Recursive properties of branching and BGG resolution

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Abstract

Recurrent relations for branching coefficients are based on a special type of singular element decomposition. We show that this decomposition can be used to construct the parabolic Verma modules and finally to obtain the generalized Weyl-Verma formulas for characters. We demonstrate how branching coefficients can determine the generalized BGG resolution sequence.

1 Introduction

Branching properties of Lie (affine Lie) algebras are highly important for applications in quantum field theory (see for example the conformal field theory models [1],[2]). In this paper we demonstrate that for an arbitrary reductive subalgebra branching is directly connected with the BGG resolution and in

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particular exhibits the resolution properties in terms of the $\mathcal{O}^p$ category [3] (the parabolic generalization of the cathegory $\mathcal{O}$ [4]).

The resolution of irreducible modules in terms of infinite-dimensional ones is important for the theory of integrable spin chains [5]. In the Baxter $\mathcal{Q}$-operator approach [6] the generic transfer matrices corresponding to the (generalized) Verma modules are factorized into the product of Baxter operators. The resolution allows to calculate the transfer matrices for finite-dimensional auxiliary spaces.

To show the connection of the BGG resolution with the branching we use the recursive approach presented in [7] (similar to the one used in [8] for maximal embeddings). We consider the subalgebra $\mathfrak{a} \hookrightarrow \mathfrak{g}$ together with its counterpart $\mathfrak{a}_\perp$ "orthogonal" to $\mathfrak{a}$ with respect to the Killing form and also $\widetilde{\mathfrak{a}_\perp} := \mathfrak{a}_\perp \oplus \mathfrak{h}_\perp$ where $\mathfrak{h} = \mathfrak{h}_\mathfrak{a} \oplus \mathfrak{h}_{\mathfrak{a}_\perp} \oplus \mathfrak{h}_\perp$. For any reductive subalgebra $\mathfrak{a}$ the subalgebra $\mathfrak{a}_\perp \hookrightarrow \mathfrak{g}$ is regular and reductive. For a highest weight integrable module $L^{(\mu)}$ and orthogonal subalgebra $\mathfrak{a}_\perp$ we consider the singular element $\Psi^{(\mu)}$ (the numerator in the Weyl character formula $ch\left(L^\mu\right) = \frac{\Psi^{(\mu)}}{\Psi^{(0)}}$, see for example [9]) and the Weyl denominator $\Psi^{(0)}_{\mathfrak{a}_\perp}$ for the orthogonal partner. It is shown that the element $\Psi^{(\mu)}_\mathfrak{g}$ can be decomposed into a combination of Weyl numerators $\Psi^{(\nu)}_{\mathfrak{a}_\perp}$ with $\nu \in P^+_{\mathfrak{a}_\perp}$. This decomposition provides the possibility to construct the set of highest weight modules $L^{\mu_{\mathfrak{a}_\perp}}_{\mathfrak{a}_\perp}$. When the injection $\mathfrak{a}_\perp \hookrightarrow \mathfrak{g}$ satisfies the "standard parabolic" conditions these modules give rise to the parabolic Verma modules $M^{\mu_{\mathfrak{a}_\perp}}_{(\mathfrak{a}_\perp \hookrightarrow \mathfrak{g})}$ so that the initial character $ch\left(L^\mu\right)$ is finally decomposed into the alternating sum of such. On the other hand when the parabolic conditions are violated the construction survives and exhibits a decomposition with respect to a set of generalized Verma modules $M^{\mu_{\widetilde{\mathfrak{a}_\perp}}}_{\left(\widetilde{\mathfrak{b}_\perp}, \mathfrak{g}\right)}$ where $\widetilde{\mathfrak{b}_\perp}$ is not a subalgebra in $\mathfrak{g}$ but a contraction of $\widetilde{\mathfrak{a}_\perp}$.

Some general properties of the proposed decompositions are formulated in terms of a specific formal element $\Gamma_{\mathfrak{a} \to \mathfrak{g}}$ called "the injection fan". Using this tool a simple and explicit algorithm for branching rules applicable for an arbitrary (maximal or nonmaximal) subalgebra in affine Lie algebras was proposed in [7].

Possible further developments are discussed in Section 4.

1.1   Notation

Consider Lie algebras (affine Lie algebras) $\mathfrak{g}$ and $\mathfrak{a}$ and an injection $\mathfrak{a} \hookrightarrow \mathfrak{g}$ such that $\mathfrak{a}$ is a reductive subalgebra $\mathfrak{a} \subset \mathfrak{g}$ with correlated root spaces: $\mathfrak{h}^*_\mathfrak{a} \subset$

$\mathfrak{h}_{\mathfrak{g}}^*$. We use the following notations:

$\mathfrak{g} = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}^+$ — the Cartan decomposition;

r , $(r_\mathfrak{a})$ — the rank of the algebra $\mathfrak{g}$ (resp.$\mathfrak{a}$) ;

Δ $(\Delta_\mathfrak{a})$— the root system; Δ^+ (resp.$\Delta_\mathfrak{a}^+$)— the positive root system (of $\mathfrak{g}$ and $\mathfrak{a}$ respectively);

mult (α) (mult$_\mathfrak{a}$ (α)) — the multiplicity of the root α in Δ (resp. in $(\Delta_\mathfrak{a})$);

S $(S_\mathfrak{a})$ — the set of simple roots (for $\mathfrak{g}$ and $\mathfrak{a}$ respectively);

α_i , $\left(\alpha_{(\mathfrak{a})j}\right)$ — the i-th (resp. j-th) simple root for $\mathfrak{g}$ (resp.$\mathfrak{a}$); $i = 0, \ldots, r$, $(j = 0, \ldots, r_\mathfrak{a})$;

$\alpha_i^\vee$, $\left(\alpha_{(\mathfrak{a})j}^\vee\right)$ – the simple coroot for $\mathfrak{g}$ (resp.$\mathfrak{a}$) , $i = 0, \ldots, r$; $(j = 0, \ldots, r_\mathfrak{a})$;

W , $(W_\mathfrak{a})$— the Weyl group;

