# Recursive algorithm and branching for nonmaximal embeddings

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**Abstract.** Recurrent relations for branching coefficients in affine Lie algebras integrable highest weight modules are studied. The decomposition algorithm based on the injection fan technique is developed for the case of an arbitrary reductive subalgebra. In particular we consider the situation where the Weyl denominator becomes singular with respect to the subalgebra. We demonstrate that for any reductive subalgebra it is possible to define the injection fan and the analogue of the Weyl numerator – the tools that describe explicitly the recurrent properties of branching coefficients. Possible applications of fan technique in CFT models are considered.

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## 1. Introduction

The branching problem for affine Lie algebras emerges in conformal field theory, for example, in the construction of modular-invariant partition functions [1]. Recently the problem of conformal embeddings was considered in [2].

There are different approaches to deal with branching coefficients. Some of them use the BGG resolution [3] (for Kac-Moody algebras the algorithm is described in [4],[5]), the Schur function series [6], the BRST cohomology [7], Kac-Peterson formulas [4, 8] or some combinatorial methods applied in [9].

In this paper we prove that for an arbitrary reductive subalgebra branching coefficients are subject to a set of recurrent properties that can be explicitly formulated and that there exists an effective and simple algorithm to solve these recurrent relations step by step. The basic idea is similar to the one used in [11] for maximal embeddings. In our case the algorithm is essentially different, new properties of singular weights are determined to deal with an arbitrary reductive injection $\mathfrak{a} \to \mathfrak{g}$.

The principal point is to consider the subalgebra $\mathfrak{a}$ together with its counterpart $\mathfrak{a}_{\perp}$ orthogonal to $\mathfrak{a}$. For any reductive algebra $\mathfrak{a}$ the subalgebra $\mathfrak{a}_{\perp} \subset \mathfrak{g}$ is regular and reductive. For a highest weight module $L^{(\mu)}$ and orthogonal pair of subalgebras $(\mathfrak{a},\mathfrak{a}_{\perp})$ we consider the so called singular element $\Psi^{(\mu)}$ (the numerator in the Weyl character formula $ch\left(L^{\mu}\right)=\frac{\Psi^{(\mu)}}{\Psi^{(0)}}$, see for example [10]) the Weyl denominator $\Psi^{(0)}_{\mathfrak{a}_{\perp}}$ and the projection $\Psi^{(\mu)}_{(\mathfrak{a},\mathfrak{a}_{\perp})}=\pi_{\mathfrak{a}}\frac{\Psi^{(\mu)}_{\mathfrak{g}}}{\Psi^{(0)}_{\mathfrak{a}_{\perp}}}$. We prove that for any highest weight $\mathfrak{h}$-diagonalizable module $L^{(\mu)}$ and orthogonal pair $(\mathfrak{a},\mathfrak{a}_{\perp})$ the element $\Psi^{(\mu)}_{(\mathfrak{a},\mathfrak{a}_{\perp})}$ has a decomposition with respect to the set of Weyl numerators $\Psi^{(\mu)}_{\mathfrak{a}_{\perp}}$ of $\mathfrak{a}_{\perp}$. This decomposition provides the possibility to construct a recurrent property for branching coefficients corresponding to the injection $\mathfrak{a}\rightarrow\mathfrak{g}$. The property is formulated in terms of the specific element $\Gamma_{\mathfrak{a}\rightarrow\mathfrak{g}}$ of the group algebra $\mathcal{E}\left(\mathfrak{g}\right)$ called "the injection fan". Using this tool we formulate a simple and explicit algorithm for branching coefficients computations applicable for an arbitrary (maximal or nonmaximal) subalgebras of finite-dimensional or affine Lie algebras. In the case of maximal embedding the corresponding fan is unsubtracted, the singular element becomes trivial $\Psi^{(\mu)}_{(\mathfrak{a},\mathfrak{a}_{\perp})}=\Psi^{(\mu)}_{(\mathfrak{g})}$ and the relations described earlier in [11] are reobtained.

We demonstrate that our algorithm is effective and can be used in studies of conformal embeddings and coset constructions in rational conformal field theory.

The paper is organized as follows. In the subsection 1.1 we fix the general notations. In the Section 2 we derive the decomposition formula based on recurrent properties of anomalous branching coefficients and describe the decomposition algorithm for integrable highest weight modules $L_{\mathfrak{g}}$ with respect to a reductive subalgebra $\mathfrak{a}\subset\mathfrak{g}$ (subsection 2.5). In Section 3 we present several simple examples for finite-dimensional Lie algebras. Affine Lie algebras and their applications in CFT models are considered in Section 4. General properties of the proposed algorithm and possible further developments are also discussed (Section 5).

### *1.1. Notation*

Consider affine Lie algebras $\mathfrak{g}$ and $\mathfrak{a}$ with underlying finite-dimensional subalgebras $\overset{\circ}{\mathfrak{g}}$ and $\overset{\circ}{\mathfrak{a}}$ and an injection $\mathfrak{a}\longrightarrow\mathfrak{g}$ such that $\mathfrak{a}$ is a reductive subalgebra $\mathfrak{a}\subset\mathfrak{g}$ with correlated root spaces: $\mathfrak{h}^{*}_{\mathfrak{a}}\subset\mathfrak{h}^{*}_{\mathfrak{g}}$ and $\mathfrak{h}^{*}_{\overset{\circ}{\mathfrak{a}}}\subset\mathfrak{h}^{*}_{\overset{\circ}{\mathfrak{g}}}$ . We use the following notations:

$L^{\mu}$ $(L^{\nu}_{\mathfrak{a}})$ — the integrable module of $\mathfrak{g}$ with the highest weight $\mu$ ; (resp. integrable $\mathfrak{a}$ -module with the highest weight $\nu$ );

$r$ , $(r_{\mathfrak{a}})$ — the rank of the algebra $\mathfrak{g}$ (resp. $\mathfrak{a}$) ;

$\Delta$ $(\Delta_{\mathfrak{a}})$— the root system; $\Delta^{+}$ (resp. $\Delta^{+}_{\mathfrak{a}}$)— the positive root system (of $\mathfrak{g}$ and $\mathfrak{a}$ respectively);

$\mathrm{mult}\,(\alpha)$ $(\mathrm{mult}_{\mathfrak{a}}\,(\alpha))$ — the multiplicity of the root $\alpha$ in $\Delta$ (resp. in $(\Delta_{\mathfrak{a}})$);

$\overset{\circ}{\Delta}$ , $\left(\overset{\circ}{\Delta}_{\mathfrak{a}}\right)$ — the finite root system of the subalgebra $\overset{\circ}{\mathfrak{g}}$ (resp. $\overset{\circ}{\mathfrak{a}}$);

$\mathcal{N}^{\mu}$ , $(\mathcal{N}^{\nu}_{\mathfrak{a}})$ — the weight diagram of $L^{\mu}$ (resp. $L^{\nu}_{\mathfrak{a}}$) ;

$W$ , $(W_{\mathfrak{a}})$— the corresponding Weyl group;

$C$ , $(C_{\mathfrak{a}})$— the fundamental Weyl chamber;

$\bar{C}$, $\left(\bar{C}_{\mathfrak{a}}\right)$ — the closure of the fundamental Weyl chamber;

$\rho$ , $(\rho_{\mathfrak{a}})$ — the Weyl vector;

$\epsilon(w) := \det(w)$ ;

$\alpha_i$ , $(\beta_j)$ — the $i$-th (resp. $j$-th) basic root for $\mathfrak{g}$ (resp. $\mathfrak{a}$); $i = 0, \dots, r$, $(j = 0, \dots, r_{\mathfrak{a}})$;

$\delta$ — the imaginary root of $\mathfrak{g}$ (and of $\mathfrak{a}$ if any);

$\alpha_i^{\vee}$ , $(\beta_j^{\vee})$— the basic coroot for $\mathfrak{g}$ (resp. $\mathfrak{a}$) , $i = 0, \dots, r$ ; $(j = 0, \dots, r_{\mathfrak{a}})$;

$\overset{\circ}{\xi}$ , $\overset{\circ}{\xi}_{(\mathfrak{a})}$ — the finite (classical) part of the weight $\xi \in P$ , $\left(\text{resp. } \xi_{(\mathfrak{a})} \in P_{\mathfrak{a}}\right)$;

$\lambda = \left(\overset{\circ}{\lambda}; k; n\right)$ — decomposition of the affine weight $\lambda$ indicating the finite part $\overset{\circ}{\lambda}$, the level $k$ and the grade $n$;

$P$ (resp. $P_{\mathfrak{a}}$) — the weight lattice;

$m_{\xi}^{(\mu)}$ , $\left(m_{\xi}^{(\nu)}\right)$ — the multiplicity of the weight $\xi \in P$ (resp. $\in P_{\mathfrak{a}}$) in the module $L^{\mu}$ , (resp. $\xi \in L_{\mathfrak{a}}^{\nu}$);

$ch\left(L^{\mu}\right)$ (resp. $ch\left(L_{\mathfrak{a}}^{\nu}\right)$)— the formal character of $L^{\mu}$ (resp. $L_{\mathfrak{a}}^{\nu}$);

$ch\left(L^{\mu}\right) = \frac{\sum_{w \in W} \epsilon(w) e^{w \circ (\mu+\rho)-\rho}}{\prod_{\alpha \in \Delta^+} (1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}$ — the Weyl-Kac formula;

$R := \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)}$ $\left(\text{resp. } R_{\mathfrak{a}} := \prod_{\alpha \in \Delta_{\mathfrak{a}}^+} (1 - e^{-\alpha})^{\mathrm{mult}_{\mathfrak{a}}(\alpha)}\right)$— the Weyl denominator.

## 2. Recurrent relations for branching coefficients.

Consider an integrable module $L^{\mu}$ of $\mathfrak{g}$ with the highest weight $\mu$ and let $\mathfrak{a} \subset \mathfrak{g}$ be a reductive subalgebra of $\mathfrak{g}$. With respect to $\mathfrak{a}$ the module $L^{\mu}$ is completely reducible,

$$L_{\mathfrak{g}\downarrow\mathfrak{a}}^{\mu} = \bigoplus_{\nu \in P_{\mathfrak{a}}^+} b_{\nu}^{(\mu)} L_{\mathfrak{a}}^{\nu}.$$

Using the projection operator $\pi_{\mathfrak{a}}$ (to the weight space $\mathfrak{h}_{\mathfrak{a}}{}^*$) one can rewrite this decomposition in terms of formal characters:

$$\pi_{\mathfrak{a}} \circ ch\left(L^{\mu}\right) = \sum_{\nu \in P_{\mathfrak{a}}^+} b_{\nu}^{(\mu)} ch\left(L_{\mathfrak{a}}^{\nu}\right). \tag{1}$$

We are interested in branching coefficients $b_{\nu}^{(\mu)}$.

### *2.1. Orthogonal subalgebra and injection fan.*

In this subsection we shall introduce some simple constructions that will be used in our studies of branching and in particular the "orthogonal partner" $\mathfrak{a}_{\perp}$ for a reductive subalgebra $\mathfrak{a}$ in $\mathfrak{g}$.

