Recursive algorithms, branching coefficients and applications

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Abstract. Recurrent relations for branching coefficients in affine Lie algebras integrable highest weight modules are studied. The decomposition algorithm based on the injection fan technique is adopted to the situation where the Weyl denominator becomes singular with respect to a reductive subalgebra. We study some modifications of the injection fan technique and demonstrate that it is possible to define the "subtracted fans" that play the role similar to the original ones. Possible applications of subtracted fans in CFT models are considered.

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1. Introduction

The branching problem for affine Lie algebras emerges in conformal field theory, for example, in the construction of modular-invariant partition functions [1]. Recently the problem of conformal embeddings was considered in the paper [2].

There exist several different approaches to deal with the branching coefficients. Some of them use the BGG resolution [3] (for Kac-Moody algebras the algorithm is described in [4],[5]), the Schur function series [6], the BRST cohomology [7], Kac-Peterson formulas [4, 8] or the combinatorial methods applied in [9].

Usually only the maximal reductive subalgebras are considered since the case of non-maximal subalgebra can be obtained using the chain of maximal injections. In this paper we find the recurrent properties for branching coefficients that generalize the relations obtained earlier (see the paper [10] and the references therein) to the case of non-maximal reductive subalgebra. The result is formulated in terms of the new injection fan called "the subtracted fan". Using this new tools we formulate a simple and explicit algorithm for computations of branching coefficients which is applicable to the non-maximal subalgebras of finite-dimensional and affine Lie algebras.

We demonstrate that our algorithm can be used in studies of conformal embeddings and coset constructions in rational conformal field theory.

The paper is organised as follows. In the subsection 1.1 we fix the notations. In the Section 2 we derive the subtracted recurrent formula for anomalous branching coefficients and describe the decomposition algorithm for integrable highest weight modules $L_{\mathfrak{g}}$ with respect to a reductive subalgebra $\mathfrak{a} \subset \mathfrak{g}$ (subsection 2.2). In the Section 3 we present several simple examples for finite-dimensional Lie algebras. The affine Lie algebras and their applications in CFT models are considered in Section 4. Possible further developments are discussed (Section 5).

1.1. Notation

Consider affine Lie algebras $\mathfrak{g}$ and $\mathfrak{a}$ with the underlying finite-dimensional subalgebras $\overset{\circ}{\mathfrak{g}}$ and $\overset{\circ}{\mathfrak{a}}$ and an injection $\mathfrak{a} \longrightarrow \mathfrak{g}$ such that $\mathfrak{a}$ is a reductive subalgebra $\mathfrak{a} \subset \mathfrak{g}$ with correlated root spaces: $\mathfrak{h}^*_{\mathfrak{a}} \subset \mathfrak{h}^*_{\mathfrak{g}}$ and $\mathfrak{h}^*_{\overset{\circ}{\mathfrak{a}}} \subset \mathfrak{h}^*_{\overset{\circ}{\mathfrak{g}}}$. We use the following notations:

L^{μ} $(L^{\nu}_{\mathfrak{a}})$ — the integrable module of $\mathfrak{g}$ with the highest weight μ ; (resp. integrable $\mathfrak{a}$ -module with the highest weight ν);

r , $(r_{\mathfrak{a}})$ — the rank of the algebra $\mathfrak{g}$ (resp. $\mathfrak{a}$) ;

Δ $(\Delta_{\mathfrak{a}})$— the root system; Δ^+ (resp. $\Delta^+_{\mathfrak{a}}$)— the positive root system (of $\mathfrak{g}$ and $\mathfrak{a}$ respectively);

$\mathrm{mult}\,(\alpha)$ $(\mathrm{mult}_{\mathfrak{a}}\,(\alpha))$ — the multiplicity of the root α in Δ (resp. in $(\Delta_{\mathfrak{a}})$);

$\overset{\circ}{\Delta}$, $\left(\overset{\circ}{\Delta}_{\mathfrak{a}}\right)$ — the finite root system of the subalgebra $\overset{\circ}{\mathfrak{g}}$ (resp. $\overset{\circ}{\mathfrak{a}}$); Θ, $(\Theta_{\mathfrak{a}})$ — the highest root of the algebra $\mathfrak{g}$ (resp. subalgebra $\mathfrak{a}$);

$\mathcal{N}^{\mu}$, $(\mathcal{N}^{\nu}_{\mathfrak{a}})$ — the weight diagram of L^{μ} (resp. $L^{\nu}_{\mathfrak{a}}$) ;

W , $(W_{\mathfrak{a}})$— the corresponding Weyl group;

C , $(C_{\mathfrak{a}})$— the fundamental Weyl chamber;

$\bar{C}$, $(\bar{C}_{\mathfrak{a}})$ — the closure of the fundamental Weyl chamber;

ρ , $(\rho_{\mathfrak{a}})$ — the Weyl vector;

$\epsilon\,(w) := \det\,(w)$;

α_i , $\left(\alpha_{(\mathfrak{a})j}\right)$ — the i-th (resp. j-th) basic root for $\mathfrak{g}$ (resp. $\mathfrak{a}$); $i = 0, \ldots, r$, $(j = 0, \ldots, r_{\mathfrak{a}})$;

δ — the imaginary root of $\mathfrak{g}$ (and of $\mathfrak{a}$ if any);

$\alpha^{\vee}_i$, $\left(\alpha^{\vee}_{(\mathfrak{a})j}\right)$— the basic coroot for $\mathfrak{g}$ (resp. $\mathfrak{a}$) , $i = 0, \ldots, r$; $(j = 0, \ldots, r_{\mathfrak{a}})$;

$\overset{\circ}{\xi}$, $\overset{\circ}{\xi}_{(\mathfrak{a})}$ — the finite (classical) part of the weight $\xi \in P$, $\left(\text{resp. } \xi_{(\mathfrak{a})} \in P_{\mathfrak{a}}\right)$;

$\lambda = \left(\overset{\circ}{\lambda}; k; n\right)$ — the decomposition of an affine weight indicating the finite part $\overset{\circ}{\lambda}$, level k and grade n;

P (resp. $P_{\mathfrak{a}}$) — the weight lattice;

M (resp. $M_{\mathfrak{a}}$) :=

$$= \left\{ \begin{array}{c} \sum_{i=1}^{r} \mathbf{Z}\alpha^{\vee}_i \ \left(\text{resp. } \sum_{i=1}^{r} \mathbf{Z}\alpha^{\vee}_{(\mathfrak{a})i}\right) \text{ for untwisted algebras or } A^{(2)}_{2r}, \\ \sum_{i=1}^{r} \mathbf{Z}\alpha_i \ \left(\text{resp. } \sum_{i=1}^{r} \mathbf{Z}\alpha_{(\mathfrak{a})i}\right) \text{ for } A^{(u\geq 2)}_{r} \text{ and } A \neq A^{(2)}_{2r}, \end{array} \right\};$$

$\Psi^{(\mu)} := \sum\limits_{w\in W} \epsilon(w) e^{w\circ(\mu+\rho)-\rho}$ — the singular weight element for the $\mathfrak{g}$-module L^{μ};

$\Psi_{(\mathfrak{a})}^{(\nu)} := \sum\limits_{w\in W_{\mathfrak{a}}} \epsilon(w) e^{w\circ(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}$ — the corresponding singular weight element for the $\mathfrak{a}$-module $L_{\mathfrak{a}}^{\nu}$;

$\widehat{\Psi^{(\mu)}}$ $\left(\widehat{\Psi_{(\mathfrak{a})}^{(\nu)}}\right)$ — the set of singular weights $\xi \in P$ (resp. $\in P_{\mathfrak{a}}$) for the module L^{μ} (resp. $L_{\mathfrak{a}}^{\nu}$) with the coordinates $\left(\overset{\circ}{\xi}, k, n, \epsilon\left(w\left(\xi\right)\right)\right) |_{\xi=w(\xi)\circ(\mu+\rho)-\rho}$, (resp. $\left(\overset{\circ}{\xi}, k, n, \epsilon\left(w_a\left(\xi\right)\right)\right) |_{\xi=w_a(\xi)\circ(\nu+\rho_a)-\rho_a}$);

$m_{\xi}^{(\mu)}$, $\left(m_{\xi}^{(\nu)}\right)$ — the multiplicity of the weight $\xi \in P$ (resp. $\in P_{\mathfrak{a}}$) in the module L^{μ} , (resp. $\xi \in L_{\mathfrak{a}}^{\nu}$);

$ch\left(L^{\mu}\right)$ (resp. $ch\left(L_{\mathfrak{a}}^{\nu}\right)$)— the formal character of L^{μ} (resp. $L_{\mathfrak{a}}^{\nu}$);

$ch\left(L^{\mu}\right) = \frac{\sum_{w\in W}\epsilon(w)e^{w\circ(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}} = \frac{\Psi^{(\mu)}}{\Psi^{(0)}}$ — the Weyl-Kac formula;

$R := \prod_{\alpha\in\Delta^+}\left(1-e^{-\alpha}\right)^{\mathrm{mult}(\alpha)} = \Psi^{(0)}$;

$\left(\text{resp. } R_{\mathfrak{a}} := \prod_{\alpha\in\Delta_{\mathfrak{a}}^+}\left(1-e^{-\alpha}\right)^{\mathrm{mult}_{\mathfrak{a}}(\alpha)} = \Psi_{\mathfrak{a}}^{(0)}\right)$— the denominator;

$L_{\mathfrak{g}\downarrow\mathfrak{a}}^{\mu} = \bigoplus\limits_{\nu\in P_{\mathfrak{a}}^+} b_{\nu}^{(\mu)} L_{\mathfrak{a}}^{\nu}$ — the module decomposition with respect to $\mathfrak{a} \longrightarrow \mathfrak{g}$;

$b_{\nu}^{(\mu)}$ — the branching coefficients,

$$\sum_{\nu\in\bar{C}_{\mathfrak{a}}} b_{\nu}^{(\mu)} \Psi_{(\mathfrak{a})}^{(\nu)} = \sum_{\lambda\in P_{\mathfrak{a}}} k_{\lambda}^{(\mu)} e^{\lambda}; \tag{1}$$

k_{λ} — the anomalous branching coefficients, notice that

$$b_{\nu}^{(\mu)} = k_{\nu}^{(\mu)} \text{ for } \nu \in \bar{C}_{\mathfrak{a}}; \tag{2}$$

$x_e = \frac{|\pi_{\mathfrak{a}}\Theta|^2}{|\Theta_{\mathfrak{a}}|^2}$ — the embedding index.

