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Recursive algorithms, branching coefficients for affine algebras and applications

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Abstract

Recurrent relations for branching coefficients in affine Lie algebras highest weight modules are studied. The decomposition algorithm for integrable highest weight modules reduction based on the injection fan technique is adopted to the situation where the Weyl denominator becomes singular with respect to a reductive subalgebra. We study some modifications of the injection fan technique and demonstrate that it is possible to define the "subtracted fans" that play the role similar to the original one. The possible applications of subtracted fans in CFT models are considered.

1 Introduction

The problem of reduction of a Lie algebra representation to a subalgebra is studied for several decades and has various applications in physics. In

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the context of finite-dimensional algebras it is important for the study of great unification models whilst the branching problem for affine Lie algebras emerges in conformal field theory, for example, in the construction of the modular-invariant partition functions [1].

There exist several approaches to the computation of the branching coefficients. Some of them use the BGG resolution [2] (for Kac-Moody algebras the algorithm is described in [3],[4]), the Schure function series [5], the BRST cohomology [6], Kac-Peterson formulas [3, 7] or the combinatorial methods applied in [8].

Usually only the embedding of the maximal reductive subalgebras is considered since the case of non-maximal subalgebra can be obtained using the chain of maximal injections. In this paper we study the recurrent properties for branching coefficients which generalise the recurrent relations obtained earlier (see the paper [9]and the references therein) to the cases of non-maximal reductive subalgebras. The result is formulated by an introduction of a new injection fan called "subtracted fan". Using this new tools we formulate simple and explicit algorithm for the computation of branching coefficients which is applicable to the non-maximal subalgebras of finite-dimensional and affine Lie algebras.

We show that our algorithm can be used in a study of conformal embeddings where the central charge of the conformal field theory is preserved, and the computations are simplified by taking into account some physical limitations.

The paper is organised as follows. In the next subsection we fix the notations used throughout the paper. In the Section 2 we derive the subtracted recurrent formula for anomalous branching coefficients and describe the decomposition algorithm for integrable highest weight modules of algebra $\mathfrak{g}$ with respect to a reductive subalgebra $\mathfrak{a}$ (subsection 2.2). In the next Section 3 we present several examples and discuss some applications in CFT models (s. 4). We conclude the paper with a review of obtained results. Possible future developments are discussed (s. 5).

1.1 Notation

Consider affine Lie algebras $\mathfrak{g}$ and $\mathfrak{a}$ with the underlying finite-dimensional subalgebras $\overset{\circ}{\mathfrak{g}}$ and $\overset{\circ}{\mathfrak{a}}$ and an injection $\mathfrak{a} \longrightarrow \mathfrak{g}$ such that $\mathfrak{a}$ is a reductive subalgebra $\mathfrak{a} \subset \mathfrak{g}$ with correlated root spaces: $\mathfrak{h}^*_{\mathfrak{a}} \subset \mathfrak{h}^*_{\mathfrak{g}}$ and $\mathfrak{h}^*_{\overset{\circ}{\mathfrak{a}}} \subset \mathfrak{h}^*_{\overset{\circ}{\mathfrak{g}}}$.

We use the following notations adopted from the paper [9].

L^{μ} $(L^{\nu}_{\mathfrak{a}})$ — the integrable module of $\mathfrak{g}$ with the highest weight μ ; (resp. integrable $\mathfrak{a}$ -module with the highest weight ν);

r , $(r_{\mathfrak{a}})$ — the rank of the algebra $\mathfrak{g}$ (resp. $\mathfrak{a}$) ;

Δ $(\Delta_{\mathfrak{a}})$— the root system; Δ^{+} (resp. $\Delta_{\mathfrak{a}}^{+}$)— the positive root system (of $\mathfrak{g}$ and $\mathfrak{a}$ respectively);

$\mathrm{mult}\,(\alpha)$ $(\mathrm{mult}_{\mathfrak{a}}\,(\alpha))$ — the multiplicity of the root α in Δ (resp. in $(\Delta_{\mathfrak{a}})$);

$\overset{\circ}{\Delta}$, $\left(\overset{\circ}{\Delta}_{\mathfrak{a}}\right)$ — the finite root system of the subalgebra $\overset{\circ}{\mathfrak{g}}$ (resp. $\overset{\circ}{\mathfrak{a}}$); Θ, $(\Theta_{\mathfrak{a}})$ — the highest root of the algebra $\mathfrak{g}$ (resp. subalgebra $\mathfrak{a}$);

$\mathcal{N}^{\mu}$, $(\mathcal{N}_{\mathfrak{a}}^{\nu})$ — the weight diagram of L^{μ} (resp. $L_{\mathfrak{a}}^{\nu}$) ;

W , $(W_{\mathfrak{a}})$— the corresponding Weyl group;

C , $(C_{\mathfrak{a}})$— the fundamental Weyl chamber;

$\bar{C}$, $\left(\bar{C}_{\mathfrak{a}}\right)$ — the closure of the fundamental Weyl chamber;

ρ , $(\rho_{\mathfrak{a}})$ — the Weyl vector;

$\epsilon\,(w) := \det\,(w)$;

α_i , $\left(\alpha_{(\mathfrak{a})j}\right)$ — the i-th (resp. j-th) basic root for $\mathfrak{g}$ (resp. $\mathfrak{a}$); $i = 0, \ldots, r$, $(j = 0, \ldots, r_{\mathfrak{a}})$;

δ — the imaginary root of $\mathfrak{g}$ (and of $\mathfrak{a}$ if any);

$\alpha_i^{\vee}$, $\left(\alpha_{(\mathfrak{a})j}^{\vee}\right)$— the basic coroot for $\mathfrak{g}$ (resp. $\mathfrak{a}$) , $i = 0, \ldots, r$; $(j = 0, \ldots, r_{\mathfrak{a}})$;

$\overset{\circ}{\xi}$, $\overset{\circ}{\xi}_{(\mathfrak{a})}$ — the finite (classical) part of the weight $\xi \in P$, $\left(\text{resp. } \xi_{(\mathfrak{a})} \in P_{\mathfrak{a}}\right)$;

$\lambda = \left(\overset{\circ}{\lambda}; k; n\right)$ — the decomposition of an affine weight indicating the finite part $\overset{\circ}{\lambda}$, level k and grade n .

P (resp. $P_{\mathfrak{a}}$) — the weight lattice;

M (resp. $M_{\mathfrak{a}}$) :=

$= \left\{ \begin{array}{c} \sum_{i=1}^{r} \mathbf{Z}\alpha_i^{\vee} \; \left(\text{resp. } \sum_{i=1}^{r} \mathbf{Z}\alpha_{(\mathfrak{a})i}^{\vee}\right) \text{ for untwisted algebras or } A_{2r}^{(2)}, \\ \sum_{i=1}^{r} \mathbf{Z}\alpha_i \; \left(\text{resp. } \sum_{i=1}^{r} \mathbf{Z}\alpha_{(\mathfrak{a})i}\right) \text{ for } A_r^{(u\geq 2)} \text{ and } A \neq A_{2r}^{(2)}, \end{array} \right\}$; $\Psi^{(\mu)} := \sum_{w\in W} \epsilon(w) e^{w\circ(\mu+\rho)-\rho}$ — the singular weight element for the $\mathfrak{g}$-module L^{μ}; $\Psi_{(\mathfrak{a})}^{(\nu)} := \sum_{w\in W_{\mathfrak{a}}} \epsilon(w) e^{w\circ(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}$ — the corresponding singular weight element for the $\mathfrak{a}$-module $L_{\mathfrak{a}}^{\nu}$;

$\widehat{\Psi^{(\mu)}}$ $\left(\widehat{\Psi_{(\mathfrak{a})}^{(\nu)}}\right)$ — the set of singular weights $\xi \in P$ (resp. $\in P_{\mathfrak{a}}$) for the module L^{μ} (resp. $L_{\mathfrak{a}}^{\nu}$) with the coordinates $\left(\overset{\circ}{\xi}, k, n, \epsilon\,(w\,(\xi))\right) |_{\xi = w(\xi)\circ(\mu+\rho)-\rho}$, (resp. $\left(\overset{\circ}{\xi}, k, n, \epsilon\,(w_a\,(\xi))\right) |_{\xi = w_a(\xi)\circ(\nu+\rho_a)-\rho_a}$), (this set is similar to $P'_{\mathrm{nice}}\,(\mu)$ in [4])

$m_{\xi}^{(\mu)}$, $\left(m_{\xi}^{(\nu)}\right)$ — the multiplicity of the weight $\xi \in P$ (resp. $\in P_{\mathfrak{a}}$) in the module L^{μ} , (resp. $\xi \in L_{\mathfrak{a}}^{\nu}$);

$ch\,(L^{\mu})$ (resp. $ch\,(L^{\nu}_{\mathfrak{a}})$)— the formal character of L^{μ} (resp. $L^{\nu}_{\mathfrak{a}}$);

$ch\,(L^{\mu}) = \frac{\sum_{w\in W}\epsilon(w)e^{w\circ(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}} = \frac{\Psi^{(\mu)}}{\Psi^{(0)}}$ — the Weyl-Kac formula.

$R := \prod_{\alpha\in\Delta^+}\left(1-e^{-\alpha}\right)^{\mathrm{mult}(\alpha)} = \Psi^{(0)}$

$\Big($resp. $R_{\mathfrak{a}} := \prod_{\alpha\in\Delta^+_{\mathfrak{a}}}\left(1-e^{-\alpha}\right)^{\mathrm{mult}_{\mathfrak{a}}(\alpha)} = \Psi^{(0)}_{\mathfrak{a}}\Big)$— the denominator.

$L^{\mu}_{\mathfrak{g}\downarrow\mathfrak{a}} = \bigoplus\limits_{\nu\in P^+_{\mathfrak{a}}} b^{(\mu)}_{\nu} L^{\nu}_{\mathfrak{a}}$ — the reduction of the representation;

$b^{(\mu)}_{\nu}$ — the branching coefficients;

$$\sum_{\nu\in\bar{C}_{\mathfrak{a}}} b^{(\mu)}_{\nu}\Psi^{(\nu)}_{(\mathfrak{a})} = \sum_{\lambda\in P_{\mathfrak{a}}} k^{(\mu)}_{\lambda} e^{\lambda} \tag{1}$$

k_{λ} — the anomalous branching coefficients;
It is important to mention that

$$b^{(\mu)}_{\nu} = k^{(\mu)}_{\nu} \text{ for } \nu\in\bar{C}_{\mathfrak{a}} \tag{2}$$

$x_e = \frac{|\pi_{\mathfrak{a}}\Theta|}{|\Theta_{\mathfrak{a}}|}$ — the embedding index.