In the Weyl-Kac formula both numerator and denominator can be considered as formal elements containing the singular weights of the Verma modules $V^{\xi}$ with the highest weights $\xi = \mu$ and $\xi = 0$ [10]. We attribute singular elements to the corresponding integrable modules $L^{\mu}$ and $L_{\mathfrak{a}}^{\nu}$:

$$\Psi^{(\mu)} := \sum_{w \in W} \epsilon(w) e^{w \circ (\mu+\rho)-\rho},$$

$$\Psi_{\mathfrak{a}}^{(\nu)} := \sum_{w\in W_{\mathfrak{a}}} \epsilon(w) e^{w\circ(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}.$$

and use the Weyl-Kac formula in the form

$$ch\left(L^{\mu}\right) = \frac{\Psi^{(\mu)}}{\Psi^{(0)}} = \frac{\Psi^{(\mu)}}{R}. \tag{2}$$

Applying formula (2) to the branching rule (1) we get the relation connecting the singular elements $\Psi^{(\mu)}$ and $\Psi_{\mathfrak{a}}^{(\nu)}$ :

$$\begin{aligned}
\pi_{\mathfrak{a}}\left(\frac{\sum_{w\in W}\epsilon(w)e^{w(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}\right) &= \sum_{\nu\in P_{\mathfrak{a}}^+} b_{\nu}^{(\mu)} \frac{\sum_{w\in W_{\mathfrak{a}}}\epsilon(w)e^{w(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}}{\prod_{\beta\in\Delta_{\mathfrak{a}}^+}(1-e^{-\beta})^{\mathrm{mult}_{\mathfrak{a}}(\beta)}},\\
\pi_{\mathfrak{a}}\left(\frac{\Psi^{(\mu)}}{R}\right) &= \sum_{\nu\in P_{\mathfrak{a}}^+} b_{\nu}^{(\mu)} \frac{\Psi_{\mathfrak{a}}^{(\nu)}}{R_{\mathfrak{a}}}.
\end{aligned} \tag{3}$$

Here $\Delta_{\mathfrak{a}}^+$ is the set of positive roots of the subalgebra $\mathfrak{a}$ (without loss of generality we consider them as vectors from the positive root space $\mathfrak{h}^{*+}$ of $\mathfrak{g}$).

Consider the root subspace $\mathfrak{h}_{\perp\mathfrak{a}}^*$ orthogonal to $\mathfrak{a}$,

$$\mathfrak{h}_{\perp\mathfrak{a}}^* := \left\{\eta\in\mathfrak{h}^* | \forall h\in\mathfrak{h}_{\mathfrak{a}}; \eta\left(h\right)=0\right\},$$

and the roots (correspondingly – positive roots) of $\mathfrak{g}$ orthogonal to $\mathfrak{a}$,

$$\begin{aligned}
\Delta_{\mathfrak{a}_\perp} &: = \left\{\beta\in\Delta_{\mathfrak{g}} | \forall h\in\mathfrak{h}_{\mathfrak{a}}; \beta\left(h\right)=0\right\},\\
\Delta_{\mathfrak{a}_\perp}^+ &: = \left\{\beta^+\in\Delta_{\mathfrak{g}}^+ | \forall h\in\mathfrak{h}_{\mathfrak{a}}; \beta^+\left(h\right)=0\right\}.
\end{aligned}$$

Let $W_{\mathfrak{a}_\perp}$ be the subgroup of $W$ generated by the reflections $w_\beta$ for the roots $\beta\in\Delta_{\mathfrak{a}_\perp}^+$ . The subsystem $\Delta_{\mathfrak{a}_\perp}$ determines the subalgebra $\mathfrak{a}_\perp$ with the Cartan subalgebra $\mathfrak{h}_{\mathfrak{a}_\perp}$. Let

$$\mathfrak{h}_\perp^* := \left\{\eta\in\mathfrak{h}_{\perp\mathfrak{a}}^* | \forall h\in\mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{a}_\perp}; \eta\left(h\right)=0\right\}$$

and consider the subalgebras

$$\begin{aligned}
\widetilde{\mathfrak{a}_\perp} &: = \mathfrak{a}_\perp\oplus\mathfrak{h}_\perp\\
\widetilde{\mathfrak{a}} &: = \mathfrak{a}\oplus\mathfrak{h}_\perp.
\end{aligned}$$

Algebras $\mathfrak{a}$ and $\mathfrak{a}_\perp$ form the "orthogonal pair" $(\mathfrak{a},\mathfrak{a}_\perp)$ of subalgebras in $\mathfrak{g}$.

For the Cartan subalgebra we have the decomposition

$$\mathfrak{h} = \mathfrak{h}_{\mathfrak{a}}\oplus\mathfrak{h}_{\mathfrak{a}_\perp}\oplus\mathfrak{h}_\perp = \mathfrak{h}_{\widetilde{\mathfrak{a}}}\oplus\mathfrak{h}_{\mathfrak{a}_\perp} = \mathfrak{h}_{\widetilde{\mathfrak{a}_\perp}}\oplus\mathfrak{h}_{\mathfrak{a}}. \tag{4}$$

For the subalgebras of an orthogonal pair $(\mathfrak{a},\mathfrak{a}_\perp)$ we consider the corresponding Weyl vectors, $\rho_{\mathfrak{a}}$ and $\rho_{\mathfrak{a}_\perp}$ , and form the so called "defects" $\mathcal{D}_{\mathfrak{a}}$ and $\mathcal{D}_{\mathfrak{a}_\perp}$ of the injection:

$$\mathcal{D}_{\mathfrak{a}} := \rho_{\mathfrak{a}} - \pi_{\mathfrak{a}}\rho, \tag{5}$$

$$\mathcal{D}_{\mathfrak{a}_\perp} := \rho_{\mathfrak{a}_\perp} - \pi_{\mathfrak{a}_\perp}\circ\rho. \tag{6}$$

For the highest weight module $L_{\mathfrak{g}}^{\mu}$ consider the singular weights $\left\{\left(w(\mu+\rho)-\rho\right)|w\in W\right\}$ and their projections to $h_{\widetilde{\mathfrak{a}_\perp}}^*$ (additionally shifted by the defect $-\mathcal{D}_{\mathfrak{a}_\perp}$):

$$\mu_{\widetilde{\mathfrak{a}_\perp}}\left(w\right) := \pi_{\widetilde{\mathfrak{a}_\perp}}\circ\left[w(\mu+\rho)-\rho\right] - \mathcal{D}_{\mathfrak{a}_\perp}, \quad w\in W.$$

Among the weights $\{\mu_{\widetilde{\mathfrak{a}_{\perp}}}(w)\,|w\in W\}$ choose those located in the fundamental chamber $\overline{C_{\widetilde{\mathfrak{a}_{\perp}}}}$ and let $U$ be the set of representatives $u$ for classes $W/W_{\mathfrak{a}_{\perp}}$ such that

$$U := \left\{u\in W| \quad \mu_{\widetilde{\mathfrak{a}_{\perp}}}(u)\in\overline{C_{\widetilde{\mathfrak{a}_{\perp}}}}\right\} \quad . \tag{7}$$

For the same set $U$ introduce the weights

$$\mu_{\mathfrak{a}}(u) := \pi_{\mathfrak{a}}\circ[u(\mu+\rho)-\rho]+\mathcal{D}_{\mathfrak{a}_{\perp}}.$$

To simplify the form of relations we shall now on omit the sign "$\circ$" in projected weights.

To describe the recurrent properties for branching coefficients $b_{\nu}^{(\mu)}$ we shall use the technique elaborated in [11]. One of the main tools is the set of weights $\Gamma_{\mathfrak{a}\to\mathfrak{g}}$ called the injection fan. As far as we consider the general situation (where the injection is not necessarily maximal) the notion of the injection fan is modified:

**Definition 1.** For the product

$$\prod_{\alpha\in\Delta^{+}\setminus\Delta^{+}_{\mathfrak{a}_{\perp}}}\left(1-e^{-\pi_{\mathfrak{a}}\alpha}\right)^{\mathrm{mult}(\alpha)-\mathrm{mult}_{\mathfrak{a}}(\pi_{\mathfrak{a}}\alpha)} = -\sum_{\gamma\in P_{\mathfrak{a}}} s(\gamma)e^{-\gamma} \tag{8}$$

consider the carrier $\Phi_{\mathfrak{a}\subset\mathfrak{g}}\subset P_{\mathfrak{a}}$ of the function $s(\gamma)=\det(\gamma)$ :

$$\Phi_{\mathfrak{a}\subset\mathfrak{g}} = \{\gamma\in P_{\mathfrak{a}}|s(\gamma)\neq 0\} \tag{9}$$

The ordering of roots in $\overset{\circ}{\Delta}_{\mathfrak{a}}$ induce the natural ordering of the weights in $P_{\mathfrak{a}}$. Denote by $\gamma_0$ the lowest vector of $\Phi_{\mathfrak{a}\subset\mathfrak{g}}$ . The set

$$\Gamma_{\mathfrak{a}\to\mathfrak{g}} = \{\xi-\gamma_0|\xi\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}\}\setminus\{0\} \tag{10}$$

is called the *injection fan.*

In the next subsection we shall see how the injection fan defines the recurrent properties of branching coefficients. It must be noticed that the injection fan is the universal instrument that depends only on the injection.

*2.2. Decomposing the singular element.*

Now we shall prove that the Weyl-Kac character formula (in terms of singular elements) describes the particular case of a more general relation:

**Lemma 1.** *Let $(\mathfrak{a},\mathfrak{a}_{\perp})$ be the orthogonal pair of reductive subalgebras in $\mathfrak{g}$, with $\widetilde{\mathfrak{a}_{\perp}}=\mathfrak{a}_{\perp}\oplus\mathfrak{h}_{\perp}$ and $\widetilde{\mathfrak{a}}=\mathfrak{a}\oplus\mathfrak{h}_{\perp}$ ,*

*$L^{\mu}$ be the highest weight module with the singular element $\Psi^{(\mu)}$ ,*

*$R_{\mathfrak{a}_{\perp}}$ be the Weyl denominator for $\mathfrak{a}_{\perp}$.*

*Then the element $\Psi^{(\mu)}_{(\mathfrak{a},\mathfrak{a}_{\perp})}=\pi_{\mathfrak{a}}\left(\frac{\Psi^{\mu}_{\mathfrak{g}}}{R_{\mathfrak{a}_{\perp}}}\right)$ can be decomposed into the sum over $u\in U$ (see (7)) of singular weights $e^{\mu_{\mathfrak{a}}(u)}$ with the coefficients $\epsilon(u)\mathrm{dim}\left(L_{\widetilde{\mathfrak{a}_{\perp}}}^{\mu_{\widetilde{\mathfrak{a}_{\perp}}}(u)}\right)$:*

$$\Psi^{(\mu)}_{(\mathfrak{a},\mathfrak{a}_{\perp})} = \quad \pi_{\mathfrak{a}}\left(\frac{\Psi^{\mu}}{R_{\mathfrak{a}_{\perp}}}\right)=\sum_{u\in U}\epsilon(u)\mathrm{dim}\left(L_{\widetilde{\mathfrak{a}_{\perp}}}^{\mu_{\widetilde{\mathfrak{a}_{\perp}}}(u)}\right)e^{\mu_{\mathfrak{a}}(u)}. \tag{11}$$

*Proof.* With $u \in U$ and $v \in W_{\mathfrak{a}_\perp}$ apply the decomposition

$$u(\mu+\rho) = \pi_{\mathfrak{a}} u(\mu+\rho) + \pi_{\widetilde{\mathfrak{a}_\perp}} u(\mu+\rho)$$

to the singular weight

$$\begin{aligned} vu(\mu+\rho)-\rho &= \pi_{\mathfrak{a}}\left(u(\mu+\rho)\right) - \rho + \rho_{\mathfrak{a}_\perp} + \pi_{\mathfrak{h}_\perp}\rho \\ &\quad + v\left(\pi_{\widetilde{\mathfrak{a}_\perp}} u(\mu+\rho) - \rho_{\mathfrak{a}_\perp} + \rho_{\mathfrak{a}_\perp}\right) - \rho_{\mathfrak{a}_\perp} - \pi_{\mathfrak{h}_\perp}\rho. \end{aligned} \tag{12}$$

