Paths in the Space of Rational Maps

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Abstract

The space of all rational maps is large and complicated as its dimension is $2n+1$ if the polynomials have degree n. In this paper we investigate this space by probing an associated energy function. We also aim to reduce this energy by using various numerical methods.

1 Introduction

Rational maps, particularly those between Riemann spheres have proved useful as a tool in probing Skyrmion solutions. Recall that a rational map is a holomorphic function from $S^2 \to S^2$ where

$$R(z) = \frac{p(z)}{q(z)} \tag{1.1}$$

with p and q having no common factors. It also is relevant to consider such maps on the Riemann sphere instead, where a point on the sphere $z \in \mathbb{C} \cup \{\infty\}$ corresponds to the $\mathbb{R}^3$ unit vector

$$\hat{\mathbf{n}}_z = \frac{1}{1+|z|^2}(2\Re(z), 2\Im(z), 1-|z|^2) \tag{1.2}$$

via standard steographic projection. Hence a rational map on the sphere corresponds to the unit vector

$$\hat{\mathbf{n}}_z = \frac{1}{1+|z|^2}(2\Re(R), 2\Im(R), 1-|R|^2) \tag{1.3}$$

. Additionally, an important characterization of these rational maps is given by their degree, defined as the maximum of the degrees of p and q:

$$\deg R = \max\{\deg p,\, \deg q\}. \tag{1.4}$$

It is known that there exists a one-to-one correspondence between maps of degree N and N-monopoles, which in turn have similiarites to Skyrmions. This unique relationship offers us a useful approach to investigating Skyrmion energies, through a rational map ansatz. Usually the Skyrme energy, for massless pions, is expressed as

$$E = \int \{-\frac{1}{2}\mathrm{Tr}(R_iR_i) - \frac{1}{16}([R_i, R_j][R_i, R_j])\}\, d^3\mathbf{x} \tag{1.5}$$

where $R_i = (\partial_i U)U^{-1}$ and $U(\mathbf{x}, t)$ is a $SU(2)$ valued scalar field, named the Skyrme field. However we can apply an ansatz to the Skyrme field using a rational map $R(z)$ and a profile function $f(r)$, where r is the radial distance from the origin. The ansatz for the Skyrme field is given by

$$U(r, z) = \exp(if(r)\hat{\mathbf{n}}_R \cdot \sigma) \tag{1.6}$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ denotes the Pauli matrices and after applying this to E with careful manipulation, we can simplify the energy to

$$E = 4\pi \int \Big(r^2 + f'^2 + 2N(f^2+1)\sin^2 f + + \mathcal{I}\frac{sin^4 f}{r^2}\Big)\, dr \tag{1.7}$$

where $\mathcal{I}$, the energy density, denotes the integral

$$\mathcal{I} = \frac{1}{4\pi}\int \left(\frac{1+|z|^2}{1+|R|^2}\right)^4 \frac{2i\, dz, d\bar{z}}{(1+|z|^2)^2}. \tag{1.8}$$

We can see that the result of using a rational map ansatz on E, is that now we can investigate minimal energies much more easily as we only need to minimise E with respect to the profile function $f(r)$ and the rational map $R(z)$. In fact, for massless pions the minimal energy Skyrmions up to $N = 22$ have all been well-approximated by the rational map ansatz.

2 Symmetries of Rational Maps and Energy Densities

In the previous section we covered the motivation behind our use of rational maps. Now we will give a brief rundown of the symmetries that are associated with skyrmions. We start off with the simplest case: maps with degree $N = 1$. The skyrmion solution for this degree is known to have $O(3)$ symmetry and since this corresponds to symmetries of a sphere, the natural rational map of degree 1 is the $R(z) = z$, known as Skyrme's hedgehog map. Plugging this into Eq 1.8 gives the value $\mathcal{I} = 1$ and gives a very good approximation of the Skyrme energy E.