C , $(C_\mathfrak{a})$— the fundamental Weyl chamber;

$\bar{C}, \left(\bar{C}_\mathfrak{a}\right)$ — the closure of the fundamental Weyl chamber;

$\epsilon(w) := (-1)^{\text{length}(w)}$;

ρ , $(\rho_\mathfrak{a})$ — the Weyl vector;

L^μ $(L_\mathfrak{a}^\nu)$ — the integrable module of $\mathfrak{g}$ with the highest weight μ ; (resp. integrable $\mathfrak{a}$-module with the highest weight ν);

$\mathcal{N}^\mu$, $(\mathcal{N}_\mathfrak{a}^\nu)$ — the weight diagram of L^μ (resp. $L_\mathfrak{a}^\nu$);

P (resp. $P_\mathfrak{a}$) — the weight lattice;

P^+ (resp. $P_\mathfrak{a}^+$) — the dominant weight lattice;

$m_\xi^{(\mu)}$, $\left(m_\zeta^{(\nu)}\right)$ — the multiplicity of the weight $\xi \in P$ (resp. $\in \mathrm{P}_\mathfrak{a}$) in L^μ, (resp. in $\zeta \in L_\mathfrak{a}^\nu$);

$ch\left(L^\mu\right)$ (resp. $ch\left(L_\mathfrak{a}^\nu\right)$)— the formal character of L^μ (resp. of $L_\mathfrak{a}^\nu$);

$ch\left(L^\mu\right) = \frac{\sum_{w \in W} \epsilon(w) e^{w \circ (\mu + \rho) - \rho}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\text{mult}(\alpha)}}$ — the Weyl-Kac formula;

$R := \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\text{mult}(\alpha)}$ (resp. $R_\mathfrak{a} := \prod_{\alpha \in \Delta_\mathfrak{a}^+} (1 - e^{-\alpha})^{\text{mult}_\mathfrak{a}(\alpha)}$)— the Weyl denominator.

2 Orthogonal subalgebra and singular elements

In this section we shall show how the recurrent approach to branching problem leads naturally to a presentation of a formal character of $\mathfrak{g}$-module in terms characters corresponding to a set of parabolic (generalized) Verma modules. Consider a reductive Lie algebra $\mathfrak{g}$ and its reductive subalgebra

$\mathfrak{a} \subset \mathfrak{g}$. Let L^μ be the highest weight integrable module of $\mathfrak{g}$, $\mu \in P^+$. Let L^μ be completely reducible with respect to $\mathfrak{a}$,

$$L^\mu_{\mathfrak{g}\downarrow\mathfrak{a}} = \bigoplus_{\nu \in P^+_\mathfrak{a}} b^{(\mu)}_\nu L^\nu_\mathfrak{a}.$$

Using the projection operator $\pi_\mathfrak{a}$ (to the weight space $\mathfrak{h}_\mathfrak{a}{}^*$) one can write this decomposition in terms of formal characters:

$$\pi_\mathfrak{a} ch\left(L^\mu\right) = \sum_{\nu \in P^+_\mathfrak{a}} b^{(\mu)}_\nu ch\left(L^\nu_\mathfrak{a}\right). \tag{1}$$

The module L^μ has the BGG resolution (see [4, 10, 11] and [12]). All the members of the filtration sequence are the direct sums of Verma modules and all their highest weights ν are strongly linked to μ:

$$\{\nu\} = \{w\left(\mu + \rho\right) - \rho | w \in W\}.$$

2.1 Orthogonal subalgebra

Let $\mathfrak{h}_\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}$. For $\mathfrak{a} \hookrightarrow \mathfrak{g}$ introduce the "orthogonal partner" $\mathfrak{a}_\perp \hookrightarrow \mathfrak{g}$.

Consider the root subspace $\mathfrak{h}^*_{\perp\mathfrak{a}}$ orthogonal to $\mathfrak{a}$,

$$\mathfrak{h}^*_{\perp\mathfrak{a}} := \{\eta \in \mathfrak{h}^* | \forall h \in \mathfrak{h}_\mathfrak{a}; \eta\left(h\right) = 0\},$$

and the roots (correspondingly – positive roots) of $\mathfrak{g}$ orthogonal to $\mathfrak{a}$,

$$\begin{aligned} \Delta_{\mathfrak{a}_\perp} &: = \{\beta \in \Delta_\mathfrak{g} | \forall h \in \mathfrak{h}_\mathfrak{a}; \beta\left(h\right) = 0\}, \\ \Delta^+_{\mathfrak{a}_\perp} &: = \{\beta^+ \in \Delta^+_\mathfrak{g} | \forall h \in \mathfrak{h}_\mathfrak{a}; \beta^+\left(h\right) = 0\}. \end{aligned} \tag{2}$$

Let $W_{\mathfrak{a}_\perp}$ be the subgroup of W generated by the reflections w_β with the roots $\beta \in \Delta^+_{\mathfrak{a}_\perp}$. The subsystem $\Delta_{\mathfrak{a}_\perp}$ determines the subalgebra $\mathfrak{a}_\perp$ with the Cartan subalgebra $\mathfrak{h}_{\mathfrak{a}_\perp}$. Let

$$\mathfrak{h}^*_\perp := \{\eta \in \mathfrak{h}^*_{\perp\mathfrak{a}} | \forall h \in \mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{a}_\perp}; \eta\left(h\right) = 0\}$$

so that $\mathfrak{g}$ has the subalgebras

$$\widetilde{\mathfrak{a}_\perp} := \mathfrak{a}_\perp \oplus \mathfrak{h}_\perp \qquad \widetilde{\mathfrak{a}} := \mathfrak{a} \oplus \mathfrak{h}_\perp. \tag{3}$$

Notice that $\mathfrak{a} \oplus \mathfrak{a}_\perp$ in general is not a subalgebra in $\mathfrak{g}$.

