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# Recursive algorithms, branching coefficients and applications

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**Abstract**

Recurrent relations for branching coefficients in affine Lie algebras integrable highest weight modules are studied. The decomposition algorithm based on the injection fan technique is adopted to the situation where the Weyl denominator becomes singular with respect to a reductive subalgebra. We study some modifications of the injection fan technique and demonstrate that it is possible to define the "subtracted fans" that play the role similar to the original ones. Possible applications of subtracted fans in CFT models are considered.

# 1 Introduction

The branching problem for affine Lie algebras emerges in conformal field theory, for example, in the construction of modular-invariant partition functions

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[1]. Recently the problem of the conformal embeddings was considered in the paper [14].

There exist several approaches to deal with the branching coefficients. Some of them use the BGG resolution [2] (for Kac-Moody algebras the algorithm is described in [3],[4]), the Schure function series [5], the BRST cohomology [6], Kac-Peterson formulas [3, 7] or the combinatorial methods applied in [8].

Usually only the maximal reductive subalgebras are considered since the case of non-maximal subalgebra can be obtained using the chain of maximal injections. In this paper we find the recurrent properties for branching coefficients that generalise the relations obtained earlier (see the paper [9] and the references therein) to the case of non-maximal reductive subalgebra. The result is formulated in terms of new injection fan called "the subtracted fan". Using this new tools we formulate simple and explicit algorithm for computations of branching coefficients which is applicable to the non-maximal subalgebras of finite-dimensional and affine Lie algebras.

We demonstrate that our algorithm can be used in the study of conformal embeddings where the central charge of the conformal field theory is preserved, and the computations are simplified by taking into account some physical limitations.

The paper is organised as follows. In the subsection 1.1 we fix the notations used throughout the paper. In the Section 2 we derive the subtracted recurrent formula for anomalous branching coefficients and describe the decomposition algorithm for integrable highest weight modules of algebra $\mathfrak{g}$ with respect to a reductive subalgebra $\mathfrak{a}$ (subsection 2.2). In the Section 3 we present several examples and discuss some applications in CFT models (Section 4). We conclude the paper with a review of obtained results. Possible future developments are discussed (Section 5).

## 1.1 Notation

Consider affine Lie algebras $\mathfrak{g}$ and $\mathfrak{a}$ with the underlying finite-dimensional subalgebras $\overset{\circ}{\mathfrak{g}}$ and $\overset{\circ}{\mathfrak{a}}$ and an injection $\mathfrak{a} \longrightarrow \mathfrak{g}$ such that $\mathfrak{a}$ is a reductive subalgebra $\mathfrak{a} \subset \mathfrak{g}$ with correlated root spaces: $\mathfrak{h}^*_{\mathfrak{a}} \subset \mathfrak{h}^*_{\mathfrak{g}}$ and $\mathfrak{h}^*_{\overset{\circ}{\mathfrak{a}}} \subset \mathfrak{h}^*_{\overset{\circ}{\mathfrak{g}}}$ .

We use the following notations adopted from the paper [9].

$L^{\mu}$ ($L^{\nu}_{\mathfrak{a}}$) — the integrable module of $\mathfrak{g}$ with the highest weight $\mu$ ; (resp. integrable $\mathfrak{a}$ -module with the highest weight $\nu$ );

$r$ , ($r_{\mathfrak{a}}$) — the rank of the algebra $\mathfrak{g}$ (resp. $\mathfrak{a}$) ;

$\Delta$ ($\Delta_{\mathfrak{a}}$)— the root system; $\Delta^+$ (resp. $\Delta^+_{\mathfrak{a}}$)— the positive root system (of $\mathfrak{g}$ and $\mathfrak{a}$ respectively);

mult $(\alpha)$ $(\mathrm{mult}_{\mathfrak{a}}(\alpha))$ — the multiplicity of the root $\alpha$ in $\Delta$ (resp. in $(\Delta_{\mathfrak{a}})$);

$\overset{\circ}{\Delta}$ , $\left(\overset{\circ}{\Delta}_{\mathfrak{a}}\right)$ — the finite root system of the subalgebra $\overset{\circ}{\mathfrak{g}}$ (resp. $\overset{\circ}{\mathfrak{a}}$); $\Theta$, $(\Theta_{\mathfrak{a}})$ — the highest root of the algebra $\mathfrak{g}$ (resp. subalgebra $\mathfrak{a}$);

$\mathcal{N}^{\mu}$ , $(\mathcal{N}_{\mathfrak{a}}^{\nu})$ — the weight diagram of $L^{\mu}$ (resp. $L_{\mathfrak{a}}^{\nu}$) ;

$W$ , $(W_{\mathfrak{a}})$— the corresponding Weyl group;

$C$ , $(C_{\mathfrak{a}})$— the fundamental Weyl chamber;

$\bar{C}$, $\left(\bar{C}_{\mathfrak{a}}\right)$ — the closure of the fundamental Weyl chamber;

$\rho$ , $(\rho_{\mathfrak{a}})$ — the Weyl vector;

$\epsilon(w) := \det(w)$ ;

$\alpha_i$ , $\left(\alpha_{(\mathfrak{a})j}\right)$ — the $i$-th (resp. $j$-th) basic root for $\mathfrak{g}$ (resp. $\mathfrak{a}$); $i = 0, \ldots, r$ , $(j = 0, \ldots, r_{\mathfrak{a}})$;

$\delta$ — the imaginary root of $\mathfrak{g}$ (and of $\mathfrak{a}$ if any);

$\alpha_i^{\vee}$ , $\left(\alpha_{(\mathfrak{a})j}^{\vee}\right)$— the basic coroot for $\mathfrak{g}$ (resp. $\mathfrak{a}$) , $i = 0, \ldots, r$ ; $(j = 0, \ldots, r_{\mathfrak{a}})$;

$\overset{\circ}{\xi}$ , $\overset{\circ}{\xi}_{(\mathfrak{a})}$ — the finite (classical) part of the weight $\xi \in P$ , $\left(\text{resp. } \xi_{(\mathfrak{a})} \in P_{\mathfrak{a}}\right)$;

$\lambda = \left(\overset{\circ}{\lambda}; k; n\right)$ — the decomposition of an affine weight indicating the finite part $\overset{\circ}{\lambda}$, level $k$ and grade $n$ .

$P$ (resp. $P_{\mathfrak{a}}$) — the weight lattice;

$M$ (resp. $M_{\mathfrak{a}}$) :=

$$= \left\{ \begin{array}{c} \sum_{i=1}^{r} \mathbf{Z}\alpha_i^{\vee} \left(\text{resp. } \sum_{i=1}^{r} \mathbf{Z}\alpha_{(\mathfrak{a})i}^{\vee}\right) \text{ for untwisted algebras or } A_{2r}^{(2)}, \\ \sum_{i=1}^{r} \mathbf{Z}\alpha_i \ \left(\text{resp. } \sum_{i=1}^{r} \mathbf{Z}\alpha_{(\mathfrak{a})i}\right) \text{ for } A_r^{(u\geq 2)} \text{ and } A \neq A_{2r}^{(2)}, \end{array} \right\}; \ \Psi^{(\mu)} :=$$

$\sum_{w\in W} \epsilon(w) e^{w\circ(\mu+\rho)-\rho}$ — the singular weight element for the $\mathfrak{g}$-module $L^{\mu}$; $\Psi_{(\mathfrak{a})}^{(\nu)} :=$ $\sum_{w\in W_{\mathfrak{a}}} \epsilon(w) e^{w\circ(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}$ — the corresponding singular weight element for the $\mathfrak{a}$-module $L_{\mathfrak{a}}^{\nu}$;

$\widehat{\Psi^{(\mu)}}$ $\left(\widehat{\Psi_{(\mathfrak{a})}^{(\nu)}}\right)$ — the set of singular weights $\xi \in P$ (resp. $\in P_{\mathfrak{a}}$) for the module $L^{\mu}$ (resp. $L_{\mathfrak{a}}^{\nu}$) with the coordinates $\left(\overset{\circ}{\xi}, k, n, \epsilon(w(\xi))\right) |_{\xi = w(\xi)\circ(\mu+\rho)-\rho}$, (resp. $\left(\overset{\circ}{\xi}, k, n, \epsilon(w_a(\xi))\right) |_{\xi = w_a(\xi)\circ(\nu+\rho_a)-\rho_a}$ ), (this set is similar to $P'_{\text{nice}}(\mu)$ in [4])

$m_{\xi}^{(\mu)}$ , $\left(m_{\xi}^{(\nu)}\right)$ — the multiplicity of the weight $\xi \in P$ (resp. $\in P_{\mathfrak{a}}$) in the module $L^{\mu}$ , (resp. $\xi \in L_{\mathfrak{a}}^{\nu}$);

$ch(L^{\mu})$ (resp. $ch(L_{\mathfrak{a}}^{\nu})$)— the formal character of $L^{\mu}$ (resp. $L_{\mathfrak{a}}^{\nu}$);

$ch(L^{\mu}) = \frac{\sum_{w\in W} \epsilon(w) e^{w\circ(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^{+}} (1-e^{-\alpha})^{\mathrm{mult}(\alpha)}} = \frac{\Psi^{(\mu)}}{\Psi^{(0)}}$ — the Weyl-Kac formula.

$R := \prod_{\alpha\in\Delta^+}\left(1-e^{-\alpha}\right)^{\mathrm{mult}(\alpha)} = \Psi^{(0)}$
$\Big($resp. $R_{\mathfrak{a}} := \prod_{\alpha\in\Delta^+_{\mathfrak{a}}}\left(1-e^{-\alpha}\right)^{\mathrm{mult}_{\mathfrak{a}}(\alpha)} = \Psi^{(0)}_{\mathfrak{a}}\Big)$— the denominator.

$L^{\mu}_{\mathfrak{g}\downarrow\mathfrak{a}} = \bigoplus\limits_{\nu\in P^+_{\mathfrak{a}}} b^{(\mu)}_{\nu} L^{\nu}_{\mathfrak{a}}$ — the module decomposition with respect to $\mathfrak{a}\longrightarrow\mathfrak{g}$;

$b^{(\mu)}_{\nu}$ — the branching coefficients;

$$\sum_{\nu\in\bar{C}_{\mathfrak{a}}} b^{(\mu)}_{\nu}\Psi^{(\nu)}_{(\mathfrak{a})} = \sum_{\lambda\in P_{\mathfrak{a}}} k^{(\mu)}_{\lambda} e^{\lambda} \tag{1}$$

$k_{\lambda}$ — the anomalous branching coefficients, notice that

$$b^{(\mu)}_{\nu} = k^{(\mu)}_{\nu} \text{ for } \nu\in\bar{C}_{\mathfrak{a}} \tag{2}$$

$x_e = \frac{|\pi_{\mathfrak{a}}\Theta|^2}{|\Theta_{\mathfrak{a}}|^2}$ — the embedding index.

# 2 Recurrent relation for branching coefficients. Singularities and subtractions

Our aim is to demonstrate that despite the zeros arriving in the Weyl denominator (when it is projected to the subalgebra root space) the injection fan technique can be properly modified. The result of this modification is that the generalized recurrent relations for anomalous branching coefficients (1) is to be formulated in the following form:

$$\begin{gathered} k^{(\mu)}_{\xi} = -\frac{1}{s(\gamma_0)}\Big(\textstyle\sum_{\omega\in W_{\perp}\setminus W}\epsilon(\omega)\dim\Big(L^{\pi_{\mathfrak{a}_{\perp}}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_{\perp}}}_{\mathfrak{a}_{\perp}}\Big)\delta_{\xi-\gamma_0,\pi_{\mathfrak{a}}(\omega(\mu+\rho)-\rho)}+ \\ +\textstyle\sum_{\gamma\in\Gamma_{\mathfrak{a}\subset\mathfrak{g}}} s(\gamma+\gamma_0)\,k^{(\mu)}_{\xi+\gamma}\Big) \end{gathered} \tag{3}$$

Here $\mathfrak{a}_{\perp}$ is the subalgebra fixed by the roots of $\mathfrak{g}$ orthogonal to the root subsystem of $\mathfrak{a}$, $W_{\perp}$ is the corresponding Weyl group, $\Gamma_{\mathfrak{a}\subset\mathfrak{g}}$ is the set of weights in the expansion of the denominator $\prod_{\alpha\in\Delta^+\setminus\Delta^+_{\mathfrak{a}_{\perp}}}\left(1-e^{-\alpha}\right)^{\mathrm{mult}(\alpha)-\mathrm{mult}_{\mathfrak{a}}(\alpha)}$ and $s(\gamma)$ is the coefficient of $e^{\gamma}$ in this expansion. In the next subsection we study the situation in details and prove the validity of this relation.

In the subsection 2.2 we shall describe the computational algorithm for branching coefficients based on this formula and present some examples.

