Recursive algorithms, branching coefficients and applications

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Abstract. Recurrent relations for branching coefficients in affine Lie algebras integrable highest weight modules are studied. The decomposition algorithm based on the injection fan technique is adopted to the situation where the Weyl denominator becomes singular with respect to a reductive subalgebra. We study some modifications of the injection fan technique and demonstrate that it is possible to define the "subtracted fans" that play the role similar to the original ones. Possible applications of subtracted fans in CFT models are considered.

Submitted to: *J. Phys. A: Math. Gen.*

1. Introduction

The branching problem for affine Lie algebras emerges in conformal field theory, for example, in the construction of modular-invariant partition functions [1]. Recently the problem of the conformal embeddings was considered in the paper [2].

There exist several approaches to deal with the branching coefficients. Some of them use the BGG resolution [3] (for Kac-Moody algebras the algorithm is described in [4],[5]), the Schure function series [6], the BRST cohomology [7], Kac-Peterson formulas [4, 8] or the combinatorial methods applied in [9].

Usually only the maximal reductive subalgebras are considered since the case of non-maximal subalgebra can be obtained using the chain of maximal injections. In this paper we find the recurrent properties for branching coefficients that generalise the relations obtained earlier (see the paper [10] and the references therein) to the case of non-maximal reductive subalgebra. The result is formulated in terms of the new injection fan called "the subtracted fan". Using this new tools we formulate a simple and explicit algorithm for computations of branching coefficients which is applicable to the non-maximal subalgebras of finite-dimensional and affine Lie algebras.

We demonstrate that our algorithm can be used in studies of conformal embeddings and coset constructions in rational conformal field theory.

The paper is organised as follows. In the subsection 1.1 we fix the notations. In the Section 2 we derive the subtracted recurrent formula for anomalous branching coefficients and describe the decomposition algorithm for integrable highest weight modules $L_{\mathfrak{g}}$ with respect to a reductive subalgebra $\mathfrak{a} \subset \mathfrak{g}$ (subsection 2.2). In the Section 3 we present several simple examples for finite-dimensional Lie algebras. The affine Lie algebras and their applications in CFT models are considered in Section 4. Possible further developments are discussed (Section 5).

1.1. Notation

Consider affine Lie algebras $\mathfrak{g}$ and $\mathfrak{a}$ with the underlying finite-dimensional subalgebras $\overset{\circ}{\mathfrak{g}}$ and $\overset{\circ}{\mathfrak{a}}$ and an injection $\mathfrak{a} \longrightarrow \mathfrak{g}$ such that $\mathfrak{a}$ is a reductive subalgebra $\mathfrak{a} \subset \mathfrak{g}$ with correlated root spaces: $\mathfrak{h}^*_{\mathfrak{a}} \subset \mathfrak{h}^*_{\mathfrak{g}}$ and $\mathfrak{h}^*_{\overset{\circ}{\mathfrak{a}}} \subset \mathfrak{h}^*_{\overset{\circ}{\mathfrak{g}}}$.

We use the following notations:

L^{μ} ($L^{\nu}_{\mathfrak{a}}$) — the integrable module of $\mathfrak{g}$ with the highest weight μ ; (resp. integrable $\mathfrak{a}$ -module with the highest weight ν);

r , ($r_{\mathfrak{a}}$) — the rank of the algebra $\mathfrak{g}$ (resp. $\mathfrak{a}$) ;

Δ ($\Delta_{\mathfrak{a}}$)— the root system; Δ^+ (resp. $\Delta^+_{\mathfrak{a}}$)— the positive root system (of $\mathfrak{g}$ and $\mathfrak{a}$ respectively);

$\mathrm{mult}\,(\alpha)$ ($\mathrm{mult}_{\mathfrak{a}}\,(\alpha)$) — the multiplicity of the root α in Δ (resp. in ($\Delta_{\mathfrak{a}}$));

$\overset{\circ}{\Delta}$, $\left(\overset{\circ}{\Delta}_{\mathfrak{a}}\right)$ — the finite root system of the subalgebra $\overset{\circ}{\mathfrak{g}}$ (resp. $\overset{\circ}{\mathfrak{a}}$); Θ, ($\Theta_{\mathfrak{a}}$) — the highest root of the algebra $\mathfrak{g}$ (resp. subalgebra $\mathfrak{a}$);

$\mathcal{N}^{\mu}$, ($\mathcal{N}^{\nu}_{\mathfrak{a}}$) — the weight diagram of L^{μ} (resp. $L^{\nu}_{\mathfrak{a}}$) ;

W , ($W_{\mathfrak{a}}$)— the corresponding Weyl group;

C , ($C_{\mathfrak{a}}$)— the fundamental Weyl chamber;

$\bar{C}$, $\left(\bar{C}_{\mathfrak{a}}\right)$ — the closure of the fundamental Weyl chamber;

ρ , ($\rho_{\mathfrak{a}}$) — the Weyl vector;

$\epsilon\,(w) := \det\,(w)$;

α_i , $\left(\alpha_{(\mathfrak{a})j}\right)$ — the i-th (resp. j-th) basic root for $\mathfrak{g}$ (resp. $\mathfrak{a}$); $i = 0, \ldots, r$, ($j = 0, \ldots, r_{\mathfrak{a}}$);

δ — the imaginary root of $\mathfrak{g}$ (and of $\mathfrak{a}$ if any);

$\alpha^{\vee}_i$, $\left(\alpha^{\vee}_{(\mathfrak{a})j}\right)$— the basic coroot for $\mathfrak{g}$ (resp. $\mathfrak{a}$) , $i = 0, \ldots, r$; ($j = 0, \ldots, r_{\mathfrak{a}}$);

$\overset{\circ}{\xi}$, $\overset{\circ}{\xi}_{(\mathfrak{a})}$ — the finite (classical) part of the weight $\xi \in P$, $\left(\text{resp. } \xi_{(\mathfrak{a})} \in P_{\mathfrak{a}}\right)$;

$\lambda = \left(\overset{\circ}{\lambda}; k; n\right)$ — the decomposition of an affine weight indicating the finite part $\overset{\circ}{\lambda}$, level k and grade n .

P (resp. $P_{\mathfrak{a}}$) — the weight lattice;

M (resp. $M_{\mathfrak{a}}$) :=

$$= \left\{ \begin{array}{c} \sum_{i=1}^{r} \mathbf{Z}\alpha^{\vee}_i \left(\text{resp. } \sum_{i=1}^{r} \mathbf{Z}\alpha^{\vee}_{(\mathfrak{a})i}\right) \text{for untwisted algebras or } A^{(2)}_{2r}, \\ \sum_{i=1}^{r} \mathbf{Z}\alpha_i \left(\text{resp. } \sum_{i=1}^{r} \mathbf{Z}\alpha_{(\mathfrak{a})i}\right) \text{for } A^{(u\geq 2)}_{r} \text{ and } A \neq A^{(2)}_{2r}, \end{array} \right\};$$

$\Psi^{(\mu)} := \sum\limits_{w\in W} \epsilon(w) e^{w\circ(\mu+\rho)-\rho}$ — the singular weight element for the $\mathfrak{g}$-module L^{μ};

$\Psi^{(\nu)}_{(\mathfrak{a})} := \sum_{w\in W_{\mathfrak{a}}} \epsilon(w) e^{w\circ(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}$ — the corresponding singular weight element for the $\mathfrak{a}$-module $L^{\nu}_{\mathfrak{a}}$;

$\widehat{\Psi^{(\mu)}}$ $\left(\widehat{\Psi^{(\nu)}_{(\mathfrak{a})}}\right)$ — the set of singular weights $\xi \in P$ (resp. $\in P_{\mathfrak{a}}$) for the module L^{μ} (resp. $L^{\nu}_{\mathfrak{a}}$) with the coordinates $\left(\overset{\circ}{\xi}, k, n, \epsilon\left(w\left(\xi\right)\right)\right) |_{\xi = w(\xi)\circ(\mu+\rho)-\rho}$, (resp. $\left(\overset{\circ}{\xi}, k, n, \epsilon\left(w_a\left(\xi\right)\right)\right) |_{\xi = w_a(\xi)\circ(\nu+\rho_a)-\rho_a}$),

$m^{(\mu)}_{\xi}$, $\left(m^{(\nu)}_{\xi}\right)$ — the multiplicity of the weight $\xi \in P$ (resp. $\in P_{\mathfrak{a}}$) in the module L^{μ} , (resp. $\xi \in L^{\nu}_{\mathfrak{a}}$);

$ch\left(L^{\mu}\right)$ (resp. $ch\left(L^{\nu}_{\mathfrak{a}}\right)$)— the formal character of L^{μ} (resp. $L^{\nu}_{\mathfrak{a}}$);

$ch\left(L^{\mu}\right) = \frac{\sum_{w\in W} \epsilon(w) e^{w\circ(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^{+}}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}} = \frac{\Psi^{(\mu)}}{\Psi^{(0)}}$ — the Weyl-Kac formula.

$R := \prod_{\alpha\in\Delta^{+}}\left(1-e^{-\alpha}\right)^{\mathrm{mult}(\alpha)} = \Psi^{(0)}$

$\left(\text{resp. } R_{\mathfrak{a}} := \prod_{\alpha\in\Delta^{+}_{\mathfrak{a}}}\left(1-e^{-\alpha}\right)^{\mathrm{mult}_{\mathfrak{a}}(\alpha)} = \Psi^{(0)}_{\mathfrak{a}}\right)$— the denominator.

$L^{\mu}_{\mathfrak{g}\downarrow\mathfrak{a}} = \bigoplus_{\nu\in P^{+}_{\mathfrak{a}}} b^{(\mu)}_{\nu} L^{\nu}_{\mathfrak{a}}$ — the module decomposition with respect to $\mathfrak{a} \longrightarrow \mathfrak{g}$;

$b^{(\mu)}_{\nu}$ — the branching coefficients;

$$\sum_{\nu\in\bar{C}_{\mathfrak{a}}} b^{(\mu)}_{\nu}\Psi^{(\nu)}_{(\mathfrak{a})} = \sum_{\lambda\in P_{\mathfrak{a}}} k^{(\mu)}_{\lambda} e^{\lambda} \tag{1}$$

k_{λ} — the anomalous branching coefficients, notice that

$$b^{(\mu)}_{\nu} = k^{(\mu)}_{\nu} \text{ for } \nu \in \bar{C}_{\mathfrak{a}}$$

$x_e = \frac{|\pi_{\mathfrak{a}}\Theta|^2}{|\Theta_{\mathfrak{a}}|^2}$ — the embedding index.

