Recursive properties of branching and Weyl-Verma formulas

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Abstract

Recurrent relations for branching coefficients are based on a special type of singular element decomposition. We show that this decomposition can be used to construct the parabolic Verma modules and finally to obtain the generalized Weyl-Verma formulas for characters. These Weyl-Verma formulas explicitly exhibit how branching and the generalized BGG-resolution properties are connected.

1 Introduction

Branching properties of Lie (affine Lie) algebras are highly important for applications in quantum field theory (see for example the conformal field theory models [1],[2]). In this paper we demonstrate that for an arbitrary reductive subalgebra branching is directly connected with the BGG resolution and in particular exhibits the resolution properties in terms of the $\mathcal{O}^p$ category [3] (the parabolic generalization of the cathegory $\mathcal{O}$ [4]).

The resolution of the irreducible modules in terms of infinite-dimensional ones is important for the theory of integrable spin chains [5]. In the Baxter $\mathcal{Q}$-operator approach the generic transfer matrices corresponding to the (generalized) Verma modules are factorized into the product of Baxter operator. Then the resolution allows to calculate the transfer matrices for finite-dimensional auxiliary spaces.

To show the connection of the BGG resolution with the branching we use the recursive approach presented in [6] (similar to the one used in [7] for maximal embeddings). We consider the subalgebra $\mathfrak{a}$ together with its counterpart $\mathfrak{a}_{\perp}$ "orthogonal" to $\mathfrak{a}$ with respect to the Killing form and also $\widetilde{\mathfrak{a}_{\perp}} := \mathfrak{a}_{\perp} \oplus \mathfrak{h}_{\perp}$ where $\mathfrak{h} = \mathfrak{h}_{\mathfrak{a}} \oplus \mathfrak{h}_{\mathfrak{a}_{\perp}} \oplus \mathfrak{h}_{\perp}$. For any reductive algebra $\mathfrak{a}$ the subalgebra $\mathfrak{a}_{\perp} \hookrightarrow \mathfrak{g}$ is regular and reductive. For a highest weight integrable module $L^{(\mu)}$ and orthogonal subalgebra $\mathfrak{a}_{\perp}$ we consider the singular element $\Psi^{(\mu)}$ (the numerator in the Weyl character formula $ch\left(L^{\mu}\right) = \frac{\Psi^{(\mu)}}{\Psi^{(0)}}$, see for example [8]) the Weyl denominator $\Psi^{(0)}_{\mathfrak{a}_{\perp}}$ for the orthogonal subalgebra and the projection $\Psi^{(\mu)}_{(\mathfrak{a},\mathfrak{a}_{\perp})} = \pi_{\mathfrak{a}} \frac{\Psi^{(\mu)}_{\mathfrak{g}}}{\Psi^{(0)}_{\mathfrak{a}_{\perp}}}$. It is shown that the element $\Psi^{(\mu)}_{\mathfrak{g}}$ can be decomposed into a combination of Weyl numerators $\Psi^{(\nu)}_{\mathfrak{a}_{\perp}}$ with $\nu \in P^{+}_{\mathfrak{a}_{\perp}}$. This decomposition provides the possibility to construct the set of highest weight modules $L^{\mu_{\widetilde{\mathfrak{a}_{\perp}}}}_{\widetilde{\mathfrak{a}_{\perp}}}$. When the injection $\mathfrak{a}_{\perp} \hookrightarrow \mathfrak{g}$ satisfies the "standard parabolic" conditions these modules give rise to the parabolic Verma modules $M^{\mu_{\widetilde{\mathfrak{a}_{\perp}}}}_{(\widetilde{\mathfrak{a}_{\perp}} \hookrightarrow \mathfrak{g})}$ so that the initial character $ch\left(L^{\mu}\right)$ is finally decomposed into the alternating sum of such. On the other hand when the parabolic conditions are violated the construction survives and exhibits a decomposition with respect to a set of Verma modules $M^{\mu_{\widetilde{\mathfrak{a}_{\perp}}}}_{(\widetilde{\mathfrak{b}_{\perp}},\mathfrak{g})}$. In such a situation the algebra $\widetilde{\mathfrak{b}_{\perp}}$ is not a subalgebra in $\mathfrak{g}$ but a contraction of $\widetilde{\mathfrak{a}_{\perp}}$.

Some general properties of the proposed decompositions are formulated in terms of a specific element $\Gamma_{\mathfrak{a}\rightarrow\mathfrak{g}}$ of the group algebra $\mathcal{E}\left(\mathfrak{g}\right)$ called "the injection fan". Using this tool the simple and explicit algorithm for branching coefficients computations applicable for an arbitrary (maximal or nonmaximal) subalgebras of finite-dimensional or affine Lie algebras was obtained in [6].

Possible further developments are presented in Section 3.

1.1 Notation

Consider Lie algebras (affine Lie algebras) $\mathfrak{g}$ and $\mathfrak{a}$ and an injection $\mathfrak{a} \hookrightarrow \mathfrak{g}$ such that $\mathfrak{a}$ is a reductive subalgebra $\mathfrak{a} \subset \mathfrak{g}$ with correlated root spaces: $\mathfrak{h}^{*}_{\mathfrak{a}} \subset \mathfrak{h}^{*}_{\mathfrak{g}}$. We use the following notations:

$\mathfrak{g} = \mathfrak{n}^{-} + \mathfrak{b} + \mathfrak{n}^{+}$ — the Cartan decomposition;

r , $(r_{\mathfrak{a}})$ — the rank of the algebra $\mathfrak{g}$ (resp.$\mathfrak{a}$) ;

Δ $(\Delta_{\mathfrak{a}})$— the root system; Δ^{+} (resp.$\Delta^{+}_{\mathfrak{a}}$)— the positive root system (of $\mathfrak{g}$ and $\mathfrak{a}$ respectively);

$\mathrm{mult}\left(\alpha\right)$ $\left(\mathrm{mult}_{\mathfrak{a}}\left(\alpha\right)\right)$ — the multiplicity of the root α in Δ (resp. in $(\Delta_{\mathfrak{a}})$);