2 Recurrent relation for branching coefficients. Singularities and subtractions

Our aim is to demonstrate that despite possible singularities arriving in the Weyl denominator when it is projected to the subalgebra root space injection fan technique can be properly modified. The result of such modification is that the generalized recurrent relations for anomalous branching coefficients (1) must be reformulated in the following form:

$$k^{(\mu)}_{\xi} = -\frac{1}{s(\gamma_0)}\Big(\sum\nolimits_{\omega\in W_{\perp}\backslash W}\epsilon(\omega)\dim\Big(L^{\pi_{\mathfrak{a}_{\perp}}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_{\perp}}}_{\mathfrak{a}_{\perp}}\Big)\delta_{\xi-\gamma_0,\pi_{\mathfrak{a}}(\omega(\mu+\rho)-\rho)} + \\ + \sum\nolimits_{\gamma\in\Gamma_{\mathfrak{a}\subset\mathfrak{g}}} s\left(\gamma+\gamma_0\right)k^{(\mu)}_{\xi+\gamma}\Big) \tag{3}$$

Here $\mathfrak{a}_{\perp}$ is the subalgebra described by the roots of $\mathfrak{g}$ orthogonal to the root subsystem of $\mathfrak{a}$, $W_{\perp}$ is the corresponding Weyl group, $\Gamma_{\mathfrak{a}\subset\mathfrak{g}}$ is the set of weights in the expansion of the denominator $\prod_{\alpha\in\Delta^+\backslash\Delta^+_{\mathfrak{a}_{\perp}}}\left(1-e^{-\alpha}\right)^{\mathrm{mult}(\alpha)-\mathrm{mult}_{\mathfrak{a}}(\alpha)}$ and $s(\gamma)$ is the coefficient of e^{γ} in this expansion. In the next subsection we study the situation in details and prove the validity of this relation.

In the section 2.2 we shall describe the computational algorithm for branching coefficients based on this formula and present some examples.

2.1 Proof of the recurrent relation

Consider the branching of a module $L^{\mu}_{\mathfrak{g}}$ in terms of formal characters and projection operators $\pi_{\mathfrak{a}}$ that bring the roots of $\mathfrak{g}$ to the weight subspace of $\mathfrak{a}$:

$$L^{\mu}_{\mathfrak{g}\downarrow\mathfrak{a}} = \bigoplus_{\nu\in P^{+}_{\mathfrak{a}}} b^{(\mu)}_{\nu} L^{\nu}_{\mathfrak{a}} \quad \Longrightarrow \quad \pi_{\mathfrak{a}}(chL^{\mu}_{\mathfrak{g}}) = \sum_{\nu\in P^{+}_{\mathfrak{a}}} b^{(\mu)}_{\nu} chL^{\nu}_{\mathfrak{a}} \tag{4}$$

The Weyl-Kac character formula leads to the equality

$$\pi_{\mathfrak{a}}\left(\frac{\sum_{\omega\in W}\epsilon(\omega)e^{\omega(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^{+}}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}\right) = \sum_{\nu\in P^{+}_{\mathfrak{a}}} b^{(\mu)}_{\nu}\frac{\sum_{\omega\in W_{\mathfrak{a}}}\epsilon(\omega)e^{\omega(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}}{\prod_{\beta\in\Delta^{+}_{\mathfrak{a}}}(1-e^{-\beta})^{\mathrm{mult}_{\mathfrak{a}}(\beta)}} \tag{5}$$

It is important to keep in mind that the projection of some of positive roots of the algebra $\mathfrak{g}$ can be equal to zero. These roots are orthogonal to the root space of the subalgebra $\mathfrak{a}$ embedded into the root space of the algebra $\mathfrak{g}$. Let's denote the subset of such roots by $\Delta^{+}_{\perp} = \left\{\alpha \in \Delta^{+}_{\mathfrak{g}} : \forall\beta \in \Delta^{+}_{\mathfrak{a}},\ \alpha\perp\beta\right\}$.

Notice that if the set $\Delta^{+}_{\perp}$ is non-empty the Weyl reflections corresponding to the positive roots of $\Delta^{+}_{\perp}$ generate a subgroup $W_{\perp}$ of the Weyl group W. Consider any two positive roots $\alpha,\ \beta \in \Delta^{+}_{\perp}$ and the corresponding Weyl reflections $\omega_{\alpha},\ \omega_{\beta} \in W_{\perp}$. Since roots of the subalgebra $\mathfrak{a}$ are invariant under $\omega_{\alpha},\ \omega_{\beta}$ they are also invariant under the action of $\omega_{\gamma} = \omega_{\alpha}\cdot\omega_{\beta}$. So the subgroup $W_{\perp}$ preserves the root system of the subalgebra $\mathfrak{a}$.

Thus we have obtained the root system $\Delta_{\perp}$ which is orthogonal to the root system $\Delta_{\mathfrak{a}}$ and invariant with respect to $W_{\perp}$. This root system can be considered as the root system of a subalgebra $\mathfrak{a}_{\perp} \subset \mathfrak{g}$.

Now we are to find out when the subset $\Delta^{+}_{\perp}$ is non-empty and the subgroup $W_{\perp}$ and subalgebra $\mathfrak{a}_{\perp}$ are non-trivial.

If $\mathfrak{a}$ is a maximal regular subalgebra of $\mathfrak{g}$ then the rank of $\mathfrak{a}$ is equal to the rank of $\mathfrak{g}$ and it is clear that $\Delta^{+}_{\perp}$ is empty. On the other hand non-maximal regular embedding of $\mathfrak{a}$ into $\mathfrak{g}$ can be obtained through the chain of maximal embeddings $\mathfrak{a} \subset \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \cdots \subset \mathfrak{g}$. The maximal regular embeddings are constructed by the exclusion of one or two roots from the extended Dynkin diagram of the algebra. Since this process can give us non-connected Dynkin diagrams we can see which roots are orthogonal to the root space of non-maximal regular subalgebra $\mathfrak{a}$.

Consider for instance regular the embedding of $A_1 \subset B_2$.

The extended Dynkin diagram of B_2 is presented in the Figure 1. Drop the central node to describe the embedding $A_1 \oplus A_1 \subset B_2$. In this case we have: $\mathfrak{a} = A_1$ and $\mathfrak{a}_{\perp} = A_1$.

The simple criterion of $\Delta^{+}_{\perp}$'s non-emptiness for a regular embedding $\mathfrak{a} \subset \mathfrak{g}$ when both $\mathfrak{a}$ and $\mathfrak{g}$ are simple can be formulated as follows: if the Dynkin diagram of $\mathfrak{g}$ can be split into the disconnected diagrams of $\mathfrak{a}$ and of some subalgebras $\{\bar{\mathfrak{a}}_j\}$ then the subset $\Delta_\perp$ is non-empty, subalgebra $\mathfrak{a}_\perp$ is non-trivial and all the $\bar{\mathfrak{a}}_j$ are the subalgebras of $\mathfrak{a}_\perp$.

Note that when we study the regular embedding obtained by dropping the nodes of the extended Dynkin diagram of the algebra $\mathfrak{g}$ and the subalgebra $\mathfrak{a}$ is one of the connected components, the subalgebra $\mathfrak{a}_\perp$ may be larger than the algebra generated by the remaining connected components. Consider for example the embedding of $B_2 \subset B_4$ (the figure 5). In this case by eliminating the simple root $\alpha_2 = e_2 - e_3$ one splits the extended Dynkin diagram of B_4 into the diagrams of the subalgebra $\mathfrak{a} = B_2$ and that of the direct sum $A_1 \oplus A_1$. But the subalgebra $\mathfrak{a}_\perp$ is equal not to $A_1 \oplus A_1$ but to B_2 (the root system of B_4 contains not only $\alpha_2 = e_2 - e_3$ but also e_2).

Such effects are due to the fact that the subalgebras $\mathfrak{a}$ and $\mathfrak{a}_\perp$ must not form a direct sum in $\mathfrak{g}$. Consider the case of such a regular embedding $\mathfrak{a} \subset \mathfrak{g}$ where both algebras are simple and the diagram of the subalgebra $\mathfrak{a}_\perp$ is not a subdiagram of the extended Dynkin diagram $\mathfrak{g}$. Drop the subdiagram of $\mathfrak{a}$ and the node α' that connects it with all the remaining nodes of the diagram of $\mathfrak{g}$. Consider the remaining diagram. This diagram is the diagram of the algebra $\bar{\mathfrak{a}}$ of $\mathrm{rank}(\bar{\mathfrak{a}}) = \mathrm{rank}(\mathfrak{g}) - \mathrm{rank}(\mathfrak{a})$. It is clear that $\bar{\mathfrak{a}} \subset \mathfrak{a}_\perp$. So the question is whether $\mathfrak{a}_\perp$ has additional roots, which are not the roots of $\bar{\mathfrak{a}}$ but are the linear combinations of them. It is possible when the set of angles between the roots of $\bar{\mathfrak{a}}$ does not contain all the angles between the roots of $\mathfrak{a}$, then reflecting the roots of $\mathfrak{a}$ by $s_{\alpha'}$ we get the additional roots of $\mathfrak{a}_\perp$.

All the cases are listed in the table 2.1.

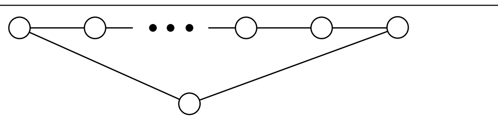
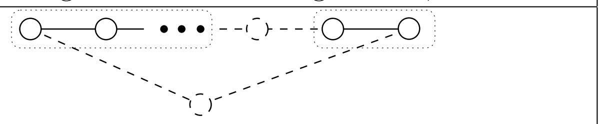
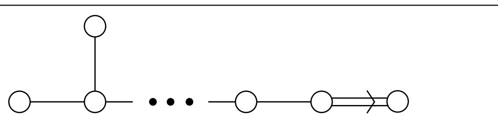
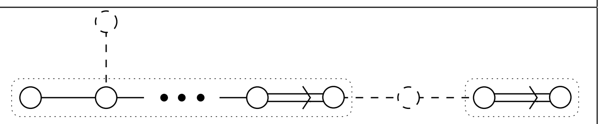
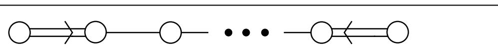
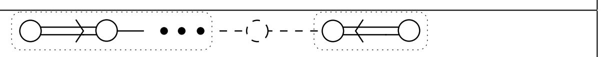
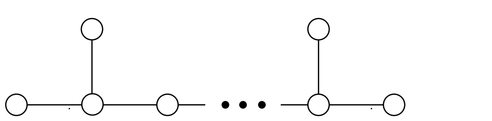
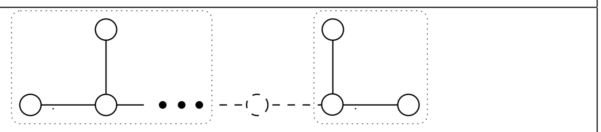
$\mathfrak{g}$	Extended diagram of the algebra $\mathfrak{g}$	Diagrams of the subalgebras $\mathfrak{a}$, $\mathfrak{a}_\perp$
A_n		
B_n		
C_n		
D_n		

Table 1: Subalgebras $\mathfrak{a}$, $\mathfrak{a}_\perp$ for the classical series

For the algebra $\mathfrak{g}$ from the series A_r the roots in the orthogonal basis $\{e_i,\ 1 \le i \le r+1\}$ are $\Delta = \{\alpha_{ij} = e_i - e_j,\ 1 \le i,j \le r+1\}$, $\Delta^+ = \{\alpha_{ij},\ i <$

$j\}$ and the set of simple roots consists of $\alpha_{1,2}, \alpha_{2,3}, \ldots, \alpha_{r,r+1}$. So for the regular subalgebra $\mathfrak{a} = A_{r_\mathfrak{a}}$ and its simple root system consisting of first $r_\mathfrak{a}$ simple roots we get $\Delta_{\mathfrak{a}_\perp} = \{\alpha_{ij},\ r_\mathfrak{a}+1 < i, j \leq r+1\}$ and $\mathfrak{a}_\perp = A_{r-r_\mathfrak{a}-1}$.