For the Cartan subalgebras we have the decomposition

$$\mathfrak{h} = \mathfrak{h}_\mathfrak{a} \oplus \mathfrak{h}_{\mathfrak{a}_\perp} \oplus \mathfrak{h}_\perp = \mathfrak{h}_{\widetilde{\mathfrak{a}}} \oplus \mathfrak{h}_{\mathfrak{a}_\perp} = \mathfrak{h}_{\widetilde{\mathfrak{a}_\perp}} \oplus \mathfrak{h}_\mathfrak{a}. \tag{4}$$

For $\mathfrak{a}$ and $\mathfrak{a}_\perp$ consider the corresponding Weyl vectors, $\rho_\mathfrak{a}$ and $\rho_{\mathfrak{a}_\perp}$. Form the so called "defects" $\mathcal{D}_\mathfrak{a}$ and $\mathcal{D}_{\mathfrak{a}_\perp}$ of the injection:

$$\mathcal{D}_\mathfrak{a} := \rho_\mathfrak{a} - \pi_\mathfrak{a}\rho, \qquad \mathcal{D}_{\mathfrak{a}_\perp} := \rho_{\mathfrak{a}_\perp} - \pi_{\mathfrak{a}_\perp}\rho. \tag{5}$$

For $\mu \in P^+$ consider the linked weights $\{(w(\mu + \rho) - \rho) \,|\, w \in W\}$. Consider the projections to $h^*_{\mathfrak{a}_\perp}$ additionally shifted by the defect $-\mathcal{D}_{\mathfrak{a}_\perp}$:

$$\mu_{\mathfrak{a}_\perp}(w) := \pi_{\mathfrak{a}_\perp}[w(\mu + \rho) - \rho] - \mathcal{D}_{\mathfrak{a}_\perp}, \quad w \in W.$$

Among the weights $\{\mu_{\mathfrak{a}_\perp}(w) \,|\, w \in W\}$ one can always choose those located in the fundamental chamber $\overline{C_{\mathfrak{a}_\perp}}$. Let U be the set of representatives u for the classes $W/W_{\mathfrak{a}_\perp}$ such that

$$U := \left\{ u \in W | \quad \mu_{\mathfrak{a}_\perp}(u) \in \overline{C_{\mathfrak{a}_\perp}} \right\} \quad . \tag{6}$$

Thus we can form the subsets:

$$\mu_{\widetilde{\mathfrak{a}}}(u) := \pi_{\widetilde{\mathfrak{a}}}[u(\mu + \rho) - \rho] + \mathcal{D}_{\mathfrak{a}_\perp}, \quad u \in U, \tag{7}$$

and

$$\mu_{\mathfrak{a}_\perp}(u) := \pi_{\mathfrak{a}_\perp}[u(\mu + \rho) - \rho] - \mathcal{D}_{\mathfrak{a}_\perp}, \quad u \in U. \tag{8}$$

Notice that the subalgebra $\mathfrak{a}_\perp$ is regular by definition since it is built on a subset of roots of the algebra $\mathfrak{g}$.

For the modules we are interested in the Weyl-Kac formula for $\mathrm{ch}\,(L^\mu)$ can be written in terms of singular elements [9],

$$\Psi^{(\mu)} := \sum_{w \in W} \epsilon(w) e^{w(\mu + \rho) - \rho},$$

namely,

$$\mathrm{ch}\,(L^\mu) = \frac{\Psi^{(\mu)}}{\Psi^{(0)}} = \frac{\Psi^{(\mu)}}{R}. \tag{9}$$

The same is true for the submodules $\mathrm{ch}\,(L^\nu_\mathfrak{a})$ in (1)

$$\mathrm{ch}\,(L^\nu_\mathfrak{a}) = \frac{\Psi^{(\nu)}_\mathfrak{a}}{\Psi^{(0)}_\mathfrak{a}} = \frac{\Psi^{(\nu)}_\mathfrak{a}}{R_\mathfrak{a}},$$

with

$$\Psi_{\mathfrak{a}}^{(\nu)} := \sum_{w \in W_{\mathfrak{a}}} \epsilon(w) e^{w(\nu + \rho_{\mathfrak{a}}) - \rho_{\mathfrak{a}}}.$$

Applying formula (9) to the branching rule (1) we get the relation connecting the singular elements $\Psi^{(\mu)}$ and $\Psi_{\mathfrak{a}}^{(\nu)}$:

$$\pi_{\mathfrak{a}} \left(\frac{\sum_{w \in W} \epsilon(w) e^{w(\mu + \rho) - \rho}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)}} \right) = \sum_{\nu \in P_{\mathfrak{a}}^+} b_{\nu}^{(\mu)} \frac{\sum_{w \in W_{\mathfrak{a}}} \epsilon(w) e^{w(\nu + \rho_{\mathfrak{a}}) - \rho_{\mathfrak{a}}}}{\prod_{\beta \in \Delta_{\mathfrak{a}}^+} (1 - e^{-\beta})^{\mathrm{mult}_{\mathfrak{a}}(\beta)}},$$

$$\pi_{\mathfrak{a}} \left(\frac{\Psi^{(\mu)}}{R} \right) = \sum_{\nu \in P_{\mathfrak{a}}^+} b_{\nu}^{(\mu)} \frac{\Psi_{\mathfrak{a}}^{(\nu)}}{R_{\mathfrak{a}}}. \tag{10}$$

2.2   Decomposing the singular element.

Now we shall perform a decomposition of the singular element $\Psi^{(\mu)}$ in terms of singular elements of the orthogonal partner modules:

Lemma 1. *Let $\mathfrak{a}_\perp$ be the orthogonal partner of a reductive subalgebra $\mathfrak{a} \hookrightarrow \mathfrak{g}$ with $\mathfrak{h} = \mathfrak{h}_{\mathfrak{a}} \oplus \mathfrak{h}_{\mathfrak{a}_\perp} \oplus \mathfrak{h}_\perp$, $\widetilde{\mathfrak{a}_\perp} = \mathfrak{a}_\perp \oplus \mathfrak{h}_\perp$ and $\widetilde{\mathfrak{a}} = \mathfrak{a} \oplus \mathfrak{h}_\perp$.*
L^μ be the highest weight integrable module with $\mu \in P^+$ and
$\Psi^{(\mu)}$ – the singular element of L^μ.
Then the element $\Psi^{(\mu)}$ can be decomposed into the sum over $u \in U$ (see (6)) of singular elements $\Psi_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}$ with the coefficients $\epsilon(u) e^{\mu_{\widetilde{\mathfrak{a}}}(u)}$:

$$\Psi^{(\mu)} = \sum_{u \in U} \epsilon(u) e^{\mu_{\widetilde{\mathfrak{a}}}(u)} \Psi_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}. \tag{11}$$