S $(S_{\mathfrak{a}})$ — the system of simple roots (of $\mathfrak{g}$ and $\mathfrak{a}$ respectively);

α_i , $(\alpha_{(\mathfrak{a})j})$ — the i-th (resp. j-th) basic root for $\mathfrak{g}$ (resp.$\mathfrak{a}$); $i = 0, \ldots, r$, $(j = 0, \ldots, r_{\mathfrak{a}})$;

δ — the imaginary root of $\mathfrak{g}$ (and of $\mathfrak{a}$ if any);

$\alpha_i^{\vee}$, $\left(\alpha_{(\mathfrak{a})j}^{\vee}\right)$— the basic coroot for $\mathfrak{g}$ (resp.$\mathfrak{a}$) , $i = 0, \ldots, r$; $(j = 0, \ldots, r_{\mathfrak{a}})$;

W , $(W_{\mathfrak{a}})$— the corresponding Weyl group;

C , $(C_{\mathfrak{a}})$— the fundamental Weyl chamber;

$\bar{C}$, $\left(\bar{C}_{\mathfrak{a}}\right)$ — the closure of the fundamental Weyl chamber;

$\epsilon(w) := (-1)^{\mathrm{length}(w)}$;

ρ , $(\rho_{\mathfrak{a}})$ — the Weyl vector;

L^{μ} $(L_{\mathfrak{a}}^{\nu})$ — the integrable module of $\mathfrak{g}$ with the highest weight μ ; (resp. integrable $\mathfrak{a}$ -module with the highest weight ν);

$\mathcal{N}^{\mu}$, $(\mathcal{N}_{\mathfrak{a}}^{\nu})$ — the weight diagram of L^{μ} (resp. $L_{\mathfrak{a}}^{\nu}$);

P (resp. $P_{\mathfrak{a}}$) — the weight lattice;

P^{+} (resp. $P_{\mathfrak{a}}^{+}$) — the dominant weight lattice;

$m_{\xi}^{(\mu)}$, $\left(m_{\xi}^{(\nu)}\right)$ — the multiplicity of the weight $\xi \in P$ (resp. $\in \mathrm{P}_{\mathfrak{a}}$) in the module L^{μ} , (resp. $\xi \in L_{\mathfrak{a}}^{\nu}$);

$ch\left(L^{\mu}\right)$ (resp. $\mathrm{ch}\left(L_{\mathfrak{a}}^{\nu}\right)$)— the formal character of L^{μ} (resp. $L_{\mathfrak{a}}^{\nu}$);

$ch\left(L^{\mu}\right) = \frac{\sum_{w \in W} \epsilon(w) e^{w \circ (\mu+\rho)-\rho}}{\prod_{\alpha \in \Delta^{+}} (1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}$ — the Weyl-Kac formula;

$R := \prod_{\alpha \in \Delta^{+}} \left(1 - e^{-\alpha}\right)^{\mathrm{mult}(\alpha)}$ (resp. $R_{\mathfrak{a}} := \prod_{\alpha \in \Delta_{\mathfrak{a}}^{+}} \left(1 - e^{-\alpha}\right)^{\mathrm{mult}_{\mathfrak{a}}(\alpha)}$)— the Weyl denominator.

2 Orthogonal subalgebra and singular elements

In this section we shall show how the recurrent approach to branching problem leads naturally to a presentation of the formal character in terms of parabolic (generalized) Verma modules. Consider a reductive Lie algebra $\mathfrak{g}$ and its reductive subalgebra $\mathfrak{a} \subset \mathfrak{g}$. Let L^{μ} be the highest weight integrable module of $\mathfrak{g}$, $\mu \in P^{+}$. Let L^{μ} be completely reducible with respect to $\mathfrak{a}$,

$$L_{\mathfrak{g}\downarrow\mathfrak{a}}^{\mu} = \bigoplus_{\nu \in P_{\mathfrak{a}}^{+}} b_{\nu}^{(\mu)} L_{\mathfrak{a}}^{\nu}.$$

Using the projection operator $\pi_{\mathfrak{a}}$ (to the weight space ${\mathfrak{h}_{\mathfrak{a}}}^{*}$) one can rewrite this decomposition in terms of formal characters:

$$\pi_{\mathfrak{a}} ch\left(L^{\mu}\right) = \sum_{\nu \in P_{\mathfrak{a}}^{+}} b_{\nu}^{(\mu)} ch\left(L_{\mathfrak{a}}^{\nu}\right). \tag{1}$$

The module L^{μ} has the BGG-resolution (see [4, 9, 10] and [11]). All the members of the filtration sequence are the direct sums of Verma modules and all their highest weights ν are strongly linked to μ:

$$\{\nu\} = \{w(\mu+\rho) - \rho | w \in W\}.$$

2.1 Orthogonal subalgebra

Let $\mathfrak{h}_{\mathfrak{a}}$ be a Cartan subalgebra of $\mathfrak{g}$. For $\mathfrak{a} \hookrightarrow \mathfrak{g}$ introduce the "orthogonal partner" $\mathfrak{a}_{\perp} \hookrightarrow \mathfrak{g}$.

Consider the root subspace $\mathfrak{h}^{*}_{\perp\mathfrak{a}}$ orthogonal to $\mathfrak{a}$,

$$\mathfrak{h}^{*}_{\perp\mathfrak{a}} := \{\eta \in \mathfrak{h}^{*} | \forall h \in \mathfrak{h}_{\mathfrak{a}}; \eta(h) = 0\},$$

and the roots (correspondingly – positive roots) of $\mathfrak{g}$ orthogonal to $\mathfrak{a}$,

$$\begin{aligned} \Delta_{\mathfrak{a}_{\perp}} \quad &: \quad = \{\beta \in \Delta_{\mathfrak{g}} | \forall h \in \mathfrak{h}_{\mathfrak{a}}; \beta(h) = 0\}, \\ \Delta^{+}_{\mathfrak{a}_{\perp}} \quad &: \quad = \left\{\beta^{+} \in \Delta^{+}_{\mathfrak{g}} | \forall h \in \mathfrak{h}_{\mathfrak{a}}; \beta^{+}(h) = 0\right\}. \end{aligned} \tag{2}$$