For the algebra $\mathfrak{g}$ from the series B_r the roots in the orthogonal basis $\{e_i,\ 1 \leq i \leq r\}$ are $\Delta = \{\alpha_{\pm i,\pm j} = \pm e_i \pm e_j,\ i < j;\ \alpha_{\pm j} = \pm e_j,\ 1 \leq j \leq r\}$, $\Delta^+ = \{\alpha_{i,-j},\ \alpha_{ij},\ \alpha_j;\ i < j,\ 1 \leq j \leq r\}$ and the set of simple roots consists of $\alpha_{1,-2}, \alpha_{2,-3}, \ldots, \alpha_{r-1,-r}, \alpha_r$. So if the regular subalgebra $\mathfrak{a} = A_{r_\mathfrak{a}}$ and its simple root system consists of first $r_\mathfrak{a}$ simple roots, then $\Delta_{\mathfrak{a}_\perp} = \{\alpha_{\pm i,\pm j},\ \alpha_j,\ r_\mathfrak{a}+1 < i < j \leq r\}$ and $\mathfrak{a}_\perp = B_{r-r_\mathfrak{a}-1}$. Otherwise if $\mathfrak{a} = B_{r_\mathfrak{a}}$ and its simple roots are $\alpha_{r-r_\mathfrak{a}+1,-r+r_\mathfrak{a}-2}, \ldots, \alpha_{r-1,r}, \alpha_r$ we see that $\Delta_{\mathfrak{a}_\perp} = \{\alpha_{\pm i,\pm j},\ \alpha_j,\ 1 < i < j \leq r - r_\mathfrak{a}\}$ and $\mathfrak{a}_\perp = B_{r-r_\mathfrak{a}}$. It is the only case when simple roots of $\mathfrak{a}_\perp$ can not be obtained from the extended Dynkin diagram, as can be seen in the Table 2.1. There exists the third possibility to get the pair of subalgebras $\mathfrak{a}, \mathfrak{a}_\perp$ with the regular subalgebra $\mathfrak{a}$ by dropping single node from the extended Dynkin diagram of B_r. It can be done by taking as the set of simple roots of $\mathfrak{a}$ the set $\{\alpha_{1,-2}, \alpha_{1,2}, \alpha_{2,-3}, \ldots, \alpha_{r_\mathfrak{a}-1,-r_\mathfrak{a}}\}$. Then $\mathfrak{a} = D_{r_\mathfrak{a}}$, $\Delta_{\mathfrak{a}_\perp} = \{\alpha_{\pm i,\pm j},\ \alpha_j,\ r_\mathfrak{a} < i < j \leq r\}$ and $\mathfrak{a}_\perp = B_{r-r_\mathfrak{a}}$.

For the algebra $\mathfrak{g}$ from the series C_r the roots in the orthogonal basis $\{e_i,\ 1 \leq i \leq r\}$ are $\Delta = \{\alpha_{\pm i,\pm j} = \pm e_i \pm e_j,\ i < j;\ \alpha_{\pm j} = \pm 2e_j,\ 1 \leq j \leq r\}$, $\Delta^+ = \{\alpha_{i,-j},\ \alpha_{ij},\ \alpha_j;\ i < j,\ 1 \leq j \leq r\}$ and the set of simple roots consists of $\alpha_{1,-2}, \alpha_{2,-3}, \ldots, \alpha_{r-1,-r}, \alpha_r$. So if the regular subalgebra $\mathfrak{a} = A_{r_\mathfrak{a}}$ and its simple root system consists of first $r_\mathfrak{a}$ simple roots, then $\Delta_{\mathfrak{a}_\perp} = \{\alpha_{\pm i,\pm j},\ \alpha_j,\ r_\mathfrak{a}+1 < i < j \leq r\}$ and $\mathfrak{a}_\perp = C_{r-r_\mathfrak{a}-1}$. Otherwise if $\mathfrak{a} = C_{r_\mathfrak{a}}$ and its simple roots are $\alpha_{r-r_\mathfrak{a}+1,-r+r_\mathfrak{a}-2}, \ldots, \alpha_{r-1,r}, \alpha_r$ we see that $\Delta_{\mathfrak{a}_\perp} = \{\alpha_{\pm i,\pm j},\ \alpha_j,\ 1 < i < j \leq r - r_\mathfrak{a}\}$ and $\mathfrak{a}_\perp = C_{r-r_\mathfrak{a}}$.

For the algebra $\mathfrak{g}$ from the series D_r the roots in the orthogonal basis $\{e_i,\ 1 \leq i \leq r\}$ are $\Delta = \{\alpha_{\pm i,\pm j} = \pm e_i \pm e_j,\ 1 \leq i < j \leq r\}$, $\Delta^+ = \{\alpha_{i,-j},\ \alpha_{ij},\ i < j,\ 1 \leq j \leq r\}$ and the set of simple roots consists of $\alpha_{1,-2}, \alpha_{2,-3}, \ldots, \alpha_{r-1,-r}, \alpha_{r-1,r}$. So if the regular subalgebra $\mathfrak{a} = A_{r_\mathfrak{a}}$ and its simple root system consists of first $r_\mathfrak{a}$ simple roots, then $\Delta_{\mathfrak{a}_\perp} = \{\alpha_{\pm i,\pm j},\ r_\mathfrak{a}+1 < i < j \leq r\}$ and $\mathfrak{a}_\perp = D_{r-r_\mathfrak{a}-1}$. Otherwise if $\mathfrak{a} = D_{r_\mathfrak{a}}$ and its simple roots are $\alpha_{r-r_\mathfrak{a}+1,-r+r_\mathfrak{a}-2}, \ldots, \alpha_{r-1,r}, \alpha_{r-1,r}$ we see that $\Delta_{\mathfrak{a}_\perp} = \{\alpha_{\pm i,\pm j},\ 1 < i < j \leq r - r_\mathfrak{a}\}$ and $\mathfrak{a}_\perp = D_{r-r_\mathfrak{a}}$.

In the case of special embeddings the set $\Delta_\perp^+$ can be empty as for the special embedding of $A_1 \subset A_2$ with the embedding index equal to 4, or non-empty for example for the embedding $A_1 \subset A_2 \subset A_3$ which is depicted at the Figure 2.1.

Using the existing classification of maximal special subalgebras [10] we

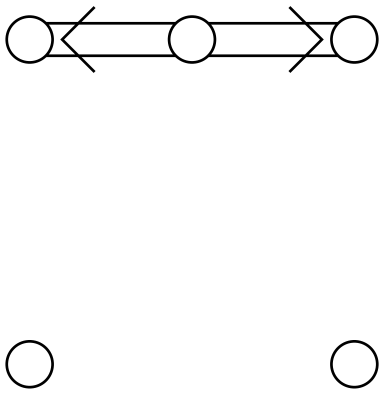


Figure 1: Extended Dynkin diagram of B_2 and embedding of A_1

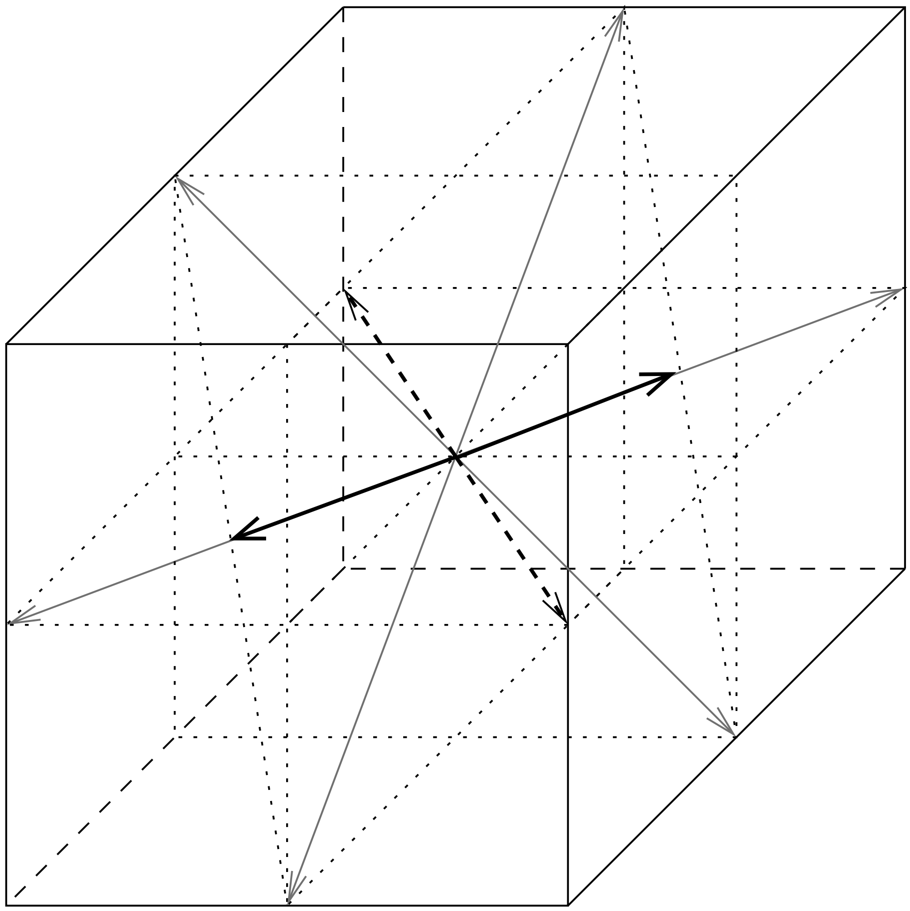


Figure 2: Special embedding $A_1 \subset A_2 \subset A_3$. Grey vectors are the roots of A_2, thick black - of $\mathfrak{a} = A_1$, dashed black are the orthogonal roots of A_1 which is contained in $\mathfrak{a}_\perp$

immediately have the following pairs of the orthogonal subalgebras $\mathfrak{a}$, $\mathfrak{a}_\perp$

$$
\begin{array}{lll}
su(p) \oplus su(q) & \subset su(pq) & \\
so(p) \oplus so(q) & \subset so(pq) & \\
sp(2p) \oplus sp(2q) & \subset so(4pq) & \\
sp(2p) \oplus so(q) & \subset sp(2pq) & \\
so(p) \oplus so(q) & \subset so(p+q) & \text{for } p \text{ and } q \text{ odd}
\end{array}
\tag{6}
$$

Exceptional Lie algebras and other non-maximal subalgebras will be considered elsewhere.

Up to this point we considered the problem of $\mathfrak{a}_\perp$-construction given $\mathfrak{a} \in \mathfrak{g}$ for regular injections in terms of Dynkin diagrams. When the root systems Δ and $\Delta_{\mathfrak{a}}$ are known explicitly all that we need is to select the roots $\Delta_\perp = \{\alpha \in \Delta : \alpha \perp \Delta_{\mathfrak{a}}\}$ and correspondingly the positive roots $\Delta_\perp^+ = \{\alpha \in \Delta^+ : \alpha \perp \Delta_{\mathfrak{a}}\}$.