For the $N = 2$ case we aim to obtain $O(2) \times \mathbb{Z}_2$ symmetry, with the general form of the degree 2 map

$$R(z) = \frac{\alpha z^2 + \beta z + \gamma}{\lambda z^2 + \mu z + \nu} \tag{2.1}$$

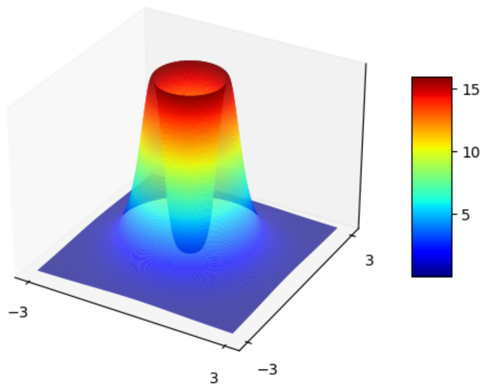
and this is satistfied with the map $R(z) = z^2$. Plugging this into Eq 1.8 yields $\mathcal{I} = 5.81$. Other symmetries for other degrees are listed in Table 1, however we will omit the full derivations. It is also useful visualise the associated skyrmion energy by plotting its energy density $\mathcal{I}$ given by Eq 1.8. We intend to plot this density as a 3D surface, so we use the fact that $dz\, d\bar{z} = -2i dx\, dy$ and using (1.1) to instead obtain $\mathcal{I}$ in the form

$$\mathcal{I} = \frac{1}{\pi} \iint (1 + |z|^2)^2 \left(\frac{p'q - q'p}{|p|^2 + |q|^2} \right)^4 dx\, dy. \tag{2.2}$$

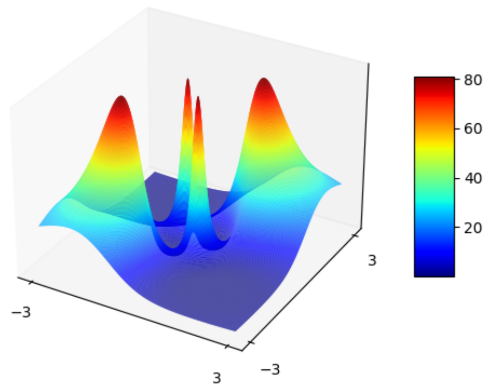
In Figure 1 we graph multiple energies and notice that the symmetries are prevalent in these plots. For example the $N = 2$ and $N = 3^*$ maps both embody $O(2) \times \mathbb{Z}_2$ symmetry and share very similar energy density plots.

N	$\mathcal{I}$	Symmetry
1	1.00	$O(3)$
2	5.81	$O(2) \times \mathbb{Z}_2$
3	13.58	T_d
4	20.65	O_h
5	35.75	D_{2d}
6	50.76	D_{4d}
7	60.87	Y_h
8	112.83	D_{6d}
9	112.83	T_d
17	367.41	Y_h
3^*	18.67	$O(2) \times \mathbb{Z}_2$
5^*	52.05	O_h
11^*	486.84	Y_h

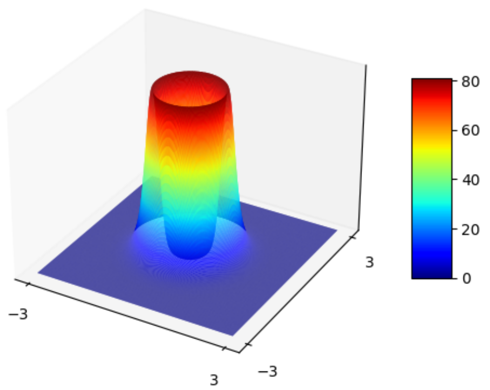
Table 1: Table of Skyrmion energy densities with their associated rational map degree number and their symmetry. A * denotes an alternative configuration, as the same degree can several different symmetries. Explanations behind these symmetries are explained in Houghton et. al [HMS98]



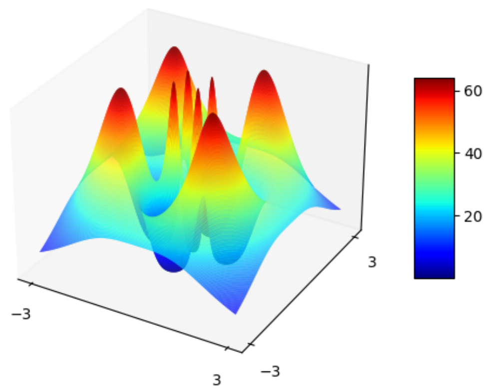
(a) $R(z) = z^2$



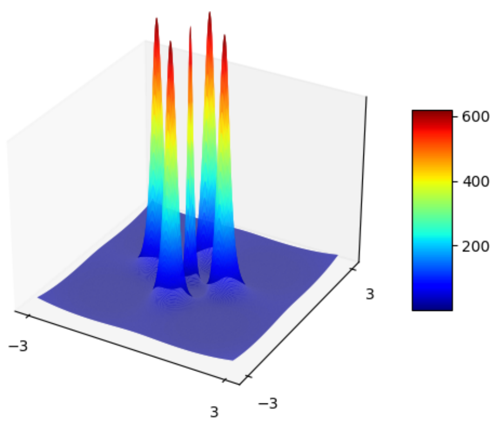
(b) $R(z) = \frac{\sqrt{3}iz^2-1}{z(z^2-\sqrt{3}i)}$