Let $W_{\mathfrak{a}_{\perp}}$ be the subgroup of W generated by the reflections w_{β} with the roots $\beta \in \Delta^{+}_{\mathfrak{a}_{\perp}}$. The subsystem $\Delta_{\mathfrak{a}_{\perp}}$ determines the subalgebra $\mathfrak{a}_{\perp}$ with the Cartan subalgebra $\mathfrak{h}_{\mathfrak{a}_{\perp}}$. Let

$$\mathfrak{h}^{*}_{\perp} := \{\eta \in \mathfrak{h}^{*}_{\perp\mathfrak{a}} | \forall h \in \mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{a}_{\perp}}; \eta(h) = 0\}$$

so that $\mathfrak{g}$ has the subalgebras

$$\begin{aligned} \widetilde{\mathfrak{a}_{\perp}} \quad &: \quad = \mathfrak{a}_{\perp} \oplus \mathfrak{h}_{\perp} \\ \widetilde{\mathfrak{a}} \quad &: \quad = \mathfrak{a} \oplus \mathfrak{h}_{\perp}. \end{aligned}$$

Notice that $\mathfrak{a} \oplus \mathfrak{a}_{\perp}$ in general is not a subalgebra of $\mathfrak{g}$

For the Cartan subalgebras we have the decomposition

$$\mathfrak{h} = \mathfrak{h}_{\mathfrak{a}} \oplus \mathfrak{h}_{\mathfrak{a}_{\perp}} \oplus \mathfrak{h}_{\perp} = \mathfrak{h}_{\widetilde{\mathfrak{a}}} \oplus \mathfrak{h}_{\mathfrak{a}_{\perp}} = \mathfrak{h}_{\widetilde{\mathfrak{a}_{\perp}}} \oplus \mathfrak{h}_{\mathfrak{a}}. \tag{3}$$

For the subalgebras $\mathfrak{a}$ and $\mathfrak{a}_{\perp}$ consider the corresponding Weyl vectors, $\rho_{\mathfrak{a}}$ and $\rho_{\mathfrak{a}_{\perp}}$. Form the so called "defects" $\mathcal{D}_{\mathfrak{a}}$ and $\mathcal{D}_{\mathfrak{a}_{\perp}}$ of the injection:

$$\mathcal{D}_{\mathfrak{a}} := \rho_{\mathfrak{a}} - \pi_{\mathfrak{a}}\rho, \tag{4}$$

$$\mathcal{D}_{\mathfrak{a}_{\perp}} := \rho_{\mathfrak{a}_{\perp}} - \pi_{\mathfrak{a}_{\perp}}\rho. \tag{5}$$

For the highest weight $\mu \in P^+$ consider the linked weights $\{(w(\mu+\rho)-\rho)\,|w \in W\}$ and their projections to $h^*_{\mathfrak{a}_\perp}$ additionally shifted by the defect $-\mathcal{D}_{\mathfrak{a}_\perp}$:

$$\mu_{\mathfrak{a}_\perp}(w) := \pi_{\mathfrak{a}_\perp}[w(\mu+\rho)-\rho] - \mathcal{D}_{\mathfrak{a}_\perp}, \quad w \in W.$$

For $\mu \in P^+$ among the weights $\{\mu_{\mathfrak{a}_\perp}(w)\,|w \in W\}$ one can always choose those located in the fundamental chamber $\overline{C_{\mathfrak{a}_\perp}}$. Let U be the set of representatives u for the classes $W/W_{\mathfrak{a}_\perp}$ such that

$$U := \left\{ u \in W | \quad \mu_{\mathfrak{a}_\perp}(u) \in \overline{C_{\mathfrak{a}_\perp}} \right\} \quad . \tag{6}$$

Thus we can select the following subsets:

$$\mu_{\widetilde{\mathfrak{a}}}(u) := \pi_{\widetilde{\mathfrak{a}}}[u(\mu+\rho)-\rho] + \mathcal{D}_{\mathfrak{a}_\perp}, \quad u \in U, \tag{7}$$

and

$$\mu_{\mathfrak{a}_\perp}(u) := \pi_{\mathfrak{a}_\perp}[u(\mu+\rho)-\rho] - \mathcal{D}_{\mathfrak{a}_\perp}, \quad u \in U. \tag{8}$$

Note that the subalgebra $\mathfrak{a}_\perp$ is regular by definition since it is built on the roots of the algebra $\mathfrak{g}$.

For the modules we are interested in the Weyl-Kac formula for $\mathrm{ch}\,(L^\mu)$ can be written in terms of singular elements [8]

$$\Psi^{(\mu)} := \sum_{w \in W} \epsilon(w) e^{w(\mu+\rho)-\rho},$$

namely,

$$\mathrm{ch}\,(L^\mu) = \frac{\Psi^{(\mu)}}{\Psi^{(0)}} = \frac{\Psi^{(\mu)}}{R}. \tag{9}$$

The same is true for the submodules $\mathrm{ch}\,(L^\nu_{\mathfrak{a}})$ in (1)

$$\mathrm{ch}\,(L^\nu_{\mathfrak{a}}) = \frac{\Psi^{(\nu)}_{\mathfrak{a}}}{\Psi^{(0)}_{\mathfrak{a}}} = \frac{\Psi^{(\nu)}_{\mathfrak{a}}}{R_{\mathfrak{a}}},$$

with

$$\Psi^{(\nu)}_{\mathfrak{a}} := \sum_{w \in W_{\mathfrak{a}}} \epsilon(w) e^{w(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}.$$