Now consider the $\mathfrak{a}_\perp \oplus \mathfrak{h}_{\mathfrak{a}}$-module with the highest weight μ. For the character of this module we have

$$
chL^\mu_{\mathfrak{a}_\perp \oplus \mathfrak{h}} = \frac{\sum_{\omega \in W_\perp} \epsilon(\omega) e^{\omega(\mu + \rho_{\mathfrak{a}_\perp}) - \rho_{\mathfrak{a}_\perp}}}{\prod_{\alpha \in \Delta_\perp^+} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)}}
\tag{7}
$$

The projection $\pi_{\mathfrak{a}}(ch\, L^\mu_{\mathfrak{a}_\perp \oplus \mathfrak{h}})$ gives us the single element $e^{\pi_{\mathfrak{a}} \cdot \mu}$ of the formal algebra $\mathcal{E}(\mathfrak{a})$ with the multiplicity equal to the dimension of the module $L^\mu_{\mathfrak{a}_\perp \oplus \mathfrak{h}}$, since all the roots of $\mathfrak{a}_\perp$ are orthogonal to that of $\Delta_{\mathfrak{a}}$.

Using this property we can reconsider the restriction $ch\, L^\mu_{\mathfrak{g} \downarrow \mathfrak{a}_\perp \oplus \mathfrak{h}}$, that is the character of the direct sum of $\mathfrak{a}_\perp \oplus \mathfrak{h}$-modules. Multiply the equation (5) by the element

$$
\pi_{\mathfrak{a}} \left(\prod_{\alpha \in \Delta^+ \setminus \Delta_\perp^+} (1 - e^{-\alpha})^{\mathrm{mult}_{\mathfrak{g}}(\alpha)} \right)
\tag{8}
$$

Taking into account that for any formal series $Q \in \mathcal{E}$ and the binomial the projection commutes with the multiplication,

$$
\pi_{\mathfrak{a}}(Q) \pi_{\mathfrak{a}}(1 - e^{-\alpha}) = \pi_{\mathfrak{a}} \left(Q \cdot (1 - e^{-\alpha}) \right),
\tag{9}
$$

we can rewrite the product of (5) and (8) in the form:

$$
\begin{aligned}
&\pi_{\mathfrak{a}} \left(\frac{\sum_{\omega \in W} \epsilon(\omega) e^{\omega(\mu + \rho) - \rho}}{\prod_{\alpha \in \Delta_\perp^+} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)}} \right) = \\
&\quad \pi_{\mathfrak{a}} \left(\prod_{\alpha \in \Delta^+ \setminus \Delta_\perp^+} (1 - e^{-\alpha})^{\mathrm{mult}_{\mathfrak{g}}(\alpha)} \right) \sum_{\nu \in P_{\mathfrak{a}}^+} b_\nu^{(\mu)} \frac{\sum_{\omega \in W_{\mathfrak{a}}} \epsilon(\omega) e^{\omega(\nu + \rho_{\mathfrak{a}}) - \rho_{\mathfrak{a}}}}{\prod_{\beta \in \Delta_{\mathfrak{a}}^+} (1 - e^{-\beta})^{\mathrm{mult}_{\mathfrak{a}}(\beta)}}
\end{aligned}
\tag{10}
$$

The right-hand side of this equation can be reorganised similarly to what was performed in the paper [9], by introducing the anomalous branching coefficients k_λ,

$$\sum_{\nu\in P_{\mathfrak{a}}} b_\nu^{(\mu)} \Psi_{(\mathfrak{a})}^{(\nu)} = \sum_{\lambda\in P_{\mathfrak{a}}} k_\lambda^{(\mu)} e^\lambda \tag{11}$$

and simplifying the multiplier:

$$\pi_{\mathfrak{a}}\left(\frac{\sum_{\omega\in W}\epsilon(\omega)e^{\omega(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta_\perp^+}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}\right) = \left(\prod_{\alpha\in\pi_{\mathfrak{a}}\left(\Delta^+\setminus\Delta_\perp^+\right)}(1-e^{-\alpha})^{\mathrm{mult}_{\mathfrak{g}}(\alpha)-\mathrm{mult}_{\mathfrak{a}}(\alpha)}\right)\sum_{\lambda\in P_{\mathfrak{a}}} k_\lambda^{(\mu)} e^\lambda \tag{12}$$

If the set $\Delta_\perp^+$ is non-empty then the Weyl reflections corresponding to the positive roots of $\Delta_\perp^+$ generate a subgroup $W_\perp$ of the Weyl group W. Let us reorganise the summation in the left-hand side of (12). Consider the factor-space $W_\perp\setminus W$. For the class $\tilde{\omega}\in W_\perp\setminus W$ choose the representative $\omega\in\tilde{\omega}$ such that $\pi_{\mathfrak{a}_\perp}\omega(\mu+\rho)\in\bar{C}_{\mathfrak{a}_\perp}$,

$$\pi_{\mathfrak{a}}\left(\frac{\sum_{\omega\in W}\epsilon(\omega)e^{\omega(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta_\perp^+}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}\right) = \pi_{\mathfrak{a}}\left(\sum_{\omega\in W_\perp\setminus W}\epsilon(\omega)\frac{\sum_{\nu\in W_\perp}\epsilon(\nu)e^{\nu\cdot\omega(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta_\perp^+}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}\right) \tag{13}$$

The fraction in the right-hand side of the equation is similar to the character of some $\mathfrak{a}_\perp$-module. Let us rewrite the shifted weights

$$\nu\cdot\omega(\mu+\rho)-\rho = \nu\cdot\left(\omega(\mu+\rho)-\pi_{\mathfrak{a}}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}+\rho_{\mathfrak{a}_\perp}+\pi_{\mathfrak{a}}(\omega(\mu+\rho))\right)-\rho \tag{14}$$

Since $\nu\cdot\pi_{\mathfrak{a}}(\omega(\mu+\rho)) = \pi_{\mathfrak{a}}(\omega(\mu+\rho))$ and $\omega(\mu+\rho)-\pi_{\mathfrak{a}}(\omega(\mu+\rho)) = \pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))$, we get

$$\begin{aligned}&\sum_{\omega\in W_\perp\setminus W}\epsilon(\omega)\frac{\sum_{\nu\in W_\perp}\epsilon(\nu)e^{\nu\cdot\omega(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta_\perp^+}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}} = \\ &\sum_{\omega\in W_\perp\setminus W}\epsilon(\omega)e^{\pi_{\mathfrak{a}}(\omega(\mu+\rho))-\rho}\frac{e^{\rho_{\mathfrak{a}_\perp}}\sum_{\nu\in W_\perp}\epsilon(\nu)e^{\nu\cdot(\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}+\rho_{\mathfrak{a}_\perp})-\rho_{\mathfrak{a}_\perp}}}{\prod_{\alpha\in\Delta_\perp^+}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}} = \\ &\sum_{\omega\in W_\perp\setminus W}\epsilon(\omega)e^{\pi_{\mathfrak{a}}(\omega(\mu+\rho))-\rho}e^{\rho_{\mathfrak{a}_\perp}}\mathrm{ch}L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}\end{aligned} \tag{15}$$

The projector $\pi_{\mathfrak{a}}$ transforms the character of the module $\mathrm{ch}L_{\mathfrak{a}_{\perp}}^{\pi_{\mathfrak{a}_{\perp}}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_{\perp}}}$ into an element equal to the dimension of the module multiplied by the unit element:

$$\pi_{\mathfrak{a}}\left(\sum_{\omega\in W_{\perp}\backslash W}\epsilon(\omega)e^{\pi_{\mathfrak{a}}(\omega(\mu+\rho))-\rho}e^{\rho_{\mathfrak{a}_{\perp}}}\mathrm{ch}L_{\mathfrak{a}_{\perp}}^{\pi_{\mathfrak{a}_{\perp}}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_{\perp}}}\right)= \sum_{\omega\in W_{\perp}\backslash W}\epsilon(\omega)\dim\left(L_{\mathfrak{a}_{\perp}}^{\pi_{\mathfrak{a}_{\perp}}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_{\perp}}}\right)e^{\pi_{\mathfrak{a}}(\omega(\mu+\rho)-\rho)} \quad (16)$$

Thus we have the equality

$$\sum_{\omega\in W_{\perp}\backslash W}\epsilon(\omega)\dim\left(L_{\mathfrak{a}_{\perp}}^{\pi_{\mathfrak{a}_{\perp}}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_{\perp}}}\right)e^{\pi_{\mathfrak{a}}(\omega(\mu+\rho)-\rho)}= \left(\prod_{\alpha\in\pi_{\mathfrak{a}}\left(\Delta^{+}\backslash\Delta_{\perp}^{+}\right)}(1-e^{-\alpha})^{\mathrm{mult}_{\mathfrak{g}}(\alpha)-\mathrm{mult}_{\alpha}}\right)\sum_{\lambda\in P_{\mathfrak{a}}}k_{\lambda}^{(\mu)}e^{\lambda} \quad (17)$$

Following the transformations performed in [9] we rewrite the multiplier in the right-hand side:

$$\prod_{\alpha\in\pi_{\mathfrak{a}}\circ(\Delta^{+}\backslash\Delta_{\perp}^{+})}\left(1-e^{-\alpha}\right)^{\mathrm{mult}(\alpha)-\mathrm{mult}_{\mathfrak{a}}(\alpha)}=-\sum_{\gamma\in P_{\mathfrak{a}}}s(\gamma)e^{-\gamma} \quad (18)$$

For the coefficient function $s(\gamma)$ define the carrier $\Phi_{\mathfrak{a}\subset\mathfrak{g}}\subset P_{\mathfrak{a}}$:

$$\Phi_{\mathfrak{a}\subset\mathfrak{g}}=\left\{\gamma\in P_{\mathfrak{a}}\mid s(\gamma)\neq 0\right\}; \quad (19)$$

In these terms the equation for the formal elements,

$$\begin{aligned}\sum_{\omega\in W_{\perp}\backslash W}\epsilon(\omega)\dim\left(L_{\mathfrak{a}_{\perp}}^{\pi_{\mathfrak{a}_{\perp}}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_{\perp}}}\right)e^{\pi_{\mathfrak{a}}(\omega(\mu+\rho)-\rho)}&=\\ =-\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}}s(\gamma)\,e^{-\gamma}\sum_{\lambda\in P_{\mathfrak{a}}}k_{\lambda}^{(\mu)}e^{\lambda}&\\ =-\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}}\sum_{\lambda\in P_{\mathfrak{a}}}s(\gamma)\,k_{\lambda}^{(\mu)}e^{\lambda-\gamma}&\end{aligned} \quad (20)$$

leads to the following equality

$$\sum_{\omega\in W_{\perp}\backslash W}\epsilon(\omega)\dim\left(L_{\mathfrak{a}_{\perp}}^{\pi_{\mathfrak{a}_{\perp}}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_{\perp}}}\right)\delta_{\xi,\pi_{\mathfrak{a}}(\omega(\mu+\rho)-\rho)}+\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}}s(\gamma)\;k_{\xi+\gamma}^{(\mu)}=0;\quad \xi\in P_{\mathfrak{a}} \quad (21)$$