2. Recurrent relation for branching coefficients.

Let $\mathfrak{a} \subset \mathfrak{g}$ be a regular subalgebra of $\mathfrak{g}$. Applying the Weyl-Kac character formula to the branching rule $\pi_{\mathfrak{a}} ch\left(L^{\mu}\right) = \sum_{\nu\in P_{\mathfrak{a}}^+} b_{\nu}^{(\mu)} ch\left(L_{\mathfrak{a}}^{\nu}\right)$ we get the relation connecting the singular elements $\Psi^{(\mu)}$ and $\Psi_{(\mathfrak{a})}^{(\nu)}$:

$$\pi_{\mathfrak{a}}\left(\frac{\sum_{w\in W}\epsilon(w)e^{w(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}\right) = \sum_{\nu\in P_{\mathfrak{a}}^+} b_{\nu}^{(\mu)} \frac{\sum_{w\in W_{\mathfrak{a}}}\epsilon(w)e^{w(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}}{\prod_{\beta\in\Delta_{\mathfrak{a}}^+}(1-e^{-\beta})^{\mathrm{mult}_{\mathfrak{a}}(\beta)}}, \tag{3}$$

$$\pi_{\mathfrak{a}}\left(\frac{\Psi^{(\mu)}}{R}\right) = \sum_{\nu\in P_{\mathfrak{a}}^+} b_{\nu}^{(\mu)} \frac{\Psi_{(\mathfrak{a})}^{(\nu)}}{R_{\mathfrak{a}}}. \tag{4}$$

where according to our notations $\Delta_{\mathfrak{a}}^+$ is the set of positive roots of the subalgebra $\mathfrak{a}$ (it is convenient to choose them from the positive root subspace of $\mathfrak{g}$). Consider the root space $\mathfrak{h}_{\perp\mathfrak{a}}^*$ orthogonal to $\Delta_{\mathfrak{a}}$,

$$\mathfrak{h}_{\perp\mathfrak{a}}^* =: \left\{\eta \in \mathfrak{h}_{\mathfrak{g}}^* | \forall\alpha \in \Delta_{\mathfrak{a}}; \alpha\perp\eta\right\},$$

and the roots (correspondingly – positive roots) of $\mathfrak{g}$ orthogonal to $\Delta_{\mathfrak{a}}$,

$$\begin{aligned}\Delta_{\mathfrak{a}_\perp} &:= \{\beta \in \Delta_{\mathfrak{g}} | \forall \alpha \in \Delta_{\mathfrak{a}}; \alpha \perp \beta\},\\ \Delta^+_{\mathfrak{a}_\perp} &:= \left\{\beta^+ \in \Delta^+_{\mathfrak{g}} | \forall \alpha^+ \in \Delta^+_{\mathfrak{a}}; \alpha^+ \perp \beta^+\right\}.\end{aligned}$$

Let $W_{\mathfrak{a}_\perp}$ be the subgroup of W generated by the reflections w_β for the roots $\beta \in \Delta^+_{\mathfrak{a}_\perp}$. The subsystem $\Delta_{\mathfrak{a}_\perp}$ determines the subalgebra $\mathfrak{a}_\perp$ with the Cartan subalgebra $\mathfrak{h}_{\mathfrak{a}_\perp}$. Let

$$\mathfrak{h}^*_\perp := \{\eta \in \mathfrak{h}^*_{\perp\mathfrak{a}} | \forall \alpha \in \Delta_{\mathfrak{a}} \cup \Delta_{\mathfrak{a}_\perp}; \alpha \perp \eta\}$$

and consider the subalgebras

$$\begin{aligned}\widetilde{\mathfrak{a}_\perp} &:= \mathfrak{a}_\perp \oplus \mathfrak{h}_\perp\\ \widetilde{\mathfrak{a}} &:= \mathfrak{a} \oplus \mathfrak{h}_\perp.\end{aligned}$$

The algebras $\mathfrak{a}$ and $\mathfrak{a}_\perp$ form an "orthogonal pair" of subalgebras in $\mathfrak{g}$. For the Cartan subalgebras we have the decomposition

$$\mathfrak{h} = \mathfrak{h}_{\mathfrak{a}} \oplus \mathfrak{h}_{\mathfrak{a}_\perp} \oplus \mathfrak{h}_\perp = \mathfrak{h}_{\widetilde{\mathfrak{a}}} \oplus \mathfrak{h}_{\mathfrak{a}_\perp} = \mathfrak{h}_{\widetilde{\mathfrak{a}_\perp}} \oplus \mathfrak{h}_{\mathfrak{a}}. \tag{5}$$

For the subalgebras of the orthogonal pair $(\mathfrak{a}, \mathfrak{a}_\perp)$ we consider the corresponding Weyl vectors, $\rho_{\mathfrak{a}}$ and $\rho_{\mathfrak{a}_\perp}$, and form the so called "defects" $\mathcal{D}_{\mathfrak{a}}$ and $\mathcal{D}_{\mathfrak{a}_\perp}$ of the injection:

$$\mathcal{D}_{\mathfrak{a}} := \rho_{\mathfrak{a}} - \pi_{\mathfrak{a}}\rho, \tag{6}$$

$$\mathcal{D}_{\mathfrak{a}_\perp} := \rho_{\mathfrak{a}_\perp} - \pi_{\mathfrak{a}_\perp}\rho. \tag{7}$$

For the highest weight module $L^\mu_{\mathfrak{g}}$ consider the singular weights $(w(\mu+\rho)-\rho) \in \widehat{\Psi^{(\mu)}}$ and their projections to $h^*_{\widetilde{\mathfrak{a}_\perp}}$ (additionally shifted by $-\mathcal{D}_{\mathfrak{a}_\perp}$):

$$\mu_{\widetilde{\mathfrak{a}_\perp}}(w) := \pi_{\widetilde{\mathfrak{a}_\perp}}[w(\mu+\rho)-\rho] - \mathcal{D}_{\mathfrak{a}_\perp}, \quad w \in W.$$

Among the weights $\mu_{\widetilde{\mathfrak{a}_\perp}}(w)|_{w\in W}$ choose those located in the fundamental chamber $\overline{C_{\widetilde{\mathfrak{a}_\perp}}}$ and let U be the set of representatives u for the classes $W/W_{\mathfrak{a}_\perp}$ such that

$$U := \left\{u \in W | \quad \mu_{\widetilde{\mathfrak{a}_\perp}}(u) \in \overline{C_{\widetilde{\mathfrak{a}_\perp}}}\right\} \quad . \tag{8}$$

For the same set U introduce the weights

$$\mu_{\mathfrak{a}}(u) := \pi_{\mathfrak{a}}[u(\mu+\rho)-\rho] + \mathcal{D}_{\mathfrak{a}_\perp}.$$

To describe the recurrent properties for branching coefficients $b^{(\mu)}_\nu$ we shall use the technique elaborated in [10] one of the main tools of which is the set of weights $\Gamma_{\mathfrak{a}\subset\mathfrak{g}}$ called the injection fan. As far as we consider more general situation (where the injection is not maximal) the notion of the injection fan is modified:

Definition 1. For the product

$$\prod_{\alpha\in\Delta^+\setminus\Delta^+_\perp} \left(1 - e^{-\pi_{\mathfrak{a}}\alpha}\right)^{\mathrm{mult}(\alpha) - \mathrm{mult}_{\mathfrak{a}}(\pi_{\mathfrak{a}}\alpha)} = -\sum_{\gamma\in P_{\mathfrak{a}}} s(\gamma) e^{-\gamma} \tag{9}$$

consider the carrier $\Phi_{\mathfrak{a}\subset\mathfrak{g}} \subset P_{\mathfrak{a}}$ of the function $s(\gamma) = \det(\gamma)$:

$$\Phi_{\mathfrak{a}\subset\mathfrak{g}} = \{\gamma \in P_{\mathfrak{a}} | s(\gamma) \neq 0\} \tag{10}$$

The ordering of roots in $\overset{\circ}{\Delta}_{\mathfrak{a}}$ induce the natural ordering of the weights in $P_{\mathfrak{a}}$. Denote by γ_0 the lowest vector of $\Phi_{\mathfrak{a}\subset\mathfrak{g}}$. The set

$$\Gamma_{\mathfrak{a}\subset\mathfrak{g}} = \{\xi - \gamma_0 | \xi \in \Phi_{\mathfrak{a}\subset\mathfrak{g}}\} \setminus \{0\} \tag{11}$$

is called the *injection fan.*

The main result is that the Weyl-Kac character formula (in terms of singular elements) describes the particular case of a more general relation:

Lemma 1. *Let* $\mathfrak{a}$ *and* $\mathfrak{a}_\perp$ *be the orthogonal pair of regular subalgebras in* $\mathfrak{g}$, *with* $\widetilde{\mathfrak{a}_\perp} = \mathfrak{a}_\perp \oplus \mathfrak{h}_\perp$ *and* $\widetilde{\mathfrak{a}} = \mathfrak{a} \oplus \mathfrak{h}_\perp$,

$L^\mu_{\mathfrak{g}}$ be the highest weight module with the singular element $\Psi^\mu_{\mathfrak{g}}$,

$R_{\mathfrak{a}_\perp}$ be the Weyl denominator for $\mathfrak{a}_\perp$.

Then the element $\pi_{\mathfrak{a}}\left(\frac{\Psi^\mu_{\mathfrak{g}}}{R_{\mathfrak{a}_\perp}}\right)$ can be decomposed into the sum over $u \in U$ (see (8)) of the singular elements $e^{\mu_{\mathfrak{a}}(u)}$ with the coefficients $\epsilon(u)\mathrm{dim}\left(L^{\mu_{\widetilde{\mathfrak{a}_\perp}}(u)}_{\widetilde{\mathfrak{a}_\perp}}\right)$:

$$\pi_{\mathfrak{a}}\left(\frac{\Psi^\mu_{\mathfrak{g}}}{R_{\mathfrak{a}_\perp}}\right) = \sum_{u\in U} \epsilon(u) e^{\mu_{\mathfrak{a}}(u)} \mathrm{dim}\left(L^{\mu_{\widetilde{\mathfrak{a}_\perp}}(u)}_{\widetilde{\mathfrak{a}_\perp}}\right). \tag{12}$$

Proof. For $u \in U$ perform the following decompositions:

$$u(\mu+\rho) = \pi_{(\mathfrak{a})} u(\mu+\rho) + \pi_{(\widetilde{\mathfrak{a}_\perp})} u(\mu+\rho)$$

$$vu(\mu+\rho) - \rho = \pi_{(\mathfrak{a})}\left(u(\mu+\rho)\right) - \rho + \rho_{\mathfrak{a}_\perp} + v\left(\pi_{(\widetilde{\mathfrak{a}_\perp})} u(\mu+\rho) - \rho_{\mathfrak{a}_\perp} + \rho_{\mathfrak{a}_\perp}\right) - \rho_{\mathfrak{a}_\perp}$$