Applying formula (9) to the branching rule (1) we get the relation connecting the singular elements $\Psi^{(\mu)}$ and $\Psi_{\mathfrak{a}}^{(\nu)}$:

$$
\begin{aligned}
\pi_{\mathfrak{a}}\left(\frac{\sum_{w\in W}\epsilon(w)e^{w(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}\right) &= \sum_{\nu\in P_{\mathfrak{a}}^+} b_{\nu}^{(\mu)}\frac{\sum_{w\in W_{\mathfrak{a}}}\epsilon(w)e^{w(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}}{\prod_{\beta\in\Delta_{\mathfrak{a}}^+}(1-e^{-\beta})^{\mathrm{mult}_{\mathfrak{a}}(\beta)}}, \\
\pi_{\mathfrak{a}}\left(\frac{\Psi^{(\mu)}}{R}\right) &= \sum_{\nu\in P_{\mathfrak{a}}^+} b_{\nu}^{(\mu)}\frac{\Psi_{\mathfrak{a}}^{(\nu)}}{R_{\mathfrak{a}}}.
\end{aligned}
\tag{10}
$$

2.2 Decomposing the singular element.

Now we shall perform a decomposition of the singular element $\Psi^{(\mu)}$ in terms of singular elements of the orthogonal subalgebra modules:

Lemma 1. *Let $\mathfrak{a}_\perp$ be the orthogonal partner of a reductive subalgebra $\mathfrak{a}\hookrightarrow\mathfrak{g}$ with $\mathfrak{h}=\mathfrak{h}_{\mathfrak{a}}\oplus\mathfrak{h}_{\mathfrak{a}_\perp}\oplus\mathfrak{h}_\perp$, $\widetilde{\mathfrak{a}_\perp}=\mathfrak{a}_\perp\oplus\mathfrak{h}_\perp$ and $\widetilde{\mathfrak{a}}=\mathfrak{a}\oplus\mathfrak{h}_\perp$.*

L^μ be the highest weight integrable module with $\mu\in P^+$ and

$\Psi^{(\mu)}$ – the singular element of L^μ.

Then the element $\Psi^{(\mu)}$ can be decomposed into the sum over $u\in U$ (see (6)) of singular elements $\Psi_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}$ with the coefficients $\epsilon(u)e^{\mu_{\widetilde{\mathfrak{a}}}(u)}$:

$$
\Psi^{(\mu)}=\sum_{u\in U}\epsilon(u)e^{\mu_{\widetilde{\mathfrak{a}}}(u)}\Psi_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}. \tag{11}
$$

Proof. With $u\in U$ and $v\in W_{\mathfrak{a}_\perp}$ perform the decomposition

$$
u(\mu+\rho)=\pi_{(\widetilde{\mathfrak{a}})}u(\mu+\rho)+\pi_{(\mathfrak{a}_\perp)}u(\mu+\rho)
$$

for the singular weight $vu(\mu+\rho)-\rho$:

$$
\begin{aligned}
vu(\mu+\rho)-\rho &= \pi_{(\mathfrak{a})}\left(u(\mu+\rho)\right)-\rho+\rho_{\mathfrak{a}_\perp} \\
&\quad+v\left(\pi_{(\widetilde{\mathfrak{a}}_\perp)}u(\mu+\rho)-\rho_{\mathfrak{a}_\perp}+\rho_{\mathfrak{a}_\perp}\right)-\rho_{\mathfrak{a}_\perp}.
\end{aligned}
\tag{12}
$$

Use the defect $\mathcal{D}_{\mathfrak{a}_\perp}$ (5) to simplify the first line in (12):

$$
\begin{aligned}
\pi_{(\widetilde{\mathfrak{a}})}\left(u(\mu+\rho)\right)-\rho+\rho_{\mathfrak{a}_\perp} &= \\
\pi_{(\widetilde{\mathfrak{a}})}\left(u(\mu+\rho)\right)-\pi_{\widetilde{\mathfrak{a}}}\rho-\pi_{\mathfrak{a}_\perp}\rho+\rho_{\mathfrak{a}_\perp} &= \\
=\pi_{(\widetilde{\mathfrak{a}})}\left(u(\mu+\rho)-\rho\right)+\mathcal{D}_{\mathfrak{a}_\perp}&,
\end{aligned}
$$

and the second one:

$$
\begin{gathered}
v\left(\pi_{(\mathfrak{a}_\perp)}u(\mu+\rho)-\rho_{\mathfrak{a}_\perp}+\rho_{\mathfrak{a}_\perp}\right)-\rho_{\mathfrak{a}_\perp}= \\
v\left(\pi_{(\mathfrak{a}_\perp)}u(\mu+\rho)-\mathcal{D}_{\mathfrak{a}_\perp}-\pi_{(\mathfrak{a}_\perp)}\rho+\rho_{\mathfrak{a}_\perp}\right)-\rho_{\mathfrak{a}_\perp}= \\
=v\left(\pi_{(\mathfrak{a}_\perp)}\left[u(\mu+\rho)-\rho\right]-\mathcal{D}_{\mathfrak{a}_\perp}+\rho_{\mathfrak{a}_\perp}\right)-\rho_{\mathfrak{a}_\perp}.
\end{gathered}
$$

These expressions provide the desired decomposition of the singular element Ψ^{μ} in terms of singular elements $\Psi^{\eta}_{\mathfrak{a}_\perp}$ of the $\mathfrak{a}_\perp$-modules $L^{\eta}_{\mathfrak{a}_\perp}$:

$$
\begin{aligned}
&\Psi^{\mu}=\sum_{u\in U}\sum_{v\in W_{\mathfrak{a}_\perp}}\epsilon(v)\epsilon(u)e^{vu(\mu+\rho)-\rho}= \\
&=\sum_{u\in U}\epsilon(u)e^{\pi_{\widetilde{\mathfrak{a}}}[u(\mu+\rho)-\rho]+\mathcal{D}_{\mathfrak{a}_\perp}}\sum_{v\in W_{\mathfrak{a}_\perp}}\epsilon(v)e^{v\left(\pi_{(\mathfrak{a}_\perp)}[u(\mu+\rho)-\rho]-\mathcal{D}_{\mathfrak{a}_\perp}+\rho_{\mathfrak{a}_\perp}\right)-\rho_{\mathfrak{a}_\perp}}= \\
&=\sum_{u\in U}\ \epsilon(u)\Psi_{\mathfrak{a}_\perp}^{\pi_{(\mathfrak{a}_\perp)}[u(\mu+\rho)-\rho]-\mathcal{D}_{\mathfrak{a}_\perp}}e^{\pi_{(\widetilde{\mathfrak{a}})}[u(\mu+\rho)-\rho]+\mathcal{D}_{\mathfrak{a}_\perp}}.
\end{aligned}
\tag{13}
$$