To get the recurrent relations for the anomalous branching coefficients we use the following procedure (similar to that in [9]). Let γ_0 be the lowest vector with respect to the natural ordering in $\overset{\circ}{\Delta}_{\mathfrak{a}}$ in the lowest grade of $\Phi_{\mathfrak{a}\subset\mathfrak{g}}$ and decompose the defining relation (18),

$$\prod_{\alpha\in\pi_{\mathfrak{a}}\circ(\Delta^+\setminus\Delta_\perp^+)} \left(1-e^{-\alpha}\right)^{\mathrm{mult}(\alpha)-\mathrm{mult}_{\mathfrak{a}}(\alpha)} = -s\left(\gamma_0\right)e^{-\gamma_0} - \sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}\setminus\{\gamma_0\}} s\left(\gamma\right)e^{-\gamma}, \tag{22}$$

then the equality (21) leads to the desired recurrent relation for the anomalous branching coefficients:

$$k_\xi^{(\mu)} = -\frac{1}{s\left(\gamma_0\right)}\left(\sum_{\omega\in W_\perp\setminus W}\epsilon(\omega)\,\dim\left(L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}\right)\delta_{\xi-\gamma_0,\pi_{\mathfrak{a}}(\omega(\mu+\rho)-\rho)}+\right.$$
$$\left.+\sum_{\gamma\in\Gamma_{\mathfrak{a}\subset\mathfrak{g}}} s\left(\gamma+\gamma_0\right)k_{\xi+\gamma}^{(\mu)}\right) \tag{23}$$

where the set

$$\Gamma_{\mathfrak{a}\subset\mathfrak{g}} = \{\xi-\gamma_0|\xi\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}\}\setminus\{0\} \tag{24}$$

was introduced that is called the injection fan.

Now let the set $\Delta_\perp^+$ be empty. There are three different reasons for $\Delta_\perp^+=0$: i) $\dim\mathfrak{h}_{\mathfrak{a}}=\dim\mathfrak{h}_{\mathfrak{g}}$, ii) $\mathfrak{a}_\perp=0$ and iii) $\mathfrak{a}_\perp\subset\mathfrak{h}_{\mathfrak{g}}$. Both the first and the second cases can be treated as corresponding to the trivial orthogonal subalgebra: $\mathfrak{a}_\perp=0$. In any of these cases instead of the formal characters in the right-hand side of (13) we obtain the formal element $e^{\pi_{\mathfrak{a}_\perp}\omega(\mu+\rho)}$ In the first two cases (equivalent to $\mathfrak{a}_\perp=0$) the projection operator retains its purely geometrical meaning: the vector $\omega(\mu+\rho)$ is projected to the subspace orthogonal to the weight space of $\mathfrak{a}$. It is clear that in any of the three variants the final vector $\pi_{\mathfrak{a}}\pi_{\mathfrak{a}_\perp}\omega(\mu+\rho)$ leads to the unit of the formal algebra $\mathcal{E}$. Thus when the set $\Delta_\perp^+$ is empty we get the more simple recurrent relation:

$$k_\xi^{(\mu)} = -\frac{1}{s\left(\gamma_0\right)}\left(\sum_{w\in W}\epsilon\left(w\right)\delta_{\xi,\pi_{\mathfrak{a}}\circ(w\circ(\mu+\rho)-\rho)+\gamma_0}+\sum_{\gamma\in\Gamma_{\mathfrak{a}\subset\mathfrak{g}}}s\left(\gamma+\gamma_0\right)k_{\xi+\gamma}^{(\mu)}\right) \tag{25}$$

It coinsides with the one obtained in [9] (formula (16)).

In the next section we describe an algorithm for the computation of branching coefficients based on the relation (23).

2.2 Algorithm for the recursive computation of the branching coefficients

We use the recurrent relation (23) to formulate an algorithm for recursive computation of the branching coefficients. It is important to mention that the computation of the branching coefficients is performed without the explicit construction of the module $L_{\mathfrak{g}}^{(\mu)}$ and any of the modules $L_{\mathfrak{a}}^{(\nu)}$.

The algorithm contains the following steps.

1. Construct the sets Δ^+ and $\Delta^+_{\mathfrak{a}}$ of positive roots for the algebras $\mathfrak{a} \subset \mathfrak{g}$.

2. Select the positive roots $\alpha \in \Delta^+$ which are orthogonal to the root subspace of $\mathfrak{a}$ and form the set $\Delta^+_{\perp}$.

3. Construct the set $\widehat{\Psi^{(\mu)}} = \{\omega(\mu+\rho) - \rho;\ \omega \in W\}$ of the anomalous weights of the $\mathfrak{g}$-module $L^{(\mu)}$.

4. Select the weights $\left\{\lambda = \omega(\mu+\rho) | \pi_{\mathfrak{a}_\perp}\lambda \in \bar{C}_{\mathfrak{a}_\perp}\right\}$ Since we have constructed the set $\Delta^+_{\perp}$ we can easily check wether the weight $\pi_{\mathfrak{a}_\perp}\lambda$ lies in the main Weyl chamber of $\mathfrak{a}_\perp$ by computing the scalar product of λ with the roots of $\Delta^+_\perp$ that must be non-negative.

5. For $\lambda = \omega(\mu+\rho),\ \pi_{\mathfrak{a}_\perp}\lambda \in \bar{C}_{\mathfrak{a}_\perp}$ calculate the dimensions of the corresponding modules $\dim\left(L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}\right)$ using the Weyl formula with the set $\Delta^+_\perp$.

6. Construct the set Γ (24).

7. Calculate the anomalous branching coefficients in the main Weyl chamber of the subalgebra $\mathfrak{a}$ using recurrent relation (23).

If we are interested in the branching coefficients for the embedding of the finite-dimensional Lie algebra into the affine Lie algebra we can construct the set of the anomalous weights up to the required grade and use the steps 4-7 of the algorithm for each grade. We can also speed up the algorithm by one-time computation of the representatives of the conjugate classes $W_\perp \backslash W$.

The next section contains several examples computed using this algorithm.

3 Examples

3.1 Finite dimensional Lie algebras

3.1.1 Regular embedding of A_1 into B_2

Consider the regular embedding of A_1 into B_2. Simple roots α_1, α_2 of B_2 are drawn as the dashed vectors at the Figure 3. We denote the corresponding Weyl reflections by ω_1, ω_2. Simple root β of the embedded A_1 is equal to $\alpha_1 + \alpha_2$ and is drawn as grey vector.

Let's describe the reduction of fundamental representation of B_2 with the highest weight (in fundamental weight basis) equal to $(1, 0)$, which is drawn as the black vector at the Figure 3. On the Figure 3 we have also

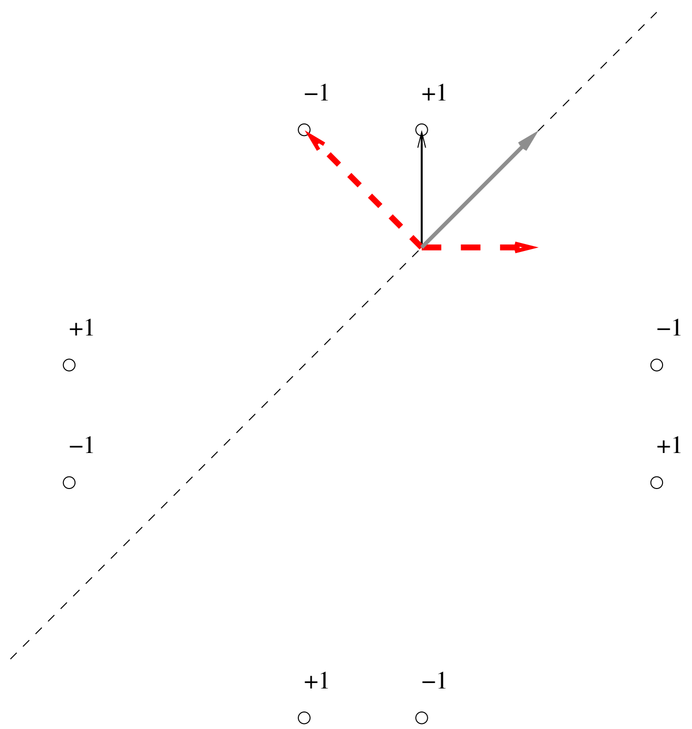


Figure 3: Regular embedding of A_1 into B_2

shown the set of points $\omega(\mu+\rho)$, $\omega \in W$ of fundamental representation of B_2 with the corresponding determinants of Weyl reflections $\epsilon(\omega)$. Now we have to factorise the Weyl group W by $W_\perp = \{\omega_1\}$. We get the following set of anomalous points $\omega(\mu+\rho) - \rho$, $\omega \in W_\perp \backslash W$: We have also depicted the corresponding $\mathfrak{a}_\perp = A_1$-modules $L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}$. Then we project these points and dimensions of modules onto the root space of subalgebra $\mathfrak{a} = A_1$

and get the following anomalous points in fundamental weights basis with corresponding multiplicities:

$$(1,2),\ (0,-3),\ (-4,3),\ (-5,-2) \tag{26}$$

For the function $s(\gamma)$ and the set Γ from the definition (19,24) we have

$$(1,2),\ (2,-1) \tag{27}$$

Here the second component denotes the value of $s(\gamma)$.

Anomalous branching coefficient $k_1^{(1,0)} = 2$, then for anomalous branching coefficient $k_0^{(1,0)}$ the formula (23) gives us

$$k_0^{(1,0)} = -1 \cdot k_2^{(1,0)} + 2 \cdot k_1^{(1,0)} - 3 \cdot \delta_{0,0} = 1 \tag{28}$$

So we have computed the branching coefficients.

3.1.2 Embedding of B_2 into B_4

Consider the the regular embedding of the subalgebra B_2 into the algebra B_4. We calculate the branching coefficients for the fundamental representation of B_4. The corresponding Dynkin diagrams are in the Figure 5.

In the orthogonal basis $e_1, \ldots, e_4$ simple roots of B_4 are

$$(e_1 - e_2,\ e_2 - e_3,\ e_3 - e_4,\ e_4) \tag{29}$$

Positive roots are

$$(e_1 - e_2,\ e_2 - e_3,\ e_3 - e_4,\ e_4,\ e_1 - e_3,\ e_2 - e_4,\ e_3 + e_4,\ e_3,\ e_1 - e_4,\\ e_2 + e_4,\ e_2,\ e_1 + e_4,\ e_2 + e_3,\ e_1,\ e_1 + e_3,\ e_1 + e_2) \tag{30}$$

Simple roots of the embedded subalgebra $\mathfrak{a} = B_2$ are

$$(e_3 - e_4, e_4) \tag{31}$$

The set $\Delta_\perp^+$ is equal to

$$\{e_1 - e_2, e_1 + e_2, e_1, e_2\} \tag{32}$$

We see that this is the set of positive roots of the algebra $\mathfrak{a}_\perp = B_2$.

To find the branching coefficients we need to compute the anomalous points of B_4, select point lying in the main Weyl chamber of $\mathfrak{a}_\perp$ and compute the dimensions of corresponding $\mathfrak{a}_\perp$-modules.