## 2.1 Proof of the recurrent relation

Consider the branching of a module $L^{\mu}_{\mathfrak{g}}$ in terms of formal characters and projection operators $\pi_{\mathfrak{a}}$ that bring the weights of $\mathfrak{g}$ to the weight subspace of

$\mathfrak{a}$:
$$L^{\mu}_{\mathfrak{g}\downarrow\mathfrak{a}} = \bigoplus_{\nu\in P^{+}_{\mathfrak{a}}} b^{(\mu)}_{\nu} L^{\nu}_{\mathfrak{a}} \quad \Longrightarrow \quad \pi_{\mathfrak{a}}(chL^{\mu}_{\mathfrak{g}}) = \sum_{\nu\in P^{+}_{\mathfrak{a}}} b^{(\mu)}_{\nu} chL^{\nu}_{\mathfrak{a}} \tag{4}$$

The Weyl-Kac character formula leads to the equality

$$\pi_{\mathfrak{a}}\left(\frac{\sum_{\omega\in W}\epsilon(\omega)e^{\omega(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^{+}}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}\right) = \sum_{\nu\in P^{+}_{\mathfrak{a}}} b^{(\mu)}_{\nu}\frac{\sum_{\omega\in W_{\mathfrak{a}}}\epsilon(\omega)e^{\omega(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}}{\prod_{\beta\in\Delta^{+}_{\mathfrak{a}}}(1-e^{-\beta})^{\mathrm{mult}_{\mathfrak{a}}(\beta)}} \tag{5}$$

It is important that the projection of some of positive roots of the algebra $\mathfrak{g}$ can be equal to zero. These roots are orthogonal to the root space of the subalgebra $\mathfrak{a}$ embedded into the root space of the algebra $\mathfrak{g}$. Let's denote the subset of such roots by $\Delta^{+}_{\perp} = \left\{\alpha\in\Delta^{+}_{\mathfrak{g}} : \forall\beta\in\Delta^{+}_{\mathfrak{a}},\ \alpha\perp\beta\right\}$.

Notice that if the set $\Delta^{+}_{\perp}$ is non-empty the Weyl reflections corresponding to the positive roots of $\Delta^{+}_{\perp}$ generate a subgroup $W_{\perp}$ of the Weyl group $W$. Consider any two positive roots $\alpha,\ \beta\in\Delta^{+}_{\perp}$ and the corresponding Weyl reflections $\omega_{\alpha},\ \omega_{\beta}\in W_{\perp}$. Since roots of the subalgebra $\mathfrak{a}$ are invariant under $\omega_{\alpha},\ \omega_{\beta}$ they are also invariant under the action of $\omega_{\gamma} = \omega_{\alpha}\cdot\omega_{\beta}$. So the subgroup $W_{\perp}$ preserves the root system of the subalgebra $\mathfrak{a}$.

Thus we have obtained the root system $\Delta_{\perp}$ which is orthogonal to the root system $\Delta_{\mathfrak{a}}$ and invariant with respect to $W_{\perp}$ . This root system can be considered as the root system of a subalgebra $\mathfrak{a}_{\perp}\subset\mathfrak{g}$.

Now we are to find out when the subset $\Delta^{+}_{\perp}$ is non-empty and the subgroup $W_{\perp}$ and subalgebra $\mathfrak{a}_{\perp}$ are non-trivial.

If $\mathfrak{a}$ is a maximal regular subalgebra of $\mathfrak{g}$ then the rank of $\mathfrak{a}$ is equal to the rank of $\mathfrak{g}$ and it is clear that $\Delta^{+}_{\perp}$ is empty. On the other hand non-maximal regular embedding of $\mathfrak{a}$ into $\mathfrak{g}$ can be obtained through the chain of maximal embeddings $\mathfrak{a}\subset\mathfrak{p}_1\subset\mathfrak{p}_2\subset\cdots\subset\mathfrak{g}$. The maximal regular embeddings are constructed by the exclusion of one or two roots from the extended Dynkin diagram of the algebra. Since this process can give us non-connected Dynkin diagrams we can see which roots are orthogonal to the root space of non-maximal regular subalgebra $\mathfrak{a}$.

Consider for instance the regular embedding of $A_1\subset B_2$. The extended Dynkin diagram of $B_2$ is presented in the Figure 1. Drop the central node to describe the embedding $A_1\oplus A_1\subset B_2$. In this case we have: $\mathfrak{a}=A_1$ and $\mathfrak{a}_{\perp}=A_1$.

The simple criterion of $\Delta^{+}_{\perp}$'s non-emptiness for a regular embedding $\mathfrak{a}\subset\mathfrak{g}$ when both $\mathfrak{a}$ and $\mathfrak{g}$ are simple can be formulated as follows: if the Dynkin diagram of $\mathfrak{g}$ can be split into the disconnected diagrams of $\mathfrak{a}$ and of some subalgebras $\{\bar{\mathfrak{a}}_j\}$ then the subset $\Delta_{\perp}$ is non-empty, subalgebra $\mathfrak{a}_{\perp}$ is non-trivial and all the $\bar{\mathfrak{a}}_j$ are the subalgebras of $\mathfrak{a}_{\perp}$.

Notice that when we study the regular embedding obtained by dropping the nodes of the extended Dynkin diagram of the algebra $\mathfrak{g}$ and the subalgebra $\mathfrak{a}$ is one of the connected components, the subalgebra $\mathfrak{a}_\perp$ may be larger than the algebra generated by the remaining connected components. Consider for example the embedding of $B_2 \subset B_4$ (the Figure 5). In this case by eliminating the simple root $\alpha_2 = e_2 - e_3$ one splits the extended Dynkin diagram of $B_4$ into the diagrams of the subalgebra $\mathfrak{a} = B_2$ and that of the direct sum $A_1 \oplus A_1$. But the subalgebra $\mathfrak{a}_\perp$ is equal not to $A_1 \oplus A_1$ but to $B_2$ (the root system of $B_4$ contains not only $\alpha_2 = e_2 - e_3$ but also $e_2$).

Such effects are due to the fact that the subalgebras $\mathfrak{a}$ and $\mathfrak{a}_\perp$ must not form a direct sum in $\mathfrak{g}$. Consider the case of such a regular embedding $\mathfrak{a} \subset \mathfrak{g}$ where both algebras are simple and the diagram of the subalgebra $\mathfrak{a}_\perp$ is not a subdiagram of the extended Dynkin diagram $\mathfrak{g}$. Drop the subdiagram of $\mathfrak{a}$ and the node $\alpha'$ that connects it with all the remaining nodes of the diagram of $\mathfrak{g}$. Consider the remaining diagram. This diagram is the diagram of the algebra $\bar{\mathfrak{a}}$ of $\mathrm{rank}(\bar{\mathfrak{a}}) = \mathrm{rank}(\mathfrak{g}) - \mathrm{rank}(\mathfrak{a})$. It is clear that $\bar{\mathfrak{a}} \subset \mathfrak{a}_\perp$. So the question is whether $\mathfrak{a}_\perp$ has additional roots, which are not the roots of $\bar{\mathfrak{a}}$ but are the linear combinations of them. The answer is positive when the set of angles between the roots of $\bar{\mathfrak{a}}$ does not contain all the angles between the roots of $\mathfrak{a}$. Then by reflecting the roots of $\mathfrak{a}$ by $s_{\alpha'}$ we get the additional roots of $\mathfrak{a}_\perp$.

All the cases are listed in the table 2.1.

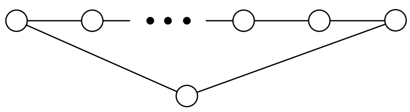
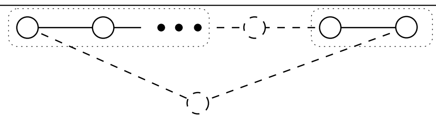
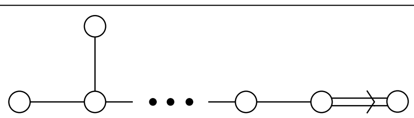
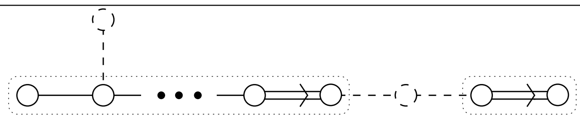
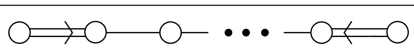
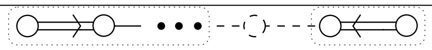
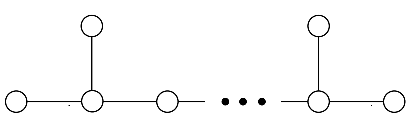
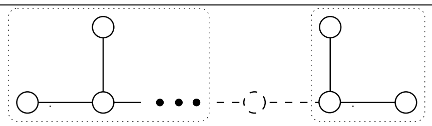
| $\mathfrak{g}$ | Extended diagram of the algebra $\mathfrak{g}$ | Diagrams of the subalgebras $\mathfrak{a}$, $\mathfrak{a}_\perp$ |
|---|---|---|
| $A_n$ |  |  |
| $B_n$ |  |  |
| $C_n$ |  |  |
| $D_n$ |  |  |

Table 1: Subalgebras $\mathfrak{a}$, $\mathfrak{a}_\perp$ for the classical series

For the algebra $\mathfrak{g}$ from the series $A_r$ the roots in the orthogonal basis $\{e_i,\ 1 \le i \le r+1\}$ are $\Delta = \{\alpha_{ij} = e_i - e_j,\ 1 \le i, j \le r+1\}$, $\Delta^+ = \{\alpha_{ij},\ i < j\}$ and the set of simple roots consists of $\alpha_{1,2}, \alpha_{2,3}, \ldots, \alpha_{r,r+1}$. So for the regular subalgebra $\mathfrak{a} = A_{r_\mathfrak{a}}$ and its simple root system consisting of first $r_\mathfrak{a}$ simple roots we get $\Delta_{\mathfrak{a}_\perp} = \{\alpha_{ij},\ r_\mathfrak{a} + 1 < i, j \leq r + 1\}$ and $\mathfrak{a}_\perp = A_{r-r_\mathfrak{a}-1}$.

For the algebra $\mathfrak{g}$ from the series $B_r$ the roots in the orthogonal basis $\{e_i,\ 1 \leq i \leq r\}$ are $\Delta = \{\alpha_{\pm i,\pm j} = \pm e_i \pm e_j,\ i < j;\ \alpha_{\pm j} = \pm e_j,\ 1 \leq j \leq r\}$, $\Delta^+ = \{\alpha_{i,-j},\ \alpha_{ij},\ \alpha_j;\ i < j,\ 1 \leq j \leq r\}$ and the set of simple roots consists of $\alpha_{1,-2}, \alpha_{2,-3}, \ldots, \alpha_{r-1,-r}, \alpha_r$. So if the regular subalgebra $\mathfrak{a} = A_{r_\mathfrak{a}}$ and its simple root system consists of first $r_\mathfrak{a}$ simple roots, then $\Delta_{\mathfrak{a}_\perp} = \{\alpha_{\pm i,\pm j},\ \alpha_j,\ r_\mathfrak{a} + 1 < i < j \leq r\}$ and $\mathfrak{a}_\perp = B_{r-r_\mathfrak{a}-1}$. Otherwise if $\mathfrak{a} = B_{r_\mathfrak{a}}$ and its simple roots are $\alpha_{r-r_\mathfrak{a}+1,-r+r_\mathfrak{a}-2}, \ldots, \alpha_{r-1,r}, \alpha_r$ we see that $\Delta_{\mathfrak{a}_\perp} = \{\alpha_{\pm i,\pm j},\ \alpha_j,\ 1 < i < j \leq r - r_\mathfrak{a}\}$ and $\mathfrak{a}_\perp = B_{r-r_\mathfrak{a}}$. This is the only case where the simple roots of $\mathfrak{a}_\perp$ can not be obtained from the extended Dynkin diagram, as can be seen in the Table 2.1. There exists the third possibility to get the pair of subalgebras $\mathfrak{a}, \mathfrak{a}_\perp$ with the regular subalgebra $\mathfrak{a}$ by dropping the single node from the extended Dynkin diagram of $B_r$. This can be done by choosing the set of simple roots of $\mathfrak{a}$ in the form $\{\alpha_{1,-2}, \alpha_{1,2}, \alpha_{2,-3}, \ldots, \alpha_{r_\mathfrak{a}-1,-r_\mathfrak{a}}\}$. Then $\mathfrak{a} = D_{r_\mathfrak{a}}$, $\Delta_{\mathfrak{a}_\perp} = \{\alpha_{\pm i,\pm j},\ \alpha_j,\ r_\mathfrak{a} < i < j \leq r\}$ and $\mathfrak{a}_\perp = B_{r-r_\mathfrak{a}}$.

For the algebra $\mathfrak{g}$ from the series $C_r$ the roots in the orthogonal basis $\{e_i,\ 1 \leq i \leq r\}$ are $\Delta = \{\alpha_{\pm i,\pm j} = \pm e_i \pm e_j,\ i < j;\ \alpha_{\pm j} = \pm 2e_j,\ 1 \leq j \leq r\}$, $\Delta^+ = \{\alpha_{i,-j},\ \alpha_{ij},\ \alpha_j;\ i < j,\ 1 \leq j \leq r\}$ and the set of simple roots consists of $\alpha_{1,-2}, \alpha_{2,-3}, \ldots, \alpha_{r-1,-r}, \alpha_r$. So if the regular subalgebra $\mathfrak{a} = A_{r_\mathfrak{a}}$ and its simple root system consists of first $r_\mathfrak{a}$ simple roots, then $\Delta_{\mathfrak{a}_\perp} = \{\alpha_{\pm i,\pm j},\ \alpha_j,\ r_\mathfrak{a} + 1 < i < j \leq r\}$ and $\mathfrak{a}_\perp = C_{r-r_\mathfrak{a}-1}$. Otherwise if $\mathfrak{a} = C_{r_\mathfrak{a}}$ and its simple roots are $\alpha_{r-r_\mathfrak{a}+1,-r+r_\mathfrak{a}-2}, \ldots, \alpha_{r-1,r}, \alpha_r$ we see that $\Delta_{\mathfrak{a}_\perp} = \{\alpha_{\pm i,\pm j},\ \alpha_j,\ 1 < i < j \leq r - r_\mathfrak{a}\}$ and $\mathfrak{a}_\perp = C_{r-r_\mathfrak{a}}$.