Use the defect $\mathcal{D}_{\mathfrak{a}_\perp}$ (6) to simplify the first summand in (12):

$$\begin{aligned} \pi_{\mathfrak{a}}\left(u(\mu+\rho)\right) - \rho + \rho_{\mathfrak{a}_\perp} + \pi_{\mathfrak{h}_\perp}\rho = \\ \pi_{\mathfrak{a}}\left(u(\mu+\rho)\right) - \pi_{\mathfrak{a}}\rho - \pi_{\mathfrak{a}_\perp}\rho + \rho_{\mathfrak{a}_\perp} = \\ = \pi_{\mathfrak{a}}\left(u(\mu+\rho)-\rho\right) + \mathcal{D}_{\mathfrak{a}_\perp}, \end{aligned}$$

and the second one:

$$\begin{aligned} v\left(\pi_{\widetilde{\mathfrak{a}_\perp}} u(\mu+\rho) - \rho_{\mathfrak{a}_\perp} + \rho_{\mathfrak{a}_\perp}\right) - \rho_{\mathfrak{a}_\perp} - \pi_{\mathfrak{h}_\perp}\rho = \\ v\left(\pi_{\widetilde{\mathfrak{a}_\perp}} u(\mu+\rho) - \mathcal{D}_{\mathfrak{a}_\perp} - \pi_{\mathfrak{a}_\perp}\rho - \pi_{\mathfrak{h}_\perp}\rho + \rho_{\mathfrak{a}_\perp}\right) - \rho_{\mathfrak{a}_\perp} = \\ = v\left(\pi_{\widetilde{\mathfrak{a}_\perp}}\left[u(\mu+\rho)-\rho\right] - \mathcal{D}_{\mathfrak{a}_\perp} + \rho_{\mathfrak{a}_\perp}\right) - \rho_{\mathfrak{a}_\perp}. \end{aligned}$$

These expressions provide a kind of a factorization in the anomalous element $\Psi^\mu$ and we find in it the combination of anomalous elements $\Psi^\eta_{\widetilde{\mathfrak{a}_\perp}}$ of the subalgebra $\widetilde{\mathfrak{a}_\perp}$-modules $L^\eta_{\widetilde{\mathfrak{a}_\perp}}$:

$$\begin{aligned} \Psi^\mu &= \sum_{u\in U}\sum_{v\in W_{\mathfrak{a}_\perp}} \epsilon(v)\epsilon(u) e^{vu(\mu+\rho)-\rho} = \\ &= \sum_{u\in U} \epsilon(u) e^{\pi_{\mathfrak{a}}[u(\mu+\rho)-\rho]+\mathcal{D}_{\mathfrak{a}_\perp}} \sum_{v\in W_{\mathfrak{a}_\perp}} \epsilon(v) e^{v\left(\pi_{\widetilde{\mathfrak{a}_\perp}}[u(\mu+\rho)-\rho]-\mathcal{D}_{\mathfrak{a}_\perp}+\rho_{\mathfrak{a}_\perp}\right)-\rho_{\mathfrak{a}_\perp}} = \\ &= \sum_{u\in U} \epsilon(u) e^{\pi_{\mathfrak{a}}[u(\mu+\rho)-\rho]+\mathcal{D}_{\mathfrak{a}_\perp}} \Psi_{\widetilde{\mathfrak{a}_\perp}}^{\pi_{\widetilde{\mathfrak{a}_\perp}}[u(\mu+\rho)-\rho]-\mathcal{D}_{\mathfrak{a}_\perp}} \end{aligned}$$

Dividing both sides by the Weyl element $R_{\mathfrak{a}_\perp} = \prod_{\beta\in\Delta_{\mathfrak{a}_\perp}}(1-e^{-\beta})^{\mathrm{mult}(\beta)}$ and projecting them to the weight space $h^*_{\mathfrak{a}}$ we obtain the desired relation:

$$\begin{aligned} \Psi^{(\mu)}_{(\mathfrak{a},\mathfrak{a}_\perp)} &= \sum_{u\in W/W_{\mathfrak{a}_\perp}} \epsilon(u) e^{\pi_{\mathfrak{a}}[u(\mu+\rho)-\rho]} \pi_{\mathfrak{a}}\left(\frac{\Psi_{\widetilde{\mathfrak{a}_\perp}}^{\pi_{\widetilde{\mathfrak{a}_\perp}}[u(\mu+\rho)-\rho]-\mathcal{D}_{\mathfrak{a}_\perp}}}{\prod_{\beta\in\Delta_{\mathfrak{a}_\perp}}(1-e^{-\beta})^{\mathrm{mult}(\beta)}}\right) \\ &= \sum_{u\in U} \epsilon(u) \dim\left(L_{\widetilde{\mathfrak{a}_\perp}}^{\mu_{\widetilde{\mathfrak{a}_\perp}}(u)}\right) e^{\pi_{\mathfrak{a}}[u(\mu+\rho)-\rho]}. \end{aligned}$$

□

**Remark 1.** This relation can be considered a generalized form of the Weyl formula for singular element $\Psi^\mu_{\mathfrak{g}}$ : the vectors $\mu_{\mathfrak{a}}(u)$ play the role of singular weights while instead of the determinants $\epsilon(u)$ we have the products $\epsilon(u)\dim\left(L_{\widetilde{\mathfrak{a}_\perp}}^{\mu_{\widetilde{\mathfrak{a}_\perp}}(u)}\right)$. In fact when $\mathfrak{a} = \mathfrak{g}$ both $\mathfrak{a}_\perp$ and $\mathfrak{h}_\perp$ are trivial, $U = W$ , and the original Weyl formula is easily reobtained.

*2.3. Constructing recurrent relations.*

Consider the right-hand side of relation (3). The numerator there describes the branching in terms of singular elements and it is reasonable to expand it as an element of $\mathcal{E}(\mathfrak{g})$:

$$\sum_{\nu\in\bar{C}_{\mathfrak{a}}} b_{\nu}^{(\mu)}\Psi_{(\mathfrak{a})}^{(\nu)} = \sum_{\lambda\in P_{\mathfrak{a}}} k_{\lambda}^{(\mu)} e^{\lambda}. \tag{13}$$

Here the coefficients $k_{\lambda}^{(\mu)}$ are integer and their signs depend on the length (see [10]) of the Weyl group elements in $\Psi_{(\mathfrak{a})}^{(\nu)}$. The important property of $k_{\lambda}^{(\mu)}$'s is that they coincide with the branching coefficients for all weights $\nu$ inside the main Weil chamber:

$$b_{\nu}^{(\mu)} = k_{\nu}^{(\mu)} \text{ for } \nu\in\bar{C}_{\mathfrak{a}}. \tag{14}$$

We call the coefficients $k_{\lambda}$ — the anomalous branching coefficients (see also [11]).

Now we can state the main theorem which gives us an instrument for the recurrent computation of branching coefficients.

**Theorem 1.** *For the anomalous branching coefficients $k_{\nu}^{(\mu)}$ (13) the following relation holds*

$$\begin{aligned} k_{\xi}^{(\mu)} = -\tfrac{1}{s(\gamma_0)}\Big(&\textstyle\sum_{u\in U}\epsilon(u)\ \dim\left(L_{\widetilde{\mathfrak{a}_{\perp}}}^{\mu_{\widetilde{\mathfrak{a}_{\perp}}}(u)}\right)\delta_{\xi-\gamma_0,\pi_{\mathfrak{a}}(u(\mu+\rho)-\rho)}+ \\ &+\textstyle\sum_{\gamma\in\Gamma_{\mathfrak{a}\to\mathfrak{g}}} s(\gamma+\gamma_0)\, k_{\xi+\gamma}^{(\mu)}\Big). \end{aligned} \tag{15}$$

*Proof.* Redress relation (3) for the element $\frac{\Psi_{\mathfrak{g}}^{\mu}}{R_{\mathfrak{a}_{\perp}}}$ using definition (9) for the carrier $\Phi_{\mathfrak{a}\subset\mathfrak{g}}$ ,

$$\begin{aligned} \Psi_{(\mathfrak{a},\mathfrak{a}_{\perp})}^{(\mu)} &= \pi_{\mathfrak{a}}\left(\frac{\Psi_{\mathfrak{g}}^{\mu}}{R_{\mathfrak{a}_{\perp}}}\right) = \\ &= \prod_{\alpha\in\Delta^{+}\setminus\Delta^{+}_{\mathfrak{a}_{\perp}}} \left(1-e^{-\pi_{\mathfrak{a}}\alpha}\right)^{\mathrm{mult}(\alpha)-\mathrm{mult}_{\mathfrak{a}}(\pi_{\mathfrak{a}}\alpha)} \left(\sum_{\nu\in P_{\mathfrak{a}}^{+}} b_{\nu}^{(\mu)} \sum_{w\in W_{\mathfrak{a}}} \epsilon(w) e^{w(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}\right) = \\ &= -\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}} s(\gamma)e^{-\gamma}\left(\sum_{\nu\in P_{\mathfrak{a}}^{+},w\in W_{\mathfrak{a}}} \epsilon(w) b_{\nu}^{(\mu)} e^{w(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}\right) \\ &= -\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}} s(\gamma)e^{-\gamma}\left(\sum_{\nu\in P_{\mathfrak{a}}^{+},w\in W_{\mathfrak{a}}} \epsilon(w) b_{\nu}^{(\mu)} e^{w(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}\right). \end{aligned}$$

Then expand the sum in brackets (with respect to the formal basis in $\mathcal{E}$):

$$\Psi_{(\mathfrak{a},\mathfrak{a}_{\perp})}^{(\mu)} = -\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}} s(\gamma)e^{-\gamma}\sum_{\lambda\in P_{\mathfrak{a}}} k_{\nu}^{(\mu)} e^{\lambda} = -\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}}\sum_{\lambda\in P_{\mathfrak{a}}} s(\gamma) k_{\nu}^{(\mu)} e^{\lambda-\gamma}.$$

Substitute the expression obtained in Lemma 1 (in the left-hand side),

$$\begin{aligned} \Psi_{(\mathfrak{a},\mathfrak{a}_{\perp})}^{(\mu)} &= \sum_{u\in U} \epsilon(u) e^{\pi_{\mathfrak{a}}(\mu_{\mathfrak{a}}(u))} \dim\left(L_{\widetilde{\mathfrak{a}_{\perp}}}^{\mu_{\widetilde{\mathfrak{a}_{\perp}}}(u)}\right) \\ &= \sum_{u\in U} \epsilon(u) e^{\pi_{\mathfrak{a}}[u(\mu+\rho)-\rho]} \dim\left(L_{\widetilde{\mathfrak{a}_{\perp}}}^{\mu_{\widetilde{\mathfrak{a}_{\perp}}}(u)}\right) \\ &= -\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}}\sum_{\lambda\in P_{\mathfrak{a}}} s(\gamma) k_{\nu}^{(\mu)} e^{\lambda-\gamma}. \end{aligned}$$

The immediate consequence of this equality is:

$$\sum_{u\in U}\epsilon(u)\dim\left(L_{\widetilde{\mathfrak{a}_\perp}}^{\mu_{\widetilde{\mathfrak{a}_\perp}}(u)}\right)\delta_{\xi,\pi_{\mathfrak{a}}[u(\mu+\rho)-\rho]}+\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}}s(\gamma)\,k_{\xi+\gamma}^{(\mu)}=0,\quad \xi\in P_{\mathfrak{a}}. \tag{16}$$

The obtained formula means that the coefficients $k_{\xi+\gamma}^{(\mu)}$ for $\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}$ are not independent, they are subject to linear relations and the form of these relations changes when the tested weight $\xi$ coincides with one of the "singular weights" $\{\pi_{\mathfrak{a}}[u(\mu+\rho)-\rho]\,|u\in U\}$ . To conclude the proof we extract the lowest weight $\gamma_0\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}$ and pass to a summation over the vectors of the injection fan $\Gamma_{\mathfrak{a}\to\mathfrak{g}}$ (see Definition 1). Thus we get the desired recurrent relation (15). □

### *2.4. Embeddings and orthogonal pairs in simple Lie algebras*

In this subsection we discuss some properties of "orthogonal pairs" of subalgebras in simple Lie algebras of classical series.