Use these formulas to decompose the anomalous element $\Psi^\mu_{\mathfrak{g}}$ of the module $L^\mu_{\mathfrak{g}}$ into the linear combination of anomalous elements $\Psi^\eta_{\widetilde{\mathfrak{a}_\perp}}$ of the subalgebra $\widetilde{\mathfrak{a}_\perp}$-modules $L^\eta_{\widetilde{\mathfrak{a}_\perp}}$:

$$\begin{aligned}
\Psi^\mu_{\mathfrak{g}} &= \textstyle\sum_{v\in W_{\mathfrak{a}_\perp}} \sum_{u\in U} \epsilon(v)\epsilon(u) e^{vu(\mu+\rho)-\rho} = \\
&= \textstyle\sum_{u\in U} \epsilon(u) e^{\pi_{\mathfrak{a}}(u(\mu+\rho))-\rho+\rho_{\mathfrak{a}_\perp}} \sum_{v\in W_{\mathfrak{a}_\perp}} \epsilon(v) e^{v\left(\pi_{(\widetilde{\mathfrak{a}_\perp})} u(\mu+\rho)-\rho_{\mathfrak{a}_\perp}+\rho_{\mathfrak{a}_\perp}\right)-\rho_{\mathfrak{a}_\perp}} = \\
&= \textstyle\sum_{u\in U} \epsilon(u) e^{\pi_{\mathfrak{a}}(u(\mu+\rho))-\pi_{\mathfrak{h}_\perp}\rho-\pi_{\mathfrak{a}}\rho+\mathcal{D}_{\mathfrak{a}_\perp}} \sum_{v\in W_{\mathfrak{a}_\perp}} \epsilon(v) e^{v\left(\pi_{(\widetilde{\mathfrak{a}_\perp})} u(\mu+\rho)-\rho_{\mathfrak{a}_\perp}+\rho_{\mathfrak{a}_\perp}\right)-\rho_{\mathfrak{a}_\perp}} = \\
&= \textstyle\sum_{u\in U} \epsilon(u) e^{\pi_{\mathfrak{a}}[u(\mu+\rho)-\rho]+\mathcal{D}_{\mathfrak{a}_\perp}} \sum_{v\in W_{\mathfrak{a}_\perp}} \epsilon(v) e^{v\left(\pi_{(\widetilde{\mathfrak{a}_\perp})} u(\mu+\rho)-\rho_{\mathfrak{a}_\perp}+\rho_{\mathfrak{a}_\perp}\right)-\pi_{\mathfrak{h}_\perp}\rho-\rho_{\mathfrak{a}_\perp}} = \\
&= \textstyle\sum_{u\in U} \epsilon(u) e^{\pi_{\mathfrak{a}}[u(\mu+\rho)-\rho]+\mathcal{D}_{\mathfrak{a}_\perp}} \sum_{v\in W_{\mathfrak{a}_\perp}} \epsilon(v) e^{v\left(\pi_{(\widetilde{\mathfrak{a}_\perp})} [u(\mu+\rho)-\rho].-\mathcal{D}_{\mathfrak{a}_\perp}+\rho_{\mathfrak{a}_\perp}\right)-\rho_{\mathfrak{a}_\perp}} = \\
&= \textstyle\sum_{u\in U} \epsilon(u) e^{\pi_{(\mathfrak{a})}[u(\mu+\rho)-\rho]+\mathcal{D}_{\mathfrak{a}_\perp}} \Psi^{\pi_{(\widetilde{\mathfrak{a}_\perp})}[u(\mu+\rho)-\rho]-\mathcal{D}_{\mathfrak{a}_\perp}}_{\widetilde{\mathfrak{a}_\perp}}
\end{aligned}$$

Dividing both sides by the Weyl element $R_{\mathfrak{a}_\perp} = \prod_{\beta\in\Delta_{\mathfrak{a}_\perp}} (1-e^{-\beta})^{\mathrm{mult}(\beta)}$ and projecting them to the weight space $h^*_{\mathfrak{a}}$ we obtain the desired relation

$$\begin{aligned}
\pi_{\mathfrak{a}}\left(\frac{\Psi^\mu_{\mathfrak{g}}}{R_{\mathfrak{a}_\perp}}\right) &= \sum_{u\in W/W_{\mathfrak{a}_\perp}} \epsilon(u) e^{\pi_{\mathfrak{a}}[u(\mu+\rho)-\rho]+\mathcal{D}_{\mathfrak{a}_\perp}} \pi_{\mathfrak{a}}\left(\frac{\Psi^{\pi_{(\widetilde{\mathfrak{a}_\perp})}[u(\mu+\rho)-\rho]-\mathcal{D}_{\mathfrak{a}_\perp}}_{\widetilde{\mathfrak{a}_\perp}}}{\prod_{\beta\in\Delta_{\mathfrak{a}_\perp}} (1-e^{-\beta})^{\mathrm{mult}(\beta)}}\right) \\
&= \sum_{u\in U} \epsilon(u) \mathrm{dim}\left(L^{\mu_{\widetilde{\mathfrak{a}_\perp}}(u)}_{\widetilde{\mathfrak{a}_\perp}}\right) e^{\mu_{\mathfrak{a}}(u)}.
\end{aligned}$$

$\square$

Remark 1. This relation is a kind of a generalization of the Weyl formula for singular element $\Psi_{\mathfrak{g}}^{\mu}$: the vectors $\mu_{\mathfrak{a}}(u)$ play the role of singular weights while the determinants $\epsilon(u)$ are changed for $\epsilon(u)\dim\left(L_{\widetilde{\mathfrak{a}_{\perp}}}^{\mu_{\widetilde{\mathfrak{a}_{\perp}}}(u)}\right)$. In fact when $\mathfrak{a}=\mathfrak{g}$ both $\mathfrak{a}_{\perp}$ and $\mathfrak{h}_{\perp}$ are trivial, $U=W$, and the original Weyl formula is easily reobtained.

Now we can state the main theorem which gives us the instrument for the recurrent computation of the branching coefficients.

Theorem 1. *For the anomalous branching coefficients (1) holds the recurrent relation*

$$\begin{aligned} k_{\xi}^{(\mu)} = -\tfrac{1}{s(\gamma_0)}\Big(\textstyle\sum_{w\in W_{\mathfrak{a}_{\perp}}\setminus W}\epsilon(w)\,\dim\left(L_{\mathfrak{a}_{\perp}}^{\pi_{\mathfrak{a}_{\perp}}(w(\mu+\rho))-\rho_{\mathfrak{a}_{\perp}}}\right)\delta_{\xi-\gamma_0,\pi_{\mathfrak{a}}(w(\mu+\rho)-\rho)}+ \\ +\textstyle\sum_{\gamma\in\Gamma_{\mathfrak{a}\subset\mathfrak{g}}} s\left(\gamma+\gamma_0\right)k_{\xi+\gamma}^{(\mu)}\Big). \end{aligned} \tag{13}$$

As we have mentioned above (see (1) and (2)) when $\zeta\in\bar{C}_{\mathfrak{a}}$ the coefficients $k_{\zeta}^{(\mu)}$ coinside with the branching coefficients thus the equation for $k_{\zeta}^{(\mu)}$ serve as recurrent relations for branching coefficients.

Proof. To prove the theorem return to the relation (3) and rewrite it for the element $\frac{\Psi_{\mathfrak{g}}^{\mu}}{R_{\mathfrak{a}_{\perp}}}$,

$$\begin{aligned} \pi_{\mathfrak{a}}\left(\frac{\Psi_{\mathfrak{g}}^{\mu}}{R_{\mathfrak{a}_{\perp}}}\right) &= \prod_{\alpha\in\Delta^{+}\setminus\Delta_{\perp}^{+}}\left(1-e^{-\pi_{\mathfrak{a}}\alpha}\right)^{\mathrm{mult}(\alpha)-\mathrm{mult}_{\mathfrak{a}}(\pi_{\mathfrak{a}}\alpha)}\left(\sum_{\nu\in P_{\mathfrak{a}}^{+}}b_{\nu}^{(\mu)}\sum_{w\in W_{\mathfrak{a}}}\epsilon(w)e^{w(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}\right)= \\ &= -\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}}s(\gamma)e^{-\gamma}\left(\sum_{\nu\in P_{\mathfrak{a}}^{+},w\in W_{\mathfrak{a}}}\epsilon(w)b_{\nu}^{(\mu)}e^{w(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}\right). \end{aligned}$$

Then decompose the last sum (with respect to the formal basis in $\mathcal{E}$):

$$\pi_{\mathfrak{a}}\left(\frac{\Psi_{\mathfrak{g}}^{\mu}}{R_{\mathfrak{a}_{\perp}}}\right) = -\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}}s(\gamma)e^{-\gamma}\sum_{\lambda\in P_{\mathfrak{a}}}k_{\nu}^{(\mu)}e^{\lambda} = -\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}}\sum_{\lambda\in P_{\mathfrak{a}}}s(\gamma)k_{\nu}^{(\mu)}e^{\lambda-\gamma}.$$

Substitute the expression obtained in the Lemma (in the left-hand side),

$$\begin{aligned} \pi_{\mathfrak{a}}\left(\frac{\Psi_{\mathfrak{g}}^{\mu}}{R_{\mathfrak{a}_{\perp}}}\right) &= \sum_{u\in U}\epsilon(u)e^{\pi_{\mathfrak{a}}(\mu_{\mathfrak{a}}(u))}\dim\left(L_{\widetilde{\mathfrak{a}_{\perp}}}^{\mu_{\widetilde{\mathfrak{a}_{\perp}}}(u)}\right) \\ &= \sum_{u\in U}\epsilon(u)e^{\pi_{\mathfrak{a}}[u(\mu+\rho)-\rho]}\dim\left(L_{\widetilde{\mathfrak{a}_{\perp}}}^{\mu_{\widetilde{\mathfrak{a}_{\perp}}}(u)}\right) \\ &= -\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}}\sum_{\lambda\in P_{\mathfrak{a}}}s(\gamma)k_{\nu}^{(\mu)}e^{\lambda-\gamma}, \end{aligned}$$

Equate the coefficients of exponents in the last equality to get the following result:

$$\sum_{u\in U}\epsilon(u)\dim\left(L_{\widetilde{\mathfrak{a}_{\perp}}}^{\mu_{\widetilde{\mathfrak{a}_{\perp}}}(u)}\right)\delta_{\xi,\pi_{\mathfrak{a}}[u(\mu+\rho)-\rho]}+\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}}s(\gamma)\,k_{\xi+\gamma}^{(\mu)}=0,\quad \xi\in P_{\mathfrak{a}}. \tag{14}$$

The obtained formula means that the coefficients $k_{\xi+\gamma}^{(\mu)}$ for $\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}$ are not independent, they are subject to the linear relations and the form of these relations changes when the tested weight ξ coinsides with one of the "anomalous weights",

$\{\pi_{\mathfrak{a}}[u(\mu+\rho)-\rho]\,|u\in U\}$. To conclude the proof we extract the lowest weight $\gamma_0 \in \Phi_{\mathfrak{a}\subset\mathfrak{g}}$ and pass to the summation over the vectors of the injection fan $\Gamma_{\mathfrak{a}\subset\mathfrak{g}}$ (see the definition 1). Thus we get the desired recurrent relation (13). □

2.1. Embeddings and orthogonal pairs in simple Lie algebras

In this subsection we discuss some properties of "orthogonal pairs" of subalgebras in simple Lie algebras of classical series.

