30th International Physics Olympiad

Padua, Italy

Experimental competition

Comments on the experimental problem.

As mentioned in the problem text, the pendulum may have two equilibrium positions, and the situation varies according to the position of the threaded rod, as shown in figure 5 in the text. The doubling of the potential energy minimum in figure 5 illustrates a phenomenon known in mathematics as bifurcation; it is also related to the various kinds of symmetry breaking that are studied in particle physics and statistical mechanics. It is unlikely that the students will be able to study — other than experimentally — the oscillation period near the bifurcation. Nevertheless, within this discussion, we shall here briefly outline the theoretical point of view.

In order to analyze the peculiar behaviour in the vicinity of the bifurcation, let's work out the mathematics: in general the restoring force is proportional to the angle (θ-θ_0), where θ is the angle between the pendulum and the normal to the plane of the stand frame, and θ_0 is a constant angle, therefore the pendulum's equation of motion is

$$I(x)\frac{d^2\theta}{dt^2} = -\kappa(\theta-\theta_0) \tag{1}$$

if the rotation axis is vertical, while it is

$$I(x)\frac{d^2\theta}{dt^2} = -\kappa(\theta-\theta_0)+(M_1+M_2)gR(x)\sin\theta \tag{2}$$

if the rotation axis is horizontal. One can define a potential energy which is a function of the angle θ, and the corresponding formulas for this potential energy are

$$U(\theta;x) = \frac{1}{2}\kappa(\theta-\theta_0)^2 \tag{3}$$

for a vertical rotation axis and

$$U(\theta;x) = \frac{1}{2}\kappa(\theta-\theta_0)^2 + (M_1+M_2)gR(x)\cos\theta \tag{4}$$

for a horizontal axis. A graph of eq. (4) is shown in figure 5 in the problem text.

As x increases, the term corresponding to the cosine in the potential energy function (4) becomes more important; at first we have a single energy minimum; then the minimum is displaced and further it separates into two different minima; in general one of them is deeper than the other one. In the most general case, it's possible to have several minima (more than two) but in practice this can't be obtained with the mechanical model in this experiment.

After the addition of mass M_3 whose center of mass is at a distance x_3 from the axis, equation (4) becomes:

$$U(\theta;x) = \frac{1}{2}\kappa(\theta-\theta_0)^2 + \left[(M_1+M_2)R(x) + M_3x_3\right]g\cos\theta \tag{5}$$

From now on, for sake of brevity, we shall write $\alpha(x) = g\left[(M_1+M_2)R(x)+M_3x_3\right]$.

For a quantitative understanding of the bifurcation due to this potential energy, let's consider a simplified equation where we replace the cosine by its series expansion up to the fourth order in θ:

$$U(\theta,x) = \frac{1}{2}\kappa\theta^2 + \alpha(x)\left(1-\frac{\theta^2}{2}+\frac{\theta^4}{24}\right) \tag{6}$$

where we have put θ_0 =0 (see figure D1 for a comparison of the two functions).

