Recursive algorithms and branching for nonmaximal embeddings

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Abstract. Recurrent relations for branching coefficients in affine Lie algebras integrable highest weight modules are studied. The decomposition algorithm based on the injection fan technique is developed for the case of reductive subalgebra. In particular we consider the situations where the Weyl denominator becomes singular with respect to the subalgebra. We study the modifications of the injection fan technique and demonstrate that for any reductive subalgebra it is possible to define the injection fan – the tool that describes explicitly the recurrent properties of branching coefficients. Possible applications of subtracted fans in CFT models are considered.

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1. Introduction

The branching problem for affine Lie algebras emerges in conformal field theory, for example, in the construction of modular-invariant partition functions [1]. Recently the problem of conformal embeddings was considered in the paper [2].

There exist different approaches to deal with the branching coefficients. Some of them use the BGG resolution [3] (for Kac-Moody algebras the algorithm is described in [4],[5]), the Schur function series [6], the BRST cohomology [7], Kac-Peterson formulas [4, 8] or the combinatorial methods applied in [9].

Usually only the maximal reductive subalgebras $\mathfrak{a} \subset \mathfrak{g}$ are considered since the case of non-maximal subalgebra can be obtained using the chain of maximal injections. In this paper we find out that the recurrent properties for branching coefficients can be explicitly formulated for an arbitrary reductive subalgebra. The principal point is to consider the subalgebra $\mathfrak{a}$ together with its counterpart $\mathfrak{a}_{\perp}$ "orthogonal" to $\mathfrak{a}$ with respect to the Killing form. For any reductive $\mathfrak{a}$ the subalgebra $\mathfrak{a}_{\perp} \subset \mathfrak{g}$ is a regular and reductive. For a highest weight module $L^{(\mu)}$ and orthogonal pair of subalgebras $(\mathfrak{a}, \mathfrak{a}_{\perp})$ we consider the so called singular element $\Psi^{(\mu)}$ (the numerator in the Weyl character formula $ch\left(L^{\mu}\right) = \frac{\Psi^{(\mu)}}{\Psi^{(0)}}$, see for example [10]) the Weyl denominator

$\Psi^{(0)}_{\mathfrak{a}_\perp}$ and the projection $\Psi^{(\mu)}_{(\mathfrak{a},\mathfrak{a}_\perp)} = \pi_{\mathfrak{a}} \frac{\Psi^{(\mu)}_{\mathfrak{g}}}{\Psi^{(0)}_{\mathfrak{a}_\perp}}$. We prove that for any highest weight $\mathfrak{h}$-diagonalizable module $L^{(\mu)}$ and orthogonal pair of subalgebras $(\mathfrak{a},\mathfrak{a}_\perp)$ the element $\Psi^{(\mu)}_{(\mathfrak{a},\mathfrak{a}_\perp)}$ has a decomposition with respect to the set of Weyl numerators $\Psi^{(\mu)}_{\mathfrak{a}_\perp}$ of $\mathfrak{a}_\perp$. This decomposition provides the possibility to construct the recurrent property for the branching coefficients corresponding to the injection $\mathfrak{a} \longrightarrow \mathfrak{g}$. This property is formulated in terms of a specific element of the group algebra $\mathcal{E}(\mathfrak{g})$ called "the injection fan". Using this tool we formulate a simple and explicit algorithm for branching coefficients computations applicable for an arbitrary (maximal or non-maximal) subalgebras of finite-dimensional and affine Lie algebras. In the case of maximal embedding the corresponding fan becomes unsubtracted and the recurrent relations described earlier in [11] are reobtained.

We demonstrate that our algorithm can be used in studies of conformal embeddings and coset constructions in rational conformal field theory.

The paper is organised as follows. In the subsection 1.1 we fix the general notations. In the Section 2 we derive the decomposition formula for the subtracted recurrent formula for anomalous branching coefficients and describe the decomposition algorithm for integrable highest weight modules $L_{\mathfrak{g}}$ with respect to a reductive subalgebra $\mathfrak{a} \subset \mathfrak{g}$ (subsection 2.5). In the Section 3 we present several simple examples for finite-dimensional Lie algebras. The affine Lie algebras and their applications in CFT models are considered in Section 4. Possible further developments are discussed (Section 5).

1.1. Notation

Consider affine Lie algebras $\mathfrak{g}$ and $\mathfrak{a}$ with the underlying finite-dimensional subalgebras $\overset{\circ}{\mathfrak{g}}$ and $\overset{\circ}{\mathfrak{a}}$ and an injection $\mathfrak{a} \longrightarrow \mathfrak{g}$ such that $\mathfrak{a}$ is a reductive subalgebra $\mathfrak{a} \subset \mathfrak{g}$ with correlated root spaces: $\mathfrak{h}^*_{\mathfrak{a}} \subset \mathfrak{h}^*_{\mathfrak{g}}$ and $\mathfrak{h}^*_{\overset{\circ}{\mathfrak{a}}} \subset \mathfrak{h}^*_{\overset{\circ}{\mathfrak{g}}}$. We use the following notations:

L^{μ} ($L^{\nu}_{\mathfrak{a}}$) — the integrable module of $\mathfrak{g}$ with the highest weight μ ; (resp. integrable $\mathfrak{a}$ -module with the highest weight ν);

r , ($r_{\mathfrak{a}}$) — the rank of the algebra $\mathfrak{g}$ (resp. $\mathfrak{a}$) ;

Δ ($\Delta_{\mathfrak{a}}$)— the root system; Δ^+ (resp. $\Delta^+_{\mathfrak{a}}$)— the positive root system (of $\mathfrak{g}$ and $\mathfrak{a}$ respectively);

$\mathrm{mult}(\alpha)$ ($\mathrm{mult}_{\mathfrak{a}}(\alpha)$) — the multiplicity of the root α in Δ (resp. in ($\Delta_{\mathfrak{a}}$));

$\overset{\circ}{\Delta}$, $\left(\overset{\circ}{\Delta}_{\mathfrak{a}}\right)$ — the finite root system of the subalgebra $\overset{\circ}{\mathfrak{g}}$ (resp. $\overset{\circ}{\mathfrak{a}}$);

$\mathcal{N}^{\mu}$, ($\mathcal{N}^{\nu}_{\mathfrak{a}}$) — the weight diagram of L^{μ} (resp. $L^{\nu}_{\mathfrak{a}}$) ;

W , ($W_{\mathfrak{a}}$)— the corresponding Weyl group;

C , ($C_{\mathfrak{a}}$)— the fundamental Weyl chamber;

$\bar{C}$, $\left(\bar{C}_{\mathfrak{a}}\right)$ — the closure of the fundamental Weyl chamber;

ρ , ($\rho_{\mathfrak{a}}$) — the Weyl vector;

$\epsilon(w) := \det(w)$;

α_i , $\left(\alpha_{(\mathfrak{a})j}\right)$ — the i-th (resp. j-th) basic root for $\mathfrak{g}$ (resp. $\mathfrak{a}$); $i = 0, \ldots, r$, ($j = 0, \ldots, r_{\mathfrak{a}}$);

δ — the imaginary root of $\mathfrak{g}$ (and of $\mathfrak{a}$ if any);

$\alpha_i^\vee$, $\left(\alpha_{(\mathfrak{a})j}^\vee\right)$— the basic coroot for $\mathfrak{g}$ (resp. $\mathfrak{a}$) , $i = 0, \ldots, r$; $(j = 0, \ldots, r_{\mathfrak{a}})$;

$\overset{\circ}{\xi}$, $\overset{\circ}{\xi}_{(\mathfrak{a})}$ — the finite (classical) part of the weight $\xi \in P$, $\left(\text{resp. } \xi_{(\mathfrak{a})} \in P_{\mathfrak{a}}\right)$;

$\lambda = \left(\overset{\circ}{\lambda}; k; n\right)$ — the decomposition of an affine weight indicating the finite part $\overset{\circ}{\lambda}$, level k and grade n;

P (resp. $P_{\mathfrak{a}}$) — the weight lattice;

$m_\xi^{(\mu)}$, $\left(m_\xi^{(\nu)}\right)$ — the multiplicity of the weight $\xi \in P$ (resp. $\in P_{\mathfrak{a}}$) in the module L^μ , (resp. $\xi \in L_{\mathfrak{a}}^\nu$);

$ch\left(L^\mu\right)$ (resp. $ch\left(L_{\mathfrak{a}}^\nu\right)$)— the formal character of L^μ (resp. $L_{\mathfrak{a}}^\nu$);

$ch\left(L^\mu\right) = \frac{\sum_{w \in W} \epsilon(w) e^{w \circ (\mu+\rho)-\rho}}{\prod_{\alpha \in \Delta^+}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}$ — the Weyl-Kac formula;

$R := \prod_{\alpha \in \Delta^+} \left(1 - e^{-\alpha}\right)^{\mathrm{mult}(\alpha)}$ $\left(\text{resp. } R_{\mathfrak{a}} := \prod_{\alpha \in \Delta_{\mathfrak{a}}^+} \left(1 - e^{-\alpha}\right)^{\mathrm{mult}_{\mathfrak{a}}(\alpha)}\right)$— the denominator.

2. Recurrent relations for branching coefficients.

Consider the integrable module L^μ of $\mathfrak{g}$ with the highest weight μ and let $\mathfrak{a} \subset \mathfrak{g}$ be a reductive subalgebra of $\mathfrak{g}$. With respect to $\mathfrak{a}$ the module L^μ is completely reducible,

$$L_{\mathfrak{g}\downarrow\mathfrak{a}}^\mu = \bigoplus_{\nu \in P_{\mathfrak{a}}^+} b_\nu^{(\mu)} L_{\mathfrak{a}}^\nu.$$

Using the projection operator $\pi_{\mathfrak{a}}$ (to the weight space $\mathfrak{h}_{\mathfrak{a}}{}^*$) one can rewrite this decomposition in terms of formal characters:

$$\pi_{\mathfrak{a}} \circ ch\left(L^\mu\right) = \sum_{\nu \in P_{\mathfrak{a}}^+} b_\nu^{(\mu)} ch\left(L_{\mathfrak{a}}^\nu\right). \tag{1}$$

We are interested in branching coefficients $b_\nu^{(\mu)}$.

2.1. Orthogonal subalgebra and injection fan.

In this subsection we shall introduce some simple constructions that will be used in our studies of branching and in particular the "orthogonal partner" $\mathfrak{a}_\perp$ for a reductive subalgebra $\mathfrak{a}$ in $\mathfrak{g}$.