When both $\mathfrak{g}$ and $\mathfrak{a}$ are finite-dimensional all the regular embeddings can be obtained by a successive elimination of nodes in the extended Dynkin diagram of $\mathfrak{g}$ (and $\Delta^+_\perp = \emptyset$ if $\mathfrak{a}$ is maximal). For the classical series A, C and D when the regular injection $\mathfrak{a} \to \mathfrak{g}$ is thus fixed, the Dynkin diagram for $\mathfrak{a}_\perp$ is obtained from the extended diagram of $\mathfrak{g}$ by eliminating the subdiagram of $\mathfrak{a}$ and the adjacent nodes:

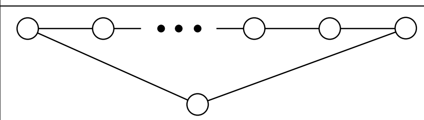
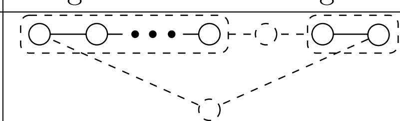
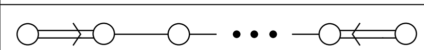
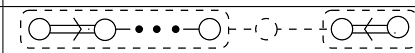
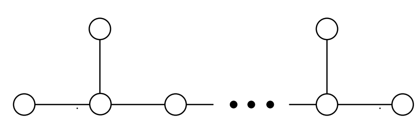
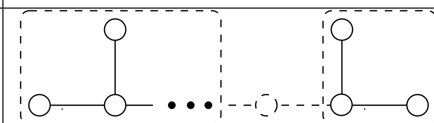
$\mathfrak{g}$	Extended diagram of $\mathfrak{g}$	Diagrams of the subalgebras $\mathfrak{a}$, $\mathfrak{a}_\perp$	
A_n			
C_n			
D_n			

Table 1. Subalgebras $\mathfrak{a}$, $\mathfrak{a}_\perp$ for the classical series

In the case of B series the situation is different. The reason is that here the subalgebra $\mathfrak{a}_\perp$ may be larger than the one obtained by elimination of the subdiagram of $\mathfrak{a}$ and the adjacent nodes. The subalgebras of the orthogonal pair, $\mathfrak{a}$ and $\mathfrak{a}_\perp$, must not form a direct sum in $\mathfrak{g}$. It can be directly checked that when $\mathfrak{g} = B_r$ and $\mathfrak{a} = B_{r_{\mathfrak{a}}}$ the orthogonal subalgebra is $\mathfrak{a}_\perp = B_{r-r_{\mathfrak{a}}}$. Consider the injection $B_{r_{\mathfrak{a}}} \to B_r, \quad 1 < r_{\mathfrak{a}} < r$. By eliminating the simple root $\alpha_{r_{\mathfrak{a}}-1} = e_{r_{\mathfrak{a}}-1} - e_{r_{\mathfrak{a}}}$ one splits the extended Dynkin diagram of B_r into the disjoint diagrams for $\mathfrak{a} = B_{r_{\mathfrak{a}}}$ and $D_{r-r_{\mathfrak{a}}}$. But the system $\Delta_{\mathfrak{a}_\perp}$ contains not only the simple roots $\{e_1-e_2, e_2-e_3, \ldots, e_{r_{\mathfrak{a}}-2}-e_{r_{\mathfrak{a}}-1}, e_1+e_2\}$ but also the root $e_{r_{\mathfrak{a}}-1}$. Thus $\Delta_{\mathfrak{a}_\perp}$ forms the subsystem of the type $B_{r-r_{\mathfrak{a}}}$ and the orthogonal pair for the injection $B_{r_{\mathfrak{a}}} \to B_r$ is $(B_{r_{\mathfrak{a}}}, B_{r-r_{\mathfrak{a}}})$. In the next Section the particular case of such orthogonal pair is presented for the injection $B_2 \to B_4$ (see Figure 3).

The complete classification of regular subalgebras for affine Lie algebras can be found in the recent paper [11]. From the complete classification of maximal special subalgebras in classical Lie algebras [12] we can deduce the following list of pairs of orthogonal subalgebras $\mathfrak{a}$, $\mathfrak{a}_\perp$:

$$
\begin{array}{lll}
su(p) \oplus su(q) & \subset su(pq) & \\
so(p) \oplus so(q) & \subset so(pq) & \\
sp(2p) \oplus sp(2q) & \subset so(4pq) & \\
sp(2p) \oplus so(q) & \subset sp(2pq) & \\
so(p) \oplus so(q) & \subset so(p+q) & \text{for p and q odd.}
\end{array}
$$

2.2. Algorithm for recursive computation of branching coefficients

The recurrent relation (13) allows us to formulate an algorithm for recursive computation of branching coefficients. In this algorithm there is no need to construct the module $L_{\mathfrak{g}}^{(\mu)}$ or any of the modules $L_{\mathfrak{a}}^{(\nu)}$.

It contains the following steps:

(i) Construct the root subsystem $\Delta_{\mathfrak{a}}$ for the embedding $\mathfrak{a} \to \mathfrak{g}$.

(ii) Select the positive roots $\alpha \in \Delta^+$ orthogonal to the roots in $\Delta_{\mathfrak{a}}$ i.e. form the set $\Delta_\perp^+$.

(iii) Construct the set $\Gamma_{\mathfrak{a}\to\mathfrak{g}}$ (11).

(iv) Construct the set $\widehat{\Psi^{(\mu)}} = \{w(\mu+\rho) - \rho;\ w \in W\}$ of the anomalous weights for the $\mathfrak{g}$-module $L^{(\mu)}$.

(v) Select the weights $\left\{\mu_{\widetilde{\mathfrak{a}_\perp}}(w) = \pi_{\widetilde{\mathfrak{a}_\perp}}\left[w(\mu+\rho) - \rho\right] - \mathcal{D}_{\mathfrak{a}_\perp} \in \overline{C_{\widetilde{\mathfrak{a}_\perp}}}\right\}$. Since the set $\Delta_\perp^+$ is fixed we can easily check wether the weight $\mu_{\widetilde{\mathfrak{a}_\perp}}(w)$ belongs to the main Weyl chamber $\overline{C_{\widetilde{\mathfrak{a}_\perp}}}$ (by computing its scalar product with the roots from $\Delta_\perp^+$).

(vi) For the weights $\mu_{\widetilde{\mathfrak{a}_\perp}}(w)$ calculate the dimensions of the corresponding modules $\dim\left(L_{\widetilde{\mathfrak{a}_\perp}}^{\mu_{\widetilde{\mathfrak{a}_\perp}}(u)}\right)$.

(vii) Calculate the anomalous branching coefficients in the main Weyl chamber $\overline{C_{\mathfrak{a}}}$ of the subalgebra $\mathfrak{a}$ using the recurrent relation (13).

When being interested in the branching coefficients for the embedding of the finite-dimensional Lie algebra into the affine Lie algebra we can construct the set of the anomalous weights up to the required grade and use the steps 4-7 of the algorithm for each grade. We can also speed up the algorithm by one-time computation of the representatives of the conjugate classes $W/W_{\mathfrak{a}_\perp}$.

The next section contains examples illustrating the application of this algorithm.

3. Branching for finite dimensional Lie algebras

3.1. Regular embedding of A_1 into B_2

Consider the regular embedding $A_1 \to B_2$. Simple roots α_1, α_2 of B_2 are presented as the dashed vectors in the Figure 1. We denote the corresponding Weyl reflections by w_1, w_2. The simple root $\beta = \alpha_1 + 2\alpha_2$ of A_1 is indicated as the grey vector.

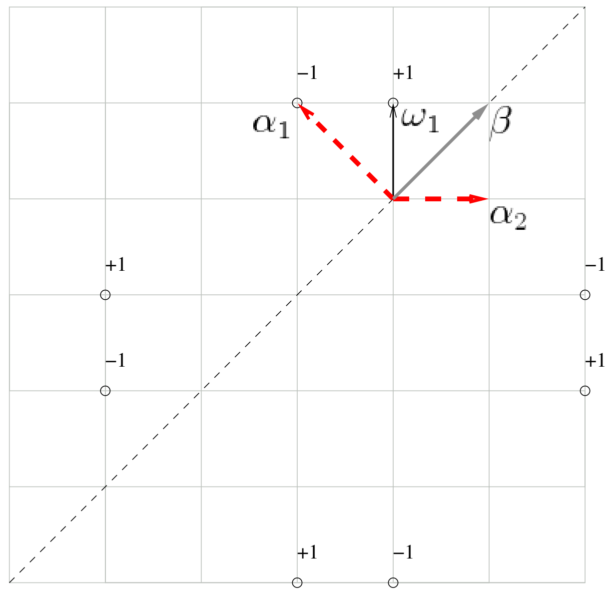


Figure 1. Regular embedding of A_1 into B_2. Simple roots α_1, α_2 of B_2 are presented as the dashed vectors. The simple root $\beta = \alpha_1 + 2\alpha_2$ of A_1 is indicated as the grey vector. The highest weight of the fundamental representation $L_{B_2}^{(1,0)=\omega_1}$ is shown by the black vector. The weights of the singular element $\Psi^{(\omega_1)}$ are indicated by the circles and the corresponding determinants $\epsilon(w)$ are shown.

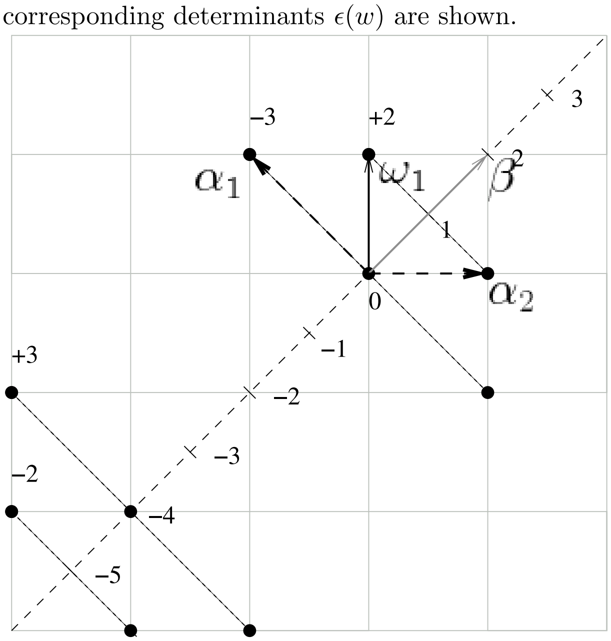


Figure 2. Regular embedding $A_1 \subset B_2$. Simple roots α_1, α_2 of B_2 are presented as the dashed vectors. The simple root $\beta = \alpha_1 + 2\alpha_2$ of A_1 is indicated as the grey vector. The highest weight of the fundamental representation $L_{B_2}^{(1,0)=\omega_1}$ is shown by the black vector. $\mathfrak{a}_{\perp} = A_1$-modules built on the weights of the singular element $\Psi^{(\omega_1)}$ are shown by dotted lines. The dimensions of these modules and the signs of the corresponding determinants $\epsilon(w)$ are shown near the anomalous weights. Fundamental weight coordinate of $\mathfrak{a} = A_1$ is indicated on the weight subspace of $\mathfrak{a}$.

