Abstract

1 Introduction

The branching properties of Lie (affine Lie) algebras are highly important for applications in quantum field theory. This concernes for example the conformal field theory models [1],[2].

In this paper we demonstrate that for an arbitrary reductive subalgebra branching is directly connected with the BGG resolution and in particular exhibits the resolution properties in terms of the $\mathcal{O}^p$ category [3] (the parabolic generalization of the cathegory $\mathcal{O}$ [4]).

To show this we use the recursive approach to branching presented in [our preprint] (similar to the one used in [5] for maximal embeddings). We consider the subalgebra $\mathfrak{a}$ together with its counterpart $\mathfrak{a}_\perp$ "orthogonal" to $\mathfrak{a}$ with respect to the Killing form and also $\widetilde{\mathfrak{a}_\perp} := \mathfrak{a}_\perp \oplus \mathfrak{h}_\perp$ where $\mathfrak{h} = \mathfrak{h}_{\mathfrak{a}} \oplus \mathfrak{h}_{\mathfrak{a}_\perp} \oplus \mathfrak{h}_\perp$. For any reductive algebra $\mathfrak{a}$ the subalgebra $\mathfrak{a}_\perp \hookrightarrow \mathfrak{g}$ is regular and reductive. For a highest weight integrable module $L^{(\mu)}$ and orthogonal pair of subalgebras $(\mathfrak{a}, \mathfrak{a}_\perp)$ we consider the singular element $\Psi^{(\mu)}$ (the numerator in the Weyl character formula $ch\left(L^{\mu}\right) = \frac{\Psi^{(\mu)}}{\Psi^{(0)}}$, see for example [6]) the Weyl denominator $R = \Psi^{(0)}_{\mathfrak{a}_\perp}$ and the projection $\Psi^{(\mu)}_{(\mathfrak{a},\mathfrak{a}_\perp)} = \pi_{\mathfrak{a}} \frac{\Psi^{(\mu)}_{\mathfrak{g}}}{\Psi^{(0)}_{\mathfrak{a}_\perp}}$. The element $\Psi^{(\mu)}_{\mathfrak{g}}$ can be decomposed with respect to the set of Weyl numerators $\Psi^{(\mu)}_{\mathfrak{a}_\perp}$ of $\mathfrak{a}_\perp$. This decomposition provides the possibility to construct the set of highest weight modules $L^{\mu_{\widetilde{\mathfrak{a}_\perp}}}_{\widetilde{\mathfrak{a}_\perp}}$. When the injection $\mathfrak{a}_\perp \hookrightarrow \mathfrak{g}$ satisfies the "standard parabolic" conditions these modules give rise to the parabolic Verma modules $M^{\mu_{\widetilde{\mathfrak{a}_\perp}}}_{(\widetilde{\mathfrak{a}_\perp}\hookrightarrow\mathfrak{g})}$ so that the initial character $ch\left(L^{\mu}\right)$ is finally decomposed into the alternating sum of such. On the other hand when the parabolic conditions are violated the construction survives and exhibites a decomposition with respect to a set of Verma modules $M^{\mu_{\widetilde{\mathfrak{a}_\perp}}}_{\left(\widetilde{\mathfrak{b}_\perp},\mathfrak{g}\right)}$. In such a situation the algebra $\widetilde{\mathfrak{b}_\perp}$ is not a subalgebra in $\mathfrak{g}$ but a contraction of $\widetilde{\mathfrak{a}_\perp}$.

Some general properties of the proposed decompositions are formulated in terms of a specific element $\Gamma_{\mathfrak{a}\to\mathfrak{g}}$ of the group algebra $\mathcal{E}\left(\mathfrak{g}\right)$ called "the injection fan". Using this tool the simple and explicit algorithm for branching coefficients computations applicable for an arbitrary (maximal or nonmaximal) subalgebras of finite-dimensional or affine Lie algebras was obtained in [our preprint].

Possible further developments are discussed in Section 7.

1.1 Notation

Consider affine Lie algebras $\mathfrak{g}$ and $\mathfrak{a}$ with the underlying finite-dimensional subalgebras $\overset{\circ}{\mathfrak{g}}$ and $\overset{\circ}{\mathfrak{a}}$ and an injection $\mathfrak{a} \hookrightarrow \mathfrak{g}$ such that $\mathfrak{a}$ is a reductive subalgebra $\mathfrak{a} \subset \mathfrak{g}$ with correlated root spaces: $\mathfrak{h}^*_{\mathfrak{a}} \subset \mathfrak{h}^*_{\mathfrak{g}}$ and $\mathfrak{h}^*_{\overset{\circ}{\mathfrak{a}}} \subset \mathfrak{h}^*_{\overset{\circ}{\mathfrak{g}}}$. We use the following notations:

L^{μ} $(L^{\nu}_{\mathfrak{a}})$ — the integrable module of $\mathfrak{g}$ with the highest weight μ ; (resp. integrable $\mathfrak{a}$ -module with the highest weight ν);

r , $(r_{\mathfrak{a}})$ — the rank of the algebra $\mathfrak{g}$ (resp.$\mathfrak{a}$) ;

Δ $(\Delta_{\mathfrak{a}})$— the root system; Δ^+ (resp.$\Delta^+_{\mathfrak{a}}$)— the positive root system (of $\mathfrak{g}$ and $\mathfrak{a}$ respectively);