When both $\mathfrak{g}$ and $\mathfrak{a}$ are finite-dimensional all regular embeddings can be obtained by a successive elimination of nodes in the extended Dynkin diagram of $\mathfrak{g}$ (and $\Delta^+_{\mathfrak{a}_\perp}=\emptyset$ if $\mathfrak{a}$ is maximal). For the classical series $A$, $C$ and $D$ when the regular injection $\mathfrak{a}\to\mathfrak{g}$ is thus fixed, the Dynkin diagram for $\mathfrak{a}_\perp$ is obtained from the extended diagram of $\mathfrak{g}$ by eliminating the subdiagram of $\mathfrak{a}$ and the adjacent nodes:

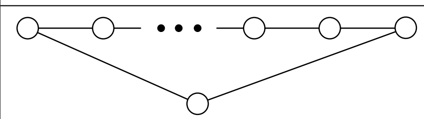
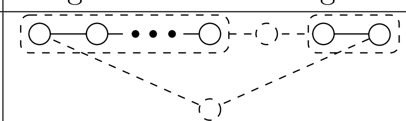
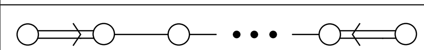
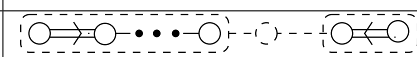
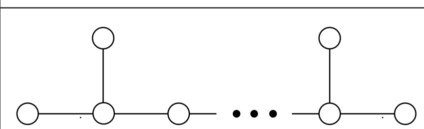
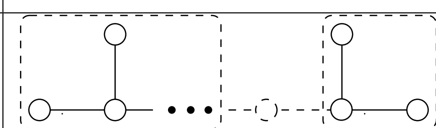
| $\mathfrak{g}$ | Extended diagram of $\mathfrak{g}$ | Diagrams of the subalgebras $\mathfrak{a}$, $\mathfrak{a}_\perp$ | |
|---|---|---|---|
| $A_n$ |  |  | |
| $C_n$ |  |  | |
| $D_n$ |  |  | |

**Table 1.** Subalgebras $\mathfrak{a}$, $\mathfrak{a}_\perp$ for the classical series

In the case of $B$ series the situation is different. The reason is that here the subalgebra $\mathfrak{a}_\perp$ may be larger than the one obtained by elimination of the subdiagram of $\mathfrak{a}$ and the adjacent nodes. The subalgebras of the orthogonal pair, $\mathfrak{a}$ and $\mathfrak{a}_\perp$, must not form a direct sum in $\mathfrak{g}$ . It can be directly checked that when $\mathfrak{g}=B_r$ and $\mathfrak{a}=B_{r_{\mathfrak{a}}}$ the orthogonal subalgebra is $\mathfrak{a}_\perp=B_{r-r_{\mathfrak{a}}}$. Consider the injection $B_{r_{\mathfrak{a}}}\to B_r$, $\quad 1<r_{\mathfrak{a}}<r$. By eliminating the simple root $\alpha_{r_{\mathfrak{a}}-1}=e_{r_{\mathfrak{a}}-1}-e_{r_{\mathfrak{a}}}$ one splits the extended Dynkin diagram of $B_r$ into the disjoint diagrams for $\mathfrak{a}=B_{r_{\mathfrak{a}}}$ and $D_{r-r_{\mathfrak{a}}}$. But the system $\Delta_{\mathfrak{a}_\perp}$ contains not only the simple roots $\{e_1-e_2,e_2-e_3,\dots,e_{r_{\mathfrak{a}}-2}-e_{r_{\mathfrak{a}}-1},e_1+e_2\}$ but also the root $e_{r_{\mathfrak{a}}-1}$. Thus $\Delta_{\mathfrak{a}_\perp}$ forms the subsystem of the type $B_{r-r_{\mathfrak{a}}}$ and the orthogonal pair for the injection $B_{r_{\mathfrak{a}}}\to B_r$ is $(B_{r_{\mathfrak{a}}},B_{r-r_{\mathfrak{a}}})$. In the next Section a particular case of such orthogonal pair is presented for the injection $B_2\to B_4$ (see Figure 3).

The complete classification of regular subalgebras for affine Lie algebras can be found in the recent paper [12]. From the complete classification of maximal special subalgebras in classical Lie algebras [13] we can deduce the following list of pairs of orthogonal subalgebras $\mathfrak{a}$, $\mathfrak{a}_\perp$:

$$\begin{array}{lll} su(p) \oplus su(q) & \subset su(pq) & \\ so(p) \oplus so(q) & \subset so(pq) & \\ sp(2p) \oplus sp(2q) & \subset so(4pq) & \\ sp(2p) \oplus so(q) & \subset sp(2pq) & \\ so(p) \oplus so(q) & \subset so(p+q) & \text{for p and q odd.} \end{array}$$

*2.5. Algorithm for recursive computation of branching coefficients*

The recurrent relation (15) allows us to formulate an algorithm for recursive computation of branching coefficients. In this algorithm there is no need to construct the module $L^{(\mu)}_{\mathfrak{g}}$ or any of the modules $L^{(\nu)}_{\mathfrak{a}}$.

It contains the following steps:

(i) Construct the root system $\Delta_{\mathfrak{a}}$ for the embedding $\mathfrak{a} \rightarrow \mathfrak{g}$.

(ii) Select all positive roots $\alpha \in \Delta^+$ orthogonal to $\mathfrak{a}$, i.e. form the set $\Delta^+_{\mathfrak{a}_\perp}$.

(iii) Construct the set $\Gamma_{\mathfrak{a}\rightarrow\mathfrak{g}}$. Relation (8) defines the sign function $s(\gamma)$ and the set $\Phi_{\mathfrak{a}\subset\mathfrak{g}}$ where the lowest weight $\gamma_0$ is to be subtracted to get the fan (10): $\Gamma_{\mathfrak{a}\rightarrow\mathfrak{g}} = \{\xi - \gamma_0 | \xi \in \Phi_{\mathfrak{a}\subset\mathfrak{g}}\} \setminus \{0\}$.

(iv) Construct the set $\widehat{\Psi^{(\mu)}} = \{w(\mu+\rho) - \rho;\ w \in W\}$ of singular weights for the $\mathfrak{g}$-module $L^{(\mu)}$.

(v) Select the weights $\left\{\mu_{\widetilde{\mathfrak{a}_\perp}}(w) = \pi_{\widetilde{\mathfrak{a}_\perp}}[w(\mu+\rho)-\rho] - \mathcal{D}_{\mathfrak{a}_\perp} \in \overline{C_{\widetilde{\mathfrak{a}_\perp}}}\right\}$. Since the set $\Delta^+_{\mathfrak{a}_\perp}$ is fixed we can easily check wether the weight $\mu_{\widetilde{\mathfrak{a}_\perp}}(w)$ belongs to the main Weyl chamber $\overline{C_{\widetilde{\mathfrak{a}_\perp}}}$ (by computing its scalar product with the fundamental weights of $\mathfrak{a}^+_\perp$).

(vi) For the weights $\mu_{\widetilde{\mathfrak{a}_\perp}}(w)$ calculate dimensions of the corresponding modules, $\dim\left(L^{\mu_{\widetilde{\mathfrak{a}_\perp}}(u)}_{\widetilde{\mathfrak{a}_\perp}}\right)$, using the Weyl dimension formula and construct the singular element $\Psi^{(\mu)}_{(\mathfrak{a},\mathfrak{a}_\perp)}$.

(vii) Calculate the anomalous branching coefficients using the recurrent relation (15) and select among them those corresponding to the weights in the main Weyl chamber $\overline{C_{\mathfrak{a}}}$.

We can speed up the algorithm by one-time computation of the representatives of the conjugate classes $W/W_{\mathfrak{a}_\perp}$.

The next section contains examples illustrating the application of this algorithm.

## 3. Branching for finite dimensional Lie algebras

### *3.1. Regular embedding of $A_1$ into $B_2$*

Consider the regular embedding $A_1 \to B_2$. Simple roots $\alpha_1, \alpha_2$ of $B_2$ are presented as dashed vectors in Figure 1. We denote the corresponding Weyl reflections by $w_1, w_2$. The simple root $\beta = \alpha_1 + 2\alpha_2$ of $A_1$ is grey.

Let's perform the reduction of the fundamental representation $L_{B_2}^{(1,0)=\omega_1}$ ($\omega_1$ – the black vector in Figure 1) according to the steps of the algorithm. The root $\alpha_1$ is orthogonal to $\beta$, so we have $\Delta^+_{\mathfrak{a}_\perp} = \{\alpha_1\}$ (step (ii)). According to Definition 1 the fan $\Gamma_{A_1 \to B_2}$ (step (iii)) consists of two weights:

$$\Gamma_{A_1 \to B_2} = \{(1;2),\ (2;-1)\},$$

where the second component is the value of the sign function $s(\gamma)$. Singular weights $\{w(\omega_1+\rho)-\rho;\ w \in W\}$ (step (iv)) are indicated by circles with the superscript $\epsilon(w)$. The space $U$ is the factor $W/W_{\mathfrak{a}_\perp}$ where $W_{\mathfrak{a}_\perp} = \{e, w_1\}$. This means that singular weights located above the $\beta$-line belong to the Weyl chamber $\overline{C_{\widetilde{\mathfrak{a}_\perp}}}$. According to formula (6) we have $\mathcal{D}_{\mathfrak{a}_\perp} = 0$ and $\mathfrak{h}_\perp = 0$, thus $\{\mu_{\mathfrak{a}_\perp}(w) = \pi_{\mathfrak{a}_\perp}[w(\mu+\rho)-\rho]\}$. We obtain four highest weights for $\mathfrak{a}_\perp$-modules. In terms of $\mathfrak{a}_\perp$-fundamental weight $\frac{1}{2}\alpha_1$ these highest weights $\{\mu_{\mathfrak{a}_\perp}(u) = \pi_{\mathfrak{a}_\perp}[u(\mu+\rho)-\rho] \,|u \in U\}$ are $\{(1)\,(2)\,(2)\,(1)\}$ (step v). In Figure (2) the corresponding weight diagrams $\left\{\mathcal{N}_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}\right\}$ are attached to the set of $\mathfrak{a}$-weights $\{\mu_{\mathfrak{a}}(u)\} = \{\pi_{\mathfrak{a}}[u(\mu+\rho)-\rho]\} = \{(1)\,(0)\,(-4)\,(-5)\}$. In fact we do not need the weight diagrams but only the dimensions of modules $L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}$ multiplied by $\epsilon(u)$ (step vi). Obtained values are to be attributed to the points $\{(1)\,(0)\,(-4)\,(-5)\}$ in $P_{\mathfrak{a}}$. The singular element $\Psi^{(\mu)}_{(\mathfrak{a},\mathfrak{a}_\perp)}$ has the set of weights with anomalous multiplicities:

$$\{(1;2),\ (0;-3),\ (-4;3),\ (-5;-2)\}. \tag{17}$$

Applying formula (15) with the fan $\Gamma_{A_1 \to B_2}$ to the set (17) (step vii) we get zeros for the weights greater than the highest anomalous vector $(1;2)$ and $k_1^{(1,0)} = 2$ for the vector $(1;2)$ itself. For the anomalous weight (0;-3) on the boundary of $\bar{C}_{\mathfrak{a}}^{(0)}$ the recurrent relation gives

$$k_0^{(1,0)} = -1 \cdot k_2^{(1,0)} + 2 \cdot k_1^{(1,0)} - 3 \cdot \delta_{0,0} = 1,$$

the branching is completed: $L^{\omega_1}_{B_2 \downarrow A_1} = 2L_{A_1}^{\omega_{(A_1)}} \bigoplus L_{A_1}^{2\omega_{(A_1)}}$.

### *3.2. Embedding $B_2$ into $B_4$*

Consider the regular embedding $B_2 \to B_4$. The corresponding Dynkin diagrams are presented in the Figure 3.