Let's describe the reduction of the fundamental representation $L_{B_2}^{(1,0)=\omega_1}$ (the black vector in Figure 1).

Circles indicate the weights of the singular element $\Psi^{(\omega_1)}$. Now we are to factorise the Weyl group W by $W_{\mathfrak{a}_\perp} = \{w_1\}$ and to construct the set $\{w(\mu+\rho)-\rho,\ w \in W_{\mathfrak{a}_\perp}\backslash W\}$. In the Figure 2 the weights of the corresponding $(\mathfrak{a}_\perp = A_1)$-modules $L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(w(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}$ are indicated. Projecting them onto the root space of the subalgebra $\mathfrak{a} = A_1$ we get the anomalous weights and their multiplicities:

$$(1,2),\ (0,-3),\ (-4,3),\ (-5,-2).$$

For the set Γ (using the definition 1) we have

$$\Gamma_{A_1\subset B_2} = \{(1,2),\ (2,-1)\}\,.$$

Here the second component denotes the value of $s(\gamma)$. Applying this fan inside the $\bar{C}_{\mathfrak{a}}^{(0)}$ we get zeros for the weights greater than the first anomalous vector (1), here $k_1^{(1,0)} = 2$. For the last weight in $\bar{C}_{\mathfrak{a}}^{(0)}$ the formula (13) gives

$$k_0^{(1,0)} = -1\cdot k_2^{(1,0)} + 2\cdot k_1^{(1,0)} - 3\cdot\delta_{0,0} = 1.$$

The recurrence property defines the branching.

3.2. Embedding of B_2 into B_4

Consider the regular embedding $B_2 \longrightarrow B_4$. The corresponding Dynkin diagrams are presented in the Figure 3.

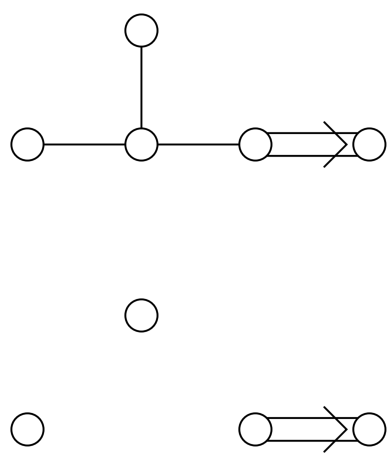


Figure 3. The regular embedding $B_2 \subset B_4$ described by dropping the node from the Dynkin diagram. Note, that $\mathfrak{a}_\perp$ is equal to B_2, not to $A_1 \oplus A_1$ seen at the diagram.

In the orthogonal basis $\{e_1,\ldots,e_4\}$ the simple roots and the positive roots of B_4 are

$$S_{B_4} = \{e_1-e_2,\ e_2-e_3,\ e_3-e_4,\ e_4\}$$

$$\Delta^+_{B_4} = \{(e_1-e_2,\ e_2-e_3,\ e_3-e_4,\ e_4,\ e_1-e_3,\ e_2-e_4,\ e_3+e_4,\ e_3,\ e_1-e_4,$$
$$e_2+e_4,\ e_2,\ e_1+e_4,\ e_2+e_3,\ e_1,\ e_1+e_3,\ e_1+e_2\}$$

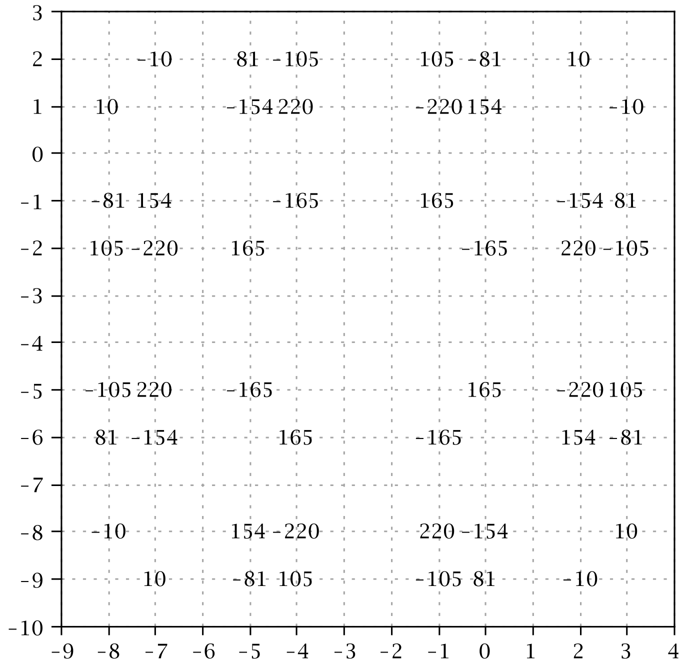


Figure 4. Projected singular weights of $-\frac{1}{s(\gamma_0)}\pi_{B_2}\left(\hat{\Psi}_{B_4}^{(0,1,0,2)}\right)$ with the dimensions of the corresponding $\mathfrak{a}_\perp = B_2$-modules.

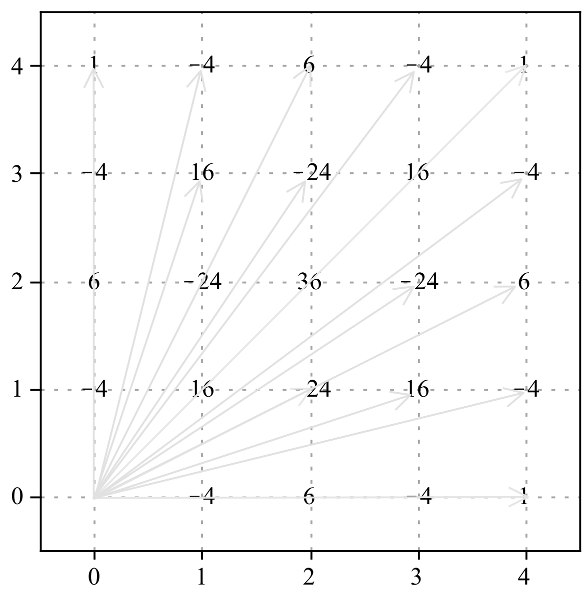


Figure 5. The fan $\Gamma_{B_2\subset B_4}$ for $B_2 \subset B_4$. The values $s(\gamma+\gamma_0)$ are shown in the corresponding weights γ.

Correspondingly for the embedded subalgebra $\mathfrak{a} = B_2$ we have

$$S_{B_2} = \{e_3 - e_4, e_4\}$$

and

$$\Delta^+_{\mathfrak{a}_\perp} = \{e_1 - e_2, e_1 + e_2, e_1, e_2\}$$

is the set of positive roots for the algebra $\mathfrak{a}_\perp = B_2$.

Using the definition (11) we obtain the fan $\Gamma_{B_2 \subset B_4}$ with the corresponding values $s(\gamma + \gamma_0)$, depicted in the Figure 5.

Consider the B_4-module L^μ with the highest weight $\mu = (0,1,0,2) = 2e_1 + 2e_2 + e_3 + e_4$; $\dim(L^{(0,1,0,2)}) = 2772$. To find the branching coefficients we need to compute the anomalous weights of $L^\mu_{B_4}$, select the weights belonging to $\bar{C}^{(0)}_{\mathfrak{a}_\perp}$ and compute the dimensions of the corresponding $\mathfrak{a}_\perp$-modules. The set of the anomalous weights $\{w(\mu+\rho) - \rho,\ w \in W\}$ contains 384 vectors.

We are to select the weights $\psi \in w(\mu + \rho)$ with the property $\pi_{\mathfrak{a}_\perp}(\psi) \in \bar{C}^{(0)}_{\mathfrak{a}_\perp}$. It means that the scalar product of these weights with all the roots in $\Delta^+_{\mathfrak{a}_\perp}$ is nonnegative.

To compute the dimensions of the corresponding $\mathfrak{a}_\perp$-modules we need to project each selected weight onto the root space $\Delta^+_{\mathfrak{a}_\perp}$, subtract $\rho_{\mathfrak{a}_\perp}$ and apply the Weyl dimension formula. The result is shown in the Figure 4.

Applying the recurrent relation (13) we obtain the following branching coefficients:

$$\pi_{\mathfrak{a}}\left(chL^{(0,1,0,2)}_{B_4}\right) = 6\ chL^{(0,0)}_{B_2} + 60\ chL^{(0,2)}_{B_2} + 30\ chL^{(1,0)}_{B_2} + 19\ chL^{(2,0)}_{B_2} + 40\ chL^{(1,2)}_{B_2} + 10\ chL^{(2,2)}_{B_2}.$$

4. Applications to the conformal field theory

4.1. Conformal embeddings

Branching coefficients for an embedding of affine Lie algebra into affine Lie algebra can be used to construct modular invariant partition functions for Wess-Zumino-Novikov-Witten models in conformal field theory ([1], [13], [14], [15]). In these models current algebras are affine Lie algebras.

The modular invariant partition function is crucial for the conformal theory to be valid on the torus and higher genus Riemann surfaces. It is important for the applications of CFT to the string theory and to the critical phenomena description.

The simplest modular-invariant partition function has the diagonal form:

$$Z(\tau) = \sum_{\mu \in P^+_{\mathfrak{g}}} \chi_\mu(\tau)\bar{\chi}_\mu(\bar{\tau})$$

Here the sum is over the set of the highest weights of integrable modules in a WZW-model and $\chi_\mu(\tau)$ are the normalized characters of these modules.

To construct the nondiagonal modular invariants is not an easy problem, although for some models the complete classification of modular invariants is known [16, 17].

