Sequence A004112* in *"The On-Line Encyclopedia of Integer Sequences"*, represents another significant unsolved problem in analysis and series. Actually, we demonstrate that this series converges, specifically when $u = 3, v = 2$, due to its convergence properties being analogous to those of the Flint-Hills series with $u = 3, v = 2$.

1.5. **Generalization of Diophantine Dirichlet Series.**

In this section, we discuss a potential theorem for the generalization of Diophantine Dirichlet series of the form $\sum_{n \geq 1} \frac{f(\pi n \alpha)}{n^s}$ based on the Abel summation formula. This generalization leverages the properties of the Riemann-Stieltjes integral and the classical results on Hölder continuity.

1.5.1. *Abel Summation Formula.* The Abel summation formula is a powerful tool in analytic number theory, used to transform and study series. For a function $f(x)$ and a sequence (a_n), the Abel summation formula is given by:

$$\sum_{n=1}^{N} a_n f(n) = A_N f(N) - \int_1^N A_x f'(x) \, dx,$$

where $A_N = \sum_{n=1}^{N} a_n$ and A_x is the continuous interpolation of A_n.

To apply this formula to the series of interest, consider the Diophantine Dirichlet series $\sum_{n \geq 1} \frac{f(\pi n \alpha)}{n^s}$. By using Abel's summation formula, we can express this series as:

$$\sum_{n=1}^{N} \frac{f(\pi n \alpha)}{n^s} = \left(\sum_{n=1}^{N} \frac{1}{n^s} \right) f(\pi N \alpha) - \int_1^N \left(\sum_{n=1}^{x} \frac{1}{n^s} \right) \frac{d}{dx} f(\pi x \alpha) \, dx.$$

1.5.2. *Hölder Continuity and Convergence.* Using the properties of the Riemann-Stieltjes integral under α-Hölder and β-Hölder continuity with $\alpha + \beta > 1$, we can analyze the convergence of such series. Recall that a function f is α-Hölder continuous if there exists a constant $C > 0$ such that:

$$|f(x) - f(y)| \leq C|x - y|^\alpha \quad \text{for all } x, y.$$

and there exists a β-Hölder continuous function that works as an integrator such as $g(x) = \lfloor x \rfloor$. For the series $\sum_{n \geq 1} \frac{f(\pi n \alpha)}{n^s}$, assume that f satisfies the following conditions:

(1) f is α-Hölder continuous.
(2) The series converges in the sense of the Riemann-Stieltjes integral.

Given these conditions, we propose the following theorem, which will be more thoroughly demonstrated and explained in Subsection *2.5 General Theorem on the Convergence of Diophantine Dirichlet Series* 2.5:

Theorem 1.3. *Let $\alpha \in \mathbb{I}$, and let $f : \mathbb{R} \to \mathbb{R}$ be a function defined on $\mathbb{R} \setminus \mathbb{Z}$ such that $\sup_{x \in \mathbb{R} \setminus \mathbb{Z}} |\sin^r(\pi x) f(x)| < \infty$ for some $r \geq 1$, but $\sup_{x \in \mathbb{R} \setminus \mathbb{Z}} |\sin^\rho(\pi x) f(x)| = +\infty$ for any $\rho < r$. Assume further that f is α-Hölder continuous, and take into consideration the integrator $g(x) = \lfloor x \rfloor$ (coming from the Riemann-Stieltjes integral representation), which is β-Hölder continuous, with $\alpha + \beta > 1$, then the series*

$$\sum_{n \geq 1} \frac{f(\pi n \alpha)}{n^s}$$

converges and is equivalent to the Riemann-Stieltjes integral representation

$$\int_1^\infty \frac{f(\pi x \alpha)}{x^s} \, d(\lfloor x \rfloor)$$

Theorem 1.3 provides a new criterion for the convergence of Diophantine Dirichlet series, grounded firstly in the Flint-Hills series as we will see in the article. Thus, diverse Diophantine Dirichlet series can be analyzed under this novel criterion. In the following sections, we will demonstrate how to appropriately adjust series of the form $\sum_{n \geq 1} \frac{f(\pi n \alpha)}{n^s}$ through a comprehensive analysis of the Flint-Hills series as a model to follow. We will explore several theorems and properties to reveal the methodology for determining the conditions under which these series converge and when they do not. We will begin by introducing the Flint-Hills series case, which is central to the unresolved status of Diophantine series and forms the cornerstone for understanding and shedding light on this research.

2. Main results

2.1 Asymptotic Analysis of Modified Bessel Functions and the Convergence of the Flint-Hills Series

Let us express using the trigonometric triple-angle identity (3) for the cosecant function, where z can be either a complex or a real argument, as follows:

$$\csc(3z) = \frac{\csc^3(z)}{3\csc^2(z) - 4} \tag{2.1}$$

and utilizing (2.1), we can represent $3\csc^2(z) - 4 = \frac{\csc^3(z)}{\csc(3z)}$. Hence, for $z = n$ where $n \in \mathbb{Z}^+$, we obtain:

$$3\csc^2(n) - 4 = \frac{\csc^3(n)}{\csc(3n)}$$

and multiplying it by $\frac{1}{n^3}$, we get:

$$\frac{3\csc^2(n)}{n^3} - \frac{4}{n^3} = \frac{\csc^3(n)}{n^3 \csc(3n)}$$

Thus, we define the corresponding distributed sums based on the integer index n as follows:

$$3\sum_{n=1}^{\infty} \frac{\csc^2(n)}{n^3} - 4\sum_{n=1}^{\infty} \frac{1}{n^3} = \sum_{n=1}^{\infty} \frac{\csc^3(n)}{n^3 \csc(3n)}, \tag{2.2}$$

where $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ represents Apéry's Constant by definition. Furthermore, there is an elegant representation for $\csc(z)$ in terms of Bessel functions of the first kind (38) and Struve functions (1). Specifically

$$\csc(z) = \sqrt{\frac{2}{\pi z}} \cdot \frac{1}{J_{\frac{1}{2}}(z)}$$

and

$$\csc(z) = \sqrt{\frac{2}{\pi z}} \cdot \frac{1}{H_{-\frac{1}{2}}(z)}$$

which lead to define (2.2) for $z = n$ as follows

$$\sum_{n=1}^{\infty} \frac{\csc^2(n)}{n^3} = \frac{4}{3}\zeta(3) + \frac{2\sqrt{3}}{3\pi} \sum_{n=1}^{\infty} \frac{J_{\frac{1}{2}}(3n)}{n^4 J_{\frac{1}{2}}^3(n)} \tag{2.3}$$

and

$$\sum_{n=1}^{\infty} \frac{\csc^2(n)}{n^3} = \frac{4}{3}\zeta(3) + \frac{2\sqrt{3}}{3\pi} \sum_{n=1}^{\infty} \frac{H_{-\frac{1}{2}}(3n)}{n^4 H_{-\frac{1}{2}}^3(n)} \tag{2.4}$$

It is routine to validate the previous equations thanks to the known relationship between the Struve function and the Bessel function of the first kind, where $H_{-m-\frac{1}{2}}(z) = (-1)^m J_{m+\frac{1}{2}}(z)$. Also, if we work on specific cases when $z = n$ in the expressions

$$\csc(n) = \sqrt{\frac{2}{\pi n}} \cdot \frac{1}{J_{\frac{1}{2}}(n)} = \sqrt{\frac{2}{\pi n}} \cdot \frac{1}{H_{-\frac{1}{2}}(n)}$$

and for $z = 3n$

$$\csc(3n) = \sqrt{\frac{2}{3\pi n}} \cdot \frac{1}{J_{\frac{1}{2}}(3n)} = \sqrt{\frac{2}{3\pi n}} \cdot \frac{1}{H_{-\frac{1}{2}}(3n)}$$

we obtain

$$\frac{\csc^3(n)}{n^3 \csc(3n)} = \frac{1}{n^3}\left(\sqrt{\frac{2}{\pi n}} \cdot \frac{1}{J_{\frac{1}{2}}(n)}\right)^3 \cdot \left(\sqrt{\frac{2}{3\pi n}} \cdot \frac{1}{J_{\frac{1}{2}}(3n)}\right)^{-1} = \frac{2\sqrt{3}}{\pi} \frac{J_{\frac{1}{2}}(3n)}{n^4 J_{\frac{1}{2}}^3(n)} \tag{2.5}$$

and

$$\frac{\csc^3(n)}{n^3 \csc(3n)} = \frac{1}{n^3}\left(\sqrt{\frac{2}{\pi n}} \cdot \frac{1}{H_{-\frac{1}{2}}(n)}\right)^3 \cdot \left(\sqrt{\frac{2}{3\pi n}} \cdot \frac{1}{H_{-\frac{1}{2}}(3n)}\right)^{-1} = \frac{2\sqrt{3}}{\pi} \frac{H_{-\frac{1}{2}}(3n)}{n^4 H_{-\frac{1}{2}}^3(n)} \tag{2.6}$$

Notice that identities (2.5) and (2.6) provide the necessary formulation for equations (2.3) and (2.4) when applied to (2.2).

Lemma 2.1. *Given $n \in \mathbb{Z}^+$, with $J_{\frac{1}{2}}(z)$ being the Bessel function of the first kind with arguments $z = 3n$ and $z = n$, there exist representations for $J_{\frac{1}{2}}(3n)$ and $J_{\frac{1}{2}}^3(n)$ based on the modified Bessel function $I_\alpha(z)$ when $\alpha = \frac{1}{2}$ and evaluated at $z = -i3n$ and $z = -in$.*

$$J_{\frac{1}{2}}(3n) = e^{i\pi/4} I_{\frac{1}{2}}(-i3n) = \frac{1+i}{\sqrt{2}} I_{\frac{1}{2}}(-i3n), \tag{2.7}$$

$$J_{\frac{1}{2}}^3(n) = \left[e^{i\pi/4} I_{\frac{1}{2}}(-in)\right]^3 = e^{i3\pi/4} I_{\frac{1}{2}}^3(-in) = \left[\frac{1+i}{\sqrt{2}}\right]^3 I_{\frac{1}{2}}^3(-in), \tag{2.8}$$

$$\frac{J_{\frac{1}{2}}(3n)}{n^4 J_{\frac{1}{2}}^3(n)} = \frac{e^{i\pi/4} \cdot I_{\frac{1}{2}}(-i3n) \cdot e^{-i3\pi/4} \cdot I_{\frac{1}{2}}^{-3}(-in)}{n^4} = -i\frac{I_{\frac{1}{2}}(-i3n)}{n^4 \cdot I_{\frac{1}{2}}^3(-in)}. \tag{2.9}$$

Where $e = 2.71828\ldots$ is Euler's number, $\pi = 3.1415\ldots$ is the mathematical constant pi, and $i = \sqrt{-1}$ is the imaginary unit.

Proof. The established identity between the Bessel function of the first kind $J_\alpha(iz)$ and the modified Bessel function $I_\alpha(z)$ prompts the derivation of the relationships (2.7), (2.8), and (2.9). This identity is

$$J_\alpha(iz) = e^{i\alpha\pi/2} I_\alpha(z), \tag{2.10}$$

because when $\alpha = \frac{1}{2}$ and $z = -i3n$, we get (2.7), step by step, as follows

$$J_{\frac{1}{2}}(3n) = J_{\frac{1}{2}}(i \cdot (-i3n)) = e^{i\left(\frac{1}{2}\right)\frac{\pi}{2}} I_{\frac{1}{2}}(-i3n) = e^{i\pi/4} I_{\frac{1}{2}}(-i3n) = \frac{1+i}{\sqrt{2}} I_{\frac{1}{2}}(-i3n),$$

and when $\alpha = \frac{1}{2}$ and $z = -in$, we get (2.8) based on the same procedure. Thus, we infer (2.9). $\qquad\square$

Theorem 2.2. *The Flint-Hills series is founded upon the novel series*

$$S_I = -i \sum_{n=1}^{\infty} \frac{1}{n^4} \frac{I_{\frac{1}{2}}(-i3n)}{I_{\frac{1}{2}}^3(-in)}$$

governed by the behavior of the modified Bessel function I, particularly through the ratio

$$\frac{I_{\frac{1}{2}}(-i3n)}{I_{\frac{1}{2}}^3(-in)} \quad and \quad -i \cdot \frac{1}{n^4}.$$

The Flint-Hills series is established as follows

$$\sum_{n=1}^{\infty} \frac{\csc^2(n)}{n^3} = \frac{4}{3}\zeta(3) - i \cdot \frac{2\sqrt{3}}{3\pi} \sum_{n=1}^{\infty} \frac{1}{n^4} \cdot \frac{I_{\frac{1}{2}}(-i3n)}{I_{\frac{1}{2}}^3(-in)}. \tag{2.11}$$

Proof. As Lemma 2.1 states

$$\frac{1}{n^4} \frac{J_{\frac{1}{2}}(3n)}{J_{\frac{1}{2}}^3(n)} = \frac{-i}{n^4} \cdot \frac{I_{\frac{1}{2}}(-i3n)}{I_{\frac{1}{2}}^3(-in)},$$

then the relationship (2.3) is just modified by the respective equivalent term leading to

$$\sum_{n=1}^{\infty} \frac{\csc^2(n)}{n^3} = \frac{4}{3}\zeta(3) - i \cdot \frac{2\sqrt{3}}{3\pi} \sum_{n=1}^{\infty} \frac{1}{n^4} \cdot \frac{I_{\frac{1}{2}}(-i3n)}{I_{\frac{1}{2}}^3(-in)}.$$

Further analysis conducted in this research will demonstrate the definitive convergence of $S_I = -i \sum_{n=1}^{\infty} \frac{1}{n^4} \frac{I_{\frac{1}{2}}(-i3n)}{I_{\frac{1}{2}}^3(-in)}$, and consequently, the convergence of the Flint-Hills series as well. To achieve this objective, it is imperative to initially investigate the asymptotic behavior of the modified Bessel functions, particularly through the ratio $\frac{I_\alpha(-i3n)}{I_\alpha^3(-in)}$, and discern the behavior of the series S_I. $\qquad\square$

We deal with the expansion of

$$\sum_{n=1}^{\infty} \frac{1}{n^4} \frac{J_{1/2}(3n)}{J_{1/2}^3(n)}$$

from $n = 1$ to a specified value $n = \sigma - 1 \in \mathbb{Z}^+$, and from $n = \sigma$ to infinity, ∞, as shown carefully by these steps:

$$\sum_{n=1}^{\infty} \frac{1}{n^4} \frac{J_{1/2}(3n)}{J_{1/2}^3(n)} = \frac{1}{1^4} \frac{J_{1/2}(3(1))}{J_{1/2}^3(1)} + \frac{1}{2^4} \frac{J_{1/2}(3(2))}{J_{1/2}^3(2)} + \cdots + \frac{1}{(\sigma-1)^4} \frac{J_{1/2}(3(\sigma-1))}{J_{1/2}^3(\sigma-1)} + \frac{1}{\sigma^4} \frac{J_{1/2}(3\sigma)}{J_{1/2}^3(\sigma)} + \cdots ,$$

$$\sum_{n=1}^{\sigma-1} \frac{1}{n^4} \frac{J_{1/2}(3n)}{J_{1/2}^3(n)} = \frac{J_{1/2}(3)}{J_{1/2}^3(1)} + \frac{1}{2^4} \frac{J_{1/2}(6)}{J_{1/2}^3(2)} + \cdots + \frac{1}{(\sigma-1)^4} \frac{J_{1/2}(3(\sigma-1))}{J_{1/2}^3(\sigma-1)} + \frac{1}{\sigma^4} \frac{J_{1/2}(3\sigma)}{J_{1/2}^3(\sigma)} + \cdots ,$$

and when contemplating the equivalent terms

$$\frac{1}{n^4} \frac{J_{1/2}(3n)}{J_{1/2}^3(n)}$$

for $n = 1, 2, 3, \ldots, \sigma - 1$, we define the sum Λ as follows:

$$\Lambda = \frac{J_{1/2}(3)}{J_{1/2}^3(1)} + \frac{1}{2^4} \frac{J_{1/2}(6)}{J_{1/2}^3(2)} + \cdots + \frac{1}{(\sigma-1)^4} \frac{J_{1/2}(3(\sigma-1))}{J_{1/2}^3(\sigma-1)} = \sum_{n=1}^{\sigma-1} \frac{1}{n^4} \frac{J_{1/2}(3n)}{J_{1/2}^3(n)}, \quad (2.12)$$

or its equivalent version based on the modified Bessel function:

$$\Lambda = -i \cdot \frac{I_{1/2}(-i3)}{I_{1/2}^3(-i)} - \frac{i}{2^4} \frac{I_{1/2}(-i6)}{I_{1/2}^3(-i2)} - \cdots - \frac{i}{(\sigma-1)^4} \frac{I_{1/2}(-i3(\sigma-1))}{I_{1/2}^3(-i(\sigma-1))}$$
$$= -i \sum_{n=1}^{\sigma-1} \frac{1}{n^4} \frac{I_{1/2}(-i3n)}{I_{1/2}^3(-in)}. \tag{2.13}$$

Notice that Λ is a finite number, as the summation in (2.12) or (2.13) implies computation extends only until the final index $n = \sigma - 1$, and the ratio

$$-\frac{i}{n^4} \frac{I_{1/2}(-i3n)}{I_{1/2}^3(-in)}$$

does not diverge when considering the modified Bessel function $I_{1/2}$ for the arguments supported by n. A detailed discussion will follow later on. Thus, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^4} \frac{J_{1/2}(3n)}{J_{1/2}^3(n)} = \Lambda + \frac{1}{\sigma^4} \frac{J_{1/2}(3\sigma)}{J_{1/2}^3(\sigma)} + \cdots \equiv \Lambda - \frac{i}{\sigma^4} \frac{I_{1/2}(-i3\sigma)}{I_{1/2}^3(-i\sigma)} - \cdots , \quad (2.14)$$

We can designate Θ for the subsequent summation, i.e., from $n = \sigma$ to infinity, ∞:

$$\Theta = \frac{1}{\sigma^4} \frac{J_{1/2}(3\sigma)}{J_{1/2}^3(\sigma)} + \cdots = -\frac{i}{\sigma^4} \frac{I_{1/2}(-i3\sigma)}{I_{1/2}^3(-i\sigma)} - \cdots = -i \sum_{n=\sigma}^{\infty} \frac{1}{n^4} \frac{I_{1/2}(-i3n)}{I_{1/2}^3(-in)}. \quad (2.15)$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{n^4} \frac{J_{1/2}(3n)}{J_{1/2}^3(n)} = \Lambda + \Theta. \tag{2.16}$$

2.1.1 Evaluation of Θ using Asymptotic Formula for Modified Bessel Function.

We begin with the asymptotic formula for the modified Bessel function $I_\nu(z)$ (38), $\nu = \frac{1}{2}$, as $|z| \to \infty$:

$$I_\nu(z) \sim \frac{1}{\sqrt{2\pi}} z^\nu (-z^2)^{-\frac{2\nu+1}{4}}$$

$$\left[\exp\left(-i \left(\frac{(2\nu+1)\pi}{4} - \sqrt{-z^2} \right) \right) \left(1 + O\left(\frac{1}{z} \right) \right) \right.$$

$$\left. + \exp\left(i \left(\frac{(2\nu+1)\pi}{4} - \sqrt{-z^2} \right) \right) \left(1 + O\left(\frac{1}{z} \right) \right) \right].$$

We need to apply this formula to $I_{1/2}(-i3n)$ and $I_{1/2}^3(-in)$ in the expression (2.15):

$$\Theta = -i \sum_{n=\sigma}^{\infty} \frac{1}{n^4} \frac{I_{1/2}(-i3n)}{I_{1/2}^3(-in)}.$$

Case 1: $z = -3in$. Applying the asymptotic formula for $I_{1/2}(-i3n)$:

$$I_{1/2}(-i3n) \sim \frac{1}{\sqrt{2\pi}} (-3in)^{\frac{1}{2}} (9n^2)^{-\frac{1}{2}}$$

$$\times \left[\exp\left(-i \left(\frac{\pi}{2} - 3n \right) \right) + \exp\left(i \left(\frac{\pi}{2} - 3n \right) \right) \right] \left(1 + O\left(\frac{1}{-3in} \right) \right).$$

This gives the asymptotic behavior of $I_{1/2}(-i3n)$ as $n \to \infty$.