In the Weyl-Kac formula both numerator and denominator can be considered as formal elements containing the singular weights of the Verma modules V^ξ with the highest weights $\xi = \mu$ and $\xi = 0$ [10]. We attribute singular numerators to the corresponding integrable modules L^μ and $L_{\mathfrak{a}}^\nu$:

$$\Psi^{(\mu)} := \sum_{w \in W} \epsilon(w) e^{w \circ (\mu+\rho)-\rho},$$

$$\Psi_{\mathfrak{a}}^{(\nu)} := \sum_{w \in W_{\mathfrak{a}}} \epsilon(w) e^{w \circ (\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}.$$

and use the Weyl-Kac formula in the form

$$ch\left(L^{\mu}\right)=\frac{\Psi^{(\mu)}}{\Psi^{(0)}}=\frac{\Psi^{(\mu)}}{R}. \tag{2}$$

Applying formula (2) to the branching rule (1) we get the relation connecting the singular elements $\Psi^{(\mu)}$ and $\Psi_{\mathfrak{a}}^{(\nu)}$:

$$\begin{aligned}
\pi_{\mathfrak{a}}\left(\frac{\sum_{w\in W}\epsilon(w)e^{w(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^{+}}\left(1-e^{-\alpha}\right)^{\mathrm{mult}(\alpha)}}\right) &= \sum_{\nu\in P_{\mathfrak{a}}^{+}} b_{\nu}^{(\mu)}\frac{\sum_{w\in W_{\mathfrak{a}}}\epsilon(w)e^{w(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}}{\prod_{\beta\in\Delta_{\mathfrak{a}}^{+}}\left(1-e^{-\beta}\right)^{\mathrm{mult}_{\mathfrak{a}}(\beta)}},\\
\pi_{\mathfrak{a}}\left(\frac{\Psi^{(\mu)}}{R}\right) &= \sum_{\nu\in P_{\mathfrak{a}}^{+}} b_{\nu}^{(\mu)}\frac{\Psi_{\mathfrak{a}}^{(\nu)}}{R_{\mathfrak{a}}}.
\end{aligned} \tag{3}$$

Here $\Delta_{\mathfrak{a}}^{+}$ is the set of positive roots of the subalgebra $\mathfrak{a}$ (without loss of generality we consider them as vectors from the positive root space $\mathfrak{h}^{*+}$ of $\mathfrak{g}$).

Consider the root subspace $\mathfrak{h}_{\perp\mathfrak{a}}^{*}$ orthogonal to the roots in $\Delta_{\mathfrak{a}}$,

$$\mathfrak{h}_{\perp\mathfrak{a}}^{*}=:\left\{\eta\in\mathfrak{h}^{*}|\forall\alpha\in\Delta_{\mathfrak{a}};\alpha\perp\eta\right\},$$

and the roots (correspondingly – positive roots) of $\mathfrak{g}$ orthogonal to $\Delta_{\mathfrak{a}}$,

$$\begin{aligned}
\Delta_{\mathfrak{a}_{\perp}} &: =\left\{\beta\in\Delta_{\mathfrak{g}}|\forall\alpha\in\Delta_{\mathfrak{a}};\alpha\perp\beta\right\},\\
\Delta_{\mathfrak{a}_{\perp}}^{+} &: =\left\{\beta^{+}\in\Delta_{\mathfrak{g}}^{+}|\forall\alpha^{+}\in\Delta_{\mathfrak{a}}^{+};\alpha^{+}\perp\beta^{+}\right\}.
\end{aligned}$$

Let $W_{\mathfrak{a}_{\perp}}$ be the subgroup of W generated by the reflections w_{β} for the roots $\beta\in\Delta_{\mathfrak{a}_{\perp}}^{+}$. The subsystem $\Delta_{\mathfrak{a}_{\perp}}$ determines the subalgebra $\mathfrak{a}_{\perp}$ with the Cartan subalgebra $\mathfrak{h}_{\mathfrak{a}_{\perp}}$. Let

$$\mathfrak{h}_{\perp}^{*}:=\left\{\eta\in\mathfrak{h}_{\perp\mathfrak{a}}^{*}|\forall\alpha\in\Delta_{\mathfrak{a}}\cup\Delta_{\mathfrak{a}_{\perp}};\alpha\perp\eta\right\}$$

and consider the subalgebras

$$\begin{aligned}
\widetilde{\mathfrak{a}_{\perp}} &: =\mathfrak{a}_{\perp}\oplus\mathfrak{h}_{\perp}\\
\widetilde{\mathfrak{a}} &: =\mathfrak{a}\oplus\mathfrak{h}_{\perp}.
\end{aligned}$$

Algebras $\mathfrak{a}$ and $\mathfrak{a}_{\perp}$ form the "orthogonal pair" $(\mathfrak{a},\mathfrak{a}_{\perp})$ of subalgebras in $\mathfrak{g}$.

For the Cartan subalgebras we have the decomposition

$$\mathfrak{h}=\mathfrak{h}_{\mathfrak{a}}\oplus\mathfrak{h}_{\mathfrak{a}_{\perp}}\oplus\mathfrak{h}_{\perp}=\mathfrak{h}_{\widetilde{\mathfrak{a}}}\oplus\mathfrak{h}_{\mathfrak{a}_{\perp}}=\mathfrak{h}_{\widetilde{\mathfrak{a}_{\perp}}}\oplus\mathfrak{h}_{\mathfrak{a}}. \tag{4}$$

For the subalgebras of an orthogonal pair $(\mathfrak{a},\mathfrak{a}_{\perp})$ we consider the corresponding Weyl vectors, $\rho_{\mathfrak{a}}$ and $\rho_{\mathfrak{a}_{\perp}}$, and form the so called "defects" $\mathcal{D}_{\mathfrak{a}}$ and $\mathcal{D}_{\mathfrak{a}_{\perp}}$ of the injection:

$$\mathcal{D}_{\mathfrak{a}}:=\rho_{\mathfrak{a}}-\pi_{\mathfrak{a}}\rho, \tag{5}$$

$$\mathcal{D}_{\mathfrak{a}_{\perp}}:=\rho_{\mathfrak{a}_{\perp}}-\pi_{\mathfrak{a}_{\perp}}\circ\rho. \tag{6}$$

For the highest weight module $L_{\mathfrak{g}}^{\mu}$ consider the singular weights $\left\{\left(w(\mu+\rho)-\rho\right)|w\in W\right\}$ and their projections to $h_{\widetilde{\mathfrak{a}_{\perp}}}^{*}$ (additionally shifted by the defect $-\mathcal{D}_{\mathfrak{a}_{\perp}}$):

$$\mu_{\widetilde{\mathfrak{a}_{\perp}}}(w):=\pi_{\widetilde{\mathfrak{a}_{\perp}}}\circ\left[w(\mu+\rho)-\rho\right]-\mathcal{D}_{\mathfrak{a}_{\perp}},\quad w\in W.$$

Among the weights $\left\{\mu_{\widetilde{\mathfrak{a}_{\perp}}}(w)|w\in W\right\}$ choose those located in the fundamental chamber $\overline{C_{\widetilde{\mathfrak{a}_{\perp}}}}$ and let U be the set of representatives u for the classes $W/W_{\mathfrak{a}_{\perp}}$ such that

$$U:=\left\{u\in W|\quad\mu_{\widetilde{\mathfrak{a}_{\perp}}}(u)\in\overline{C_{\widetilde{\mathfrak{a}_{\perp}}}}\right\}\quad. \tag{7}$$

For the same set U introduce the weights

$$\mu_{\mathfrak{a}}(u) := \pi_{\mathfrak{a}} \circ [u(\mu+\rho)-\rho] + \mathcal{D}_{\mathfrak{a}_\perp}.$$

To simplify the form of relations we shall now on omit the sign "$\circ$" in projected weights.

To describe the recurrent properties for branching coefficients $b_\nu^{(\mu)}$ we shall use the technique elaborated in [11]. One of the main tools is the set of weights $\Gamma_{\mathfrak{a}\subset\mathfrak{g}}$ called the injection fan. As far as we consider more general situation (where the injection is not maximal) the notion of the injection fan is modified:

Definition 1. For the product

$$\prod_{\alpha\in\Delta^+\setminus\Delta_\perp^+} \left(1-e^{-\pi_{\mathfrak{a}}\alpha}\right)^{\mathrm{mult}(\alpha)-\mathrm{mult}_{\mathfrak{a}}(\pi_{\mathfrak{a}}\alpha)} = -\sum_{\gamma\in P_{\mathfrak{a}}} s(\gamma)e^{-\gamma} \tag{8}$$

consider the carrier $\Phi_{\mathfrak{a}\subset\mathfrak{g}} \subset P_{\mathfrak{a}}$ of the function $s(\gamma) = \det(\gamma)$:

$$\Phi_{\mathfrak{a}\subset\mathfrak{g}} = \{\gamma \in P_{\mathfrak{a}} | s(\gamma) \neq 0\} \tag{9}$$

The ordering of roots in $\overset{\circ}{\Delta}_{\mathfrak{a}}$ induce the natural ordering of the weights in $P_{\mathfrak{a}}$. Denote by γ_0 the lowest vector of $\Phi_{\mathfrak{a}\subset\mathfrak{g}}$. The set

$$\Gamma_{\mathfrak{a}\subset\mathfrak{g}} = \{\xi-\gamma_0 | \xi \in \Phi_{\mathfrak{a}\subset\mathfrak{g}}\} \setminus \{0\} \tag{10}$$

is called the *injection fan*.

In the next subsection we shall see how the injection fan defines the recurrent properties of branching coefficients. It must be noticed that the injection fan is the universal instrument that depend only on the injection.

2.2. Decomposing the singular element.

Now we shall prove that the Weyl-Kac character formula (in terms of singular elements) describes the particular case of a more general relation:

Lemma 1. *Let* $(\mathfrak{a}, \mathfrak{a}_\perp)$ *be the orthogonal pair of reductive subalgebras in* $\mathfrak{g}$*, with* $\widetilde{\mathfrak{a}_\perp} = \mathfrak{a}_\perp \oplus \mathfrak{h}_\perp$ *and* $\widetilde{\mathfrak{a}} = \mathfrak{a} \oplus \mathfrak{h}_\perp$ *,*

L^μ be the highest weight module with the singular element $\Psi^{(\mu)}$,

$R_{\mathfrak{a}_\perp}$ be the Weyl denominator for $\mathfrak{a}_\perp$.