$\mathrm{mult}\,(\alpha)$ $(\mathrm{mult}_{\mathfrak{a}}\,(\alpha))$ — the multiplicity of the root α in Δ (resp. in $(\Delta_{\mathfrak{a}})$);

$\overset{\circ}{\Delta}$, $\left(\overset{\circ}{\Delta}_{\mathfrak{a}}\right)$ — the finite root system of the subalgebra $\overset{\circ}{\mathfrak{g}}$ (resp. $\overset{\circ}{\mathfrak{a}}$);

$\mathcal{N}^{\mu}$, $(\mathcal{N}^{\nu}_{\mathfrak{a}})$ — the weight diagram of L^{μ} (resp.$\mathrm{L}^{\nu}_{\mathfrak{a}}$) ;

W , $(W_{\mathfrak{a}})$— the corresponding Weyl group;

C , $(C_{\mathfrak{a}})$— the fundamental Weyl chamber;

$\bar{C}$, $\left(\bar{C}_{\mathfrak{a}}\right)$ — the closure of the fundamental Weyl chamber;

ρ , $(\rho_{\mathfrak{a}})$ — the Weyl vector;

$\epsilon\,(w) := (-1)^{\mathrm{length}(w)}$;

α_i , $\left(\alpha_{(\mathfrak{a})j}\right)$ — the i-th (resp. j-th) basic root for $\mathfrak{g}$ (resp.$\mathfrak{a}$); $i = 0, \ldots, r$, $(j = 0, \ldots, r_{\mathfrak{a}})$;

δ — the imaginary root of $\mathfrak{g}$ (and of $\mathfrak{a}$ if any);

$\alpha^{\vee}_i$, $\left(\alpha^{\vee}_{(\mathfrak{a})j}\right)$— the basic coroot for $\mathfrak{g}$ (resp.$\mathfrak{a}$) , $i = 0, \ldots, r$; $(j = 0, \ldots, r_{\mathfrak{a}})$;

$\overset{\circ}{\xi}$, $\overset{\circ}{\xi}_{(\mathfrak{a})}$ — the finite (classical) part of the weight $\xi \in P$, $\left(\text{resp.}\xi_{(\mathfrak{a})} \in \mathrm{P}_{\mathfrak{a}}\right)$;

$\lambda = \left(\overset{\circ}{\lambda}; k; n\right)$ — the decomposition of an affine weight indicating the finite part $\overset{\circ}{\lambda}$, level k and grade n;

P (resp.$\mathrm{P}_{\mathfrak{a}}$) — the weight lattice;

P^+ (resp.$\mathrm{P}^+_{\mathfrak{a}}$) — the dominant weight lattice;

$m^{(\mu)}_{\xi}$, $\left(m^{(\nu)}_{\xi}\right)$ — the multiplicity of the weight $\xi \in P$ (resp. $\in \mathrm{P}_{\mathfrak{a}}$) in the module L^{μ} , (resp. $\xi \in L^{\nu}_{\mathfrak{a}}$);

$ch\,(L^{\mu})$ (resp.$\mathrm{ch}\,(\mathrm{L}^{\nu}_{\mathfrak{a}})$)— the formal character of L^{μ} (resp.$\mathrm{L}^{\nu}_{\mathfrak{a}}$);

$ch\left(L^{\mu}\right)=\frac{\sum_{w\in W}\epsilon(w)e^{w\circ(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^{+}}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}$ — the Weyl-Kac formula;

$R\ :=\ \prod_{\alpha\in\Delta^{+}}\left(1-e^{-\alpha}\right)^{\mathrm{mult}(\alpha)}\qquad\left(\mathrm{resp.}R_{\mathfrak{a}}:=\prod_{\alpha\in\Delta_{\mathfrak{a}}^{+}}\left(1-e^{-\alpha}\right)^{\mathrm{mult}_{\mathfrak{a}}(\alpha)}\right)$— the Weyl denominator.

2 Orthogonal subalgebra and the singular element

Consider the highest weight $\mu\in P^{+}$ of an integrable module L^{μ} of $\mathfrak{g}$ and let $\mathfrak{a}\subset\mathfrak{g}$ be a reductive subalgebra of $\mathfrak{g}$. Let L^{μ} be completely reducible with respect to $\mathfrak{a}$,

$$L_{\mathfrak{g}\downarrow\mathfrak{a}}^{\mu}=\bigoplus_{\nu\in P_{\mathfrak{a}}^{+}}b_{\nu}^{(\mu)}L_{\mathfrak{a}}^{\nu}.$$

Using the projection operator $\pi_{\mathfrak{a}}$ (to the weight space ${\mathfrak{h}_{\mathfrak{a}}}^{*}$) one can rewrite this decomposition in terms of formal characters:

$$\pi_{\mathfrak{a}}ch\left(L^{\mu}\right)=\sum_{\nu\in P_{\mathfrak{a}}^{+}}b_{\nu}^{(\mu)}ch\left(L_{\mathfrak{a}}^{\nu}\right). \tag{1}$$

The module L^{μ} has the BGG-resolution (see [BGG] and [Humphrys]). All the members of the filtration sequence are the direct sums of Verma modules and all their highest weights ν are strongly linked to μ:

$$\{\nu\}=\{w\left(\mu+\rho\right)-\rho|w\in W\}.$$

2.1 Orthogonal subalgebra

Let us introduce the "orthogonal partner" $\mathfrak{a}_{\perp}$ for a reductive subalgebra $\mathfrak{a}$ in $\mathfrak{g}$.

