# CONVERGENCE ERROR ANALYSIS

## 1. Exponential

Let's say that $\frac{P_n}{Q_n} \to T$ as $n \to \infty$, and that it behaves as:

$$\frac{P_n}{Q_n} = \Omega\left(T - C \cdot b^{-n}\right) \ , \ b > 1 \ , \ C \in \mathbb{R}$$

from our heuristic experience, either an equality holds starting from some $n$, or it holds for pair and odd $n$'s separately. So in any case we have:

$$\begin{aligned}
\frac{P_{2n}}{Q_{2n}} &= T - C_p \cdot b_p^{-n} \\
\frac{P_{2n+1}}{Q_{2n+1}} &= T - C_o \cdot b_o^{-n} \\
b_p, b_o &> 1
\end{aligned}$$

let's define:

$$\begin{aligned}
\Pi_n &= \frac{P_n}{Q_n} \\
\Delta_n &= \Pi_{n+2} - \Pi_n
\end{aligned}$$

and evaluate the latter:

$$\begin{aligned}
\Delta_n &= T - C_\alpha b_\alpha^{-n-2} - T + C_\alpha b_\alpha^{-n} \\
&= C_\alpha b_\alpha^{-n}\left(1 - b_\alpha^{-2}\right) \\
&\Downarrow
\end{aligned}$$

$$\log|\Delta_n| = \log\left(|C_\alpha|\left(1 - b_\alpha^{-2}\right)\right) - n \log b_\alpha \tag{1.1}$$

$$\frac{\Delta_n}{\Delta_{n+2}} = \frac{b_\alpha^{-n}}{b_\alpha^{-n-2}} = b_\alpha^2 > 1 \tag{1.2}$$

we chose a delta of 2 to keep the odd and pair separated, in case they're divided to different characteristics. We can now characterize the behaviour of the system with the linear fitting for $\log \Delta_n$ and with the ratio $\frac{\Delta_{n+2}}{\Delta_n}$. From eq. (1.1) we can see that $\log b_\alpha$ is the slope of the linear fitting of $\log \Delta_n$ vs. $n$. From eq. (1.2) we see that the ratio approaches (or equals) to $b_\alpha^2 > 1$.

## 2. Polynomial

Now let's assume that $\Pi_n$ behaves as:

$$\begin{aligned}
\Pi_{2n} &= T + C_p n^{-\kappa_p} \\
\Pi_{2n+1} &= T + C_o n^{-\kappa_o} \\
\kappa_p, \kappa_o &> 0
\end{aligned}$$

as before, we define and evaluate:

$$\begin{aligned}
\Delta_n &= \Pi_{n+2} - \Pi_n \\
&= C_\alpha (n+2)^{-\kappa_\alpha} - C_\alpha n^{-\kappa_\alpha} \\
&= C_\alpha n^{-\kappa_\alpha} \left( \left(1 + \frac{2}{n}\right)^{-\kappa_\alpha} - 1 \right) \\
&= C_\alpha n^{-\kappa_\alpha} \left( 1 - \frac{2}{n}\kappa_\alpha + o\left(\frac{1}{n^2}\right) - 1 \right) \\
&= -C_\alpha n^{-\kappa_\alpha} \left( \frac{2}{n}\kappa_\alpha + o\left(\frac{1}{n^2}\right) \right) \\
\Delta_{n+2} &= C_\alpha (n+4)^{-\kappa_\alpha} - C_\alpha (n+2)^{-\kappa_\alpha} \\
&= C_\alpha n^{-\kappa_\alpha} \left( \left(1 + \frac{4}{n}\right)^{-\kappa_\alpha} - \left(1 + \frac{2}{n}\right)^{-\kappa_\alpha} \right) \\
&= C_\alpha n^{-\kappa_\alpha} \left( 1 - \frac{4}{n}\kappa_\alpha + o\left(\frac{1}{n^2}\right) - 1 + \frac{2}{n}\kappa_\alpha + o\left(\frac{1}{n^2}\right) \right) \\
&= -C_\alpha n^{-\kappa_\alpha} \left( \frac{2}{n}\kappa_\alpha + o\left(\frac{1}{n^2}\right) \right) \\
\frac{\Delta_n}{\Delta_{n+2}} &= \frac{-C_\alpha n^{-\kappa_\alpha} \left( \frac{2}{n}\kappa_\alpha + o\left(\frac{1}{n^2}\right) \right)}{-C_\alpha n^{-\kappa_\alpha} \left( \frac{2}{n}\kappa_\alpha + o\left(\frac{1}{n^2}\right) \right)} \\
&= \frac{\kappa_\alpha + \frac{n}{2} \cdot o\left(\frac{1}{n^2}\right)}{\kappa_\alpha + \frac{n}{2} \cdot o\left(\frac{1}{n^2}\right)} \\
&\Downarrow \\
\lim_{n\to\infty} \frac{\Delta_n}{\Delta_{n+2}} &= \frac{\kappa_\alpha}{\kappa_\alpha} = 1 \qquad (2.1)
\end{aligned}$$

in addition:

$$\begin{aligned}
\log|\Delta_n| &\approx \log|C_a| - \kappa_\alpha \log n + \log\frac{2\kappa_\alpha}{n} \\
&= \log|C_a| - \kappa_\alpha \log n + \log\frac{2\kappa_\alpha}{n} \\
\log|\Delta_n| &\underset{n\gg 1}{\approx} \log\left(|C_\alpha|\, 2\kappa_\alpha\right) - (\kappa_\alpha + 1)\log n \qquad (2.2)
\end{aligned}$$

As before, we can characterize the behaviour of the system using the ratio and the logarithm. From eq. (2.1) we see that for a polynomial approach the ratio approaches to 1, in contrary to the exponential approach in which the ratio approaches to $b_\alpha^2 > 1$. From eq. (2.2) we see that for $n$ big enough, the slope of $\log|\Delta_n|$ vs. $\log n$ is $-(\kappa_\alpha + 1)$.