Then the element $\pi_{\mathfrak{a}}\left(\frac{\Psi_{\mathfrak{g}}^\mu}{R_{\mathfrak{a}_\perp}}\right)$ can be decomposed into the sum over $u \in U$ (see (7)) of the singular weights $e^{\mu_{\mathfrak{a}}(u)}$ with the coefficients $\epsilon(u)\mathrm{dim}\left(L_{\widetilde{\mathfrak{a}_\perp}}^{\mu_{\widetilde{\mathfrak{a}_\perp}}(u)}\right)$:

$$\pi_{\mathfrak{a}}\left(\frac{\Psi^\mu}{R_{\mathfrak{a}_\perp}}\right) = \sum_{u\in U} \epsilon(u)\mathrm{dim}\left(L_{\widetilde{\mathfrak{a}_\perp}}^{\mu_{\widetilde{\mathfrak{a}_\perp}}(u)}\right) e^{\mu_{\mathfrak{a}}(u)}. \tag{11}$$

Proof. With $u \in U$ and $v \in W_{\mathfrak{a}_\perp}$ perform the decomposition

$$u(\mu+\rho) = \pi_{(\mathfrak{a})}u(\mu+\rho) + \pi_{(\widetilde{\mathfrak{a}_\perp})}u(\mu+\rho)$$

for the singular weight $vu(\mu+\rho)-\rho$:

$$\begin{aligned} vu(\mu+\rho)-\rho \;=\;& \pi_{(\mathfrak{a})}\left(u(\mu+\rho)\right)-\rho+\rho_{\mathfrak{a}_\perp}+\pi_{(\mathfrak{h}_\perp)}\rho \\ &+v\left(\pi_{(\widetilde{\mathfrak{a}_\perp})}u(\mu+\rho)-\rho_{\mathfrak{a}_\perp}+\rho_{\mathfrak{a}_\perp}\right)-\rho_{\mathfrak{a}_\perp}-\pi_{(\mathfrak{h}_\perp)}\rho. \end{aligned} \tag{12}$$

Use the defect $\mathcal{D}_{\mathfrak{a}_\perp}$ (6) to simplify the first summand in (12):

$$\begin{aligned} \pi_{(\mathfrak{a})}\left(u(\mu+\rho)\right)-\rho+\rho_{\mathfrak{a}_\perp}+\pi_{(\mathfrak{h}_\perp)}\rho &= \\ \pi_{(\mathfrak{a})}\left(u(\mu+\rho)\right)-\pi_{\mathfrak{a}}\rho-\pi_{\mathfrak{a}_\perp}\rho+\rho_{\mathfrak{a}_\perp} &= \\ \pi_{(\mathfrak{a})}\left(u(\mu+\rho)-\rho\right)+\mathcal{D}_{\mathfrak{a}_\perp}&, \end{aligned}$$

and the second one:

$$\begin{aligned} v\left(\pi_{(\widetilde{\mathfrak{a}_\perp})}u(\mu+\rho)-\rho_{\mathfrak{a}_\perp}+\rho_{\mathfrak{a}_\perp}\right)-\rho_{\mathfrak{a}_\perp}-\pi_{(\mathfrak{h}_\perp)}\rho &= \\ v\left(\pi_{(\widetilde{\mathfrak{a}_\perp})}u(\mu+\rho)-\mathcal{D}_{\mathfrak{a}_\perp}-\pi_{(\mathfrak{a}_\perp)}\rho-\pi_{(\mathfrak{h}_\perp)}\rho+\rho_{\mathfrak{a}_\perp}\right)-\rho_{\mathfrak{a}_\perp} &= \\ v\left(\pi_{(\widetilde{\mathfrak{a}_\perp})}\left[u(\mu+\rho)-\rho\right]-\mathcal{D}_{\mathfrak{a}_\perp}+\rho_{\mathfrak{a}_\perp}\right)-\rho_{\mathfrak{a}_\perp}&. \end{aligned}$$

These expressions provide a kind of a factorization in the anomalous element Ψ^{μ} and we find in it the combination of anomalous elements $\Psi^{\eta}_{\widetilde{\mathfrak{a}_\perp}}$ of the subalgebra $\widetilde{\mathfrak{a}_\perp}$-modules $L^{\eta}_{\widetilde{\mathfrak{a}_\perp}}$:

$$\begin{aligned} &\Psi^{\mu}=\sum_{u\in U}\sum_{v\in W_{\mathfrak{a}_\perp}}\epsilon(v)\epsilon(u)e^{vu(\mu+\rho)-\rho}= \\ &=\sum_{u\in U}\epsilon(u)e^{\pi_{\mathfrak{a}}[u(\mu+\rho)-\rho]+\mathcal{D}_{\mathfrak{a}_\perp}}\sum_{v\in W_{\mathfrak{a}_\perp}}\epsilon(v)e^{v\left(\pi_{(\widetilde{\mathfrak{a}_\perp})}[u(\mu+\rho)-\rho]-\mathcal{D}_{\mathfrak{a}_\perp}+\rho_{\mathfrak{a}_\perp}\right)-\rho_{\mathfrak{a}_\perp}}= \\ &=\sum_{u\in U}\epsilon(u)e^{\pi_{(\mathfrak{a})}[u(\mu+\rho)-\rho]+\mathcal{D}_{\mathfrak{a}_\perp}}\Psi_{\widetilde{\mathfrak{a}_\perp}}^{\pi_{(\widetilde{\mathfrak{a}_\perp})}[u(\mu+\rho)-\rho]-\mathcal{D}_{\mathfrak{a}_\perp}} \end{aligned}$$

Dividing both sides by the Weyl element $R_{\mathfrak{a}_\perp}=\prod_{\beta\in\Delta_{\mathfrak{a}_\perp}}(1-e^{-\beta})^{\mathrm{mult}(\beta)}$ and projecting them to the weight space $h^*_{\mathfrak{a}}$ we obtain the desired relation:

$$\begin{aligned} \pi_{\mathfrak{a}}\left(\frac{\Psi^{\mu}_{\mathfrak{g}}}{R_{\mathfrak{a}_\perp}}\right) &= \sum_{u\in W/W_{\mathfrak{a}_\perp}}\epsilon(u)e^{\pi_{\mathfrak{a}}[u(\mu+\rho)-\rho]}\pi_{\mathfrak{a}}\left(\frac{\Psi_{\widetilde{\mathfrak{a}_\perp}}^{\pi_{(\widetilde{\mathfrak{a}_\perp})}[u(\mu+\rho)-\rho]-\mathcal{D}_{\mathfrak{a}_\perp}}}{\prod_{\beta\in\Delta_{\mathfrak{a}_\perp}}(1-e^{-\beta})^{\mathrm{mult}(\beta)}}\right) \\ &= \sum_{u\in U}\epsilon(u)\mathrm{dim}\left(L_{\widetilde{\mathfrak{a}_\perp}}^{\mu_{\widetilde{\mathfrak{a}_\perp}}(u)}\right)e^{\pi_{\mathfrak{a}}[u(\mu+\rho)-\rho]}. \end{aligned}$$

□

Remark 1. This relation can be considered a generalized form of the Weyl formula for singular element $\Psi^{\mu}_{\mathfrak{g}}$: the vectors $\mu_{\mathfrak{a}}(u)$ play the role of singular weights while instead of the determinants $\epsilon(u)$ we have the products $\epsilon(u)\mathrm{dim}\left(L_{\widetilde{\mathfrak{a}_\perp}}^{\mu_{\widetilde{\mathfrak{a}_\perp}}(u)}\right)$. In fact when $\mathfrak{a}=\mathfrak{g}$ both $\mathfrak{a}_\perp$ and $\mathfrak{h}_\perp$ are trivial, $U=W$, and the original Weyl formula is easily reobtained.

2.3. Constructing recurrent relations.

Consider the right-hand side of relation (3). The numerator there describes the branching in terms of singular elements and it is reasonable to expand it as an element of $\mathcal{E}\left(\mathfrak{g}\right)$:

$$\sum_{\nu\in\bar{C}_{\mathfrak{a}}}b^{(\mu)}_{\nu}\Psi^{(\nu)}_{(\mathfrak{a})}=\sum_{\lambda\in P_{\mathfrak{a}}}k^{(\mu)}_{\lambda}e^{\lambda}. \tag{13}$$

Here the coefficients $k_\lambda^{(\mu)}$ are integer and their signes depend on the length (see [10]) of the Weyl group elements in $\Psi^{(\nu)}_{(\mathfrak{a})}$. The important property of $k_\lambda^{(\mu)}$'s is that they coinside with the branching coefficients for all weights ν inside the main Weil chamber:

$$b_\nu^{(\mu)} = k_\nu^{(\mu)} \text{ for } \nu \in \bar{C}_{\mathfrak{a}}. \tag{14}$$

We call the coefficients k_λ — the anomalous branching coefficients (see also [11]).

Now we can state the main theorem which gives us an instrument for the recurrent computation of branching coefficients.