Case 2: $z = -in$. For $I_{1/2}(-in)$:

$$I_{\frac{1}{2}}(-in) \sim \frac{1}{\sqrt{2\pi}} (-i.n)^{\frac{1}{2}} n^{-1}$$

$$\times \left[\exp\left(-i \left(\frac{\pi}{2} - n \right) \right) + \exp\left(i \left(\frac{\pi}{2} - n \right) \right) \right] \left(1 + O\left(\frac{1}{-in} \right) \right).$$

The ratio of the asymptotic behavior of $\frac{I_{1/2}(-i3n)}{(I_{1/2}(-in))^3}$ as $n \to \infty$ is given by:

$$\frac{I_{1/2}(-i3n)}{\left[I_{\frac{1}{2}}(-in) \right]^3} \sim \frac{\left[\frac{1}{\sqrt{2\pi}} (-3in)^{\frac{1}{2}} (9n^2)^{-\frac{1}{2}} \left[\exp\left(-i \left(\frac{\pi}{2} - 3n \right) \right) + \exp\left(i \left(\frac{\pi}{2} - 3n \right) \right) \right] \right]}{\left[\frac{1}{\sqrt{2\pi}} (-in)^{\frac{1}{2}} n^{-1} \left[\exp\left(-i \left(\frac{\pi}{2} - n \right) \right) + \exp\left(i \left(\frac{\pi}{2} - n \right) \right) \right] \right]^3}$$

$$\times \left(1 + O\left(\frac{1}{-3in} \right) \right) \left(1 + O\left(\frac{1}{-in} \right) \right)^{-3}. \tag{2.17}$$

or without invoking the big O notation

$$\frac{I_{1/2}(-i3n)}{\left[I_{\frac{1}{2}}(-in)\right]^3} \sim \frac{\frac{1}{\sqrt{2\pi}}(-3in)^{\frac{1}{2}}(9n^2)^{-\frac{1}{2}}\left[\exp\left(-i\left(\frac{\pi}{2}-3n\right)\right)+\exp\left(i\left(\frac{\pi}{2}-3n\right)\right)\right]}{\left(\frac{1}{\sqrt{2\pi}}(-in)^{\frac{1}{2}}n^{-1}\left[\exp\left(-i\left(\frac{\pi}{2}-n\right)\right)+\exp\left(i\left(\frac{\pi}{2}-n\right)\right)\right]\right)^3}.$$

Then, after routine algebraic steps, we simplify equation (2.17) to include it in the ratio $\frac{I_{1/2}(-i3n)}{(I_{1/2}(-in))^3}$ of (2.15). Thus, (2.15) is simplified, using the convention for asymptotic approximation behavior as follows:

$$\Theta \cong \frac{-2\pi\sqrt{3}}{3} \cdot \sum_{n=\sigma}^{\infty} \frac{1}{n^3} \frac{\left[-\exp(i\,(3n))+\exp\left(i\,(-3n)\right)\right]}{\left[-\exp(i\,(n))+\exp\left(i\,(-n)\right)\right]^3}. \tag{2.18}$$

In fact, (2.18), after utilizing the associated identities of Euler's formula, is equivalent to

$$\Theta \cong \left(-\frac{2\pi\sqrt{3}}{3}\right)\sum_{n=\sigma}^{\infty}\left(\frac{1}{n^3}-\frac{3}{4}\cdot\frac{1}{n^3}\cdot\csc^2(n)\right) = -\frac{2\pi\sqrt{3}}{3}\left(\sum_{n=\sigma}^{\infty}\frac{1}{n^3}-\frac{3}{4}\sum_{n=\sigma}^{\infty}\frac{\csc^2(n)}{n^3}\right), \tag{2.19}$$

because Euler's formula states that:

$$e^{ix}=\cos(x)+i\sin(x) \quad \text{and} \quad e^{-ix}=\cos(x)-i\sin(x)$$

Using these identities, we can convert exponentials to trigonometric functions in (2.18).

Let's consider the numerator in (2.18) first:

$$-\exp(i3n)+\exp(-i3n)$$

Using Euler's formula:

$$-\cos(3n)-i\sin(3n)+\cos(3n)-i\sin(3n)=-2i\sin(3n)$$

Now, let's consider the denominator:

$$[-\exp(in)+\exp(-in)]^3$$

Using Euler's formula:

$$[-\cos(n)-i\sin(n)+\cos(n)-i\sin(n)]^3=[-2i\sin(n)]^3$$

Since $i^3=-i$, this becomes:

$$-8(-i)\sin^3(n)=8i\sin^3(n)$$

So, the expression:

$$\frac{[-\exp(i3n)+\exp(-i3n)]}{[-\exp(in)+\exp(-in)]^3}$$

Becomes:

$$\frac{-2i\sin(3n)}{8i\sin^3(n)}$$

Simplifying this:

$$\frac{-2i\sin(3n)}{8i\sin^3(n)}=\frac{-2\sin(3n)}{8\sin^3(n)}=\frac{-\sin(3n)}{4\sin^3(n)}$$

To simplify the expression:
$$\frac{-\sin(3n)}{4\sin^3(n)}$$
we start by using the triple angle identity for sine:
$$\sin(3n) = 3\sin(n) - 4\sin^3(n).$$
Substituting $\sin(3n)$ into the numerator, we get:
$$\frac{-(3\sin(n) - 4\sin^3(n))}{4\sin^3(n)}.$$
Distributing the negative sign in the numerator, we obtain:
$$\frac{-3\sin(n) + 4\sin^3(n)}{4\sin^3(n)}.$$
Separating the fraction into two terms, we have:
$$\frac{-3\sin(n)}{4\sin^3(n)} + \frac{4\sin^3(n)}{4\sin^3(n)}.$$
Simplifying each term, we find:
$$\frac{-3\sin(n)}{4\sin^3(n)} = -\frac{3}{4} \cdot \frac{1}{\sin^2(n)} = -\frac{3}{4}\csc^2(n),$$
$$\frac{4\sin^3(n)}{4\sin^3(n)} = 1.$$
Combining the simplified terms, we get:
$$1 - \frac{3}{4}\csc^2(n).$$
that is why we see the use of
$$\frac{-\sin(3n)}{4\sin^3(n)} = 1 - \frac{3}{4}\csc^2(n)$$
in the structure of (2.19) as
$$\Theta \stackrel{\simeq}{=} \left(-\frac{2\pi\sqrt{3}}{3}\right)\sum_{n=\sigma}^{\infty}\left(\frac{1}{n^3} - \frac{3}{4}\cdot\frac{1}{n^3}\cdot\csc^2(n)\right) = -\frac{2\pi\sqrt{3}}{3}\left(\sum_{n=\sigma}^{\infty}\frac{1}{n^3} - \frac{3}{4}\sum_{n=\sigma}^{\infty}\frac{\csc^2(n)}{n^3}\right).$$

This contains the term $\sum_{n=\sigma}^{\infty}\frac{1}{n^3}$ which is equal to $-\frac{\psi''(\sigma)}{2}$ (33). The sum $\sum_{n=\sigma}^{\infty}\frac{1}{n^3}$ can be linked to the polygamma function, which is the derivative of the digamma function $\psi(x)$. The polygamma function of order m, denoted as $\psi^{(m)}(x)$, is defined as the $(m+1)$-th derivative of the logarithm of the gamma function $\Gamma(x)$:
$$\psi^{(m)}(x) = \frac{d^{m+1}}{dx^{m+1}}\ln\Gamma(x).$$

Specifically, the trigamma function is the second derivative of the digamma function. By definition, we have:
$$\sum_{n=\sigma}^{\infty}\frac{1}{n^3} = \frac{\zeta(3,\sigma)}{2},$$

where $\zeta(s, \sigma)$ is the Hurwitz zeta function. For large σ, this can be approximated in terms of the trigamma function as:

$$\sum_{n=\sigma}^{\infty} \frac{1}{n^3} \sim \frac{1}{2} \psi^{(2)}(\sigma),$$

where $\psi^{(2)}(\sigma)$ or $\psi''(\sigma)$ is the second derivative of the digamma function (also known as the trigamma function) evaluated at σ.

This property is useful in various contexts, such as in the evaluation of series and in approximations involving the gamma and digamma functions.

This leads to the definition

$$\Theta \stackrel{\simeq}{=} -\frac{2\pi\sqrt{3}}{3}\left(-\frac{\psi''(\sigma)}{2} - \frac{3}{4}\sum_{n=\sigma}^{\infty} \frac{\csc^2(n)}{n^3}\right), \tag{2.20}$$

or

$$\Theta \stackrel{\simeq}{=} \frac{\pi\sqrt{3}}{3}\psi''(\sigma) + \frac{\pi\sqrt{3}}{2}\sum_{n=\sigma}^{\infty} \frac{\csc^2(n)}{n^3}, \tag{2.21}$$

2.1.2 Partial Summation of the Flint-Hills Series via Asymptotic Behavior of Modified Bessel Functions.

As the Flint-Hills series is established by (2.11) via the Theorem 2.2, we just utilize the forms given by (2.13), (2.14), (2.15), (2.16) and (2.21) in (2.11) as follows

$$\sum_{n=1}^{\infty} \frac{\csc^2(n)}{n^3} = \frac{4}{3}\zeta(3) - i \cdot \frac{2\sqrt{3}}{3\pi} \sum_{n=1}^{\infty} \frac{1}{n^4} \cdot \frac{I_{\frac{1}{2}}(-i3n)}{I_{\frac{1}{2}}^3(-in)}$$

$$= \frac{4}{3}\zeta(3) + \frac{2\sqrt{3}}{3\pi}\left(-i \sum_{n=1}^{\infty} \frac{1}{n^4} \cdot \frac{I_{\frac{1}{2}}(-i3n)}{I_{\frac{1}{2}}^3(-in)}\right)$$

$$= \frac{4}{3}\zeta(3) + \frac{2\sqrt{3}}{3\pi}\left(-i \sum_{n=1}^{\sigma-1} \frac{1}{n^4} \cdot \frac{I_{\frac{1}{2}}(-i3n)}{I_{\frac{1}{2}}^3(-in)} - i \sum_{n=\sigma}^{\infty} \frac{1}{n^4} \cdot \frac{I_{\frac{1}{2}}(-i3n)}{I_{\frac{1}{2}}^3(-in)}\right)$$

where we identify Λ of (2.13) and Θ of (2.15) respectively

$$\Lambda = -i \cdot \frac{I_{1/2}(-i3)}{I_{1/2}^3(-i)} - \frac{i}{2^4} \frac{I_{1/2}(-i6)}{I_{1/2}^3(-i2)} - \cdots - \frac{i}{(\sigma-1)^4} \frac{I_{1/2}(-i3(\sigma-1))}{I_{1/2}^3(-i(\sigma-1))}$$

$$= -i \sum_{n=1}^{\sigma-1} \frac{1}{n^4} \frac{I_{1/2}(-i3n)}{I_{1/2}^3(-in)}$$

and

$$\Theta = \frac{1}{\sigma^4} \frac{J_{1/2}(3\sigma)}{J_{1/2}^3(\sigma)} + \cdots = -\frac{i}{\sigma^4} \frac{I_{1/2}(-i3\sigma)}{I_{1/2}^3(-i\sigma)} - \cdots = -i \sum_{n=\sigma}^{\infty} \frac{1}{n^4} \frac{I_{1/2}(-i3n)}{I_{1/2}^3(-in)}.$$

Based on these expressions we proceed to recognize that

$$\sum_{n=1}^{\infty} \frac{\csc^2(n)}{n^3} =$$

$$= \frac{4}{3}\zeta(3) + \frac{2\sqrt{3}}{3\pi}\left(-i\sum_{n=1}^{\sigma-1}\frac{1}{n^4}\cdot\frac{I_{\frac{1}{2}}(-i3n)}{I_{\frac{1}{2}}^3(-in)} - i\sum_{n=\sigma}^{\infty}\frac{1}{n^4}\cdot\frac{I_{\frac{1}{2}}(-i3n)}{I_{\frac{1}{2}}^3(-in)}\right)$$

$$= \frac{4}{3}\zeta(3) + \frac{2\sqrt{3}}{3\pi}(\Lambda + \Theta)$$

and recalling (2.21) we replace Θ to obtain

$$\sum_{n=1}^{\infty}\frac{\csc^2(n)}{n^3} \cong \frac{4}{3}\zeta(3) + \frac{2\sqrt{3}}{3\pi}\left(\Lambda + \frac{\pi\sqrt{3}}{3}\psi''(\sigma) + \frac{\pi\sqrt{3}}{2}\sum_{n=\sigma}^{\infty}\frac{\csc^2(n)}{n^3}\right). \qquad (2.22)$$

which is manipulated algebraically such as

$$\sum_{n=1}^{\infty}\frac{\csc^2(n)}{n^3} \simeq \frac{4}{3}\zeta(3) + \frac{2\sqrt{3}}{3\pi}\Lambda + \frac{2}{3}\psi''(\sigma) + \sum_{n=\sigma}^{\infty}\frac{\csc^2(n)}{n^3}. \qquad (2.23)$$

Notice that after combining the sums in n from (2.23) and subtracting one from the other, we obtain the partial sum indicated by $\sum_{n=1}^{\sigma-1}\frac{\csc^2(n)}{n^3} = \sum_{n=1}^{\sigma-1}\frac{1}{n^3\sin^2(n)}$ introduced in (1.1)

$$\sum_{n=1}^{\infty}\frac{\csc^2(n)}{n^3} - \sum_{n=\sigma}^{\infty}\frac{\csc^2(n)}{n^3} = \sum_{n=1}^{\sigma-1}\frac{1}{n^3\sin^2(n)} \simeq \frac{4}{3}\zeta(3) + \frac{2\sqrt{3}}{3\pi}\Lambda(\sigma) + \frac{2}{3}\psi''(t)\Big|_{t=\sigma}.$$
$$(2.24)$$

In (2.23), we have adjusted the terms such that $\Lambda = \Lambda(\sigma)$ and $\psi''(\sigma) = \psi''(t)\Big|_{t=\sigma}$ to obtain (1.1) or (2.24). These terms will later be considered in relation to the behavior of general functions with the variable $t \in \mathbb{R}$ and $\sigma \in \mathbb{Z}^+$.

Theorem 2.3. *The partial sum of the Flint-Hills series, expanded from $n = 1$ to a high sample $n = \sigma - 1$, is defined as*

$$\sum_{n=1}^{\sigma-1}\frac{1}{n^3\sin^2(n)} \cong \frac{4}{3}\zeta(3) + \frac{2\sqrt{3}}{3\pi}\Lambda(\sigma) + \frac{2}{3}\psi''(t)\Big|_{t=\sigma},$$

where $t \in \mathbb{R}$, $\sigma \in \mathbb{Z}^+$, Apéry's constant $\zeta(3)$, $\psi''(t)$ is the second derivative of the digamma function with respect to t, and the series $\Lambda(\sigma)$ (later adjusted as a function $\Lambda(t)$) is defined as

$$\Lambda(\sigma) = -i\sum_{n=1}^{\sigma-1}\frac{1}{n^4}\frac{I_{1/2}(-i3n)}{I_{1/2}^3(-in)},$$

as given in (2.13).

Proof. The proof comes from the analysis of the current subsection *2.1.2 Partial Summation of the Flint-Hills Series via Asymptotic Behavior of Modified Bessel Functions* starting here, which demonstrates this theorem.

$\square$

2.1.3 Abel Summation Method for Analyzing Convergence of $\Lambda = \Lambda(\sigma)$.

An advanced method to establish the convergence of the series $\Lambda(\sigma)$ (2.13) is through the use of the Abel summation method (20).

Consider the series $\Lambda(\sigma)$ defined as:

$$\Lambda(\sigma) = -i \sum_{n=1}^{\sigma-1} \frac{1}{n^4} \frac{I_{1/2}(-i3n)}{I_{1/2}^3(-in)},$$

where $I_{1/2}$ denotes the modified Bessel function of the first kind, σ is an integer significantly larger than 1 ($\sigma \gg 8000$), and i represents the imaginary unit.

Abel's summation formula relates the series $\Lambda(\sigma)$ to the behavior of the sequence $\sum_{n=1}^{\sigma-1} \frac{1}{n^4}$. It states:

$$\sum_{n=1}^{\sigma-1} a_n b_n = A(\sigma)b_\sigma - \int_1^\sigma A(x)b'(x)\,dx,$$

where $a_n = \frac{1}{n^4}$ and $b_n = \frac{I_{1/2}(-i3n)}{I_{1/2}^3(-in)}$, $A(x) = \sum_{n=1}^x a_n$, and $b(x) = b_x$.

Convergence of $\sum_{n=1}^{\sigma-1} \frac{1}{n^4}$: The series $\sum_{n=1}^{\sigma-1} \frac{1}{n^4}$ is a p-series with $p = 4$, which converges since $p > 1$.

Decay of $\frac{I_{1/2}(-i3n)}{I_{1/2}^3(-in)}$: From the asymptotic analysis of modified Bessel functions, $\frac{I_{1/2}(-i3n)}{I_{1/2}^3(-in)}$ decays sufficiently rapidly as $n \to \infty$.

The decay behavior of $\frac{I_{1/2}(-i3n)}{I_{1/2}^3(-in)}$ can be rigorously analyzed using asymptotic properties of modified Bessel functions. Recall that $I_{1/2}(z)$, the modified Bessel function of the first kind of order $\frac{1}{2}$, has the asymptotic form:

$$I_{1/2}(z) \sim \frac{e^z}{\sqrt{2\pi z}} \quad \text{as} \quad |z| \to \infty.$$

Applying this asymptotic approximation to $I_{1/2}(-in)$ and $I_{1/2}(-i3n)$, we obtain:

$$I_{1/2}(-in) \sim \frac{e^{-in}}{\sqrt{-2\pi in}},$$

$$I_{1/2}(-i3n) \sim \frac{e^{-i3n}}{\sqrt{-2\pi i3n}}.$$

Thus, the ratio $\frac{I_{1/2}(-i3n)}{I_{1/2}^3(-in)}$ simplifies as follows:

$$\frac{I_{1/2}(-i3n)}{I_{1/2}^3(-in)} \sim \frac{\frac{e^{-i3n}}{\sqrt{-2\pi i3n}}}{\left(\frac{e^{-in}}{\sqrt{-2\pi in}}\right)^3}.$$

This expression indicates that as $n \to \infty$, $\frac{I_{1/2}(-i3n)}{I_{1/2}^3(-in)}$ exhibits rapid decay. Specifically, the dominant exponential factors contribute to the decay rate, ensuring that the series $\sum_{n=1}^\infty \frac{1}{n^4} \frac{I_{1/2}(-i3n)}{I_{1/2}^3(-in)}$ converges for large σ.

We arrive to the Conclusion by Abel's Method: By applying Abel's summation method, we formally connect the convergence of $\Lambda(\sigma)$ to the convergence of $\sum_{n=1}^{\sigma-1} \frac{1}{n^4}$.

Since $\sum_{n=1}^{\sigma-1} \frac{1}{n^4}$ converges and $\frac{I_{1/2}(-i3n)}{I_{1/2}^3(-in)}$ decays sufficiently rapidly, $\Lambda(\sigma)$ converges for $\sigma \gg 8000$.

Numerical Verification. While the theoretical analysis strongly suggests convergence, numerical verification of partial sums can provide additional confirmation. By computing the partial sums

$$\sum_{n=1}^{N} \left(-i \cdot \frac{n^{-4} \cdot I_{1/2}(-3in)}{\left(I_{1/2}(-in)\right)^3} \right)$$

for increasing values of N, one can observe the behavior of the series and verify its convergence numerically. As presented in the introduction of this paper, c_1 is derived from (1.2) and is denoted by expression (1.6), where

$$c_1 = \lim_{\sigma \to \infty} \left(-i \sum_{n=1}^{\sigma-1} \frac{I_{1/2}(-3i \cdot n)}{n^4 I_{1/2}^3(-i \cdot n)} \right) \approx 78.1160806386.$$

Typical computational tools like WolframAlpha(37) provide accurate computations. For instance, when $\sigma \geq 10001 - 1 = 10000$,

Sum[- i ∗ Divide [BesselI \(4 0) Divide [1 , 2]\(4 4) − 3∗ i ∗n \(4 1) , Power [n , 4] ∗ Power [BesselI \(4 0) Divide [1 , 2]\(4 4) −n∗ i \(4 1) , 3]] , { n , 1 , 1 0 0 0 0 }]

Thus, we refer back to (1.5) to deduce the Flint-Hills series

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \sin^2(n)} = \mathcal{O}(\zeta(w)) = \mathcal{O}(1) \cong \frac{4}{3}\zeta(3) + \frac{2\sqrt{3}}{3\pi}c_1 \lesssim \frac{\pi^2}{6\delta^2}.$$

As a result, the expected convergent value $\mathcal{O}(1)$ for the Flint-Hills series closely approximates 30.314510, aligning with the mathematical community's expectations regarding the series' convergence. Later, we will demonstrate how Hölder's inequality further supports this observation for $\mathcal{O}(1) \cong 30.314510 \lesssim \frac{\pi^2}{6\delta^2}$, where $0.23 \lesssim \delta \lesssim 0.232942$. Furthermore, we illustrate how calculus can model a novel function $\Psi(t) = \frac{4}{3}\zeta(3) + \frac{2\sqrt{3}}{3\pi}\Lambda(t) + \frac{2}{3}\psi''(t)$, based on $\Lambda(t)$ (1.2)

$$\Lambda(t) = c_1 - \frac{\pi}{\sqrt{3}}\psi''(t) = -i\sum_{n=1}^{t-1} \frac{I_{1/2}(-3i \cdot n)}{n^4 I_{1/2}^3(-i \cdot n)}$$

over $t \in [\sigma, \infty)$, such that its derivatives $\Lambda^{(m)}(t) = -\frac{\pi}{\sqrt{3}}\psi^{(m+2)}(t)$ exist for $m \geq 1$

2.1.4 The bounded functions $\Psi(t) = \frac{4}{3}\zeta(3) + \frac{2\sqrt{3}}{3\pi}\Lambda(t) + \frac{2}{3}\psi''(t)$ and $\Lambda(t)$.