Consider the root subspace $\mathfrak{h}_{\perp\mathfrak{a}}^{*}$ orthogonal to $\mathfrak{a}$,

$$\mathfrak{h}_{\perp\mathfrak{a}}^{*}:=\{\eta\in\mathfrak{h}^{*}|\forall h\in\mathfrak{h}_{\mathfrak{a}};\eta\left(h\right)=0\},$$

and the roots (correspondingly – positive roots) of $\mathfrak{g}$ orthogonal to $\mathfrak{a}$,

$$\begin{aligned}\Delta_{\mathfrak{a}_{\perp}}\ &:\ =\{\beta\in\Delta_{\mathfrak{g}}|\forall h\in\mathfrak{h}_{\mathfrak{a}};\beta\left(h\right)=0\},\\ \Delta_{\mathfrak{a}_{\perp}}^{+}\ &:\ =\left\{\beta^{+}\in\Delta_{\mathfrak{g}}^{+}|\forall h\in\mathfrak{h}_{\mathfrak{a}};\beta^{+}\left(h\right)=0\right\}.\end{aligned}$$

Let $W_{\mathfrak{a}_\perp}$ be the subgroup of W generated by the reflections w_β with the roots $\beta \in \Delta^+_{\mathfrak{a}_\perp}$. The subsystem $\Delta_{\mathfrak{a}_\perp}$ determines the subalgebra $\mathfrak{a}_\perp$ with the Cartan subalgebra $\mathfrak{h}_{\mathfrak{a}_\perp}$. Let

$$\mathfrak{h}^*_\perp := \{\eta \in \mathfrak{h}^*_{\perp\mathfrak{a}} | \forall h \in \mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{a}_\perp}; \eta(h) = 0\}$$

so that $\mathfrak{g}$ has the subalgebras

$$\begin{aligned} \widetilde{\mathfrak{a}_\perp} &: = \mathfrak{a}_\perp \oplus \mathfrak{h}_\perp \\ \widetilde{\mathfrak{a}} &: = \mathfrak{a} \oplus \mathfrak{h}_\perp. \end{aligned}$$

Algebras $\mathfrak{a}$ and $\mathfrak{a}_\perp$ form the "orthogonal pair" $(\mathfrak{a}, \mathfrak{a}_\perp)$ of subalgebras in $\mathfrak{g}$.

For the Cartan subalgebras we have the decomposition

$$\mathfrak{h} = \mathfrak{h}_{\mathfrak{a}} \oplus \mathfrak{h}_{\mathfrak{a}_\perp} \oplus \mathfrak{h}_\perp = \mathfrak{h}_{\widetilde{\mathfrak{a}}} \oplus \mathfrak{h}_{\mathfrak{a}_\perp} = \mathfrak{h}_{\widetilde{\mathfrak{a}_\perp}} \oplus \mathfrak{h}_{\mathfrak{a}}. \tag{2}$$

For the subalgebras of an orthogonal pair $(\mathfrak{a}, \mathfrak{a}_\perp)$ we consider the corresponding Weyl vectors, $\rho_{\mathfrak{a}}$ and $\rho_{\mathfrak{a}_\perp}$, and form the so called "defects" $\mathcal{D}_{\mathfrak{a}}$ and $\mathcal{D}_{\mathfrak{a}_\perp}$ of the injection:

$$\mathcal{D}_{\mathfrak{a}} := \rho_{\mathfrak{a}} - \pi_{\mathfrak{a}}\rho, \tag{3}$$

$$\mathcal{D}_{\mathfrak{a}_\perp} := \rho_{\mathfrak{a}_\perp} - \pi_{\mathfrak{a}_\perp}\rho. \tag{4}$$

For the highest weight $\mu \in P^+$ consider the linked weights $\{(w(\mu+\rho)-\rho) | w \in W\}$ and their projections to $h^*_{\widetilde{\mathfrak{a}_\perp}}$ additionally shifted by the defect $-\mathcal{D}_{\mathfrak{a}_\perp}$:

$$\mu_{\widetilde{\mathfrak{a}_\perp}}(w) := \pi_{\widetilde{\mathfrak{a}_\perp}}[w(\mu+\rho)-\rho] - \mathcal{D}_{\mathfrak{a}_\perp}, \quad w \in W.$$

In the case we do consider here among the weights $\{\mu_{\widetilde{\mathfrak{a}_\perp}}(w) | w \in W\}$ one can always choose those located in the fundamental chamber $\overline{C_{\widetilde{\mathfrak{a}_\perp}}}$. Let U be the set of representatives u for the classes $W/W_{\mathfrak{a}_\perp}$ such that

$$U := \left\{u \in W | \quad \mu_{\widetilde{\mathfrak{a}_\perp}}(u) \in \overline{C_{\widetilde{\mathfrak{a}_\perp}}}\right\} \quad . \tag{5}$$

For such a set U introduce the weights

$$\mu_{\mathfrak{a}}(u) := \pi_{\mathfrak{a}}[u(\mu+\rho)-\rho] + \mathcal{D}_{\mathfrak{a}_\perp}.$$

Note that the subalgebra $\mathfrak{a}_\perp$ is regular by definition since it is built on the roots of the algebra $\mathfrak{g}$.