Theorem 1. *For the anomalous branching coefficients $k_\nu^{(\mu)}$ (13) the following relation holds*

$$\begin{aligned} k_\xi^{(\mu)} = -\tfrac{1}{s(\gamma_0)} \Big(& \textstyle\sum_{u\in U} \epsilon(u)\ \dim\left(L_{\widetilde{\mathfrak{a}_\perp}}^{\widetilde{\mu_{\mathfrak{a}_\perp}}(u)}\right) \delta_{\xi-\gamma_0,\pi_{\mathfrak{a}}(u(\mu+\rho)-\rho)} + \\ & + \textstyle\sum_{\gamma\in\Gamma_{\mathfrak{a}\longrightarrow\mathfrak{g}}} s\left(\gamma+\gamma_0\right) k_{\xi+\gamma}^{(\mu)} \Big). \end{aligned} \tag{15}$$

Proof. Redress the relation (3) for the element $\frac{\Psi_{\mathfrak{g}}^{\mu}}{R_{\mathfrak{a}_\perp}}$ using defenition (9) of the carrier $\Phi_{\mathfrak{a}\subset\mathfrak{g}}$,

$$\begin{aligned} &\pi_{\mathfrak{a}}\left(\frac{\Psi_{\mathfrak{g}}^{\mu}}{R_{\mathfrak{a}_\perp}}\right) = \\ &= \textstyle\prod_{\alpha\in\Delta^+\setminus\Delta_\perp^+} \left(1-e^{-\pi_{\mathfrak{a}}\alpha}\right)^{\mathrm{mult}(\alpha)-\mathrm{mult}_{\mathfrak{a}}(\pi_{\mathfrak{a}}\alpha)} \left(\sum_{\nu\in P_{\mathfrak{a}}^+} b_\nu^{(\mu)} \sum_{w\in W_{\mathfrak{a}}} \epsilon(w) e^{w(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}\right) = \\ &= -\textstyle\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}} s(\gamma) e^{-\gamma} \left(\sum_{\nu\in P_{\mathfrak{a}}^+, w\in W_{\mathfrak{a}}} \epsilon(w) b_\nu^{(\mu)} e^{w(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}\right). \end{aligned}$$

Then expand the sum in brackets (with respect to the formal basis in $\mathcal{E}$):

$$\pi_{\mathfrak{a}}\left(\frac{\Psi_{\mathfrak{g}}^{\mu}}{R_{\mathfrak{a}_\perp}}\right) = -\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}} s(\gamma) e^{-\gamma} \sum_{\lambda\in P_{\mathfrak{a}}} k_\nu^{(\mu)} e^{\lambda} = -\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}} \sum_{\lambda\in P_{\mathfrak{a}}} s(\gamma) k_\nu^{(\mu)} e^{\lambda-\gamma}.$$

Substitute the expression obtained in the Lemma (in the left-hand side),

$$\begin{aligned} \pi_{\mathfrak{a}}\left(\frac{\Psi_{\mathfrak{g}}^{\mu}}{R_{\mathfrak{a}_\perp}}\right) &= \sum_{u\in U} \epsilon(u) e^{\pi_{\mathfrak{a}}(\mu_{\mathfrak{a}}(u))} \dim\left(L_{\widetilde{\mathfrak{a}_\perp}}^{\widetilde{\mu_{\mathfrak{a}_\perp}}(u)}\right) \\ &= \sum_{u\in U} \epsilon(u) e^{\pi_{\mathfrak{a}}[u(\mu+\rho)-\rho]} \dim\left(L_{\widetilde{\mathfrak{a}_\perp}}^{\widetilde{\mu_{\mathfrak{a}_\perp}}(u)}\right) \\ &= -\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}} \sum_{\lambda\in P_{\mathfrak{a}}} s(\gamma) k_\nu^{(\mu)} e^{\lambda-\gamma}. \end{aligned}$$

The immediate consequence of this equality is:

$$\sum_{u\in U} \epsilon(u) \dim\left(L_{\widetilde{\mathfrak{a}_\perp}}^{\widetilde{\mu_{\mathfrak{a}_\perp}}(u)}\right) \delta_{\xi,\pi_{\mathfrak{a}}[u(\mu+\rho)-\rho]} + \sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}} s(\gamma)\, k_{\xi+\gamma}^{(\mu)} = 0, \quad \xi\in P_{\mathfrak{a}}. \tag{16}$$

The obtained formula means that the coefficients $k_{\xi+\gamma}^{(\mu)}$ for $\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}$ are not independent, they are subject to the linear relations and the form of these relations changes when the tested weight ξ coinsides with one of the "singular weights" $\left\{\pi_{\mathfrak{a}}\left[u(\mu+\rho)-\rho\right] | u\in U\right\}$. To conclude the proof we extract the lowest weight $\gamma_0\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}$ and pass to the summation over the vectors of the injection fan $\Gamma_{\mathfrak{a}\subset\mathfrak{g}}$ (see the definition 1). Thus we get the desired recurrent relation (15). $\square$

2.4. Embeddings and orthogonal pairs in simple Lie algebras

In this subsection we discuss some properties of "orthogonal pairs" of subalgebras in simple Lie algebras of classical series.

When both $\mathfrak{g}$ and $\mathfrak{a}$ are finite-dimensional all the regular embeddings can be obtained by a successive elimination of nodes in the extended Dynkin diagram of $\mathfrak{g}$ (and $\Delta^+_\perp = \emptyset$ if $\mathfrak{a}$ is maximal). For the classical series A, C and D when the regular injection $\mathfrak{a} \to \mathfrak{g}$ is thus fixed, the Dynkin diagram for $\mathfrak{a}_\perp$ is obtained from the extended diagram of $\mathfrak{g}$ by eliminating the subdiagram of $\mathfrak{a}$ and the adjacent nodes:

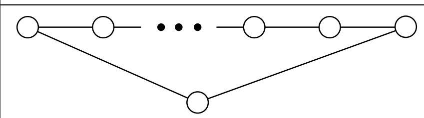
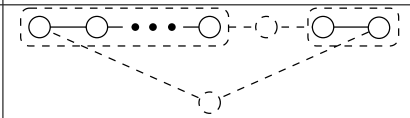
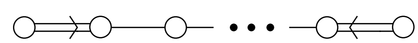
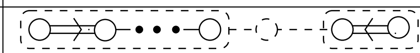
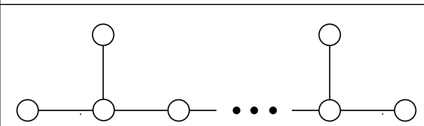
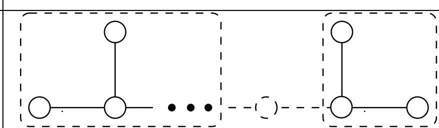
$\mathfrak{g}$	Extended diagram of $\mathfrak{g}$	Diagrams of the subalgebras $\mathfrak{a}$, $\mathfrak{a}_\perp$	
A_n			
C_n			
D_n			

Table 1. Subalgebras $\mathfrak{a}$, $\mathfrak{a}_\perp$ for the classical series

In the case of B series the situation is different. The reason is that here the subalgebra $\mathfrak{a}_\perp$ may be larger than the one obtained by elimination of the subdiagram of $\mathfrak{a}$ and the adjacent nodes. The subalgebras of the orthogonal pair, $\mathfrak{a}$ and $\mathfrak{a}_\perp$, must not form a direct sum in $\mathfrak{g}$. It can be directly checked that when $\mathfrak{g} = B_r$ and $\mathfrak{a} = B_{r_\mathfrak{a}}$ the orthogonal subalgebra is $\mathfrak{a}_\perp = B_{r-r_\mathfrak{a}}$. Consider the injection $B_{r_\mathfrak{a}} \to B_r, \quad 1 < r_\mathfrak{a} < r$. By eliminating the simple root $\alpha_{r_\mathfrak{a}-1} = e_{r_\mathfrak{a}-1} - e_{r_\mathfrak{a}}$ one splits the extended Dynkin diagram of B_r into the disjoint diagrams for $\mathfrak{a} = B_{r_\mathfrak{a}}$ and $D_{r-r_\mathfrak{a}}$. But the system $\Delta_{\mathfrak{a}_\perp}$ contains not only the simple roots $\{e_1 - e_2, e_2 - e_3, \dots, e_{r_\mathfrak{a}-2} - e_{r_\mathfrak{a}-1}, e_1 + e_2\}$ but also the root $e_{r_\mathfrak{a}-1}$. Thus $\Delta_{\mathfrak{a}_\perp}$ forms the subsystem of the type $B_{r-r_\mathfrak{a}}$ and the orthogonal pair for the injection $B_{r_\mathfrak{a}} \to B_r$ is $(B_{r_\mathfrak{a}}, B_{r-r_\mathfrak{a}})$. In the next Section the particular case of such orthogonal pair is presented for the injection $B_2 \to B_4$ (see Figure 3).

The complete classification of regular subalgebras for affine Lie algebras can be found in the recent paper [12]. From the complete classification of maximal special subalgebras in classical Lie algebras [13] we can deduce the following list of pairs of orthogonal subalgebras $\mathfrak{a}$, $\mathfrak{a}_\perp$:

$$\begin{array}{lll}
su(p) \oplus su(q) & \subset su(pq) & \\
so(p) \oplus so(q) & \subset so(pq) & \\
sp(2p) \oplus sp(2q) & \subset so(4pq) & \\
sp(2p) \oplus so(q) & \subset sp(2pq) & \\
so(p) \oplus so(q) & \subset so(p+q) & \text{for p and q odd.}
\end{array}$$

2.5. Algorithm for recursive computation of branching coefficients

The recurrent relation (15) allows us to formulate an algorithm for recursive computation of branching coefficients. In this algorithm there is no need to construct the module $L_{\mathfrak{g}}^{(\mu)}$ or any of the modules $L_{\mathfrak{a}}^{(\nu)}$.

It contains the following steps:

(i) Construct the root system $\Delta_{\mathfrak{a}}$ for the embedding $\mathfrak{a} \to \mathfrak{g}$.

(ii) Select the positive roots $\alpha \in \Delta^+$ orthogonal to the roots in $\Delta_{\mathfrak{a}}$ i.e. form the set $\Delta_{\perp}^+$.

(iii) Construct the set $\Gamma_{\mathfrak{a}\to\mathfrak{g}}$ (10).

(iv) Construct the set $\widehat{\Psi^{(\mu)}} = \{w(\mu+\rho)-\rho;\ w \in W\}$ of singular weights for the $\mathfrak{g}$-module $L^{(\mu)}$.

(v) Select the weights $\left\{\mu_{\widetilde{\mathfrak{a}_{\perp}}}(w) = \pi_{\widetilde{\mathfrak{a}_{\perp}}}[w(\mu+\rho)-\rho] - \mathcal{D}_{\mathfrak{a}_{\perp}} \in \overline{C_{\widetilde{\mathfrak{a}_{\perp}}}}\right\}$. Since the set $\Delta_{\perp}^+$ is fixed we can easily check wether the weight $\mu_{\widetilde{\mathfrak{a}_{\perp}}}(w)$ belongs to the main Weyl chamber $\overline{C_{\widetilde{\mathfrak{a}_{\perp}}}}$ (by computing its scalar product with the roots from $\Delta_{\perp}^+$).

(vi) For the weights $\mu_{\widetilde{\mathfrak{a}_{\perp}}}(w)$ calculate the dimensions of the corresponding modules $\dim\left(L_{\widetilde{\mathfrak{a}_{\perp}}}^{\mu_{\widetilde{\mathfrak{a}_{\perp}}}(u)}\right)$.

(vii) Calculate the anomalous branching coefficients in the main Weyl chamber $\overline{C_{\mathfrak{a}}}$ of the subalgebra $\mathfrak{a}$ using the recurrent relation (15).