$\square$

Remark 1. This relation can be considered as a generalized form of the Weyl formula for the singular element $\Psi^{\mu}_{\mathfrak{g}}$: the vectors $\mu_{\widetilde{\mathfrak{a}}}(u)$ play the role of singular weights while the alternating factors $\epsilon(u)$ are extended to $\epsilon(u)\Psi_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}$. In fact when $\mathfrak{a}=\mathfrak{g}$ both $\mathfrak{a}_\perp$and $\mathfrak{h}_\perp$are zeros, $U=W$, and the original Weyl formula is reobtained via the trivilization of the singular elements: $\epsilon(u)\Psi_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}=\epsilon(u)$. In the opposite limit when $\mathfrak{a}=\mathfrak{o}$, $\Delta_{\mathfrak{a}_\perp}=\Delta_{\mathfrak{g}}$, $\mathfrak{h}^*_\perp=0$, $\mathfrak{a}_\perp=\mathfrak{g}$, $\mathcal{D}_{\mathfrak{a}_\perp}=0$, $U=W/W_{\mathfrak{a}_\perp}=e$ and Ψ^{μ} is again reobtained, now via the trivilization of the set of vectors: $\mu_{\mathfrak{a}}(e)=0$.

2.3 Reduction of Verma modules

In [6] the decomposition analogous to (13) was used to construct the recurrent relations for branching coefficients $k_{\xi}^{(\mu)}$ corresponding to the injection $\mathfrak{a}\hookrightarrow\mathfrak{g}$:

$$
\begin{aligned}
k_{\xi}^{(\mu)}=-\tfrac{1}{s(\gamma_0)}\Big(&\sum_{u\in U}\epsilon(u)\ \dim\left(L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}\right)\delta_{\xi-\gamma_0,\pi_{\mathfrak{a}}(u(\mu+\rho)-\rho)}+ \\
&+\sum_{\gamma\in\Gamma_{\mathfrak{a}\to\mathfrak{g}}}s\left(\gamma+\gamma_0\right)k_{\xi+\gamma}^{(\mu)}\Big).
\end{aligned}
\tag{14}
$$

The recursion is goverened by the set $\Gamma_{\mathfrak{a}\to\mathfrak{g}}$ called the injection fan. The later is defined as the carrier set $\{\xi\}_{\mathfrak{a}\to\mathfrak{g}}$ for the coefficient function $s(\xi)$

$$\{\xi\}_{\mathfrak{a}\to\mathfrak{g}}:=\{\xi\in P_{\mathfrak{a}}|s(\xi)\neq 0\}$$

appearing in the expansion

$$\prod_{\alpha\in\Delta^{+}\setminus\Delta^{+}_{\mathfrak{a}_{\perp}}}\left(1-e^{-\pi_{\mathfrak{a}}\alpha}\right)^{\mathrm{mult}(\alpha)-\mathrm{mult}_{\mathfrak{a}}(\pi_{\mathfrak{a}}\alpha)}=-\sum_{\gamma\in P_{\mathfrak{a}}}s(\gamma)e^{-\gamma}; \tag{15}$$

The weights in $\{\xi\}_{\mathfrak{a}\rightarrow\mathfrak{g}}$ are to be shifted by γ_0 – the lowest vector in $\{\xi\}$ – and the zero element is to be eliminated:

$$\Gamma_{\mathfrak{a}\rightarrow\mathfrak{g}}=\{\xi-\gamma_0|\xi\in\{\xi\}\}\setminus\{0\}\,. \tag{16}$$

The recursion relation (14) was originally used to described branchings for integrable modules. There exists an important class of modules that also can be reduced with the help of an injection fan – these are the Verma modules.

For a regular embedding $\mathfrak{a}\hookrightarrow\mathfrak{g}$ we have the corresponding embedding for the root systems $\Delta^{+}_{\mathfrak{a}}\hookrightarrow\Delta^{+}$. Moreover, in this case it is reasonable to enlarge the subalgebra $\mathfrak{a}$ and consider $\mathfrak{b}=\mathfrak{a}\oplus\mathfrak{h}_{\perp\mathfrak{a}}$. In this case the weights of a $\mathfrak{b}$-module can be considered in $\mathfrak{h}^{*}$. Verma module is decomposable for any highest weight:

$$\mathrm{ch}M^{\mu}=\sum_{\nu}b^{(\mu)}_{\nu}\mathrm{ch}M^{\nu}_{\mathfrak{b}}. \tag{17}$$

Verma module $\mathrm{ch}M^{\mu}$ have one singular weight μ with the multiplicity $(+1)$. The Weyl groups can be considered to be trivial. The orthogonal partner $\mathfrak{a}_{\perp}$ is also trivial and the recursion relation is simplified:

$$k^{(\mu)}_{\xi}=-\frac{1}{s\left(\gamma_0\right)}\left(\delta_{\xi-\gamma_0,\mu}+\sum_{\gamma\in\Gamma_{\mathfrak{a}\rightarrow\mathfrak{g}}}s\left(\gamma+\gamma_0\right)k^{(\mu)}_{\xi+\gamma}\right).$$

Remark 2. Notice that inside the main Weyl chamber branching coefficients $k^{(\mu)}_{\xi}$ for Verma modules reduction coincide with the coefficients for reduction of integrable simple modules.