Consider the Wess-Zumino-Witten model with the affine Lie algebra $\mathfrak{a}$. Nondiagonal modular invariants for this model can be constructed from the diagonal invariant if there exists an affine algebra $\mathfrak{g}$ such that $\mathfrak{a} \subset \mathfrak{g}$. Then we can replace the characters of the $\mathfrak{g}$-modules in the diagonal modular invariant partition function (15) by the decompositions

$$\sum_{\nu \in P_{\mathfrak{a}}^{+}} b_{\nu}^{(\mu)} \chi_{\nu}$$

containing the modified characters χ_ν of the corresponding $\mathfrak{a}$-modules. Thus we obtain the nondiagonal modular-invariant partition function for the theory with the current algebra $\mathfrak{a}$,

$$Z_{\mathfrak{a}}(\tau) = \sum_{\nu,\lambda \in P_{\mathfrak{a}}^{+}} \chi_{\nu}(\tau) M_{\nu\lambda} \bar{\chi}_{\lambda}(\bar{\tau}). \tag{15}$$

The effective reduction procedure is crucial for this construction. The embedding is required to preserve the conformal invariance. Let $X_{-n_j}^{\alpha_j}$ and $\tilde{X}_{-n_j}^{\alpha'_j}$ be the lowering generators for $\mathfrak{g}$ and for $\mathfrak{a} \subset \mathfrak{g}$ correspondingly. Let $\pi_{\mathfrak{a}}$ be the projection operator of $\pi_{\mathfrak{a}} : \mathfrak{g} \longrightarrow \mathfrak{a}$. In the theory attributed to $\mathfrak{g}$ with the vacuum $|\lambda\rangle$ the states can be described as

$$X_{-n_1}^{\alpha_1} X_{-n_2}^{\alpha_2} \ldots |\lambda\rangle \quad n_1 \geq n_2 \geq \ldots > 0.$$

And for the sub-algebra $\mathfrak{a}$ the corresponding states are

$$\tilde{X}_{-n_1}^{\alpha'_1} \tilde{X}_{-n_2}^{\alpha'_2} \ldots |\pi_{\mathfrak{a}}(\lambda)\rangle .$$

The $\mathfrak{g}$-invariance of the vacuum entails its $\mathfrak{a}$-invariance, but this is not the case for the energy-momentum tensor. So the energy-momentum tensor of the larger theory should contain only the generators $\tilde{X}$. Then the relation

$$T_{\mathfrak{g}}(z) = T_{\mathfrak{a}}(z) \tag{16}$$

leads to the equality of the central charges

$$c(\mathfrak{g}) = c(\mathfrak{a})$$

and to the equation

$$\frac{k \dim \mathfrak{g}}{k+g} = \frac{x_e k \dim \mathfrak{a}}{x_e k + a}. \tag{17}$$

Here x_e is the embedding index and g, a are the dual Coxeter numbers for the corresponding algebras.

It can be demonstrated that the solutions of the equation (17) exist only for the level $k = 1$ [1].

The complete classification of conformal embeddings is given in [15].

The relation (17) and the asymptotics of the branching functions can be used to prove the finite reducibility theorem [18]. It states that for the conformal embedding $\mathfrak{a} \subset \mathfrak{g}$ only finite number of branching coefficients have nonzero values.

Note 4.1. *The orthogonal subalgebra $\mathfrak{a}_\perp$ is always empty for the conformal embeddings $\mathfrak{a} \subset \mathfrak{g}$.*

Proof. Consider the modes expansion of the energy-momentum tensor

$$T(z) = \frac{1}{2(k+h^v)} \sum_n z^{-n-1} L_n.$$

The modes L_n are constructed as combination of normally-ordered products of the generators of $\mathfrak{g}$,

$$L_n = \frac{1}{2(k+h^v)} \sum_\alpha \sum_m : X^\alpha_m X^\alpha_{n-m} : \ .$$

In the case of a conformal embedding energy-momentum tensors are to be equal (16).

The substitution of the generators of $\mathfrak{a}$ in terms of the generators of $\mathfrak{g}$ into these combinations should give the energy-momentum tensor $T_{\mathfrak{g}}$. But if the set of the generators $\Delta_{\mathfrak{a}_\perp}$ is not empty this is not possible, since $T_{\mathfrak{g}}$ contains the combinations of the generators X^α_n, $\alpha \in \Delta_{\mathfrak{a}_\perp}$. □

4.1.1. Special embedding $\hat{A}_1 \subset \hat{A}_2$. Consider the case where both $\mathfrak{g}$ and $\mathfrak{a}$ are affine Lie algebras: $\hat{A}_1 \longrightarrow \hat{A}_2$ and the injection is the affine extension of the special injection $A_1 \longrightarrow A_2$ with the embedding index $x_e = 4$. As far as the $\mathfrak{g}$-modules to be considered are of level one, the $\mathfrak{g}$-modules will be of level $\tilde{k} = kx_e = 4$.

There exist three level one fundamental weights in the weight space of $\hat{A}_2$. It is easy to see that the set $\Delta_{\mathfrak{a}_\perp}$ is empty and the subalgebra $\mathfrak{a}_\perp = 0$.

Using the definition (11) we construct the fan $\Gamma_{\hat{A}_1 \to \hat{A}_2}$ and the function $s(\gamma + \gamma_0)$ (see the Figure 6).

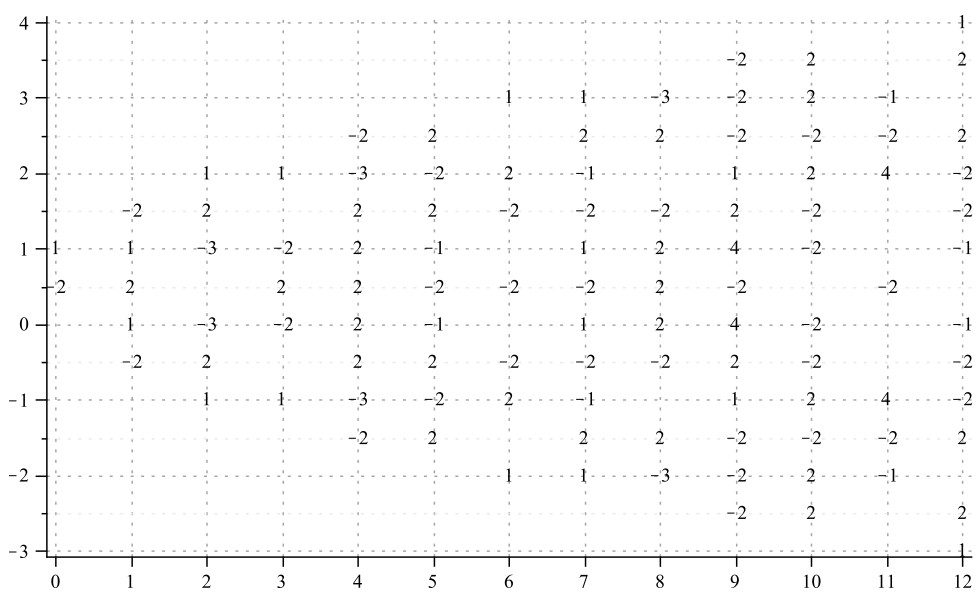


Figure 6. The fan $\Gamma_{\hat{A}_1 \longrightarrow \hat{A}_2}$ for $\hat{A}_1 \longrightarrow \hat{A}_2$. Values of $s(\gamma + \gamma_0)$ are shown for the weights $\gamma \in \Gamma_{\hat{A}_1 \longrightarrow \hat{A}_2}$

Let us consider the module $L^{\omega_0=(0,0;1;0)}$. Here we use the (finite part; level; grade) presentation of the highest weight and the finite part coordinates are the Dynkin indices (see section(1.1)).

The set $\widehat{\Psi^{(\omega_0)}}$ is depicted in the Figure 7 up to the sixth grade.

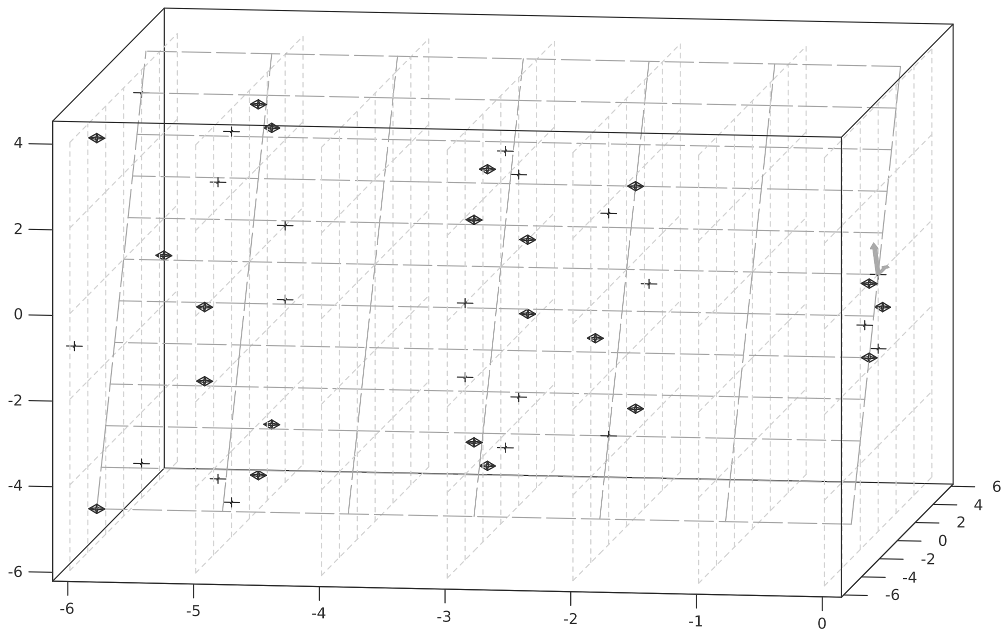


Figure 7. The anomalous weights of the module $L^{\omega_0}_{\hat{A}_2} = L^{(0,0;1;0)}_{\hat{A}_2}$. The weights $w(\omega_0+\rho)-\rho$ are marked by crosses when $\epsilon(w)=1$ and by diamond when $\epsilon(w)=-1$. Simple roots of the classical subalgebra A_2 are grey and the grey diagonal plane corresponds to the Cartan subalgebra of the embedded algebra $\hat{A}_1$.

The next step is to project the anomalous weights to $P_{\hat{A}_1}$. The result is presented in the Figure 8 up to the twelfth grade.