Proof. Let

$$u(\mu + \rho) = \pi_{(\widetilde{\mathfrak{a}})} u(\mu + \rho) + \pi_{(\mathfrak{a}_\perp)} u(\mu + \rho)$$

with $u \in U$. For any $v \in W_{\mathfrak{a}_\perp}$ consider the singular weight $vu(\mu + \rho) - \rho$ and perform the decomposition:

$$
\begin{aligned}
vu(\mu + \rho) - \rho = {} & \pi_{(\mathfrak{a})} \left(u(\mu + \rho) \right) - \rho + \rho_{\mathfrak{a}_\perp} \\
& + v \left(\pi_{(\widetilde{\mathfrak{a}_\perp})} u(\mu + \rho) - \rho_{\mathfrak{a}_\perp} + \rho_{\mathfrak{a}_\perp} \right) - \rho_{\mathfrak{a}_\perp}.
\end{aligned} \tag{12}
$$

Use the defect $\mathcal{D}_{\mathfrak{a}_\perp}$ (5) to simplify the first line in (12):

$$
\begin{aligned}
\pi_{(\widetilde{\mathfrak{a}})} \left(u(\mu + \rho) \right) - \rho + \rho_{\mathfrak{a}_\perp} = {} & \\
\pi_{(\widetilde{\mathfrak{a}})} \left(u(\mu + \rho) \right) - \pi_{\widetilde{\mathfrak{a}}} \rho - \pi_{\mathfrak{a}_\perp} \rho + \rho_{\mathfrak{a}_\perp} = {} & \\
= \pi_{(\widetilde{\mathfrak{a}})} \left(u(\mu + \rho) - \rho \right) + \mathcal{D}_{\mathfrak{a}_\perp}, &
\end{aligned}
$$

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and the second one:

$$v\left(\pi_{(\mathfrak{a}_\perp)}u(\mu+\rho)-\rho_{\mathfrak{a}_\perp}+\rho_{\mathfrak{a}_\perp}\right)-\rho_{\mathfrak{a}_\perp}=$$
$$v\left(\pi_{(\mathfrak{a}_\perp)}u(\mu+\rho)-\mathcal{D}_{\mathfrak{a}_\perp}-\pi_{(\mathfrak{a}_\perp)}\rho+\rho_{\mathfrak{a}_\perp}\right)-\rho_{\mathfrak{a}_\perp}=$$
$$=v\left(\pi_{(\mathfrak{a}_\perp)}\left[u(\mu+\rho)-\rho\right]-\mathcal{D}_{\mathfrak{a}_\perp}+\rho_{\mathfrak{a}_\perp}\right)-\rho_{\mathfrak{a}_\perp}.$$

This provides the desired decomposition of the singular element Ψ^μ in terms of singular elements $\Psi^\eta_{\mathfrak{a}_\perp}$ of the $\mathfrak{a}_\perp$-modules $L^\eta_{\mathfrak{a}_\perp}$:

$$\Psi^\mu=\sum_{u\in U}\sum_{v\in W_{\mathfrak{a}_\perp}}\epsilon(v)\epsilon(u)e^{vu(\mu+\rho)-\rho}=$$
$$=\sum_{u\in U}\epsilon(u)e^{\pi_{\widetilde{\mathfrak{a}}}[u(\mu+\rho)-\rho]+\mathcal{D}_{\mathfrak{a}_\perp}}\sum_{v\in W_{\mathfrak{a}_\perp}}\epsilon(v)e^{v\left(\pi_{(\mathfrak{a}_\perp)}[u(\mu+\rho)-\rho]-\mathcal{D}_{\mathfrak{a}_\perp}+\rho_{\mathfrak{a}_\perp}\right)-\rho_{\mathfrak{a}_\perp}}=$$
$$=\sum_{u\in U}\epsilon(u)\Psi^{\pi_{(\mathfrak{a}_\perp)}[u(\mu+\rho)-\rho]-\mathcal{D}_{\mathfrak{a}_\perp}}_{\mathfrak{a}_\perp}e^{\pi_{(\widetilde{\mathfrak{a}})}[u(\mu+\rho)-\rho]+\mathcal{D}_{\mathfrak{a}_\perp}}.$$

$$(13)$$

$\square$

Remark 1. This relation can be considered as a generalized form of the Weyl formula for the singular element $\Psi^\mu_\mathfrak{g}$: the vectors $\mu_{\widetilde{\mathfrak{a}}}(u)$ play the role of singular weights while the alternating factors $\epsilon(u)$ are extended to $\epsilon(u)\Psi^{\mu_{\mathfrak{a}_\perp}(u)}_{\mathfrak{a}_\perp}$. In fact when $\mathfrak{a}=\mathfrak{g}$ both $\mathfrak{a}_\perp$ and $\mathfrak{h}_\perp$ are zeros, $U=W$, and the original Weyl formula is reobtained so far as the singular elements $\epsilon(u)\Psi^{\mu_{\mathfrak{a}_\perp}(u)}_{\mathfrak{a}_\perp}=\epsilon(u)$ become trivial. In the opposite limit when $\mathfrak{a}=\mathfrak{o}$, $\Delta_{\mathfrak{a}_\perp}=\Delta_\mathfrak{g}$, $\mathfrak{h}^*_\perp=0$, $\mathfrak{a}_\perp=\mathfrak{g}$, $\mathcal{D}_{\mathfrak{a}_\perp}=0$ and $U=W/W_{\mathfrak{a}_\perp}=e$ the singular element Ψ^μ is again reobtained, now via the trivilization of the set of vectors $\mu_\mathfrak{a}(e)=0$.