2. Recurrent relation for branching coefficients. Singularities and subtractions

Our aim is to demonstrate that despite the zeros arriving in the Weyl denominator R (when it is projected to the $P_{\mathfrak{a}}$) the injection fan technique [10] can be properly modified. The result is the generalized recurrent relation for anomalous branching coefficients 1:

$$\begin{aligned} k^{(\mu)}_{\xi} = -\frac{1}{s(\gamma_0)}\Bigg(&\sum_{\omega\in W_{\perp}\backslash W}\epsilon(\omega)\dim\left(L^{\pi_{\mathfrak{a}_{\perp}}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_{\perp}}}_{\mathfrak{a}_{\perp}}\right)\delta_{\xi-\gamma_0,\pi_{\mathfrak{a}}(\omega(\mu+\rho)-\rho)}+ \\ &+\sum_{\gamma\in\Gamma_{\mathfrak{a}\subset\mathfrak{g}}} s\left(\gamma+\gamma_0\right) k^{(\mu)}_{\xi+\gamma}\Bigg) \end{aligned} \tag{2}$$

Here $\mathfrak{a}_{\perp}$ is the subalgebra fixed by those roots of $\mathfrak{g}$ that are orthogonal $P_{\mathfrak{a}}$, $W_{\perp}$ is the corresponding Weyl group, $\Gamma_{\mathfrak{a}\subset\mathfrak{g}}$ is the set of weights in the expansion of the product $\prod_{\alpha\in\Delta^{+}\backslash\Delta^{+}_{\mathfrak{a}_{\perp}}}\left(1-e^{-\alpha}\right)^{\mathrm{mult}(\alpha)-\mathrm{mult}_{\mathfrak{a}}(\alpha)}$ and $s(\gamma)$ is the multiplicity of e^{γ} in this expansion. In the next subsection we study the situation in details and prove the relation 2.

2.1. Proof of the recurrent relation

Consider the branching of a module $L^{\mu}_{\mathfrak{g}}$ in terms of formal characters and projection operators $\pi_{\mathfrak{a}} : P \to P_{\mathfrak{a}}$:

$$L^{\mu}_{\mathfrak{g}\downarrow\mathfrak{a}} = \bigoplus_{\nu\in P^{+}_{\mathfrak{a}}} b^{(\mu)}_{\nu} L^{\nu}_{\mathfrak{a}} \quad \Longrightarrow \quad \pi_{\mathfrak{a}}(chL^{\mu}_{\mathfrak{g}}) = \sum_{\nu\in P^{+}_{\mathfrak{a}}} b^{(\mu)}_{\nu} chL^{\nu}_{\mathfrak{a}}$$

The Weyl-Kac character formula leads to the relation

$$\pi_{\mathfrak{a}}\left(\frac{\sum_{\omega\in W}\epsilon(\omega)e^{\omega(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^{+}}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}\right) = \sum_{\nu\in P^{+}_{\mathfrak{a}}} b^{(\mu)}_{\nu}\frac{\sum_{\omega\in W_{\mathfrak{a}}}\epsilon(\omega)e^{\omega(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}}{\prod_{\beta\in\Delta^{+}_{\mathfrak{a}}}(1-e^{-\beta})^{\mathrm{mult}_{\mathfrak{a}}(\beta)}} \tag{3}$$

Consider the positive roots $\alpha \in \Delta$ orthogonal to $P_{\mathfrak{a}}$. Denote the subset of such roots by $\Delta^{+}_{\perp} = \left\{\alpha \in \Delta^{+}_{\mathfrak{g}} : \forall\beta \in \Delta^{+}_{\mathfrak{a}},\ \alpha\perp\beta\right\}$. Notice that if $\Delta^{+}_{\perp} \neq \emptyset$ the corresponding Weyl reflections generate a subgroup $W_{\perp}$ of the group W. Consider the roots $\alpha,\ \beta \in \Delta^{+}_{\perp}$ and the corresponding reflections $\omega_{\alpha},\ \omega_{\beta} \in W_{\perp}$. Since the roots in $\Delta_{\mathfrak{a}}$ are invariant with respect to $\omega_{\alpha},\ \omega_{\beta}$ they are also invariant under the product $\omega_{\gamma} = \omega_{\alpha}\cdot\omega_{\beta}$ and the subgroup $W_{\perp}$ preserves the roots in $\Delta_{\mathfrak{a}}$. The system $\Delta_{\perp}$ can be considered as the root system of a subalgebra $\mathfrak{a}_{\perp} \subset \mathfrak{g}$.

Now we are to find out when the subset $\Delta^{+}_{\perp}$ is non-empty with the non-trivial subgroup $W_{\perp}$ and the subalgebra $\mathfrak{a}_{\perp}$.

If $\mathfrak{a}$ is a maximal regular subalgebra of $\mathfrak{g}$ then $r_{\mathfrak{a}} = r$ and $\Delta^{+}_{\perp}$ is empty. On the other hand non-maximal regular embedding of $\mathfrak{a}$ into $\mathfrak{g}$ can be obtained through the chain of maximal embeddings $\mathfrak{a} \subset \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \ldots \subset \mathfrak{g}$. The maximal regular embeddings can be described by the elimination of one or two roots from the extended Dynkin diagram of the algebra. When the process gives us non-connected diagrams we immediately fix the roots orthogonal to $P_{\mathfrak{a}}$. The complete classification of the regular subalgebras for affine Lie algebras can be found in the recent paper [11].

For a regular embedding $\mathfrak{a} \subset \mathfrak{g}$ when both $\mathfrak{a}$ and $\mathfrak{g}$ are simple the sufficient condition for $\Delta^{+}_{\perp}$ to be nontrivial can be formulated as follows: if the Dynkin diagram of $\mathfrak{g}$ contains the disconnected subdiagrams of $\mathfrak{a}$ and of some subalgebras $\{\mathfrak{a}_j\}$ then the subset $\Delta_{\perp}$ is non-empty, subalgebra $\mathfrak{a}_{\perp}$ is non-trivial and $\{\mathfrak{a}_j\}$ are the subalgebras of $\mathfrak{a}_{\perp}$.

Notice that when we study the regular embedding obtained by dropping the nodes of the extended Dynkin diagram of the algebra $\mathfrak{g}$ and the subalgebra $\mathfrak{a}$ is one of the connected components, the subalgebra $\mathfrak{a}_{\perp}$ may be larger than the algebra generated by the remaining connected components of the diagram. Consider for example the embedding of $B_2 \subset B_4$ (the Figure 3). In this case by eliminating the simple root $\alpha_2 = e_2 - e_3$ one splits the extended Dynkin diagram of B_4 into the diagrams of the subalgebra $\mathfrak{a} = B_2$ and that of the direct sum $A_1 \oplus A_1$. But the subalgebra $\mathfrak{a}_{\perp}$ is equal to B_2 (the root system of B_4 contains not only $\alpha_2 = e_2 - e_3$ but also e_2).

Such effects are due to the fact that the subalgebras $\mathfrak{a}$ and $\mathfrak{a}_{\perp}$ must not form a direct sum in $\mathfrak{g}$. Consider the case of such a regular embedding $\mathfrak{a} \subset \mathfrak{g}$ where both algebras are simple and the diagram of the subalgebra $\mathfrak{a}_{\perp}$ is not a subdiagram of the extended Dynkin diagram $\mathfrak{g}$. Drop the subdiagram of $\mathfrak{a}$ and the node α' that connects it with all the remaining nodes of the diagram of $\mathfrak{g}$. The remaining diagram is the diagram of the algebra $\mathfrak{a}$ with $\mathrm{rank}(\mathfrak{a}) = \mathrm{rank}(\mathfrak{g}) - \mathrm{rank}(\mathfrak{a})$. It is clear that $\mathfrak{a} \subset \mathfrak{a}_\perp$. So the question is whether $\mathfrak{a}_\perp$ has additional roots, not the roots of $\mathfrak{a}$ but the linear combinations of them. The answer is positive when the set of angles between the roots of $\mathfrak{a}$ does not contain all the angles between the roots of $\mathfrak{a}$. Then by reflecting the roots of $\mathfrak{a}$ by $s_{\alpha'}$ we get the additional roots of $\mathfrak{a}_\perp$. If $\mathfrak{a} = B_{r_\mathfrak{a}}$ we see that $\Delta_{\mathfrak{a}_\perp} = \{\alpha_{\pm i,\pm j},\ \alpha_j,\ 1 < i < j \leq r - r_\mathfrak{a}\}$

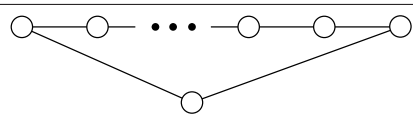
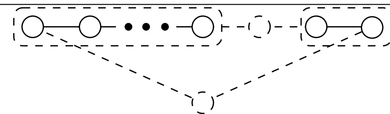
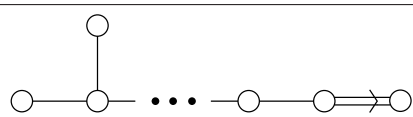
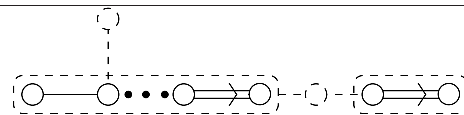
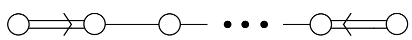
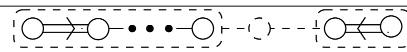
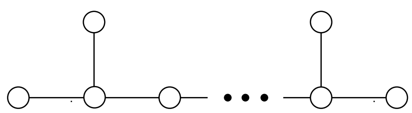
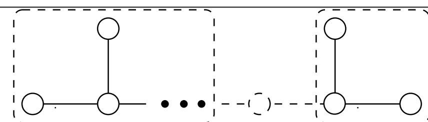
$\mathfrak{g}$	Extended diagram of $\mathfrak{g}$	Diagrams of the subalgebras $\mathfrak{a}$, $\mathfrak{a}_\perp$
A_n		
B_n		
C_n		
D_n		

Table 1. Subalgebras $\mathfrak{a}$, $\mathfrak{a}_\perp$ for the classical series

and $\mathfrak{a}_\perp = B_{r-r_\mathfrak{a}}$. This is the only case where the simple roots of $\mathfrak{a}_\perp$ can not be obtained from the extended Dynkin diagram, as can be seen in the Table 2.1.