For the algebra $\mathfrak{g}$ from the series $D_r$ the roots in the orthogonal basis $\{e_i,\ 1 \leq i \leq r\}$ are $\Delta = \{\alpha_{\pm i,\pm j} = \pm e_i \pm e_j,\ 1 \leq i < j \leq r\}$, $\Delta^+ = \{\alpha_{i,-j},\ \alpha_{ij},\ i < j,\ 1 \leq j \leq r\}$ and the set of simple roots consists of $\alpha_{1,-2}, \alpha_{2,-3}, \ldots, \alpha_{r-1,-r}, \alpha_{r-1,r}$. So if the regular subalgebra $\mathfrak{a} = A_{r_\mathfrak{a}}$ and its simple root system consists of first $r_\mathfrak{a}$ simple roots, then $\Delta_{\mathfrak{a}_\perp} = \{\alpha_{\pm i,\pm j},\ r_\mathfrak{a} + 1 < i < j \leq r\}$ and $\mathfrak{a}_\perp = D_{r-r_\mathfrak{a}-1}$. Otherwise if $\mathfrak{a} = D_{r_\mathfrak{a}}$ and its simple roots are $\alpha_{r-r_\mathfrak{a}+1,-r+r_\mathfrak{a}-2}, \ldots, \alpha_{r-1,r}, \alpha_{r-1,r}$ we see that $\Delta_{\mathfrak{a}_\perp} = \{\alpha_{\pm i,\pm j},\ 1 < i < j \leq r - r_\mathfrak{a}\}$ and $\mathfrak{a}_\perp = D_{r-r_\mathfrak{a}}$.

In the case of special embeddings the set $\Delta_\perp^+$ can be empty as for the special embedding of $A_1 \subset A_2$ with the embedding index equal to 4, or non-empty for example for the embedding $A_1 \subset A_2 \subset A_3$ which is depicted at the Figure 2.1.

Using the existing classification of maximal special subalgebras [10] we

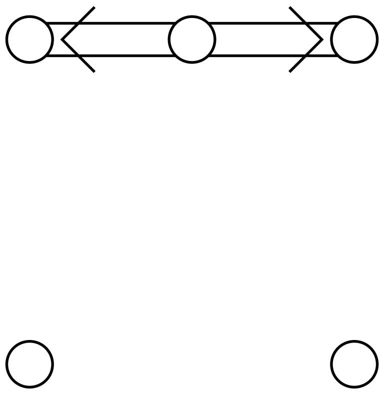


Figure 1: Extended Dynkin diagram of $B_2$ and embedding of $A_1$

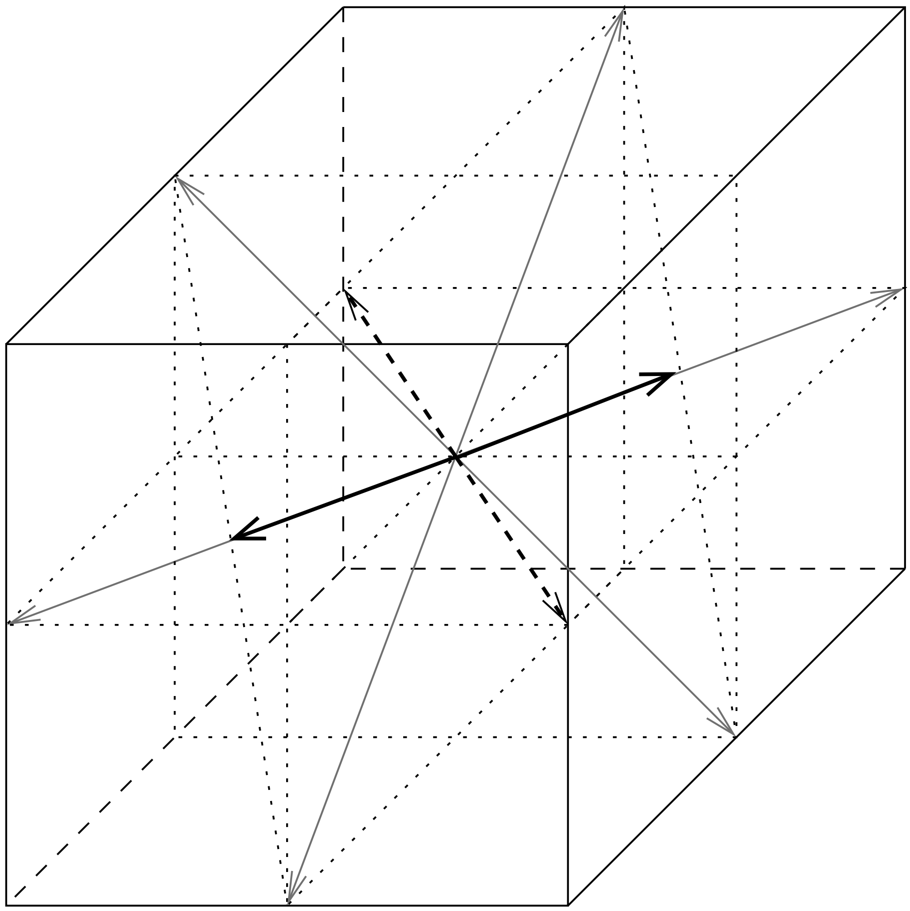


Figure 2: Special embedding $A_1 \subset A_2 \subset A_3$. Grey vectors are the roots of $A_2$, thick black - of $\mathfrak{a} = A_1$, dashed black are the orthogonal roots of $A_1$ which is contained in $\mathfrak{a}_{\perp}$

immediately have the following pairs of the orthogonal subalgebras $\mathfrak{a}$, $\mathfrak{a}_\perp$

$$\begin{array}{lll} su(p)\oplus su(q) & \subset su(pq) & \\ so(p)\oplus so(q) & \subset so(pq) & \\ sp(2p)\oplus sp(2q) & \subset so(4pq) & \\ sp(2p)\oplus so(q) & \subset sp(2pq) & \\ so(p)\oplus so(q) & \subset so(p+q) & \text{for } p \text{ and } q \text{ odd} \end{array} \tag{6}$$

Exceptional Lie algebras and other non-maximal subalgebras will be considered elsewhere.

Up to this point we considered the problem of constructing the subalgebra $\mathfrak{a}_\perp$ for a given regular injection $\mathfrak{a} \in \mathfrak{g}$ in terms of Dynkin diagrams. When the root systems $\Delta$ and $\Delta_\mathfrak{a}$ are known explicitly all that we need is to select the roots $\Delta_\perp = \{\alpha \in \Delta : \alpha \perp \Delta_\mathfrak{a}\}$ and correspondingly the positive roots $\Delta_\perp^+ = \{\alpha \in \Delta^+ : \alpha \perp \Delta_\mathfrak{a}\}$.

Now consider the direct sum $\mathfrak{a}_\perp \oplus \mathfrak{h}_\mathfrak{a}$ and its module $L^\mu_{\mathfrak{a}_\perp\oplus\mathfrak{h}}$ with the highest weight $\mu$. The character $chL^\mu_{\mathfrak{a}_\perp\oplus\mathfrak{h}}$ can be written as

$$chL^\mu_{\mathfrak{a}_\perp\oplus\mathfrak{h}} = \frac{\sum_{\omega\in W_\perp}\epsilon(\omega)e^{\omega(\mu+\rho_{\mathfrak{a}_\perp})-\rho_{\mathfrak{a}_\perp}}}{\prod_{\alpha\in\Delta_\perp^+}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}. \tag{7}$$

Its projection $\pi_\mathfrak{a}(ch\, L^\mu_{\mathfrak{a}_\perp\oplus\mathfrak{h}})$ is the single element $e^{\pi_\mathfrak{a}\cdot\mu}$ of the formal algebra $\mathcal{E}\,(\mathfrak{a})$ with the multiplicity equal to the dimension of the module $L^\mu_{\mathfrak{a}_\perp\oplus\mathfrak{h}}$, since all the roots of $\mathfrak{a}_\perp$ are orthogonal to that of $\Delta_\mathfrak{a}$.

Using this property we can consider the restriction $ch\, L^\mu_{\mathfrak{g}\downarrow\mathfrak{a}_\perp\oplus\mathfrak{h}}$, that is the character of the direct sum of $(\mathfrak{a}_\perp\oplus\mathfrak{h})$-modules. Multiply the equation (5) by the element

$$\pi_\mathfrak{a}\left(\prod_{\alpha\in\Delta^+\setminus\Delta_\perp^+}(1-e^{-\alpha})^{\mathrm{mult}_\mathfrak{g}(\alpha)}\right) \tag{8}$$

Taking into account that the projection commutes with the multiplication,

$$\pi_\mathfrak{a}(Q)\pi_\mathfrak{a}(1-e^{-\alpha}) = \pi_\mathfrak{a}\left(Q\cdot(1-e^{-\alpha})\right), \tag{9}$$

we can rewrite the product of (5) and (8):

$$\pi_\mathfrak{a}\left(\frac{\sum_{\omega\in W}\epsilon(\omega)e^{\omega(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta_\perp^+}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}\right) = \\ \pi_\mathfrak{a}\left(\prod_{\alpha\in\Delta^+\setminus\Delta_\perp^+}(1-e^{-\alpha})^{\mathrm{mult}_\mathfrak{g}(\alpha)}\right)\sum_{\nu\in P_\mathfrak{a}^+} b_\nu^{(\mu)}\frac{\sum_{\omega\in W_\mathfrak{a}}\epsilon(\omega)e^{\omega(\nu+\rho_\mathfrak{a})-\rho_\mathfrak{a}}}{\prod_{\beta\in\Delta_\mathfrak{a}^+}(1-e^{-\beta})^{\mathrm{mult}_\mathfrak{a}(\beta)}}. \tag{10}$$

The right-hand side of this equation can be reorganised similarly to what was performed in the paper [9], by introducing the anomalous branching coefficients $k_\lambda$,

$$\sum_{\nu\in P_{\mathfrak{a}}} b_\nu^{(\mu)}\Psi_{(\mathfrak{a})}^{(\nu)} = \sum_{\lambda\in P_{\mathfrak{a}}} k_\lambda^{(\mu)} e^\lambda \tag{11}$$

and simplifying the multiplier:

$$\pi_{\mathfrak{a}}\left(\frac{\sum_{\omega\in W}\epsilon(\omega)e^{\omega(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta_\perp^+}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}\right) = \left(\prod_{\alpha\in\pi_{\mathfrak{a}}\left(\Delta^+\setminus\Delta_\perp^+\right)}(1-e^{-\alpha})^{\mathrm{mult}_{\mathfrak{g}}(\alpha)-\mathrm{mult}_{\mathfrak{a}}(\alpha)}\right)\sum_{\lambda\in P_{\mathfrak{a}}} k_\lambda^{(\mu)} e^\lambda \tag{12}$$

If the set $\Delta_\perp^+$ is non-empty then the Weyl reflections corresponding to the positive roots of $\Delta_\perp^+$ generate a subgroup $W_\perp$ of the Weyl group $W$. Let us reorganise the summation in the left-hand side of (12). Consider the factor-space $W_\perp\setminus W$. For the class $\tilde{\omega}\in W_\perp\setminus W$ choose the representative $\omega\in\tilde{\omega}$ such that $\pi_{\mathfrak{a}_\perp}\omega(\mu+\rho)\in\bar{C}_{\mathfrak{a}_\perp}$,

$$\pi_{\mathfrak{a}}\left(\frac{\sum_{\omega\in W}\epsilon(\omega)e^{\omega(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta_\perp^+}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}\right) = \pi_{\mathfrak{a}}\left(\sum_{\omega\in W_\perp\setminus W}\epsilon(\omega)\frac{\sum_{\nu\in W_\perp}\epsilon(\nu)e^{\nu\cdot\omega(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta_\perp^+}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}\right) \tag{13}$$

The fraction in the right-hand side of the equation is similar to the character of some $\mathfrak{a}_\perp$-module. Let us rewrite the shifted weights

$$\nu\cdot\omega(\mu+\rho)-\rho = \nu\cdot\big(\omega(\mu+\rho)-\pi_{\mathfrak{a}}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}+\rho_{\mathfrak{a}_\perp}+\pi_{\mathfrak{a}}(\omega(\mu+\rho))\big)-\rho \tag{14}$$

Since $\nu\cdot\pi_{\mathfrak{a}}(\omega(\mu+\rho)) = \pi_{\mathfrak{a}}(\omega(\mu+\rho))$ and $\omega(\mu+\rho)-\pi_{\mathfrak{a}}(\omega(\mu+\rho)) = \pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))$, we get

$$\begin{aligned}
&\sum_{\omega\in W_\perp\setminus W}\epsilon(\omega)\frac{\sum_{\nu\in W_\perp}\epsilon(\nu)e^{\nu\cdot\omega(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta_\perp^+}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}} = \\
&\sum_{\omega\in W_\perp\setminus W}\epsilon(\omega)e^{\pi_{\mathfrak{a}}(\omega(\mu+\rho))-\rho}\frac{e^{\rho_{\mathfrak{a}_\perp}}\sum_{\nu\in W_\perp}\epsilon(\nu)e^{\nu\cdot(\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}+\rho_{\mathfrak{a}_\perp})-\rho_{\mathfrak{a}_\perp}}}{\prod_{\alpha\in\Delta_\perp^+}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}} = \\
&\sum_{\omega\in W_\perp\setminus W}\epsilon(\omega)e^{\pi_{\mathfrak{a}}(\omega(\mu+\rho))-\rho}e^{\rho_{\mathfrak{a}_\perp}}\mathrm{ch}L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}
\end{aligned} \tag{15}$$