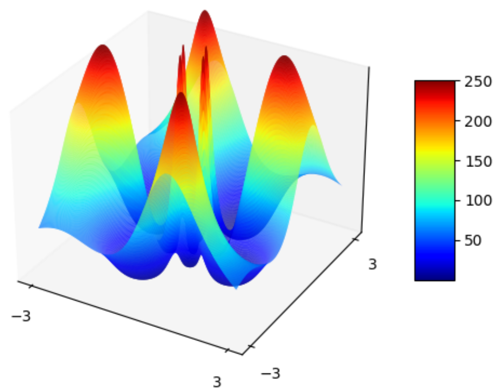
(c) $R(z) = -\frac{1}{z^3}$



(d) $R(z) = \frac{z^4+2\sqrt{3}iz^2+1}{z^4-2\sqrt{3}iz^2+1}$



(e) $R(z) = \frac{z^5-5}{-5z^4+1}$



(f) $R(z) = \frac{z^4+0.16i}{0.16iz^6+z^2}$

Figure 1: Energy density plots for lowest energies (a) $N = 2$, (b) $N = 3$, (c) $N = 3^*$, (d) $N = 4$,(e) $N = 5$, (f) $= N = 6$. We exclude the $N = 1$ case, as the corresponding energy density is constant.

3 Interpolation between Rational Maps

Following from the previous section, we will further compute energy densities but now we are interested in how densities vary when we transition from one map to another. The simplest method to do this, is to generate a path from a map f to g using linear interpolation. In other words for rational maps $R_0(z) = \frac{a(z)}{b(z)}$ and $R_1(z) = \frac{c(z)}{d(z)}$, we set $f_i = a_i(1-t) + c_i t$ and $g_i = b_i(1-t) + d_i t$ to obtain

$$\gamma(t) = R_t(z) = \sum_{i=0}^{N} \frac{f_i}{g_i} = \sum_{i=0}^{N} \frac{a_i(1-t) + c_i t}{b_i(1-t) + d_i t} \quad (3.1)$$

so that $\gamma(0) = R_0$ and $\gamma(1) = R_1$. Now that we have a tool to probe energy densities between rational maps, we begin plotting the $N = 3$ to $N = 3^*$ map in Figure 2, as it is interesting to see the energy barriers of two very similar minimal energies with differing symmetries of T_d and $O(2) \times (Z)_2$ respectively.

We will also define the energy barrier of the path as

$$E_{\text{barrier}} = \max\{\mathcal{I}[R_t] : 0 \leq t \leq 1\}. \quad (3.2)$$

We emphasis that the energy barrier involves the energy density and not the energy itself. Interestingly the energy barrier of Figure 2 appears to be almost minimal. We will now investigate the maps given by Manko, Manton and Wood [] for the $B = 5$ minimal-energy Skyrmion map.

$$R(z) = \frac{z(z^4 + ibz^2 + a)}{az^4 + ibz^2 + 1}, a = -3.07, b = 3.94 \quad (3.3)$$

This has D_{2d} symmetry and we will compare this with another maps, by changing a and b. By setting $b = -3.94$, we should effectively obtain a rotation and hence should be have the same energy as the case for $b = 3.94$. As expected, this is true and is shown in Figure 3a. By setting $b = 0$, we obtain D_{4d} symmetry, acquiring octehedral symmetry when $a = 5$ and this is shown in Figure 3b.