Consider the Weyl-Kac formula for $ch\,(L^{\mu})$ in terms of singular elements [6]:

$$ch\,(L^{\mu}) = \frac{\Psi^{(\mu)}}{\Psi^{(0)}} = \frac{\Psi^{(\mu)}}{R}, \tag{6}$$

where

$$\Psi^{(\mu)} := \sum_{w \in W} \epsilon(w) e^{w \circ (\mu+\rho) - \rho}.$$

The same form will be used for the submodules $ch\,(L^{\nu}_{\mathfrak{a}})$

$$ch\,(L^{\nu}_{\mathfrak{a}}) = \frac{\Psi^{(\nu)}_{\mathfrak{a}}}{\Psi^{(0)}_{\mathfrak{a}}} = \frac{\Psi^{(\nu)}_{\mathfrak{a}}}{R_{\mathfrak{a}}},$$

with

$$\Psi^{(\nu)}_{\mathfrak{a}} := \sum_{w \in W_{\mathfrak{a}}} \epsilon(w) e^{w \circ (\nu+\rho_{\mathfrak{a}}) - \rho_{\mathfrak{a}}}.$$

Applying formula (6) to the branching rule (1) we get the relation connecting the singular elements $\Psi^{(\mu)}$ and $\Psi^{(\nu)}_{\mathfrak{a}}$:

$$\begin{aligned}
\pi_{\mathfrak{a}} \left(\frac{\sum_{w \in W} \epsilon(w) e^{w(\mu+\rho)-\rho}}{\prod_{\alpha \in \Delta^{+}} (1-e^{-\alpha})^{\mathrm{mult}(\alpha)}} \right) &= \sum_{\nu \in P^{+}_{\mathfrak{a}}} b^{(\mu)}_{\nu} \frac{\sum_{w \in W_{\mathfrak{a}}} \epsilon(w) e^{w(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}}{\prod_{\beta \in \Delta^{+}_{\mathfrak{a}}} (1-e^{-\beta})^{\mathrm{mult}_{\mathfrak{a}}(\beta)}}, \\
\pi_{\mathfrak{a}} \left(\frac{\Psi^{(\mu)}}{R} \right) &= \sum_{\nu \in P^{+}_{\mathfrak{a}}} b^{(\mu)}_{\nu} \frac{\Psi^{(\nu)}_{\mathfrak{a}}}{R_{\mathfrak{a}}}.
\end{aligned} \tag{7}$$

Here $\Delta^{+}_{\mathfrak{a}}$ is the set of positive roots of the subalgebra $\mathfrak{a}$ (without loss of generality we consider them as vectors from the positive root space $\mathfrak{h}^{*+}$ of $\mathfrak{g}$).

To describe the recurrent properties for branching coefficients $b^{(\mu)}_{\nu}$ we shall use the technique first proposed in [5]. One of the main tools is the set of weights $\Gamma_{\mathfrak{a} \to \mathfrak{g}}$ called the injection fan:

Definition 1. For the product

$$\prod_{\alpha \in \Delta^{+} \setminus \Delta^{+}_{\perp}} \left(1 - e^{-\pi_{\mathfrak{a}} \alpha}\right)^{\mathrm{mult}(\alpha) - \mathrm{mult}_{\mathfrak{a}}(\pi_{\mathfrak{a}} \alpha)} = - \sum_{\gamma \in P_{\mathfrak{a}}} s(\gamma) e^{-\gamma} \tag{8}$$

consider the carrier $\Phi_{\mathfrak{a} \subset \mathfrak{g}} \subset P_{\mathfrak{a}}$ of the function $s(\gamma) = \det(\gamma)$:

$$\Phi_{\mathfrak{a} \subset \mathfrak{g}} = \{ \gamma \in P_{\mathfrak{a}} | s(\gamma) \neq 0 \} \tag{9}$$

The ordering of roots in $\overset{\circ}{\Delta}_{\mathfrak{a}}$ induce the natural ordering of the weights in $P_{\mathfrak{a}}$. Denote by γ_0 the lowest vector of $\Phi_{\mathfrak{a}\subset\mathfrak{g}}$. The set

$$\Gamma_{\mathfrak{a}\to\mathfrak{g}} = \{\xi - \gamma_0 | \xi \in \Phi_{\mathfrak{a}\subset\mathfrak{g}}\} \setminus \{0\} \tag{10}$$

is called the *injection fan.*

It must be noticed that the injection fan is the universal instrument that depend only on the injection.

3 Branching for Verma modules

The formal character of Verma module is given by the expression

$$\mathrm{ch} M^{\mu} = \frac{e^{\mu}}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})} \tag{11}$$

The denominator $R = \prod_{\alpha\in\Delta^+}(1-e^{-\alpha})$ can be rewritten as the sum over the Weyl group of the algebra

$$R = \sum_{w\in W} \epsilon(w) e^{w\rho-\rho}. \tag{12}$$

If we consider the regular embeddings, all the positive roots of the subalgebra $\mathfrak{a}$ are in the set of positive roots of the algebra $\mathfrak{g}$, $\Delta_{\mathfrak{a}}^+ \subset \Delta^+$. Verma module can be decomposed into the set of Verma modules of the subalgebra:

$$\mathrm{ch} M^{\mu} = \sum_{\nu} b_{\nu}^{(\mu)} e^{\nu - \pi_{\mathfrak{a}}\nu} \mathrm{ch} M_{\mathfrak{a}}^{\pi_{\mathfrak{a}}\nu} \tag{13}$$

Using the equation (**??**) we can write the recurrent relation for the branching coefficients $b_{\nu}^{(\mu)}$:

$$b_{\xi}^{(\mu)} = -\frac{1}{s(\gamma_0)} \left(\delta_{\xi-\gamma_0,\mu} + \sum_{\gamma\in\Gamma_{\mathfrak{a}\to\mathfrak{g}}} s(\gamma+\gamma_0)\, b_{\xi+\gamma}^{(\mu)} \right). \tag{14}$$

Inside the main Weyl chamber branching coefficients for Verma module coincide with the coefficients for reduction of irreducible representations.