When being interested in the branching coefficients for the embedding of a finite-dimensional Lie algebra into an affine Lie algebra we can construct the set of anomalous weights up to a required grade and use the steps 4-7 of the algorithm for each grade. We can also speed up the algorithm by one-time computation of the representatives of the conjugate classes $W/W_{\mathfrak{a}_{\perp}}$.

The next section contains examples illustrating the application of this algorithm.

3. Branching for finite dimensional Lie algebras

3.1. Regular embedding of A_1 into B_2

Consider the regular embedding $A_1 \to B_2$. Simple roots α_1, α_2 of B_2 are presented as the dashed vectors in the Figure 1. We denote the corresponding Weyl reflections by w_1, w_2. The simple root $\beta = \alpha_1 + 2\alpha_2$ of A_1 is indicated as the grey vector.

Let's perform the reduction of the fundamental representation $L_{B_2}^{(1,0)=\omega_1}$ (ω_1 – the black vector in Figure 1). The root α_1 is orthogonal to β, so we have $\Delta_{\perp}^+ = \{\alpha_1\}$. According to the defenition (1) the fan $\Gamma_{A_1\to B_2}$ consists of two weights:

$$\Gamma_{A_1\to B_2} = \{(1;2),\ (2;-1)\},$$

where the second component is the value of the sign function $s(\gamma)$. The singular weights $\{w(\omega_1+\rho)-\rho;\ w \in W\}$ are indicated by circles with the superscript $\epsilon(w)$. The space U is the factor $W/W_{\mathfrak{a}_{\perp}}$ where $W_{\mathfrak{a}_{\perp}} = \{e, w_1\}$. This means that the singular weights

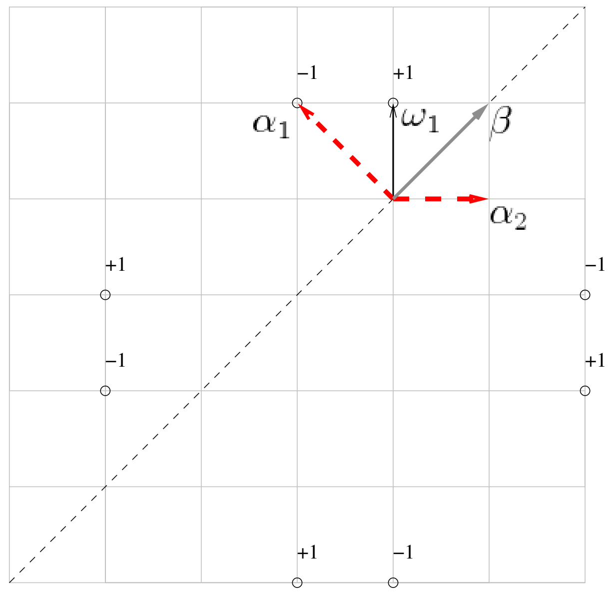


Figure 1. Regular embedding of A_1 into B_2. Simple roots α_1, α_2 of B_2 are presented as the dashed vectors. The simple root $\beta = \alpha_1 + 2\alpha_2$ of A_1 is indicated as the grey vector. The highest weight of the fundamental representation $L_{B_2}^{(1,0)=\omega_1}$ is shown by the black vector. The superscripts of the highest weights $\mathfrak{a}_{\perp}^{\mu_{\mathfrak{a}_\perp}}(u)$ are the products $\epsilon(w)\dim\left(L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}\right)$ The weights of the singular element $\Psi^{(\omega_1)}$ are marked by circles with superscripts indicating the corresponding determinants $\epsilon(w)$.

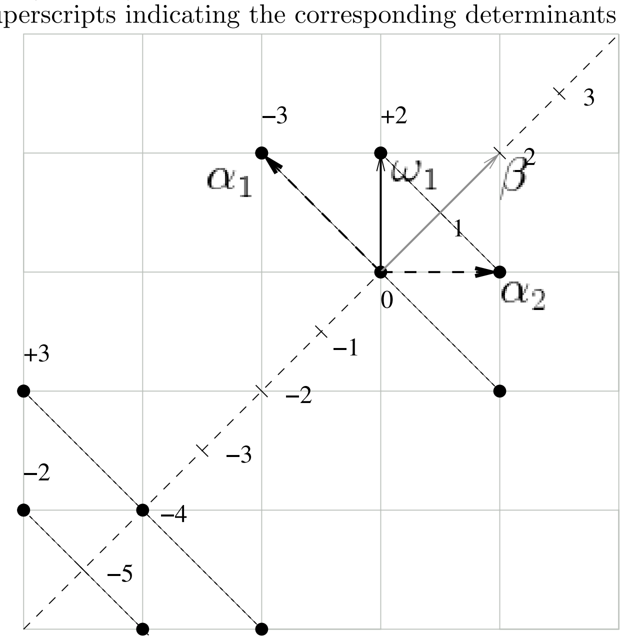


Figure 2. Here in addition to the diagram presented above (Figure(1)) the weights of $\mathfrak{a}_\perp = A_1$-modules $L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}$ originating in the points $\pi_{\mathfrak{a}}\left[u(\mu+\rho)-\rho\right]$ are shown by dotted lines. The superscripts over the highest weights $\mu_{\mathfrak{a}_\perp}(u)$ are now the products $\epsilon(u)\dim\left(L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}\right)$. Coordinates along the root β are counted in terms of the fundamental weight of $\mathfrak{a}$.

located above the line generated by β belong to the Weyl chamber $\overline{C_{\widetilde{\mathfrak{a}_\perp}}}$. According to formula (6) in our case $\mathcal{D}_{\mathfrak{a}_\perp} = 0$ and $\mathfrak{h}_\perp = 0$, thus $\{\mu_{\mathfrak{a}_\perp}(w) = \pi_{\mathfrak{a}_\perp}[w(\mu+\rho)-\rho]\}$. We have four highest weights of $\mathfrak{a}_\perp$-modules. In terms of $\mathfrak{a}_\perp$-fundamental weight $\frac{1}{2}\alpha_1$ these highest weights $\{\mu_{\mathfrak{a}_\perp}(u) = \pi_{\mathfrak{a}_\perp}[u(\mu+\rho)-\rho]\,|u \in U\}$ are $\{(1)\,(2)\,(2)\,(1)\}$. To visualize the procedure we indicate explicitly in Figure (2) how the corresponding weight diagrams $\left\{\mathcal{N}_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}\right\}$ are attached to the set of $\mathfrak{a}$-weights $\{\mu_{\mathfrak{a}}(u)\} = \{\pi_{\mathfrak{a}}[u(\mu+\rho)-\rho]\} = \{(1)\,(0)\,(-4)\,(-5)\}$. In fact we do not need the weight diagrams but only the dimensions of the corresponding modules $L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}$ multiplied by $\epsilon(u)$. The obtained values are to be placed in $P_{\mathfrak{a}}$ at the points $\{(1)\,(0)\,(-4)\,(-5)\}$ to produce the set of singular weights with anomalous multiplicities:

$$\{(1;2),\ (0;-3),\ (-4;3),\ (-5;-2)\}\,. \tag{17}$$

According to the defenition (1) the fan $\Gamma_{A_1\to B_2}$ consists of two weights:

$$\Gamma_{A_1\to B_2} = \{(1;2),\ (2;-1)\}\,,$$

where the second component of the weight is the value of the sign function $s(\gamma)$.

Applying formula (15) with the fan $\Gamma_{A_1\to B_2}$ to the set (17) we get zeros for the weights greater than the highest anomalous vector $(1;2)$ and $k_1^{(1,0)} = 2$ for the vector $(1;2)$ itself. For the anomalous weight (0;-3) on the boundary of $\bar{C}_{\mathfrak{a}}^{(0)}$ the recurrent relation gives

$$k_0^{(1,0)} = -1\cdot k_2^{(1,0)} + 2\cdot k_1^{(1,0)} - 3\cdot\delta_{0,0} = 1,$$

the branching is completed: $L_{B_2\downarrow A_1}^{\omega_1} = 2L_{A_1}^{\omega_{(A_1)1}} \bigoplus L_{A_1}^{2\omega_{(A_1)1}}$.

3.2. Embedding B_2 into B_4

Consider the regular embedding $B_2 \longrightarrow B_4$. The corresponding Dynkin diagrams are presented in the Figure 3.

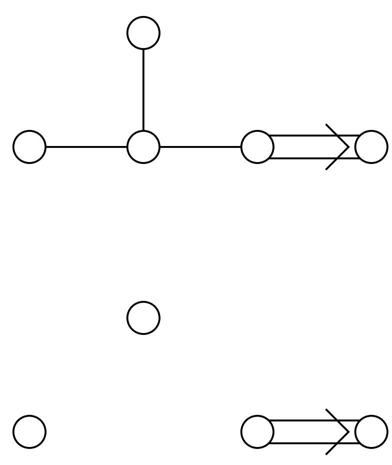


Figure 3. The regular embedding $B_2 \longrightarrow B_4$ described by dropping the node from the Dynkin diagram. Remember that here $\mathfrak{a}_\perp$ is equal to B_2 while the diagram shows only $A_1 \oplus A_1$ (see Subsection 2.4).

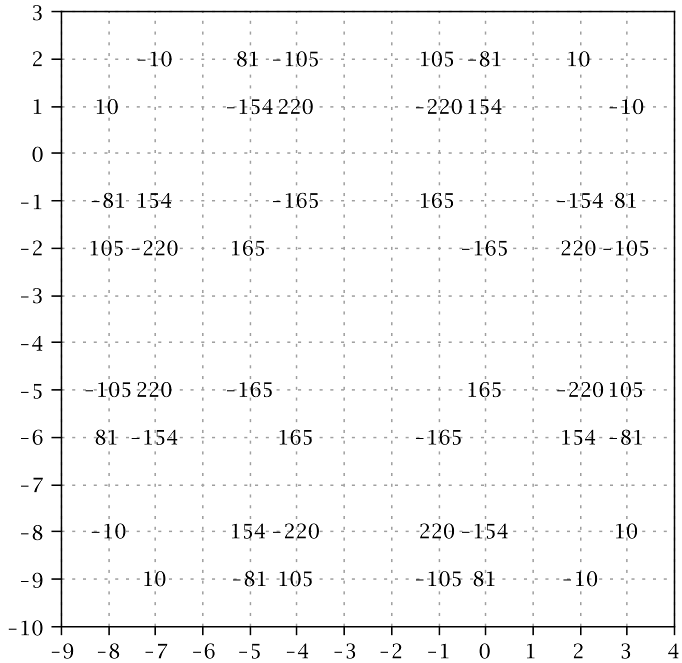


Figure 4. The weight subspace $P_{\mathfrak{a}}$ for $\mathfrak{a} = B_2$ with the basis $\{e_3, e_4\}$. The set of projected singular weights $\{\pi_{\mathfrak{a}}\left[u(\mu+\rho)-\rho\right]+\gamma_0 | u \in U\}$ shifted by γ_0. Each weight is supplied by the multiplier $\epsilon(u)\dim\left(L^{\mu_{\mathfrak{a}_\perp}(u)}_{\mathfrak{a}_\perp}\right)$.