Using the existing classification of maximal special subalgebras in classical Lie algebras [12] we have the following pairs of the orthogonal subalgebras $\mathfrak{a}$, $\mathfrak{a}_\perp$

$$\begin{array}{lll} su(p) \oplus su(q) & \subset su(pq) & \\ so(p) \oplus so(q) & \subset so(pq) & \\ sp(2p) \oplus sp(2q) & \subset so(4pq) & \\ sp(2p) \oplus so(q) & \subset sp(2pq) & \\ so(p) \oplus so(q) & \subset so(p+q) & \text{for } p \text{ and } q \text{ odd} \end{array}$$

Exceptional Lie algebras and other non-maximal subalgebras will be considered elsewhere.

Now consider the direct sum $\mathfrak{a}_\perp \oplus \mathfrak{h}_\mathfrak{a}$ and its module $L^\mu_{\mathfrak{a}_\perp\oplus\mathfrak{h}}$ with the highest weight μ. The character $chL^\mu_{\mathfrak{a}_\perp\oplus\mathfrak{h}}$ can be written as

$$chL^\mu_{\mathfrak{a}_\perp\oplus\mathfrak{h}} = \frac{\sum_{\omega\in W_\perp} \epsilon(\omega) e^{\omega(\mu+\rho_{\mathfrak{a}_\perp})-\rho_{\mathfrak{a}_\perp}}}{\prod_{\alpha\in\Delta^+_\perp}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}.$$

Its projection $\pi_\mathfrak{a}(ch\,L^\mu_{\mathfrak{a}_\perp\oplus\mathfrak{h}})$ is the element $e^{\pi_\mathfrak{a}\cdot\mu}$ of the formal algebra $\mathcal{E}(\mathfrak{a})$ with the multiplicity equal to the dimension of the module $L^\mu_{\mathfrak{a}_\perp\oplus\mathfrak{h}}$, since all the roots of $\mathfrak{a}_\perp$ are orthogonal to that of $\Delta_\mathfrak{a}$.

Using this property we can consider the restriction $ch\, L^{\mu}_{\mathfrak{g}\downarrow \mathfrak{a}_{\perp}\oplus\mathfrak{h}}$, that is the character of the direct sum of $(\mathfrak{a}_{\perp}\oplus\mathfrak{h})$-modules. Multiply the equation (3) by the element

$$\pi_{\mathfrak{a}}\left(\prod_{\alpha\in\Delta^{+}\setminus\Delta^{+}_{\perp}}(1-e^{-\alpha})^{\mathrm{mult}_{\mathfrak{g}}(\alpha)}\right)$$

and take into account that the projection commutes with the multiplication:

$$\pi_{\mathfrak{a}}\left(\frac{\sum_{\omega\in W}\epsilon(\omega)e^{\omega(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^{+}_{\perp}}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}\right)=$$
$$\pi_{\mathfrak{a}}\left(\prod_{\alpha\in\Delta^{+}\setminus\Delta^{+}_{\perp}}(1-e^{-\alpha})^{\mathrm{mult}_{\mathfrak{g}}(\alpha)}\right)\sum_{\nu\in P^{+}_{\mathfrak{a}}}b^{(\mu)}_{\nu}\frac{\sum_{\omega\in W_{\mathfrak{a}}}\epsilon(\omega)e^{\omega(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}}{\prod_{\beta\in\Delta^{+}_{\mathfrak{a}}}(1-e^{-\beta})^{\mathrm{mult}_{\mathfrak{a}}(\beta)}}.$$

Similarly to the transformation proposed in [10] we introduce the anomalous branching coefficients k_{λ},

$$\sum_{\nu\in P_{\mathfrak{a}}}b^{(\mu)}_{\nu}\Psi^{(\nu)}_{(\mathfrak{a})}=\sum_{\lambda\in P_{\mathfrak{a}}}k^{(\mu)}_{\lambda}e^{\lambda}$$

and simplify the first factor,

$$\pi_{\mathfrak{a}}\left(\frac{\sum_{\omega\in W}\epsilon(\omega)e^{\omega(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^{+}_{\perp}}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}\right)=$$
$$\left(\prod_{\alpha\in\pi_{\mathfrak{a}}\left(\Delta^{+}\setminus\Delta^{+}_{\perp}\right)}(1-e^{-\alpha})^{\mathrm{mult}_{\mathfrak{g}}(\alpha)-\mathrm{mult}_{\mathfrak{a}}(\alpha)}\right)\sum_{\lambda\in P_{\mathfrak{a}}}k^{(\mu)}_{\lambda}e^{\lambda}$$

If the set $\Delta^{+}_{\perp}$ is non-empty then the Weyl reflections corresponding to the positive roots of $\Delta^{+}_{\perp}$ generate a subgroup $W_{\perp}\subset W$. Consider the factor-space $W_{\perp}\setminus W$. For the class $\tilde{\omega}\in W_{\perp}\setminus W$ choose the representative $\omega\in\tilde{\omega}$ such that $\pi_{\mathfrak{a}_{\perp}}\omega(\mu+\rho)\in\bar{C}_{\mathfrak{a}_{\perp}}$,

$$\begin{aligned}&\pi_{\mathfrak{a}}\left(\frac{\sum_{\omega\in W}\epsilon(\omega)e^{\omega(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^{+}_{\perp}}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}\right)=\\&=\pi_{\mathfrak{a}}\left(\sum_{\omega\in W_{\perp}\setminus W}\epsilon(\omega)\frac{\sum_{\nu\in W_{\perp}}\epsilon(\nu)e^{\nu\cdot\omega(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^{+}_{\perp}}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}\right)\end{aligned}\tag{4}$$

The fraction in the right-hand side of the equation looks like the character of some $\mathfrak{a}_{\perp}$-module. Since $\nu\cdot\pi_{\mathfrak{a}}(\omega(\mu+\rho))=\pi_{\mathfrak{a}}(\omega(\mu+\rho))$ and $\omega(\mu+\rho)-\pi_{\mathfrak{a}}(\omega(\mu+\rho))=\pi_{\mathfrak{a}_{\perp}}(\omega(\mu+\rho))$, we can rewrite the shifted weights

$$\nu\cdot\omega(\mu+\rho)-\rho=\nu\cdot\left(\pi_{\mathfrak{a}_{\perp}}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_{\perp}}+\rho_{\mathfrak{a}_{\perp}}+\pi_{\mathfrak{a}}(\omega(\mu+\rho))\right)-\rho$$

$$\sum_{\omega\in W_{\perp}\setminus W}\epsilon(\omega)\frac{\sum_{\nu\in W_{\perp}}\epsilon(\nu)e^{\nu\cdot\omega(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^{+}_{\perp}}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}=$$
$$\sum_{\omega\in W_{\perp}\setminus W}\epsilon(\omega)e^{\pi_{\mathfrak{a}}(\omega(\mu+\rho))-\rho}\frac{e^{\rho_{\mathfrak{a}_{\perp}}}\sum_{\nu\in W_{\perp}}\epsilon(\nu)e^{\nu\cdot(\pi_{\mathfrak{a}_{\perp}}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_{\perp}}+\rho_{\mathfrak{a}_{\perp}})-\rho_{\mathfrak{a}_{\perp}}}}{\prod_{\alpha\in\Delta^{+}_{\perp}}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}=$$
$$\sum_{\omega\in W_{\perp}\setminus W}\epsilon(\omega)e^{\pi_{\mathfrak{a}}(\omega(\mu+\rho))-\rho}e^{\rho_{\mathfrak{a}_{\perp}}}\mathrm{ch}L^{\pi_{\mathfrak{a}_{\perp}}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_{\perp}}}_{\mathfrak{a}_{\perp}}$$

The projector $\pi_{\mathfrak{a}}$ sends the character $\mathrm{ch} L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}$ to the unit element of $\mathcal{E}$ multiplied by $\dim L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}$:

$$\pi_{\mathfrak{a}}\left(\sum_{\omega\in W_\perp\backslash W}\epsilon(\omega)e^{\pi_{\mathfrak{a}}(\omega(\mu+\rho))-\rho}e^{\rho_{\mathfrak{a}_\perp}}\mathrm{ch}L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}\right)=$$
$$\sum_{\omega\in W_\perp\backslash W}\epsilon(\omega)\,\dim\left(L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}\right)e^{\pi_{\mathfrak{a}}(\omega(\mu+\rho)-\rho)}$$

Thus we obtain the relation

$$\begin{aligned}&\textstyle\sum_{\omega\in W_\perp\backslash W}\epsilon(\omega)\dim\left(L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}\right)e^{\pi_{\mathfrak{a}}(\omega(\mu+\rho)-\rho)}=\\&\qquad\left(\textstyle\prod_{\alpha\in\pi_{\mathfrak{a}}\left(\Delta^+\backslash\Delta_\perp^+\right)}(1-e^{-\alpha})^{\mathrm{mult}_{\mathfrak{g}}(\alpha)-\mathrm{mult}_\alpha}\right)\textstyle\sum_{\lambda\in P_{\mathfrak{a}}}k_\lambda^{(\mu)}e^\lambda.\end{aligned}\tag{5}$$

Let us rewrite the factor in the right-hand side:

$$\prod_{\alpha\in\pi_{\mathfrak{a}}\circ(\Delta^+\backslash\Delta_\perp^+)}\left(1-e^{-\alpha}\right)^{\mathrm{mult}(\alpha)-\mathrm{mult}_{\mathfrak{a}}(\alpha)}=-\sum_{\gamma\in P_{\mathfrak{a}}}s(\gamma)e^{-\gamma}\tag{6}$$

For the coefficient function $s(\gamma)$ define the carrier $\Phi_{\mathfrak{a}\subset\mathfrak{g}}\subset P_{\mathfrak{a}}$:

$$\Phi_{\mathfrak{a}\subset\mathfrak{g}}=\{\gamma\in P_{\mathfrak{a}}\mid s(\gamma)\neq 0\}\,.\tag{7}$$