The projector $\pi_{\mathfrak{a}}$ transforms the character $\mathrm{ch}L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}$ into the unit element of $\mathcal{E}$ multiplied by $\dim L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}$ :

$$\pi_{\mathfrak{a}}\left(\sum_{\omega\in W_\perp\setminus W}\epsilon(\omega)e^{\pi_{\mathfrak{a}}(\omega(\mu+\rho))-\rho}e^{\rho_{\mathfrak{a}_\perp}}\mathrm{ch}L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}\right)= \sum_{\omega\in W_\perp\setminus W}\epsilon(\omega)\dim\left(L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}\right)e^{\pi_{\mathfrak{a}}(\omega(\mu+\rho)-\rho)} \quad (16)$$

Thus we obtained the relation

$$\sum_{\omega\in W_\perp\setminus W}\epsilon(\omega)\mathrm{dim}\left(L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}\right)e^{\pi_{\mathfrak{a}}(\omega(\mu+\rho)-\rho)}= \left(\prod_{\alpha\in\pi_{\mathfrak{a}}\left(\Delta^+\setminus\Delta_\perp^+\right)}(1-e^{-\alpha})^{\mathrm{mult}_{\mathfrak{g}}(\alpha)-\mathrm{mult}_\alpha}\right)\sum_{\lambda\in P_{\mathfrak{a}}}k_\lambda^{(\mu)}e^\lambda. \quad (17)$$

Let us rewrite the multiplier in the right-hand side:

$$\prod_{\alpha\in\pi_{\mathfrak{a}}\circ(\Delta^+\setminus\Delta_\perp^+)}\left(1-e^{-\alpha}\right)^{\mathrm{mult}(\alpha)-\mathrm{mult}_{\mathfrak{a}}(\alpha)}=-\sum_{\gamma\in P_{\mathfrak{a}}}s(\gamma)e^{-\gamma} \quad (18)$$

For the coefficient function $s(\gamma)$ define the carrier $\Phi_{\mathfrak{a}\subset\mathfrak{g}}\subset P_{\mathfrak{a}}$:

$$\Phi_{\mathfrak{a}\subset\mathfrak{g}}=\{\gamma\in P_{\mathfrak{a}}\mid s(\gamma)\neq 0\}. \quad (19)$$

From the obtained equation for the formal elements,

$$\begin{aligned}\sum_{\omega\in W_\perp\setminus W}\epsilon(\omega)\mathrm{dim}\left(L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}\right)e^{\pi_{\mathfrak{a}}(\omega(\mu+\rho)-\rho)}&= \\ =-\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}}s(\gamma)\,e^{-\gamma}\sum_{\lambda\in P_{\mathfrak{a}}}k_\lambda^{(\mu)}e^\lambda& \\ =-\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}}\sum_{\lambda\in P_{\mathfrak{a}}}s(\gamma)\,k_\lambda^{(\mu)}e^{\lambda-\gamma}&\end{aligned} \quad (20)$$

we can deduce the following property of the anomalous branching coefficients,

$$\sum_{\omega\in W_\perp\setminus W}\epsilon(\omega)\mathrm{dim}\left(L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}\right)\delta_{\xi,\pi_{\mathfrak{a}}(\omega(\mu+\rho)-\rho)}+\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}}s(\gamma)\,k_{\xi+\gamma}^{(\mu)}=0;\quad \xi\in P_{\mathfrak{a}}. \quad (21)$$

To get the recurrent relations for the coefficients $k_{\xi+\gamma}^{(\mu)}$ we use the following procedure (similar to that in [9]). Let $\gamma_0$ be the lowest vector with respect to the natural ordering in $\overset{\circ}{\Delta}_{\mathfrak{a}}$ in the lowest grade of $\Phi_{\mathfrak{a}\subset\mathfrak{g}}$ and decompose the defining relation (18),

$$\prod_{\alpha\in\pi_{\mathfrak{a}}\circ(\Delta^{+}\setminus\Delta_{\perp}^{+})} \left(1-e^{-\alpha}\right)^{\mathrm{mult}(\alpha)-\mathrm{mult}_{\mathfrak{a}}(\alpha)} = -s\left(\gamma_0\right)e^{-\gamma_0} - \sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}\setminus\{\gamma_0\}} s\left(\gamma\right)e^{-\gamma}, \tag{22}$$

then the equality (21) leads to the desired recurrent relation for the anomalous branching coefficients:

$$k_{\xi}^{(\mu)} = -\frac{1}{s\left(\gamma_0\right)}\left(\sum_{\omega\in W_{\perp}\setminus W} \epsilon(\omega)\,\dim\left(L_{\mathfrak{a}_{\perp}}^{\pi_{\mathfrak{a}_{\perp}}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_{\perp}}}\right)\delta_{\xi-\gamma_0,\pi_{\mathfrak{a}}(\omega(\mu+\rho)-\rho)} + \right.$$

$$\left. + \sum_{\gamma\in\Gamma_{\mathfrak{a}\subset\mathfrak{g}}} s\left(\gamma+\gamma_0\right)k_{\xi+\gamma}^{(\mu)}\right) \tag{23}$$

where the set

$$\Gamma_{\mathfrak{a}\subset\mathfrak{g}} = \left\{\xi-\gamma_0|\xi\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}\right\}\setminus\{0\} \tag{24}$$

was introduced that is called the injection fan.

Now consider the case $\Delta_{\perp}^{+}=0$. There are three different reasons for $\Delta_{\perp}^{+}$ to be empty: i) $\dim\mathfrak{h}_{\mathfrak{a}}=\dim\mathfrak{h}_{\mathfrak{g}}$, ii) $\mathfrak{a}_{\perp}=0$ and iii) $\mathfrak{a}_{\perp}\subset\mathfrak{h}_{\mathfrak{g}}$. Both the first and the second cases can be treated as corresponding to the trivial orthogonal subalgebra: $\mathfrak{a}_{\perp}=0$. In any of these cases instead of the formal characters in the right-hand side of (13) we obtain the formal element $e^{\pi_{\mathfrak{a}_{\perp}}\omega(\mu+\rho)}$. In the first two cases (equivalent to $\mathfrak{a}_{\perp}=0$) the projection operator retains its purely geometrical meaning: the vector $\omega(\mu+\rho)$ is projected to the subspace orthogonal to the weight space of $\mathfrak{a}$. It is clear that in any of the three variants the final vector $\pi_{\mathfrak{a}}\pi_{\mathfrak{a}_{\perp}}\omega(\mu+\rho)$ leads to the unit of the formal algebra $\mathcal{E}$. Thus when the set $\Delta_{\perp}^{+}$ is empty the recurrent relation is simplified:

$$k_{\xi}^{(\mu)} = -\frac{1}{s\left(\gamma_0\right)}\left(\sum_{w\in W}\epsilon\left(w\right)\delta_{\xi,\pi_{\mathfrak{a}}\circ(w\circ(\mu+\rho)-\rho)+\gamma_0} + \sum_{\gamma\in\Gamma_{\mathfrak{a}\subset\mathfrak{g}}} s\left(\gamma+\gamma_0\right)k_{\xi+\gamma}^{(\mu)}\right), \tag{25}$$

the latter coinsides with the one obtained in [9] (formula (16)).

In the next section we describe a computation algorithm for branching coefficients based on the relation (23).

## 2.2 Algorithm for the recursive computation of the branching coefficients

The recurrent relation (23) allows us to formulate an algorithm for recursive computation of the branching coefficients. In this algorithm there is no need to construct the module $L_{\mathfrak{g}}^{(\mu)}$ or any of the modules $L_{\mathfrak{a}}^{(\nu)}$.

It contains the following steps:

1. Construct the sets $\Delta^+$ and $\Delta_{\mathfrak{a}}^+$ of positive roots for the algebras $\mathfrak{a} \subset \mathfrak{g}$.

2. Select the positive roots $\alpha \in \Delta^+$ which are orthogonal to the root subspace of $\mathfrak{a}$ and form the set $\Delta_{\perp}^+$.

3. Construct the set $\Gamma$ (24).

4. Construct the set $\widehat{\Psi^{(\mu)}} = \{\omega(\mu+\rho)-\rho;\ \omega \in W\}$ of the anomalous weights of the $\mathfrak{g}$-module $L^{(\mu)}$.

5. Select the weights $\left\{\lambda = \omega(\mu+\rho) | \pi_{\mathfrak{a}_\perp}\lambda \in \bar{C}_{\mathfrak{a}_\perp}\right\}$ Since we have constructed the set $\Delta_{\perp}^+$ we can easily check wether the weight $\pi_{\mathfrak{a}_\perp}\lambda$ lies in the main Weyl chamber of $\mathfrak{a}_\perp$ by computing the scalar product of $\lambda$ with the roots of $\Delta_\perp^+$, it must be non-negative.

6. For $\lambda = \omega(\mu+\rho),\ \pi_{\mathfrak{a}_\perp}\lambda \in \bar{C}_{\mathfrak{a}_\perp}$ calculate the dimensions of the corresponding modules $\dim\left(L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}\right)$ using the Weyl formula with the set $\Delta_\perp^+$.

7. Calculate the anomalous branching coefficients in the main Weyl chamber of the subalgebra $\mathfrak{a}$ using the recurrent relation (23).

If we are interested in the branching coefficients for the embedding of the finite-dimensional Lie algebra into the affine Lie algebra we can construct the set of the anomalous weights up to the required grade and use the steps 4-7 of the algorithm for each grade. We can also speed up the algorithm by one-time computation of the representatives of the conjugate classes $W_\perp \backslash W$.

The next section contains several examples computed using this algorithm.

# 3 Examples

## 3.1 Finite dimensional Lie algebras

### 3.1.1 Regular embedding of $A_1$ into $B_2$

Consider the regular embedding of $A_1$ into $B_2$. Simple roots $\alpha_1, \alpha_2$ of $B_2$ are drawn as the dashed vectors at the Figure 3. We denote the corresponding Weyl reflections by $\omega_1, \omega_2$. Simple root $\beta$ of the embedded $A_1$ is equal to $\alpha_1 + 2\alpha_2$ and is drawn as the grey vector.

Let's describe the reduction of the fundamental representation of $B_2$ with the highest weight equal to $(1, 0)$ (in the fundamental weight basis), it is drawn as the black vector in the Figure 3. There we have also shown the

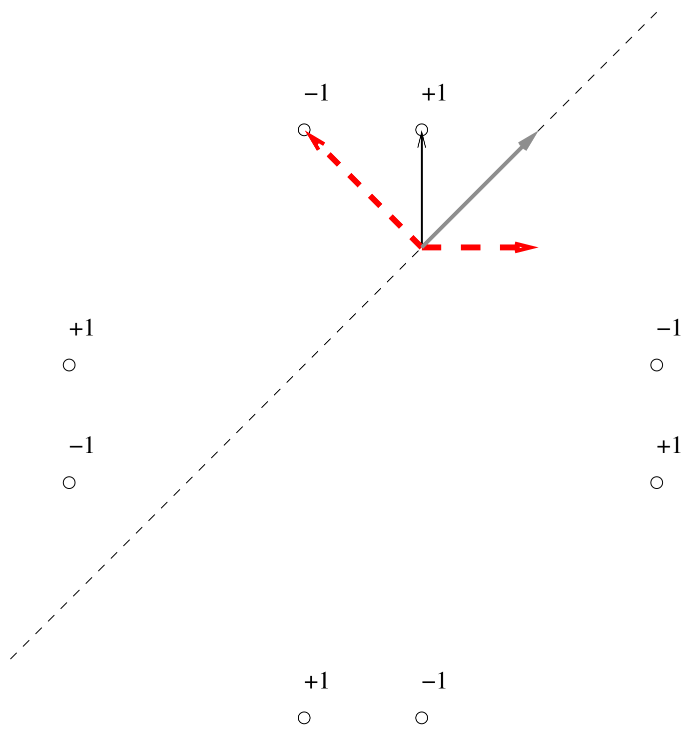



Figure 3: Regular embedding of $A_1$ into $B_2$

set of weights $\omega(\mu + \rho) - \rho$, $\omega \in W$ of the fundamental representation of $B_2$ with the corresponding determinants $\epsilon(\omega)$ of Weyl transformations. Now we have to factorise the Weyl group $W$ by $W_\perp = \{\omega_1\}$. We get the following set of anomalous weights $\omega(\mu + \rho) - \rho$, $\omega \in W_\perp \backslash W$: We have also depicted the corresponding $\mathfrak{a}_\perp = A_1$-modules $L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho)) - \rho_{\mathfrak{a}_\perp}}$. Then we project these weights and dimensions of modules onto the root space of subalgebra $\mathfrak{a} = A_1$

and get the following anomalous weights in fundamental weights basis with corresponding multiplicities:

$$(1,2),\ (0,-3),\ (-4,3),\ (-5,-2) \tag{26}$$

For the function $s(\gamma)$ and the set $\Gamma$ (using the definition (19,24)) we have

$$(1,2),\ (2,-1) \tag{27}$$

Here the second component denotes the value of $s(\gamma)$.

As far as the anomalous branching coefficient $k_1^{(1,0)} = 2$ for the coefficient $k_0^{(1,0)}$ the formula (23) gives the value

$$k_0^{(1,0)} = -1 \cdot k_2^{(1,0)} + 2 \cdot k_1^{(1,0)} - 3 \cdot \delta_{0,0} = 1. \tag{28}$$

The recurrence property defines the branching.

### 3.1.2 Embedding of $B_2$ into $B_4$

Consider the regular embedding $B_2 \longrightarrow B_4$. We calculate the branching coefficients for the fundamental vector representation of $B_4$. The corresponding Dynkin diagrams are in the Figure 5.