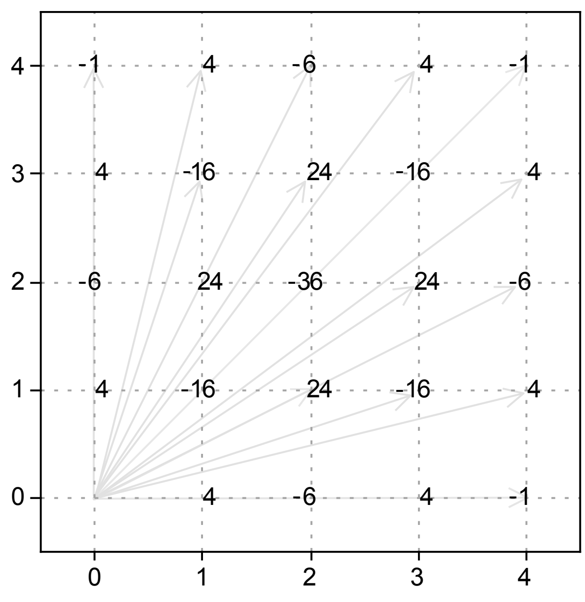


Figure 5. The fan Γ for $B_2 \longrightarrow B_4$ and the values of $s(\gamma+\gamma_0)$ for the weights γ.

In the orthogonal basis $\{e_1, \ldots, e_4\}$ simple roots and positive roots of B_4 are

$$S_{B_4} = \{e_1 - e_2,\ e_2 - e_3,\ e_3 - e_4,\ e_4\},$$

$$\Delta^+_{B_4} = \{(e_1 - e_2,\ e_2 - e_3,\ e_3 - e_4,\ e_4,\ e_1 - e_3,\ e_2 - e_4,\ e_3 + e_4,\ e_3,\ e_1 - e_4,\\ e_2 + e_4,\ e_2,\ e_1 + e_4,\ e_2 + e_3,\ e_1,\ e_1 + e_3,\ e_1 + e_2\}$$

The subalgebra $\mathfrak{a} = B_2$ is fixed by the simple roots

$$S_{B_2} = \{e_3 - e_4, e_4\}$$

Its orthogonal counterpart $\mathfrak{a}_\perp = B_2$ has

$$S_{\mathfrak{a}_\perp} = \{e_1 - e_2, e_2\},\\ \Delta^+_{\mathfrak{a}_\perp} = \{e_1 - e_2, e_1 + e_2, e_1, e_2\}.$$

As far as the set $\Delta^+_{B_4} \setminus \left(\Delta^+_{\mathfrak{a}} \cup \Delta^+_{\mathfrak{a}_\perp}\right)$ is fixed the injection fan $\Gamma_{B_2 \to B_4}$ can be constructed (using formulas (8,9,10)). As far as for this injection $s(\gamma_0) = -1$ in the recursion formula we need just the factor $s(\gamma + \gamma_0)$. The result is presented in Figure 5.

Consider the B_4-module L^μ with the highest weight $\mu = 2e_1 + 2e_2 + e_3 + e_4$; $\dim(L^{[0,1,0,2]}) = 2772$. The set of singular weights for B_4 contains 384 vectors. Here the defect is nontrivial, $\mathcal{D}_{\mathfrak{a}_\perp} = -2(e_1 + e_2)$, while $\mathfrak{h}_\perp = 0$. Taking this into account we find among the singular weights 48 vectors with the property $\left\{\mu_{\mathfrak{a}_\perp}(u) = \pi_{\mathfrak{a}_\perp}[u(\mu + \rho) - \rho] - \mathcal{D}_{\mathfrak{a}_\perp} \in \overline{C_{\mathfrak{a}_\perp}}\right\}$, scalar products of these weights with all the roots in $\Delta^+_{\mathfrak{a}_\perp}$ are nonnegative. The set $U = \{u\}$ is thus fixed. Compute the dimensions of the corresponding $\mathfrak{a}_\perp$-modules with the highest weights $\mu_{\mathfrak{a}_\perp}(u)$ (using the Weyl dimension formula) and multiply them by $\epsilon(u)$. The results are shown in the Figure 4.

Now one can place the fan Γ from Figure 5 in the highest of the weights presented in Figure 4 and start the recursive determination of the branching coefficients (using relation (15)):

$$\pi_{\mathfrak{a}}\left(chL^{[0,1,0,2]}_{B_4}\right) = 6\ chL^{[0,0]}_{B_2} + 60\ chL^{[0,2]}_{B_2} + 30\ chL^{[1,0]}_{B_2} + 19\ chL^{[2,0]}_{B_2} +\\ 40\ chL^{[1,2]}_{B_2} + 10\ chL^{[0,4]}_{B_2}.$$

4. Applications to the conformal field theory

4.1. Conformal embeddings

Branching coefficients for an embedding of affine Lie algebra into affine Lie algebra can be used to construct modular invariant partition functions for Wess-Zumino-Novikov-Witten models in conformal field theory ([1], [14], [15], [16]). In these models current algebras are affine Lie algebras.

The modular invariant partition function is crucial for the conformal theory to be valid on the torus and higher genus Riemann surfaces. It is important for the applications of CFT to the string theory and to the critical phenomena description.

The simplest modular-invariant partition function has the diagonal form:

$$Z(\tau) = \sum_{\mu \in P_{\mathfrak{g}}^{+}} \chi_\mu(\tau) \bar{\chi}_\mu(\bar{\tau})$$

Here the sum is over the set of the highest weights of integrable modules in a WZW-model and $\chi_\mu(\tau)$ are the normalized characters of these modules.

To construct the nondiagonal modular invariants is not an easy problem, although for some models the complete classification of modular invariants is known [17, 18].

Consider the Wess-Zumino-Witten model with the affine Lie algebra $\mathfrak{a}$. Nondiagonal modular invariants for this model can be constructed from the diagonal invariant if there exists an affine algebra $\mathfrak{g}$ such that $\mathfrak{a} \subset \mathfrak{g}$. Then we can replace the characters of the $\mathfrak{g}$-modules in the diagonal modular invariant partition function (18) by the decompositions

$$\sum_{\nu \in P_{\mathfrak{a}}^{+}} b_\nu^{(\mu)} \chi_\nu$$

containing the modified characters χ_ν of the corresponding $\mathfrak{a}$-modules. Thus we obtain the nondiagonal modular-invariant partition function for the theory with the current algebra $\mathfrak{a}$,

$$Z_{\mathfrak{a}}(\tau) = \sum_{\nu, \lambda \in P_{\mathfrak{a}}^{+}} \chi_\nu(\tau) M_{\nu\lambda} \bar{\chi}_\lambda(\bar{\tau}). \tag{18}$$

The effective reduction procedure is crucial for this construction. The embedding is required to preserve the conformal invariance. Let $X_{-n_j}^{\alpha_j}$ and $\tilde{X}_{-n_j}^{\alpha'_j}$ be the lowering generators for $\mathfrak{g}$ and for $\mathfrak{a} \subset \mathfrak{g}$ correspondingly. Let $\pi_{\mathfrak{a}}$ be the projection operator of $\pi_{\mathfrak{a}} : \mathfrak{g} \longrightarrow \mathfrak{a}$. In the theory attributed to $\mathfrak{g}$ with the vacuum $|\lambda\rangle$ the states can be described as

$$X_{-n_1}^{\alpha_1} X_{-n_2}^{\alpha_2} \ldots |\lambda\rangle \quad n_1 \geq n_2 \geq \ldots > 0.$$

And for the sub-algebra $\mathfrak{a}$ the corresponding states are

$$\tilde{X}_{-n_1}^{\alpha'_1} \tilde{X}_{-n_2}^{\alpha'_2} \ldots |\pi_{\mathfrak{a}}(\lambda)\rangle .$$

The $\mathfrak{g}$-invariance of the vacuum entails its $\mathfrak{a}$-invariance, but this is not the case for the energy-momentum tensor. So the energy-momentum tensor of the larger theory should contain only the generators $\tilde{X}$. Then the relation

$$T_{\mathfrak{g}}(z) = T_{\mathfrak{a}}(z) \tag{19}$$

leads to the equality of the central charges

$$c(\mathfrak{g}) = c(\mathfrak{a})$$

and to the equation

$$\frac{k \dim \mathfrak{g}}{k+g} = \frac{x_e k \dim \mathfrak{a}}{x_e k + a}. \tag{20}$$

Here x_e is the so called "embedding index": $x_e = \frac{|\pi_{\mathfrak{a}}\Theta|^2}{|\Theta_{\mathfrak{a}}|^2}$ with Θ, $\Theta_{\mathfrak{a}}$ being the highest roots of $\mathfrak{g}$ and $\mathfrak{a}$ while g and a are the corresponding dual Coxeter numbers.

It can be demonstrated that the solutions of the equation (20) exist only for the level $k = 1$ [1].

The complete classification of conformal embeddings is given in [16].

The relation (20) and the asymptotics of the branching functions can be used to prove the finite reducibility theorem [19]. It states that for the conformal embedding $\mathfrak{a} \subset \mathfrak{g}$ only finite number of branching coefficients have nonzero values.

Note 4.1. *The orthogonal subalgebra* $\mathfrak{a}_\perp$ *is always empty for the conformal embeddings* $\mathfrak{a} \subset \mathfrak{g}$.

Proof. Consider the modes expansion of the energy-momentum tensor

$$T(z) = \frac{1}{2(k+h^v)} \sum_n z^{-n-1} L_n.$$

The modes L_n are constructed as combination of normally-ordered products of the generators of $\mathfrak{g}$,

$$L_n = \frac{1}{2(k+h^v)} \sum_\alpha \sum_m : X^\alpha_m X^\alpha_{n-m} : \ .$$

In the case of a conformal embedding energy-momentum tensors are to be equal (19).