Now consider the following example. Let Lie algebra $\mathfrak{g}$ be $so(5)$ and subalgebra $\mathfrak{a} \subset \mathfrak{g}$ be $so(3)$. There are different embeddings of $so(3) \to so(5)$.

Here we limit ourselves to the regular ones. The root system of the algebra $so(5)$ consists of 8 roots. We denote simple roots by $\alpha_1 = e_1 - e_2$ and $\alpha_2 = e_2$, where e_1, e_2 form standard basis. The set of the positive roots is $\Delta^+ = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_2\}$. Each of these roots can be taken as the simple root of the subalgebra $so(3)$. At first we consider the subalgebra $so(3)$ with the root space spanned on $\beta = \alpha_1 + 2\alpha_2$. Verma module M^μ with the highest weight $\mu = \omega_1 + 2\omega_2 = 2\alpha_1 + 3\alpha_2$ is shown at Figure 1.

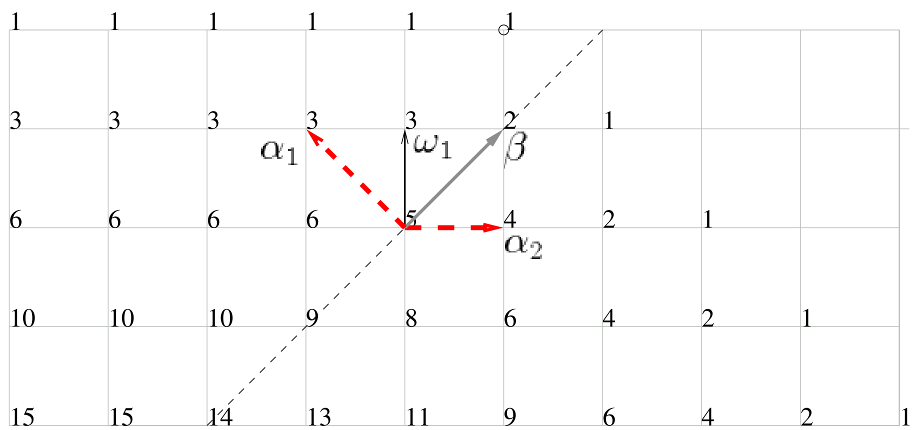

Figure 1: Regular embedding of A_1 into B_2. Simple roots α_1, α_2 of B_2 are presented as the dashed vectors. The simple root $\beta = \alpha_1 + 2\alpha_2$ of A_1 is indicated as the grey vector. Dimensions of weight subspaces of Verma module $M^{(1,2)}$ are shown.

The branching coefficients are shown at Figure 2 for the different regular embeddings $so(3) \to so(5)$. Here we can see that the picture depends upon the embedding and looks similar to the Verma modules of some subalgebra. In present paper we interpret this branching coefficients as the dimensions of the weight subspaces of the modules of contracted algebras.

4 Regular subalgebras and generalized Verma modules

Let us state the following lemma.

Lemma 1. *The quotient of the singular weights element* $\Psi^{(\mu)}$ *by the denominator* $R_{\mathfrak{a}_\perp}$ *is given by the formal element corresponding to the modified set*