Example 1. Let $\mathfrak{g}=B_2$ be and consider $A_1=\mathfrak{a}\hookrightarrow\mathfrak{g}$. There are different embeddings $A_1\hookrightarrow B_2$. Here we limit ourselves by the regular ones. The root system Δ_{B_2} consists of 8 roots. Let $S_{B_2}=\{\alpha_1=e_1-e_2,\alpha_2=e_2\}$ where e_1,e_2 form the standard basis in $\mathbf{R}^2$. The set of the positive roots is $\Delta^{+}=\{\alpha_1,\alpha_1+\alpha_2,\alpha_1+2\alpha_2,\alpha_2\}$. Each of these roots can be taken as the simple root of the A_1-subalgebra. Firstly we consider the subalgebra A_1 with the simple root $\beta=\alpha_1+2\alpha_2$. Consider the Verma module M^{μ} with the highest weight $\mu=\omega_1+2\omega_2=2\alpha_1+3\alpha_2$ as it is shown at Figure 1.

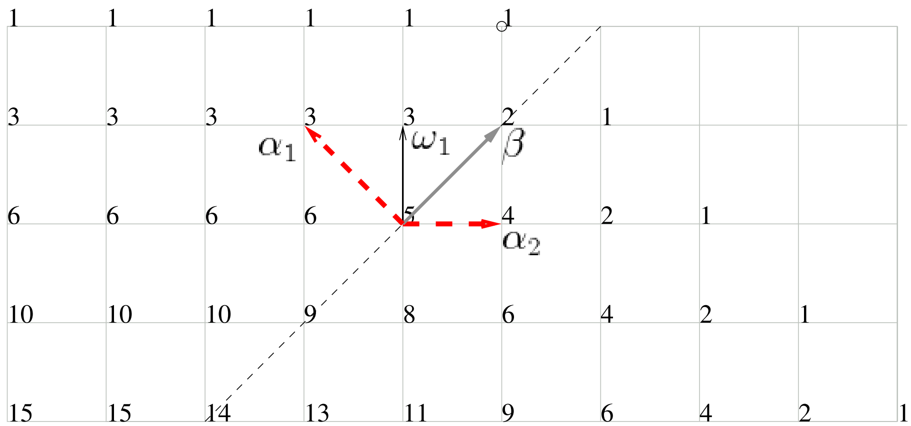

Figure 1: Regular embedding of A_1 into B_2. Simple roots α_1, α_2 of B_2 are presented as the dashed vectors. The simple root $\beta = \alpha_1 + 2\alpha_2$ of A_1 is indicated as the grey vector. Dimensions of weight subspaces of Verma module $M^{(1,2)}$ are shown.

The branching coefficients are shown at Figure 2 for the regular embedding $A_1 \to B_2$. Here we can see that the picture depends upon the embedding and looks similar to the Verma modules of some subalgebra.

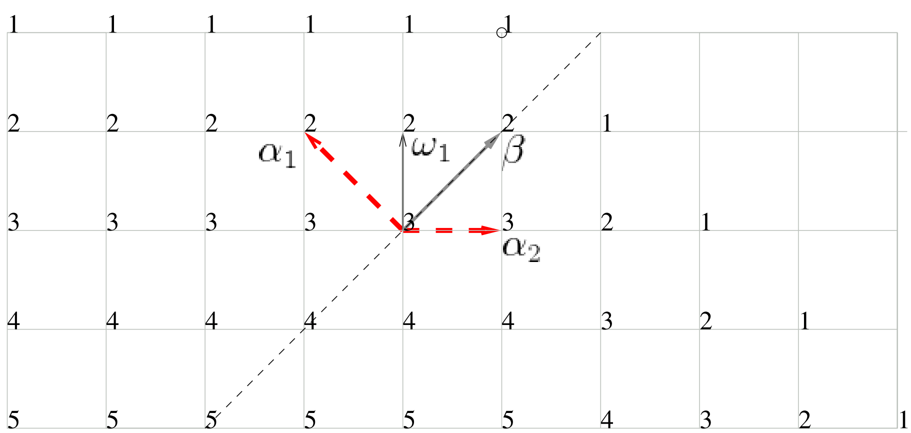

Figure 2: Branching of the Verma module $M^{(1,2)}$ for the regular embedding of A_1 into B_2. Simple roots α_1, α_2 of B_2 are presented as the dashed vectors. The simple root $\beta = \alpha_1 + 2\alpha_2$ of A_1 is indicated as the grey vector. Branching coefficients for Verma module of the subalgebra are shown. Dashed line indicates the direction of subalgebra modules.

In the present paper we interpret this branching coefficients as the dimensions of the weight subspaces of modules of contracted algebras.

2.4 Weyl-Verma formulas.

Statement 1. *For an orthogonal subalgebra $\mathfrak{a}_{\perp}$ in $\mathfrak{g}$ (orthogonal partner of reductive $\mathfrak{a} \hookrightarrow \mathfrak{g}$) the character of an integrable highest weight module L^{μ} can be presented as a combination (with integral coefficients) of parabolic Verma modules distributed by the set of weights $e^{\mu_{\widetilde{\mathfrak{a}}}(u)}$:*

$$\mathrm{ch}\left(L^{\mu}\right)=\sum_{u\in U}\epsilon(u)e^{\mu_{\widetilde{\mathfrak{a}}}(u)}\mathrm{ch}M_{I}^{\mu_{\mathfrak{a}_{\perp}}(u)},$$

where $U:=\left\{u\in W|\quad \mu_{\mathfrak{a}_{\perp}}(u)\in\overline{C_{\mathfrak{a}_{\perp}}}\right\}$ and Δ_{I}^{+} is equivalent to $\Delta_{\mathfrak{a}_{\perp}}^{+}$.