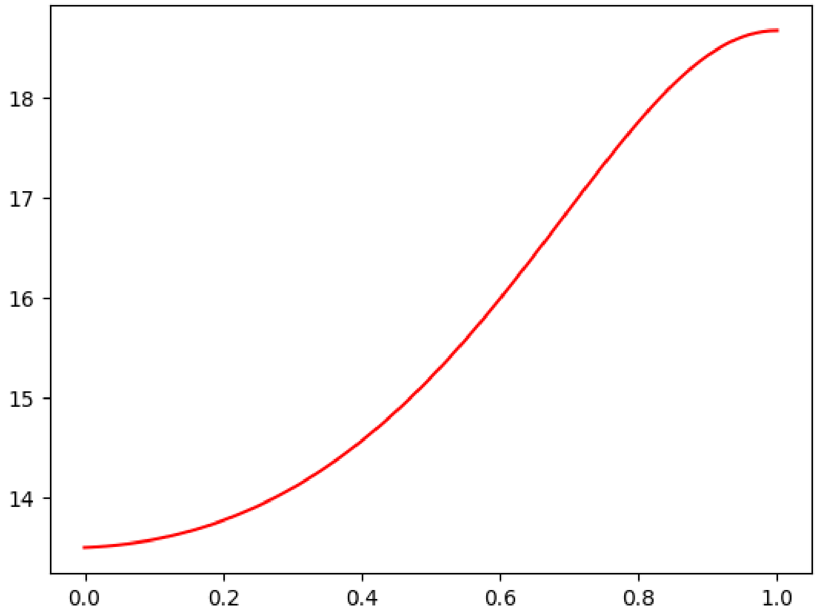
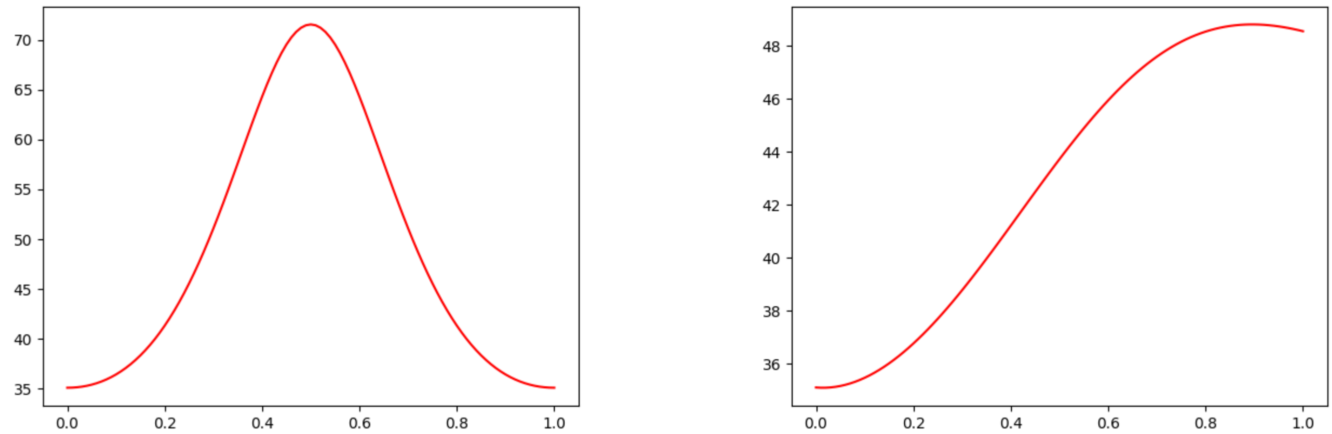


Figure 2: Energy density from $N = 3$ to $N = 3^*$ maps using linear interpolation



(a) $a = -3.07$, $b = -3.94$ (b) $a = -5$, $b = 0$

Figure 3: Energy density plots for lowest energies (a) $N = 2$, (b) $N = 3$, (c) $N = 3*$, (d) $N = 4$,(e) $N = 5$, (f) $= N = 6$. We exclude the $N = 1$ case, as the corresponding energy density is constant.

4 Reducing Energy Barriers

In this section we will attempt two methods in reducing the energy barriers between paths of rational maps.

4.1 Quadratic Interpolation

The most natural way to change the energy barrier is to change the path itself and this can be done easily quadratically interpolating the two maps. Hence, instead of Equation 3.1, we can obtain the path

$$\gamma_Q(t) = \sum_{i=0}^{N} \frac{a_i(1-t) + c_i t + t(1-t)r}{b_i(1-t) + d_i t + t(1-t)s} \tag{4.1}$$

where $r, s \in \mathbb{C}$ so that $\gamma_Q(0)$ and $\gamma_Q(t)$ are the starting and ending maps respectively. In 4 we compute the path for different values of $r = s$ and we are able to reduce the energy barrier by a substantial amount.