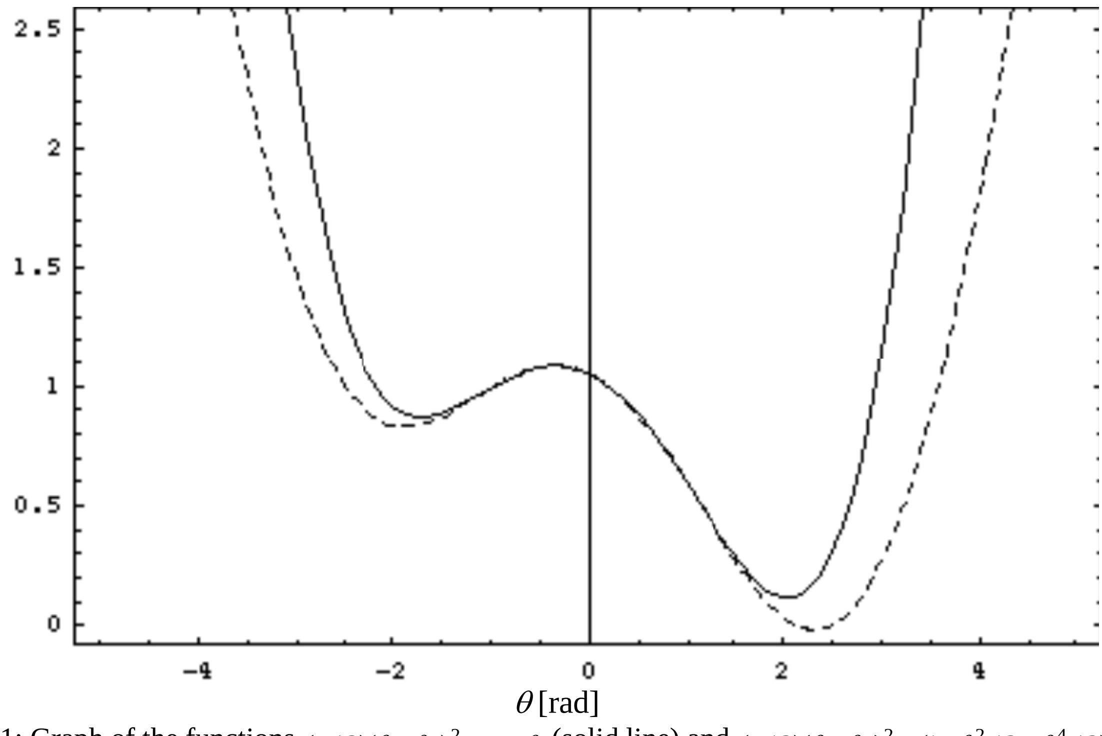


Figure D1: Graph of the functions $(a/2)(\theta-\theta_0)^2+\cos\theta$ (solid line) and $(a/2)(\theta-\theta_0)^2+(1-\theta^2/2+\theta^4/2)$ (dashed line), with a=0.4, θ_0=0.5. The two functions are slightly different, but the minima structure is the same.

Then the first and second derivatives are:

$$\frac{dU}{d\theta} = \kappa\theta + \alpha(x)\left(-\theta + \frac{\theta^3}{6}\right)$$
$$\frac{d^2U}{d\theta^2} = \kappa + \alpha(x)\left(-1 + \frac{\theta^2}{2}\right) \tag{7}$$

We see that when $\alpha(x)$ increases, the second derivative at the origin passes from positive to negative values (that means that we pass at the origin from a situation of a stable minimum to an unstable maximum). The equilibrium position can be found as usual by equating the first derivative (with respect to θ) to zero:

$$\theta\left[\kappa + \alpha(x)\left(-1 + \frac{\theta^2}{6}\right)\right] = 0 \tag{8}$$

and this equation has a solution at the origin (but we already know that at the onset of the bifurcation the origin becomes an unstable maximum) and two other solutions

$$\theta_\pm = \pm\sqrt{6\left(1 - \frac{\kappa}{\alpha(x)}\right)} \tag{9}$$

(these angles are imaginary before the bifurcation so that they do not represent physical solutions). Let's now go back to the equation of motion:

$$I_3(x)\frac{d^2\theta}{dt^2} = -\frac{dU}{d\theta} = -\kappa\theta - \alpha(x)\left(-\theta + \frac{\theta^3}{6}\right) \tag{10}$$

where $I_3(x)$ is the total moment of inertia after including the additional mass; without bifurcation we neglect the cubic term and we find that near the origin

$$\frac{d^2\theta}{dt^2} = -\frac{(\kappa - \alpha(x))}{I_3(x)}\theta \tag{11}$$

and therefore the angular frequency of the small oscillations in the (single) stable potential energy minimum without bifurcation is given by