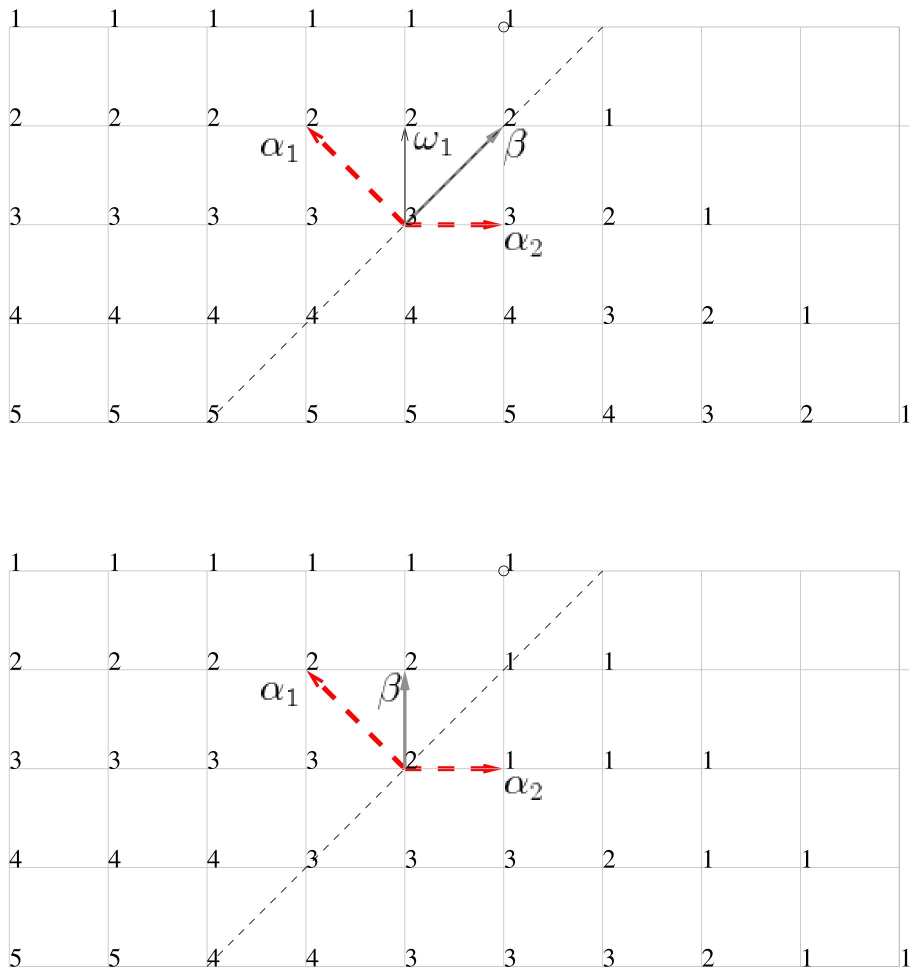

Figure 2: Branching for the Verma modules

of the singular weights of $L_{\mathfrak{a}\oplus\mathfrak{h}_d}^{(\pi_{\mathfrak{a}\oplus\mathfrak{h}_d}\mu)}$ *with the multiplicities given by the dimensions of* $\mathfrak{a}_\perp$*-modules.*

$$\frac{\Psi_{\mathfrak{g}}^{(\mu)}}{\prod_{\alpha\in\Delta^+_{\mathfrak{a}_\perp}}\left(1-e^{-\alpha}\right)^{\mathrm{mult}(\alpha)}} = \sum_{u\in W/W_{\mathfrak{a}_\perp}} \epsilon(u) e^{\pi_{\tilde{\mathfrak{a}}}[(u(\mu+\rho)-\rho)]+\mathcal{D}} \cdot \mathrm{ch}_{\mathfrak{a}_\perp} L^{\pi_{\mathfrak{a}_\perp}(u(\mu+\rho)-\rho)-\mathcal{D}} \tag{15}$$

Proof. Consider $\Psi_{\mathfrak{g}}^{(\mu)}$. It is given by the sum over the Weyl group.

$$\Psi_{\mathfrak{g}}^{(\mu)} = \sum_{w\in W} \epsilon(w) e^{w(\mu+\rho)-\rho} \tag{16}$$

We can divide the summation to the sum over $W_{\mathfrak{a}_\perp}$ and sum over all the classes of $W/W_{\mathfrak{a}_\perp}$: $w = vu,\ v \in W_{\mathfrak{a}_\perp},\ u \in W/W_{\mathfrak{a}_\perp}$. We have the following obvious property of the projections:

$$u(\mu+\rho) = \pi_{\mathfrak{a}}(u(\mu+\rho)) + \pi_{\mathfrak{a}_\perp}(u(\mu+\rho)) + \pi_{h^*_\perp}(u(\mu+\rho)) \tag{17}$$

Since $\mathfrak{h}^*_{\mathfrak{a}}, \mathfrak{h}^*_\perp$ are invariant under the action of $v \in W_{\mathfrak{a}_\perp}$, we have

$$vu(\mu+\rho)-\rho = v\cdot(\pi_{\mathfrak{a}_\perp}(u(\mu+\rho))-\rho_{\mathfrak{a}_\perp}+\rho_{\mathfrak{a}_\perp})-\rho_{\mathfrak{a}_\perp}+\pi_{\mathfrak{a}}(u(\mu+\rho))-\rho+\rho_{\mathfrak{a}_\perp}+\pi_{\mathfrak{h}^*_\perp}(u(\mu+\rho)) \tag{18}$$

Only the first term depends on v. So

$$\frac{\Psi^{(\mu)}_{\mathfrak{g}}}{\mathrm{R}_{\mathfrak{a}_\perp}} = \frac{\sum_{u\in W/W_{\mathfrak{a}_\perp}} \epsilon(u) e^{\pi_{\mathfrak{a}}(u(\mu+\rho))-\rho+\rho_{\mathfrak{a}_\perp}+\pi_{\mathfrak{h}_\perp}(u(\mu+\rho))} \cdot \sum_{v\in W_{\mathfrak{a}_\perp}} \epsilon(v) e^{v(\pi_{\mathfrak{a}_\perp} u(\mu+\rho)-\rho_{\mathfrak{a}_\perp}+\rho_{\mathfrak{a}_\perp})-\rho_{\mathfrak{a}_\perp}}}{\mathrm{R}_{\mathfrak{a}_\perp}} \tag{19}$$

We can rewrite the r.h.s. as

$$\sum_{u\in W/W_{\mathfrak{a}_\perp}} \epsilon(u) e^{\pi_{(\mathfrak{a}\oplus\mathfrak{h}_d)}[u(\mu+\rho)-\rho]+\pi_{(\mathfrak{a}\oplus\mathfrak{h}_d)}\cdot\rho-\rho+\rho_{\mathfrak{a}_\perp}} \cdot \mathrm{ch}_{\mathfrak{a}_\perp} L^{\pi_{\mathfrak{a}_\perp} u(\mu+\rho)-\rho_{\mathfrak{a}_\perp}}. \tag{20}$$

Since $\pi_{\mathfrak{a}_\perp}(u(\mu+\rho)-\rho)-(\rho_{\mathfrak{a}_\perp}-\pi_{\mathfrak{a}_\perp}\rho) = \pi_{\mathfrak{a}_\perp}(u(\mu+\rho)-\rho)-\mathcal{D}$ and $\pi_{\tilde{\mathfrak{a}}}\rho-\rho = -\pi_{\mathfrak{a}_\perp}\rho$ we get the statement of the Lemma.