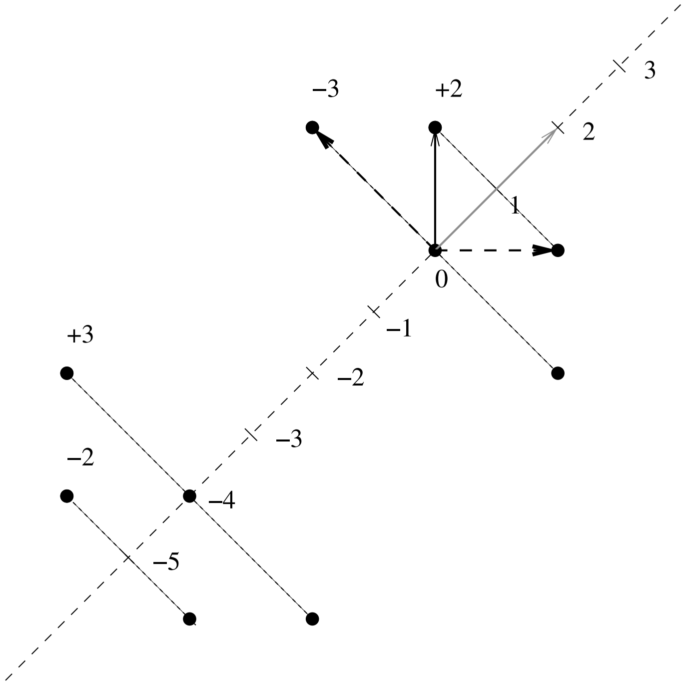


Figure 4: Anomalous points and the corresponding $\mathfrak{a}_\perp = A_1$-modules

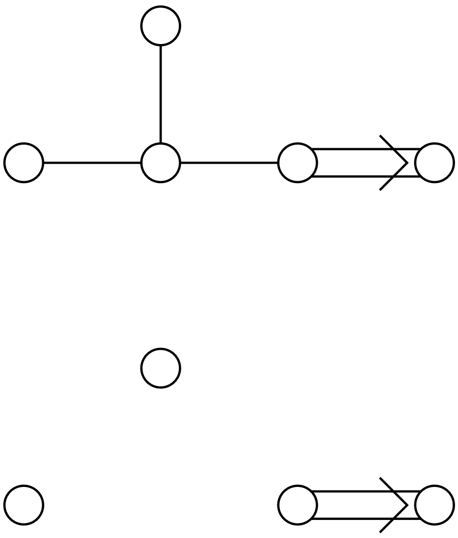


Figure 5: Dynkin diagrams

We consider the B_4-module with the highest weight $\mu = (0,1,0,2) = 2e_1 + 2e_2 + e_3 + e_4$.

The set of the anomalous points $\omega(\mu+\rho) - \rho,\ \omega \in W$ consists of 384 points. We do not show it here.

We need to select those points $\omega(\mu+\rho)$ which are projected into the main chamber of the embedded algebra $\mathfrak{a}_{\perp}$. It means that scalar product of these points with all the roots from $\Delta^{+}_{\perp}$ is non-negative.

To compute dimensions of the corresponding $\mathfrak{a}_{\perp}$-modules we need to project each selected point onto the root space $\Delta^{+}_{\perp}$ and substract $\rho_{\mathfrak{a}_{\perp}}$, then use Weyl dimension formula.

We show the result of this procedure on the Figure 6.

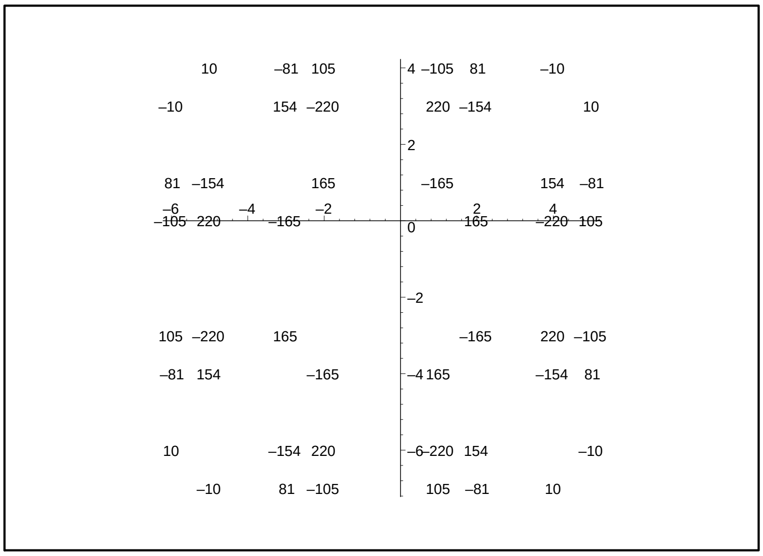


Figure 6: Anomalous points with the dimensions of corresponding $\mathfrak{a}_{\perp}$-modules.

Then we should construct "the fan" and use the recurrent relation for the computation of anomalous branching coefficients.

Using the definition (24) we get the following set of the points Γ with the corresponding values $s(\gamma+\gamma_0)$, depicted at the Figure 7. We use the recurrent relation (23) and get following branching coefficients:

$$\pi_{\mathfrak{a}}\left(chL^{(0,1,0,2)}_{B_4}\right) = 6\ chL^{(0,0)}_{B_2} + 60\ chL^{(0,2)}_{B_2} + 30\ chL^{(1,0)}_{B_2} + 19\ chL^{(2,0)}_{B_2} + 40\ chL^{(1,2)}_{B_2} + 10\ chL^{(2,2)}_{B_2} \tag{33}$$

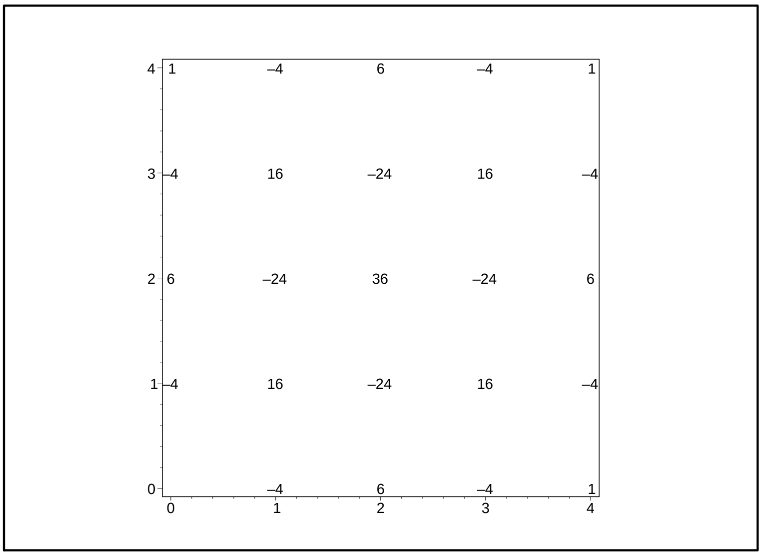


Figure 7: Fan for $B_2 \subset B_4$

The dimension of the highest-weight B_4-module $L_{B_4}^{(0,1,0,2)}$ is equal to 2772. It is easy to see, that right-hand side of the equation (33) gives the same result.

3.2 Affine Lie algebras

3.2.1 Embedding of the affine algebra into affine algebra

Consider the affine extension of the example 3.1.1. Since this embedding is regular, the level of the representations of the subalgebra is equal to the level of the representation of the algebra.

The set $\Delta_\perp^+$ of the orthogonal positive roots with the zero projection on the root space of the subalgebra $\hat{A}_1$ is the same as in the finite-dimensional case.

Consider the level one representation of the algebra $\mathfrak{g} = \hat{B}_2$ with the highest weight $w_1 = (1, 0, 1, 0)$, where the first two components are the coordinates of the classical part in the orthogonal basis e_1, e_2, the third is is the level of the weight and the fourth the grade.

The set of the anomalous points of this representation up to sixth grade is depicted in the Figure 8 and in each grade it looks like in the Figure 3.

As the next step of our algorithm 2.2 we project the anomalous points to

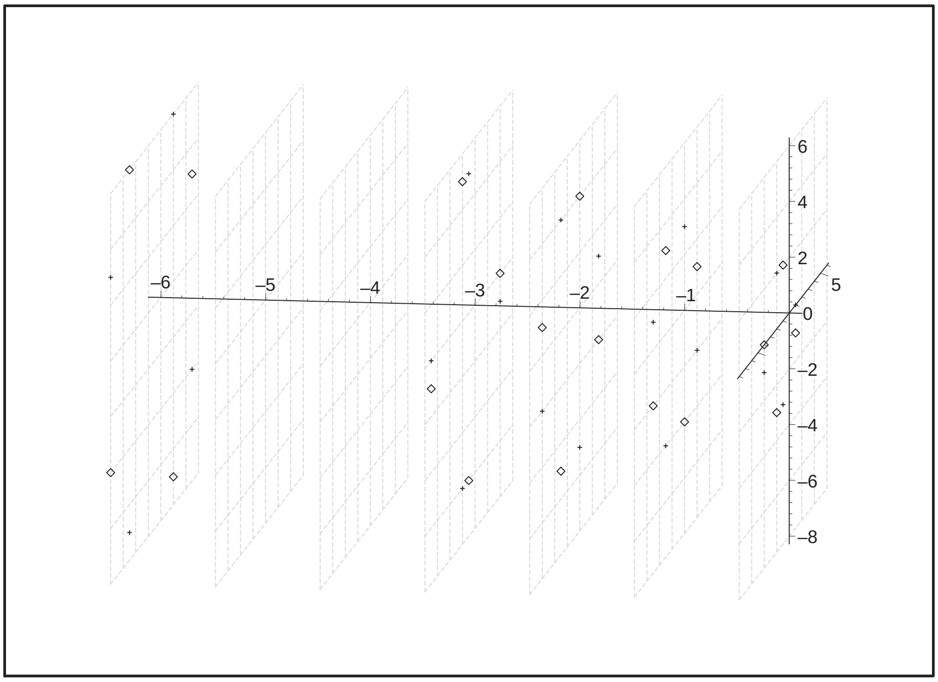


Figure 8: The anomalous points of the $(1,0,1,0)$ representation of the algebra $\hat{B}_2$

the weight space of the subalgebra $\hat{A}_1$ and calculate the dimensions of the corresponding $\mathfrak{a}_\perp$-modules $L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}$. The result of this computation up to the twelfth grade is presented at the Figure

Then we should construct "the fan" and use the recurrent relation for the computation of anomalous branching coefficients.

Using the definition (24) we get the following set of the points Γ with the corresponding values $s(\gamma+\gamma_0)$: 10. Here we restricted the computation to the twelfth grade.

Also we should mention that the lowest vector of the fan γ_0 is equal to zero, since we have excluded all the roots of $\Delta_\perp^+$ from the defining relation (24).