From the relation 5 the obtained equation for the formal elements,

$$\begin{aligned}&\sum_{\omega\in W_\perp\backslash W}\epsilon(\omega)\dim\left(L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}\right)e^{\pi_{\mathfrak{a}}(\omega(\mu+\rho)-\rho)}=\\&=-\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}}s(\gamma)\,e^{-\gamma}\sum_{\lambda\in P_{\mathfrak{a}}}k_\lambda^{(\mu)}e^\lambda=\\&=-\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}}\sum_{\lambda\in P_{\mathfrak{a}}}s(\gamma)\,k_\lambda^{(\mu)}e^{\lambda-\gamma}\end{aligned}$$

we can deduce the following property of the anomalous branching coefficients,

$$\sum_{\omega\in W_\perp\backslash W}\epsilon(\omega)\dim\left(L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}\right)\delta_{\xi,\pi_{\mathfrak{a}}(\omega(\mu+\rho)-\rho)}+\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}}s(\gamma)\,k_{\xi+\gamma}^{(\mu)}=0\tag{8}$$

$$\xi\in P_{\mathfrak{a}}.\tag{9}$$

To get the recurrent relations for the coefficients $k_{\xi+\gamma}^{(\mu)}$ we use the following procedure (similar to that in [10]). Let γ_0 be the lowest vector with respect to the natural ordering in $\overset{\circ}{\Delta}_{\mathfrak{a}}$ in the lowest grade of $\Phi_{\mathfrak{a}\subset\mathfrak{g}}$ and decompose the defining relation (6),

$$\prod_{\alpha\in\pi_{\mathfrak{a}}\circ(\Delta^+\backslash\Delta_\perp^+)}\left(1-e^{-\alpha}\right)^{\mathrm{mult}(\alpha)-\mathrm{mult}_{\mathfrak{a}}(\alpha)}=-s\left(\gamma_0\right)e^{-\gamma_0}-\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}\backslash\{\gamma_0\}}s\left(\gamma\right)e^{-\gamma},$$

the equality (8) leads to the desired recurrent relation for the anomalous branching coefficients:

$$\begin{aligned}k_\xi^{(\mu)}=-\frac{1}{s\left(\gamma_0\right)}&\left(\sum_{\omega\in W_\perp\backslash W}\epsilon(\omega)\,\dim\left(L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}\right)\delta_{\xi-\gamma_0,\pi_{\mathfrak{a}}(\omega(\mu+\rho)-\rho)}+\right.\\&\left.+\sum_{\gamma\in\Gamma_{\mathfrak{a}\subset\mathfrak{g}}}s\left(\gamma+\gamma_0\right)k_{\xi+\gamma}^{(\mu)}\right)\end{aligned}$$

where the set

$$\Gamma_{\mathfrak{a}\subset\mathfrak{g}} = \{\xi - \gamma_0 | \xi \in \Phi_{\mathfrak{a}\subset\mathfrak{g}}\} \setminus \{0\} \tag{10}$$

was introduced that is called the injection fan.

Consider the case $\Delta_{\perp}^{+} = 0$. There are three different reasons for $\Delta_{\perp}^{+}$ to be empty: i) $\dim\mathfrak{h}_{\mathfrak{a}} = \dim\mathfrak{h}_{\mathfrak{g}}$, ii) $\mathfrak{a}_{\perp} = 0$ and iii) $\mathfrak{a}_{\perp} \subset \mathfrak{h}_{\mathfrak{g}}$. Both the first and the second cases can be treated as corresponding to the trivial orthogonal subalgebra: $\mathfrak{a}_{\perp} = 0$. In any of these cases the formal characters in the right-hand side of (4) reduces to the formal element $e^{\pi_{\mathfrak{a}_{\perp}}\omega(\mu+\rho)}$. In the first two cases the vector $\omega(\mu+\rho)$ is projected to the subspace orthogonal to the weight space of $\mathfrak{a}$. It is clear that in any of the three variants the final vector $\pi_{\mathfrak{a}}\pi_{\mathfrak{a}_{\perp}}\omega(\mu+\rho)$ leads to the unit of the formal algebra $\mathcal{E}$. Thus when the set $\Delta_{\perp}^{+}$ is empty the recurrent relation is simplified:

$$k_{\xi}^{(\mu)} = -\frac{1}{s\left(\gamma_0\right)}\left(\sum_{w\in W}\epsilon\left(w\right)\delta_{\xi,\pi_{\mathfrak{a}}\circ(w\circ(\mu+\rho)-\rho)+\gamma_0} + \sum_{\gamma\in\Gamma_{\mathfrak{a}\subset\mathfrak{g}}} s\left(\gamma+\gamma_0\right) k_{\xi+\gamma}^{(\mu)}\right),$$

the latter coinsides with the one obtained in [10] (formula (16)).

2.2. Algorithm for recursive computation of branching coefficients

The recurrent relation 10 allows us to formulate an algorithm for recursive computation of the branching coefficients. In this algorithm there is no need to construct the module $L_{\mathfrak{g}}^{(\mu)}$ or any of the modules $L_{\mathfrak{a}}^{(\nu)}$.

It contains the following steps:

(i) Construct the sets Δ^{+} and $\Delta_{\mathfrak{a}}^{+}$ of positive roots for the algebras $\mathfrak{g} \supset \mathfrak{a}$.

(ii) Select the positive roots $\alpha \in \Delta^{+}$ orthogonal to the root subspace of $\mathfrak{a}$ and form the set $\Delta_{\perp}^{+}$.

(iii) Construct the set Γ 10.

(iv) Construct the set $\widehat{\Psi^{(\mu)}} = \{\omega(\mu+\rho) - \rho;\ \omega \in W\}$ of the anomalous weights for the $\mathfrak{g}$-module $L^{(\mu)}$.

(v) Select the weights $\left\{\lambda = \omega(\mu+\rho) | \pi_{\mathfrak{a}_{\perp}}\lambda \in \bar{C}_{\mathfrak{a}_{\perp}}\right\}$ Since we have constructed the set $\Delta_{\perp}^{+}$ we can easily check wether the weight $\pi_{\mathfrak{a}_{\perp}}\lambda$ lies in the main Weyl chamber of $\mathfrak{a}_{\perp}$ by computing the scalar product of λ with the roots of $\Delta_{\perp}^{+}$.

(vi) For $\lambda = \omega(\mu+\rho)$, $\pi_{\mathfrak{a}_{\perp}}\lambda \in \bar{C}_{\mathfrak{a}_{\perp}}$ calculate the dimensions of the corresponding modules $\dim\left(L_{\mathfrak{a}_{\perp}}^{\pi_{\mathfrak{a}_{\perp}}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_{\perp}}}\right)$ using the Weyl formula with the set $\Delta_{\perp}^{+}$.

(vii) Calculate the anomalous branching coefficients in the main Weyl chamber $\bar{C}_{\mathfrak{a}}$ of the subalgebra $\mathfrak{a}$ using the recurrent relation (10).

When are interested in the branching coefficients for the embedding of the finite-dimensional Lie algebra into the affine Lie algebra we can construct the set of the anomalous weights up to the required grade and use the steps 4-7 of the algorithm for each grade. We can also speed up the algorithm by one-time computation of the representatives of the conjugate classes $W_{\perp}\backslash W$.

The next section contains several examples computed using this algorithm.

3. Branching for finite dimensional Lie algebras

3.1. Regular embedding of A_1 into B_2

Consider the regular embedding of $A_1 \to B_2$. Simple roots α_1, α_2 of B_2 are presented as the dashed vectors in the Figure 1. We denote the corresponding Weyl reflections by ω_1, ω_2. The simple root $\beta = \alpha_1 + 2\alpha_2$ of A_1 is indicated as the grey vector.

Let's describe the reduction of the fundamental representation of $L_{B_2}^{(1,0)=w_1}$ (the black vector in Figure 1). Circles indicate the weights of the singular element $\Psi^{(w_1)}$.

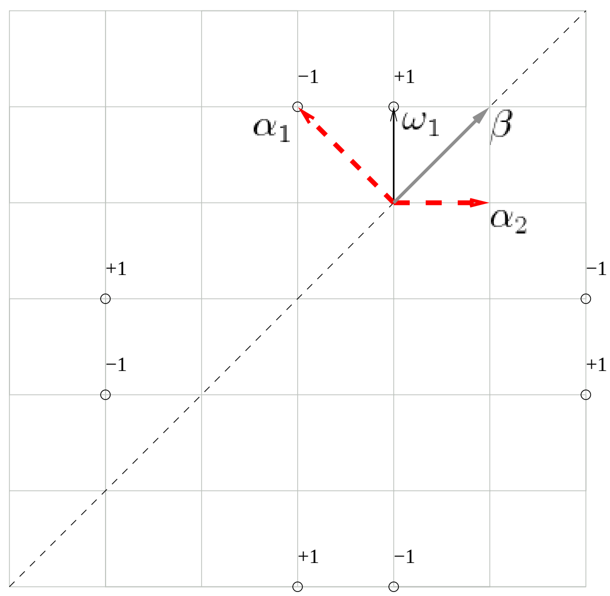


Figure 1. Regular embedding of A_1 into B_2

Now we are to factorise the Weyl group W by $W_\perp = \{\omega_1\}$ and to construct the set $\{\omega(\mu+\rho)-\rho,\ \omega \in W_\perp \backslash W\}$. On the Figure 2 the weights of the corresponding $\mathfrak{a}_\perp = A_1$-modules $L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}$ are indicated. Projecting them onto the root space of the subalgebra $\mathfrak{a} = A_1$ we get the anomalous weights with the corresponding multiplicities:

$$(1,2),\ (0,-3),\ (-4,3),\ (-5,-2)$$

For the set Γ (using the definition (7,10)) we have

$$\Gamma_{A_1 \subset B_2} = \{(1,2),\ (2,-1)\}\,.$$

Here the second component denotes the value of $s(\gamma)$. Applying this fan inside the $\bar{C}_{\mathfrak{a}}^{(0)}$ we get zeros for the weights greater than the first anomalous vector (1), here $k_1^{(1,0)} = 2$. For the last weight in $\bar{C}_{\mathfrak{a}}^{(0)}$ the formula (10) gives

$$k_0^{(1,0)} = -1 \cdot k_2^{(1,0)} + 2 \cdot k_1^{(1,0)} - 3 \cdot \delta_{0,0} = 1.$$

The recurrence property defines the branching.