Corollary 2.4. *Given a bounded function, i.e., $\Psi(t) = \frac{4}{3}\zeta(3) + \frac{2\sqrt{3}}{3\pi}\Lambda(t) + \frac{2}{3}\psi''(t)$, associated with the convergent behavior of the partial summation for the Flint-Hills series, then the first derivative of $\Psi(t)$ with respect to t, or $\Psi'(t) = \frac{d\Psi}{dt}$, equated to zero gives:*

$$\Lambda(t) = c_1 - \frac{\pi}{\sqrt{3}}\psi''(t),$$

where $\psi''(t)$ is the second derivative of the digamma function with respect to t, and c_1 is a constant derived from the slope field of a first-order ordinary differential equation, for which the solution to find is $\Lambda(t)$, i.e.,

$$\Lambda'(t) = -\frac{\pi}{\sqrt{3}}\psi'''(t),$$

where $\psi'''(t)$ is the third derivative of the digamma function with respect to t.

Proof. Since the first derivative of a constant value approached by the partial sum of the Flint-Hills series from $n = 1$ to a limit $n = \sigma - 1$ (where $t \in [\sigma, \infty)$, $\sigma \gg 8000 \in \mathbb{Z}^+$), i.e., $\frac{d}{dt}\left(\sum_{n=1}^{\sigma-1}\frac{1}{n^3\sin^2(n)}\right) = \frac{d}{dt}(S_\sigma) = 0$, it is expected to obtain the same consistency when $\Psi'(t)\Big|_{t=\sigma} = 0$ or specifically $\Psi'(t)\Big|_{t=\sigma} = \frac{d}{dt}\left(\sum_{n=1}^{\sigma-1}\frac{1}{n^3\sin^2(n)}\right) = \frac{d}{dt}(S_\sigma) = 0$. This does not infer a maximum, minimum, or saddle point, but rather the situation where the derivative of the partial sums of the Flint-Hills series is the derivative of a constant S_σ, which is zero.

Thus, considering that

$$\frac{d\Psi}{dt} = \frac{d}{dt}\left(\frac{4}{3}\zeta(3)\right) + \frac{d}{dt}\left(\frac{2\sqrt{3}}{3\pi}\Lambda(t)\right) + \frac{d}{dt}\left(\frac{2}{3}\psi''(t)\right) = 0,$$

we have

$$0 + \frac{2\sqrt{3}}{3\pi}\Lambda'(t) + \frac{2}{3}\psi'''(t) = 0.$$

After routine algebraic manipulation, we obtain a first-order ordinary differential equation defining a slope field based on the constant c_1:

$$\Lambda'(t) = -\frac{\pi}{\sqrt{3}}\psi'''(t).$$

Integrating this, we get $\Lambda(t) = -\frac{\pi}{\sqrt{3}}\int \psi'''(t)\,dt + c_1$, then by unifying constants into a single c_1 it leads to $\Lambda(t) = c_1 - \frac{\pi}{\sqrt{3}}\psi''(t)$.

This constant c_1 must be finite and consistent, given that $\Lambda(t) = -i\sum_{n=1}^{t-1}\frac{I_{1/2}(-3i\cdot n)}{n^4[I_{1/2}(-i\cdot n)]^3}$ converges (see (1.2)). We have established the convergence of the sum in (1.2), ensuring it does not tend towards infinity or become indeterminate. This consistency is supported by the fact that $\lim_{\sigma\to\infty}\psi''(t)\Big|_{t=\sigma} = 0$ and $\lim_{\sigma\to\infty}\Lambda(t)\Big|_{t=\sigma} = c_1$, or finally, by the conclusion in (1.6) that $c_1 = \lim_{\sigma\to\infty}\left(-i\sum_{n=1}^{\sigma-1}\frac{I_{1/2}(-3i\cdot n)}{n^4 I_{1/2}^3(-i\cdot n)}\right) \approx 78.1160806386$. That fulfills the requirement of the existence of a constant $c_1 \approx 78.1160806386$, derived from the slope field of the first-order ordinary differential equation we have analyzed. As shown in Figure 1, the slope field illustrates the behavior of the differential equation and supports our analysis.
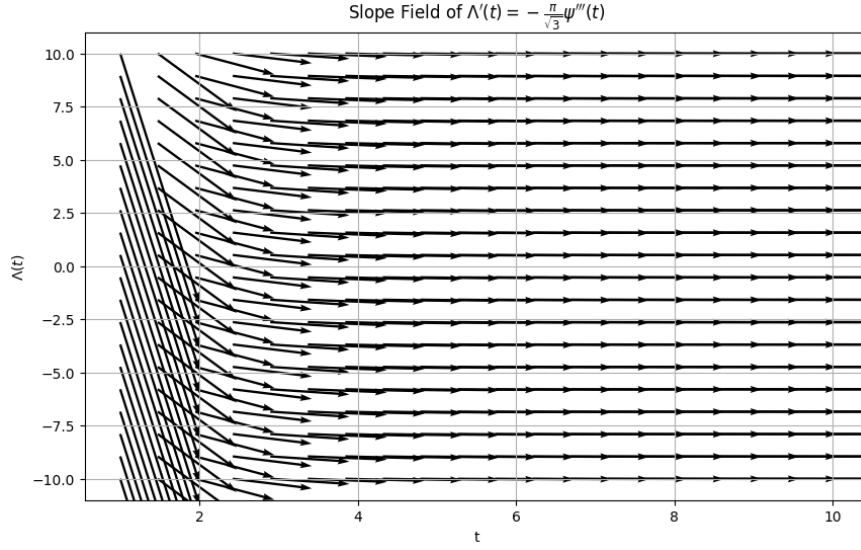
FIGURE 1. Slope field of the first-order ordinary differential equation $\Lambda'(t) = -\frac{\pi}{\sqrt{3}}\psi'''(t)$, where $\psi'''(t)$ is the third derivative of the digamma function with respect to t.

Corollary 2.5. *Given $\Lambda(t)$ from Corollary 2.4, its derivatives are defined by*

$$\Lambda^{(m)}(t) = -\frac{\pi}{\sqrt{3}}\psi^{(m+2)}(t),$$

where $m \geq 1$, and $\psi^{(m+2)}(t)$ denotes the m-th derivative of the digamma function with respect to t.

Proof.

As proved via Corollary 2.4, we can apply the first derivative to the left side of (1.2) and successively generate the higher-order derivatives from

$$\Lambda'(t) = -\frac{\pi}{\sqrt{3}}\psi'''(t),$$

i.e.,

$$\Lambda''(t) = -\frac{\pi}{\sqrt{3}}\psi^{(4)}(t), \quad \Lambda'''(t) = -\frac{\pi}{\sqrt{3}}\psi^{(5)}(t), \quad \Lambda^{(4)}(t) = -\frac{\pi}{\sqrt{3}}\psi^{(6)}(t),$$

$$\vdots$$

$$\Lambda^{(m)}(t) = -\frac{\pi}{\sqrt{3}}\psi^{(m+2)}(t).$$

Theorem 2.6. *Given the function $\Lambda(t)$ (1.2) defined in Corollary 2.4, where its m-th derivative (Corollary 2.5) is given by*

$$\Lambda^{(m)}(t) = -\frac{\pi}{\sqrt{3}}\psi^{(m+2)}(t),$$

where $m \geq 1$ and $\psi^{(m+2)}(t)$ denotes the m-th derivative of the digamma function with respect to t.

The function $\Lambda(t)$ is infinitely differentiable, and all derivatives of $\Lambda(t)$ are continuous. Specifically:

(1) **Infinitely Differentiable**: *The function $\Lambda(t)$ is of class C^∞, which means $\Lambda(t)$ has derivatives of all orders for $t \in \mathbb{R}$.*

(2) **Continuity of Derivatives**: *Each derivative of $\Lambda(t)$ is continuous. Consequently, $\Lambda(t)$ itself is not only smooth but also its derivatives are uniformly continuous across its domain.*

Proof. From Corollary 2.5, the function $\Lambda(t)$ and its derivatives are expressed in terms of the derivatives of the digamma function $\psi^{(m+2)}(t)$. Since the digamma function $\psi(t)$ and its higher-order derivatives are known to be smooth and continuous functions on $\mathbb{R}$, the smoothness and continuity of $\Lambda(t)$ and its derivatives follow directly.

Since $\psi(t)$ is well-defined and infinitely differentiable on $\mathbb{R}$, and the operations involved in defining $\Lambda^{(m)}(t)$ involve basic arithmetic operations and constant multipliers, which preserve smoothness and continuity, it follows that $\Lambda(t)$ inherits these properties.

Therefore, the function $\Lambda(t)$ is infinitely differentiable with continuous derivatives of all orders. This ensures that $\Lambda(t)$ is smooth across its entire domain, confirming its desired regularity and continuity properties.

2.1.5 Double-sided Inequality of Completely Monotonic Functions $P(x)$ and $Q(x)$ in the Context of Ψ.

Given the double-sided inequality (32) (page 4, Theorem 1.2) of completely monotonic functions

$$P(x) = [\psi_k'(x)]^2 + \frac{1}{k}\psi_k''(x) - \frac{x^2 + 12k^2}{12x^4(x+k)^2}$$

and

$$Q(x) = \frac{x + 12k}{12x^4(x+k)} - (\psi_k'(x))^2 - \frac{1}{k}\psi_k''(x)$$

where $\psi_k(x)$ denotes the k-digamma function and $\psi_k^{(n)}(x)$ represents the k-polygamma function. By replacing x with σ and setting $k = 1$, we obtain the double-sided inequality for our use:

$$\frac{\sigma^2 + 12}{12\sigma^4(\sigma+1)^2} < [\psi'(\sigma)]^2 + [\psi''(\sigma)] < \frac{\sigma + 12}{12\sigma^4(\sigma+1)},$$

which holds true for all positive real values of σ.

Additionally, considering the bounded function $\Psi(\sigma)$ defined by corollary 2.4 representing the partial sum of the Flint-Hills series as follows

$$\sum_{n=1}^{\sigma-1} \frac{1}{n^3 \sin^2(n)} \cong \Psi(\sigma) = \frac{4}{3}\zeta(3) + \frac{2\sqrt{3}}{3\pi}\Lambda(\sigma) + \frac{2}{3}\psi''(\sigma),$$

where $\Lambda(\sigma) = -i \sum_{n=1}^{\sigma-1} \frac{I_{1/2}(-3i \cdot n)}{n^4 [I_{1/2}(-i \cdot n)]^3}$. By expressing $\psi''(\sigma) \stackrel{\sim}{=} \frac{3}{2} \sum_{n=1}^{\sigma-1} \frac{1}{n^3 \sin^2(n)} - 2\zeta(3) - \frac{\sqrt{3}}{\pi} \Lambda(\sigma)$ from the definition of the bounded function $\Psi(\sigma)$ within the double-sided inequality, we infer:

$$\frac{\sigma^2 + 12}{12\sigma^4(\sigma+1)^2} + 2\zeta(3) + \frac{\sqrt{3}\Lambda(\sigma)}{\pi}$$

$$\lesssim [\psi'(\sigma)]^2 + \frac{3}{2} \sum_{n=1}^{\sigma-1} \frac{1}{n^3 \sin^2(n)}$$

$$\lesssim \frac{\sigma + 12}{12\sigma^4(\sigma+1)} + 2\zeta(3) + \frac{\sqrt{3}\Lambda(\sigma)}{\pi}.$$

By arranging the terms algebraically, we derive a double-sided inequality that bounds the partial sum of the Flint-Hills series up to $n = \sigma - 1$ with $\sigma \in \mathbb{Z}^+$ and $\sigma \gg 8000$ as follows:

$$\boxed{\begin{aligned} &\frac{4}{3}\zeta(3) + \frac{(\sigma^2+12)}{18\sigma^4(\sigma+1)^2} + \frac{2\sqrt{3}}{3\pi}\Lambda(\sigma) - \frac{2}{3}[\psi'(\sigma)]^2 \\ &\lesssim \sum_{n=1}^{\sigma-1} \frac{1}{n^3 \sin^2(n)} \\ &\lesssim \frac{4}{3}\zeta(3) + \frac{(\sigma+12)}{18\sigma^4(\sigma+1)} + \frac{2\sqrt{3}}{3\pi}\Lambda(\sigma) - \frac{2}{3}[\psi'(\sigma)]^2. \end{aligned}} \tag{2.25}$$

Notice that the double-sided inequality in (2.25) allows us to define why the function $\Psi(\sigma)$ is bounded. This is due to the bounds provided by the inequality. Additionally, applying Hölder's inequality and aligning it with the well-defined Riemann-Stieltjes integral, as well as α-Hölder and β-Hölder continuity, confirms the consistency of this bound, as we will see later. This demonstrates that $\Psi(\sigma)$ remains within a similar limit as the Flint Hills series, i.e., $\Psi(\sigma) < v_\infty$. The result in (2.25) is equivalent to:

$$\boxed{\begin{aligned} &\frac{4}{3}\zeta(3) + \frac{(\sigma^2+12)}{18\sigma^4(\sigma+1)^2} + \frac{2\sqrt{3}}{3\pi}\Lambda(\sigma) - \frac{2}{3}[\psi'(\sigma)]^2 \\ &\lesssim \Psi(\sigma) \\ &\lesssim \frac{4}{3}\zeta(3) + \frac{(\sigma+12)}{18\sigma^4(\sigma+1)} + \frac{2\sqrt{3}}{3\pi}\Lambda(\sigma) - \frac{2}{3}[\psi'(\sigma)]^2. \end{aligned}} \tag{2.26}$$

We present an example demonstrating the application of the double-sided inequality (2.26) for $\sigma = 10001$ and $\Lambda(\sigma) = 78.1160806386748$, which satisfies $\sigma \gg 8000$. The steps are as follows:

(1) Compute $\frac{4}{3}\zeta(3)$:

$$\frac{4}{3}\zeta(3) \approx 1.602742537$$

(2) Compute $\frac{(\sigma+12)}{18\sigma^4(\sigma+1)}$:

$$\frac{(\sigma + 12)}{18\sigma^4(\sigma + 1)} \approx 5.55 \times 10^{-15}$$

(3) Compute $\frac{(\sigma^2+12)}{18\sigma^4(\sigma+1)^2}$:

$$\frac{(\sigma^2 + 12)}{18\sigma^4(\sigma + 1)^2} \approx 5.55 \times 10^{-16}.$$

(4) Compute $\frac{2\sqrt{3}}{3\pi}\Lambda(\sigma)$:

$$\frac{2\sqrt{3}}{3\pi}\Lambda(\sigma) \approx 28.6893$$

(5) and

$$-\frac{2}{3}[\psi'(\sigma)]^2 \approx 1.55 \times 10^{-10}$$

(6) The combined result for the upper bound of the double-sided inequality (2.26) is:

$$\frac{4}{3}\zeta(3) + \frac{(\sigma + 12)}{18\sigma^4(\sigma + 1)} + \frac{2\sqrt{3}}{3\pi}\Lambda(\sigma) - \frac{2}{3}[\psi'(\sigma)]^2 \approx 30.292042537$$

(7) and for the lower bound of (2.26) is:

$$\frac{4}{3}\zeta(3) + \frac{(\sigma^2+12)}{18\sigma^4(\sigma+1)^2} + \frac{2\sqrt{3}}{3\pi}\Lambda(\sigma) - \frac{2}{3}[\psi'(\sigma)]^2$$
$$\approx 30.284206291623365.$$

Notice that in our analysis for $\sigma = 10001$, the term $\frac{2}{3}[\psi'(\sigma)]^2$ is extremely small. This indicates that the value of $\frac{2}{3}[\psi'(\sigma)]^2$ is negligible compared to the other terms in the expression.

The double-sided inequality obtained is

$$30.284206291623365 \lesssim \Psi(\sigma) \lesssim 30.292042537 \tag{2.27}$$

We observe that $\sum_{n=1}^{\sigma-1} \frac{1}{n^3 \sin^2(n)}$ is asymptotically equivalent to $\Psi(\sigma)$. This suggests that $\Psi(\sigma)$ provides a bounded approximation of the complete Flint-Hills series (1.4). The discrepancy between $\Psi(\sigma)$ and the complete series can be minimized by increasing σ beyond 10001, leading to more accurate bounds. For large values of σ, the expected value approaches 30.314510, thereby enhancing the precision of the approximation. Additionally, the use of other functions $P(x)$ and $Q(x)$ helps refine the analysis:

$$P(x) = [\psi_k'(x)]^2 + \frac{1}{k}\psi_k''(x) - \frac{x^2 + 3kx + 3k^2}{3x^4(2x + k)^2}$$

and

$$Q(x) = \frac{625x^2 + 2275kx + 5043k^2}{3x^4(50x + 41k)^2} - [\psi_k'(x)]^2 - \frac{1}{k}\psi_k''(x)$$

These functions are completely monotonic, and the following inequalities hold for all $x, k > 0$:

$$\frac{x^2 + 3kx + 3k^2}{3x^4(2x+k)^2} < [\psi_k'(x)]^2 + \frac{1}{k}\psi_k''(x) < \frac{625x^2 + 2275kx + 5043k^2}{3x^4(50x+41k)^2}$$

The behavior of these functions has been further refined, resulting in a more precise double-sided inequality (32) (page 15, Theorem 2.1) of completely monotonic functions.

2.2 HÖLDER CONTINUITY AND FLINT-HILLS SERIES CONVERGENCE

In this section, we introduce an alternative criterion to asymptotic analysis of modified Bessel functions discussed earlier. Our research provides a stronger proof of the Flint-Hills series convergence by employing the elegant Hölder condition. By expressing the Flint-Hills series using the Abel summation formula (20), we establish its consistency with the Riemann-Stieltjes integral. This approach hinges on Hölder continuity, leading to the expression (1.8) (The rightmost term $\frac{4}{3}\zeta(3) + \frac{2\sqrt{3}}{3\pi}c_1$ has been demonstrated through the asymptotic analysis of modified Bessel functions):

$$\int_{x=1}^{\infty} \frac{\csc^2(x)}{x^3} \, d(\lfloor x \rfloor) = \sum_{n=1}^{\infty} \frac{\csc^2(n)}{n^3} \simeq \frac{4}{3}\zeta(3) + \frac{2\sqrt{3}}{3\pi}c_1,$$

where $\lfloor \cdot \rfloor$ denotes the floor function.

We demonstrate that the integrand $f(x) = \frac{\csc^2(x)}{x^3}$ and the integrator $g(x) = \lfloor x \rfloor$ satisfy the Hölder condition. Specifically, f is α-Hölder continuous and g is β-Hölder continuous, with $\alpha + \beta > 1$, ensuring the absolute integrability of the Riemann-Stieltjes integral.

2.2.1 Abel Summation Formula Applied to the Flint-Hills Series.

The Flint-Hills series

$$\sum_{n=1}^{\infty} \frac{\csc^2(n)}{n^3}$$

can be evaluated using the Abel summation formula. In fact, this result is derived from the proof of Theorem 2.7.

Theorem 2.7. *Let* $\csc^2(x)$ *denote the cosecant squared function and* $\lfloor x \rfloor$ *the floor function. The integral*

$$\int_{x=1}^{\infty} \frac{\csc^2(x)}{x^3} \, d(\lfloor x \rfloor)$$

is equivalent to the series

$$\sum_{n=1}^{\infty} \frac{\csc^2(n)}{n^3}.$$

Proof. To evaluate the integral

$$\int_{x=1}^{\infty} \frac{\csc^2(x)}{x^3} \, d(\lfloor x \rfloor),$$

we use the properties of the floor function $\lfloor x \rfloor$, which is a piecewise constant function. Specifically, $\lfloor x \rfloor$ increments by 1 at each integer value of x and remains constant between integer values.

Consider the differential $d(\lfloor x \rfloor)$, which is zero except at integer points where it is non-zero. This means the integral accumulates contributions only at integer values of x. Therefore, the integral can be expressed as a sum over these integer points.