Remark 2. In [7] the decomposition analogous to (13) was used to construct the recurrent relations for branching coefficients $k^{(\mu)}_\xi$ corresponding to the injection $\mathfrak{a}\hookrightarrow\mathfrak{g}$:

$$k^{(\mu)}_\xi=-\frac{1}{s(\gamma_0)}\left(\sum_{u\in U}\epsilon(u)\,\dim\left(L^{\mu_{\mathfrak{a}_\perp}(u)}_{\mathfrak{a}_\perp}\right)\delta_{\xi-\gamma_0,\pi_{\widetilde{\mathfrak{a}}}(u(\mu+\rho)-\rho)}+\right.$$
$$\left.+\sum_{\gamma\in\Gamma_{\widetilde{\mathfrak{a}}\to\mathfrak{g}}}s\left(\gamma+\gamma_0\right)k^{(\mu)}_{\xi+\gamma}\right).$$

$$(14)$$

The recursion is goverened by the set $\Gamma_{\widetilde{\mathfrak{a}}\to\mathfrak{g}}$ called the injection fan. The latter is defined by the carrier set $\{\xi\}_{\mathfrak{a}\to\mathfrak{g}}$ for the coefficient function $s(\xi)$

$$\{\xi\}_{\widetilde{\mathfrak{a}}\to\mathfrak{g}}:=\{\xi\in P_{\widetilde{\mathfrak{a}}}|s(\xi)\neq0\}$$

appearing in the expansion

$$\prod_{\alpha\in\Delta^+\backslash\Delta_\perp^+}\left(1-e^{-\pi_{\widetilde{\mathfrak{a}}}\alpha}\right)^{\mathrm{mult}(\alpha)-\mathrm{mult}_{\mathfrak{a}}(\pi_{\widetilde{\mathfrak{a}}}\alpha)}=-\sum_{\gamma\in P_{\widetilde{\mathfrak{a}}}}s(\gamma)e^{-\gamma};\qquad(15)$$

The weights in $\{\xi\}_{\widetilde{\mathfrak{a}}\to\mathfrak{g}}$ are to be shifted by γ_0 – the lowest vector in $\{\xi\}$ – and the zero element is to be eliminated:

$$\Gamma_{\widetilde{\mathfrak{a}}\to\mathfrak{g}}=\{\xi-\gamma_0|\xi\in\{\xi\}\}\backslash\{0\}.\qquad(16)$$

The recursion relation (14) was originally used to describe branchings for integrable modules. Notice that there exists an important class of modules that also can be reduced with the help of the injection fan – these are Verma modules.

2.3 Weyl-Verma formulas.

Statement 1. *For an orthogonal subalgebra $\mathfrak{a}_\perp$ in $\mathfrak{g}$ (an orthogonal partner of a reductive $\mathfrak{a}\hookrightarrow\mathfrak{g}$) the character of an integrable highest weight module L^μ can be presented as a combination (with integral coefficients) of parabolic Verma modules distributed by the set of weights $e^{\mu_{\widetilde{\mathfrak{a}}}(u)}$:*

$$\mathrm{ch}\,(L^\mu)=\sum_{u\in U}\epsilon(u)e^{\mu_{\widetilde{\mathfrak{a}}}(u)}\mathrm{ch}M_I^{\mu_{\mathfrak{a}_\perp}(u)},\qquad(17)$$

where $U:=\left\{u\in W|\quad \mu_{\mathfrak{a}_\perp}(u)\in\overline{C_{\mathfrak{a}_\perp}}\right\}$ and I is such a subset of S that Δ_I^+ is equivalent to $\Delta_{\mathfrak{a}_\perp}^+$.

Proof. By the definition (2) the subalgebra $\mathfrak{a}_\perp$ is regular and reductive. Consider its Weyl denominator $R_{\mathfrak{a}_\perp}:=\prod_{\alpha\in\Delta_{\mathfrak{a}_\perp}^+}\left(1-e^{-\alpha}\right)^{\mathrm{mult}_{\mathfrak{a}}(\alpha)}$ and the element $R_J:=\prod_{\alpha\in\Delta^+\backslash\Delta_{\mathfrak{a}_\perp}^+}\left(1-e^{-\alpha}\right)^{\mathrm{mult}(\alpha)}$ as the factors in R:

$$R=R_JR_{\mathfrak{a}_\perp}.$$

According to this factorization and the decomposition (11) the character $\mathrm{ch}\,(L^\mu)$ can be written as

$$\begin{aligned}\mathrm{ch}\,(L^\mu)&=&\left(R_J\right)^{-1}\left(R_{\mathfrak{a}_\perp}\right)^{-1}\Psi^\mu=\left(R_J\right)^{-1}\sum_{u\in U}e^{\mu_{\widetilde{\mathfrak{a}}}(u)}\epsilon(u)\left(R_{\mathfrak{a}_\perp}\right)^{-1}\Psi_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}\\ &=&\left(R_J\right)^{-1}\sum_{u\in U}e^{\mu_{\widetilde{\mathfrak{a}}}(u)}\epsilon(u)\mathrm{ch}\left(L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}\right),\end{aligned}$$

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where $\left\{L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}|u \in U\right\}$ is the set of finite-dimensional $\mathfrak{a}_\perp$-modules with the highest weights $\mu_{\mathfrak{a}_\perp}(u)$. We are interested in nontrivial subalgebras $\mathfrak{a}$ and correspondingly in nontrivial $\mathfrak{a}_\perp$ (the case of a trivial orthogonal subalgebra was considered above (see Remark 1)). This means that $r_{\mathfrak{a}} \geq 1$ and $r_{\mathfrak{a}_\perp} < r$. Due to the fact that any maximal regular subalgebra has the Dynkin scheme obtained by one or two node subtractions from the extended Dynkin scheme and the extended scheme has at most one dependent root (the highest root) the set of roots $\Delta_{\mathfrak{a}_\perp}^+$ is always equivalent to the one Δ_I^+ generated by some subset $I \subset S$ of simple roots.