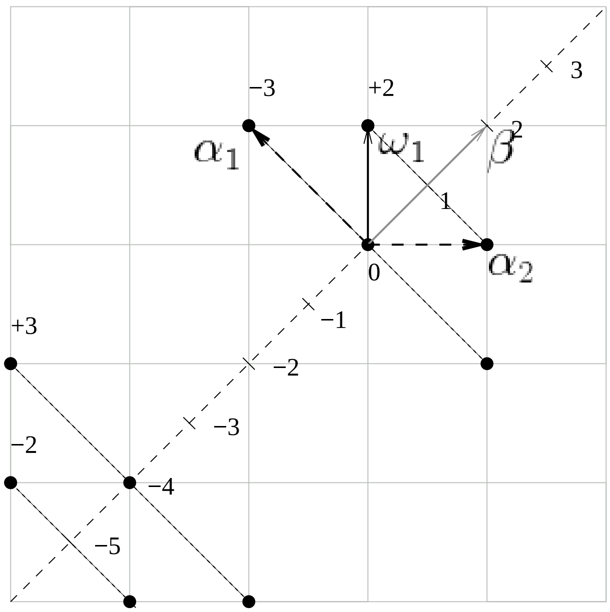


Figure 2. Anomalous weights and the corresponding $\mathfrak{a}_{\perp} = A_1$-modules for the embedding $A_1 \subset B_2$

3.2. Embedding of B_2 into B_4

Consider the regular embedding $B_2 \longrightarrow B_4$. The corresponding Dynkin diagrams are presented in the Figure 3.

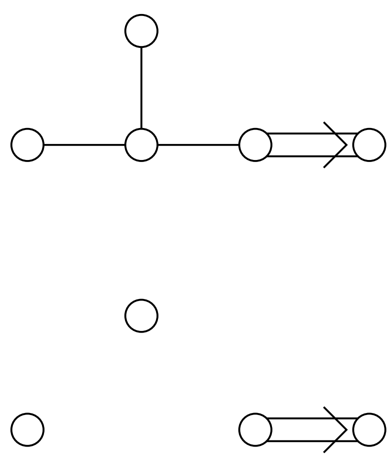


Figure 3. The regular embedding $B_2 \subset B_4$ described by dropping the node.

In the orthogonal basis $\{e_1, \ldots, e_4\}$ the simple roots and the positive roots of B_4 are

$$S_{B_4} = \{e_1 - e_2,\ e_2 - e_3,\ e_3 - e_4,\ e_4\}$$

$$\Delta^+_{B_4} = \{(e_1 - e_2,\ e_2 - e_3,\ e_3 - e_4,\ e_4,\ e_1 - e_3,\ e_2 - e_4,\ e_3 + e_4,\ e_3,\ e_1 - e_4,\\ e_2 + e_4,\ e_2,\ e_1 + e_4,\ e_2 + e_3,\ e_1,\ e_1 + e_3,\ e_1 + e_2\}$$

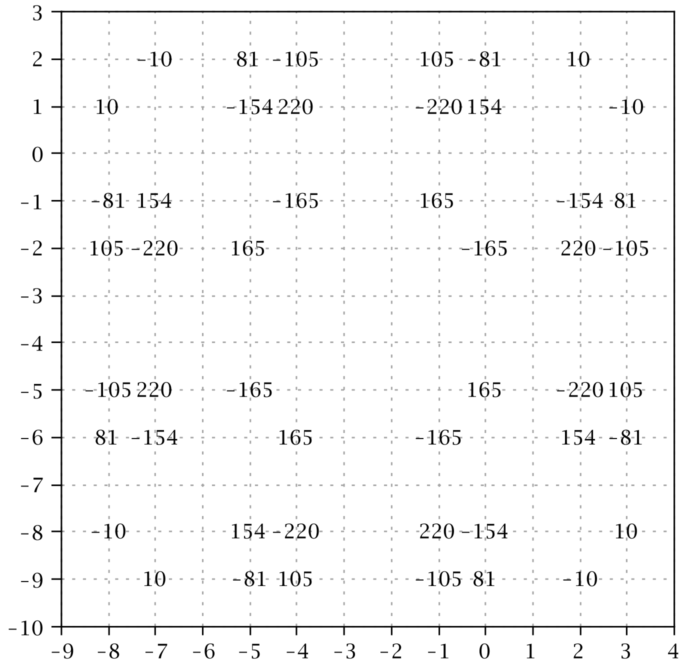


Figure 4. Projected weights $-\frac{1}{s(\gamma_0)}\pi_{B_2}\left(\hat{\Psi}_{B_4}^{(0,1,0,2)}\right)$ with the dimensions of the corresponding $\mathfrak{a}_\perp$-modules.

Correspondingly for the embedded subalgebra $\mathfrak{a} = B_2$ we have

$$S_{B_2} = \{e_3 - e_4, e_4\}$$

$$\Delta_\perp^+ = \{e_1 - e_2, e_1 + e_2, e_1, e_2\}$$

and is the set of positive roots for the algebra $\mathfrak{a}_\perp = B_2$.

Using the definition (10) we obtain the fan $\Gamma_{B_2 \subset B_4}$ with the corresponding values $s(\gamma + \gamma_0)$, depicted in the Figure 5.

Consider the B_4-module L^μ with the highest weight $\mu = (0,1,0,2) = 2e_1 + 2e_2 + e_3 + e_4$; $\dim(L^{(0,1,0,2)}) = 2772$.

To find the branching coefficients we need to compute the anomalous weights of $L^\mu_{B_4}$, select the weights belonging to $\bar{C}_{\mathfrak{a}_\perp}^{(0)}$ and compute the dimensions of the corresponding $\mathfrak{a}_\perp$-modules. The set of the anomalous weights $\{\omega(\mu+\rho) - \rho,\ \omega \in W\}$ contains 384 vectors.

We need to select the weights $\psi \in \omega(\mu+\rho)$ with the property $\pi_{\mathfrak{a}_\perp}(\psi) \in C_{\mathfrak{a}_\perp}^{(0)}$. It means that the scalar product of these weights with all the roots in $\Delta_\perp^+$ is non-negative.

To compute the dimensions of the corresponding $\mathfrak{a}_\perp$-modules we need to project each selected weight onto the root space $\Delta_\perp^+$, substract $\rho_{\mathfrak{a}_\perp}$ and apply the Weyl dimension formula. The result is shown in the Figure 4.

Applying the recurrent relation (10) we obtain the following branching coefficients:

$$\pi_{\mathfrak{a}}\left(chL_{B_4}^{(0,1,0,2)}\right) = 6\ chL_{B_2}^{(0,0)} + 60\ chL_{B_2}^{(0,2)} + 30\ chL_{B_2}^{(1,0)} + 19\ chL_{B_2}^{(2,0)} + 40\ chL_{B_2}^{(1,2)} + 10\ chL_{B_2}^{(2,2)}.$$

4. Applications to the conformal field theory

4.1. Conformal embeddings

Branching coefficients for an embedding of affine Lie subalgebra into affine Lie algebra can be used to construct modular invariant partition functions for Wess-Zumino-Novikov-Witten models of conformal field theory ([1], [13], [14], [15]). In these models current algebras are affine Lie algebras.

The modular invariant partition function is crucial for the conformal theory to be valid on the torus and higher genus Riemann surfaces. It is important for the applications of CFT to the string theory and to the critical phenomena description.

The simplest modular-invariant partition function has the diagonal form:

$$Z(\tau) = \sum_{\mu \in P_{\mathfrak{g}}^{+}} \chi_{\mu}(\tau) \bar{\chi}_{\mu}(\bar{\tau})$$

Here the sum is over the set of the highest weights of integrable modules of WZW-model and $\chi_{\mu}(\tau)$ are the normalized characters of these modules.

To construct the non-diagonal modular invariants is not an easy problem, although for some model the complete classification of modular invariants is known [16, 17].

Consider the Wess-Zumino-Witten model with the affine Lie algebra $\mathfrak{a}$. Non-diagonal modular invariants for this model can be constructed from the diagonal invariant if there exists affine algebra $\mathfrak{g}$ such that $\mathfrak{a} \subset \mathfrak{g}$. Then we can replace the characters of the $\mathfrak{g}$-modules in the diagonal modular-invariant partition function (11) by the decompositions

$$\sum_{\nu \in P_{\mathfrak{a}}^{+}} b_{\nu}^{(\mu)} \chi_{\nu}$$

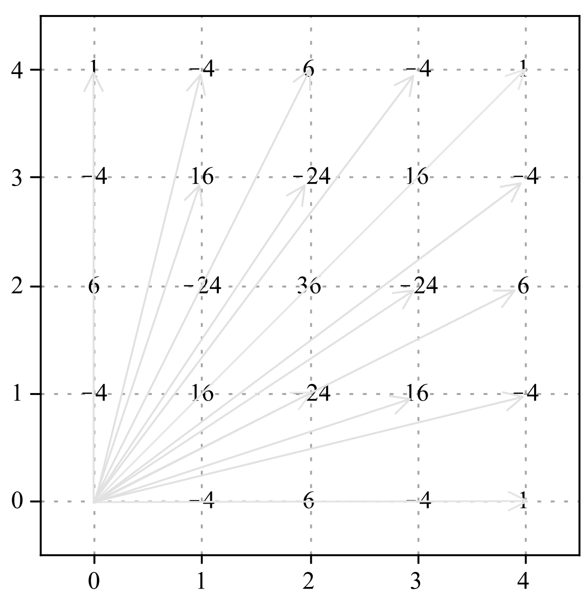


Figure 5. The fan for $B_2 \subset B_4$

containing the modified characters χ_ν of the corresponding $\mathfrak{a}$-modules. Thus we obtain the non-diagonal modular-invariant partition function for the theory with the current algebra $\mathfrak{a}$,

$$Z_{\mathfrak{a}}(\tau) = \sum_{\nu,\lambda \in P_{\mathfrak{a}}^+} \chi_\nu(\tau) M_{\nu\lambda} \bar{\chi}_\lambda(\bar{\tau}) \tag{11}$$

The effective reduction procedure is crucial for this construction.