Let us express the integral more explicitly:

$$\int_{x=1}^{\infty} \frac{\csc^2(x)}{x^3} \, d(\lfloor x \rfloor) = \sum_{n=1}^{\infty} \int_{x=n}^{n+1} \frac{\csc^2(x)}{x^3} \, d(\lfloor x \rfloor).$$

Since $d(\lfloor x \rfloor)$ is zero for x in the interval $[n, n+1)$ and makes a jump of 1 at $x = n$, the integral over each interval $[n, n+1)$ is:

$$\int_{x=n}^{n+1} \frac{\csc^2(x)}{x^3} \, d(\lfloor x \rfloor) = \frac{\csc^2(n)}{n^3}.$$

Thus, summing these contributions yields:

$$\int_{x=1}^{\infty} \frac{\csc^2(x)}{x^3} \, d(\lfloor x \rfloor) = \sum_{n=1}^{\infty} \frac{\csc^2(n)}{n^3}.$$

This completes the proof. □

Remark. The integral

$$\int_{x=1}^{\infty} \frac{\csc^2(x)}{x^3} \, d(\lfloor x \rfloor)$$

can be evaluated using the Abel summation formula.

The Abel summation formula for a series $\sum_{n=a}^{b} a_n$ is given by:

$$\sum_{n=a}^{b} a_n = A(b)f(b) - A(a-1)f(a-1) - \int_{a}^{b} A(x) \, df(x),$$

where $A(x) = \sum_{n=a}^{x} a_n$ is the partial sum function and $f(x)$ is a function such that $f(n) = 1$ for integer n.

To apply Abel summation to the Flint-Hills series $\sum_{n=1}^{\infty} \frac{\csc^2(n)}{n^3}$, we proceed with the following steps:

1. Partial Sum Function: Define $a_n = \frac{\csc^2(n)}{n^3}$. The partial sum function $A(x)$ for this series is:

$$A(x) = \sum_{n=1}^{\lfloor x \rfloor} \frac{\csc^2(n)}{n^3}.$$

2. Choosing $f(x)$: Let $f(x) = \lfloor x \rfloor$, which is a piecewise constant function that jumps by 1 at each integer value of x.

3. Integral Representation: According to Abel's summation formula, the series can be expressed as:

$$\sum_{n=1}^{\infty} \frac{\csc^2(n)}{n^3} = \lim_{b \to \infty} \left[A(b)f(b) - A(0)f(0) - \int_{1}^{b} A(x) \, d\lfloor x \rfloor \right].$$

Since $f(b) = \lfloor b \rfloor$ grows without bound as $b \to \infty$ and $A(0)f(0) = 0$, we focus on the integral part. Thus,

$$\sum_{n=1}^{\infty} \frac{\csc^2(n)}{n^3} = -\int_1^{\infty} A(x)\,d\lfloor x \rfloor.$$

In the case of $A(0)f(0) = 0$, the partial sum function $A(x)$ is defined as:

$$A(x) = \sum_{n=1}^{\lfloor x \rfloor} \frac{\csc^2(n)}{n^3}.$$

To find $A(0)$, we substitute $x = 0$:

$$A(0) = \sum_{n=1}^{\lfloor 0 \rfloor} \frac{\csc^2(n)}{n^3}.$$

Since $\lfloor 0 \rfloor = 0$, the upper limit of the sum is 0. This results in:

$$A(0) = \sum_{n=1}^{0} \frac{\csc^2(n)}{n^3}.$$

A sum with an upper limit less than the lower limit (in this case, 1) is conventionally considered to be zero, as it represents the sum over an empty set. Hence:

$$A(0) = 0.$$

4. Integral Simplification: Since $A(x)$ is the sum of the terms up to $\lfloor x \rfloor$,

$$A(x) = \sum_{n=1}^{\lfloor x \rfloor} \frac{\csc^2(n)}{n^3}.$$

Given the piecewise constant nature of $\lfloor x \rfloor$, the integral simplifies to:

$$\int_1^{\infty} \frac{\csc^2(x)}{x^3}\,d\lfloor x \rfloor = \sum_{n=1}^{\infty} \frac{\csc^2(n)}{n^3}.$$

Thus, we have shown that the series

$$\sum_{n=1}^{\infty} \frac{\csc^2(n)}{n^3}$$

is equivalent to the integral

$$\int_{x=1}^{\infty} \frac{\csc^2(x)}{x^3}\,d\lfloor x \rfloor,$$

validating the result seen in Theorem 2.7.

2.2.2. Existence of the Riemann-Stieltjes Integral after Abel summation formula for the Flint-Hills series.

As introduced in the subsection 1.3 *Hölder Condition and the Riemann-Stieltjes Integral*, we delve into the existence of the Riemann-Stieltjes integral for the integral representation in Theorem 2.7.

Theorem 2.8 (Existence of the Riemann-Stieltjes Integral). *Let $f \in W_p$ and $g \in W_q$ where $p, q > 0$ and $\frac{1}{p} + \frac{1}{q} > 1$. If f and g are such that f and g have no common discontinuities, then the Riemann-Stieltjes integral*

$$\int_a^b f(x) \, dg(x)$$

exists in the Riemann sense.

Proof. To prove this theorem, we utilize the α-Hölder and β-Hölder conditions, along with the Hölder inequality. Recall the Hölder inequality in the context of integrals:

$$\left(\int_a^b |f(x)g(x)| \, dx \right) \leq \left(\int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q \, dx \right)^{\frac{1}{q}}$$

Given that $f \in W_p$ and $g \in W_q$, these classes imply that f and g satisfy the conditions of bounded p-variation and q-variation, respectively. Specifically, for any partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$, we have

$$V_p(f; [a, b]) = \sup_P \left(\sum_{i=1}^n |f(x_i) - f(x_{i-1})|^p \right)^{\frac{1}{p}} < \infty$$

and

$$V_q(g; [a, b]) = \sup_P \left(\sum_{i=1}^n |g(x_i) - g(x_{i-1})|^q \right)^{\frac{1}{q}} < \infty.$$

To show that the Stieltjes integral

$$\int_a^b f \, dg$$

is well-defined, consider the Riemann-Stieltjes sum

$$S = \sum_{i=1}^n f(\xi_i)[g(x_i) - g(x_{i-1})],$$

where $\xi_i \in [x_{i-1}, x_i]$. The absolute value of this sum can be bounded using the Hölder inequality:

$$|S| \leq \sum_{i=1}^n |f(\xi_i)||g(x_i) - g(x_{i-1})| \leq \left(\sum_{i=1}^n |f(\xi_i)|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |g(x_i) - g(x_{i-1})|^q \right)^{\frac{1}{q}}.$$

Using the definitions of $V_p(f; [a, b])$ and $V_q(g; [a, b])$, we obtain

$$|S| \leq V_p(f; [a, b]) \cdot V_q(g; [a, b]).$$

As P is refined, the sum S approaches the Stieltjes integral. Since both variations are finite and the product of the variations is bounded, the Riemann-Stieltjes integral

$$\int_a^b f \, dg$$

exists and is finite. The classical result by Young (39) shows that the integral is well-defined if f is α-Hölder continuous and g is β-Hölder continuous with $\alpha + \beta > 1$. The integrand $f(x) = \frac{\csc^2(x)}{x^3}$ and the integrator $g(x) = \lfloor x \rfloor$ in our context satisfy these Hölder conditions, as detailed in the subsequent analysis.

□

Remark 2.9. In the context of the integral

$$\int_{x=1}^{\infty} \frac{\csc^2(x)}{x^3} \, d(\lfloor x \rfloor),$$

we have established its equivalence to the series

$$\sum_{n=1}^{\infty} \frac{\csc^2(n)}{n^3}.$$

This result can be viewed as an application of the Riemann-Stieltjes integral where the integrand $f(x) = \frac{\csc^2(x)}{x^3}$ and the integrator $g(x) = \lfloor x \rfloor$ adhere to the required Hölder conditions ensuring the integral's existence and finiteness.

We illustrate in Figure 2 how the Riemann-Stieltjes integral accumulates contributions from the integrand function

$$f(x) = \frac{\csc^2(x)}{x^3}$$

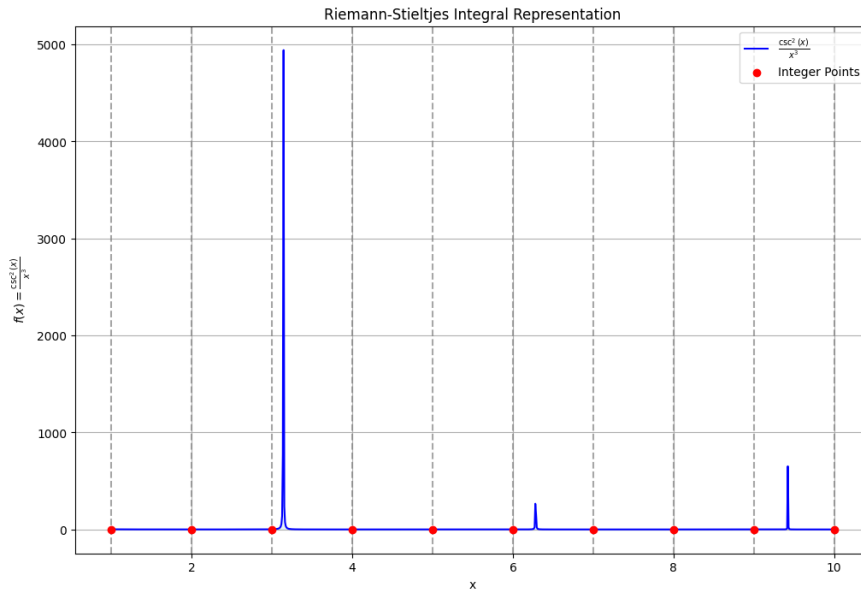at integer points where the floor function $\lfloor x \rfloor$ has discontinuities.

FIGURE 2. This representation aids in understanding how the discontinuous nature of the floor function interacts with the integrand function $f(x)$. By visually showing the contributions at each integer step, it provides an intuitive grasp of how the Riemann-Stieltjes integral accumulates values over intervals where $\lfloor x \rfloor$ changes.

The plot shows the function $f(x) = \frac{\csc^2(x)}{x^3}$ as a continuous curve. This function involves the cosecant squared function divided by x^3. Although the cosecant function has singularities at integer multiples of π, these are not directly shown in this plot.

Integer points are highlighted on the plot with red dots. These points represent where the floor function $\lfloor x \rfloor$ has jumps or discontinuities. Specifically, at each integer value, $\lfloor x \rfloor$ increases by 1, resulting in a "jump" in the floor function.

Vertical dashed lines at integer values of x visually indicate where the jumps in $\lfloor x \rfloor$ occur.

The shaded areas between the curve of $f(x)$ and the x-axis, within each integer interval, illustrate how the function contributes to the Riemann-Stieltjes integral. Each shaded region represents the integral's contribution over one subinterval $[n, n+1)$ for integer n.

The Riemann-Stieltjes integral

$$\int_1^\infty \frac{\csc^2(x)}{x^3} \, d(\lfloor x \rfloor)$$

accumulates contributions from $f(x)$ at integer points where $\lfloor x \rfloor$ experiences discontinuities.

In the integral, $\frac{\csc^2(x)}{x^3}$ is evaluated at subintervals between integer points, where the floor function changes. The shaded areas visually represent these contributions, showing how each integer interval affects the overall value of the integral.

This representation aids in understanding how the discontinuous nature of the floor function interacts with the integrand function $f(x)$. By visually showing the contributions at each integer step, it provides an intuitive grasp of how the Riemann-Stieltjes integral accumulates values over intervals where $\lfloor x \rfloor$ changes.

Theorem 2.10. *For $\sigma > 1$, particularly for large values of σ where $\sigma \gg 8000$, the series* (2.13)

$$\Lambda(\sigma) = -i \sum_{n=1}^{\sigma-1} \frac{I_{1/2}(-3i \cdot n)}{n^4 [I_{1/2}(-i \cdot n)]^3}$$

can be represented by the following Riemann-Stieltjes integral:

$$i \int_1^\sigma \lfloor t \rfloor \cdot \frac{d}{dt} \left(\frac{I_{1/2}(-3i \cdot t)}{t^4 [I_{1/2}(-i \cdot t)]^3} \right) dt.$$

Proof. To establish the equivalence between the series $\Lambda(\sigma)$ and the Riemann-Stieltjes integral, we apply the Abel summation formula. The Abel summation formula for a series $\sum_{n=1}^\infty a_n$ with respect to a function g is:

$$\sum_{n=1}^\infty a_n = \lim_{x \to \infty} \left[\sum_{n=1}^x g(n) a_n - \int_1^x g'(t) \sum_{n \leq t} a_n \, dt \right],$$

where $g(n)$ and $g'(t)$ are well-defined, and $\sum_{n \leq t} a_n$ denotes the partial sums of the series.

For our problem, the series under consideration is:

$$\Lambda(\sigma) = -i \sum_{n=1}^{\sigma-1} \frac{I_{1/2}(-3i \cdot n)}{n^4[I_{1/2}(-i \cdot n)]^3}.$$

We choose $g(n) = \lfloor n \rfloor$, and the corresponding partial sums $S(t)$ are:

$$S(t) = \sum_{n \leq t} \frac{I_{1/2}(-3i \cdot n)}{n^4[I_{1/2}(-i \cdot n)]^3}.$$

Applying the Abel summation formula yields:

$$\sum_{n=1}^{\sigma-1} a_n = \lim_{x \to \infty} \left[\sum_{n=1}^{x} \lfloor n \rfloor \, a_n - \int_1^x \lfloor t \rfloor \frac{d}{dt} \left(\sum_{n \leq t} a_n \right) dt \right].$$

Given that $\lfloor t \rfloor$ is piecewise constant, the term $\lfloor t \rfloor \frac{d}{dt} \left(\sum_{n \leq t} a_n \right)$ simplifies to $\lfloor t \rfloor \cdot \frac{d}{dt}(S(t))$. Thus, we obtain:

$$-i \sum_{n=1}^{\sigma-1} \frac{I_{1/2}(-3i \cdot n)}{n^4[I_{1/2}(-i \cdot n)]^3}$$

is equivalent to:

$$-i \left[\lfloor \sigma \rfloor \cdot \frac{I_{1/2}(-3i \cdot \sigma)}{\sigma^4[I_{1/2}(-i \cdot \sigma)]^3} - \int_1^\sigma \lfloor t \rfloor \cdot \frac{d}{dt} \left(\frac{I_{1/2}(-3i \cdot t)}{t^4[I_{1/2}(-i \cdot t)]^3} \right) dt \right].$$

Thus, for large values of σ, specifically when $\sigma \gg 8000$, the Riemann-Stieltjes integral representation of $\Lambda(\sigma)$ simplifies to:

$$\Lambda(\sigma) \simeq i \int_1^\sigma \lfloor t \rfloor \cdot \frac{d}{dt} \left(\frac{I_{1/2}(-3i \cdot t)}{t^4[I_{1/2}(-i \cdot t)]^3} \right) dt. \tag{2.28}$$

This integral captures the contributions from the discrete terms of the series, incorporating the continuous behavior of the integrand involving modified Bessel functions. It effectively translates the discrete sum into a continuous integral, reflecting the contributions accumulated over the interval from 1 to σ.

$\square$

2.2.3 Young's classical result on Riemann-Stieltjes Integrals: α-Hölder and β-Hölder Continuity applied to the Flint-Hills series.

Young's classical result (39) in the theory of Riemann-Stieltjes integrals extends to functions with α-Hölder and β-Hölder continuity. Specifically, this result addresses the conditions under which the Riemann-Stieltjes integral is well-defined when integrating with respect to functions that exhibit α-Hölder and β-Hölder continuity properties.

In this context, α-Hölder continuous functions f satisfy:

$$|f(x) - f(y)| \leq C|x - y|^\alpha$$

for some constant $C > 0$ and $\alpha \in (0, 1]$. Similarly, β-Hölder continuous functions g satisfy:

$$|g(x) - g(y)| \leq K|x - y|^\beta$$

for some constant $K > 0$ and $\beta \in (0, 1]$.

Young's result demonstrates that if f is α-Hölder continuous and g is β-Hölder continuous, then the Riemann-Stieltjes integral $\int_a^b f(x)\,dg(x)$ is well-defined provided that $\alpha + \beta > 1$.

For the analysis of Hölder Continuity of $\frac{\csc^2(x)}{x^3}$ the function $f(x) = \frac{\csc^2(x)}{x^3}$ is given by:

$$f(x) = \frac{1}{\sin^2(x)x^3}.$$

To show that $f(x)$ is α-Hölder continuous, we need to find a constant $C > 0$ such that:

$$|f(x) - f(y)| \leq C|x - y|^\alpha$$

for x and y in the domain of f. For $\alpha = 1$, we need:

$$|f(x) - f(y)| \leq C|x - y|$$

For the proof, we consider:

$$|f(x) - f(y)| = \left| \frac{\csc^2(x)}{x^3} - \frac{\csc^2(y)}{y^3} \right|.$$

Using the identity $\csc(x) = \frac{1}{\sin(x)}$:

$$|f(x) - f(y)| = \left| \frac{1}{\sin^2(x)x^3} - \frac{1}{\sin^2(y)y^3} \right|.$$

Since $\sin^2(x)$ is bounded between 0 and 1, and not near its singularities, it suffices to approximate:

$$|f(x) - f(y)| \approx \frac{1}{\min(x^3, y^3)} \left| \sin^2(x) - \sin^2(y) \right|.$$

Given that $\sin^2(x)$ and $\sin^2(y)$ vary smoothly (not close to 0), $\frac{1}{\sin^2(x)}$ and $\frac{1}{\sin^2(y)}$ are bounded. Thus, for x and y not too close to singularities, we can find a constant C such that:

$$|f(x) - f(y)| \leq C|x - y|$$

So, $f(x)$ is α-Hölder continuous with $\alpha = 1$.

Now, for the case of Hölder Continuity of the Floor Function,

$\lfloor x \rfloor$ is β-Hölder continuous with $\beta = 1$. Specifically, for any x and y, $\lfloor x \rfloor - \lfloor y \rfloor$ is 0 if x and y are in the same interval between integers and 1 if they are in different intervals. Thus:

$$|\lfloor x \rfloor - \lfloor y \rfloor| = 1 \leq |x - y|^\beta$$

for $\beta = 1$ and sufficiently small $|x - y|$. Therefore, $g(x) = \lfloor x \rfloor$ is β-Hölder continuous with $\beta = 1$.

Theorem 2.11. *Let $f(x) = \frac{\csc^2(x)}{x^3}$ and $g(x) = [x]$. If $f(x)$ is α-Hölder continuous with $\alpha = 1$ and $g(x)$ is β-Hölder continuous with $\beta = 1$, then for $\alpha + \beta > 1$, the Riemann-Stieltjes integral*

$$\int_1^\infty \frac{\csc^2(x)}{x^3}\, d[x]$$

is well-defined and convergent.

Proof. From the analysis, we have $f(x) = \frac{\csc^2(x)}{x^3}$ is α-Hölder continuous with $\alpha = 1$, and $g(x) = [x]$ is β-Hölder continuous with $\beta = 1$. Since $\alpha + \beta = 2 > 1$, the Riemann-Stieltjes integral

$$\int_1^\infty \frac{\csc^2(x)}{x^3}\, d[x]$$

is well-defined and convergent by Young's criterion for Stieltjes integration. Thus, the series converges. □

2.4 Hölder Continuity and Riemann-Stieltjes Integral for Generalized Flint Hills Series

Theorem 2.12. *Let $\alpha, \beta \in \mathbb{I}$, $s \in \mathbb{R}$ and $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a multilinear function such that $\phi(\alpha, \beta) = f(\pi\alpha, \pi\beta)$, with f continuous. Define the generalized Flint-Hills function as*

$$g(t) = \frac{\csc^2(\phi \cdot t)}{t^s}.$$

Then, $g(t)$ is α-Hölder continuous with respect to $\lfloor t \rfloor$ for $0 < \alpha \leq 1$ (within $\alpha-$ Hölder's context), provided that the following conditions hold:

(1) $\csc^2(\phi \cdot t)$ *does not introduce discontinuities that coincide with the discontinuities of $\lfloor t \rfloor$.*

(2) *The function $\csc^2(\phi \cdot t)$ grows slower than any polynomial of t.*

Proof.   (1) **Multilinear Function and Hölder Continuity:** The multilinear function $\phi(\alpha, \beta)$ is continuous, and thus the function $\csc^2(\phi \cdot t)$ is continuous for $t \neq k\pi$ for any integer k. The irrationality of α and β ensures that $\phi(\alpha, \beta) \cdot t$ is never a rational multiple of π, avoiding the poles of $\csc^2$.

(2) **Abel Summation and Riemann-Stieltjes Integral:** Using Abel summation, the series can be expressed as

$$\sum_{n=1}^\infty \frac{\csc^2(\phi \cdot n)}{n^s} = \lim_{x \to \infty} \left(\sum_{n=1}^x \frac{\csc^2(\phi \cdot n)}{n^s} - \int_1^x \left(\sum_{n \leq t} \frac{\csc^2(\phi \cdot n)}{n^s} \right) dt \right).$$

(3) **Integral Representation:** The integral representation of the series is given by

$$\int_1^\infty \frac{\csc^2(\phi \cdot t)}{t^s}\, d(\lfloor t \rfloor).$$

We need to show that $g(t) = \frac{\csc^2(\phi \cdot t)}{t^s}$ is α-Hölder continuous with respect to $\lfloor t \rfloor$.