It follows that we can (by redefining the set Δ^+) identify $\Delta_{\mathfrak{a}_\perp}^+$ with the subset Δ_I^+ where $I \subset S$. This allows us to introduce the elements necessary to compose the generalized Verma modules [3, 12]. We have two sets of root vectors $\left\{x_\xi \in \mathfrak{g}_\xi | \xi \in \Delta_I^+\right\}$ and $\left\{x_\eta \in \mathfrak{g}_\eta | \eta \in \Delta^+ \setminus \Delta_I^+\right\}$ and the corresponding nilpotent subalgebras in $\mathfrak{n}^+$:

$$\mathfrak{n}_I^+ := \sum_{\xi \in \Delta_I^+} \mathfrak{g}_\xi, \quad \mathfrak{u}_I^+ := \sum_{\eta \in \Delta^+ \setminus \Delta_I^+} \mathfrak{g}_\eta.$$

The first subalgebra together with its negative counterpart $\mathfrak{n}_I^-$ generates a simple subalgebra

$$\mathfrak{s}_I = \mathfrak{n}_I^- + \mathfrak{h}_I + \mathfrak{n}_I^+.$$

We enlarge it with the remaining Cartan generators:

$$\mathfrak{l}_I = \mathfrak{n}_I^- + \mathfrak{h} + \mathfrak{n}_I^+.$$

The semidirect product of $\mathfrak{l}_I$ and $\mathfrak{u}_I^+$ gives a parabolic subalgebra $\mathfrak{p}_I \hookrightarrow \mathfrak{g}$:

$$\mathfrak{p}_I = \mathfrak{l}_I \rhd \mathfrak{u}_I^+. \tag{18}$$

Its universal enveloping $U(\mathfrak{p}_I)$ is a subalgebra in $U(\mathfrak{g})$. The $\mathfrak{l}_I$-modules $L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}$ can be easily lifted to $\mathfrak{p}_I$-modules using the trivial action of the nilradical $\mathfrak{u}_I^+$. The latter induce $U(\mathfrak{g})$-modules in a standard way:

$$M_I^{\mu_{\mathfrak{a}_\perp}(u)} = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_I)} L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}.$$

These are the *generalized Verma modules* [3] generated by the highest weights $\mu_{\mathfrak{a}_\perp}(u)$. As a $U(\mathfrak{u}_I^-)$-module each $M_I^{\mu_{\mathfrak{a}_\perp}(u)}$ is isomorphic to $U(\mathfrak{u}_I^-) \otimes$

$L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}$ and thus its character can be written in terms of Kostant-Heckman function [13] corresponding to the injection of the orthogonal partner $\mathfrak{a}_\perp \hookrightarrow \mathfrak{g}$:

$$\mathrm{ch}M_I^{\mu_{\mathfrak{a}_\perp}(u)} = \mathcal{KH}_{\mathfrak{a}_\perp \hookrightarrow \mathfrak{g}} \mathrm{ch}L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}.$$

The function $\mathcal{KH}_{\mathfrak{a}_\perp \hookrightarrow \mathfrak{g}}$ is generated by the denominator R_I thus the last expression can be written in the form

$$\mathrm{ch}M_I^{\mu_{\mathfrak{a}_\perp}(u)} = \frac{1}{R_I}\mathrm{ch}L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}.$$

This means that we have obtained the generalized Weyl-Verma character formula – the decomposition of ch (L^μ) in terms of generalized Verma module characters:

$$\mathrm{ch}\,(L^\mu) = \sum_{u \in U} e^{\mu_{\widetilde{\mathfrak{a}}}(u)}\epsilon(u)\mathrm{ch}M_I^{\mu_{\mathfrak{a}_\perp}(u)}. \tag{19}$$

$\square$

Remark 3. Here the generalized Weyl-Verma character formula (called the alternating sum formula in [12]) appears in a special form: the weights $\mu_{\widetilde{\mathfrak{a}}}$ and the generalized Verma module highest weights $\mu_{\mathfrak{a}_\perp}$ are separated. The reason is that the highest weight of M_I-module is not equal to the projection of its maximal weight to $h_{\mathfrak{a}_\perp}^*$ (but must be additionally shifted by the defect).

Example 1. Consider the generalized Verma modules for the embedding $A_1 \hookrightarrow B_2$ with the subalgebra $\mathfrak{a}_\perp$ attributed to the root α_1 of B_2. The generalized Verma module $M_I^{\omega_1}$ with the highest weight $\omega_1 = e_1$ is shown in Figure 1.

Remark 4. As it was proved in [12] (see Proposition 9.6) characters of the generalized Verma modules $M_I^{\mu_{\mathfrak{a}_\perp}(u)}$ can be described as linear combinations of ordinary Verma modules of $\mathfrak{g}$:

$$\mathrm{ch}M_I^{\mu_{\mathfrak{a}_\perp}(u)} = \sum_{w \in W_{\mathfrak{a}_\perp}} \epsilon\,(w)\,\mathrm{ch}M^{w\left(\mu_{\mathfrak{a}_\perp}(u)+\rho_{\mathfrak{a}_\perp}\right)-\rho_{\mathfrak{a}_\perp}}$$

Substituting this expression in (19) and using the definitions (7,8) and (5) we reconstruct the standard Weyl-Verma decomposition of the character:

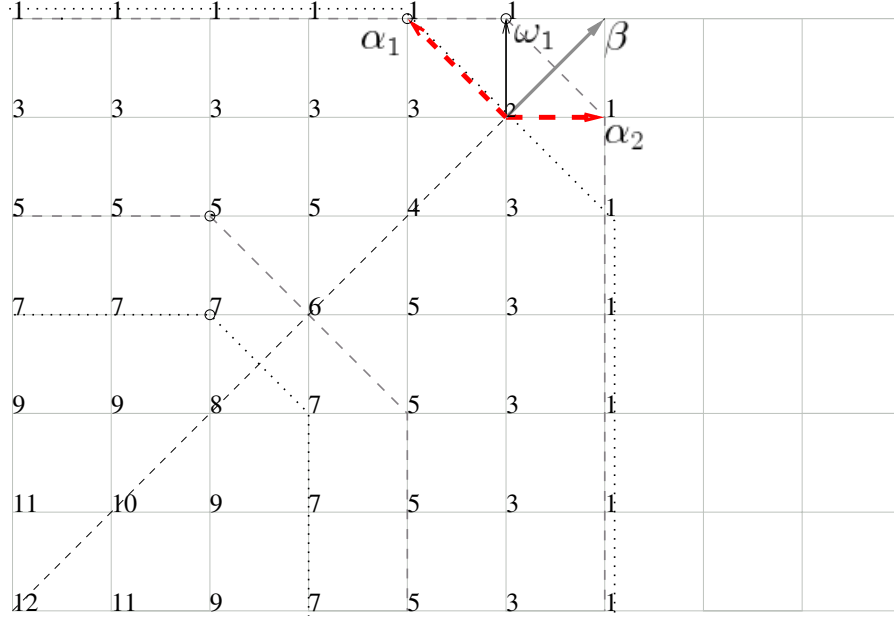$$\mathrm{ch}\,(L^\mu) = \sum_{w \in W} \epsilon(u)\mathrm{ch}M^{w(\mu+\rho)-\rho}.$$

Figure 1: Generalized Verma modules for the regular embedding of A_1 into B_2. Simple roots α_1, α_2 of B_2 are presented as the dashed vectors. The simple root $\beta = \alpha_1 + 2\alpha_2$ of A_1 is indicated as the grey vector. The decomposition of L^{ω_1} is indicated by the set of contours of the involved generalized Verma modules. Dashed contours correspond to positive $\epsilon(u)$ and dotted to negative.

3   BGG resolution and branching

In [3] it was demonstrated that for the highest weight module L^μ with $\mu \in P^+$ the sequence

$$0 \to M_r^I \overset{\delta_r}{\to} M_{r-1}^I \overset{\delta_{r-1}}{\to} \ldots \overset{\delta_1}{\to} M_0^I \overset{\varepsilon}{\to} L^\mu \to 0, \qquad (20)$$

with

$$M_k^I = \bigoplus_{u \in U,\ \text{length}(u)=k} M_I^{u(\mu+\rho)-\rho}, \quad M_0^I = M_I^\mu \qquad (21)$$

(the generalized BGG resolution) is exact and formula (17) is a cosequence of this resolution.

Statement 2. *Let L^μ be the highest weight $\mathfrak{g}$-module with $\mu \in P^+$, let its regular subalgebra $\mathfrak{a}_\perp \hookrightarrow \mathfrak{g}$ be orthogonal to a reductive subalgebra $\mathfrak{a} \hookrightarrow \mathfrak{g}$.*
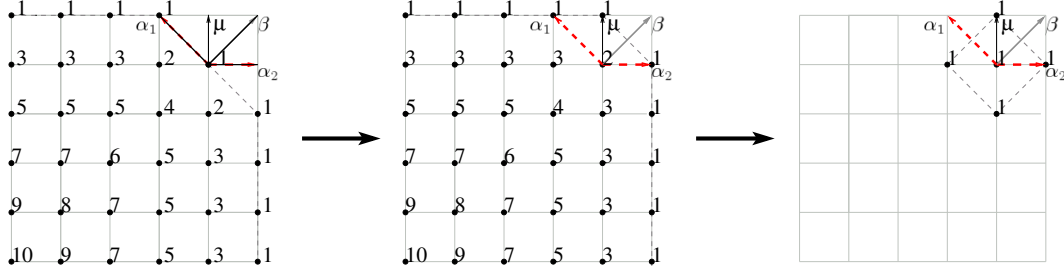
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Figure 2: Injection $A_1 \hookrightarrow B_2$ (see Figure 1). The orthogonal partner is A_1 corresponding to the root α_1. The resolution of the simple module L^{ω_1}. Presented is the central part of the exact sequence $0 \to Im(\delta_2) \to \left(e^{\mu_{\widetilde{\mathfrak{a}}}(e)} \mathrm{ch} M_I^{\pi_{\mathfrak{a}_\perp}[\omega_1] - \mathcal{D}_{\mathfrak{a}_\perp}} = M_I^{\omega_1} \right) \to L^{\omega_1} \to 0$. Here $\mu_{\widetilde{\mathfrak{a}}}(e) = \pi_{\widetilde{\mathfrak{a}}}[\mu] + \mathcal{D}_{\mathfrak{a}_\perp}$.

Then the decomposition (11) defines both the generalized resolution of L^μ with respect to $\mathfrak{a}_\perp$ and the branching rules for L^μ with respect to $\mathfrak{a}$.

Proof. Put

$$\mathrm{ch} M_I^{u(\mu+\rho)-\rho} = e^{\mu_{\widetilde{\mathfrak{a}}}(u)} \mathrm{ch} M_I^{\mu_{\mathfrak{a}_\perp}(u)}, \mathrm{ch} M_I^\mu = e^{\mu_{\widetilde{\mathfrak{a}}}(e)} \mathrm{ch} M_I^{\pi_{\mathfrak{a}_\perp}[\mu] - \mathcal{D}_{\mathfrak{a}_\perp}}$$

with $\mu_{\widetilde{\mathfrak{a}}}(u), \mu_{\mathfrak{a}_\perp}(u)$ and $\mathcal{D}_{\mathfrak{a}_\perp}$ as in Lemma 1 and $u \in U$ defined by (6). This gives the elements of the filtration sequence (20).