□

Corollary 0.1. *If we apply the projection $\pi_{\mathfrak{a}}$ to the subalgebra $\mathfrak{a}$ to the equation (15) we obtain the "Lemma 1" from the paper [7] in the case of regular subalgebras $\mathfrak{a}, \mathfrak{a}_\perp$. However this Lemma 2 is more general then Lemma 1, since it holds for the special subalgebra $\mathfrak{a}$.*

Lemma 2. *Let $\mathfrak{a}$ and $\mathfrak{a}_\perp$ be the orthogonal pair of regular subalgebras in $\mathfrak{g}$, with $\widetilde{\mathfrak{a}_\perp} = \mathfrak{a}_\perp \oplus \mathfrak{h}_\perp$ and $\widetilde{\mathfrak{a}} = \mathfrak{a} \oplus \mathfrak{h}_\perp$,*

$L^\mu_{\mathfrak{g}}$ be the highest weight module with the singular element $\Psi^\mu_{\mathfrak{g}}$,

$R_{\mathfrak{a}_\perp}$ be the Weyl denominator for $\mathfrak{a}_\perp$.

Then the element $\pi_{\mathfrak{a}}\left(\frac{\Psi^\mu_{\mathfrak{g}}}{R_{\mathfrak{a}_\perp}}\right)$ can be decomposed into the sum over $u \in U$ (see (5)) of the singular elements $e^{\mu_{\mathfrak{a}}(u)}$ with the coefficients $\epsilon(u)\dim\left(L^{\mu_{\widetilde{\mathfrak{a}_\perp}}(u)}_{\widetilde{\mathfrak{a}_\perp}}\right)$:

$$\pi_{\mathfrak{a}}\left(\frac{\Psi^\mu_{\mathfrak{g}}}{R_{\mathfrak{a}_\perp}}\right) = \sum_{u\in U} \epsilon(u) e^{\mu_{\mathfrak{a}}(u)} \dim\left(L^{\mu_{\widetilde{\mathfrak{a}_\perp}}(u)}_{\widetilde{\mathfrak{a}_\perp}}\right). \tag{21}$$

This lemma can be used to state the recurrent relation for the branching coefficients for the reduction of $\mathfrak{g}$-module to $\mathfrak{a}$-modules. (See [7] for the details). It will be also useful for the discussion of special subalgebras.

Corollary 0.2. *If $\mathfrak{a}, \mathfrak{a}_{\perp}$ are regular subalgebras the simple root system $\alpha_1, \ldots, \alpha_r$ of $\mathfrak{g}$ sometimes can be chosen in such a way that simple roots $\beta_1, \ldots, \beta_{r_{\mathfrak{a}_{\perp}}}$ of $\mathfrak{a}_{\perp}$ are contained in it. Denote the set of roots β_i by I. This notation is in the agreement with the book [8].*

$$I = \{\beta_1, \ldots, \beta_{r_{\mathfrak{a}_{\perp}}}\} \subset \{\alpha_1, \ldots, \alpha_r\}$$

Then the subset I determines not only the subalgebra $\mathfrak{a}_{\perp}$, but also the minimal parabolic subalgebra $\mathfrak{p}_I$, such that $\mathfrak{a}_{\perp} \subset \mathfrak{p}$. Then we can consider the modules of the subcategory $\mathcal{O}^{\mathfrak{p}}$, for example the generalized Verma modules $M_{\mathfrak{p}}^{(\mu)}$. This modules are constructed from the irreducible highest-weight modules $L_{\mathfrak{p}}^{(\mu)}$ as follows

$$M_{\mathfrak{p}}^{(\mu)} = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L_{\mathfrak{p}}^{(\mu)} \tag{22}$$

Denote by $\tilde{R}$ the denominator constructed of the positive roots of $\mathfrak{g}$ which are not the roots of $\mathfrak{a}_{\perp}$.

$$\tilde{R} = \prod_{\alpha \in \Delta^+ \setminus \Delta^+_{\mathfrak{a}_{\perp}}} \left(1 - e^{-\alpha}\right)^{\mathrm{mult}_{\mathfrak{g}} \alpha} \tag{23}$$

If we expand $\frac{1}{\tilde{R}}$ the coefficients near the terms e^{ξ} are given by the Kostant partition function p^I constructed from the roots of $\mathfrak{g}$ which are not in I. The character of the generalized Verma module then can be formally written as

$$\mathrm{ch} M_{\mathfrak{p}}^{(\mu)} = \frac{1}{\tilde{R}} \mathrm{ch} L_{\mathfrak{p}}^{(\mu)} \tag{24}$$