In the orthogonal basis $e_1, \ldots, e_4$ the simple roots of $B_4$ are

$$(e_1 - e_2,\ e_2 - e_3,\ e_3 - e_4,\ e_4) \tag{29}$$

The positive roots are

$$\begin{aligned}(e_1 - e_2,\ e_2 - e_3,\ e_3 - e_4,\ e_4,\ e_1 - e_3,\ e_2 - e_4,\ e_3 + e_4,\ e_3,\ e_1 - e_4,\\ e_2 + e_4,\ e_2,\ e_1 + e_4,\ e_2 + e_3,\ e_1,\ e_1 + e_3,\ e_1 + e_2)\end{aligned} \tag{30}$$

The simple roots of the embedded subalgebra $\mathfrak{a} = B_2$ are

$$(e_3 - e_4, e_4) \tag{31}$$

The set $\Delta_\perp^+$ contains the roots

$$\{e_1 - e_2, e_1 + e_2, e_1, e_2\} \tag{32}$$

and is the set of positive roots for the algebra $\mathfrak{a}_\perp = B_2$.

Firstly we construct the fan for this injection. Using the definition (24) we obtain the set $\Gamma$ with the corresponding values $s(\gamma + \gamma_0)$, depicted at the Figure 7.

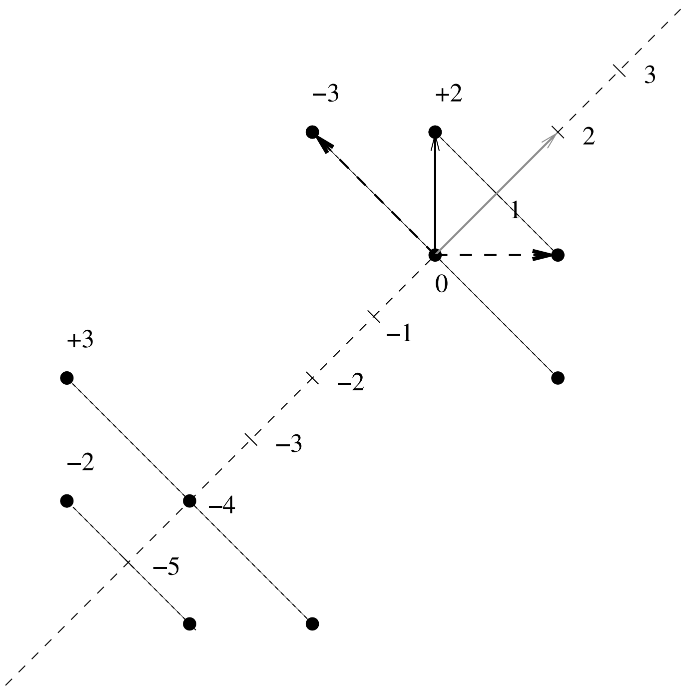



Figure 4: Anomalous weights and the corresponding $\mathfrak{a}_\perp = A_1$-modules

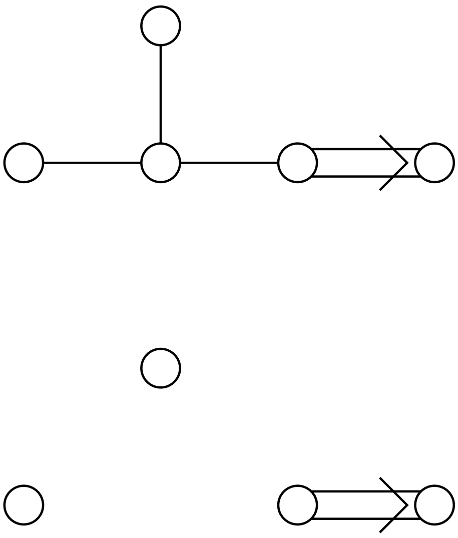


Figure 5: Dynkin diagrams

To find the branching coefficients we need to compute the anomalous weights of $B_4$, select the weights belonging to $C^{(0)}_{\mathfrak{a}_\perp}$ and compute the dimensions of the corresponding $\mathfrak{a}_\perp$-modules.

Consider the $B_4$-module $L^\mu$ with the highest weight $\mu = (0,1,0,2) = 2e_1 + 2e_2 + e_3 + e_4$; $dim(L^{(0,1,0,2)}) = 2772$.

The set of the anomalous weights $\{\omega(\mu+\rho) - \rho,\ \omega \in W\}$ contains of 384 vectors.

We need to select the weights $\psi \in \omega(\mu+\rho)$ with the property $\pi_{\mathfrak{a}_\perp}(\psi) \in C^{(0)}_{\mathfrak{a}_\perp}$. It means that the scalar product of these weights with all the roots in $\Delta^+_\perp$ is non-negative.

To compute the dimensions of the corresponding $\mathfrak{a}_\perp$-modules we need to project each selected weight onto the root space $\Delta^+_\perp$, substract $\rho_{\mathfrak{a}_\perp}$ and apply the Weyl dimension formula.

The result of this procedure is shown in the Figure 6.

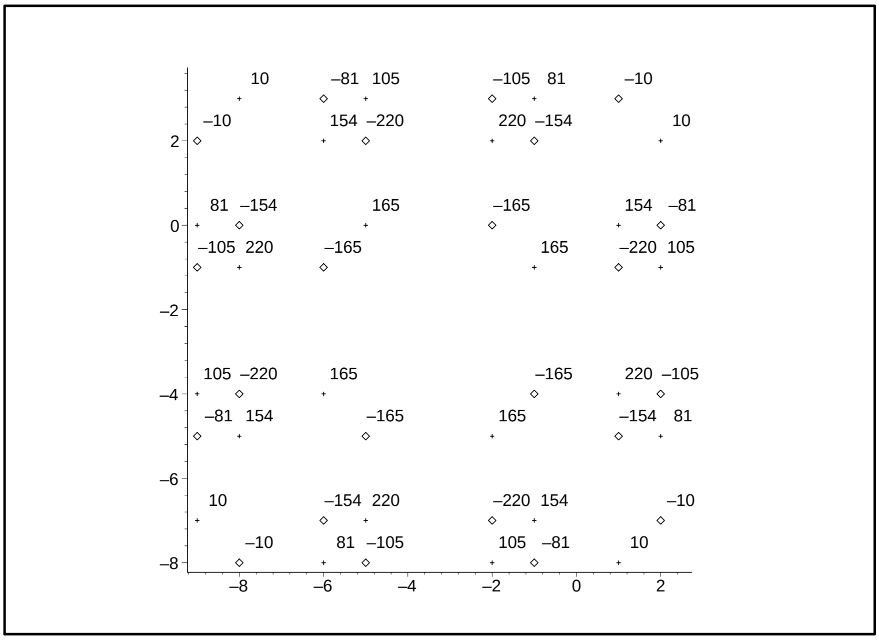



Figure 6: Anomalous weights with the dimensions of the corresponding $\mathfrak{a}_\perp$-modules.

Applying the recurrent relation (23) we obtain the following branching

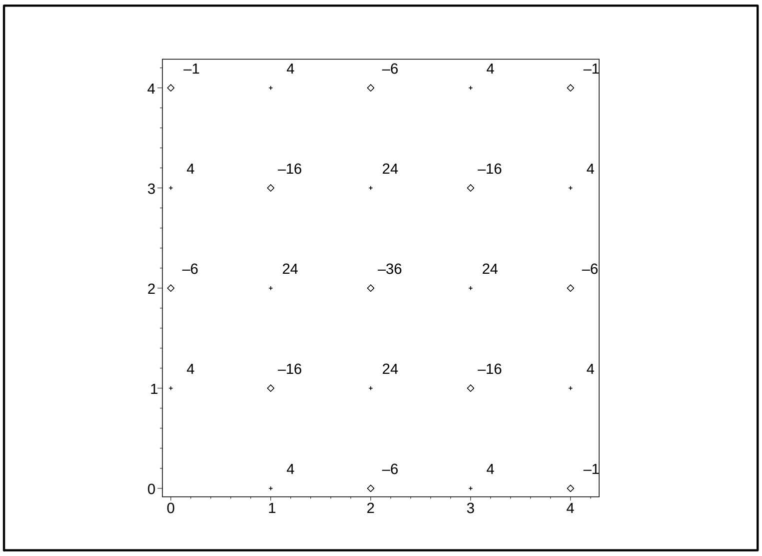



Figure 7: Fan for $B_2 \subset B_4$

coefficients:

$$\pi_{\mathfrak{a}}\left(chL_{B_4}^{(0,1,0,2)}\right) = 6\ chL_{B_2}^{(0,0)} + 60\ chL_{B_2}^{(0,2)} + 30\ chL_{B_2}^{(1,0)} + 19\ chL_{B_2}^{(2,0)} + 40\ chL_{B_2}^{(1,2)} + 10\ chL_{B_2}^{(2,2)}. \tag{33}$$

## 3.2 Affine Lie algebras

### 3.2.1 Affine into affine embedding

Consider the affine extension of the example 3.1.1. Since this embedding is regular, the level of the subalgebra modules is equal to that of the initial module.

The set $\Delta_{\perp}^{+}$ of the orthogonal positive roots with the zero projection on the root space of the subalgebra $\hat{A}_1$ is the same as in the finite-dimensional case.

Using the definition (24) we get the fan $\Gamma_{\hat{A}_1 \longrightarrow \hat{B}_2}$ with the corresponding values $s(\gamma + \gamma_0)$(see the Figure 8). Here we restricted the computation to the twelfth grade.

Consider the level one module $L_{\hat{B}_2}^{(1,0;1;0)}$ with the highest weight $w_1 = (1, 0; 1; 0)$, where the first two components are the coordinates of the classical part in the orthogonal basis $e_1, e_2$, the third is the level of the weight and the fourth is the grade.

The set of the anomalous weights for this module up to the sixth grade is depicted in the Figure 9 and in each grade its form is similar to that in the Figure 3.

Performing the next step of the algorithm 2.2 we project the anomalous weights to the weight space of the subalgebra $\hat{A}_1$ and find the dimensions of the corresponding $\mathfrak{a}_\perp$-modules $L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}$. The result of this computation up to the twelfth grade is presented in the Figure 10.

Notice that here the lowest weight $\gamma_0$ of the fan is zero, since we have excluded all the roots of $\Delta_\perp^+$ from the defining relation (24).

Selecting the elements inside the Weyl chamber $\bar{C}_{\hat{A}_1}^{(0)}$ we get the following results for the branching coefficients up to twelfth grade

$$\begin{aligned} L^{w_1}_{\hat{B}_2\downarrow\hat{A}_1} = {} & 2L^{w_1}_{\hat{A}_1}(0) \oplus 1L^{w_0}_{\hat{A}_1}(0) \oplus 4L^{w_0}_{\hat{A}_1}(-1)\oplus \\ & 2L^{w_1}_{\hat{A}_1}(-1) \oplus 8L^{w_0}_{\hat{A}_1}(-2) \oplus 8L^{w_1}_{\hat{A}_1}(-2) \oplus 15L^{w_0}_{\hat{A}_1}(-3)\oplus \\ & 12L^{w_1}_{\hat{A}_1}(-3) \oplus 26L^{w_1}_{\hat{A}_1}(-4) \oplus 29L^{w_0}_{\hat{A}_1}(-4) \oplus 51L^{w_0}_{\hat{A}_1}(-5)\oplus \\ & 42L^{w_1}_{\hat{A}_1}(-5) \oplus 78L^{w_1}_{\hat{A}_1}(-6) \oplus 85L^{w_0}_{\hat{A}_1}(-6) \oplus 120L^{w_1}_{\hat{A}_1}(-7)\oplus \\ & 139L^{w_0}_{\hat{A}_1}(-7) \oplus 202L^{w_1}_{\hat{A}_1}(-8) \oplus 222L^{w_0}_{\hat{A}_1}(-8) \oplus 306L^{w_1}_{\hat{A}_1}(-9)\oplus \\ & 346L^{w_0}_{\hat{A}_1}(-9) \oplus 530L^{w_0}_{\hat{A}_1}(-10) \oplus 482L^{w_1}_{\hat{A}_1}(-10) \oplus 714L^{w_1}_{\hat{A}_1}(-11)\oplus \\ & 797L^{w_0}_{\hat{A}_1}(-11) \oplus 1080L^{w_1}_{\hat{A}_1}(-12) \oplus 1180L^{w_0}_{\hat{A}_1}(-12) \end{aligned} \tag{34}$$

This result can be presented as the set of branching functions [3].

$$\begin{aligned} b_0^{(w_1)} = {} & 1 + 4\,q^1 + 8\,q^2 + 15\,q^3 + 29\,q^4 + 51\,q^5 + 85\,q^6 + 139\,q^7 + \\ & 222\,q^8 + 346\,q^9 + 530\,q^{10} + 797\,q^{11} + 1180\,q^{12} + \dots \end{aligned} \tag{35}$$

$$\begin{aligned} b_1^{(w_1)} = {} & 2 + 2\,q^1 + 8\,q^2 + 12\,q^3 + 26\,q^4 + 42\,q^5 + 78\,q^6 + 120\,q^7 + \\ & 202\,q^8 + 306\,q^9 + 482\,q^{10} + 714\,q^{11} + 1080\,q^{12} + \dots \end{aligned} \tag{36}$$

Here the lower index enumerates the branching function according to their highest weights in $P^+_{\hat{A}_1}$, these are the fundamental weights $w_0 = \lambda_0 = (0,1,0)$, $w_1 = \alpha/2 = (1,1,0)$.