The embedding is required to preserve the conformal invariance. Let $X_{-n_j}^{\alpha_j}$ and $\tilde{X}_{-n_j}^{\alpha'_j}$ be the lowering generators for $\mathfrak{g}$ and for $\mathfrak{a} \subset \mathfrak{g}$ correspondingly. Let $\pi_{\mathfrak{a}}$ be the projection operator of $\pi_{\mathfrak{a}} : \mathfrak{g} \longrightarrow \mathfrak{a}$. In the theory attributed to $\mathfrak{g}$ with the vacuum $|\lambda\rangle$ the states can be described as

$$X_{-n_1}^{\alpha_1} X_{-n_2}^{\alpha_2} \dots |\lambda\rangle \quad n_1 \geq n_2 \geq \dots > 0.$$

And for the sub-algebra $\mathfrak{a}$ the corresponding states are

$$\tilde{X}_{-n_1}^{\alpha'_1} \tilde{X}_{-n_2}^{\alpha'_2} \dots |\pi_{\mathfrak{a}}(\lambda)\rangle \,.$$

The $\mathfrak{g}$-invariance of the vacuum entails its $\mathfrak{a}$-invariance, but this is not the case for the energy-momentum tensor. So the energy-momentum tensor of the larger theory should consist only of the generators of $\tilde{X}$. Then

$$T_{\mathfrak{g}}(z) = T_{\mathfrak{a}}(z), \tag{12}$$

leads to the equality of the central charges

$$c(\mathfrak{g}) = c(\mathfrak{a})$$

and to the equation

$$\frac{k \dim \mathfrak{g}}{k+g} = \frac{x_e k \dim \mathfrak{a}}{x_e k + a} \tag{13}$$

Here x_e is the embedding index and g, a are the dual Coxeter numbers for the corresponding algebras.

It can be demonstrated that the solutions of the equation (13) exist only for the level $k = 1$ [1].

The complete classification of conformal embeddings is given in [15].

The relation (13) and the asymptotics of the branching functions can be used to prove the finite reducibility theorem [18]. It states that for the conformal embedding $\mathfrak{a} \subset \mathfrak{g}$ only finite number of branching coefficients have non-zero values.

Note 4.1. *The orthogonal subalgebra $\mathfrak{a}_\perp$ is always empty for the conformal embeddings $\mathfrak{a} \subset \mathfrak{g}$.*

Proof. Consider the modes expansion of the energy-momentum tensor

$$T(z) = \frac{1}{2(k+h^v)} \sum_n z^{-n-1} L_n$$

The modes L_n are constructed as the combination of normally-ordered products of the generators of $\mathfrak{g}$.

$$L_n = \frac{1}{2(k+h^v)} \sum_{\alpha} \sum_{m} : X^{\alpha}_{m} X^{\alpha}_{n-m} :$$

In the case of the conformal embedding energy-momentum tensors are to be equal (12).

The substitution of the generators of $\mathfrak{a}$ in terms of the generators of $\mathfrak{g}$ into these combinations should give the energy-momentum tensor $T_{\mathfrak{g}}$. But if the set of the generators $\Delta_{\perp}$ is not empty this is not possible, since $T_{\mathfrak{g}}$ contains the combinations of the generators X^{α}_{n}, $\alpha \in \Delta_{\perp}$. □

4.1.1. Special embedding $\hat{A}_1 \subset \hat{A}_2$ Consider the case where both $\mathfrak{g}$ and $\mathfrak{a}$ are affine Lie algebras: $\hat{A}_1 \longrightarrow \hat{A}_2$ and the injection is the affine extension of the special injection $A_1 \longrightarrow A_2$ with the embedding index $x_e = 4$. As far as the $\mathfrak{g}$-modules to be considered are of level one, the $\mathfrak{g}$-modules will be of level $\tilde{k} = kx_e = 4$.

There exist three level 1 fundamental weights in the weight space of $\hat{A}_2$. It is easy to see that the set $\Delta_{\perp}$ is empty and the subalgebra $\mathfrak{a}_{\perp} = 0$.

Using the definition (10) we construct the fan $\Gamma_{\hat{A}_1 \rightarrow \hat{A}_2}$ and the function $s(\gamma + \gamma_0)$ (see the Figure 6).

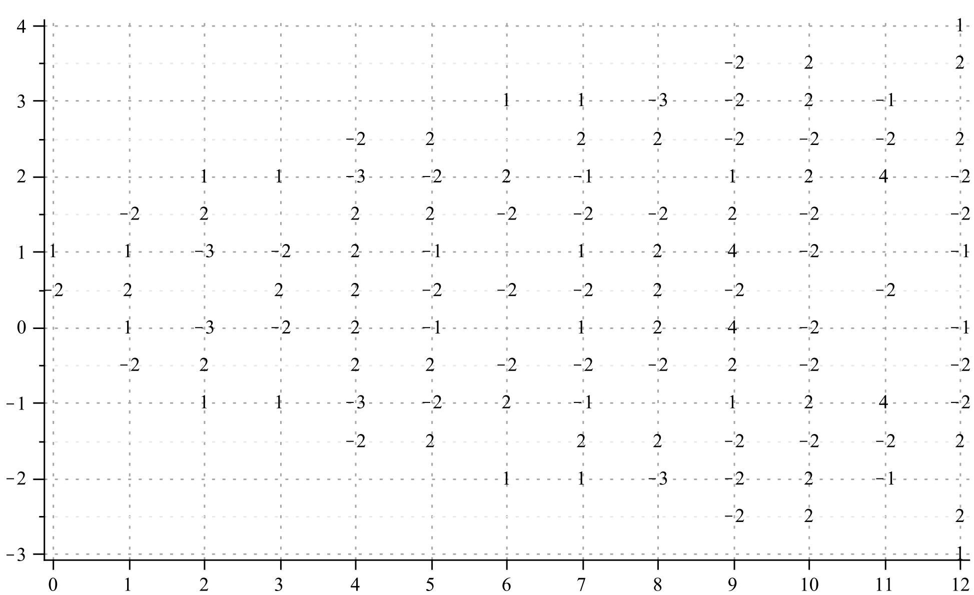


Figure 6. The fan and $s(\gamma + \gamma_0)$ for $\hat{A}_1 \longrightarrow \hat{A}_2$

Let us consider the module $L^{w_0=(0,0;1;0)}$. Here we use the (finite part; level; grade) presentation (see section(1.1)) of the highest weight and the finite part coordinates are the Dynkin indices.

The set $\widehat{\Psi^{(w_0)}}$ is depicted in the Figure 7 up to the sixth grade. The weights $\omega(w_0 + \rho) - \rho$ are marked by crosses when $\epsilon(\omega) = 1$ and by diamond when $\epsilon(\omega) = -1$.

Simple roots of the classical subalgebra A_2 are grey and the grey diagonal plane corresponds to the Cartan subalgebra of the embedded algebra $\hat{A}_1$.

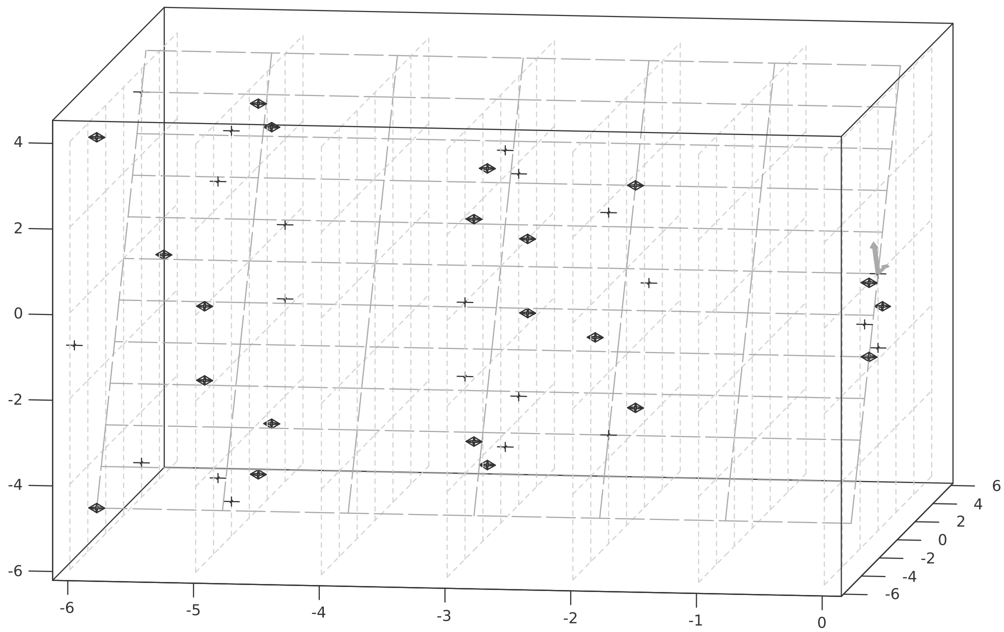


Figure 7. The anomalous weights of the module $L_{\hat{A}_2}^{(0,0;1;0)}$

The next step is to project the anomalous weights to $P_{\hat{A}_1}$. The result up to the twelfth grade is presented in the Figure 8.