In the orthonormal basis $\{e_1, \ldots, e_4\}$ simple roots and positive roots of $B_4$ are

$$S_{B_4} = \{e_1 - e_2,\ e_2 - e_3,\ e_3 - e_4,\ e_4\},$$

$$\Delta^+_{B_4} = \{(e_1 - e_2,\ e_2 - e_3,\ e_3 - e_4,\ e_4,\ e_1 - e_3,\ e_2 - e_4,\ e_3 + e_4,\ e_3,\ e_1 - e_4,\\ e_2 + e_4,\ e_2,\ e_1 + e_4,\ e_2 + e_3,\ e_1,\ e_1 + e_3,\ e_1 + e_2\}$$

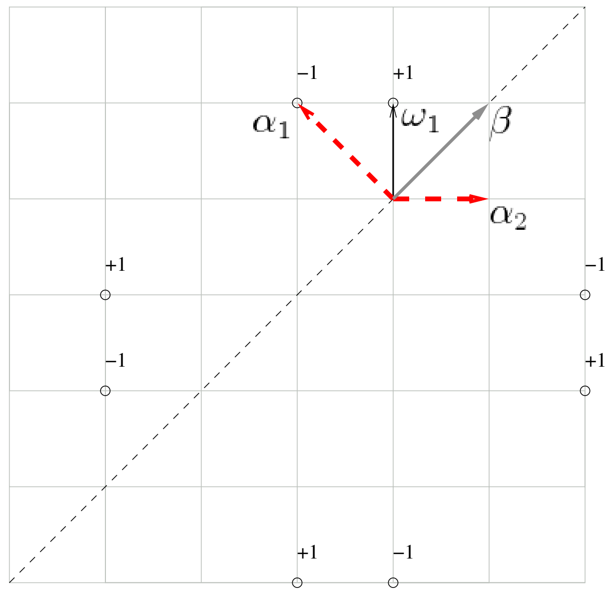



**Figure 1.** Regular embedding of $A_1$ into $B_2$. Simple roots $\alpha_1, \alpha_2$ of $B_2$ are presented as dashed vectors. The simple root $\beta = \alpha_1 + 2\alpha_2$ of $A_1$ is grey. The highest weight of the fundamental representation $L_{B_2}^{(1,0)=\omega_1}$ is black. The weights of the singular element $\Psi^{(\omega_1)}$ are marked by circles with superscripts indicating the corresponding determinants $\epsilon(w)$.

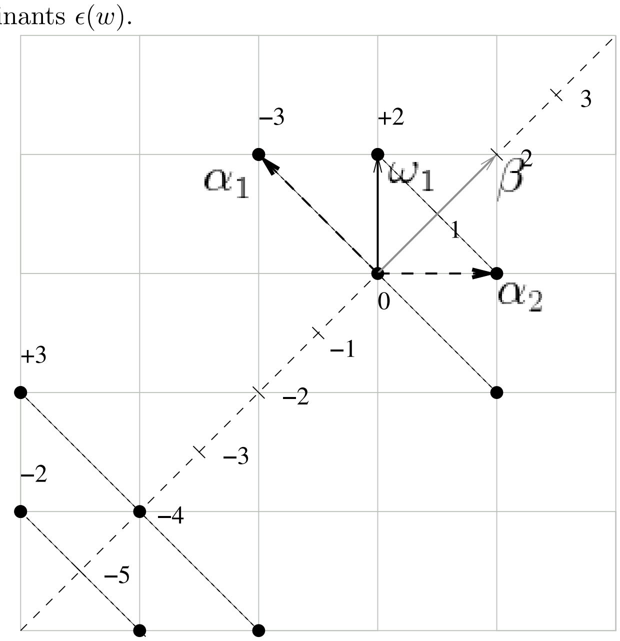



**Figure 2.** Here in addition to the diagram presented above (Figure(1)) the weights of $(\mathfrak{a}_\perp = A_1)$-modules $L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}$ originating in the points $\pi_{\mathfrak{a}}\left[u(\mu+\rho)-\rho\right]$ are shown by dotted lines. The superscripts over the highest weights $\mu_{\mathfrak{a}_\perp}(u)$ are now the products $\epsilon(u)\dim\left(L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}\right)$. Coordinates along the root $\beta$ are counted in terms of the fundamental weight of $\mathfrak{a}$.

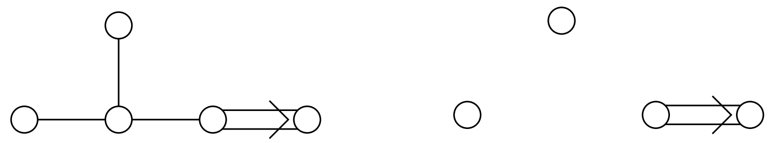


**Figure 3.** The regular embedding $B_2 \to B_4$ described by dropping the node from the Dynkin diagram. Remember that here $\mathfrak{a}_\perp$ is equal to $B_2$ while the diagram shows only $A_1 \oplus A_1$ (see Subsection 2.4).

The subalgebra $\mathfrak{a} = B_2$ is fixed by the simple roots

$$S_{B_2} = \{e_3 - e_4, e_4\}.$$

Its orthogonal counterpart $\mathfrak{a}_\perp = B_2$ has

$$S_{\mathfrak{a}_\perp} = \{e_1 - e_2, e_2\},$$
$$\Delta^+_{\mathfrak{a}_\perp} = \{e_1 - e_2, e_1 + e_2, e_1, e_2\}.$$

As far as the set $\Delta^+_{B_4} \setminus \Delta^+_{\mathfrak{a}_\perp}$ is fixed the injection fan $\Gamma_{B_2 \to B_4}$ can be constructed using Definition 1. As far as for this injection $s(\gamma_0) = -1$ in the recursion formula we need only the factor $s(\gamma + \gamma_0)$. The result is presented in Figure 5.

Consider the $B_4$-module $L^\mu$ with the highest weight $\mu = 2e_1 + 2e_2 + e_3 + e_4$; $\dim(L^{[0,1,0,2]}) = 2772$. Here the defect is nontrivial, $\mathcal{D}_{\mathfrak{a}_\perp} = -2(e_1 + e_2)$, while $\mathfrak{h}_\perp = 0$. Among the singular weights there are 48 vectors with the property $\left\{\mu_{\mathfrak{a}_\perp}(u) = \pi_{\mathfrak{a}_\perp}[u(\mu+\rho) - \rho] - \mathcal{D}_{\mathfrak{a}_\perp} \in \overline{C_{\mathfrak{a}_\perp}}\right\}$. The set $U = \{u\}$ is thus fixed. Compute the dimensions of the corresponding $\mathfrak{a}_\perp$-modules with the highest weights $\mu_{\mathfrak{a}_\perp}(u)$ (using the Weyl dimension formula) and multiply them by $\epsilon(u)$. The result is the singular element $\Psi^{(\mu)}_{(\mathfrak{a},\mathfrak{a}_\perp)}$ shown in Figure 4.

Now one can place the fan $\Gamma$ (see Figure 5) in the highest of the weights presented in Figure 4 and start the recursive determination of the branching coefficients (using relation (15)):

$$\pi_{\mathfrak{a}}\left(chL^{[0,1,0,2]}_{B_4}\right) = 6\, chL^{[0,0]}_{B_2} + 60\, chL^{[0,2]}_{B_2} + 30\, chL^{[1,0]}_{B_2} + 19\, chL^{[2,0]}_{B_2} +$$
$$40\, chL^{[1,2]}_{B_2} + 10\, chL^{[0,4]}_{B_2}.$$

## 4. Applications to conformal field theory

### *4.1. Conformal embeddings*

Branching coefficients for an embedding of affine Lie algebra into affine Lie algebra can be used to construct modular invariant partition functions for Wess-Zumino-Novikov-Witten models in conformal field theory ([1], [14], [15], [16]). In these models current algebras are affine Lie algebras.

The modular invariant partition function is crucial for the conformal theory to be valid on the torus and higher genus Riemann surfaces. It is important for the applications of CFT to string theory and to critical phenomena description.

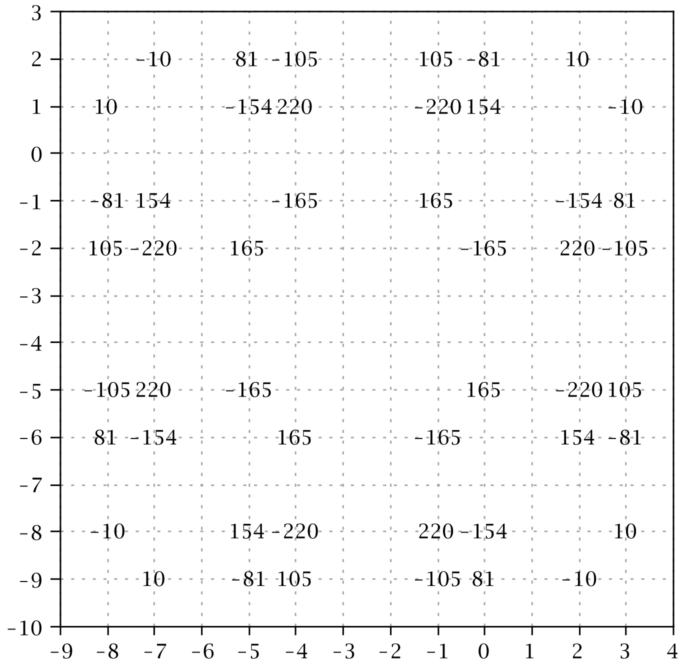


**Figure 4.** The singular element $e^{\gamma_0}\Psi^{(\mu)}_{(\mathfrak{a},\mathfrak{a}_\perp)}$ displayed in the weight subspace $P_{\mathfrak{a}}$ for $\mathfrak{a} = B_2$ with the basis $\{e_3, e_4\}$. We see the projected singular weights $\{\pi_{\mathfrak{a}}[u(\mu+\rho)-\rho]+\gamma_0|u \in U\}$ shifted by $\gamma_0$ and supplied by multipliers $\epsilon(u)\dim\left(L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}\right)$.

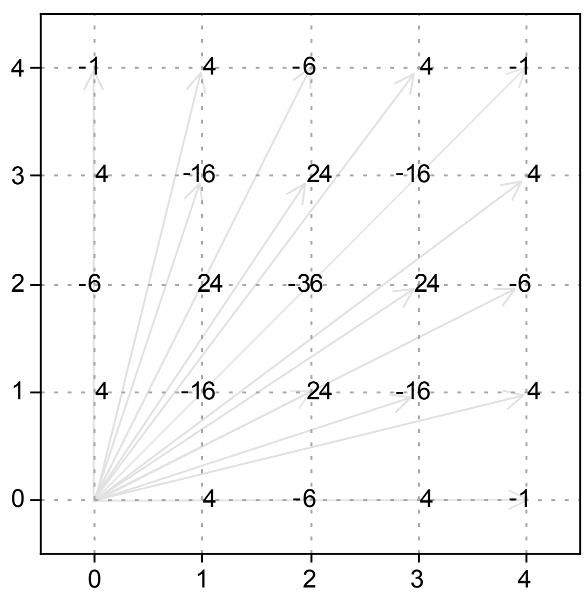


**Figure 5.** The fan $\Gamma$ for $B_2 \to B_4$ with values of $s(\gamma+\gamma_0)$ attributed to each $\gamma \in \Gamma$.

The simplest modular-invariant partition function has the diagonal form:

$$Z(\tau)=\sum_{\mu\in P^+_{\mathfrak{g}}}\chi_\mu(\tau)\bar{\chi}_\mu(\bar{\tau}) \tag{18}$$

Here the sum is over the set of highest weights for integrable modules in a WZW-model and $\chi_\mu(\tau)$ are the normalized characters (see [1]) of these modules.

To construct nondiagonal modular invariants is not an easy problem, although for some models the complete classification of modular invariants is known [17, 18].