# 4 Applications to the conformal field theory

Branching coefficients for an embedding of affine Lie subalgebra into affine Lie algebra can be used to construct modular invariant partition functions for Wess-Zumino-Novikov-Witten models of conformal field theory ([1], [11], [12], [13]). In these models currents algebras are affine Lie algebras. For the construction to be valid the embedding is required to be conformal, which means that the central charge of the subalgebra is equal to the central charge of the algebra:

$$c(\mathfrak{a}) = c(\mathfrak{g}) \tag{37}$$

Let $X^{a_j}_{-n_j}$ and $\tilde{X}^{a'_j}_{-n_j}$ be the lowering generators for $\mathfrak{g}$ and for $\mathfrak{a} \subset \mathfrak{g}$ correspondingly. Let $\pi_{\mathfrak{a}}$ be the projection operator of $\pi_{\mathfrak{a}} : \mathfrak{g} \longrightarrow \mathfrak{a}$. In the theory attributed to $\mathfrak{g}$ with the vacuum $|\lambda\rangle$ the states can be described as

$$X^{a_1}_{-n_1} X^{a_2}_{-n_2} \dots |\lambda\rangle \quad n_1 \geq n_2 \geq \dots > 0. \tag{38}$$

And for the sub-algebra $\mathfrak{a}$ the corresponding states are

$$\tilde{X}^{a'_1}_{-n_1} \tilde{X}^{a'_2}_{-n_2} \dots |\pi_{\mathfrak{a}}(\lambda)\rangle \,. \tag{39}$$

The $\mathfrak{g}$-invariance of the vacuum entails its $\mathfrak{a}$-invariance, but this is not the case for the energy-momentum tensor. So the energy-momentum tensor of the larger theory should consist only of the generators of $\tilde{X}$. Then $T_{\mathfrak{g}} = T_{\mathfrak{a}} \Rightarrow c(\mathfrak{g}) = c(\mathfrak{a})$. This leads to the equation

$$\frac{k \dim \mathfrak{g}}{k+g} = \frac{x_e k \dim \mathfrak{a}}{x_e k + a} \tag{40}$$

Here $x_e$ is the embedding index and $g$, $a$ are the dual Coxeter numbers for the corresponding algebras.

It can be demonstrated that the solutions of the equation (40) exist only for the level $k = 1$ [1].

The complete classification of conformal embeddings is given in the paper [13]. The requirement (37) allows to reduce the problem of finding the branching coefficients for affine Lie algebras to the computation of branching coefficients for finite-dimensional Lie algebras.

If we have modular-invariant partition function for the fields described by a representation of the algebra $\mathfrak{g}$ this modular invariance is preserved by the projection on the subalgebra $\mathfrak{a}$, but we need also the preservation of the conformal invariance. So we should select only those highest-weight modules of the algebra $\mathfrak{g}$ for which the relation (40) holds.

This relation and the asymptotics of the branching functions can be used to prove the finite reducibility theorem. It states that for the conformal embedding $\mathfrak{a} \subset \mathfrak{g}$ only finite number of branching coefficients have non-zero values.

Having found all such weights $\nu$ and the corresponding branching coefficients $b_\nu^{(\mu)}$ we can replace the characters of the $\mathfrak{g}$-modules in the diagonal modular-invariant partition function

$$Z(\tau) = \sum_{\mu \in P_{\mathfrak{g}}^+} \chi_\mu(\tau) \bar{\chi}_\mu(\bar{\tau}) \tag{41}$$

by the decompositions $\sum_{\nu \in P_{\mathfrak{a}}^+} b_\nu^{(\mu)} \chi_\nu$ containing the modified characters $\chi_\nu$ of the corresponding $\mathfrak{a}$-modules. Thus we obtain the non-diagonal modular-invariant partition function for the theory with the current algebra $\mathfrak{a}$.

$$Z_{\mathfrak{a}}(\tau) = \sum_{\nu, \lambda \in P_{\mathfrak{a}}^+} \chi_\nu(\tau) M_{\nu\lambda} \bar{\chi}_\lambda(\bar{\tau}) \tag{42}$$

It is important to note that the orthogonal subalgebra $\mathfrak{a}_\perp$ is always empty for the conformal embeddings $\mathfrak{a} \subset \mathfrak{g}$. It can be seen from the following consideration. In the case of the conformal embedding energy-momentum tensors are to be equal

$$T_{\mathfrak{a}}(z) = T_{\mathfrak{g}}(z) \tag{43}$$

The energy-momentum tensor can be expanded into the modes $L_n$

$$T(z) = \frac{1}{2(k + h^v)} \sum_n z^{-n-1} L_n \tag{44}$$

The modes are constructed as the combination of normally-ordered products of the generators of $\mathfrak{g}$.

$$L_n = \frac{1}{2(k + h^v)} \sum_a \sum_m : X_m^a X_{n-m}^a : \tag{45}$$

The substitution of the generators of the algebra $\mathfrak{a}$ in terms of the generators of $\mathfrak{g}$ into these combinations of $T_{\mathfrak{a}}$ should give the energy-momentum tensor $T_{\mathfrak{g}}$. But if the set of the generators $\Delta_\perp$ is not empty it is not possible, since $T_{\mathfrak{g}}$ contains the combinations of the generators $X_n^\alpha$, $\alpha \in \Delta_\perp$, which can not be obtained in the expansion of components of $T_{\mathfrak{a}}$.

If we limit the consideration by the affine extensions of simple classical Lie algebras, the number of possible non-maximal conformal embeddings is rather small [13].

## 4.1 Examples

### 4.1.1 Special embedding $\hat{A}_1 \subset \hat{A}_2$

Consider the embedding of the affine Lie algebra $\hat{A}_1$ into $\hat{A}_2$ constructed as the affine extension of the special embedding $A_1 \subset A_2$ with the embedding index $x_e = 4$. The level of the representations of the algebra $\mathfrak{g} = \hat{A}_2$ is equal to one, so the level of the modules of the subalgebra is equal $\tilde{k} = kx_e = 4$.

There exist three level one dominant weights in the weight space of $\hat{A}_2$. It is easy to see that the set $\Delta_\perp$ is empty and the subalgebra $\mathfrak{a}_\perp = 0$.

Let us consider the representation with the highest weight $w_0 = (0, 0; 1; 0)$ in details. Here the first two components are the Dynkin indices (coordinates in the fundamental weights basis) of the finite part of the weight.

The set of the anomalous weights of this representation is depicted in the Figure 12 up to the sixth grade. We also show the root subspace of the subalgebra $\hat{A}_1$ under consideration as the dotted diagonal plane.

Using the definition (24) we construct "the fan" $\Gamma_{\hat{A}_1 \to \hat{A}_2}$ with the corresponding values of the function $s(\gamma + \gamma_0)$ (see the Figure 13). Here we restricted the calculations to the first eight grades.

The next step is to project the anomalous weights to $P_{\hat{A}_1}$. The result up to the twelfth grade is presented in the Figure 14

Using the recurrent relation for the anomalous branching coefficients we get the result presented in Figure 15.

We see that inside the main Weyl chamber of $\hat{A}_1$, which is dotted at the Figure 15, there are only two non-zero anomalous weights. These are the branching coefficients. So the finite reducibility theorem holds and we get the decomposition

$$L^{(0,0;1;0)}_{\hat{A}_2 \downarrow \hat{A}_1} = L^{(0;4;0)}_{\hat{A}_1} \oplus L^{(4;4;0)}_{\hat{A}_1}. \tag{46}$$

For the other level one irreducible modules of $\hat{A}_2$ we get the trivial branching

$$L^{(1,0;1;0)}_{\hat{A}_2 \downarrow \hat{A}_1} = L^{(2;4;0)}_{\hat{A}_1}, \tag{47}$$

$$L^{(0,1;1;0)}_{\hat{A}_2 \downarrow \hat{A}_1} = L^{(2;4;0)}_{\hat{A}_1}. \tag{48}$$

Using these results the modular-invariant partition function is easily found,

$$Z = \left| \chi_{(4;4;0)} + \chi_{(0;4;0)} \right|^2 + 2\chi^2_{(2;4;0)}. \tag{49}$$

### 4.1.2 Embedding of semisimple subalgebras

Consider the conformal embedding constructed as the affine extension $\mathfrak{a}_1 \oplus \mathfrak{a}_2 \subset \mathfrak{g}$ of the embedding of semisimple subalgebra $\mathring{\mathfrak{a}}_1 \oplus \mathring{\mathfrak{a}}_2$ into simple Lie algebra $\mathring{\mathfrak{g}}$.

Substituting the branching rules into (41) we get non-diagonal mass matrix

$$M_{\lambda\xi,\mu\eta} \tag{50}$$

where $\lambda, \mu$ are the highest weights of the representations of $\mathfrak{a}_1$ and $\xi, \eta$ of $\mathfrak{a}_2$.

Consider the embedding $\mathfrak{a} \subset \mathfrak{g}$ where $\mathfrak{a} = \widehat{su(2)} \oplus \widehat{su(2)}$ and $\mathfrak{g} = \widehat{su(4)}$, which is the affine extension of the special embedding $su(2) \oplus su(2) \subset su(4)$. Let's construct the special embedding $su(2) \oplus su(2) \subset su(4)$ using the method of [15]. We start with 4-dimensional representation of $su(2) \oplus su(2)$ with the highest weight $(1,1)$. The weights of this representation are numbered as depicted at the Figure 16 and have the following coordinates in the fundamental weights basis: $\nu_1 = (1,1),\ \nu_2 = (-1,1),\ \nu_3 = (1,-1),\ \nu_4 = (-1,-1)$.

Then for the matrix elements of representation of Cartan subalgebra generators $b_1, b_2$ in Weyl basis we have $d(b_i) = \mathrm{diag}\left(\frac{2(\nu_1,\alpha_i)}{(\alpha_i,\alpha_i)}, \frac{2(\nu_2,\alpha_i)}{(\alpha_i,\alpha_i)}, \frac{2(\nu_3,\alpha_i)}{(\alpha_i,\alpha_i)}, \frac{2(\nu_4,\alpha_i)}{(\alpha_i,\alpha_i)}\right)$ [15], so $d(b_1) = \mathrm{diag}(1,-1,1,-1),\ d(b_2) = \mathrm{diag}(1,1,-1,-1)$. The embedded roots $\alpha_1, \alpha_2$ of $su(2) \oplus su(2)$ in terms of roots $\tilde{\alpha}_i$ of $su(4)$ are

$$\begin{array}{l} \alpha_1 = \frac{1}{2}(\tilde{\alpha}_1 + \tilde{\alpha}_3) \\ \alpha_2 = \frac{1}{2}(\tilde{\alpha}_1 + 2\tilde{\alpha}_2 + \tilde{\alpha}_3) \end{array} \tag{51}$$

They are depicted at the Figure 17.

This embedding is characterised by the embedding indexes $(2,2)$ and is conformal, since $c(A_1 \oplus A_1) = c(A_1) + c(A_1) = 2\frac{x_e \dim(A_1)}{x_e+2} = \frac{\dim A_3}{5} = c(A_3)$.

We are interested in the reduction of the fundamental representations of $\widehat{su(4)}$. Four dominant weights of level one have the following coordinates in the orthogonal basis:

$$\begin{array}{lcl} \omega_0 & = & (0,0,0,0;1;0) \\ \omega_1 & = & (\frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}; 1; 0) \\ \omega_2 & = & (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}; 1; 0) \\ \omega_3 & = & (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{3}{4}; 1; 0) \end{array} \tag{52}$$

These representation could be easily reduced by the method described in [12], which is based upon the properties of $A_n$-series of Lie algebras, but we use our general method and discuss its features.

For the sets of positive roots $\Delta^+$ and $\Delta^+_{\mathfrak{a}}$ we have

$$\begin{array}{lcl} \Delta^+ & = & \left\{ \overset{\circ}{\Delta}{}^+;\ \overset{\circ}{\Delta} - n\delta;\ -n\delta \text{ with multiplicity } 3;\ n = 1,2,\ldots, \right. \\ & & \left. \quad \overset{\circ}{\Delta}{}^+ = \{\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_1 + \tilde{\alpha}_2, \tilde{\alpha}_2 + \tilde{\alpha}_3, \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3\} \right\} \\ \Delta^+_{\mathfrak{a}} & = & \{\alpha_1, \alpha_2;\ \pm\alpha_1 - n\delta, \pm\alpha_2 - n\delta;\ -n\delta \text{ with multiplicity } 2;\ n = 1,2,\ldots\} \end{array} \tag{53}$$

The set $\Delta^+_\perp$ is empty. The fan $\Gamma_{\mathfrak{a}\subset\mathfrak{g}}$ is shown at the Figure 18. Coordinates of fan element's finite part are given in the basis of fundamental weights of

$su(2) \oplus su(2)$. Element $\gamma$ is shown by cross if $s(\gamma) = 1$ and by diamond if $s(\gamma) = -1$.

We limit our computation by the fifth grade.

The set $\widehat{\Psi^{(\mu)}} = \{\omega(\mu+\rho) - \rho;\ \omega \in W\}$ of the anomalous weights of the representation of the algebra $\hat{A}_3$ consists of 192 elements and its projection $\pi_{\mathfrak{a}}\left(\widehat{\Psi^{(\mu)}}\right)$ for $\mu = \omega_2 = (0,0,1;1;0)$ is shown at the Figure 19. Coordinates of anomalous weight's finite part are given in the basis of fundamental weights of $su(2) \oplus su(2)$. Weight $g$ are shown by cross if the corresponding value $\epsilon(\omega) = 1$ and by diamond otherwise.

We omit similar pictures for $\omega_0, \omega_1, \omega_3$ and include only the branching coefficients for these modules to save the space.