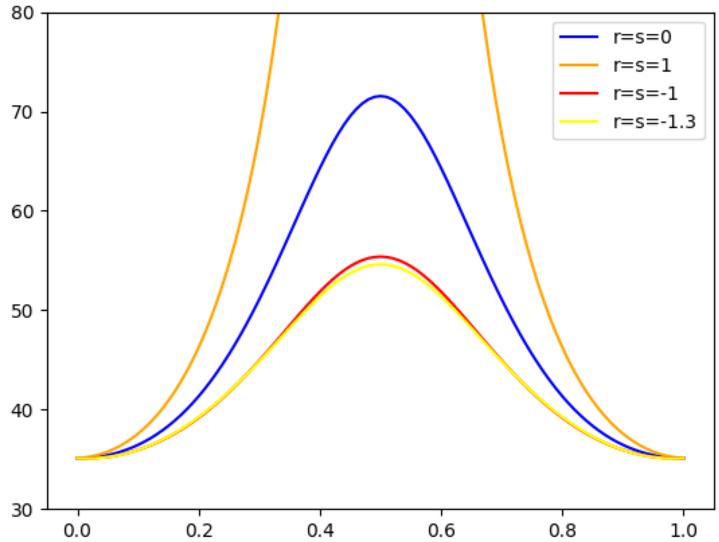


Figure 4: Energy density from $N = 3$ to $N = 3^*$ maps using linear interpolation

4.2 Möbius Transformations

Möbius transformations are another natural way of altering the energy barrier as they are effectively a $SU(2)$ rotation and should give the same $\mathcal{I}$ value. There are two types of Möbius transformations: one that acts on the domain space and one that acts on the target space. The rotation of the domain by a Möbius transformation is given by

$$\frac{p(z)}{q(z)} \mapsto \frac{p(\frac{\alpha z+\beta}{-\bar{\beta}z+\bar{\alpha}})}{q(\frac{\alpha z+\beta}{-\bar{\beta}z+\bar{\alpha}})} \tag{4.2}$$

where α, β take complex values and $|\alpha|^2 + |\beta|^2 = 1$. For the rotation of the target space, the Möbius transformation is given by

$$\frac{p(z)}{q(z)} \mapsto \frac{\gamma\left(\frac{p(z)}{q(z)}\right)+\delta}{-\bar{\gamma}\left(\frac{p(z)}{q(z)}\right)+\bar{\delta}} = \frac{\gamma p(z)+\delta q(z)}{-\bar{\delta}p(z)+\bar{\delta}q(z)} \tag{4.3}$$

where similarly γ, δ are complex and $|\gamma|^2 + |\delta|^2 = 1$. We turn our attention to 2 maps of degree 9 that different symmetry but very close minimal energies. Firstly the map with D_{4d} symmetry

$$R = \frac{z(a+ibz^4+z^8)}{1+ibz^4+az^8} \tag{4.4}$$

where $a = -3.38$ and $b = -11.19$ and the map with T_d tetrahedral symmetry

$$R = \frac{5i\sqrt{3}z^6 - 9z^4 + 3i\sqrt{3}z^2 + 1 + az^2(z^6 - i\sqrt{3}z^4 - z^2 + i\sqrt{3})}{z^3(-z^6 - 3i\sqrt{3}z^4 + 9z^2 - 5i\sqrt{3}) + az(-i\sqrt{3}z^6 + z^4 + i\sqrt{3}z^2 - 1)}, \tag{4.5}$$

where $a = -1.98$. In Figure 5 we try combination of Möbius transformations to reduce the energy barrier and we see that it is a very effective method in the case of these two maps.