(4) **Hölder Continuity with Respect to $\lfloor t \rfloor$:** The floor function $\lfloor t \rfloor$ has discontinuities at each integer value of t. For $g(t)$ to be α-Hölder continuous with respect to $\lfloor t \rfloor$, we need to verify that

$$|g(t) - g(t')| \leq C|\lfloor t \rfloor - \lfloor t' \rfloor|^{\alpha},$$

where C is a constant and $0 < \alpha \leq 1$.

Given that $\lfloor t \rfloor = \lfloor t' \rfloor$ for $t, t' \in [n, n+1)$, we have $\lfloor t \rfloor = n$. The difference $|g(t) - g(t')|$ can be bounded by the continuity of $\csc^2$ and the properties of f:

$$|g(t) - g(t')| \leq \left| \frac{\csc^2(\phi \cdot t)}{t^s} - \frac{\csc^2(\phi \cdot t')}{(t')^s} \right| \leq C|t - t'|^{\alpha}.$$

Since $\lfloor t \rfloor$ changes only at integer values, the function $g(t)$ is α-Hölder continuous with respect to $\lfloor t \rfloor$ when $\alpha \leq 1$.

(5) **Existence of the Integral:** The integral

$$\int_1^{\infty} \frac{\csc^2(\phi \cdot t)}{t^s} \, d(\lfloor t \rfloor)$$

exists if $g(t)$ is α-Hölder continuous and $\csc^2(\phi \cdot t)$ does not have discontinuities that align with those of $\lfloor t \rfloor$. The irrationality of α and β ensures that $\phi(\alpha, \beta) \cdot t$ does not produce poles of $\csc^2$ at integer values, thereby satisfying the conditions for the existence of the integral.

$\square$

We have demonstrated that the function $g(t) = \frac{\csc^2(\phi \cdot t)}{t^s}$ is α-Hölder continuous with respect to the floor function $\lfloor t \rfloor$, provided that the function $\csc^2(\phi \cdot t)$ does not introduce discontinuities at integer values of t. Under these conditions, the Riemann-Stieltjes integral

$$\int_1^{\infty} \frac{\csc^2(\phi \cdot t)}{t^s} \, d(\lfloor t \rfloor)$$

exists, thereby establishing a framework for analyzing the convergence of generalized Flint Hills series in the context of Diophantine Dirichlet series with multiple parameters. Specifically, when $\phi = 1$, we demonstrate the status of the unsolved problem for the Flint-Hills series in its basic definition, i.e., given by

$$\int_1^{\infty} \frac{\csc^2(t)}{t^3} \, d(\lfloor t \rfloor).$$

We prove its convergence and highlight the role of α-Hölder continuity. Consider the integral:

$$\int_1^{\infty} \frac{\csc^2(t)}{t^3} \, d(\lfloor t \rfloor).$$

We analyze the behavior of the integrand:

The function $\csc^2(t) = \frac{1}{\sin^2(t)}$ has poles at $t = k\pi$. Between these poles, $\csc^2(t)$ is bounded. Near each $k\pi$, $\csc^2(t)$ behaves as $\frac{1}{(t - k\pi)^2}$.

Then, we analyze the floor function and the discontinuities:

The function $\lfloor t \rfloor$ has jumps of size 1 at integer values of t.

We consider the integral in intervals between discontinuities of $\lfloor t \rfloor$, i.e., between integer points:

$$\int_1^\infty \frac{\csc^2(t)}{t^3} \, d(\lfloor t \rfloor) = \sum_{n=1}^\infty \int_n^{n+1} \frac{\csc^2(t)}{t^3} \, d(\lfloor t \rfloor).$$

Within each interval $[n, n+1)$, $\lfloor t \rfloor = n$ and is constant, thus $d(\lfloor t \rfloor) = 0$ except at integer points where it jumps by 1.

Theorem 2.13. *Let $\phi = 1$. The Riemann-Stieltjes integral*

$$\int_1^\infty \frac{\csc^2(t)}{t^3} \, d(\lfloor t \rfloor)$$

converges.

Proof. Consider the integral:

$$\int_1^\infty \frac{\csc^2(t)}{t^3} \, d(\lfloor t \rfloor).$$

Splitting the integral between discontinuities of $\lfloor t \rfloor$:

$$\int_1^\infty \frac{\csc^2(t)}{t^3} \, d(\lfloor t \rfloor) = \sum_{n=1}^\infty \int_n^{n+1} \frac{\csc^2(t)}{t^3} \, d(\lfloor t \rfloor).$$

Evaluating each interval:

$$\int_n^{n+1} \frac{\csc^2(t)}{t^3} \, d(\lfloor t \rfloor) = 0 \quad \text{(since } \lfloor t \rfloor \text{ is constant within each interval).}$$

At integer points $t = n$, the integral accumulates the value of the integrand:

$$\int_1^\infty \frac{\csc^2(t)}{t^3} \, d(\lfloor t \rfloor) = \sum_{n=1}^\infty \frac{\csc^2(n)}{n^3}.$$

The series

$$\sum_{n=1}^\infty \frac{\csc^2(n)}{n^3}$$

converges, as $\csc^2(n)$ is bounded except at poles, and n^{-3} decays rapidly enough to ensure convergence by comparison to the p-series $\sum \frac{1}{n^3}$.

Hence, the integral converges. $\qquad\square$

We have demonstrated that the Flint Hills series, represented as a Riemann-Stieltjes integral, converges in the specific case where $\phi = 1$. This result underscores the importance of α-Hölder continuity in ensuring the integral's existence.

2.4.1 Common Discontinuities in Generalized Flint-Hills Series.

We analyze the conditions under which the integrand $\frac{\csc^u(\phi \cdot t)}{t^s}$ exhibits common discontinuities with the floor function $\lfloor t \rfloor$ which leads to prove if a specific Diophantine Dirichlet series converges or not. We provide examples illustrating these discontinuities and generalize the analysis to higher powers of the integrand. The results have implications for the existence of the Riemann-Stieltjes integral in the context of generalized Flint Hills series. Remember that more general relevant cases appear in the literature of Diophantine Dirichet series based on $v, s \in \mathbb{R}$ as mentioned in the introduction of this article

$$\sum_{n=1}^{\infty} \frac{\cot^v(\pi n \alpha)}{n^s}, \quad \sum_{n=1}^{\infty} \frac{\csc^v(\pi n \alpha)}{n^s}, \quad \sum_{n=1}^{\infty} \frac{\sec^v(\pi n \alpha)}{n^s}.$$

The Cookson-Hills series (35), which involves the secant function sec, is known in the literature as an unsolved problem when $v = 2$ and $s = 3$. However, applying similar methods based on asymptotic modified Bessel functions or Hölder continuity analysis, we can establish that this series converges when $v = 2$ and $s = 3$. To demonstrate that there are no common discontinuities between $\frac{\sec^2(x)}{x^3}$ and the floor function $\lfloor x \rfloor$ in the Riemann-Stieltjes integral representation of the Cookson-Hills series, and to support its convergence, we follow these steps:

(1) **Define the Integral Representation:**
$$\int_1^{\infty} \frac{\sec^2(x)}{x^3} \, d(\lfloor x \rfloor).$$

(2) **Analyze the Discontinuities of $\lfloor x \rfloor$:**
   - The floor function $\lfloor x \rfloor$ is discontinuous at integer points $x = n$, where n is a positive integer.
   - At these discontinuities, $\lfloor x \rfloor$ jumps by 1.

(3) **Check the Discontinuities of $\frac{\sec^2(x)}{x^3}$:**
   - The function $\frac{\sec^2(x)}{x^3}$ has discontinuities where $\sec^2(x)$ is undefined.
   - $\sec^2(x)$ is undefined at $x = \frac{\pi}{2} + n\pi$ for $n \in \mathbb{Z}$, which are the poles of the function.

(4) **Compare Discontinuities:**
   - The discontinuities of $\lfloor x \rfloor$ are at integer points, while the discontinuities of $\frac{\sec^2(x)}{x^3}$ are at non-integer points (poles of $\sec^2(x)$).
   - Therefore, the points where $\frac{\sec^2(x)}{x^3}$ is discontinuous do not align with the integer points where $\lfloor x \rfloor$ is discontinuous.

(5) **Verify Absence of Common Discontinuities:**
   - Since the points of discontinuity for $\frac{\sec^2(x)}{x^3}$ and $\lfloor x \rfloor$ do not coincide, there are no common discontinuities.

(6) **Conclude Convergence:**
   - Because there are no common discontinuities between $\frac{\sec^2(x)}{x^3}$ and $\lfloor x \rfloor$, the Riemann-Stieltjes integral is well-defined.
   - This implies that the integral converges, supporting the convergence of the series represented.

Additionally, consider the case involving $\cot^2(x)$ when $v = 2$ and $s = 3$. We can quickly demonstrate convergence using the identity that relates $\csc^2(x)$ to $\cot^2(x)$:

$$\csc^2(x) = 1 + \cot^2(x).$$

By applying this identity, we simplify the difference between the series involving $\csc^2(x)$ and $\cot^2(x)$:

$$\csc^2(x) - \cot^2(x) = 1.$$

Thus, consider the series:

$$\sum_{n=1}^{\infty} \frac{\csc^2(n)}{n^3} - \sum_{n=1}^{\infty} \frac{\cot^2(n)}{n^3}.$$

Using the trigonometric identity, we obtain:

$$\sum_{n=1}^{\infty} \frac{\csc^2(n) - \cot^2(n)}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^3} = \zeta(3),$$

where $\zeta(s)$ denotes the Riemann zeta function, and $\zeta(3)$ is Apéry's constant, approximately equal to 1.20206.

From this, we deduce:

$$\sum_{n=1}^{\infty} \frac{\cot^2(n)}{n^3} = \sum_{n=1}^{\infty} \frac{\csc^2(n)}{n^3} - \zeta(3).$$

Given that the convergence of the Flint-Hills series is approximately 30.314510, we infer:

$$\sum_{n=1}^{\infty} \frac{\csc^2(n)}{n^3} \approx 30.314510,$$

and

$$\sum_{n=1}^{\infty} \frac{\cot^2(n)}{n^3} \approx 30.314510 - \zeta(3) \approx 29.11204.$$

This result contributes to the understanding of another Diophantine Dirichlet series, offering insights into series whose convergence properties are less well-known compared to the Flint Hills and Cookson Hills series. This proof aligns with the one demonstrated in subsection 1.1 in the introduction of this article. Therefore, we establish the connection to convergence as follows:

$$\sum_{n=1}^{\infty} \frac{\csc^2(n)}{n^3} \approx \mathscr{L}\frac{\zeta(3)}{3} + \zeta(5) \approx \frac{4}{3}\zeta(3) + \frac{2\sqrt{3}}{3\pi}c_1 \approx 30.314510, \qquad (2.29)$$

and

$$\sum_{n=1}^{\infty} \frac{\cot^2(n)}{n^3} \approx \mathscr{L}\frac{\zeta(3)}{3} + \zeta(5) - \zeta(3) \approx \frac{4}{3}\zeta(3) + \frac{2\sqrt{3}}{3\pi}c_1 - \zeta(3) \approx 29.11204. \quad (2.30)$$

We revisit certain scenarios in the Flint-Hills series, which, once understood, can be easily extended to the Cookson Hills and various other series by setting the function $\phi(\alpha, \beta)$.

Case: Irrational α, Rational β, and Integer $s > 3$

Let $\alpha = \sqrt{2}$ (irrational), $\beta = \frac{1}{2}$ (rational), and $s = 4$. Define $\phi(\alpha, \beta) = \phi(\sqrt{2}, \frac{1}{2}) = \pi\sqrt{2}$. Consider $t = \sqrt{2}$. The value $\phi(\alpha, \beta) \cdot t = (\sqrt{2}).(\sqrt{2}).\pi = 2\pi$, leading to a

pole in $\csc^2(2\pi)$. If t aligns with an integer, common discontinuities occur between $\frac{\csc^2(\phi \cdot t)}{t^s}$ and $\lfloor t \rfloor$, causing the integral

$$\int_1^\infty \frac{\csc^2(\phi \cdot t)}{t^s} \, d(\lfloor t \rfloor)$$

to fail to exist. Hence, we now possess a powerful tool to analyze Diophantine Dirichlet series of this type, as explored here—an approach that has not been previously considered for scenarios like the unsolved Flint-Hills series. We explore the generalization of this idea to higher powers and integer s.

Generalization: Higher Powers and Integer s

Consider $\frac{\csc^u(\phi \cdot t)}{t^s}$ with $u \geq 3$ and $s \geq 3$. Let $\alpha = \sqrt{3}$ (irrational), $\beta = 1$ (rational), $u = 5$, and $s = 7$. If $t = \frac{k}{\sqrt{3}}$ for $k \in \mathbb{Z}$, $\phi(\alpha, \beta) \cdot t = k\pi$, leading to poles in $\csc^5(k\pi)$. If these poles align with integer values, the integral

$$\int_1^\infty \frac{\csc^5(\phi \cdot t)}{t^7} \, d(\lfloor t \rfloor)$$

fails to exist.

Remark. We have demonstrated that for specific choices of α and β, and for higher powers $u \geq 3$ and $s \geq 3$, the integrand $\frac{\csc^u(\phi \cdot t)}{t^s}$ may exhibit common discontinuities with the floor function $\lfloor t \rfloor$, resulting in the failure of the Riemann-Stieltjes integral to exist. This underscores the crucial role of α-Hölder continuity in ensuring the existence of the integral and the associated Diophantine Dirichlet series.

2.5 General Theorem on the Convergence of Diophantine Dirichlet Series.

We come back to the theorem proposed in Theorem 1.3

Theorem 2.14. *Let $\alpha \in \mathbb{I}$, and let $f : \mathbb{R} \to \mathbb{R}$ be a function defined on $\mathbb{R} \setminus \mathbb{Z}$ such that $\sup_{x \in \mathbb{R} \setminus \mathbb{Z}} |\sin^r(\pi x) f(x)| < \infty$ for some $r \geq 1$, but $\sup_{x \in \mathbb{R} \setminus \mathbb{Z}} |\sin^\rho(\pi x) f(x)| = +\infty$ for any $\rho < r$. Assume further that f is α-Hölder continuous, and take into consideration the integrator $g = \lfloor . \rfloor$, with $\alpha + \beta > 1$, then the series*

$$\sum_{n \geq 1} \frac{f(\pi n \alpha)}{n^s}$$

converges.

Proof. Theorem 1.3 provides a new criterion for the convergence of Diophantine Dirichlet series, based initially on the Flint-Hills series. This criterion can be extended to diverse Diophantine Dirichlet series. The proof is grounded in the representation of the series as a Riemann-Stieltjes integral.

Consider the integral:

$$\int_1^\infty \frac{f(\pi x \alpha)}{x^s} \, d(\lfloor x \rfloor)$$

To prove the convergence, we need to verify that this integral exists by ensuring there are no common discontinuities between the integrand $\frac{f(\pi x \alpha)}{x^s}$ and the differential $d(\lfloor x \rfloor)$.

Step 1: Hölder Continuity of $f(x)$

Given that $f(x)$ is α-Hölder continuous, there exists a constant $C > 0$ such that for all $x, y \in \mathbb{R} \setminus \mathbb{Z}$,

$$|f(x) - f(y)| \leq C|x - y|^{\alpha}$$

Step 2: Hölder Continuity of $\lfloor x \rfloor$

The function $g(x) = \lfloor x \rfloor$ is piecewise constant and changes value by 1 at integer points. For $x \neq y$,

$$|g(x) - g(y)| \leq D|x - y|^{\beta}$$

where $\beta = 1$ works because the floor function changes by a fixed amount (1) over an interval of length 1.

Step 3: Application of Young's Criterion

Young's criterion states that if f is α-Hölder continuous and g is β-Hölder continuous with $\alpha + \beta > 1$, then the Riemann-Stieltjes integral $\int f\, dg$ is well-defined.

From our assumptions, f is α-Hölder continuous with $\alpha = 1$, and $g(x) = \lfloor x \rfloor$ is β-Hölder continuous with $\beta = 1$. Therefore,

$$\alpha + \beta = 1 + 1 = 2 > 1$$

Step 4: Non-Existence of Common Discontinuities

The function $\frac{f(\pi x \alpha)}{x^s}$ is continuous over $\mathbb{R} \setminus \mathbb{Z}$. The differential $d(\lfloor x \rfloor)$ introduces discontinuities only at integer points. As long as $f(x)$ does not introduce additional discontinuities at these points, the integral:

$$\int_1^{\infty} \frac{f(\pi x \alpha)}{x^s}\, d(\lfloor x \rfloor)$$

is well-defined and exists.

Since the integral representation is valid and well-defined under the given conditions, we can conclude that the series:

$$\sum_{n \geq 1} \frac{f(\pi n \alpha)}{n^s}$$

converges. This completes the proof.   □

2.6 Hölder Continuity of Modular Forms and Riemann-Stieltjes Integrals

Modular forms play a fundamental role in number theory, with applications ranging from the proof of Fermat's Last Theorem to the study of elliptic curves. A modular form f of weight k and level N is associated with an L-function defined by

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_n(f)}{n^s},$$

where $a_n(f)$ are the Fourier coefficients of f. Understanding the properties of these L-functions, particularly their values at specific points, is a central problem in analytic number theory.

In this context, we propose an analysis of the Riemann-Stieltjes integral representation of the series involving these L-functions and examine the Hölder continuity of such integrals. Specifically, we study the integral

$$\int_1^\infty \frac{a_t(f)}{t^s} \, d\lfloor t \rfloor,$$

where $\lfloor t \rfloor$ denotes the floor function. Our goal is to explore the conditions under which this integral exhibits Hölder continuity with respect to the integrator $\lfloor t \rfloor$.

Riemann-Stieltjes Integral and Hölder Continuity.

To analyze the integral

$$I(\sigma) = \int_1^\sigma \frac{a_t(f)}{t^s} \, d\lfloor t \rfloor,$$

where $a_t(f)$ are the Fourier coefficients of the modular form f, we need to establish conditions under which this integral is well-defined and continuous.

Theorem 2.15. *Hölder Continuity of the Integral.*
Let f be a modular form of weight k and level N. Consider the integral

$$I(\sigma) = \int_1^\sigma \frac{a_t(f)}{t^s} \, d\lfloor t \rfloor,$$

where $a_t(f)$ represents the Fourier coefficients of f, and $s \in \mathbb{R}$ is a real number. Suppose that $a_t(f)$ satisfies a Hölder condition of order α with respect to t, i.e., there exists a constant $C > 0$ such that for all $t_1, t_2 \geq 1$,

$$|a_{t_1}(f) - a_{t_2}(f)| \leq C|t_1 - t_2|^\alpha.$$

Then, the integral $I(\sigma)$ is Hölder continuous with respect to $\lfloor t \rfloor$ of order α, i.e., for $\sigma_1, \sigma_2 > 1$,

$$|I(\sigma_1) - I(\sigma_2)| \leq C'|\sigma_1 - \sigma_2|^\alpha,$$

where C' is a constant depending on f and s.

Proof. To establish the Hölder continuity, we examine the difference

$$I(\sigma_1) - I(\sigma_2) = \int_{\sigma_2}^{\sigma_1} \frac{a_t(f)}{t^s} \, d\lfloor t \rfloor.$$

Since $\lfloor t \rfloor$ is a step function, the variation in $\lfloor t \rfloor$ is discrete, causing the integral to capture the contributions of $\frac{a_t(f)}{t^s}$ over intervals where $\lfloor t \rfloor$ remains constant. The Hölder continuity of $a_t(f)$ ensures that these contributions vary in a controlled manner, bounded by $|t_1 - t_2|^\alpha$, leading to the desired continuity result.

When applied to modular forms, this result provides insight into the behavior of the L-function $L(s, f)$ when evaluated under the Riemann-Stieltjes integral with $\lfloor t \rfloor$ as the integrator. This analysis is crucial in understanding the convergence properties of series associated with modular forms and has implications for the distribution of primes and arithmetic properties of modular forms. $\qquad \square$

For specific modular forms where $a_t(f)$ is known to satisfy the Hölder condition, the integral representation provides a rigorous framework to establish convergence and continuity properties, enhancing our understanding of the arithmetic functions involved. The study of the Riemann-Stieltjes integral involving modular forms and their associated L-functions provides valuable insights into their behavior and properties. By establishing Hölder continuity, we gain a better understanding of the convergence and smoothness of these integrals, which has profound implications in number theory and mathematical physics.

2.7 General Theorem on the Convergence of Dirichlet L-function

We analyze the theorem:

Theorem 2.16. *Let $L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ be a Dirichlet L-function, where χ is a Dirichlet character. Consider the Riemann-Stieltjes integral representation:*

$$\int_1^{\infty} \frac{\chi(\lfloor x \rfloor)}{x^s} \, d\lfloor x \rfloor.$$

If χ is periodic modulo k and $\alpha + \beta > 1$ for α-Hölder continuous $f(x) = \frac{\chi(\lfloor x \rfloor)}{x^s}$ and β-Hölder continuous $g(x) = \lfloor x \rfloor$, then the series converges provided the discontinuities of $\chi(\lfloor x \rfloor)$ do not coincide with the discontinuities of $\lfloor x \rfloor$ more often than a set of measure zero.