The anomalous branching coefficients for the module $L^{(\omega_1)}$ are shown at the Figure 20.

We get the following branching rules

$$\begin{array}{rcl} L^{(0,0,0;1;0)}_{\hat{A}_3 \downarrow \hat{A}_1 \oplus \hat{A}_1} & = & L^{(0;2;0)}_{\hat{A}_1} \otimes L^{(0;2;0)}_{\hat{A}_1} \\ L^{(1,0,0;1;0)}_{\hat{A}_3 \downarrow \hat{A}_1 \oplus \hat{A}_1} & = & L^{(1;2;0)}_{\hat{A}_1} \otimes L^{(1;2;0)}_{\hat{A}_1} \\ L^{(0,1,0;1;0]}_{\hat{A}_3 \downarrow \hat{A}_1 \oplus \hat{A}_1} & = & \left( L^{(2;2;0)}_{\hat{A}_1} \otimes L^{(0;2;0)}_{\hat{A}_1} \right) \oplus \left( L^{(0;2;0)}_{\hat{A}_1} \otimes L^{(2;2;0)}_{\hat{A}_1} \right) \\ L^{(0,0,1;1;0)}_{\hat{A}_3 \downarrow \hat{A}_1 \oplus \hat{A}_1} & = & L^{(1;2;0)}_{\hat{A}_1} \otimes L^{(1;2;0)}_{\hat{A}_1} \end{array} \tag{54}$$

Now we can obtain modular invariant partition function for WZW-model with the chiral algebra $\hat{A}_1 \oplus \hat{A}_1$.

$$\begin{aligned} Z = \left|\chi_{(0;2;0)}\chi_{(0;2;0)}\right|^2 + 2\left|\chi_{(1;2;0)}\chi_{(1;2;0)}\right|^2 + \left|\chi_{(2;2;0)}\chi_{(0;2;0)} + \chi_{(0;2;0)}\chi_{(2;2;0)}\right|^2 = \\ \left|\chi_{(0;2;0)}\right|^4 + 2\left|\chi_{(1;2;0)}\right|^4 + 4\left|\chi_{(2;2;0)}\chi_{(0;2;0)}\right|^2 \end{aligned} \tag{55}$$

# 5 Conclusion

We have proved that the injection fan technique can be used to deal with the nonmaximal subalgebras. It was demonstrated that in such cases in the set of positive roots $\Delta^+_{\mathfrak{g}}$ it is necessary to separate an additional subset $\Delta^+_{\perp}$. The injection fan is formed by the weights $\Delta^+_{\mathfrak{g}} \setminus \Delta^+_{\mathfrak{a}} \setminus \Delta^+_{\perp}$ and the additional role of the subset $\Delta^+_{\perp}$ is to modify the anomalous weights of the initial module. This modification reduces to a simple procedure: the anomalous weights are to be substituted by the dimensions of the corresponding $\mathfrak{a}_{\perp}$-modules.

We have demonstrated the effectiveness of the proposed generalizations of the injection fan algorithm and discussed its possible application to some physical problems. In particular we considered the construction modular-invariant partition functions in the conformal field theory in the framework of conformal embedding method. This method is widely used in the study of WZW-models emerging in the context of the AdS/CFT correspondence [16, 17, 18].

# 6 Acknowledgements

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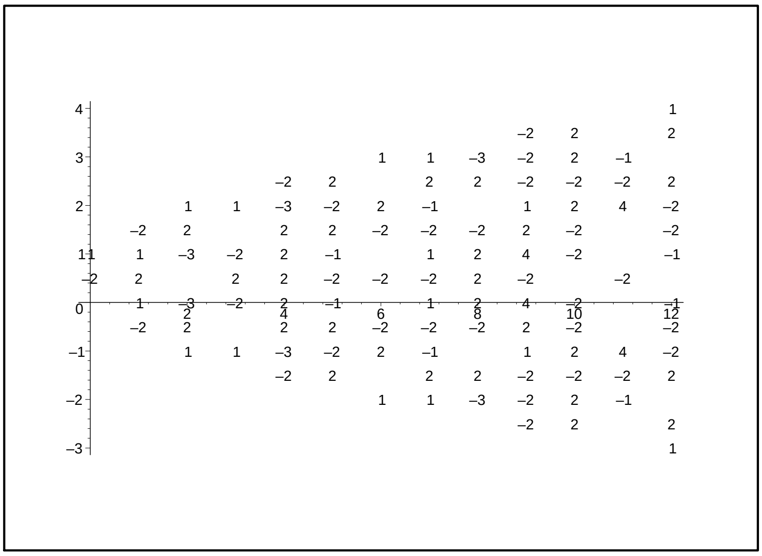

Figure 8: Fan for $\hat{A}_1 \subset \hat{B}_2$

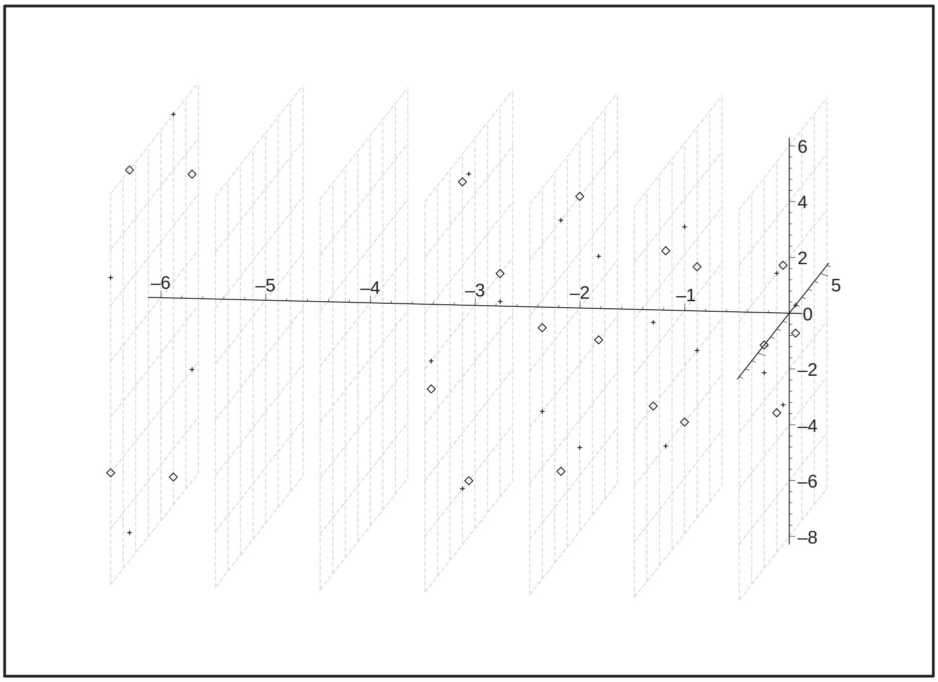


Figure 9: The anomalous weights of the $(1,0;1;0)$ representation of the algebra $\hat{B}_2$

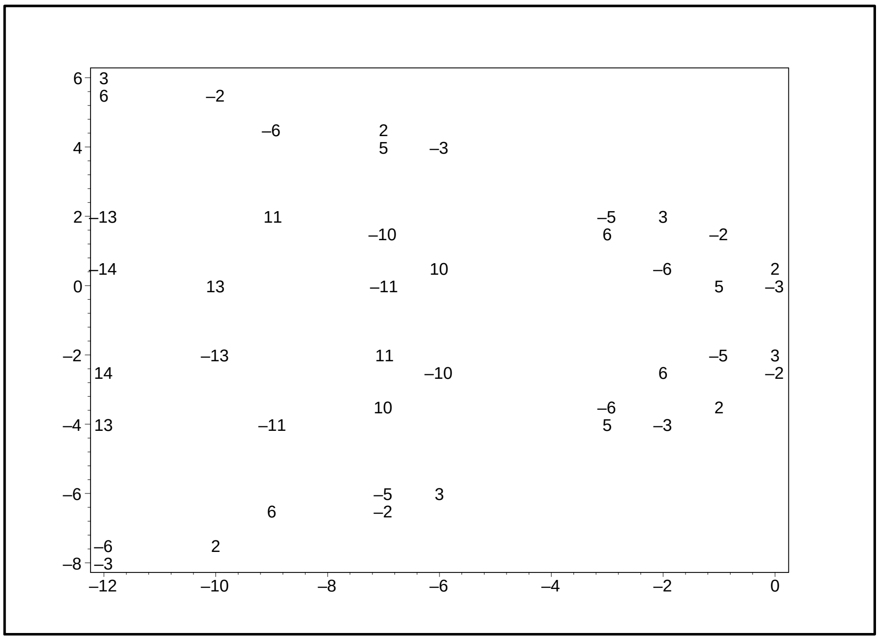


Figure 10: Projected anomalous weights and the dimensions of $\mathfrak{a}_\perp$-modules.