Proof. By the definition (2) the subalgebra $\mathfrak{a}_{\perp}$ is regular and reductive. Consider its Weyl denominator $R_{\mathfrak{a}_{\perp}}:=\prod_{\alpha\in\Delta_{\mathfrak{a}_{\perp}}^{+}}\left(1-e^{-\alpha}\right)^{\mathrm{mult}_{\mathfrak{a}}(\alpha)}$ and the element $R_{J}:=\prod_{\alpha\in\Delta^{+}\setminus\Delta_{\mathfrak{a}_{\perp}}^{+}}\left(1-e^{-\alpha}\right)^{\mathrm{mult}(\alpha)}$ as the factors in R:

$$R=R_{J}R_{\mathfrak{a}_{\perp}}.$$

According to this factorization and the decomposition (11) the character $\mathrm{ch}\left(L^{\mu}\right)$ can be written as

$$\begin{aligned}\mathrm{ch}\left(L^{\mu}\right) &= \left(R_{J}\right)^{-1}\left(R_{\mathfrak{a}_{\perp}}\right)^{-1}\Psi^{\mu}=\left(R_{J}\right)^{-1}\sum_{u\in U}e^{\mu_{\widetilde{\mathfrak{a}}}(u)}\epsilon(u)\left(R_{\mathfrak{a}_{\perp}}\right)^{-1}\Psi_{\mathfrak{a}_{\perp}}^{\mu_{\mathfrak{a}_{\perp}}(u)}\\ &= \left(R_{J}\right)^{-1}\sum_{u\in U}e^{\mu_{\widetilde{\mathfrak{a}}}(u)}\epsilon(u)L_{\mathfrak{a}_{\perp}}^{\mu_{\mathfrak{a}_{\perp}}(u)},\end{aligned}$$

where $\left\{L_{\mathfrak{a}_{\perp}}^{\mu_{\mathfrak{a}_{\perp}}(u)}|u\in U\right\}$ is the set of finite-dimensional $\mathfrak{a}_{\perp}$-modules with the highest weights $\mu_{\mathfrak{a}_{\perp}}(u)$. We are interested in nontrivial subalgebras $\mathfrak{a}$ and correspondingly in nontrivial $\mathfrak{a}_{\perp}$ (the case of a trivial orthogonal subalgebra arise when $\mathfrak{a}=\mathfrak{g}$ and was considered above (see Remark 1)). This means that $r_{\mathfrak{a}}\geq 1$ and $r_{\mathfrak{a}_{\perp}}<r$. Due to the fact that any maximal regular subalgebra has the Dynkin scheme obtained by one or two node subtractions from the extended Dynkin scheme and the extended scheme has at most one dependent

root (the highest root) the set of roots $\Delta^+_{\mathfrak{a}_\perp}$ is always equivalent to the one Δ^+_I generated by some subset $I \subset S$ of simple roots.

It follows that we can (by redefining the set Δ^+) identify $\Delta^+_{\mathfrak{a}_\perp}$ with the subset Δ^+_I where $I \subset S$. This allows us to introduce the elements necessary to compose the generalized Verma modules [3, 11]. We have two sets of root vectors $\left\{x_\xi \in \mathfrak{g}_\xi | \xi \in \Delta^+_I\right\}$ and $\left\{x_\eta \in \mathfrak{g}_\eta | \eta \in \Delta^+ \setminus \Delta^+_I\right\}$. They generate nilpotent subalgebras of $\mathfrak{n}^+$:

$$\mathfrak{n}^+_I := \sum_{\xi \in \Delta^+_I} \mathfrak{g}_\xi, \quad \mathfrak{u}^+_I := \sum_{\eta \in \Delta^+ \setminus \Delta^+_I} \mathfrak{g}_\eta.$$

The first subalgebra together with its negative counterpart $\mathfrak{n}^-_I$ generates a simple subalgebra

$$\mathfrak{s}_I = \mathfrak{n}^-_I + \mathfrak{h}_I + \mathfrak{n}^+_I.$$

We enlarge it with the remaining Cartan generators and introduce the subalgebra:

$$\mathfrak{l}_I = \mathfrak{n}^-_I + \mathfrak{h} + \mathfrak{n}^+_I.$$

The semidirect product of subalgebras $\mathfrak{l}_I$ and $\mathfrak{u}^+_I$ gives a parabolic subalgebra $\mathfrak{p}_I \hookrightarrow \mathfrak{g}$:

$$\mathfrak{p}_I = \mathfrak{l}_I \rhd \mathfrak{u}^+_I, \tag{18}$$

Its universal enveloping $U(\mathfrak{p}_I)$ is a subalgebra in $U(\mathfrak{g})$. According to the obtained structure (18) the $\mathfrak{l}_I$-modules $L^{\mu_{\mathfrak{a}_\perp}(u)}_{\mathfrak{a}_\perp}$ can be easily lifted to $\mathfrak{p}_I$-modules using the trivial action of the nilradical $\mathfrak{u}^+_I$. The latter induce $U(\mathfrak{g})$-modules in a standard way:

$$M^{\mu_{\mathfrak{a}_\perp}(u)}_I = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_I)} L^{\mu_{\mathfrak{a}_\perp}(u)}_{\mathfrak{a}_\perp}.$$

According to the defenition presented in [3] these are the *generalized Verma modules* generated by the highest weights $\mu_{\mathfrak{a}_\perp}(u)$. As a $U\left(\mathfrak{u}^-_I\right)$-module each $M^{\mu_{\mathfrak{a}_\perp}(u)}_I$ is isomorphic to $U\left(\mathfrak{u}^-_I\right) \otimes L^{\mu_{\mathfrak{a}_\perp}(u)}_{\mathfrak{a}_\perp}$ and thus its character can be written with the help of Kostant-Heckman function [12] corresponding to the injection of the orthogonal subalgebra $\mathfrak{a}_\perp \hookrightarrow \mathfrak{g}$:

$$\mathrm{ch} M^{\mu_{\mathfrak{a}_\perp}(u)}_I = \mathcal{KH}_{\mathfrak{a}_\perp \hookrightarrow \mathfrak{g}} \mathrm{ch} L^{\mu_{\mathfrak{a}_\perp}(u)}_{\mathfrak{a}_\perp}.$$

As far as the function $\mathcal{KH}_{\mathfrak{a}_\perp \hookrightarrow \mathfrak{g}}$ is generated by the denominator R_I the last expression can be rewritten in the form

$$\mathrm{ch} M^{\mu_{\mathfrak{a}_\perp}(u)}_I = \frac{1}{R_I} \mathrm{ch} L^{\mu_{\mathfrak{a}_\perp}(u)}_{\mathfrak{a}_\perp}.$$

This means that we have obtained the generalized Weyl-Verma character formula – the decomposition of $\mathrm{ch}\,(L^{\mu})$ in terms of generalized Verma modules characters:

$$\mathrm{ch}\,(L^{\mu}) = \sum_{u \in U} e^{\mu_{\widetilde{\mathfrak{a}}}(u)} \epsilon(u) \mathrm{ch} M_I^{\mu_{\mathfrak{a}_\perp}(u)}. \tag{19}$$

□

Remark 3. Here the generalized Weyl-Verma character formula appear in a detailed form: the weights $\mu_{\widetilde{\mathfrak{a}}}$ and the generalized Verma modules highest weights $\mu_{\mathfrak{a}_\perp}$ are indicated separately. The reason is that the highest weight of M_I-module is not equal to the projection of the maximal weight to $h^*_{\mathfrak{a}_\perp}$ (but must be additionally shifted by the defect).