$$\omega^2 = \frac{(\kappa - \alpha(x))}{I_3(x)} \tag{12}$$

and it equals zero at the bifurcation itself, whereas after the onset of the bifurcation the equation of motion becomes

$$\frac{d^2\theta}{dt^2} = -\frac{2(\alpha(x) - \kappa)}{I_3(x)}(\theta - \theta_\pm) \tag{13}$$

and therefore the angular frequency of the small oscillations in each stable minimum is given by

$$\omega^2 = \frac{2(\alpha(x) - \kappa)}{I_3(x)} \tag{14}$$

and it also equals zero at the bifurcation value of x. Since the period is given by $T = 2\pi/\omega$, we can compute it with equations (12) and (14).

If the angle θ_0 is not zero, the computations are considerably more complicated, and can only be performed numerically (some results are show in figures D2 and D3).

The oscillation period has a local maximum near the onset of the bifurcation: the shape of this maximum does not change very much for different misalignment angles θ_0, but the peak value is lower for greater misalignments.

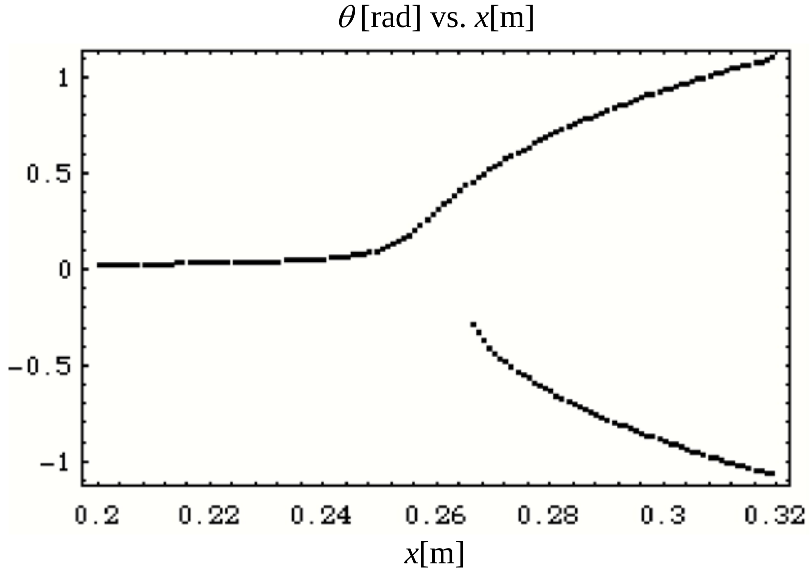


Figure D2: This figure shows the result of a numerical calculation of the stable minima of the pendulum performed using the data measured in a test run and a misalignement angle θ_0 = 0.0035 rad. The onset of the bifurcation is at $x \approx 0.266$ m.