Consider the Wess-Zumino-Witten model with the affine Lie algebra $\mathfrak{a}$. Nondiagonal modular invariants for this model can be constructed from the diagonal invariant if there exists an affine algebra $\mathfrak{g}$ such that $\mathfrak{a}\subset\mathfrak{g}$. Then we can replace the characters of the $\mathfrak{g}$-modules in the diagonal modular invariant partition function (18) by the decompositions

$$\sum_{\nu\in P^+_{\mathfrak{a}}} b^{(\mu)}_\nu\chi_\nu$$

containing normalized characters $\chi_\nu$ of the corresponding $\mathfrak{a}$-modules. Thus we obtain a nondiagonal modular-invariant partition function for the theory with the current algebra $\mathfrak{a}$,

$$Z_{\mathfrak{a}}(\tau)=\sum_{\nu,\lambda\in P^+_{\mathfrak{a}}}\chi_\nu(\tau)M_{\nu\lambda}\bar{\chi}_\lambda(\bar{\tau}). \tag{19}$$

The effective reduction procedure is crucial for this construction. The embedding is required to preserve the conformal invariance. Let $X^{\alpha_j}_{-n_j}$ and $\tilde{X}^{\alpha'_j}_{-n_j}$ be the lowering generators for $\mathfrak{g}$ and for $\mathfrak{a}\subset\mathfrak{g}$ correspondingly. Let $\pi_{\mathfrak{a}}$ be the projection operator of $\pi_{\mathfrak{a}}:\mathfrak{g}\longrightarrow\mathfrak{a}$. In the theory attributed to $\mathfrak{g}$ with the vacuum $|\lambda\rangle$ the states can be described as

$$X^{\alpha_1}_{-n_1}X^{\alpha_2}_{-n_2}\dots|\lambda\rangle \quad n_1\geq n_2\geq\ldots>0.$$

And for the sub-algebra $\mathfrak{a}$ the corresponding states are

$$\tilde{X}^{\alpha'_1}_{-n_1}\tilde{X}^{\alpha'_2}_{-n_2}\dots|\pi_{\mathfrak{a}}(\lambda)\rangle\,.$$

The $\mathfrak{g}$-invariance of the vacuum entails its $\mathfrak{a}$-invariance, but this is not the case for the energy-momentum tensor. So the energy-momentum tensor of the larger theory should contain only the generators $\tilde{X}$. Then the relation

$$T_{\mathfrak{g}}(z)=T_{\mathfrak{a}}(z) \tag{20}$$

leads to the equality of central charges

$$c(\mathfrak{g})=c(\mathfrak{a})$$

and to the relation

$$\frac{k\,\dim\mathfrak{g}}{k+g}=\frac{x_ek\,\dim\mathfrak{a}}{x_ek+a}. \tag{21}$$

Here $x_e$ is the so called "embedding index": $x_e=\frac{|\pi_{\mathfrak{a}}\Theta|^2}{|\Theta_{\mathfrak{a}}|^2}$ with $\Theta$, $\Theta_{\mathfrak{a}}$ being the highest roots of $\mathfrak{g}$ and $\mathfrak{a}$ while $g$ and $a$ are the corresponding dual Coxeter numbers.

It can be demonstrated that solutions of equation (21) exist only for the level $k=1$ [1].

The complete classification of conformal embeddings is given in [16].

The relation (21) and the asymptotics of the branching functions can be used to prove the finite reducibility theorem [19]. It states that for a conformal embedding $\mathfrak{a} \longrightarrow \mathfrak{g}$ only finite number of branching coefficients have nonzero values.

**Note 4.1.** *The orthogonal subalgebra $\mathfrak{a}_{\perp}$ is always trivial for conformal embeddings $\mathfrak{a} \longrightarrow \mathfrak{g}$.*

*Proof.* Consider the modes expansion of the energy-momentum tensor

$$T(z) = \frac{1}{2(k+h^v)} \sum_n z^{-n-1} L_n.$$

The modes $L_n$ are constructed as combinations of normally-ordered products of $\mathfrak{g}$-algebra generators,

$$L_n = \frac{1}{2(k+h^v)} \sum_{\alpha} \sum_{m} : X^{\alpha}_m X^{\alpha}_{n-m} : \; .$$

In case of a conformal embedding energy-momentum tensors $T_{\mathfrak{g}}(z)$ and $T_{\mathfrak{a}}(z)$ are equal (see (20)).

In these combinations we are to substitute $\mathfrak{a}$-generators in terms of $\mathfrak{g}$-generators and obtain the energy-momentum tensor $T_{\mathfrak{g}}$. But if the set of generators attributed to $\Delta_{\mathfrak{a}_{\perp}}$ is not empty this is not possible, since $T_{\mathfrak{g}}$ contains generators $X^{\alpha}_n$ for $\alpha \in \Delta_{\mathfrak{a}_{\perp}}$. □

*4.1.1. Special embedding $\hat{A}_1 \to \hat{A}_2$.* Consider the case where both $\mathfrak{g}$ and $\mathfrak{a}$ are affine Lie algebras: $\hat{A}_1 \to \hat{A}_2$ and the injection is the affine extension of the special injection $A_1 \to A_2$ with the embedding index $x_e = 4$. As far as $\mathfrak{g}$-modules to be considered are of level one, the necessary $\mathfrak{a}$-modules will be of level $\tilde{k} = k x_e = 4$.

There exist three level one fundamental weights of $\hat{A}_2$. It is easy to see that the set $\Delta_{\mathfrak{a}_{\perp}}$ is empty and the subalgebra $\mathfrak{a}_{\perp} = 0$. Then $\mathcal{D}_{\mathfrak{a}_{\perp}} = 0$, $\mathfrak{h}_{\perp}$ is one-dimensional abelian subalgebra and the dimension of $\tilde{\mathfrak{a}}_{\perp} = \mathfrak{a}_{\perp} \oplus \mathfrak{h}_{\perp}$ is also 1. It is convenient to choose the classical root for $\hat{A}_1$ to be $\beta = \frac{1}{2}(\alpha_1 + \alpha_2)$.

Using Definition (1) we construct the fan $\Gamma_{\hat{A}_1 \to \hat{A}_2}$. In this case $\gamma_0 = 0$ and its sign $s\,(0) = -1$ thus we are to use the sign function $s(\gamma)$ (see Figure 6).

Consider the module $L^{\omega_0 = (0,0;1;0)}$. Here we use the (finite part; level; grade) presentation of the highest weight and the finite part coordinates are the Dynkin indices (see section(1.1)).

The set $\widehat{\Psi^{(\omega_0)}}$ is displayed in Figure 7 up to the sixth grade.

The next step is to project the anomalous weights to $P_{\hat{A}_1}$. The result is the element $\Psi^{(\omega_0)}_{\left(\hat{A}_1\,,\,\mathfrak{a}_{\perp}=0\right)}$ presented in Figure 8 up to the twelfth grade.

Using the recurrent relation (15) with the fan $\Gamma_{\hat{A}_1 \to \hat{A}_2}$ and the singular weights in $\Psi^{(\omega_0)}_{\left(\hat{A}_1\,,\,\mathfrak{a}_{\perp}=0\right)}$ we get the anomalous branching coefficients presented in Figure 9. Inside

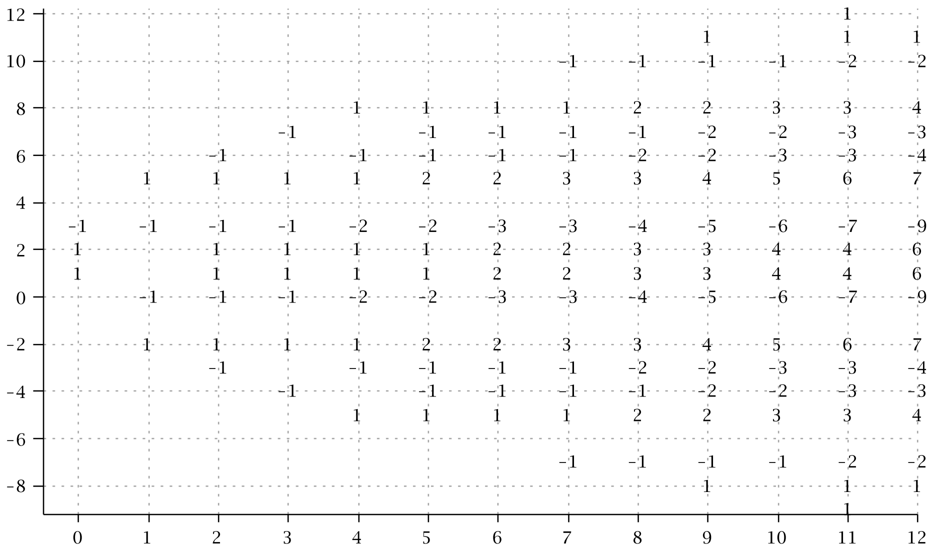


**Figure 6.** The fan $\Gamma_{\hat{A}_1 \to \hat{A}_2}$ for $\hat{A}_1 \to \hat{A}_2$ in the basis $\{\beta, \delta\}$. Notice that $\gamma_0 = 0$, so values of $s(\gamma)$ are prescribed to the weights $\gamma \in \Gamma_{\hat{A}_1 \to \hat{A}_2}$

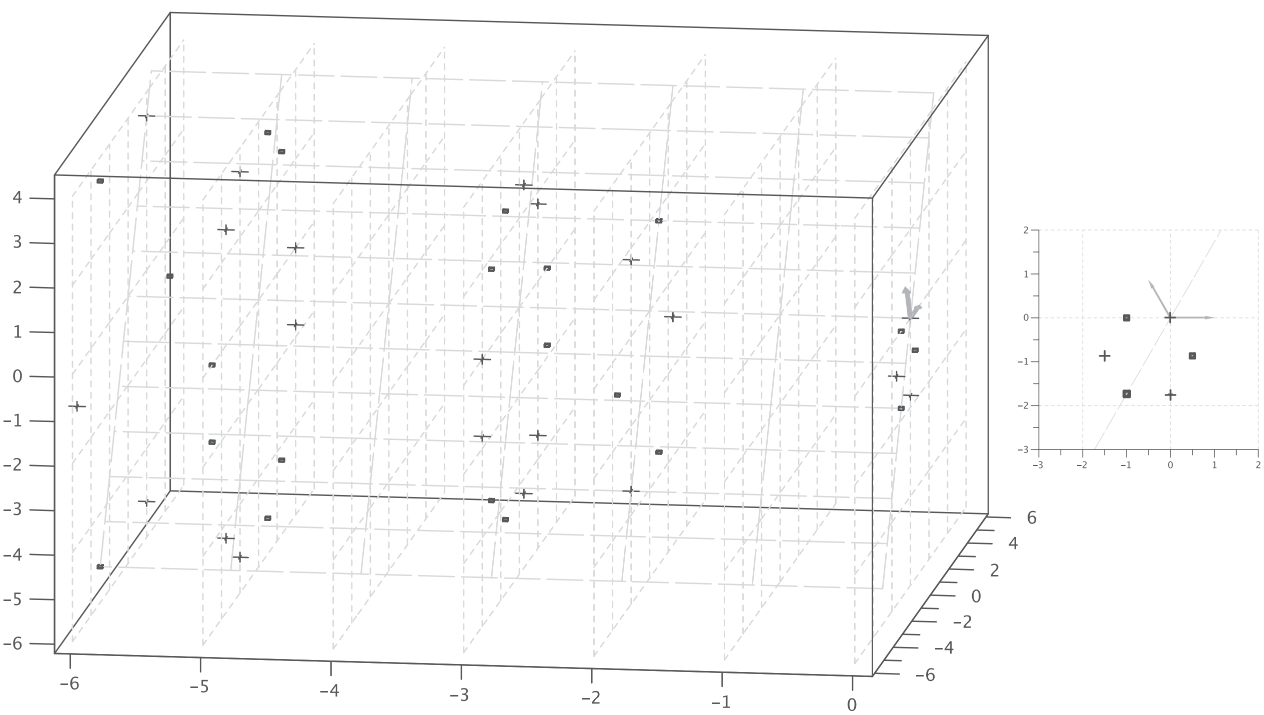


**Figure 7.** Singular weights of the module $L^{\omega_0}_{\hat{A}_2} = L^{(0,0;1;0)}_{\hat{A}_2}$. The classical (grade zero) cross-section of the diagram is shown separately in the right part of the figure. We use the orthogonal basis with the unit vector equal to $\alpha_1$. The weights $w(\omega_0 + \rho) - \rho$ are marked by crosses when $\epsilon(w) = 1$ and by box when $\epsilon(w) = -1$. Simple roots of the classical subalgebra $A_2$ are grey and the grey diagonal plane corresponds to the Cartan subalgebra of the embedded algebra $\hat{A}_1$.