Using the recurrent relation for the anomalous branching coefficients we get the result presented in Figure 9. Inside the Weyl chamber $\bar{C}_{\hat{A}_1}$ (its boundaries are indicated in the Figure 9) there are only two nonzero anomalous weights and both have multiplicity 1. These are the highest weights of $\mathfrak{a}$-submodules and their branching coefficients. So the finite reducibility theorem holds and we get the decomposition

$$L^{(0,0;1;0)}_{\hat{A}_2\downarrow\hat{A}_1} = L^{(0;4;0)}_{\hat{A}_1} \oplus L^{(4;4;0)}_{\hat{A}_1}.$$

For the other irreducible modules of level one we get the trivial branching

$$L^{(1,0;1;0)}_{\hat{A}_2\downarrow\hat{A}_1} = L^{(2;4;0)}_{\hat{A}_1}, L^{(0,1;1;0)}_{\hat{A}_2\downarrow\hat{A}_1} = L^{(2;4;0)}_{\hat{A}_1}.$$

Using these results the modular-invariant partition function is easily found,

$$Z = \left|\chi_{(4;4;0)} + \chi_{(0;4;0)}\right|^2 + 2\chi^2_{(2;4;0)}.$$

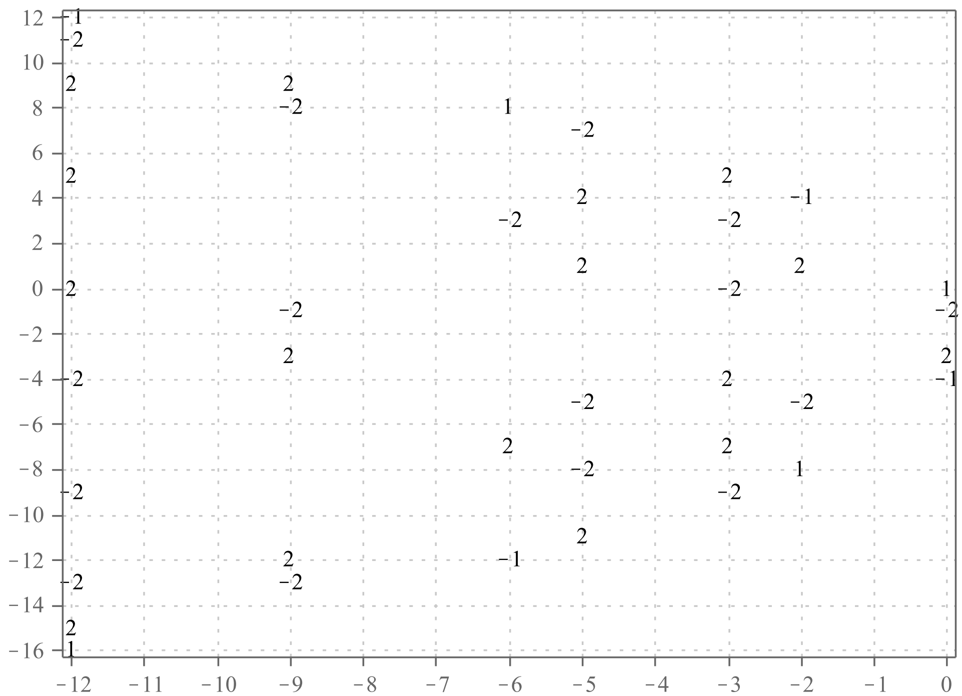


Figure 8. Projected anomalous weights of $L^{(0,0;1;0)}_{\hat{A}_2}$. The multiplicities of projected weights and the corresponding signs are shown.

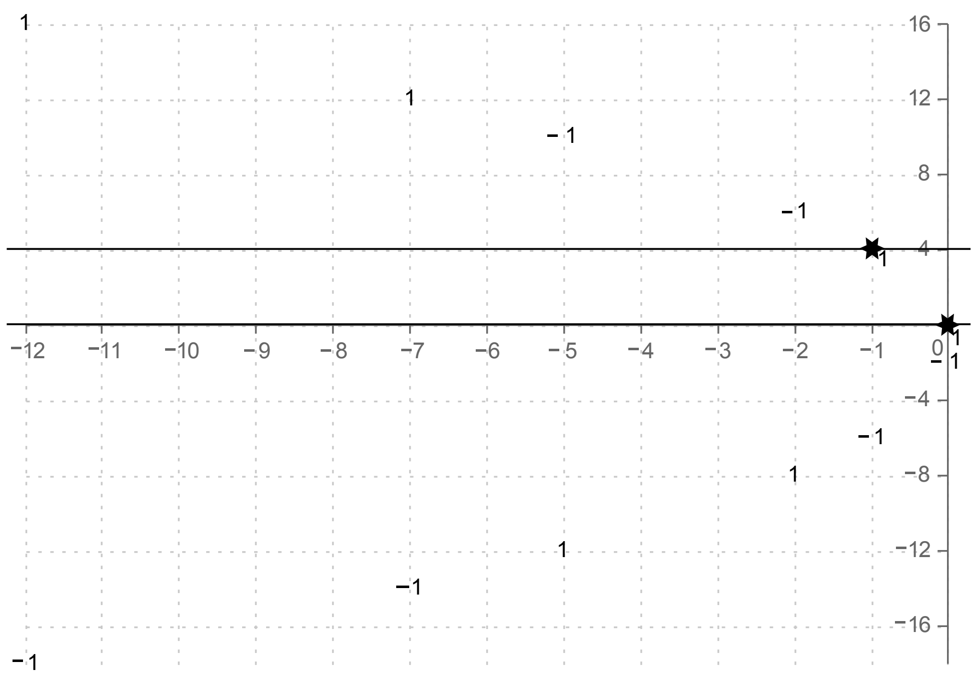


Figure 9. Anomalous branching coefficients for $\hat{A}_1 \subset \hat{A}_2$. The boundaries of the main Weyl chamber $\bar{C}_{\hat{A}_1}$ are indicated by the black lines. Two nonzero anomalous weights in the main Weyl chamber are marked by stars. Both weights have multiplicity 1, so the branching coefficients are equal to 1.

4.2. Coset models

Coset models [19] tightly connected with the gauged WZW-models are actively studied in string theory, especially in string models on anti-de-Sitter space [20, 21, 22, 23, 24]. The characters in coset models are proportional to the branching functions,

$$\chi_{\nu}^{(\mu)}(\tau) = e^{2\pi i\tau(m_\mu - m_\nu)} b_{\nu}^{(\mu)}(\tau), \tag{18}$$

with

$$m_\mu = \frac{|\mu+\rho|^2}{2(k+g)} - \frac{|\rho|^2}{2g}.$$

The problem of the branching functions construction in the coset models was considered in [25], [7], [26].

Let us return to the example 3.1 and consider the affine extension of the injection $A_1 \longrightarrow B_2$. Since this embedding is regular and $x_e = 1$, the subalgebra modules and the initial module are of the same level. The set $\Delta^+_{\mathfrak{a}_\perp}$ of the orthogonal positive roots with the zero projection on the root space of the subalgebra $\hat{A}_1$ is the same as in the finite-dimensional case.

Using the definition (11) we get the fan $\Gamma_{\hat{A}_1 \longrightarrow \hat{B}_2}$ with the corresponding values $s(\gamma+\gamma_0)$ (see the Figure 10). We restricted the computation to the twelfth grade.

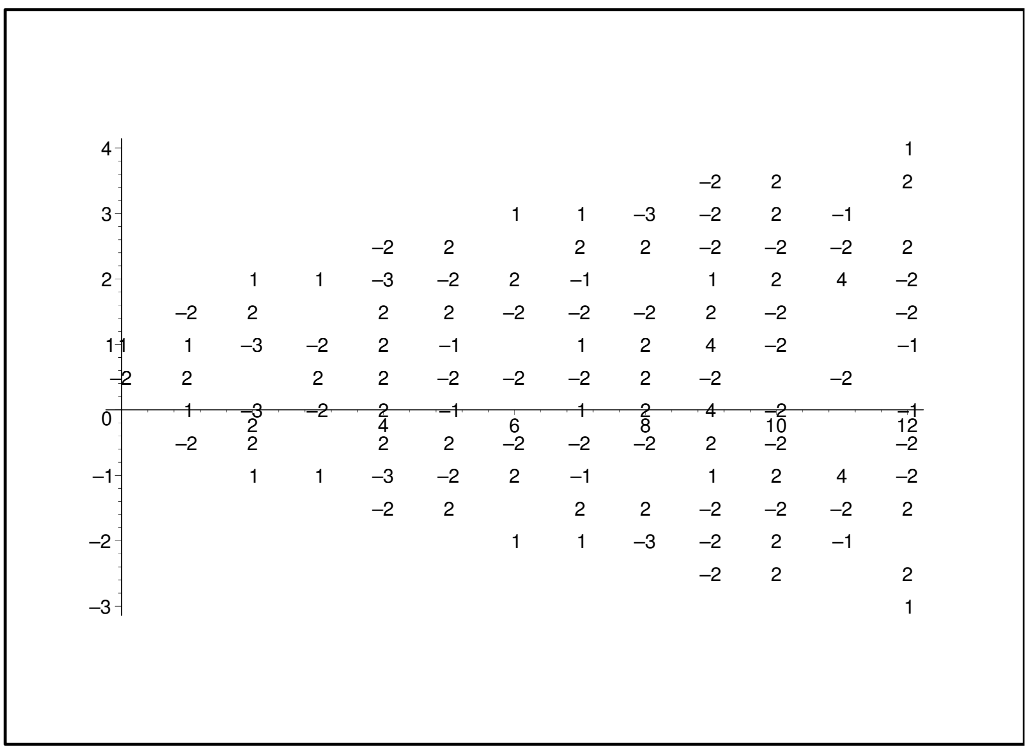


Figure 10. The fan $\Gamma_{\hat{A}_1 \longrightarrow \hat{B}_2}$ for $\hat{A}_1 \longrightarrow \hat{B}_2$. Values of $s(\gamma+\gamma_0)$ are shown for the weights $\gamma \in \Gamma_{\hat{A}_1 \longrightarrow \hat{B}_2}$

Consider the level one module $L^{(1,0;1;0)}_{\hat{B}_2}$ with the highest weight $\omega_1 = (1,0;1;0)$, where the finite part coordinates are in the orthogonal basis e_1, e_2. The set of anomalous weights for this module up to the sixth grade is presented in the Figure 11. In the grade zero it is exactly the set of the anomalous weights for the embedding of the classical Lie algebras $A_1 \subset B_2$ that can be seen in the Figure 1.

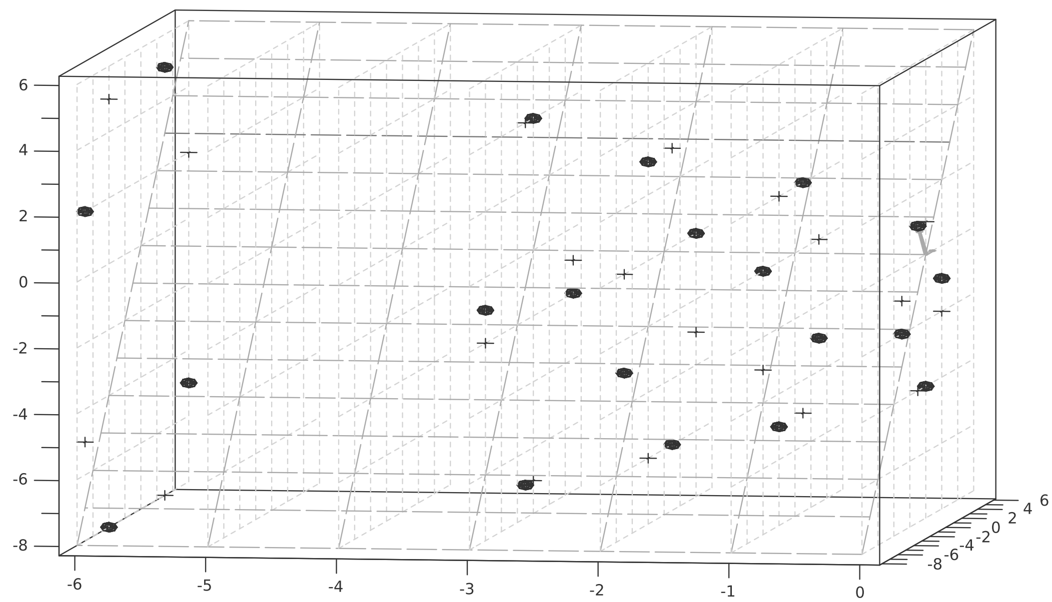


Figure 11. The anomalous weights of $L^{(1,0;1;0)}_{\hat{B}_2}$. The weights in the zero grade can be seen in the Figure 1. The weights $w(\omega_1+\rho)-\rho$ are marked by crosses if $\epsilon(w)=1$ and by circles otherwise. Simple roots of the classical subalgebra B_2 are grey and grey diagonal plane corresponds to the Cartan subalgebra of the embedded algebra $\hat{A}_1$.