Example 2. Continuing Example 1 we consider the generalized Verma modules for the embedding $A_1 \hookrightarrow B_2$ with the subalgebra $\mathfrak{a}_\perp$ built on the root α_1 of B_2. The generalized Verma module $M_I^{\omega_1}$ with the highest weight $\omega_1 = e_1$ is shown in Figure 3. The decomposition of the corresponding irreducible module L^{ω_1} is indicated by the set of dashed contours of the involved generalized Verma modules.

Remark 4. As it was proved in [11] (see Proposition 9.6) the characters of the generalized Verma modules $M_I^{\mu_{\mathfrak{a}_\perp}(u)}$ can be described as linear combinations of ordinary Verma modules of $\mathfrak{g}$:

$$\mathrm{ch} M_I^{\mu_{\mathfrak{a}_\perp}(u)} = \sum_{w \in W_{\mathfrak{a}_\perp}} \epsilon\,(w)\, \mathrm{ch} M^{w\left(\mu_{\mathfrak{a}_\perp}(u) + \rho_{\mathfrak{a}_\perp}\right) - \rho_{\mathfrak{a}_\perp}}$$

Substituting this expression in (19) and using the definitions (7,8) and (5) we reobtain the standard Weyl-Verma decomposition of the character:

$$\mathrm{ch}\,(L^{\mu}) = \sum_{w \in W} \epsilon(u) \mathrm{ch} M^{w(\mu+\rho)-\rho}.$$

3 Conclusions

In [6] it was demonstrated that the injection fan recursive mechanism works also for special injections. Here we must stress that the Weyl-Verma decomposition can be obtained in this case. The Weyl-Verma relations corresponding to the special subalgebras will describe the projections of characters of

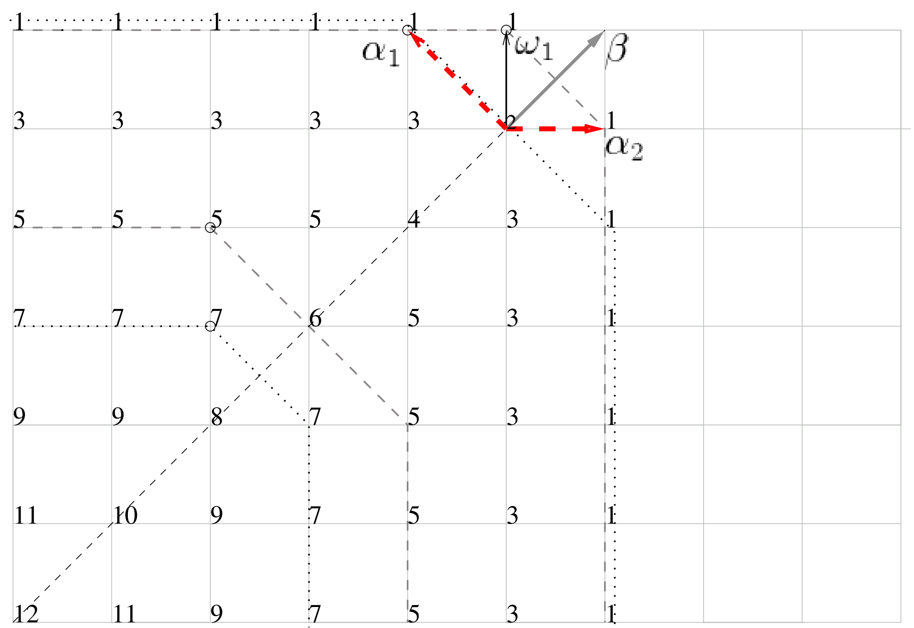

Figure 3: Generalized Verma module for the regular embedding of A_1 into B_2. Simple roots α_1, α_2 of B_2 are presented as the dashed vectors. The simple root $\beta = \alpha_1 + 2\alpha_2$ of A_1 is indicated as the grey vector. The decomposition of L^{ω_1} is indicated by the set of dashed contours of the involved generalized Verma modules. Dashed contours correspond to positive $\epsilon(u)$ and dotted to negative.

the initial module in terms of generalized Verma modules distributed in the subspace of h^*.

Consider the situation where the simple roots are prescribed by some external forces (originating in physical applications conditions, for example). In this case the orthogonal partner cannot be generated by simple root vectors only. The elements $\mathfrak{u}_I^+ := \sum_{\eta \in \Delta^+ \setminus \Delta_I^+} \mathfrak{g}_\eta$ do not form a subalgebra in $\mathfrak{g}$ because some nonsimple roots are lost in $\Delta^+ \setminus \Delta_I^+$. It is important to indicate that in this case the Weyl-Verma formula still exists. In it the generalized Verma modules correspond to the contractions [13] of the algebra $\mathfrak{n}^+$ and the Weyl-Verma relations describes the decomposition of the representation space of L^μ into the set of generalized Verma modules of contracted algebra $U(\mathfrak{n}_c^+)$. The weight vectors are formed by the PBW-basis of $U(\mathfrak{n}_c^+)$ and of $U(\mathfrak{a}_\perp)$. To consider such space as a $\mathfrak{g}$-module we must perform the deformation [14] of the algebra $\mathfrak{n}_c^+$ (restore the initial composition law). The space survives and after such a deformation the initial algebra generators will act properly on it.

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