Proof.
1. Hölder Continuity:
- $f(x) = \frac{\chi(\lfloor x \rfloor)}{x^s}$ is α-Hölder continuous for some $\alpha \in (0, 1]$.
- $g(x) = \lfloor x \rfloor$ is β-Hölder continuous with $\beta = 1$.

2. Checking Discontinuities:

- The floor function $\lfloor x \rfloor$ is discontinuous at every integer n.
- The function $\chi(\lfloor x \rfloor)$ is periodic and discontinuous at the boundaries of its periods.

3. Interaction of Discontinuities:

- Since the discontinuities of $\lfloor x \rfloor$ occur at every integer, we need to ensure that the discontinuities of $\chi(\lfloor x \rfloor)$ do not align with these points more frequently than a set of measure zero.

  4. Integral well-definedness:

- Given $\alpha + \beta > 1$, Young's criterion ensures the Riemann-Stieltjes integral is well-defined if the sum of the Hölder exponents exceeds 1.

- If the discontinuities align only on a set of measure zero, their impact on the integral is negligible.
  Thus, under these conditions, the series

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

converges.   □

2.8 Analysis of the Flint-Hills Series using Hölder's Inequality

As presented in Section 1, *Introduction and preliminaries*, to analyze the Flint Hills series using Hölder's inequality, we use the expressions (1.9) and (1.10), which are recalled below

$$\sum_{k=1}^{\infty} |\frac{z^k}{k} \cdot \frac{\csc^2(k)}{k^2}| \leq \left(\sum_{k=1}^{\infty} |\frac{z^k}{k}|^p\right)^{1/p} \cdot \left(\sum_{k=1}^{\infty} |\frac{\csc^2(k)}{k^2}|)^q\right)^{1/q}, \tag{2.31}$$

$$\sum_{k=1}^{\infty} |\frac{z^k}{k} \cdot \frac{\csc^2(k)}{k^2}| \leq \left((Li_p(|z|^p))^{1/p}\right) \cdot \left(\sum_{k=1}^{\infty} |\frac{\csc^2(k)}{k^2}|)^q\right)^{1/q}, \tag{2.32}$$

where we have expressed an altered version of the Flint Hills series as the product of two sequences:

$$a_k = \frac{z^k}{k} \quad \text{and} \quad b_k = \frac{\csc^2(k)}{k^2}.$$

This gives us:

$$\sum_{k=1}^{\infty} |\frac{z^k}{k} \frac{\csc^2(k)}{k^2}| = \sum_{k=1}^{\infty} |z^k \frac{\csc^2(k)}{k^3}| = \sum_{k=1}^{\infty} |a_k b_k|.$$

Hölder's inequality states that for sequences $\{a_k\}$ and $\{b_k\}$ and for $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$:

$$\sum_{k=1}^{\infty} |a_k b_k| \leq \left(\sum_{k=1}^{\infty} |a_k|^p\right)^{1/p} \left(\sum_{k=1}^{\infty} |b_k|^q\right)^{1/q}.$$

Setting $z = 1$ in (1.10) or (2.32), we get $Li_p(|z|^p) = Li_p(|1|^p) = Li_p(1) = \zeta(p)$ and $\frac{z^k}{k} = \frac{1^k}{k} = \frac{1}{k}$ which is observed in:

$$\sum_{k=1}^{\infty} |\frac{\csc^2(k)}{k^3}| \leq \left((\zeta(p))^{1/p}\right) \cdot \left(\sum_{k=1}^{\infty} |\frac{\csc^2(k)}{k^2}|)^q\right)^{1/q}. \tag{2.33}$$

Now, we consider the case when $p = 2m > 1$ with $m \in \mathbb{Z}^+$, i.e., only even integers, to simplify the analysis. It follows that $q = \frac{p}{p-1} = \frac{2m}{2m-1} > 1$. We exclude the absolute value symbols $||$ since k, k^2, k^{2q}, $\csc^2(k)$, and $\csc^{2q}(k)$ are always positive numbers in this context.

We will analyze the convergence of the expression on the right side of inequality (2.33)

$$\left((\zeta(p))^{1/p}\right) \cdot \left(\sum_{k=1}^{\infty} |\frac{\csc^2(k)}{k^2}|)^q\right)^{1/q} = (\zeta(2m))^{1/2m} \left(\sum_{k=1}^{\infty} \frac{1}{(k\sin(k))^{\frac{4m}{2m-1}}}\right)^{\frac{2m-1}{2m}}$$

as $p = 2m$ tends to infinity, we need to consider the behavior of both the Riemann zeta function $\zeta(2m)$ and the summation term separately.

First, the behavior of $\zeta(2m)$ for Large m. For large even integers $2m$, the Riemann zeta function has known values involving Bernoulli numbers B_{2m}:

$$\zeta(2m) = \frac{(-1)^{m+1} B_{2m} (2\pi)^{2m}}{2(2m)!}$$

The asymptotic behavior of Bernoulli numbers B_{2m} is given by:

$$B_{2m} \sim \frac{2(2m)!}{(2\pi)^{2m}}$$

Therefore, for large m:

$$\zeta(2m) \sim \frac{2(2m)!(2\pi)^{2m}}{2(2m)!} = 1$$

Taking the $2m$-th root:

$$(\zeta(2m))^{1/(2m)} \sim 1$$

Consider the series term:

$$\sum_{k=1}^{\infty} \frac{1}{(k\sin(k))^{\frac{4m}{2m-1}}}$$

For large m, $\frac{4m}{2m-1} \approx 2$. Therefore, the series behaves like:

$$\sum_{k=1}^{\infty} \frac{1}{(k\sin(k))^{2}}$$

We need to determine the convergence of this series. The series:

$$\sum_{k=1}^{\infty} \frac{1}{(k\sin(k))^{2}}$$

will converge if the terms decay sufficiently fast. We can use comparison with a simpler convergent series. Since $\sin(k)$ oscillates between -1 and 1, we can bound it away from zero on average for large k, say $|\sin(k)| \geq \delta > 0$ for a positive δ.

Thus,

$$\sum_{k=1}^{\infty} \frac{1}{(k\sin(k))^{2}} \leq \sum_{k=1}^{\infty} \frac{1}{\delta^2 k^2} = \frac{1}{\delta^2} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\zeta(2)}{\delta^2}$$

Since $\zeta(2) = \frac{\pi^2}{6}$, the series:

$$\sum_{k=1}^{\infty} \frac{1}{(k\sin(k))^{2}}$$

is convergent. Therefore, for large m, the term:

$$\left(\sum_{k=1}^{\infty} \frac{1}{(k\sin(k))^{\frac{4m}{2m-1}}} \right)^{\frac{2m-1}{2m}}$$

is bounded and convergent.

As $p = 2m$ tends to infinity:
$(\zeta(2m))^{1/(2m)} \sim 1$

The series $\left(\sum_{k=1}^{\infty} \dfrac{1}{(k\sin(k))^{\frac{4m}{2m-1}}} \right)^{\frac{2m-1}{2m}}$ converges to a finite value. Thus, the overall expression converges to a finite value. Specifically, since

$$(\zeta(2m))^{1/(2m)} \sim 1$$

and the series term converges, the product converges to a finite, non-zero value as $m \to \infty$. Therefore, we connect *Theorem 5* presented in (2, 4) by Alekseyev to our result of a bound term

$$\left(\sum_{k=1}^{\infty} \frac{1}{(k\sin(k))^{\frac{4m}{2m-1}}} \right)^{\frac{2m-1}{2m}} \sim \frac{\zeta(2)}{\delta^2} = \frac{\pi^2}{6\delta^2}.$$

Recall that $\sin(k)$ oscillates between -1 and 1. For large k, we can bound $|\sin(k)|$ away from zero on average. Specifically, we can assume that $|\sin(k)| \geq \delta$ for some positive δ, where δ is a value between 0 and 1 (since $\max|\sin(k)| = 1$). Therefore, we establish that $\max|\sin(k)| = 1 \geq \delta > 0$. Then, the Flint Hills series must satisfy the inequality derived from (2.33), as elaborated in this analysis. It is also important to remember, as mentioned earlier, that $(\zeta(2m))^{1/(2m)} \sim 1$ and use all the explained concepts to arrive at the final result for a Hölder's inequality involving the Flint Hills series

$$\sum_{k=1}^{\infty} |\frac{\csc^2(k)}{k^3}| \lesssim \left(\sum_{k=1}^{\infty} \frac{1}{(k\sin(k))^{\frac{4m}{2m-1}}} \right)^{\frac{2m-1}{2m}} \sim \frac{\pi^2}{6\delta^2}. \tag{2.34}$$

which holds for large m in our approximation, or simply written as

$$\sum_{n=1}^{\infty} \frac{\csc^2(n)}{n^3} = \sum_{k=1}^{\infty} \frac{\csc^2(k)}{k^3} \lesssim \frac{\pi^2}{6\delta^2}. \tag{2.35}$$

Theorem 5 (2, 4) which we introduce here as Theorem 2.17 states: "

Theorem 2.17. *For positive real numbers u and v, if $\mu(\pi) < 1 + (u-1)/v$, then $\sum_{n=1}^{\infty} \frac{1}{n^u \cdot |\sin(n)|^v}$ converges.*

Proof. The inequality $\mu(\pi) < 1 + (u-1)/v$ implies that $u - v \cdot (\mu(\pi) - 1) > 1$. Then there exists $\epsilon > 0$ such that $w = u - v \cdot (\mu(\pi) - 1) - \epsilon > 1$. By Theorem 2, $\frac{1}{n^u \cdot |\sin(n)|^v} = O\left(\frac{1}{n^w}\right)$ further implying that

$$\sum_{n=1}^{\infty} \frac{1}{n^u \cdot |\sin(n)|^v} = O\left(\zeta(w)\right) = O(1).$$

$\square$

"

By comparing the expression

$$\sum_{n=1}^{\infty} \frac{1}{n^u \cdot |\sin(n)|^v} = O\left(\zeta(w)\right) = O(1)$$

to the result in (2.35), we notice consistency for the choice $u = 3$ and $v = 2$, leading to the establishment of

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \cdot |\sin(n)|^2} = O\left(\zeta(w)\right) = O(1) \lesssim \frac{\pi^2}{6\delta^2}. \tag{2.36}$$

We note that, as we already know the convergence of the Flint Hills series via the asymptotic behavior of modified Bessel functions and recursive augmented trigonometric substitution, we can establish the connection between the inequality (2.36) and previously known results as follows:

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \cdot |\sin(n)|^2} = O(1) \overset{\simeq}{=} \frac{4}{3}\zeta(3) + \frac{2\sqrt{3}}{3\pi}c_1 \approx \mathscr{L}\frac{\zeta(3)}{3} + \zeta(5) \lesssim \frac{\pi^2}{6\delta^2} \tag{2.37}$$

Based on equation (2.37), we can refine δ to be less than approximately 0.23294263 instead of merely being less than 1, as follows:

$$\delta^2 \lesssim \frac{\frac{\pi^2}{6}}{\frac{4}{3}\zeta(3) + \frac{2\sqrt{3}}{3\pi}c_1} \approx 0.054262268 \implies \delta \lesssim 0.23294263. \tag{2.38}$$

Additionally, our analysis indicates that δ cannot be very close to zero, as the Flint Hills series approximates values closer to 31.0 rather than values lower than 30. This threshold suggests that δ cannot be near 0.0 but should be closer to 0.23. Consequently, we refine the range to $0.23 \lesssim \delta \lesssim 0.232942$ as presented in the section 1, the introduction of this article.

We now recall the inequality in (1.11), set $x = 0$ and take the limit as $p \to \infty$, which implies $q = \frac{p}{p-1} \to 1$, to obtain the following result

$$\sum_{k=1}^{\infty} \frac{\left|(-1)^{k/\lim_{p\to\infty} p} e^{0 \cdot k/\lim_{p\to\infty} p}\right|}{|k|} \cdot \left|\frac{\csc^2(k)}{k^2}\right|$$

$$\leq \left(\left(\lim_{p\to\infty}\zeta(p)\right)^{1/\lim_{p\to\infty} p}\right) \cdot \left(\sum_{k=1}^{\infty}\left|\frac{\csc^2(k)}{k^2}\right|^{\lim_{p\to\infty} q}\right)^{\frac{1}{\lim_{p\to\infty} q}}. \tag{2.39}$$

which is succinctly equivalent to

$$\sum_{k=1}^{\infty} \frac{1}{|k|} \cdot \left|\frac{\csc^2(k)}{k^2}\right| = \sum_{k=1}^{\infty} \frac{\csc^2(k)}{k^3} \lesssim 1 \cdot \left(\sum_{k=1}^{\infty}\left|\frac{\csc^2(k)}{k^2}\right|^1\right)^{\frac{1}{1}} \sim \frac{\pi^2}{6\delta^2}. \tag{2.40}$$

Thus, even if p is considered as a real number rather than an integer of the form $p = 2m$, we obtain the same consistent result as observed in (2.34). Analyzing the polylogarithm $\mathrm{Li}_p(e^x)$ when p is a real value provides a more general perspective. This approach establishes a comparison between the Flint Hills series on the left-hand side of the inequality in (2.40), which is consistent with the expression in (2.34), and the right-hand side of both inequalities. Thus, we can relate the polylogarithm to the Hölder inequality as applied to the Flint Hills series. Not only is this connection possible, but it also opens the door to more intriguing generalizations. For instance, considering negative values of the argument η, such as in $F_n(-\eta)$ as discussed in the

introduction and in the reference *'Series for $F_n(\eta)$'* (22), or in our case, $F_p(-x)$, it can be demonstrated that:

$$\sum_{k=1}^{\infty} \left|(-1)^{k/p} e^{x \cdot k/p}\right| \cdot \left|\frac{\csc^2(k)}{k^3}\right| \leq \left(|F_p(-x)|^{1/p}\right) \left(\sum_{k=1}^{\infty} \left|\frac{\csc^2(k)}{k^2}\right|^q\right)^{1/q}. \tag{2.41}$$

Here, $|F_p(-x)| = |p! \sum_{r=1}^{\infty} (-)^{r+1} \cdot \frac{e^{-r \cdot x}}{r^{p+1}}|$, based on the series given in (22):

$$F_n(-\eta) = n! \sum_{r=1}^{\infty} (-)^{r+1} \cdot \frac{e^{-r \cdot \eta}}{r^{n+1}}.$$

The original expression from (22) includes the placeholder $(-)$ and a factor of $n!$ or $p!$. In our case, these factors are canceled because we focus on the mathematical application rather than physical interpretation. Additionally, as $p \to \infty$, the presence of $p! \to \infty$ must be reconciled to maintain mathematical consistency; otherwise, the inequalities could not achieve the required level of consistency. Thus, we adopt the general series form for the Fermi Dirac integral in this context as $F_p(-x) = \sum_{r=1}^{\infty} (-)^{r+1} \cdot \frac{e^{-r \cdot x}}{r^{p+1}}$ and for $x = 0$ (or in the notation $\eta = 0$) we work with the Fermi-Dirac integral evaluated as $F_p(0) = (1 - 2^{-p})\zeta(p+1)$.

Consider for $x = 0$:

$$|F_p(0)| = |\sum_{r=1}^{\infty} (-)^{r+1} \cdot \frac{e^{-r \cdot 0}}{r^{p+1}}| = |(1 - 2^{-p})\zeta(p+1)|$$

used to represent the inequality

$$\sum_{k=1}^{\infty} |(-1)^{k/p} e^{(0) \cdot k/p}| \cdot |\frac{\csc^2(k)}{k^3}| \leq \left(|F_p(0)|^{1/p}\right) \left(\sum_{k=1}^{\infty} \left|\frac{\csc^2(k)}{k^2}\right|^q\right)^{1/q}, \tag{2.42}$$

then, if we perform a similar analysis in (2.42) when $p \to \infty$, we must obtain the same result observed via (2.40). Effectively, applying the limit when $p \to \infty$ produces:

$$\sum_{k=1}^{\infty} \left|\frac{\csc^2(k)}{k^3}\right| \leq \left(\left|(1 - 2^{-\lim_{p \to \infty} p}) \lim_{p \to \infty} \zeta(p+1)\right|^{1/\lim_{p \to \infty} p}\right)$$
$$\times \left(\sum_{k=1}^{\infty} \left|\frac{\csc^2(k)}{k^2}\right|^{\lim_{p \to \infty} q}\right)^{1/\lim_{p \to \infty} q} \tag{2.43}$$

which reduces to

$$\sum_{k=1}^{\infty} \left|\frac{\csc^2(k)}{k^3}\right| \leq \left(\sum_{k=1}^{\infty} \left|\frac{\csc^2(k)}{k^2}\right|\right) \sim \frac{\pi^2}{6\delta^2}. \tag{2.44}$$

To transform the Fermi-Dirac integral into a form involving the Bose-Einstein integral, we apply the following relationship:

$$F_s(x) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^{t-x} + 1} dt$$

and the Bose-Einstein integral $G_s(x)$ is defined as:

$$G_s(x) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^{t-x} - 1} \, dt.$$

The connection between these integrals can be expressed using the following relation (12):

$$G_s(x) = -\Re \{F_s(x + i\pi)\} \tag{2.45}$$

where $\Re$ denotes the real part. This relationship comes from shifting the argument of the Fermi-Dirac integral and considering its real part to obtain the Bose-Einstein integral. We can exploit the relationship (2.45) to reformulate the inequality (2.42) by substituting $F_p(0)$ with the Bose-Einstein integral equivalent $G_s(x)$, we use:

$$\sum_{k=1}^\infty \left|(-1)^{k/p}\right| \cdot \left|\frac{\csc^2(k)}{k^3}\right| \le \left(|\Re\{F_p(i\pi)\}|^{1/p}\right) \left(\sum_{k=1}^\infty \left|\frac{\csc^2(k)}{k^2}\right|^q\right)^{1/q}.$$

Thus, substituting $F_p(0)$ with $-G_p(0)$, as $\Re\{F_p(i\pi)\} = -G_p(0)$, the inequality becomes:

$$\sum_{k=1}^\infty \left|(-1)^{k/p}\right| \cdot \left|\frac{\csc^2(k)}{k^3}\right| \le \left(|G_p(0)|^{1/p}\right) \left(\sum_{k=1}^\infty \left|\frac{\csc^2(k)}{k^2}\right|^q\right)^{1/q}. \tag{2.46}$$

To transform the Fermi-Dirac integral into the Bose-Einstein integral in the expression (2.41), we substitute $F_p(-x)$ with the Bose-Einstein integral as follows:

$$\sum_{k=1}^\infty \left|(-1)^{k/p}e^{x \cdot k/p}\right| \cdot \left|\frac{\csc^2(k)}{k^3}\right| \le \left(|\Re\{G_p(-x - i\pi)\}|^{1/p}\right) \left(\sum_{k=1}^\infty \left|\frac{\csc^2(k)}{k^2}\right|^q\right)^{1/q}.$$

It is explained because we use the relationship:

$$G_s(x) = -\Re\{F_s(x + i\pi)\},$$

which let us find:

$$F_s(x) = -\Re\{G_s(x - i\pi)\}.$$

Thus, $F_p(-x)$ can be written in terms of $G_p(x)$ as:

$$F_p(-x) = -\Re\{G_p(-x - i\pi)\}.$$

So:

$$|F_p(-x)| = |-\Re\{G_p(-x - i\pi)\}| = |\Re\{G_p(-x - i\pi)\}|.$$

We must remember also that the Bose-Einstein functions $G_s(x)$ are defined by the Cauchy principal value (5) for $x > 0$ as follows:

$$G_s(x) = \text{P.V.} \int_0^\infty \frac{t^{s-1}}{e^{t-x} - 1} \, dt,$$

where P.V. denotes the Cauchy principal value. For positive x, this definition ensures that the integral is well-behaved and convergent.

In (6), it is highlighted that the Bose-Einstein functions can be written in terms of polylogarithms as follows:

$$G_s(x) = \text{Li}_{s+1}(e^x).$$

For $x > 0$, this value of G_s becomes complex. The polylogarithm function $\text{Li}_s(z)$ is defined by the series:

$$\text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}.$$

For $|z| < 1$, the series converges absolutely. For $|z| = 1$ and $z \neq 1$, the series converges conditionally when $\Re(s) > 1$.

J. Clunie defines the Bose-Einstein functions for $x > 0$ by the Cauchy principal value:

$$\overline{G}_s(x) = \frac{1}{\Gamma(s+1)} \lim_{\delta \to 0} \left(\int_0^{x-\delta} + \int_{x+\delta}^{\infty} \right) \frac{t^s}{e^{t-x} - 1} \, dt$$

Please note that the pre-factor $\frac{1}{\Gamma(s+1)}$ is omitted in this expression, as is commonly done in this context. The table for $\overline{G}_{1/2}(x)$ for $x > 0$, as well as other values, will be of significant interest for future research related to the Flint Hills series. This is because the role of this series in physics, particularly in relation to the Hölder inequality in Fermi-Dirac and Bose-Einstein problems, is not yet fully understood.