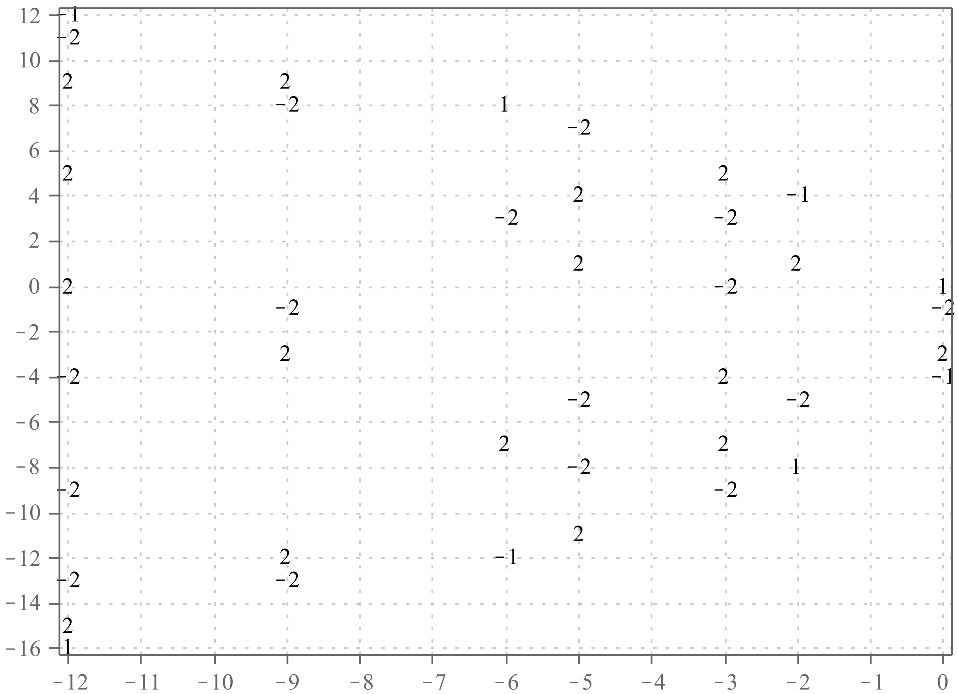



**Figure 8.** The singular element $\Psi^{(\omega_0)}_{(\hat{A}_1\,,\,\mathfrak{a}_\perp=0)}$ displayed in $P_{\hat{A}_1}$ with the basis $\{\beta, \delta\}$.

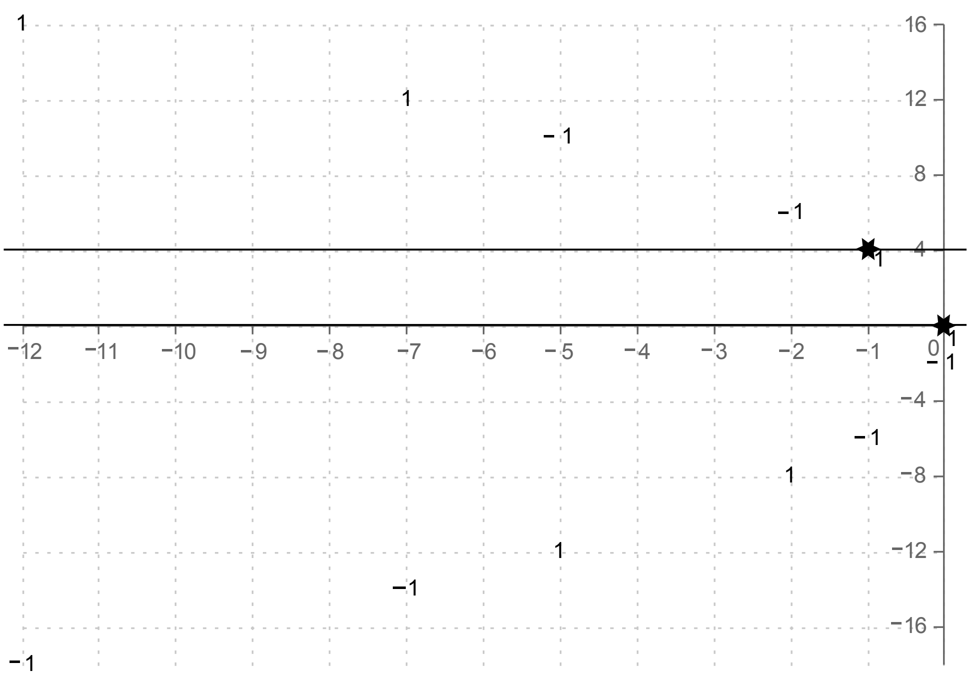



**Figure 9.** Anomalous branching coefficients for $\hat{A}_1 \subset \hat{A}_2$. The boundaries of the main Weyl chamber $\bar{C}_{\hat{A}_1}$ are indicated by black lines. Two anomalous highest weights located in the main Weyl chamber are marked by stars. Both have multiplicity 1, so the branching coefficients for them are equal 1.

the Weyl chamber $\bar{C}_{\hat{A}_1}$ (its boundaries are indicated in Figure 9) there are only two nonzero anomalous weights and both have multiplicity 1. These are the highest weights of $\mathfrak{a}$-submodules and the multiplicities are their branching coefficients. Thus we get the decomposition

$$L^{(0,0;1;0)}_{\hat{A}_2\downarrow\hat{A}_1} = L^{(0;4;0)}_{\hat{A}_1} \oplus L^{(4;4;0)}_{\hat{A}_1}.$$

Notice that the finite reducibility theorem holds.

The same fan $\Gamma_{\hat{A}_1\to\hat{A}_2}$ can be used for any other highest weight module $L^{\mu}_{\hat{A}_2}$. In particular for irreducible modules of level one we get the trivial branching:

$$L^{(1,0;1;0)}_{\hat{A}_2\downarrow\hat{A}_1} = L^{(2;4;0)}_{\hat{A}_1}, L^{(0,1;1;0)}_{\hat{A}_2\downarrow\hat{A}_1} = L^{(2;4;0)}_{\hat{A}_1}.$$

Using these results the modular-invariant partition function is easily found,

$$Z = \left|\chi_{(4;4;0)} + \chi_{(0;4;0)}\right|^2 + 2\chi^2_{(2;4;0)}.$$

## *4.2. Coset models*

Coset models [20] tightly connected with the gauged WZW-models are actively studied in string theory, especially in string models on anti-de-Sitter space [21, 22, 23, 24, 25]. The characters in coset models are proportional to branching functions,

$$\chi^{(\mu)}_{\nu}(\tau) = e^{2\pi i\tau(m_\mu - m_\nu)} b^{(\mu)}_{\nu}(\tau), \tag{22}$$

with

$$m_\mu = \frac{|\mu+\rho|^2}{2(k+g)} - \frac{|\rho|^2}{2g}.$$

The problem of branching functions construction in the coset models was considered in [26], [7], [27].

Let us return to our example 3.1 and consider the affine extension of the injection $A_1 \to B_2$. Since this embedding is regular and $x_e = 1$, the subalgebra modules and the initial module are of the same level. The set of positive roots with zero projection on the root space of the subalgebra $\hat{A}_1$ is the same as in the finite-dimensional case: $\Delta^+_{\mathfrak{a}_\perp} = \{\alpha_1\}$ and $\mathfrak{a}_\perp = A_1$. It is easy to see that here $\mathfrak{h}_\perp$ is trivial and $\mathcal{D}_{\mathfrak{a}_\perp} = 0$.

Using Definition (1) we obtain the fan $\Gamma_{\hat{A}_1\longrightarrow\hat{B}_2}$. Notice that here the lowest weight $\gamma_0$ of the fan is zero and $s(\gamma_0) = -1$. Values of the sign function $s(\gamma)$ for $\gamma \in \Gamma_{\hat{A}_1\longrightarrow\hat{B}_2}$ are presented in Figure 10. We restricted the computation to the twelfth grade.

Consider the level one module $L^{(1,0;1;0)}_{\hat{B}_2}$ with the highest weight $\omega_1 = (1,0;1;0)$, where the finite part coordinates are in the orthogonal basis $e_1, e_2$. The set of anomalous weights for this module up to the sixth grade is presented in Figure 11.

According to the recursive algorithm 2.5 we project these anomalous weights to $P_{\hat{A}_1}$ and find the dimensions of the corresponding $\mathfrak{a}_\perp$-modules $L^{\pi_{\mathfrak{a}_\perp}(w(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}_{\mathfrak{a}_\perp}$. In the grade zero this projection gives exactly the set $\Psi^{(\mu)}_{(A_1,A_1)}$ for the embedding of the classical Lie algebra $A_1 \to B_2$. To see this compare Figure 1 with Figure 12 where the singular element $\Psi^{(\mu)}_{\left(\widehat{A_1},A_1\right)}$ for the affine embedding $\hat{A}_1$ is presented up to the twelfth grade.

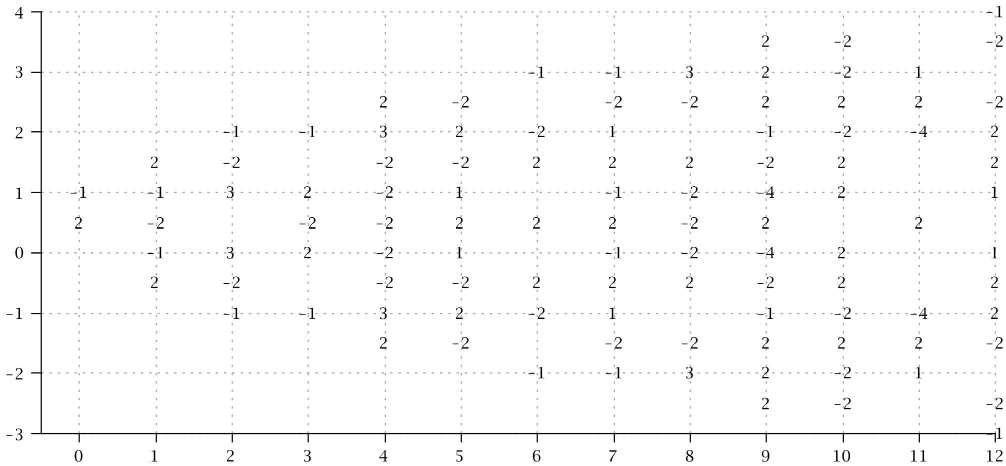


**Figure 10.** The fan $\Gamma_{\hat{A}_1 \to \hat{B}_2}$ for $\hat{A}_1 \to \hat{B}_2$ in the basis $\{\beta, \delta\}$. Values of $s(\gamma)$ are shown for the weights $\gamma \in \Gamma_{\hat{A}_1 \to \hat{B}_2}$

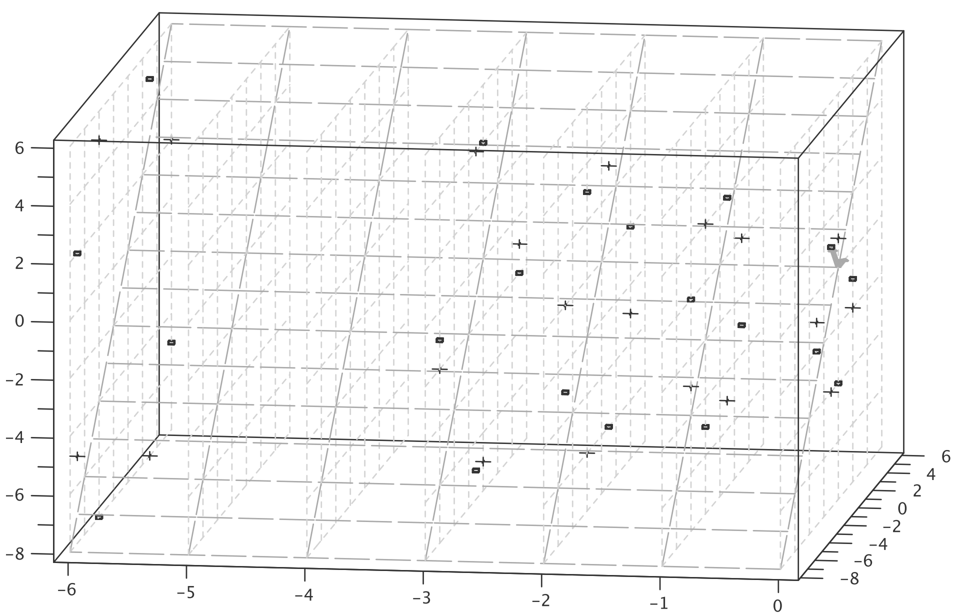


**Figure 11.** Singular weights for $L^{(1,0;1;0)}_{\hat{B}_2}$. The standard basis $\{e_1, e_2\}$ is used for the classical cross-section. The weights in the zero grade are the same as in Figure 1. The weights $w(\omega_1 + \rho) - \rho$ are marked by crosses if $\epsilon(w) = 1$ and by boxes for $\epsilon(w) = -1$. Simple roots of the classical subalgebra $B_2$ are grey and grey diagonal plane corresponds to the Cartan subalgebra of the embedded algebra $\hat{A}_1$.