The character of the irreducible finite-dimensional $\mathfrak{p}$-module $L_{\mathfrak{p}}^{(\mu)}$ differs from the character of the irreducible finite-dimensional $\mathfrak{a}_{\perp}$-module by the factor $e^{\pi_{\tilde{\mathfrak{a}}}(\mu)}$, since $\mathfrak{p}$ contains the full Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Now if we multiply the statement (15) of the lemma 1 by $\frac{1}{\tilde{R}}$ we get the character of irreducible highest-weight module $L^{(\mu)}$ on the left and the sum of the characters of generalized Verma modules on the right-hand side of the equation:

$$\mathrm{ch} L_{\mathfrak{g}}^{(\mu)} = \sum_{u \in W/W_{\mathfrak{a}_{\perp}}} \epsilon(u) e^{\pi_{\mathfrak{a} \oplus \mathfrak{h}_d}[(u(\mu+\rho)-\rho)] + \mathcal{D}} \cdot \frac{1}{\tilde{R}} \mathrm{ch} L_{\mathfrak{a}_{\perp}}^{\pi_{\mathfrak{a}_{\perp}}(u(\mu+\rho)-\rho) - \mathcal{D}} =$$

$$\sum_{u \in W/W_{\mathfrak{a}_{\perp}}} \epsilon(u) \mathrm{ch} M_{\mathfrak{p}}^{(u(\mu+\rho)-\rho)} \tag{25}$$

So we have recovered the Bernstein-Gelfand-Gelfand resolution [9] for the parabolic category $\mathcal{O}^{\mathfrak{p}}$.

Note, that not all the examples of branching lead to the generalized Verma modules. For example, consider the embedding $\mathfrak{a}_1 = B_4 \subset \mathfrak{g} = B_6$. As we mentioned in [7] the orthogonal subalgebra $\mathfrak{a}_{\perp} = B_2$. Then consider the regular subalgebra $\mathfrak{a} = A_1 \subset \mathfrak{a}_1 \subset \mathfrak{g}$ with the root system spanned over the first long root of $\mathfrak{a}_1$. Its orthogonal subalgebra in $\mathfrak{g}$ contains the root systems of two algebras B_2. But those two root systems cannot be built on the simple roots of $\mathfrak{g}$.

5 Generalizations

1. If $\mathfrak{a}$ is a special subalgebra of $\mathfrak{g}$, the equation (15) and the Lemma 1 no longer holds since the module of the algebra can not be presented as the composition of the modules of the subalgebra. Only the projection of this module can be decomposed. So we have to rely on the Lemma 2, which holds in this case too [7].

   Again as in the Corollary 0.2 we can multiply the equation (21) by the (projection of) the denominator $\frac{1}{\tilde{R}}$. This projection if well-defined due to the construction of $\mathfrak{a}_{\perp}$.

$$\pi_{\mathfrak{a}} \left(\mathrm{ch} L_{\mathfrak{g}}^{(\mu)} \right) = \sum_{u \in U} \epsilon(u) \pi_{\mathfrak{a}} \left(\frac{1}{\tilde{R}} \right) e^{\mu_{\mathfrak{a}}(u)} \mathrm{dim} \left(L_{\widetilde{\mathfrak{a}_{\perp}}}^{\mu_{\widetilde{\mathfrak{a}_{\perp}}}(u)} \right). \tag{26}$$

   Since the subalgebra $\mathfrak{a}_{\perp}$ is regular by construction, at the right-hand side of the equation we see the generalized Verma module character projected on the root space of $\mathfrak{a}$. So we have recovered the equation (25) in the projected form.

$$\pi_{\mathfrak{a}} \left(\mathrm{ch} L_{\mathfrak{g}}^{(\mu)} \right) = \pi_{\mathfrak{a}} \left(\sum_{u \in W/W_{\mathfrak{a}_{\perp}}} \epsilon(u) \mathrm{ch} M_{\mathfrak{p}}^{(u(\mu+\rho)-\rho)} \right) \tag{27}$$

   We have also lost any trace of subalgebra $\mathfrak{a}$. But we can rewrite the left-hand side as the combination of the irreducible highest weight modules of subalgebra $\mathfrak{a}$

$$\sum_{\nu \in P_{\mathfrak{a}}^{+}} b_{\nu}^{(\mu)} \mathrm{ch} L_{\mathfrak{a}}^{(\nu)} = \sum_{u \in W/W_{\mathfrak{a}_{\perp}}} \epsilon(u) \pi_{\mathfrak{a}} \left(\mathrm{ch} M_{\mathfrak{p}}^{(u(\mu+\rho)-\rho)} \right) \tag{28}$$

Using the Bernstein-Gelfand-Gelfand resolution for the subalgebra $\mathfrak{a}$ we get the expression which links the projected characters of generalized Verma modules with the characters of Verma modules of special subalgebra:

$$\sum_{\nu\in P_{\mathfrak{a}}^{+}} b_{\nu}^{(\mu)} \sum_{w\in W_{\mathfrak{a}}} \epsilon(w)\mathrm{ch}M_{\mathfrak{a}}^{w(\mu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}} = \sum_{u\in W/W_{\mathfrak{a}_{\perp}}} \epsilon(u)\pi_{\mathfrak{a}}\left(\mathrm{ch}M_{\mathfrak{p}}^{(u(\mu+\rho)-\rho)}\right) \tag{29}$$

6 Applications

7 Conclusion

8 Acknowledgements

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