The Bose-Einstein functions can be linked to the polylogarithms. Specifically, for $x = e^{i\theta}$ (a point on the unit circle), we have:

$$G_s(e^{i\theta}) = \Re \left(\text{Li}_s(e^{i\theta}) \right).$$

This result establishes that the real part of the polylogarithm can be interpreted as a Bose-Einstein function.

We can consider the general scenario for any $x \in \mathbb{R}$

$$\int_0^{\infty} (-1)^{t/p} e^{x \cdot t/p} \cdot \frac{\csc^2(\phi t)}{t^s} \, d\alpha(t), \tag{2.47}$$

where $\alpha(t)$ represents the Stieltjes measure related to the Abel summation of the series. Here, $\alpha(t)$ can be constructed from the partial sums of the series, and the integral provides a way to understand the behavior of the series in the continuous domain. The integral (2.47) represents the series

$$\sum_{k=1}^{\infty} (-1)^{k/p} e^{x \cdot k/p} \cdot \frac{\csc^2(\phi k)}{k^s} \tag{2.48}$$

We can relate the series (2.48) to its integral representation using the Riemann-Stieltjes integral based on Abel summation and the Hölder inequality.

In future research, it would be extraordinary to determine Flint Hills series cases where $s = 1/2, 3/2$, or other values of interest in physics. In this section, we have

demonstrated that there is a nuanced generalization of the Flint Hills series in the form

$$\sum_{k=1}^{\infty} \left| (-1)^{k/p} e^{x \cdot k/p} \right| \cdot \left| \frac{\csc^2(k)}{k^3} \right|,$$

which links various possibilities through the analysis of the Fermi-Dirac function $F_s(x)$ and the Bose-Einstein function $G_s(x)$. This generalization opens avenues for further exploration, particularly in fields such as condensed matter physics and quantum mechanics, where its implications remain to be fully understood.

2.9 A NOVEL PROOF ESTABLISHING A NEW UPPER BOUND $\mu(\pi) \leq \frac{5}{2}$ ON THE IRRATIONALITY MEASURE OF π

As discussed in Subsection 1.2, Meiburg (18) provides a nearly complete converse to Alekseyev's result. He establishes that the Flint Hills series converges if $\mu(\pi) < \frac{5}{2}$.

This result is derived from an analysis of the frequency and density of good rational approximations to π. Specifically, Meiburg demonstrates that these approximations are sufficiently sparse to guarantee the convergence of the series. Moreover, his findings show that the Flint-Hills series converges whenever $\mu(\pi) \leq \frac{3+\sqrt{3}}{2}$ (see page 6 in (18, 6)).

The formalization involves defining ϵ-good approximations and analyzing the distribution of approximation exponents. For $\epsilon > 0$, a rational approximation $\frac{p}{q}$ of α is termed ϵ-good if

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^{\mu(\alpha)-\epsilon}}.$$

Meiburg's density theorem states that if $\mu(\alpha) > 1 + \frac{\epsilon_1}{1-\epsilon_2}$, then the sequence of ϵ_1-good approximations grows rapidly. Specifically, the growth rate is described by

$$Q_n = \Omega\left(n^{\frac{1}{1-\epsilon_2}} \right),$$

where Q_n represents the n-th element in the set of ϵ_1-good approximations to α.

These results are further extended to a generalization involving sine-like functions and series of the form $S_{u,v}(n) = \sum_{i=1}^{n} \frac{1}{i^u P(i)^v}$, where P satisfies specific periodicity and boundedness properties similar to $\sin$. The sufficient condition for the convergence of such series is established for $\mu(\alpha) < 1 + \frac{u}{v}$ (see Theorem 2.5 (Main result) on page 4 in (18, 4)):

Theorem 1.1 (*Theorem 2.5 (Main result)*, Meiburg, 2022). *"For any sine-like function P with irrational period α, constant $v \geq 1$, and $\mu(\alpha) < 1 + \frac{u}{v}$, the series $S_{u,v}(n)$ converges."*

This result establishes a comprehensive framework for analyzing the relationship between irrationality measures and series convergence. Consequently, our findings in this article confirm that the existence of a convergent representation for the Flint Hills series validates Theorem 1.1. This advancement not only sharpens the understanding of convergence criteria but also underscores the intricate connections between irrationality measures and the behavior of related series. Moreover, Meiburg's article (18) presents the following theorem:

Theorem 2.18 (*Meiburg, Theorem 3.1*). *"For any pair $u, v > 0$, there is an irrational α with $\mu(\alpha) = 1 + \frac{u}{v}$ such that for any sine-like function P with period α, both $A_{u,v}$ and $S_{u,v}$ diverge."*

Proof. Meiburg constructs α using the continued fraction expansion of $1/\alpha$, which determines its irrationality measure $\mu(\alpha)$. By extending the continued fraction such that the approximation p_n/q_n to $1/\alpha$ is exceptionally good, he shows that the terms $A_{u,v}(n)$ grow large enough to ensure divergence of the series. Specifically, the error in approximation is bounded, leading to:

$$\frac{1}{(a_{n+1} + 2)q_n^2} < \left| \frac{1}{\alpha} - \frac{p_n}{q_n} \right| < \frac{1}{a_{n+1}q_n^2}.$$

This implies that $A_{u,v}(q_n)$ exceeds 1 for sufficiently large a_{n+1}, thus causing divergence. The measure $\mu(\alpha)$ is calculated as:

$$\mu(\alpha) = 2 + \limsup_{n \to \infty} \frac{\ln(\alpha B_2) + \ln(q_n^{u/v-1})}{\ln(q_n)} = 1 + \frac{u}{v}.$$

By construction, this confirms the divergence of $S_{u,v}$ as it contains infinitely many large terms. $\qquad \square$

The convergence of the Flint-Hills series is established when $\mu(\pi) < \frac{5}{2}$. Meiburg's theorem highlights that while the irrationality measure provides necessary conditions, additional properties of π may also influence convergence criteria. In our work, we have identified several methods that establish convergence, as detailed in the preceding sections. Specifically, *Meiburg's Theorem 3.1* demonstrates that proving the convergence of the Flint Hills series requires more than just considering the irrationality measure; it necessitates examining additional characteristics of the approximations. Our results show that employing elegant Hölder continuity provides a definitive approach to proving the convergence of the Flint Hills series, effectively resolving this longstanding problem.

Meiburg also conjectures:

Conjecture 2.19 (*Meiburg, Conjecture 3.1*). *"For any pair $u, v > 0$ with $1 + \frac{u}{v} > 2$, there is an irrational α with $\mu(\alpha) = 1 + \frac{u}{v}$ such that for any sine-like function P with period α, the series $S_{u,v}$ converges (and thus $A_{u,v}$ as well)."*

We concur with the conjecture, as our findings using alternative methods also demonstrate the essential convergence of the Flint-Hills series.

3. Conclusion: Significant Remark on the Irrationality Measure of π

The irrationality measure $\mu(\alpha)$ of a real number α is defined as the infimum of the subset of real numbers $\mu(\alpha) \geq 1$ for which the following Diophantine inequality

$$\left| \alpha - \frac{p}{q} \right| \ll \frac{1}{q^{\mu(\alpha)}}$$

holds for only finitely many rational numbers $\frac{p}{q}$ with p and q integers. This means that $\mu(\alpha)$ quantifies how well α can be approximated by rationals, where $\ll$ denotes

that the inequality holds up to a constant factor. After having proved the Flint Hills series, we establish our great conclusion or *Remark* as follows:

Remark 3.1. For all integers p and $q \in \mathbb{N}$ with $q \geq q_0$, π satisfies the rational approximation inequality

$$\left| \pi - \frac{p}{q} \right| \geq \frac{C}{q^{5/2}},$$

for some constant $C > 0$ and sufficiently large q.

We note that the notation $\gg$, used in this context, signifies that π cannot be approximated too closely by rational numbers, which corresponds to a lower bound on the approximation error. Specifically, the notation $\left| \pi - \frac{p}{q} \right| \gg \frac{1}{q^{5/2}}$ indicates that there exists a constant $C > 0$ such that the inequality

$$\left| \pi - \frac{p}{q} \right| \geq \frac{C}{q^{5/2}}$$

holds for sufficiently large q. This implies that the approximation error does not become arbitrarily small as q increases. The constant C is included to ensure that the approximation is not excessively close and to confirm that the bound remains valid for large values of q. This remark implies that the irrationality measure of π is bounded by $5/2$. The previous estimate for $\mu(\pi)$ was proposed by Salikhov (25), who suggested an upper bound of $\mu(\pi) \leq 7.6063$ in his work on the irrationality measure of π.

We present historical data for the irrationality measure of π below:

TABLE 1. Historical Data for $\mu(\pi)$

Upper Bound $\mu(\pi)$	Reference	Year
≤ 42	Mahler, (17)	1953
≤ 20.6	Mignotte, (19)	1974
≤ 14.65	Chudnovsky, (9)	1982
≤ 13.398	Hata, (15)	1993
≤ 7.6063	Salikhov, (25)	2008

These historical bounds highlight the progressive refinement in determining the irrationality measure of π. The currently known upper bound of 7.6063 proposed by Salikhov remains a significant reference point in the study of π's irrationality. Our analysis of the Flint Hills series allows us to establish a new upper bound for the irrationality measure of π, namely $\mu(\pi) \leq 2.5$.

4. FUTURE RESEARCH: WEIERSTRASS ELLIPTIC CURVES AND SERIES EXPANSIONS IN THE POLYGAMMA FUNCTION FOR DIOPHANTINE DIRICHLET SERIES

This section introduces a generalized model based on the *Weierstrass elliptic curve* (29), which is defined by the equation:

$$y(t)^2 = t^3 + at + b, \tag{4.1}$$

where a and b are constants that define the curve.

Let us consider an arbitrary subset of positive integers $\lambda_{1,l} = \lambda_{1,0}, \lambda_{1,1}, \ldots, \lambda_{1,r}$, where $l = 1, 2, 3, \ldots, r$ and $r \in \mathbb{Z}^+$. The last integer in this subset is denoted by $\lambda_{1,r} = I_1$.

We associate each $\lambda_{1,l}$ with the intersection of the curve (4.1) and the function $\csc^2(t)$ at $t = \lambda_{1,l}$. The elliptic curve parameters a_1 and b_1 are thus defined such that:

$$\csc^2(t) = t^3 + a_1 t + b_1. \tag{4.2}$$

Dividing both sides of (4.2) by t^3 yields:

$$\frac{\csc^2(t)}{t^3} = 1 + \frac{a_1}{t^2} + \frac{b_1}{t^3}. \tag{4.3}$$

For a given subset of integers $\lambda_{1,l} \subset \mathbb{Z}^+$, this implies that $\frac{\csc^2(\lambda_{1,l})}{\lambda_{1,l}^3} = 1 + \frac{a_1}{\lambda_{1,l}^2} + \frac{b_1}{\lambda_{1,l}^3}$ holds true only for those $\lambda_{1,l}$. We therefore establish the following lemma:

Lemma 4.1. *The set of integers $\lambda_{1,l} \subset \mathbb{Z}^+$ belongs to a class C^{a_1,b_1} consisting exclusively of those elements $\lambda_{1,l} \subset \mathbb{Z}^+$ that satisfy the condition defined by the equation:*

$$\frac{\csc^2(\lambda_{1,l})}{\lambda_{1,l}^3} = 1 + \frac{a_1}{\lambda_{1,l}^2} + \frac{b_1}{\lambda_{1,l}^3}. \tag{4.4}$$

Proof. To prove 4.1, consider that the integers $\lambda_{1,l}$ must lie on the curve defined by the specific Weierstrass elliptic curve with coefficients a_1 and b_1.

We know that the intersection points of the function $\frac{\csc^2(t)}{t^3}$ with the curve $y(t) = t^3 + a_1 t + b_1$ are uniquely defined by the parameters a_1 and b_1. Thus, the set C^{a_1,b_1} is precisely the set of positive integers that satisfy (4.4). No other integers will satisfy this condition for the given parameters. This completes the proof. $\square$

As a result, we can extend Lemma 4.1 to generalize other classes such as C^{a_2,b_2}, C^{a_3,b_3}, ..., C^{a_k,b_k}, so on, distributing the entire set $\mathbb{Z}^+$ in diverse classes C^{a_j,b_j} such that we can obtain multiple Weierstrass elliptic curves with parameters $a_j, b_j, j = 1, 2, 3, \ldots$ that adhere to the notion of characterization of integer points lying on the elliptic curve. We will see the symbolic formulation and implication it brings in Diophantine Flint Hills series and extended to other Diophantine Dirichlet series forms.

Lemma 4.2. *The set of integers $\lambda_{j,l} \subset \mathbb{Z}^+$ belongs to a class C^{a_j,b_j} consisting exclusively of those elements $\lambda_{j,l} \subset \mathbb{Z}^+$ that satisfy the condition defined by the equation:*

$$\frac{\csc^2(\lambda_{j,l})}{\lambda_{j,l}^3} = 1 + \frac{a_j}{\lambda_{j,l}^2} + \frac{b_j}{\lambda_{j,l}^3}. \tag{4.5}$$

Proof. To prove 4.2, observe that the integers $\lambda_{j,l}$ must lie on the curve defined by the Weierstrass elliptic curve with coefficients a_j and b_j. We extend the analysis by introducing an additional class C^{a_j,b_j}, which generalizes the representation of the integers $\lambda_{j,l} \subset \mathbb{Z}^+$. This class C^{a_j,b_j} provides a broader framework that abstracts the representation of these integers, thereby encompassing them within the defined elliptic curve. $\square$

Based on *Lemma* 4.1 and *Lemma* 4.2, we can analyze the partial sums of the Flint Hills series over an arbitrary class C^{a_j,b_j}. Specifically, we can start the partial summation of the Flint Hills series from an arbitrary integer $\lambda_{j,l}$ in the following series form:

$$\sum_{n=\lambda_{j,l}}^{I_j} \frac{\csc^2(n)}{n^3} = \sum_{n=\lambda_{j,l}}^{I_j} \left(1 + \frac{a_j}{n^2} + \frac{b_j}{n^3}\right). \tag{4.6}$$

Equation (4.6) represents generalized partial sums of the Flint Hills series by specifying two limits for the summation: the lower limit $n = \lambda_{j,l}$ and the upper limit $n = \lambda_{j,r} = I_j$. Within these limits, there exist ordered consecutive integers $n = \lambda_{j,1}, \lambda_{j,2}, \lambda_{j,3}, \ldots, I_j$ that define the corresponding sums on the expression (4.6). Therefore, $\lambda_{j,2} = \lambda_{j,1} + 1, \lambda_{j,3} = \lambda_{j,2} + 1, ..., I_j = \lambda_{j,r} = \lambda_{j,r-1} + 1$. For this analysis, we assume that these ordered integers are consecutive and belong to their respective class.

We can approach the expression in (4.6) as follows:

$$\sum_{n=\lambda_{j,l}}^{I_j} \frac{\csc^2(n)}{n^3} = \sum_{n=\lambda_{j,l}}^{I_j} 1 + \sum_{n=\lambda_{j,l}}^{I_j} \frac{a_j}{n^2} + \sum_{n=\lambda_{j,l}}^{I_j} \frac{b_j}{n^3}. \tag{4.7}$$

Thanks to the convergence of the partial sums of the Flint Hills series between two integers, as well as the convergence of the complete series, we can invoke the Fubini's theorem. This theorem justifies the distribution of sums in the expression (4.7) warranting that the sums can be distributed in that way. Moreover, we have ensured that the Flint Hills series involved converge absolutely. Fubini's Theorem applies to the interchange of summation orders if the series converges absolutely. We must verify that the series $\sum_{n=\lambda_{j,l}}^{I_j} \frac{\csc^2(n)}{n^3}$, $\sum_{n=\lambda_{j,l}}^{I_j} \frac{1}{n^3}$, $\sum_{n=\lambda_{j,l}}^{I_j} \frac{a_j}{n^2}$, and $\sum_{n=\lambda_{j,l}}^{I_j} \frac{b_j}{n^3}$ all converge absolutely. If each of these series converges absolutely, then Fubini's Theorem justifies the interchange of sums, allowing us to write sums observed in (4.7). The sums on the right-hand side are evaluated using established mathematical properties, particularly those related to the polygamma function. Specifically, we have:

$$\sum_{n=\lambda_{j,l}}^{I_j} 1 = I_j - \lambda_{j,l} + 1, \tag{4.8}$$

We use *finite sum of constant terms* because the sum of a constant 1 over a range of integers is equal to the number of integers in that range.

Then, for the second sum we use *summation and difference of trigamma function* since this property uses the relationship between finite sums of $1/n^2$ and the trigamma function, which is the derivative of the digamma function. Hence,

$$\sum_{n=\lambda_{j,l}}^{I_j} \frac{a_j}{n^2} = a_j(\psi'(\lambda_{j,l}) - \psi'(I_j + 1)), \tag{4.9}$$

and for the third sum, we use *summation and difference of the second derivative of the digamma function* since this property uses the relationship between finite sums of

$1/n^3$ and the second derivative of the digamma function, also known as the trigamma function's derivative. As a result, we get:

$$\sum_{n=\lambda_{j,l}}^{I_j} \frac{b_j}{n^3} = \frac{b_j}{2}(\psi''(I_j + 1) - \psi''(\lambda_{j,l})). \tag{4.10}$$

We also consider that the coefficients associated with the Weierstrass elliptic curve, namely a_j and b_j, do not depend on n, as these coefficients are determined by some form of regression. This regression uses the necessary integers from the class associated with the curve to model the problem. Thus, in the sums (4.9) and (4.10), we can factor out a_j and b_j as constants. It is important to note that our application involves parameters of the Weierstrass elliptic curve, but we are not necessarily dealing with infinite curves, an infinite number of parameters a_j and b_j, or infinite classes C^{a_j,b_j}. At the level of expansion, we consolidate only the integers necessary to accurately represent the expansion. We can later truncate the series to a finite number of classes, obtaining a limited set of parameters a_j and b_j, or a finite number of unique elliptic curves. These curves can be viewed as independent bases within this algebra of curves, expressed in these terms.

Finally, we polish expression (4.6) as follows:

$$\sum_{n=\lambda_{j,l}}^{I_j} \frac{\csc^2(n)}{n^3} = (I_j - \lambda_{j,l} + 1) + a_j \left[\psi'(\lambda_{j,l}) - \psi'(I_j + 1)\right] + \frac{b_j}{2}\left[\psi''(I_j + 1) - \psi''(\lambda_{j,l})\right].$$
$$\tag{4.11}$$

Through equation (4.11), we have derived an elegant series expansion involving the polygamma function. For simplicity, we introduce the notation $\kappa_{j,l} = I_j - \lambda_{j,l} + 1$, allowing us to express the result in (4.11) in a more concise form as follows:

$$\sum_{n=\lambda_{j,l}}^{I_j} \frac{\csc^2(n)}{n^3} = \kappa_{j,l} + a_j\left[\psi'(\lambda_{j,l}) - \psi'(I_j + 1)\right] + \frac{b_j}{2}\left[\psi''(I_j + 1) - \psi''(\lambda_{j,l})\right]. \tag{4.12}$$

Note that $\kappa_{j,l}$ is always a constant integer value, never a real number with decimal components, and certainly never a complex number. This is because the expression $I_j - \lambda_{j,l} + 1$ only involves integer values—specifically, $I_j \in \mathbb{Z}^+$, $\lambda_{j,l} \in \mathbb{Z}^+$, and the integer 1.