Using the recurrent relation for the anomalous branching coefficients we get the following result

Selecting the points inside the main Weyl chamber of the subalgebra $\hat{A}_1$

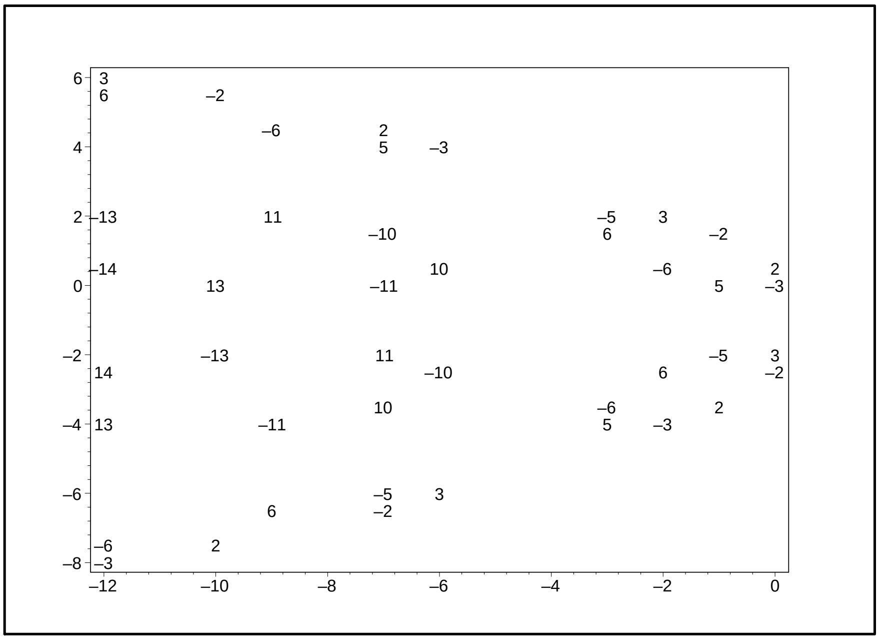


Figure 9: Projected anomalous points and the dimensions of $\mathfrak{a}_{\perp}$-modules.

we get the following results for the branching coefficients up to twelfth grade

$$\begin{aligned} L^{w_1}_{\hat{B}_2\downarrow\hat{A}_1} = {} & 2L^{w_1}_{\hat{A}_1}(0) \oplus 1L^{w_0}_{\hat{A}_1}(0) \oplus 4L^{w_0}_{\hat{A}_1}(-1)\oplus \\ & 2L^{w_1}_{\hat{A}_1}(-1) \oplus 8L^{w_0}_{\hat{A}_1}(-2) \oplus 8L^{w_1}_{\hat{A}_1}(-2) \oplus 15L^{w_0}_{\hat{A}_1}(-3)\oplus \\ & 12L^{w_1}_{\hat{A}_1}(-3) \oplus 26L^{w_1}_{\hat{A}_1}(-4) \oplus 29L^{w_0}_{\hat{A}_1}(-4) \oplus 51L^{w_0}_{\hat{A}_1}(-5)\oplus \\ & 42L^{w_1}_{\hat{A}_1}(-5) \oplus 78L^{w_1}_{\hat{A}_1}(-6) \oplus 85L^{w_0}_{\hat{A}_1}(-6) \oplus 120L^{w_1}_{\hat{A}_1}(-7)\oplus \\ & 139L^{w_0}_{\hat{A}_1}(-7) \oplus 202L^{w_1}_{\hat{A}_1}(-8) \oplus 222L^{w_0}_{\hat{A}_1}(-8) \oplus 306L^{w_1}_{\hat{A}_1}(-9)\oplus \\ & 346L^{w_0}_{\hat{A}_1}(-9) \oplus 530L^{w_0}_{\hat{A}_1}(-10) \oplus 482L^{w_1}_{\hat{A}_1}(-10) \oplus 714L^{w_1}_{\hat{A}_1}(-11)\oplus \\ & 797L^{w_0}_{\hat{A}_1}(-11) \oplus 1080L^{w_1}_{\hat{A}_1}(-12) \oplus 1180L^{w_0}_{\hat{A}_1}(-12) \end{aligned} \tag{34}$$

This result can be expressed using the power series expansion of the branching

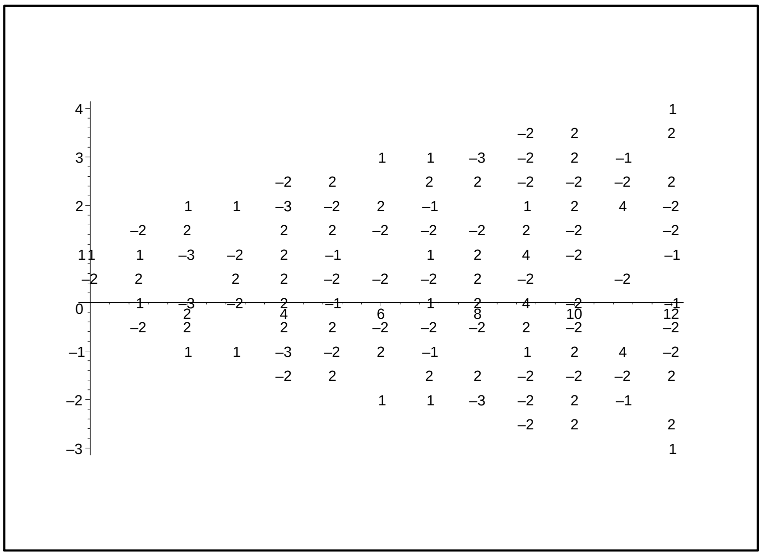


Figure 10: Fan for $\hat{A}_1 \subset \hat{B}_2$

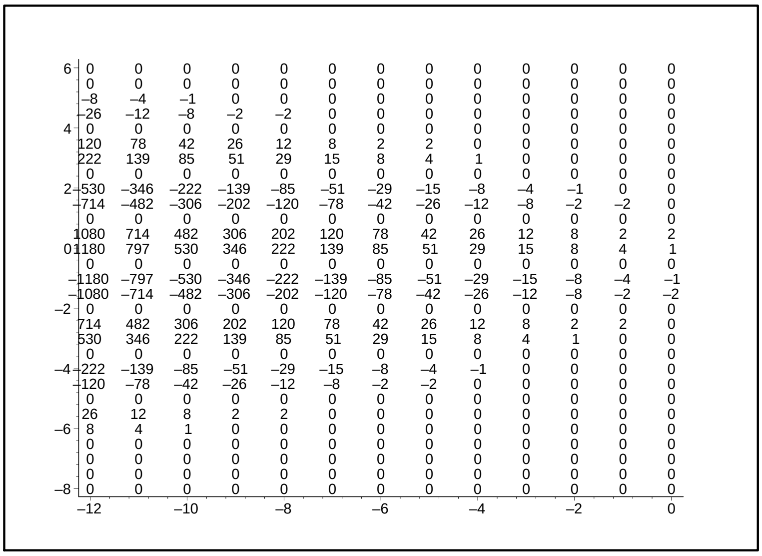


Figure 11: Anomalous branching coefficients for $\hat{A}_1 \subset \hat{B}_2$

functions [3].

$$b_0^{(w_1)} = \begin{array}{l} 1 + 4\,q^1 + 8\,q^2 + 15\,q^3 + 29\,q^4 + 51\,q^5 + 85\,q^6 + 139\,q^7 + \\ \quad 222\,q^8 + 346\,q^9 + 530\,q^{10} + 797\,q^{11} + 1180\,q^{12} + \ldots \end{array} \tag{35}$$

$$b_1^{(w_1)} = \begin{array}{l} 2 + 2\,q^1 + 8\,q^2 + 12\,q^3 + 26\,q^4 + 42\,q^5 + 78\,q^6 + 120\,q^7 + \\ \quad 202\,q^8 + 306\,q^9 + 482\,q^{10} + 714\,q^{11} + 1080\,q^{12} + \ldots \end{array} \tag{36}$$

Here the lower index of the branching function denotes the number of the corresponding $\hat{A}_1$ fundamental weight $w_0 = \lambda_0 = (0,1,0)$, $w_1 = \alpha/2 = (1,1,0)$.

4 Physical applications

Here we want to discuss possible applications of the described techniques in physical models.

Branching coefficients for the embedding of affine Lie subalgebra into affine Lie algebra can be used to construct modular invariant partition functions of Wess-Zumino-Novikov-Witten models ([1], [11], [12], [13]).

But for this construction to work the embedding is required to be conformal, which means that the central charge of the subalgebra is equal to the central charge of the algebra.

$$c(\mathfrak{a}) = c(\mathfrak{g}) \tag{37}$$

The class of the conformal embeddings is rather small, the complete classification is given in the paper [13]. The requirement (37) allows to reduce the task of the computation of the branching coefficients of affine Lie algebras to the computation of the branching coefficients of the finite-dimensional Lie algebras.

Here we describe this procedure and discuss how the requirement (37) can be used to simplify the algorithm 2.2.

Conformal embeddings should preserve conformal invariance, so Sugawara central charge should be the same for the original and the embedded theory.

The states for the theory that corresponds to the algebra $\mathfrak{g}$

$$J^{a_1}_{-n_1} J^{a_2}_{-n_2} \ldots |\lambda\rangle \quad n_1 \geq n_2 \geq \cdots > 0 \tag{38}$$

For sub-algebra $\mathfrak{a} \subset \mathfrak{g}$

$$\tilde{J}^{a'_1}_{-n_1} \tilde{J}^{a'_2}_{-n_2} \ldots |\pi_{\mathfrak{a}}(\lambda)\rangle \tag{39}$$

Here $\tilde{J}^{a'_j}_{-n_j}$ are the generators of $\mathfrak{a}$ and $\pi_{\mathfrak{a}}$ is the projection of $\mathfrak{g}$ to $\mathfrak{a}$. $\mathfrak{g}$-invariance of vacuum entails its $\mathfrak{a}$-invariance, but it is not the case for energy-momentum tensor. So energy-momentum tensor of bigger theory should consist only of generators of $\mathfrak{a}$. Then $T_{\mathfrak{g}} = T_{\mathfrak{a}} \Rightarrow c(\mathfrak{g}) = c(\mathfrak{a})$. This leads to equation

$$\frac{k \dim \mathfrak{g}}{k+g} = \frac{x_e k \dim \mathfrak{a}}{x_e k + a} \tag{40}$$

Here x_e is the embedding index and g, a are dual Coxeter numbers of corresponding algebras.

It can be shown that solutions of equation (40) exist only for level 1 $k = 1$ [1].

If we have modular-invariant partition function for the fields described by the representation of the algebra $\mathfrak{g}$ this modular invariance is preserved by the projection on the subalgebra $\mathfrak{a}$, but we need also the preservation of the conformal invariance. So we should select only those highest-weight modules of the subalgebra $\mathfrak{a}$ for which the relation (40) holds.

Since in the decomposition (4) the highest weight ν of the subalgebra module belongs to some grade n of projected algebra module $\pi_{\mathfrak{a}} \cdot L^{(\mu)}_{\mathfrak{g}}$, from the relation (40) one can obtain the following requirement on the conformal dimensions of the corresponding fields

$$\Delta_{\pi_{\mathfrak{a}}\mu} + n = \Delta_\nu \tag{41}$$

It leads to the relation on the classical parts of the corresponding weights:

$$\frac{(\overset{\circ}{\mu}, \overset{\circ}{\mu} + 2\rho)}{2(1+g)} + n = \frac{(\overset{\circ}{\nu}, \overset{\circ}{\nu} + 2\rho_{\mathfrak{a}})}{2(x_e + a)} \tag{42}$$

There exits the finite reducibility theorem for the conformal embeddings which states that only finite number of the branching coefficients is non-zero in the case of the conformal embedding $\mathfrak{a} \subset \mathfrak{a}$.