The substitution of the generators of $\mathfrak{a}$ in terms of the generators of $\mathfrak{g}$ into these combinations should give the energy-momentum tensor $T_\mathfrak{g}$. But if the set of the generators $\Delta_{\mathfrak{a}_\perp}$ is not empty this is not possible, since $T_\mathfrak{g}$ contains the combinations of the generators $X^\alpha_n, \ \alpha \in \Delta_{\mathfrak{a}_\perp}$. □

4.1.1. Special embedding $\hat{A}_1 \subset \hat{A}_2$. Consider the case where both $\mathfrak{g}$ and $\mathfrak{a}$ are affine Lie algebras: $\hat{A}_1 \longrightarrow \hat{A}_2$ and the injection is the affine extension of the special injection $A_1 \longrightarrow A_2$ with the embedding index $x_e = 4$. As far as the $\mathfrak{g}$-modules to be considered are of level one, the $\mathfrak{g}$-modules will be of level $\tilde{k} = kx_e = 4$.

There exist three level one fundamental weights in the weight space of $\hat{A}_2$. It is easy to see that the set $\Delta_{\mathfrak{a}_\perp}$ is empty and the subalgebra $\mathfrak{a}_\perp = 0$.

Using the definition (10) we construct the fan $\Gamma_{\hat{A}_1 \to \hat{A}_2}$ and the function $s(\gamma + \gamma_0)$ (see the Figure 6).

Let us consider the module $L^{\omega_0=(0,0;1;0)}$. Here we use the (finite part; level; grade) presentation of the highest weight and the finite part coordinates are the Dynkin indices (see section(1.1)).

The set $\widehat{\Psi^{(\omega_0)}}$ is depicted in the Figure 7 up to the sixth grade.

The next step is to project the anomalous weights to $P_{\hat{A}_1}$. The result is presented in the Figure 8 up to the twelfth grade.

Using the recurrent relation for the anomalous branching coefficients we get the result presented in Figure 9. Inside the Weyl chamber $\bar{C}_{\hat{A}_1}$ (its boundaries are indicated in the Figure 9) there are only two nonzero anomalous weights and both have multiplicity

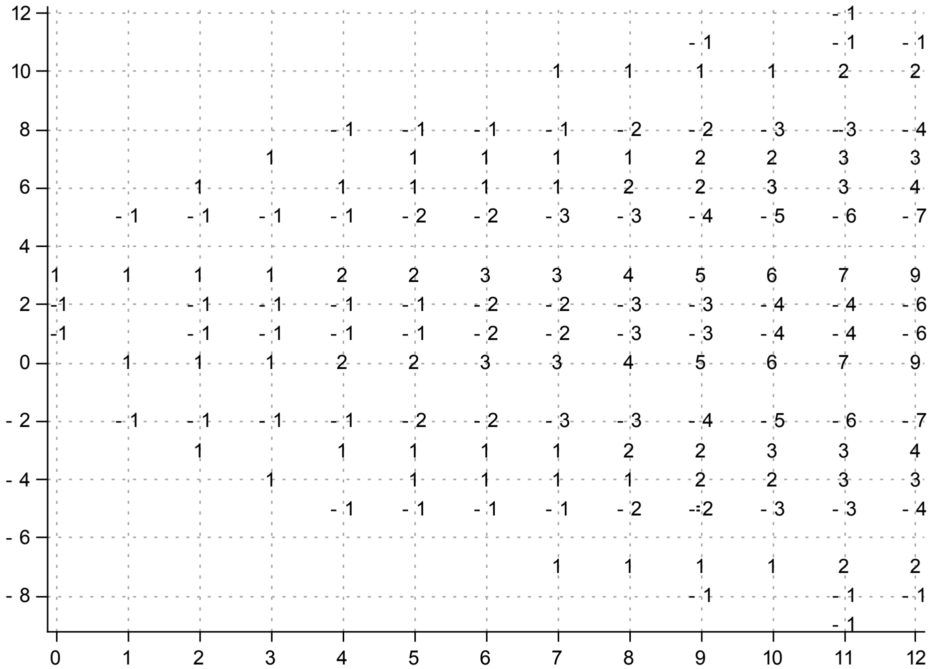


Figure 6. The fan $\Gamma_{\hat{A}_1 \longrightarrow \hat{A}_2}$ for $\hat{A}_1 \longrightarrow \hat{A}_2$. The grade is along the horizontal axis and the simple root α of A_1 is used as the unit of measurement of vertical axis. Values of $s(\gamma + \gamma_0)$ are shown for the weights $\gamma \in \Gamma_{\hat{A}_1 \longrightarrow \hat{A}_2}$

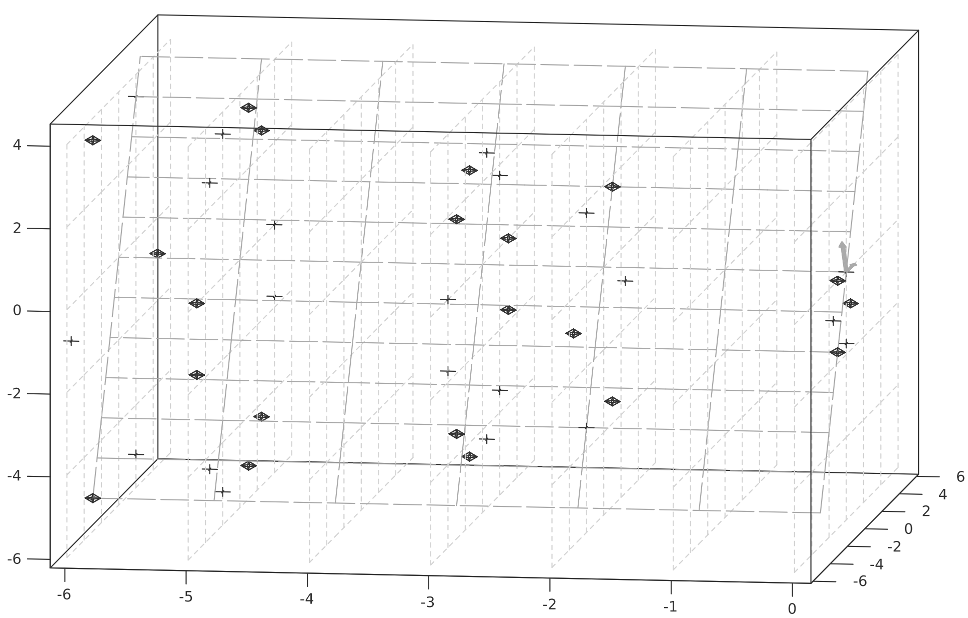


Figure 7. The anomalous weights of the module $L^{\omega_0}_{\hat{A}_2} = L^{(0,0;1;0)}_{\hat{A}_2}$. The standard basis $\{e_1, e_2\}$ is used for the finite-dimensional subalgebra and the grade is along the horizontal axis. The weights $w(\omega_0 + \rho) - \rho$ are marked by crosses when $\epsilon(w) = 1$ and by diamond when $\epsilon(w) = -1$. Simple roots of the classical subalgebra A_2 are grey and the grey diagonal plane corresponds to the Cartan subalgebra of the embedded algebra $\hat{A}_1$.

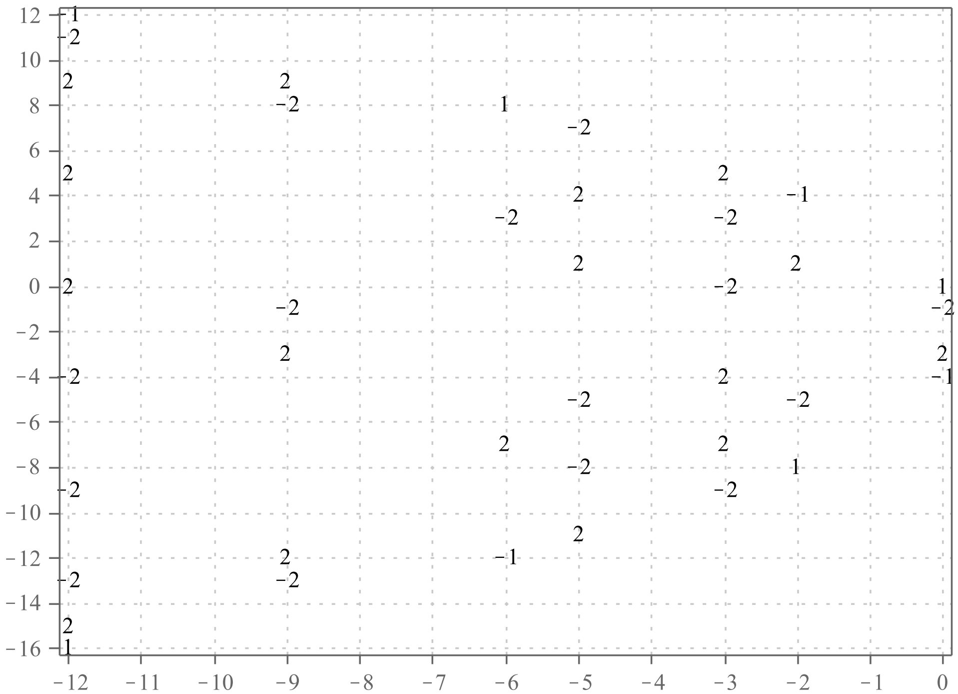


Figure 8. Projected anomalous weights of $L^{(0,0;1;0)}_{\hat{A}_2}$. The multiplicities of projected weights and the corresponding signs are shown.