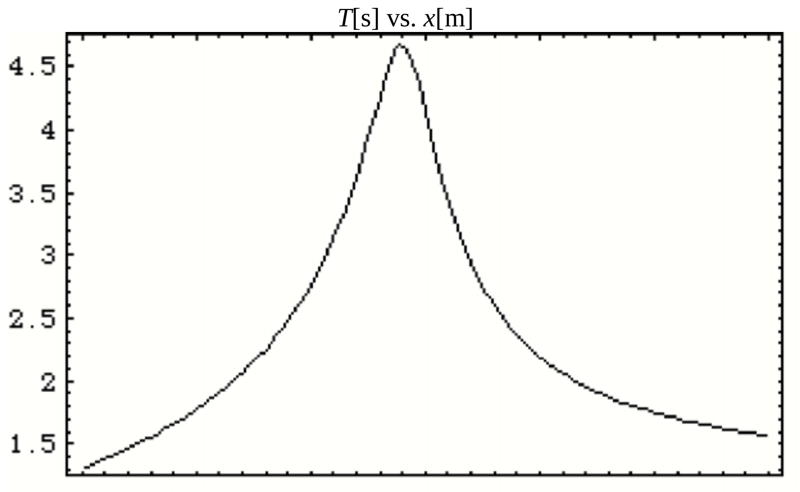


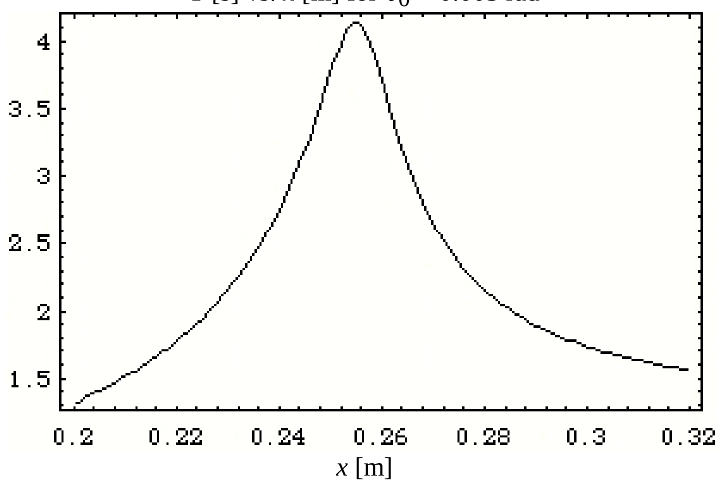
Figure D3: This is a plot of the period T computed with the same data as figure D2. Notice that the period has a local maximum while there is still just one stable position.

The shape of the oscillation time peak is influenced by many parameters, but it is especially sensitive to the angle θ_0. Here are some example plots, calculated with the same data as figures D2 and D3, namely:

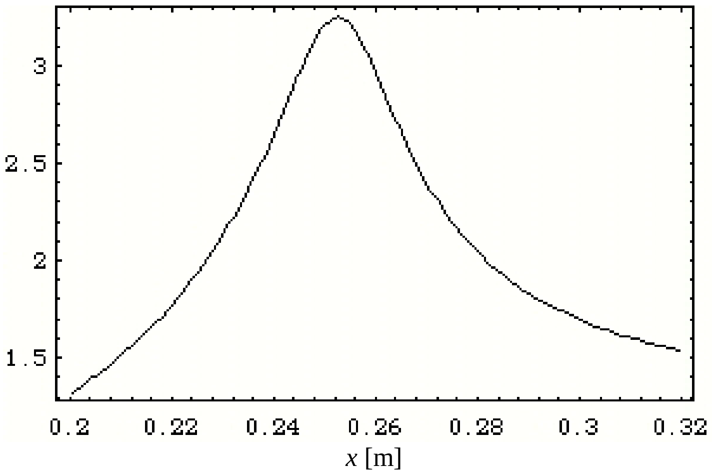
$g = 9.81$ m/s^2;
$\kappa = 0.056$ J;
$M_1 = 0.0261$ kg;
$M_2 = 0.0150$ kg;
$M_3 = 0.00664$ kg; mass of the final long nut
$I_1 = 1.\cdot 10^{-4}$ kg·m^2;
$\ell = 0.21$ m;
$\ell_3 = 0.025$ m; length of the final long nut
$a = 0.365$;
$b = 0.0022$ m;
$R(x) = a\cdot x+b$;
$I_3(x) = I_1+M_2\cdot (x^2-\ell\cdot x+\ell^2/3)+(M_3/(3\cdot\ell_3))\cdot((x+\ell_3/2)^3-(x-\ell_3/2)^3)$;

Since it is explicitly requested that the pendulum be as vertical as possible near equilibrium, the shape of the final plot may be used to estimate the experimental prowess of each participant.

T [s] vs. x [m] for $\theta_0 = 0.005$ rad



T [s] vs. x [m] for θ_0 = 0.01 rad



T [s] vs. x [m] for θ_0 = 0.03 rad