Then after we have found all such weights ν and the corresponding branching coefficients $b^{(\mu)}_\nu$ we can substitute the sums $\sum_{\nu \in P^+_{\mathfrak{a}}} b^{(\mu)}_\nu \chi_\nu$ over the modified characters χ_ν of the corresponding $\mathfrak{a}$-modules in place of the characters of the $\mathfrak{g}$-modules in the diagonal modular-invariant partition function

$$Z(\tau) = \sum_{\mu \in P^+_{\mathfrak{g}}} \chi_\mu(\tau) \bar{\chi}_\mu(\bar{\tau}) \tag{43}$$

Thus we obtain the non-diagonal modular-invariant partition function for the theory with the current algebra $\mathfrak{a}$.

$$Z_{\mathfrak{a}}(\tau) = \sum_{\nu,\lambda \in P^+_{\mathfrak{a}}} \chi_\nu(\tau) M_{\nu\lambda} \bar{\chi}_\lambda(\bar{\tau}) \tag{44}$$

4.1 Example

4.1.1 Special embedding $\hat{A}_1 \subset \hat{A}_2$

Consider the embedding of the affine Lie algebra $\hat{A}_1$ into $\hat{A}_2$ built as the affine extension of the special embedding $su(2) \subset su(3)$ with the embedding index $x_e = 4$. The level of the algebra $\mathfrak{g} = \hat{A}_2$ is equal to one, so the level of the subalgebra $\tilde{k} = kx_e = 4$.

There exist three level one dominant weights $[1,0,0]$, $[0,1,0]$, $[0,0,1]$ in the weight space of $\hat{A}_2$. It is easy to see that the set $\Delta_\perp$ is empty.

Here we view the representation with the highest weight $w_0 = (1,0,0)$ in detail.

The set of the anomalous points of this representation up to sixth grade is depicted in the Figure 12. We also show the root subspace of the subalgebra $\hat{A}_1$ under consideration.

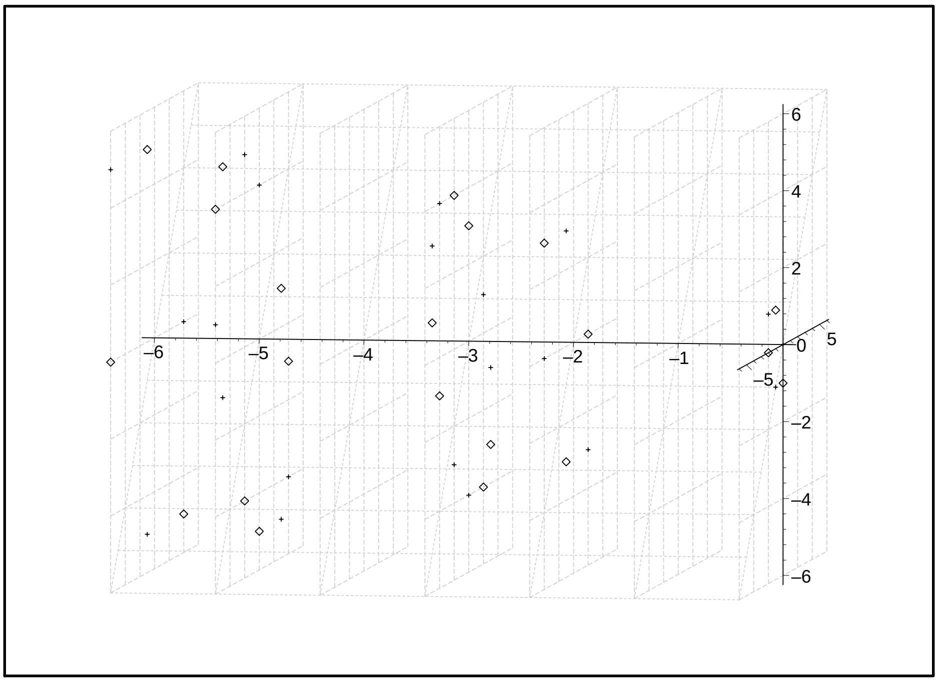


Figure 12: The anomalous points of the $(1,0,1,0)$ representation of the algebra $\hat{A}_2$

As the next step of our algorithm 2.2 we project the anomalous points to the weight space of the subalgebra $\hat{A}_1$ and calculate the dimensions of the corresponding $\mathfrak{a}_\perp$-modules $L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}$. The result of this computation up to the twelfth grade is presented at the Figure 13

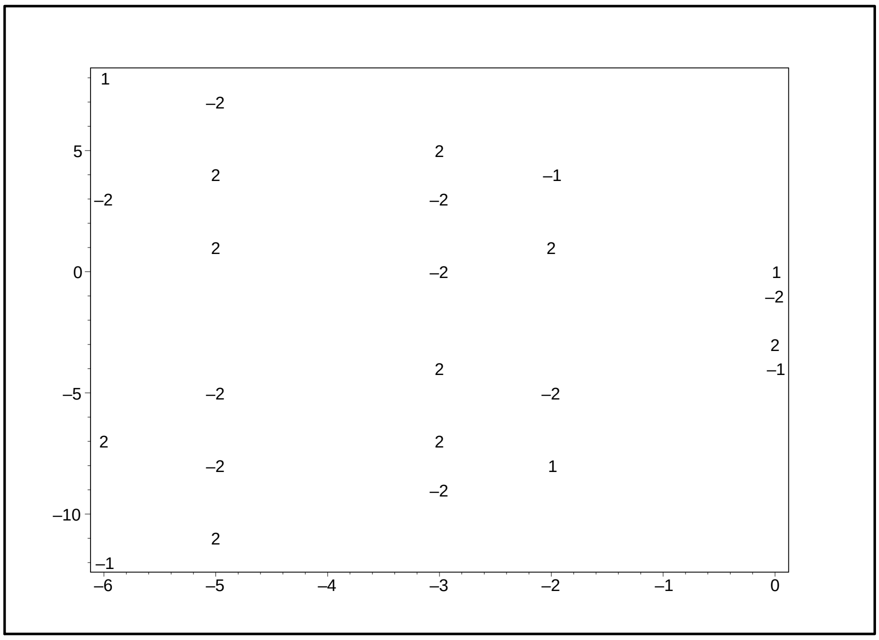


Figure 13: Projected anomalous points and the dimensions of $\mathfrak{a}_\perp$-modules.

Then we should construct "the fan" and use the recurrent relation for the computation of anomalous branching coefficients.

Using the definition (24) we get the following set of the points Γ with the corresponding values $s(\gamma+\gamma_0)$, depicted at the Figure 14. Here we restricted the computation to the twelfth grade.

Using the recurrent relation for the anomalous branching coefficients we get the following result

We see that only two anomalous branching coefficients inside the main Weyl chamber of $\hat{A}_1$ are non-zero and equal to the branching coefficients. So the finite reducibility theorem holds and we get

$$L_{\hat{A}_2\downarrow\hat{A}_1}^{[1,0,0]} = L_{\hat{A}_1}^{[0,4]} \oplus L_{\hat{A}_1}^{[4,0]} \tag{45}$$

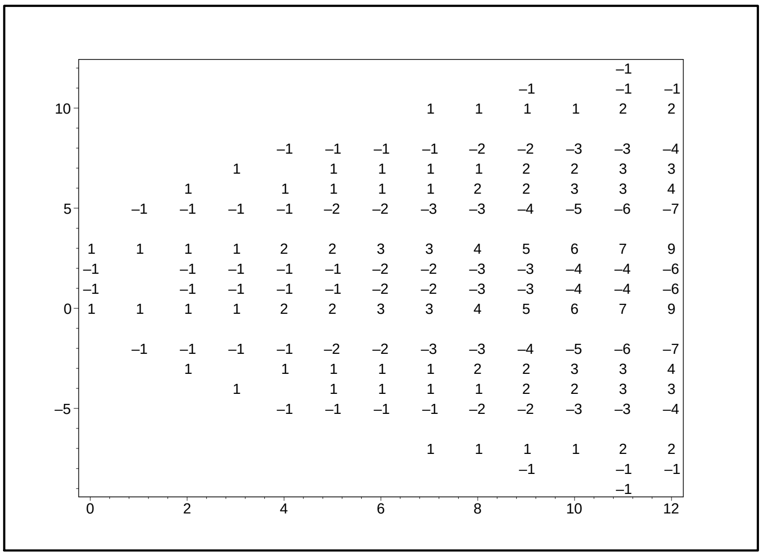


Figure 14: Fan for $\hat{A}_1 \subset \hat{A}_2$

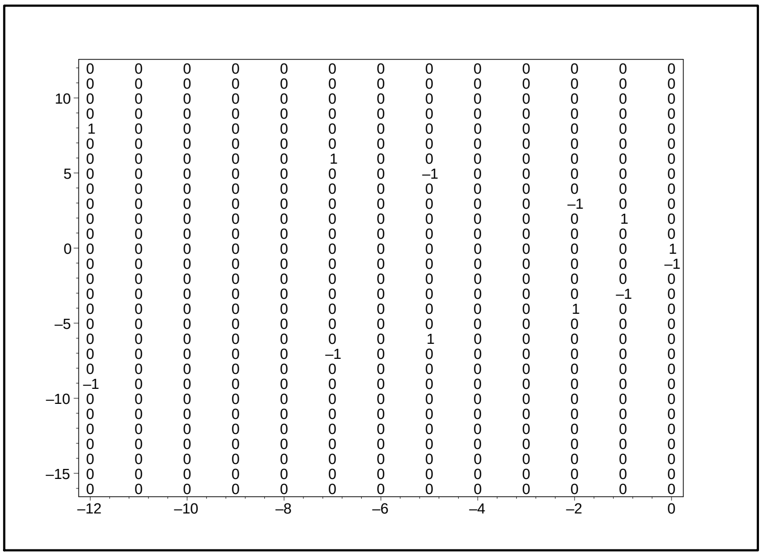


Figure 15: Anomalous branching coefficients for $\hat{A}_1 \subset \hat{A}_2$

For the other level one dominant weights of $\hat{A}_2$ we get trivial branching rules

$$L^{[0,1,0]}_{\hat{A}_2\downarrow\hat{A}_1} = L^{[2,2]}_{\hat{A}_1} \tag{46}$$

$$L^{[0,0,1]}_{\hat{A}_2\downarrow\hat{A}_1} = L^{[2,2]}_{\hat{A}_1} \tag{47}$$

Using this result we can construct modular-invariant partition function

$$Z = \left|\chi_{[4,0]} + \chi_{[0,4]}\right|^2 + 2\chi^2_{[2,2]} \tag{48}$$

5 Conclusion

We have constructed the recurrent relation for branching coefficients and proposed practical algorithm for the reduction procedure. Also we have discussed the application of this algorithm to the physical problem of construction the modular-invariant partition functions in the conformal field theory. This method of conformal embeddings is well-known but may be actual in the study of WZW-models emerging in the context of the AdS/CFT correspondence [14, 15, 16].

6 Acknowledgements

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