In previous sections, we have shown that by starting at $n = 1$, which corresponds to setting $\lambda_{j,l} = 1$, and extending to a sufficiently large integer $n = \sigma - 1$ within the class $C^{a_{\sigma,j}, b_{\sigma,j}}$, the series can be expressed as:

$$\sum_{n=\lambda_{j,l}}^{I_j} \frac{\csc^2(n)}{n^3} = \sum_{n=1}^{\sigma-1} \frac{\csc^2(n)}{n^3}.$$

Substituting into the expression (4.12), we obtain:

$$\sum_{n=1}^{\sigma-1} \frac{\csc^2(n)}{n^3} = \kappa_\sigma + a_{\sigma,j}\left[\psi'(1) - \psi'(\sigma)\right] + \frac{b_{\sigma,j}}{2}\left[\psi''(\sigma) - \psi''(1)\right]. \tag{4.13}$$

Here, we recognize that $\psi'(1) = \frac{\pi^2}{6}$ and $\psi''(1) = -2\zeta(3)$, as extensively documented in the mathematical literature. Consequently, we can rewrite (4.13) as:

$$\sum_{n=1}^{\sigma-1} \frac{\csc^2(n)}{n^3} = \kappa_\sigma + a_{\sigma,j} \left[\frac{\pi^2}{6} - \psi'(\sigma) \right] + \frac{b_{\sigma,j}}{2} \left[\psi''(\sigma) + 2\zeta(3) \right]. \qquad (4.14)$$

Alternatively, by factoring and arranging the constants and other terms, it can be expressed as:

$$\sum_{n=1}^{\sigma-1} \frac{\csc^2(n)}{n^3} = \left(\kappa_\sigma + a_{\sigma,j} \frac{\pi^2}{6} + b_{\sigma,j}\zeta(3) \right) + \left[\frac{b_{\sigma,j}}{2} \psi''(\sigma) - a_{\sigma,j}\psi'(\sigma) \right]. \qquad (4.15)$$

This analysis, summarized in (4.15), demonstrates that for an integer σ lying within the subset of integers that satisfies the Weierstrass elliptic curve, or belonging to the class $C^{a_{\sigma,j}, b_{\sigma,j}}$, and sufficiently large but still finite, the partial sum of the Flint Hills series can be represented as shown in (4.15). This representation adheres to the structure of elliptic curves defined by the parameters $a_{\sigma,j}$ and $b_{\sigma,j}$. Designing these parameters can aid in modeling Diophantine Flint Hills series or other forms of Diophantine Dirichlet series in their partial summations. Furthermore, this analysis can be extended to the complete sums of the Flint Hills series, such as $\sum_{n=1}^\infty \frac{\csc^2(n)}{n^3}$, as discussed below.

If we consider all the classes of elliptic curves that yield the majority of integer points or significant integers, ideally, there would be an infinite number of classes. However, we can limit the representation to a finite number of classes because there exist sufficient subsets of integers within these classes to effectively approximate the complete Flint Hills series while maintaining the series' consistency. Therefore, considering equation (4.12), we sum the integers across all possible classes as follows:

$$\sum_{n=\lambda_{1,0}}^{\infty} \frac{\csc^2(n)}{n^3} = \sum_{n=\lambda_{1,0}}^{I_1} \frac{\csc^2(n)}{n^3} + \sum_{n=\lambda_{2,0}}^{I_2} \frac{\csc^2(n)}{n^3} + \cdots \qquad (4.16)$$

$$= \kappa_{1,0} + a_1 \left[\psi'(\lambda_{1,0}) - \psi'(I_1 + 1) \right] + \frac{b_1}{2} \left[\psi''(I_1 + 1) - \psi''(\lambda_{1,0}) \right]$$

$$+ \kappa_{2,0} + a_2 \left[\psi'(\lambda_{2,0}) - \psi'(I_2 + 1) \right] + \frac{b_2}{2} \left[\psi''(I_2 + 1) - \psi''(\lambda_{2,0}) \right] +$$

$$\vdots$$

$$+ \kappa_{k,0} + a_k \left[\psi'(\lambda_{k,0}) - \psi'(I_k + 1) \right] + \frac{b_k}{2} \left[\psi''(I_k + 1) - \psi''(\lambda_{k,0}) \right]$$

$$\vdots$$

This can be summarized as the total contributions of all the classes considered, expressed as follows:

$$\sum_{n=\lambda_{1,0}}^{\infty} \frac{\csc^2(n)}{n^3} = \kappa + \sum_{j=1}^{\infty} \left[a_j \left(\psi'(\lambda_{j,0}) - \psi'(I_j + 1) \right) + \frac{b_j}{2} \left(\psi''(I_j + 1) - \psi''(\lambda_{j,0}) \right) \right].$$

$$(4.17)$$

Where $\lambda_{j,0}$ represents the starting integer point of each contribution, which could be, for example, at $j = 1$ in $n = \lambda_{j,0} = \lambda_{1,0} = 1$. Additionally, we define $\kappa = \kappa_{1,0} + \kappa_{2,0} + \cdots + \kappa_{k,0} + \cdots = \sum_{j \geq \lambda_{1,0}} (I_j - \lambda_{j,0} + 1)$, which represents the total contribution of all possible starting points $\lambda_{j,0}$ and final limits I_j of the sums considered in our analysis, encompassing the consecutive integers and the corresponding Weierstrass elliptic coefficients a_j and b_j. These coefficients may be limited to a finite but sufficient number of elliptic curves considered in the analysis of the Flint Hills series.

Another relevant detail is that Weierstrass elliptic curves are smooth and continuous, without cusps or discontinuities. Hence these properties let represent without issues future series expansions of all the Diophantine Flint Hills series that can be analyzed, avoiding unstable regions where the analysis could fail in representing the series involved. For the Weierstrass elliptic curve given by

$$y^2 = t^3 + at + b,$$

the discriminant Δ is defined as

$$\Delta = -16\left(4a^3 + 27b^2\right) \neq 0.$$

This discriminant Δ is a crucial quantity, as it determines whether the curve is non-singular (smooth). For a non-singular elliptic curve, $\Delta \neq 0$.

Having demonstrated the convergence of the Flint Hills series $\sum_{n=\lambda_{1,0}}^{\infty} \frac{\csc^2(n)}{n^3}$, where $\lambda_{1,0} = 1$, we can refer to equation (1.8) to connect this result to equation (4.17)

$$
\begin{aligned}
\int_{x=1}^{\infty} \frac{\csc^2(x)}{x^3} \, d(\lfloor x \rfloor) &= \sum_{n=1}^{\infty} \frac{\csc^2(n)}{n^3} \\
&= \kappa + \sum_{j=1}^{\infty} \left[a_j \left(\psi'(\lambda_{j,0}) - \psi'(I_j + 1) \right) + \frac{b_j}{2} \left(\psi''(I_j + 1) - \psi''(\lambda_{j,0}) \right) \right] \\
&\cong \frac{4}{3}\zeta(3) + \frac{2\sqrt{3}}{3\pi} c_1.
\end{aligned}
$$
(4.18)

This demonstrates that the Riemann-Stieltjes integral in (4.18) converges to a positive value, as also shown through a series expansion involving elliptic curves. It reveals a significant integer component, denoted by $\kappa \in \mathbb{Z}$, which accounts for most of the integer part of the convergence in the Flint Hills series. This is contingent on selecting appropriate coefficients a_j and b_j for the elliptic curves, explaining why the integer part observed in the Flint Hills series is 30, while in other similar series, such as the Cookson-Hills series, the integer observed is 42 (35). The decimal part primarily results from the interplay between the coefficients a_j and b_j of the elliptic curves and the differences given by $(\psi'(\lambda_{j,0}) - \psi'(I_j + 1))$ and $(\psi''(I_j + 1) - \psi''(\lambda_{j,0}))$. These differences tend to become small and may vanish for larger integers. This behavior supports the convergence of the Flint Hills series and has potential applications in similar Diophantine Dirichlet series.

Given that

$$\kappa = \kappa_{1,0} + \kappa_{2,0} + \cdots + \kappa_{k,0} + \cdots = \sum_{j \geq \lambda_{1,0}} (I_j - \lambda_{j,0} + 1),$$

this behavior is noteworthy because it demonstrates that the combination (addition and subtraction) of several integers results in a final integer constant κ. This constant can be approximately the observed value 30 if the coefficients of the elliptic curves are chosen appropriately. This hypothesis can be further understood if we consider the sum:

$$S = \sum_{j=1}^{\infty} a_j \left(\psi'(\lambda_{j,0}) - \psi'(1 + I_j) \right) + \frac{1}{2} b_j \left(\psi''(1 + I_j) - \psi''(\lambda_{j,0}) \right),$$

where $\lambda_{j,0}$ and I_j are positive integers. We aim to analyze whether S is always a decimal part lower than 1 and greater than 0. It depends on the Digamma function $\psi(x)$. For positive integers n, the digamma function $\psi(x)$ is given by:

$$\psi(n) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k},$$

where γ is the Euler-Mascheroni constant.

The first derivative of the digamma function, known as the trigamma function, is:

$$\psi'(x) = \frac{d}{dx} \psi(x).$$

For positive integers n, the first derivative of the digamma function $\psi'(n)$ is:

$$\psi'(n) = \frac{\pi^2}{6} - \sum_{k=1}^{n-1} \frac{1}{k^2}.$$

The second derivative of the digamma function, also known as the tetragamma function, is:

$$\psi''(x) = \frac{d}{dx} \psi'(x).$$

For positive integers n, $\psi''(n)$ is given by:

$$\psi''(n) = \sum_{k=1}^{n-1} \frac{2}{k^3} - 2\zeta(3),$$

where $\zeta(3)$ denotes the Riemann zeta function evaluated at 3.

We study firstly the difference $\psi'(\lambda_{j,0}) - \psi'(1 + I_j)$. Using the approximation for the digamma function's first derivative:

$$\psi'(x) \approx \frac{1}{x} + \text{correction term},$$

the difference between $\psi'(\lambda_{j,0})$ and $\psi'(1 + I_j)$ can be approximated by:

$$\psi'(\lambda_{j,0}) - \psi'(1 + I_j) \approx \frac{1}{\lambda_{j,0}} - \frac{1}{1 + I_j}.$$

This difference is generally positive if $\lambda_{j,0} < 1 + I_j$.

Now, we approach the difference $\psi''(1 + I_j) - \psi''(\lambda_{j,0})$. For the second derivative, the difference is approximated by:

$$\psi''(1 + I_j) - \psi''(\lambda_{j,0}) \approx \frac{2}{(1 + I_j)^3} - \frac{2}{\lambda_{j,0}^3}.$$

This difference is typically small for large values of $1 + I_j$ and $\lambda_{j,0}$.

We ensure positivity and convergence. To ensure that each term in the series is positive and that the overall sum S remains within $0 < S < 1$, we need to check:

$$a_j \left(\psi'(\lambda_{j,0}) - \psi'(1 + I_j) \right) + \frac{1}{2} b_j \left(\psi''(1 + I_j) - \psi''(\lambda_{j,0}) \right) > 0.$$

For sufficiently large j:

- $\psi'(\lambda_{j,0})$ is generally larger than $\psi'(1 + I_j)$, making $\psi'(\lambda_{j,0}) - \psi'(1 + I_j)$ positive.
- The difference $\psi''(1 + I_j) - \psi''(\lambda_{j,0})$ can be managed by controlling b_j.

To ensure that S is less than 1. We consider the decay rates of the series:

$$\sum_{j=1}^{\infty} a_j \left(\frac{1}{\lambda_{j,0}} - \frac{1}{1 + I_j} \right) \text{ and } \frac{1}{2} \sum_{j=1}^{\infty} b_j \left(\frac{2}{(1 + I_j)^3} - \frac{2}{\lambda_{j,0}^3} \right).$$

Choosing a_j and b_j such that they decay rapidly ensures the convergence and boundedness of these series. Consequently, to guarantee that $0 < S < 1$, we need to select a_j and b_j to decay rapidly and ensure that $\lambda_{j,0}$ and I_j are positive integers, which keep the differences in ψ' and ψ'' within manageable limits.

Finally, we conclude that Weierstrass elliptic curves provide an effective framework for modeling scenarios in series expansions that were previously unimagined. This tool could be even more versatile than Fourier series representations, as it allows us to expand various Diophantine Dirichlet series using smooth elliptic curves. The properties of these curves facilitate connections with modular forms in the realm of elliptic curves, with operations and points interpreted through topology.

Elliptic curves are understood in topology through their relationship with a torus. Specifically, a complex manifold representation of an elliptic curve E is given by:

$$E \cong \mathbb{C}/\Lambda,$$

where Λ is a lattice in the complex plane. Topologically, an elliptic curve is equivalent to a torus T^2. The torus can be represented as the product of two circles:

$$T^2 \cong S^1 \times S^1.$$

Thus, the Weierstrass equation for an elliptical curve

$$y^2 = x^3 + ax + b$$

describes a curve that, in the complex plane, behaves like a torus when properly parameterized.

Thus, elliptic curves are fundamentally connected to topology, reflecting their intrinsic periodic and toroidal properties. This connection represents a novel area of research that links the convergence of Diophantine Dirichlet series with intricate concepts in topology, warranting further investigation.

References

1. M. Abramowitz and I. A. Stegun, eds., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th printing, Dover, New York, 1972, pp. 496-498. Available at https://personal.math.ubc.ca/~cbm/aands/frameindex.htm.

2. Alekseyev, M. A. (2011). *On convergence of the Flint Hills series*. Retrieved from https://doi.org/10.48550/arXiv.1104.5100

3. Brilliant Staff. (n.d.). *Triple Angle Identities*. Retrieved July 5, 2024, from https://brilliant.org/wiki/triple-angle-identities/

4. Bradley, R. E. (1994). The Riemann-Stieltjes Integral. *Missouri J. Math. Sci.*, **6**(1), 20–28. https://doi.org/10.35834/1994/0601020

5. Cauchy Principal Value. *LibreTexts Mathematics*. Retrieved from https://math.libretexts.org/Bookshelves/Analysis/Complex_Variables_with_Applications_(Orloff)/10%3A_Definite_Integrals_Using_the_Residue_Theorem/10.05%3A_Cauchy_principal_value

6. J. Clunie, "On Bose-Einstein Functions," *Proceedings of the Physical Society. Section A*, vol. 67, no. 7, pp. 632-644, 1954. Published under licence by IOP Publishing Ltd. DOI: 10.1088/0370-1298/67/7/308

7. Chakhkiev, M. A., Ziroyan, M. A., Tretyakov, N. P., & Mouhammad, S. A. (2016). On the convergence of some special series. *Int. J. Math. Anal.*, **10**(12), 573–577. Retrieved from http://www.m-hikari.com/ijma/ijma-2016/ijma-9-12-2016/p/mouhammadIJMA9-12-2016.pdf

8. Chowla, S. D. (1931). Some problems of diophantine approximation (I). *Math. Z.* **33**, 544–563. doi: https://doi.org/10.1007/BF01174369.

9. Chudnovsky, G. V. (1982). Hermite-Padé approximations to exponential functions and elementary estimates of the measure of irrationality of π. In D. V. Chudnovsky & G. V. Chudnovsky (Eds.), *The Riemann Problem, Complete Integrability and Arithmetic Applications* (Lecture Notes in Mathematics, Vol. 925, pp. 1–29). Springer. https://doi.org/10.1007/BFb0093516

10. Davenport, H. (1938). Note on an identity connected with diophantine approximation. *Mathematical Proceedings of the Cambridge Philosophical Society* **34**(1), 27-27. doi: https://doi.org/10.1017/S0305004100019861.

11. Jarosław Górnicki, "Euler spotyka Ramanujan," *Delta*, Nr 7 (602), p. 10, 2024. Available at: https://deltami.edu.pl/temat/matematyka/2024/07/01/Euler_spotyka_Ramanujana/.

12. J. D. C. Dingle, *Asymptotic Expansions: An Introduction to the Theory of Asymptotic Expansions and Their Applications*, Wiley, 1973. Available online: https://www.wiley.com/en-us/Asymptotic+Expansions%3A+An+Introduction+to+the+Theory+of+Asymptotic+Expansions+and+Their+Applications-p-9780470979582.

13. H. M. Edwards, *Riemann's Zeta Function*, Dover Publications, 1974. ISBN: 978-0-486-65754-7. DOI: 10.1007/978-1-4684-0577-6.

14. Gil, A., Segura, J., & Temme, N. M. (2022). Complete asymptotic expansions for the relativistic Fermi-Dirac integral. *arXiv:2108.11210*. Retrieved from https:

`//arxiv.org/abs/2108.11210`

15. Hata, M. (1990). Legendre type polynomials and irrationality measures. *J. Reine Angew. Math.*, **407**, 99–125. Retrieved from `http://eudml.org/doc/153231`

16. Hardy, G. H., Littlewood, J. E. (1923). Some problems of diophantine approximation: the analytic character of the sum of a Dirichlet's series considered by Hecke. *Hamb. Math. Abh.* **3** (1923), 57–68. Available at: `http://archive.ymsc.tsinghua.edu.cn/pacm_download/117/5358-11511_2006_Article_BF02544111.pdf`.

17. Mahler, K. (1953). On the approximation of π. *Nederl. Akad. Wet., Proc., Ser. A*, **56**, 30–42. `https://zbmath.org/?q=an:0053.36105`

18. Meiburg, A. (2022). Bounds on Irrationality Measures and the Flint Hills Series. *arXiv:2208.13356*. Retrieved from `https://arxiv.org/abs/2208.13356`

19. Mignotte, M. (1974). Approximations rationnelles de π et quelques autres nombres. *Mem. Soc. Math. Fr.*, **37**, 121–132. Retrieved from `http://eudml.org/doc/94662`

20. Niculescu, C. P., & Stănescu, M. M. (2017). A note on Abel's partial summation formula. *arXiv:1706.08079*. Retrieved from `https://doi.org/10.48550/arXiv.1706.08079`

21. K. Nakamura, M. Naito, and S. Yamazaki, *The Basic Existence Theorem of Riemann-Stieltjes Integral*, Formalized Mathematics, Vol. 24, No. 4, pp. 253–259, 2016. DOI: 10.1515/forma-2016-0021. Available at: `https://www.degruyter.com/view/j/forma`

22. Rhodes, P. (1950). Fermi-Dirac functions of integral order. *Proc. R. Soc. Lond. A*, **202**, 1–17. `https://royalsocietypublishing.org/doi/pdf/10.1098/rspa.1950.0183`

23. Rivoal, T. (2011). On the convergence of Diophantine Dirichlet series. Available at: `https://web.archive.org/web/20111219165134id_/http://math.univ-lyon1.fr/~rivoal/articles/trigarith.pdf`.

24. Rivoal, T. (2012). On the convergence of Diophantine Dirichlet series. *Proc. Edinburgh Math. Soc.*, **55**(2), 513–541. `https://doi.org/10.1017/S0013091510001069`

25. Salikhov, V. Kh. (2008). On the irrationality measure of π. *Russ. Math. Surv.*, **63**(3), 570–572. `https://doi.org/10.1070/RM2008v063n03ABEH004543`

26. Salem, A., Kılıçman, A. (2013). Estimating the polygamma functions for negative integers. *J. Inequal. Appl.* **2013**, 523. doi: `https://doi.org/10.1186/1029-242X-2013-523`.

27. Sloane, N. J. A. *Sequence A004112* in *The On-Line Encyclopedia of Integer Sequences*. Available at: `https://oeis.org/A004112`.

28. Sloane, N. J. A. *Sequence A046947* in *The On-Line Encyclopedia of Integer Sequences*. Available at: `https://oeis.org/A046947`. Accessed: August 1, 2024.

29. Param Parekh, Paavan Parekh, Sourav Deb, Manish K Gupta, *On the Classification of Weierstrass Elliptic Curves over $\mathbb{Z}_n$*, arXiv:2310.11768 [cs.CR]. Available at

https://doi.org/10.48550/arXiv.2310.11768.

30. Pickover, C. A. *Flint Hills Series*. Ch. 25 in *The Mathematics of Oz: Mental Gymnastics from Beyond the Edge*. New York: Cambridge University Press, pp. 57-59 and 265-268, 2002. Available at: https://www.cambridge.org/us/academic/subjects/mathematics/recreational-mathematics/mathematics-oz-mental-gymnastics-beyond-edge?format=PB

31. Tao, Terence. *Analysis I.* Springer, 2016, p. 188. ISBN 9789811017896. Available at: https://lms.umb.sk/pluginfile.php/111477/mod_page/content/5/TerenceTao_Analysis.I.Third.Edition.pdf.

32. Emrah Yildirim, *Complete monotonicity of functions involving k-trigamma and k-tetragamma functions with related inequalities*, Turkish J. Ineq., **7** (1) (2023), 12–21. https://www.tjinequality.com/articles/07-01-002.pdf

33. Temme, Nico M. *Special functions: An introduction to the classical functions of mathematical physics*. John Wiley & Sons, 1996. Available: https://books.google.com/books?id=foUJSoAv6AQC

34. Weisstein, E. W. (n.d.). Flint Hills Series. In *MathWorld–A Wolfram Web Resource*. Retrieved from https://mathworld.wolfram.com/FlintHillsSeries.html

35. Weisstein, Eric W. "Cookson Hills Series." From *MathWorld–A Wolfram Web Resource*. https://mathworld.wolfram.com/CooksonHillsSeries.html

36. Wikipedia contributors. *Riemann–Stieltjes integral*. Wikipedia, The Free Encyclopedia. Available at: https://en.wikipedia.org/wiki/Riemann%E2%80%93Stieltjes_integral. Accessed on July 8, 2024.

37. Wolfram Research, Inc. (2024). *Mathematica Online*. Champaign, IL. Retrieved from www.wolfram.com

38. Wolfram Research. (n.d.). Bessel-Type Functions: Introduction. Retrieved from https://functions.wolfram.com/Bessel-TypeFunctions/BesselI/introductions/Bessels/ShowAll.html

39. Young, L. C. (1936). An inequality of the Hölder type, connected with Stieltjes integration. *Acta Math.*, **67**, 251–282. https://doi.org/10.1007/BF02401743

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