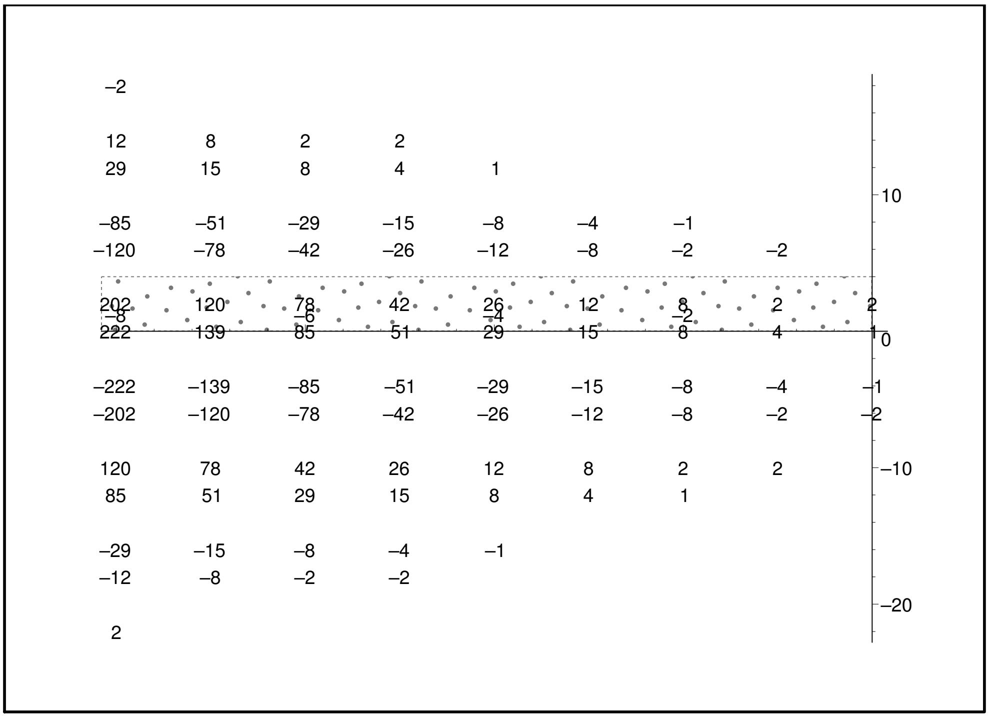

Figure 11: Anomalous branching coefficients for $\hat{A}_1 \subset \hat{B}_2$

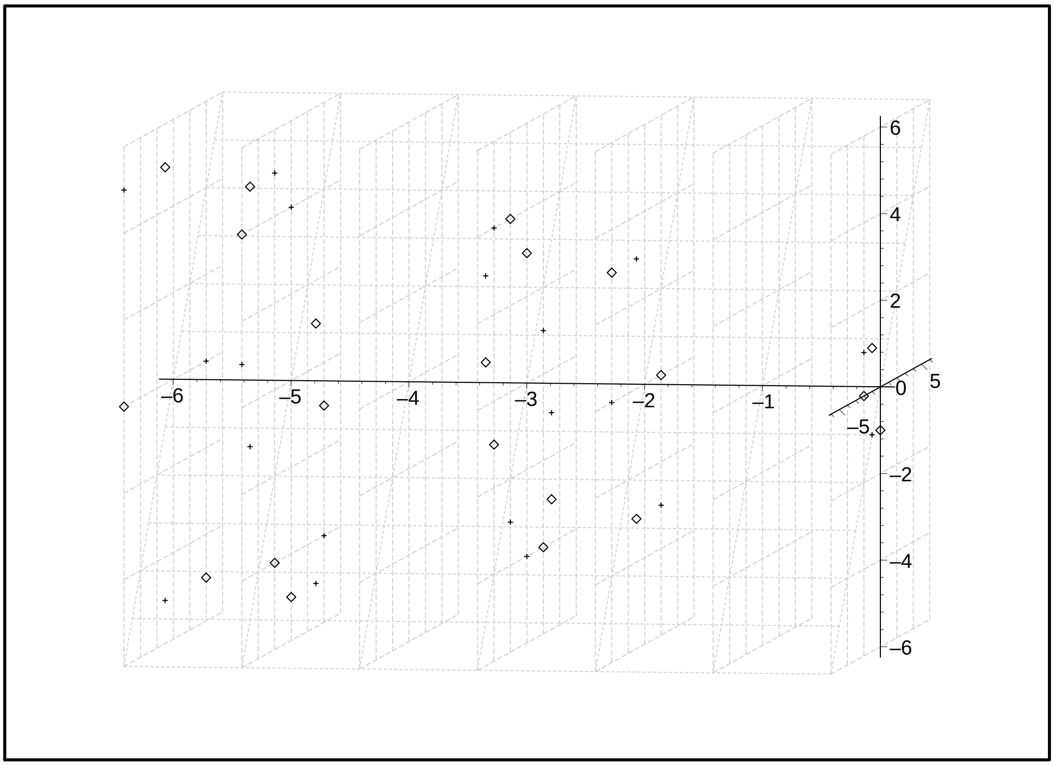


Figure 12: The anomalous weights of the module $L_{\hat{A}_2}^{(0,0;1;0)}$

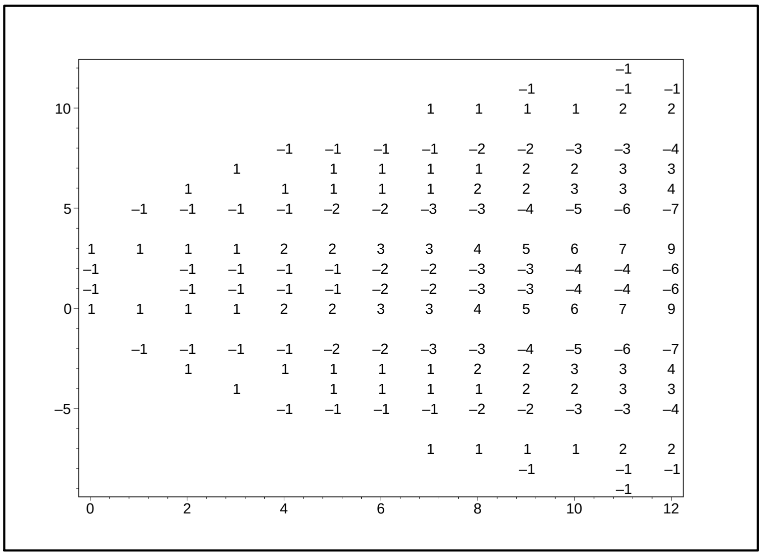

Figure 13: The fan $\Gamma_{\hat{A}_1 \subset \hat{A}_2}$

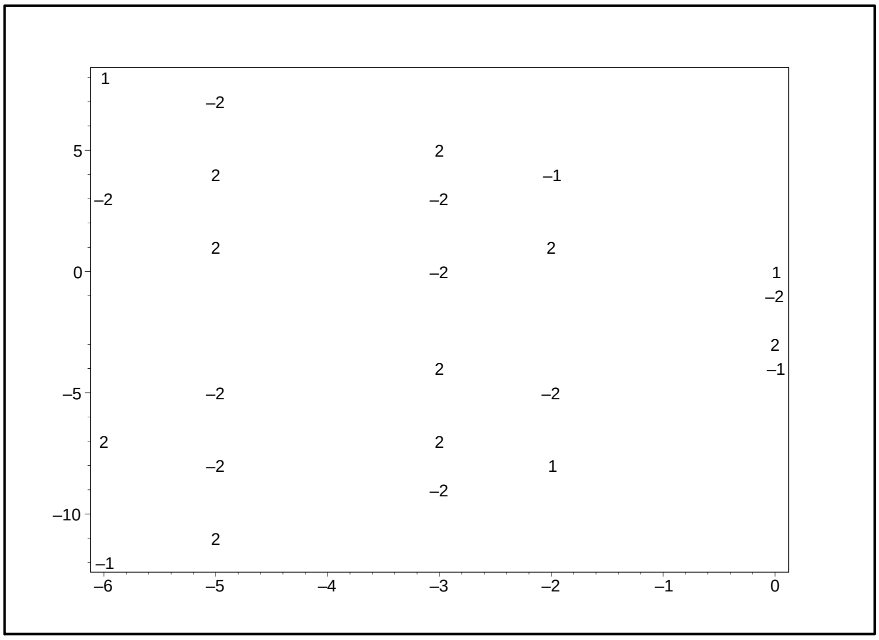


Figure 14: Projected anomalous weights and the dimensions of $\mathfrak{a}_{\perp}$-modules.

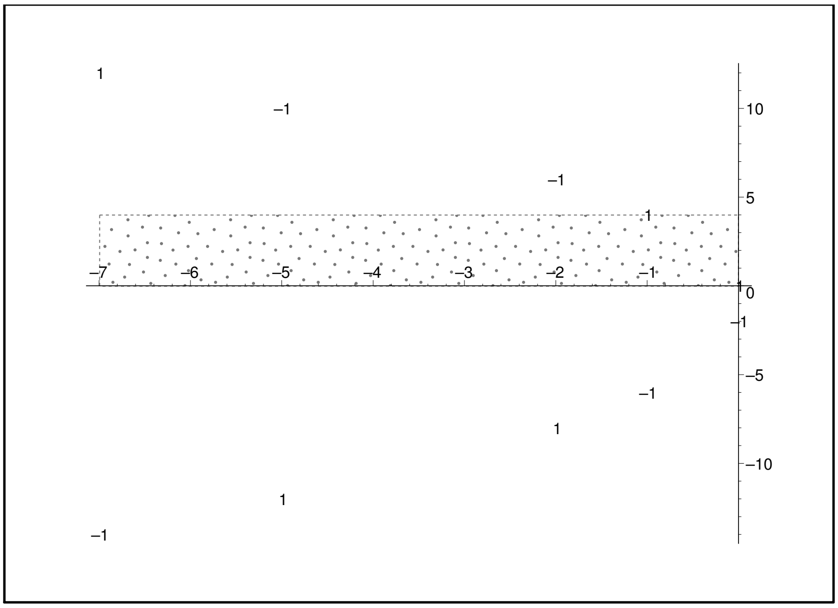


Figure 15: Anomalous branching coefficients for $\hat{A}_1 \subset \hat{A}_2$

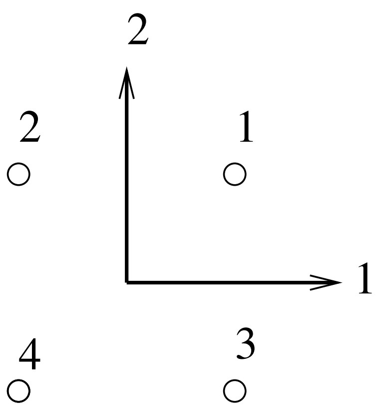


Figure 16: Representation for the special embedding $su(2) \oplus su(2) \subset su(4)$

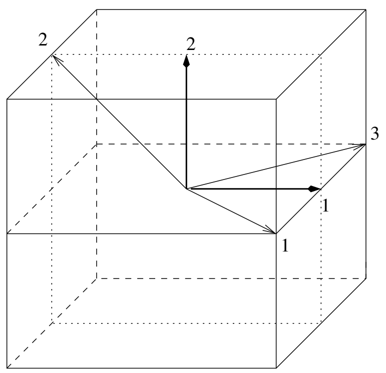


Figure 17: Embedded roots for the special embedding $su(2) \oplus su(2) \subset su(4)$

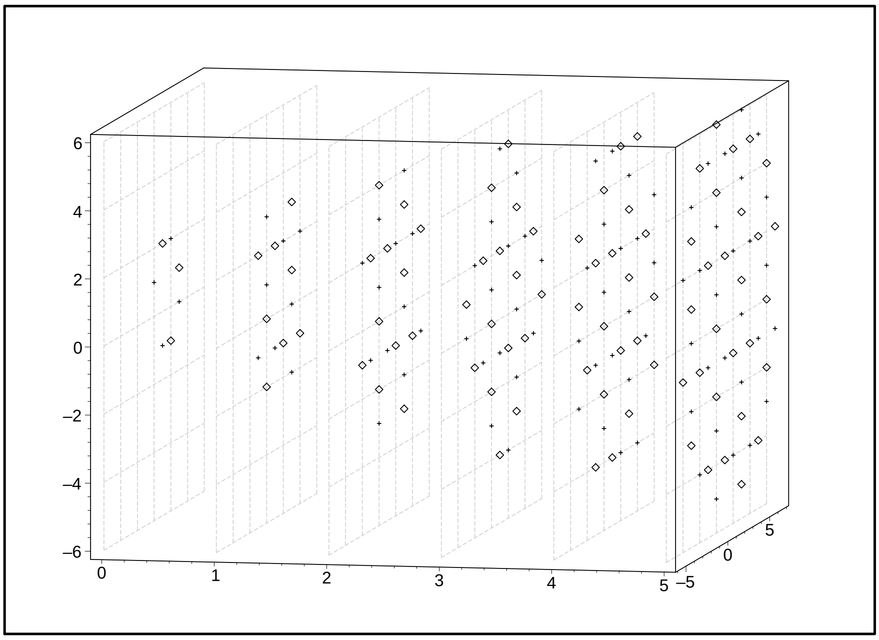


Figure 18: Fan for the special embedding $\widehat{su(2)} \oplus \widehat{su(2)} \subset \widehat{su(4)}$

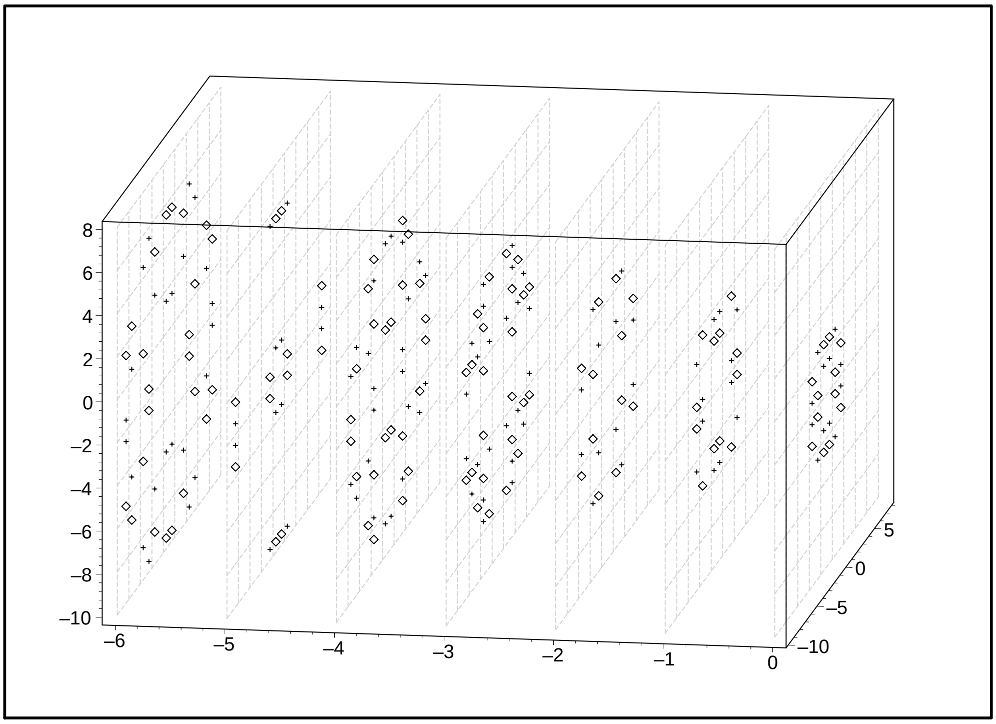


Figure 19: Projected anomalous weights for $L_{A_3}^{\omega_1}$

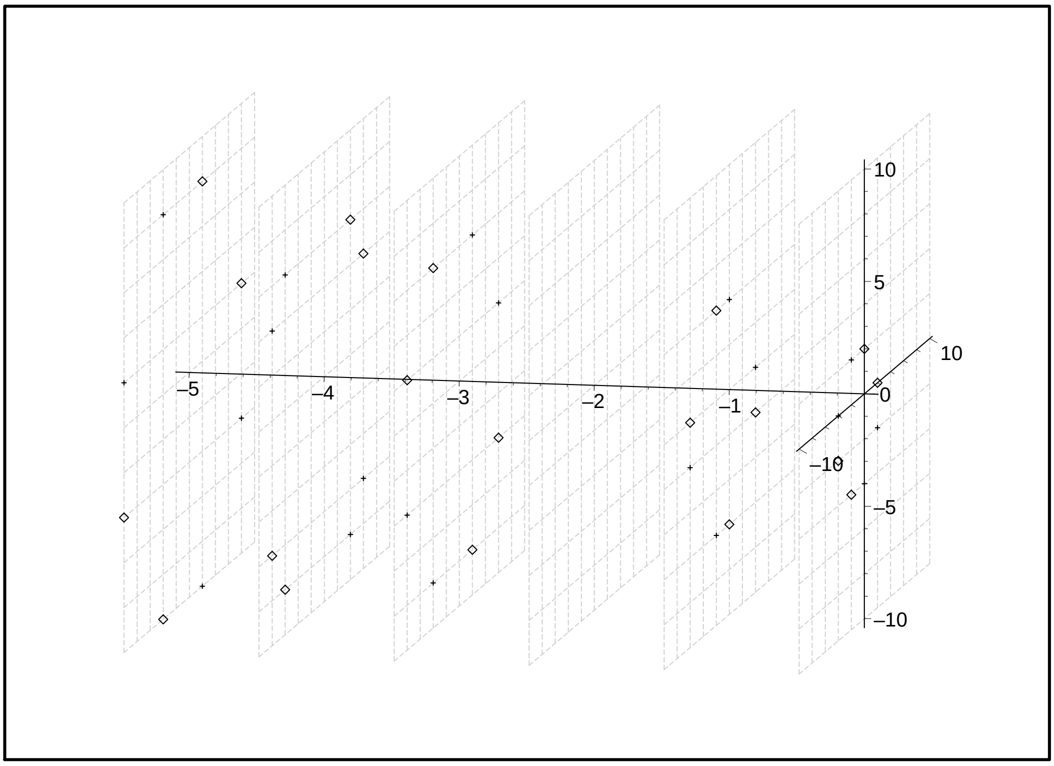


Figure 20: Anomalous branching coefficients for $L_{A_3}^{(0,0,1;1;0)}$