Consider the set $\{\mu_{\mathfrak{a}_\perp}(u) \,|\, u \in U\}$ as the highest weights for the simple modules $L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}$ and evaluate their dimensions. Together with $\{\mu_{\widetilde{\mathfrak{a}}}(u) \,|\, u \in U\}$ this gives the set of singular weights

$$\left\{ \epsilon(u) \, e^{\mu_{\widetilde{\mathfrak{a}}}(u)} \dim \left(L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)} \right) \right\}.$$

The branching $L_{\mathfrak{g}\downarrow\mathfrak{a}}^\mu = \bigoplus_{\nu \in P_{\mathfrak{a}}^+} b_\nu^{(\mu)} L_{\mathfrak{a}}^\nu$ is then fixed by the injection fan $\Gamma_{\mathfrak{a}\to\mathfrak{g}}$ and the relation (14). The latter gives us the coefficients $k_\xi^{(\mu)}$ and thus defines $b_\nu^{(\mu)}$ due to the property $b_\nu^{(\mu)} = k_\nu^{(\mu)}$ for $\nu \in \overline{C_{\mathfrak{a}}}$. $\qquad\square$

Corollary 0.1. *Let L^μ be the highest weight $\mathfrak{g}$-module with $\mu \in P^+$ and $\mathfrak{a} \hookrightarrow \mathfrak{g}$ – a reductive subalgebra in $\mathfrak{g}$. Let $\mathfrak{a}_\perp$, the orthogonal partner for $\mathfrak{a}$, be equivalent to A_1, $\mathfrak{a}_\perp \approx A_1$, and $\widetilde{\mathfrak{a}} = \mathfrak{a} \oplus \mathfrak{h}_\perp$ with $\mathfrak{h} = \mathfrak{h}_{\mathfrak{a}} \oplus \mathfrak{h}_{\mathfrak{a}_\perp} \oplus \mathfrak{h}_\perp$. Let $L_{\mathfrak{g}\downarrow\widetilde{\mathfrak{a}}}^\mu = \bigoplus_{\nu \in P_{\widetilde{\mathfrak{a}}}^+} b_\nu^{(\mu)} L_{\widetilde{\mathfrak{a}}}^\nu$ be the branching of L^μ with respect to $\widetilde{\mathfrak{a}}$. Then the*

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branching coefficients $b_\nu^{(\mu)}$ *define the generalized resolution (20) of* L^μ *with respect to* $\mathfrak{a}_\perp$.

Proof. Let α be the simple root of A_1. Use the Weyl transformations to identify it with some simple root of $\mathfrak{g}$, say α_1. Construct the singular element for the module $L^\mu_{\mathfrak{g}\downarrow\tilde{\mathfrak{a}}}$, i.e. the $\Psi_{\tilde{\mathfrak{a}}}^{\left(L^\mu_{\mathfrak{g}\downarrow\tilde{\mathfrak{a}}}\right)} = \sum_{\nu\in P_{\tilde{\mathfrak{a}}}^+, b_\nu^{(\mu)}>0} b_\nu^{(\mu)} \Psi_{\tilde{\mathfrak{a}}}^{(\nu)}$, and decompose it $\Psi_{\tilde{\mathfrak{a}}}^{\left(L^\mu_{\mathfrak{g}\downarrow\tilde{\mathfrak{a}}}\right)} = k_\xi^{(\mu)} e^\xi$. In our case the representatives u in the recurrent relation (14) are uniquely determined by the weight ξ:

$$\epsilon\left(u\left(\xi\right)\right) \dim\left(L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u(\xi))}\right) = -s\left(\gamma_0\right) k_\xi^{(\mu)} - \sum_{\gamma\in\Gamma_{\tilde{\mathfrak{a}}\to\mathfrak{g}}} s\left(\gamma+\gamma_0\right) k_{\xi+\gamma}^{(\mu)}.$$

We have

$$\dim\left(L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u(\xi))}\right) = \left| s\left(\gamma_0\right) k_\xi^{(\mu)} + \sum_{\gamma\in\Gamma_{\tilde{\mathfrak{a}}\to\mathfrak{g}}} s\left(\gamma+\gamma_0\right) k_{\xi+\gamma}^{(\mu)} \right|$$

and

$$\mu_{\mathfrak{a}_\perp}\left(u\left(\xi\right)\right) = \frac{1}{2}\left(\dim\left(L_{A_1}^{\mu(\xi)}\right) - 1\right)\alpha_1$$

The set of generalized Verma modules $e^{\xi+\mathcal{D}_{\mathfrak{a}_\perp}} \mathrm{ch} M_I^{\mu_{\mathfrak{a}_\perp}(u(\xi))}$ is thus fixed:

$$\left\{ e^{\mu_{\tilde{\mathfrak{a}}}(u)} \mathrm{ch} M_I^{\mu_{\mathfrak{a}_\perp}(u)} | u \in U \right\}.$$

Classifying these modules according to the length of u we get the components (21) of the resolution (20). $\qquad\square$

4 Conclusions

In [7] it was demonstrated that the injection fan recursive mechanism works also for special injections. It must be mentioned that in this case the Weyl-Verma decompositions can also be obtained. The resolutions corresponding to special subalgebras describe the relations between the projections of characters of the initial module and the generalized Verma modules with highest weights in the subspace of h^*.

Consider the situation where the simple roots are prescribed by some external factors (originating in physical applications conditions, for example).

In this case the orthogonal partner cannot be generated by simple root vectors only. The elements $\mathfrak{u}_I^+ := \sum_{\eta \in \Delta^+ \setminus \Delta_I^+} \mathfrak{g}_\eta$ do not form a subalgebra in $\mathfrak{g}$ because some nonsimple roots are lost in $\Delta^+ \setminus \Delta_I^+$. It is important to indicate that in this case the Weyl-Verma formula still exists. In it the generalized Verma modules correspond to the contractions [14] of the algebra $\mathfrak{n}^+$ and the Weyl-Verma relations describe the decomposition of the representation space of L^μ into the set of generalized Verma modules of contracted algebra $U(\mathfrak{n}_c^+)$. The weight vectors are formed by the PBW-basis of $U(\mathfrak{n}_c^+)$ and of $U(\mathfrak{a}_\perp)$. To consider such space as a $\mathfrak{g}$-module we must perform the deformation [15] of the algebra $\mathfrak{n}_c^+$ (and thus restore the initial composition law). The space survives and after such a deformation the initial algebra generators will act properly on it.

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