## 3. Conclusions

We can differentiate between polynomial and exponential approaches to the convergence value by examining and verify: $\lim_{n\to\infty} \frac{\Delta_n}{\Delta_{n+2}} > 1$.

Moreover, we can find the relevant parameters $b_\alpha$ and $\kappa_\alpha$ using either the ratio or the log characterization for the exponential approach, and the log characterization for the polynomial approach.

## 4. Appendix I - Methods Consistency

One might wonder whether the results for $b_\alpha$ from eq. (1.1) and eq. (1.2) agree one with each other. For the continuous fraction:

$$\frac{4}{\pi} = 1 + \frac{1^2}{3 + \frac{2^2}{5 + \ldots}}$$

using an equation similar to eq. (1.1), where the difference is from the target value $T$ (which is known) rather than between neighbour $\Delta_n$, we receive the result:

$$b = 1.7624\ldots$$

while from eq. (1.2) we receive the result:

$$b = 1.7627\ldots$$

as we see, the results agree.

For the polynomial approach too, when comapring eq. (2.2) to the slope of

$$\log\left(\Pi_n - T\right) = \log C_\alpha n^{\kappa_\alpha} = \log C_\alpha + \kappa_\alpha \log n \tag{4.1}$$

vs. $\log n$ for the continuous fraction:

$$\frac{4}{x} + 1 = 2 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \ldots}}}$$

we receive the following results:

eq. (2.2):

$$\begin{aligned} \kappa &= 2.9970 \\ R^2 &= 1 \pm 2 \times 10^{-8} \end{aligned}$$

eq. (4.1):

$$\begin{aligned} \kappa &= 2.9971 \\ R^2 &= 1 \pm 1 \times 10^{-8} \end{aligned}$$

## 5. Appendix II - Characterization over integers

In order to differentiate quickly between different types of convergences fastly over integers, to make it relevant for massive integers calculations, we develop the following method. Notice that instead of storing $\Pi_n$, we're storing separately $P_n$ and $Q_n$.

$$\begin{aligned} \Delta_n &= \Pi_{n+2} - \Pi_n \\ &= \frac{P_{n+2}}{Q_{n+2}} - \frac{P_n}{Q_n} \\ &= \frac{P_{n+2}Q_n - P_nQ_{n+2}}{Q_nQ_{n+2}} \\ &= \frac{P_{n+2}Q_n - P_nQ_{n+2}}{Q_nQ_{n+2}} \cdot \frac{Q_{n+2}Q_{n+4}}{P_{n+4}Q_{n+2} - P_{n+2}Q_{n+4}} \\ &= \frac{Q_{n+4}\left(P_{n+2}Q_n - P_nQ_{n+2}\right)}{Q_n\left(P_{n+4}Q_{n+2} - P_{n+2}Q_{n+4}\right)} \\ &\Downarrow \\ Q_{n+4}\left(P_{n+2}Q_n - P_nQ_{n+2}\right) &= \frac{\Delta_n}{\Delta_{n+2}} Q_n\left(P_{n+4}Q_{n+2} - P_{n+2}Q_{n+4}\right) \end{aligned}$$

We now remind that if $\frac{\Delta_n}{\Delta_{n+2}} \rightarrow \alpha$ and $\alpha > 1$ then the convergence is exponential. We'll mention, without a proof, that if $\frac{\Delta_n}{\Delta_{n+2}} \rightarrow \infty$ then the convergence is over-exponential. Thus, if for several sequential $n$'s we obatin $Q_{n+4}\left(P_{n+2}Q_n - P_nQ_{n+2}\right) > Q_n\left(P_{n+4}Q_{n+2} - P_{n+2}Q_{n+4}\right)$ then the convergence is at least exponential. To distinguish an exponential approach from a polynomial approach for which $\frac{\Delta_n}{\Delta_{n+2}} \rightarrow 1$ from above, and use only fast integer calculations, we can choose a threshold $\alpha = 1 + 2^{-\eta}$. Now we check whether the following inequality holds:

$$Q_{n+4}\left(P_{n+2}Q_n - P_nQ_{n+2}\right) \quad -Q_n\left(P_{n+4}Q_{n+2} - P_{n+2}Q_{n+4}\right) > \quad Q_n\left(P_{n+4}Q_{n+2} - P_{n+2}Q_{n+4}\right) \times 2^{-\eta}$$

where $Q_n\left(P_{n+4}Q_{n+2} - P_{n+2}Q_{n+4}\right) \times 2^{-\eta}$ can be calculated efficiently using $\eta$ right shifts (e.g. $c >> \eta$).