According to the algorithm 2.2 we project the anomalous weights to $P_{\hat{A}_1}$ and find the dimensions of the corresponding $\mathfrak{a}_\perp$-modules $L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(w(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}$. The result is presented in the Figure 12 up to the twelfth grade.

Notice that here the lowest weight γ_0 of the fan is zero, since we have excluded all the roots of $\Delta^+_{\mathfrak{a}_\perp}$ from the defining relation (11).

The multiplicities of the weights inside the Weyl chamber $\bar{C}^{(0)}_{\hat{A}_1}$ are the branching coefficients (up to the twelfth grade),

$$
\begin{aligned}
&L^{\omega_1}_{\hat{B}_2\downarrow\hat{A}_1} = 2L^{\omega_1}_{\hat{A}_1} \oplus 1L^{\omega_0}_{\hat{A}_1} \oplus 4L^{\omega_0-\delta}_{\hat{A}_1} \oplus \\
&2L^{\omega_1-\delta}_{\hat{A}_1} \oplus 8L^{\omega_0-2\delta}_{\hat{A}_1} \oplus 8L^{\omega_1-2\delta}_{\hat{A}_1} \oplus 15L^{\omega_0-3\delta}_{\hat{A}_1} \oplus \\
&12L^{\omega_1-3\delta}_{\hat{A}_1} \oplus 26L^{\omega_1-4\delta}_{\hat{A}_1} \oplus 29L^{\omega_0-4\delta}_{\hat{A}_1} \oplus 51L^{\omega_0-5\delta}_{\hat{A}_1} \oplus \\
&42L^{\omega_1-5\delta}_{\hat{A}_1} \oplus 78L^{\omega_1-6\delta}_{\hat{A}_1} \oplus 85L^{\omega_0-6\delta}_{\hat{A}_1} \oplus 120L^{\omega_1-7\delta}_{\hat{A}_1} \oplus \\
&139L^{\omega_0-7\delta}_{\hat{A}_1} \oplus 202L^{\omega_1-8\delta}_{\hat{A}_1} \oplus 222L^{\omega_0-8\delta}_{\hat{A}_1} \oplus 306L^{\omega_1-9\delta}_{\hat{A}_1} \oplus \\
&346L^{\omega_0-9\delta}_{\hat{A}_1} \oplus 530L^{\omega_0-10\delta}_{\hat{A}_1} \oplus 482L^{\omega_1-10\delta}_{\hat{A}_1} \oplus 714L^{\omega_1-11\delta}_{\hat{A}_1} \oplus \\
&797L^{\omega_0-11\delta}_{\hat{A}_1} \oplus 1080L^{\omega_1-12\delta}_{\hat{A}_1} \oplus 1180L^{\omega_0-12\delta}_{\hat{A}_1} \oplus \ldots
\end{aligned}
$$

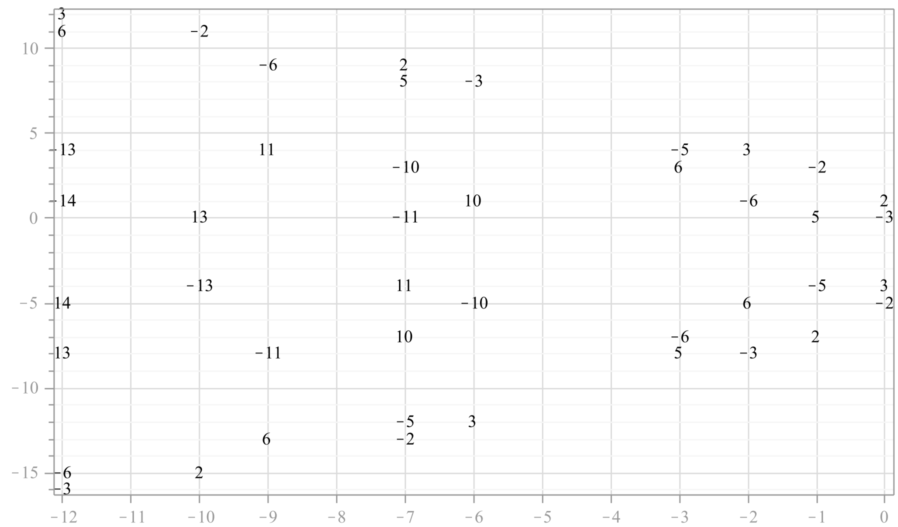


Figure 12. The projected anomalous weights $\pi_{\hat{A}_1}\left(\Psi^{(1,0;1;0)}_{\hat{B}_2}\right)$. The dimensions of the corresponding $\mathfrak{a}_\perp = A_1$-modules with the signs $\epsilon(w)$ are shown.

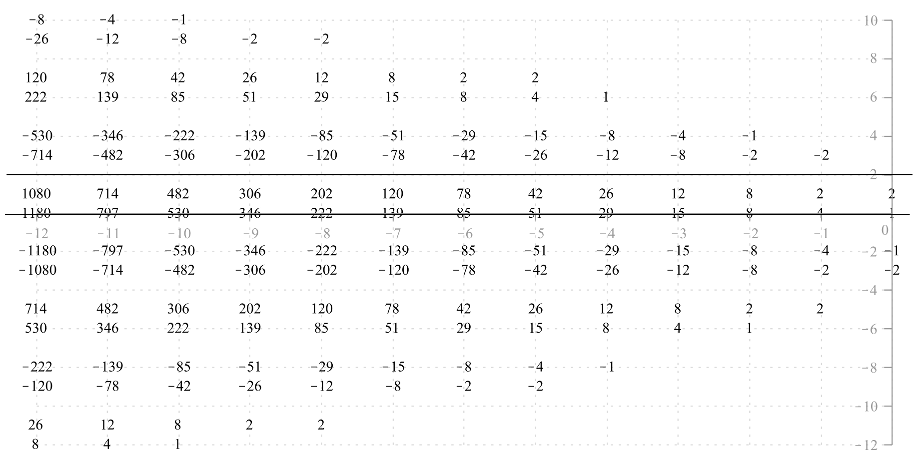


Figure 13. Anomalous branching coefficients for $\hat{A}_1 \subset \hat{B}_2$. The boundaries of the main Weyl chamber $\bar{C}_{\hat{A}_1}$ are indicated by the black lines. The anomalous branching coefficients inside the main Weyl chamber are equal to the branching coefficients of the embedding $\hat{A}_1 \longrightarrow \hat{B}_2$.

This result can be presented as the set of branching functions:

$$\begin{aligned} b_0^{(\omega_1)} = &\; 1 + 4\,q^1 + 8\,q^2 + 15\,q^3 + 29\,q^4 + 51\,q^5 + 85\,q^6 + 139\,q^7 + \\ &\; 222\,q^8 + 346\,q^9 + 530\,q^{10} + 797\,q^{11} + 1180\,q^{12} + \dots \\ b_1^{(\omega_1)} = &\; 2 + 2\,q^1 + 8\,q^2 + 12\,q^3 + 26\,q^4 + 42\,q^5 + 78\,q^6 + 120\,q^7 + \\ &\; 202\,q^8 + 306\,q^9 + 482\,q^{10} + 714\,q^{11} + 1080\,q^{12} + \dots \end{aligned}$$

Here $q = \exp(2\pi i \tau)$ and the lower index enumerates the branching functions according to their highest weights in $P^+_{\hat{A}_1}$. These are the fundamental weights $\omega_0 = \lambda_0 = (0,1,0)$, $\omega_1 = \alpha/2 = (1,1,0)$.

Now we can use the relation (18) to get the expansion of the B_2/A_1-coset characters:

$$\begin{aligned}\chi_1^{(\omega_1)}(q) = & \; q^{\frac{7}{12}}\,(2 + 2\,q^1 + 8\,q^2 + 12\,q^3 + 26\,q^4 + 42\,q^5 + 78\,q^6 + 120\,q^7 + \\ & \; 202\,q^8 + 306\,q^9 + 482\,q^{10} + 714\,q^{11} + 1080\,q^{12} + \ldots) \\ \chi_0^{(\omega_1)}(q) = & \; q^{\frac{5}{6}}\,(1 + 4\,q^1 + 8\,q^2 + 15\,q^3 + 29\,q^4 + 51\,q^5 + 85\,q^6 + 139\,q^7 + \\ & \; 222\,q^8 + 346\,q^9 + 530\,q^{10} + 797\,q^{11} + 1180\,q^{12} + \ldots)\end{aligned}$$

Further amelioration of the algorithm can be achieved by using the folded fan technique [27] to get the explicit expression for the branching functions and the corresponding coset characters in conformal field theory.

5. Conclusion

We have demonstrated that the injection fan technique can be used to deal with the nonmaximal subalgebras. It was found out that for such subalgebras an auxiliary subset $\Delta^+_{\mathfrak{a}_\perp}$ must be extracted from the set of positive roots $\Delta^+_{\mathfrak{g}}$. The role of the subset $\Delta^+_{\mathfrak{a}_\perp}$ is to modify both the injection fan (formed here by the weights $\left(\Delta^+_{\mathfrak{g}} \setminus \Delta^+_{\mathfrak{a}}\right) \setminus \Delta^+_{\mathfrak{a}_\perp}$) and the anomalous weights of the initial module. This modification reduces to a simple procedure: the anomalous weights multiplicities are to be substituted by the dimensions of the corresponding $\mathfrak{a}_\perp$-modules.

The efficiency of the injection fan algorithm was verified. Its possible applications to some physical problems were discussed. In particular we considered the construction of modular-invariant partition functions in the framework of conformal embedding method and the coset construction in the rational conformal field theory. This construction is useful in the study of WZW-models emerging in the context of the AdS/CFT correspondence [20, 21, 22].

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