Multiplicities of the highest weights inside the Weyl chamber $\bar{C}^{(0)}_{\hat{A}_1}$ define the following branching coefficients (up to the twelfth grade),

$$
\begin{aligned}
L^{\omega_1}_{\hat{B}_2 \downarrow \hat{A}_1} = {} & 2L^{\omega_1}_{\hat{A}_1} \oplus 1L^{\omega_0}_{\hat{A}_1} \oplus 4L^{\omega_0-\delta}_{\hat{A}_1} \oplus \\
& 2L^{\omega_1-\delta}_{\hat{A}_1} \oplus 8L^{\omega_0-2\delta}_{\hat{A}_1} \oplus 8L^{\omega_1-2\delta}_{\hat{A}_1} \oplus 15L^{\omega_0-3\delta}_{\hat{A}_1} \oplus \\
& 12L^{\omega_1-3\delta}_{\hat{A}_1} \oplus 26L^{\omega_1-4\delta}_{\hat{A}_1} \oplus 29L^{\omega_0-4\delta}_{\hat{A}_1} \oplus 51L^{\omega_0-5\delta}_{\hat{A}_1} \oplus
\end{aligned}
$$

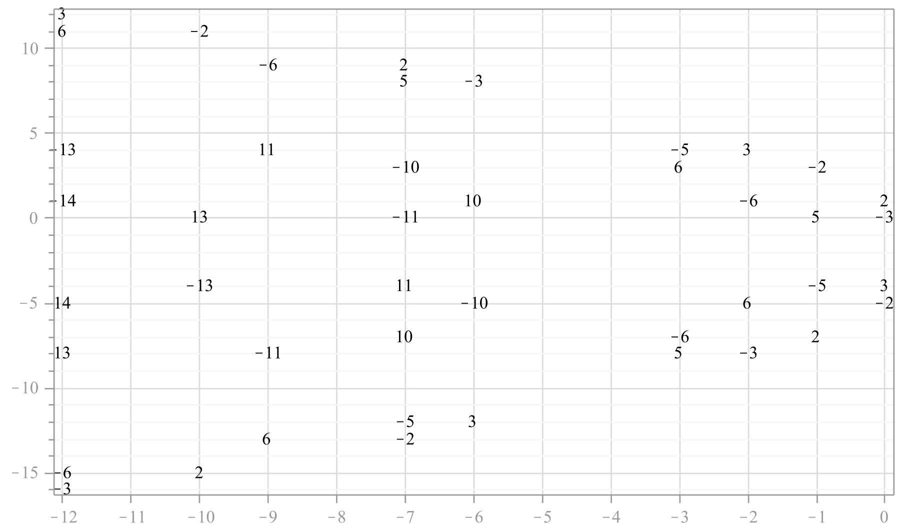


**Figure 12.** Singular element $\Psi^{(\omega_1)}_{(\widehat{A_1},A_1)}$ in the basis $\{\beta,\delta\}$. Dimensions of the corresponding $\mathfrak{a}_{\perp}=A_1$-modules with the signs $\epsilon(u)$ are indicated.

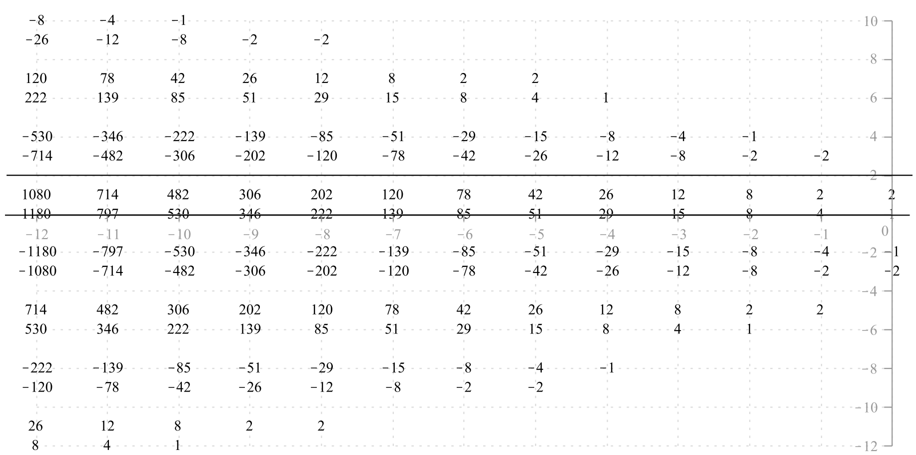


**Figure 13.** Anomalous branching coefficients for $\hat{A}_1 \to \hat{B}_2$. The basis $\{\beta,\delta\}$ is used. Boundaries of the main Weyl chamber $\bar{C}_{\hat{A}_1}$ are indicated by black lines. Anomalous branching coefficients inside the main Weyl chamber are equal to branching coefficients of the embedding $\hat{A}_1 \to \hat{B}_2$.

$$
\begin{aligned}
&42L^{\omega_1-5\delta}_{\hat{A}_1} \oplus 78L^{\omega_1-6\delta}_{\hat{A}_1} \oplus 85L^{\omega_0-6\delta}_{\hat{A}_1} \oplus 120L^{\omega_1-7\delta}_{\hat{A}_1} \oplus \\
&139L^{\omega_0-7\delta}_{\hat{A}_1} \oplus 202L^{\omega_1-8\delta}_{\hat{A}_1} \oplus 222L^{\omega_0-8\delta}_{\hat{A}_1} \oplus 306L^{\omega_1-9\delta}_{\hat{A}_1} \oplus \\
&346L^{\omega_0-9\delta}_{\hat{A}_1} \oplus 530L^{\omega_0-10\delta}_{\hat{A}_1} \oplus 482L^{\omega_1-10\delta}_{\hat{A}_1} \oplus 714L^{\omega_1-11\delta}_{\hat{A}_1} \oplus \\
&797L^{\omega_0-11\delta}_{\hat{A}_1} \oplus 1080L^{\omega_1-12\delta}_{\hat{A}_1} \oplus 1180L^{\omega_0-12\delta}_{\hat{A}_1} \oplus \dots
\end{aligned}
$$

This result can be presented as the set of branching functions:

$$
\begin{aligned}
b^{(\omega_1)}_0 = \; & 1+4\,q^1+8\,q^2+15\,q^3+29\,q^4+51\,q^5+85\,q^6+139\,q^7+ \\
& 222\,q^8+346\,q^9+530\,q^{10}+797\,q^{11}+1180\,q^{12}+\dots
\end{aligned}
$$

$$b_1^{(\omega_1)} = \; 2+2\,q^1+8\,q^2+12\,q^3+26\,q^4+42\,q^5+78\,q^6+120\,q^7+ \\ 202\,q^8+306\,q^9+482\,q^{10}+714\,q^{11}+1080\,q^{12}+\dots$$

Here $q=\exp(2\pi i\tau)$ and the lower index enumerates the branching functions according to their highest weights in $P^+_{\hat{A}_1}$. These are the fundamental weights $\omega_0 = \lambda_0 = (0,1,0)$, $\omega_1 = \alpha/2 = (1,1,0)$.

Now we can return to (22),

$$\chi_1^{(\omega_1)}(q) = \; q^{\frac{7}{12}}\,(2+2\,q^1+8\,q^2+12\,q^3+26\,q^4+42\,q^5+78\,q^6+120\,q^7+ \\ 202\,q^8+306\,q^9+482\,q^{10}+714\,q^{11}+1080\,q^{12}+\dots)\,,$$
$$\chi_0^{(\omega_1)}(q) = \; q^{\frac{5}{6}}\,(1+4\,q^1+8\,q^2+15\,q^3+29\,q^4+51\,q^5+85\,q^6+139\,q^7+ \\ 222\,q^8+346\,q^9+530\,q^{10}+797\,q^{11}+1180\,q^{12}+\dots)\,,$$

and finally obtain expansions for the $B_2/A_1$-coset characters.

## 5. Conclusion

We have demonstrated that the injection fan technique can be used to deal with arbitrary reductive subalgebras (maximal as well as nonmaximal). It was shown that the branching problem for $\mathfrak{a} \subset \mathfrak{g}$ is tightly connected with the properties of the orthogonal partner $\mathfrak{a}_\perp$ of $\mathfrak{a}$. The subalgebra $\mathfrak{a}_\perp$ corresponds to the subset $\Delta^+_{\mathfrak{a}_\perp}$ of positive roots in $\Delta^+_{\mathfrak{g}}$ that trivialize the Cartan subalgebra $\mathfrak{h}_{\mathfrak{a}_\perp}$. Both the injection fan and the sets of singular weights for highest weight $\mathfrak{g}$-modules depend substantially on the structure of $\mathfrak{a}_\perp$ and its submodules. For the fan $\Gamma_{\mathfrak{a}\to\mathfrak{g}}$ this dependence is almost obvious: in the element $\Phi_{\mathfrak{a}\to\mathfrak{g}}$ the factors corresponding to the roots of $\Delta^+_{\mathfrak{a}_\perp}$ are eliminated. The transformation in the set of projected singular weights is more interesting. We have found out that in the new singular element $\Psi^{(\mu)}_{(\mathfrak{a},\mathfrak{a}_\perp)}$ the coefficients depend on the $\mathfrak{a}_\perp$-submodules (their highest weights $\mu_{\widetilde{\mathfrak{a}_\perp}}(u)$ are fixed by the injection and by the weights of the initial element $\Psi^\mu$). Fortunately no more information on $L^{\mu_{\widetilde{\mathfrak{a}_\perp}}(u)}_{\{\mathfrak{a}_\perp\}}$-submodules is necessary than their dimensions. In the new singular element $\Psi^{(\mu)}_{(\mathfrak{a},\mathfrak{a}_\perp)}$ weight multiplicities are equal to dimensions $\dim\left(L^{\mu_{\widetilde{\mathfrak{a}_\perp}}(u)}_{\{\mathfrak{a}_\perp\}}\right)$ of the corresponding $\mathfrak{a}_\perp$-modules multiplied by the values $\epsilon(u)$. As a result the highest weights of $\mathfrak{a}$-submodules and their multiplicities are subject to the set of linear equations (16). These properties are valid for any reductive subalgebra $\mathfrak{a} \to \mathfrak{g}$ and the set can be redressed to the form of recurrent relations to be solved step by step.

The efficiency of the obtained algorithm was illustrated in various examples. In particular we considered the construction of modular-invariant partition functions in the framework of conformal embedding method and the coset construction in rational conformal field theory. This construction is useful in the study of WZW-models emerging in the context of the AdS/CFT correspondence [21, 22, 23].

Further amelioration of the algorithm can be achieved by using the folded fan technique [28]. It must be mentioned that even in the case of string functions the explicit solution of the corresponding recurrent relations is a difficult problem (see [28] for details). Nevertheless we hope that by developing the procedure of folding one could get explicit solutions for at least some of branching functions and the corresponding coset characters.

## 6. Acknowledgements

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