Using the recurrent relation for the anomalous branching coefficients we get the result presented in Figure 9. There we see that inside the chamber $\bar{C}_{\hat{A}_1}$, which is isolated in the Figure 9, there are only two non-zero anomalous weights both with multiplicities 1. These are the highest weights of the $\mathfrak{a}$-submodules and their branching coefficients. So the finite reducibility theorem holds and we get the decomposition

$$L_{\hat{A}_2 \downarrow \hat{A}_1}^{(0,0;1;0)} = L_{\hat{A}_1}^{(0;4;0)} \oplus L_{\hat{A}_1}^{(4;4;0)}.$$

For the other level 1 irreducible modules of $\hat{A}_2$ we get the trivial branching

$$L_{\hat{A}_2 \downarrow \hat{A}_1}^{(1,0;1;0)} = L_{\hat{A}_1}^{(2;4;0)}, L_{\hat{A}_2 \downarrow \hat{A}_1}^{(0,1;1;0)} = L_{\hat{A}_1}^{(2;4;0)}.$$

Using these results the modular-invariant partition function is easily found,

$$Z = \left|\chi_{(4;4;0)} + \chi_{(0;4;0)}\right|^2 + 2\chi_{(2;4;0)}^2.$$

4.2. Coset models

Another natural setting where the reduction problem for affine Lie algebras appears is the coset construction in rational conformal field theories [20]. Coset models are tightly connected with the gauged WZW-models and are actively studied in string theory

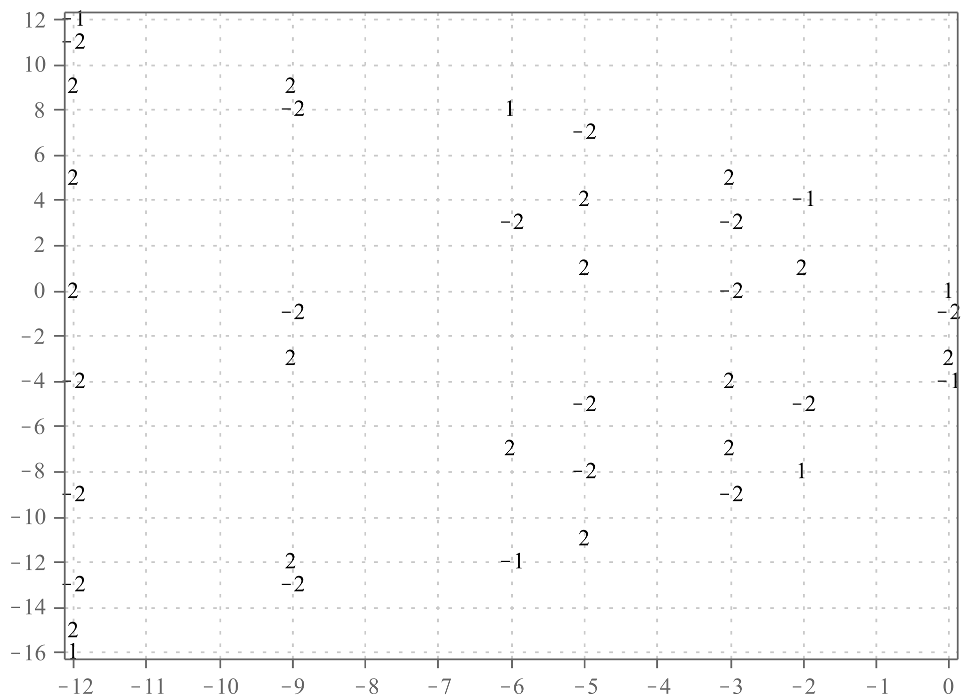


Figure 8. Projected anomalous weights of $L^{(0,0;1;0)}_{\hat{A}_2}$.

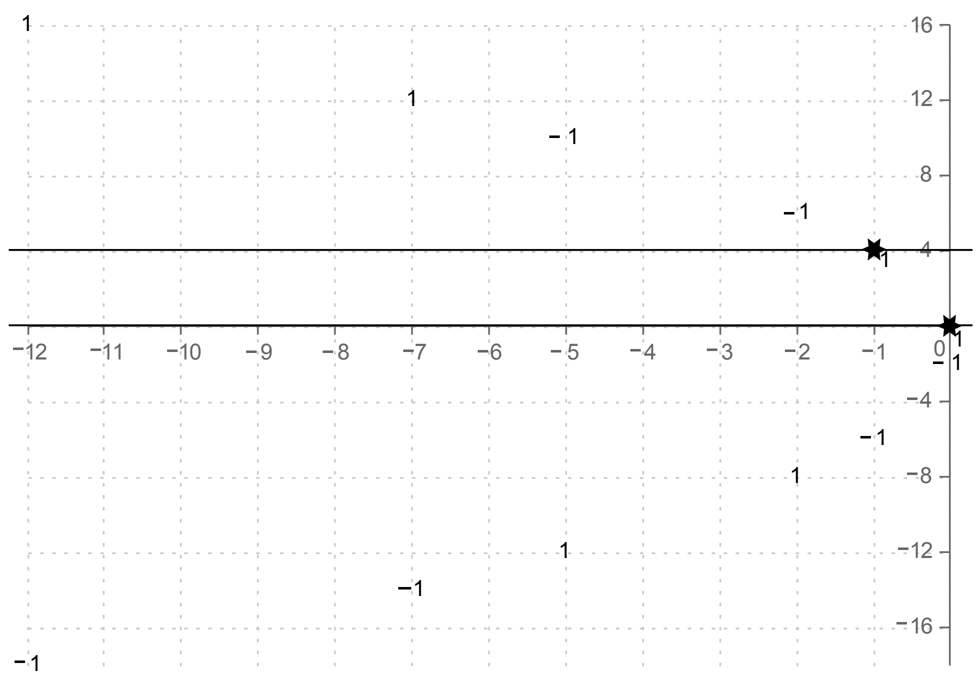


Figure 9. Anomalous branching coefficients for $\hat{A}_1 \subset \hat{A}_2$

especially in string models on anti-de-Sitter space [21, 22, 23, 24, 25]. The characters of the coset model are connected with the branching functions,

$$\chi^{(\mu)}_{\nu}(\tau) = e^{2\pi i\tau(m_\mu - m_\nu)} b^{(\mu)}_{\nu}(\tau), \qquad (14)$$

where

$$m_\mu = \frac{|\mu+\rho|^2}{2(k+g)} - \frac{|\rho|^2}{2g}.$$

The problem of the branching functions construction in the coset models was considered in [26], [7], [27].

Let us return to the example 3.1 and consider the affine extension of the injection $A_1 \subset B_2$. Since this embedding is regular and $x_e = 1$, the level of the subalgebra modules is equal to that of the initial module. The set $\Delta_\perp^+$ of the orthogonal positive roots with the zero projection on the root space of the subalgebra $\hat{A}_1$ is the same as in the finite-dimensional case.

Using the definition (10) we get the fan $\Gamma_{\hat{A}_1 \longrightarrow \hat{B}_2}$ with the corresponding values $s(\gamma+\gamma_0)$(see the Figure 10). We restricted the computation to the twelfth grade.

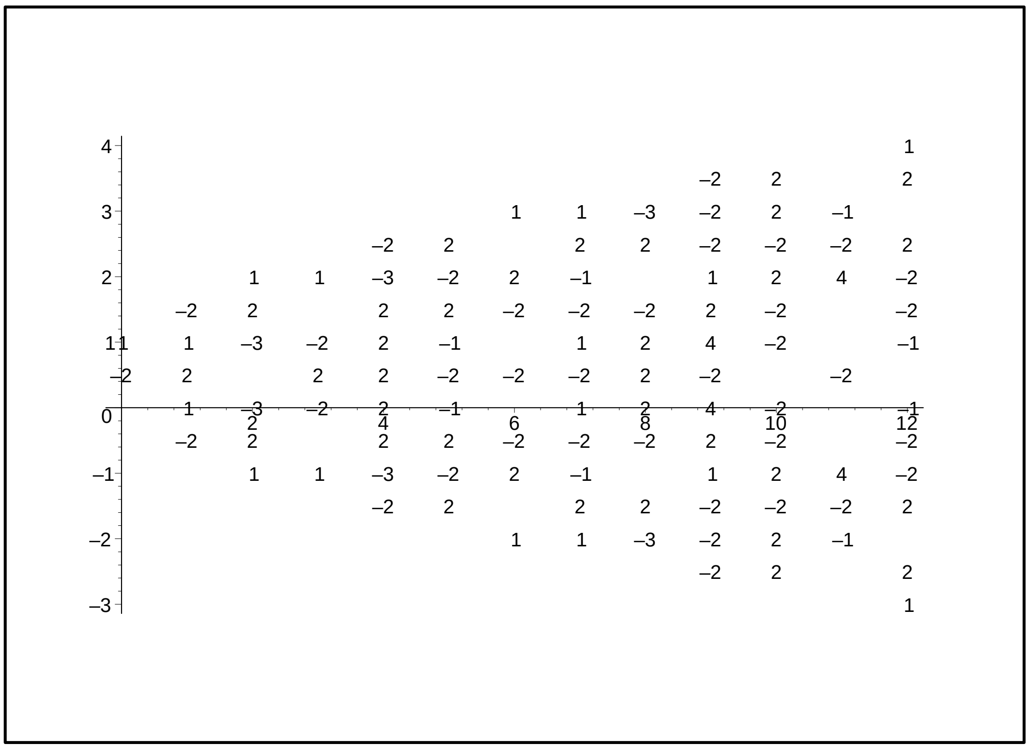


Figure 10. The fan for $\hat{A}_1 \subset \hat{B}_2$

Consider the level one module $L^{(1,0;1;0)}_{\hat{B}_2}$ with the highest weight $w_1 = (1,0;1;0)$, where the finite part coordinates are in the orthogonal basis e_1, e_2.

The set of anomalous weights for this module up to the sixth grade is presented in the Figure 11. In the grade zero it is exactly the set of the anomalous weights for the embedding of the classical Lie algebras $A_1 \subset B_2$ depicted in the Figure 1. The weights $\omega(w_1+\rho)-\rho$ are marked by crosses if $\epsilon(\omega) = 1$ and by circles otherwise. Simple roots of the classical subalgebra B_2 are grey and grey diagonal plane corresponds to the Cartan subalgebra of the embedded algebra $\hat{A}_1$.

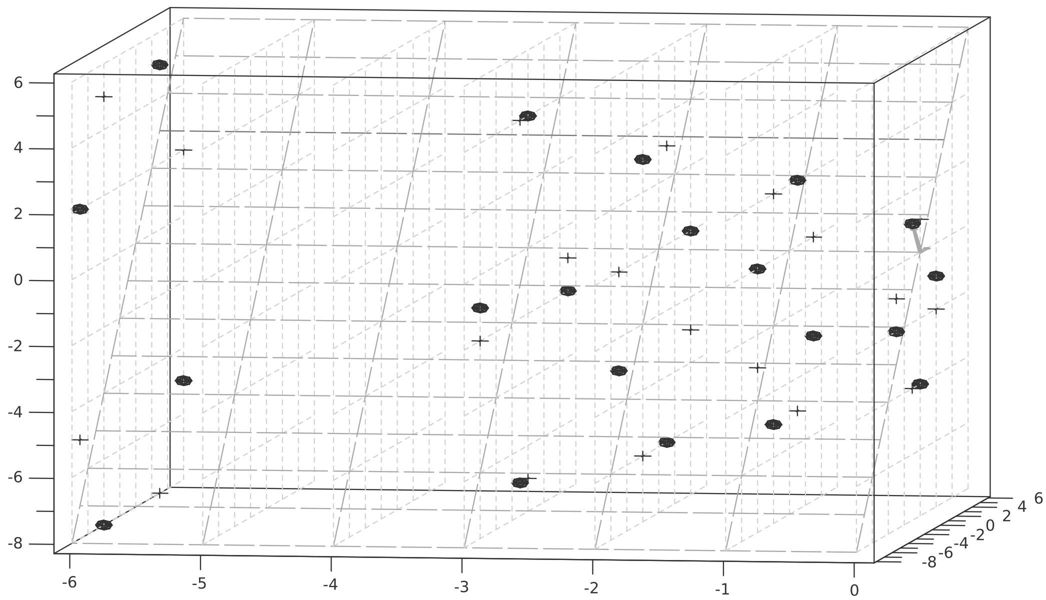


Figure 11. The anomalous weights of $L_{\hat{B}_2}^{(1,0;1;0)}$. The weights in the zero grade can be seen in the Figure 1

.