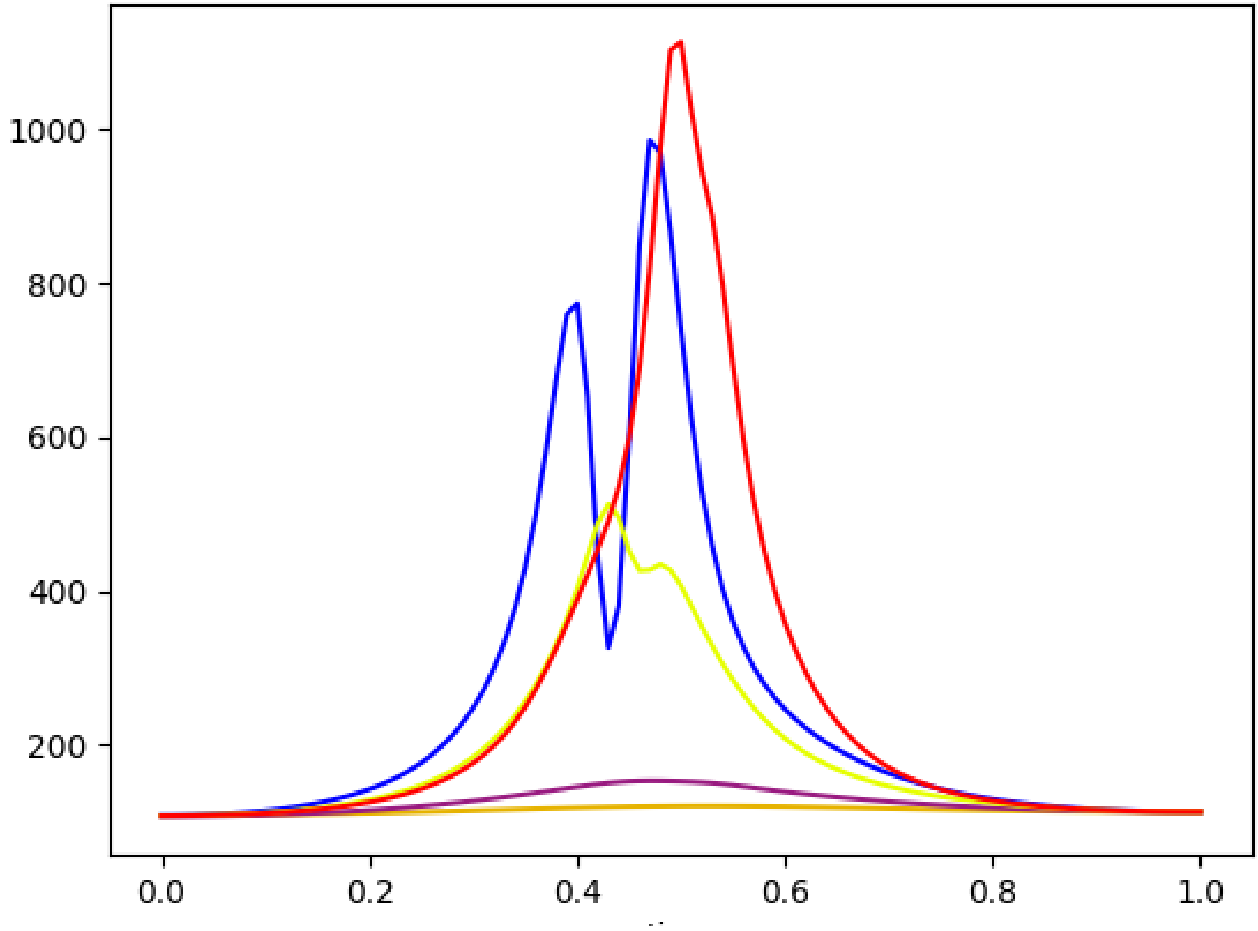


Figure 5: Plot of energy densities through linear interpolation. Different colours correspond to different Möbius transformations. The table below gives the values for such transformations and the corresponding energy barriers.

Colour	α	β	γ	δ	$\mathcal{I}_{max}$	$\mathcal{I}_{min}$	E Barrier
Blue	1	0	1	0	986.04	108.41	877.63
Orange	$-0.171 + 0.288i$	$-0.119 + 0.624i$	$-0.579 + 0.609i$	$-0.395 - 0.372i$	120.43	107.72	12.70
Purple	$0.520 - 0.190i$	$0.634 - 0.539i$	$-0.339 - 0.531i$	$-0.186 - 0.754i$	153.56	106.80	46.76
Yellow	$-0.408 - 0.106i$	$-0.582 - 0.695i$	$-0.145 - 0.989i$	$-0.02 - 0.013i$	514.02	107.98	406.04
Red	$-0.252 + 0.836i$	$0.267 - 0.408i$	$0.261 - 0.413i$	$-0.794 - 0.362i$	1,130.14	107.80	1022.34

References

[HMS98] Conor J. Houghton, Nicholas S. Manton, and Paul M. Sutcliffe. "Rational maps, monopoles and skyrmions". In: *Nuclear Physics B* 510.3 (Jan. 1998), pp. 507–537. DOI: 10.1016/s0550-3213(97)00619-6. URL: https://doi.org/10.1016%2Fs0550-3213%2897%2900619-6.