Performing the next step of the algorithm 2.2 we project the anomalous weights to $P_{\hat{A}_1}$ and find the dimensions of the corresponding $\mathfrak{a}_{\perp}$-modules $L_{\mathfrak{a}_{\perp}}^{\pi_{\mathfrak{a}_{\perp}}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_{\perp}}}$. The result is presented in the Figure 12 up to the twelfth grade. 12.

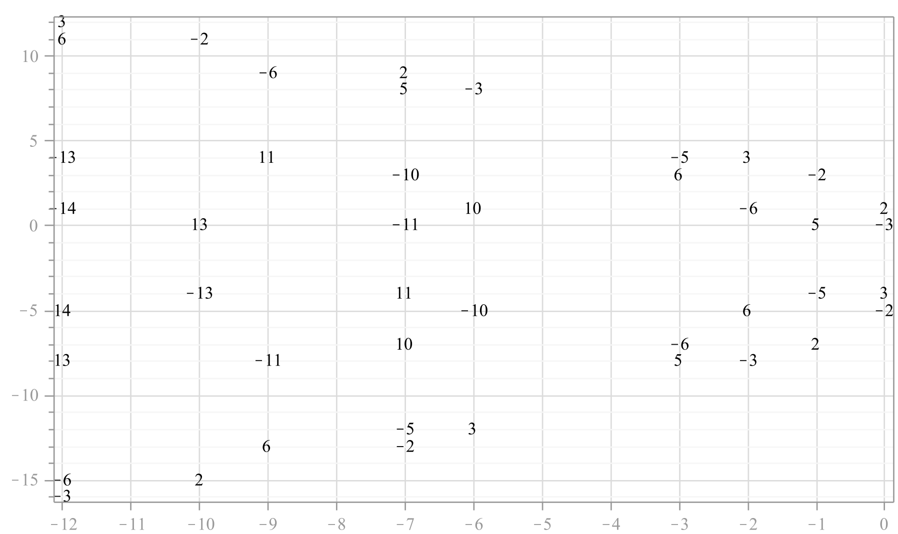


Figure 12. The projected anomalous weights $\pi_{\hat{A}_1}\left(\Psi_{\hat{B}_2}^{(1,0;1;0)}\right)$ and the dimensions of $\mathfrak{a}_{\perp}$-modules.

Notice that here the lowest weight γ_0 of the fan is zero, since we have excluded all the roots of $\Delta_{\perp}^{+}$ from the defining relation (10).

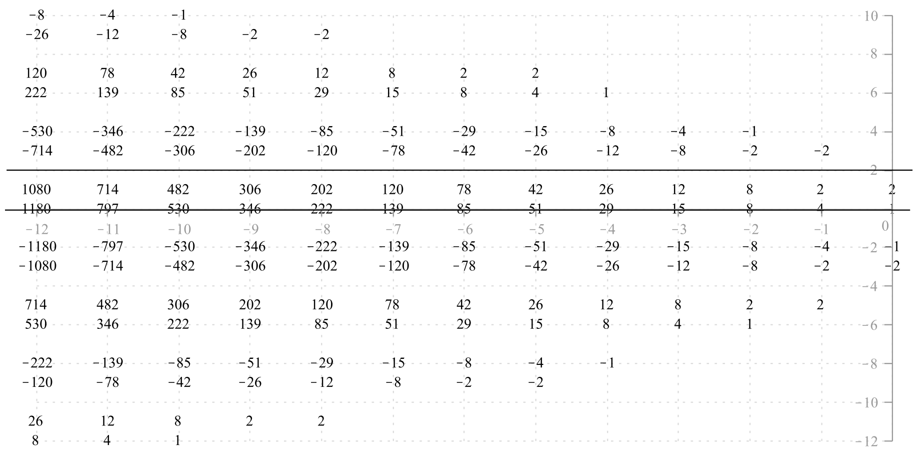


Figure 13. Anomalous branching coefficients for $\hat{A}_1 \subset \hat{B}_2$

Selecting the elements inside the Weyl chamber $\bar{C}^{(0)}_{\hat{A}_1}$ we get the branching coefficients up to the twelfth grade,

$$
\begin{aligned}
& L^{w_1}_{\hat{B}_2 \downarrow \hat{A}_1} = 2L^{w_1}_{\hat{A}_1} \oplus 1L^{w_0}_{\hat{A}_1} \oplus 4L^{w_0-\delta}_{\hat{A}_1} \oplus \\
& 2L^{w_1-\delta}_{\hat{A}_1} \oplus 8L^{w_0-2\delta}_{\hat{A}_1} \oplus 8L^{w_1-2\delta}_{\hat{A}_1} \oplus 15L^{w_0-3\delta}_{\hat{A}_1} \oplus \\
& 12L^{w_1-3\delta}_{\hat{A}_1} \oplus 26L^{w_1-4\delta}_{\hat{A}_1} \oplus 29L^{w_0-4\delta}_{\hat{A}_1} \oplus 51L^{w_0-5\delta}_{\hat{A}_1} \oplus \\
& 42L^{w_1-5\delta}_{\hat{A}_1} \oplus 78L^{w_1-6\delta}_{\hat{A}_1} \oplus 85L^{w_0-6\delta}_{\hat{A}_1} \oplus 120L^{w_1-7\delta}_{\hat{A}_1} \oplus \\
& 139L^{w_0-7\delta}_{\hat{A}_1} \oplus 202L^{w_1-8\delta}_{\hat{A}_1} \oplus 222L^{w_0-8\delta}_{\hat{A}_1} \oplus 306L^{w_1-9\delta}_{\hat{A}_1} \oplus \\
& 346L^{w_0-9\delta}_{\hat{A}_1} \oplus 530L^{w_0-10\delta}_{\hat{A}_1} \oplus 482L^{w_1-10\delta}_{\hat{A}_1} \oplus 714L^{w_1-11\delta}_{\hat{A}_1} \oplus \\
& 797L^{w_0-11\delta}_{\hat{A}_1} \oplus 1080L^{w_1-12\delta}_{\hat{A}_1} \oplus 1180L^{w_0-12\delta}_{\hat{A}_1}
\end{aligned}
$$

This result can be presented as the set of branching functions [4]:

$$
\begin{aligned}
b^{(w_1)}_0 = & \; 1 + 4\,q^1 + 8\,q^2 + 15\,q^3 + 29\,q^4 + 51\,q^5 + 85\,q^6 + 139\,q^7 + \\
& 222\,q^8 + 346\,q^9 + 530\,q^{10} + 797\,q^{11} + 1180\,q^{12} + \ldots \\
b^{(w_1)}_1 = & \; 2 + 2\,q^1 + 8\,q^2 + 12\,q^3 + 26\,q^4 + 42\,q^5 + 78\,q^6 + 120\,q^7 + \\
& 202\,q^8 + 306\,q^9 + 482\,q^{10} + 714\,q^{11} + 1080\,q^{12} + \ldots
\end{aligned}
$$

Here $q = \exp(2\pi i \tau)$, the lower index enumerates the branching functions according to their highest weights in $P^+_{\hat{A}_1}$, these are the fundamental weights $w_0 = \lambda_0 = (0,1,0)$, $w_1 = \alpha/2 = (1,1,0)$.

Now we can use the relation 14 to get the expansion of the B_2/A_1-coset characters:

$$
\begin{aligned}
\chi^{(w_1)}_1(q) = & \; q^{\frac{7}{12}}\,(2 + 2\,q^1 + 8\,q^2 + 12\,q^3 + 26\,q^4 + 42\,q^5 + 78\,q^6 + 120\,q^7 + \\
& 202\,q^8 + 306\,q^9 + 482\,q^{10} + 714\,q^{11} + 1080\,q^{12} + \ldots) \\
\chi^{(w_1)}_0(q) = & \; q^{\frac{5}{6}}\,(1 + 4\,q^1 + 8\,q^2 + 15\,q^3 + 29\,q^4 + 51\,q^5 + 85\,q^6 + 139\,q^7 + \\
& 222\,q^8 + 346\,q^9 + 530\,q^{10} + 797\,q^{11} + 1180\,q^{12} + \ldots)
\end{aligned}
$$

Further amelioration of the algorithm can be achieved by using the folded fan technique [28] to get the explicit expression for the branching functions and the corresponding coset characters in conformal field theory.

5. Conclusion

We have shown that the injection fan technique can be used to deal with the nonmaximal subalgebras. It was demonstrated that for such subalgebras an auxiliary subset $\Delta_{\perp}^{+}$ must be extracted from the set of positive roots $\Delta_{\mathfrak{g}}^{+}$. The role of the subset $\Delta_{\perp}^{+}$ is to modify both the injection fan (formed here by the weights $\left(\Delta_{\mathfrak{g}}^{+} \setminus \Delta_{\mathfrak{a}}^{+}\right) \setminus \Delta_{\perp}^{+}$) and the anomalous weights of the initial module. This modification reduces to a simple procedure: the anomalous weights multiplicities are to be substituted by the dimensions of the corresponding $\mathfrak{a}_{\perp}$-modules.

We have demonstrated the efficiency of the proposed generalizations of the injection fan algorithm. Its possible applications to some physical problems were discussed. In particular we considered the construction of modular-invariant partition functions in the framework of conformal embedding method and the coset construction in the rational conformal field theory. This construction is useful in the study of WZW-models emerging in the context of the AdS/CFT correspondence [21, 22, 23].

The presented technique to the reduction problem can also be applied in the study of integrable spin-chains.

6. Acknowledgements

The work was supported in part by RFFI grant N 09-01-00504 and the National Project RNP.2.1.1./1575.

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