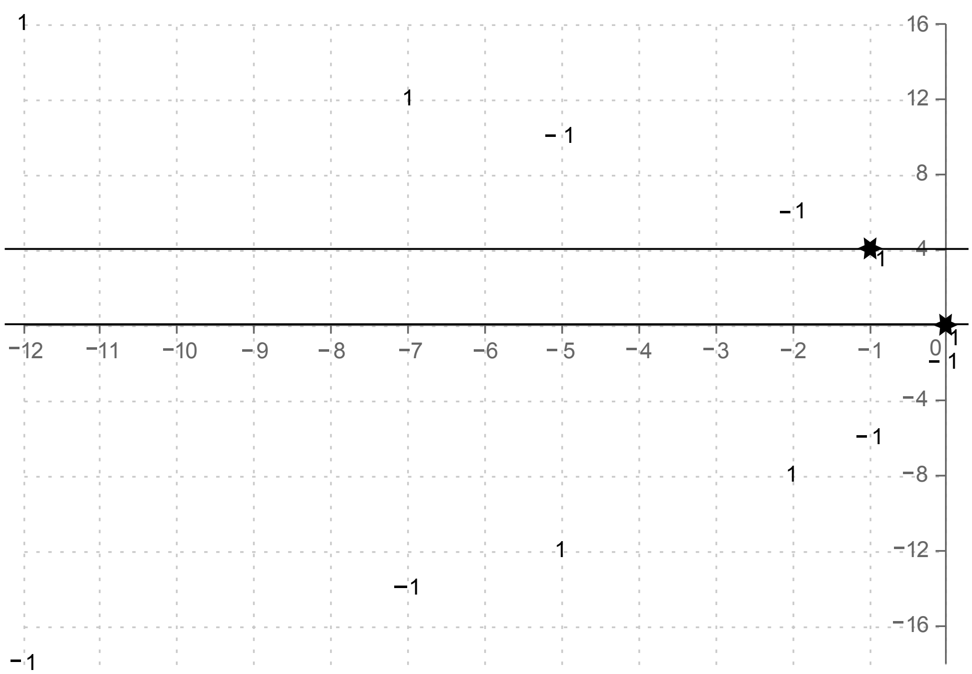


Figure 9. Anomalous branching coefficients for $\hat{A}_1 \subset \hat{A}_2$. The boundaries of the main Weyl chamber $\bar{C}_{\hat{A}_1}$ are indicated by the black lines. Two nonzero anomalous weights in the main Weyl chamber are marked by stars. Both weights have multiplicity 1, so the branching coefficients are equal to 1.

1. These are the highest weights of $\mathfrak{a}$-submodules and their branching coefficients. So the finite reducibility theorem holds and we get the decomposition

$$L^{(0,0;1;0)}_{\hat{A}_2\downarrow\hat{A}_1} = L^{(0;4;0)}_{\hat{A}_1} \oplus L^{(4;4;0)}_{\hat{A}_1}.$$

For the other irreducible modules of level one we get the trivial branching

$$L^{(1,0;1;0)}_{\hat{A}_2\downarrow\hat{A}_1} = L^{(2;4;0)}_{\hat{A}_1}, L^{(0,1;1;0)}_{\hat{A}_2\downarrow\hat{A}_1} = L^{(2;4;0)}_{\hat{A}_1}.$$

Using these results the modular-invariant partition function is easily found,

$$Z = \left|\chi_{(4;4;0)} + \chi_{(0;4;0)}\right|^2 + 2\chi^2_{(2;4;0)}.$$

4.2. Coset models

Coset models [20] tightly connected with the gauged WZW-models are actively studied in string theory, especially in string models on anti-de-Sitter space [21, 22, 23, 24, 25]. The characters in coset models are proportional to the branching functions,

$$\chi^{(\mu)}_{\nu}(\tau) = e^{2\pi i\tau(m_\mu - m_\nu)} b^{(\mu)}_{\nu}(\tau), \tag{21}$$

with

$$m_\mu = \frac{|\mu+\rho|^2}{2(k+g)} - \frac{|\rho|^2}{2g}.$$

The problem of the branching functions construction in the coset models was considered in [26], [7], [27].

Let us return to the example 3.1 and consider the affine extension of the injection $A_1 \longrightarrow B_2$. Since this embedding is regular and $x_e = 1$, the subalgebra modules and the initial module are of the same level. The set $\Delta^+_{\mathfrak{a}_\perp}$ of the orthogonal positive roots with the zero projection on the root space of the subalgebra $\hat{A}_1$ is the same as in the finite-dimensional case.

Using the definition (10) we get the fan $\Gamma_{\hat{A}_1\longrightarrow\hat{B}_2}$ with the corresponding values $s(\gamma+\gamma_0)$ (see the Figure 10). We restricted the computation to the twelfth grade.

Consider the level one module $L^{(1,0;1;0)}_{\hat{B}_2}$ with the highest weight $\omega_1 = (1,0;1;0)$, where the finite part coordinates are in the orthogonal basis e_1, e_2. The set of anomalous weights for this module up to the sixth grade is presented in the Figure 11. In the grade zero it is exactly the set of the anomalous weights for the embedding of the classical Lie algebras $A_1 \subset B_2$ that can be seen in the Figure 1.

According to the algorithm 2.5 we project the anomalous weights to $P_{\hat{A}_1}$ and find the dimensions of the corresponding $\mathfrak{a}_\perp$-modules $L^{\pi_{\mathfrak{a}_\perp}(w(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}_{\mathfrak{a}_\perp}$. The result is presented in the Figure 12 up to the twelfth grade.

Notice that here the lowest weight γ_0 of the fan is zero, since we have excluded all the roots of $\Delta^+_{\mathfrak{a}_\perp}$ from the defining relation (10).

The multiplicities of the weights inside the Weyl chamber $\bar{C}^{(0)}_{\hat{A}_1}$ are the branching coefficients (up to the twelfth grade),

$$L^{\omega_1}_{\hat{B}_2\downarrow\hat{A}_1} = 2L^{\omega_1}_{\hat{A}_1} \oplus 1L^{\omega_0}_{\hat{A}_1} \oplus 4L^{\omega_0-\delta}_{\hat{A}_1} \oplus$$

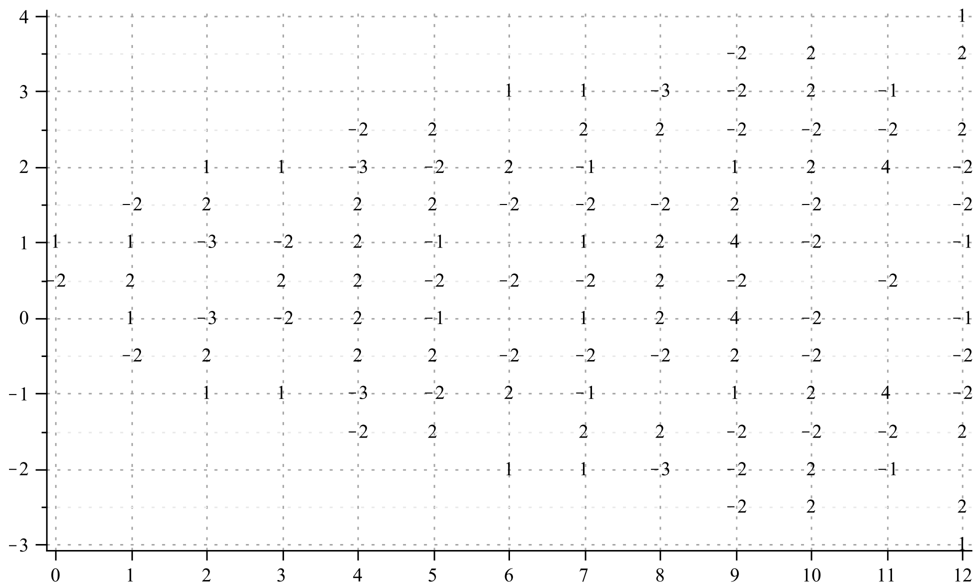


Figure 10. The fan $\Gamma_{\hat{A}_1 \longrightarrow \hat{B}_2}$ for $\hat{A}_1 \longrightarrow \hat{B}_2$. Values of $s(\gamma+\gamma_0)$ are shown for the weights $\gamma \in \Gamma_{\hat{A}_1 \longrightarrow \hat{B}_2}$

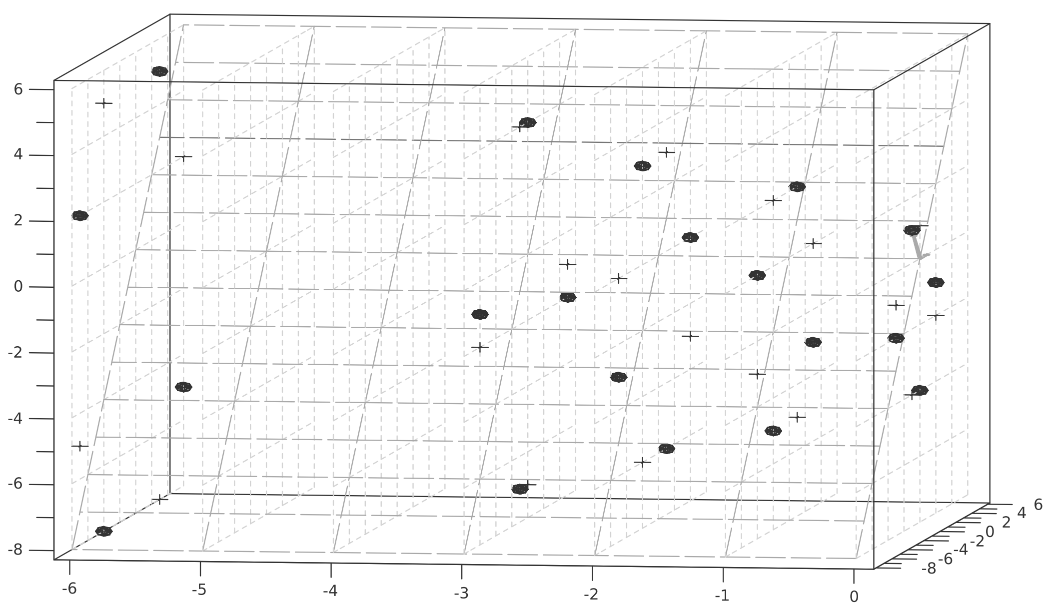


Figure 11. The anomalous weights of $L^{(1,0;1;0)}_{\hat{B}_2}$. The weights in the zero grade can be seen in the Figure 1. The weights $w(\omega_1+\rho)-\rho$ are marked by crosses if $\epsilon(w)=1$ and by circles otherwise. Simple roots of the classical subalgebra B_2 are grey and grey diagonal plane corresponds to the Cartan subalgebra of the embedded algebra $\hat{A}_1$.

$$
2L^{\omega_1-\delta}_{\hat{A}_1} \oplus 8L^{\omega_0-2\delta}_{\hat{A}_1} \oplus 8L^{\omega_1-2\delta}_{\hat{A}_1} \oplus 15L^{\omega_0-3\delta}_{\hat{A}_1} \oplus
$$
$$
12L^{\omega_1-3\delta}_{\hat{A}_1} \oplus 26L^{\omega_1-4\delta}_{\hat{A}_1} \oplus 29L^{\omega_0-4\delta}_{\hat{A}_1} \oplus 51L^{\omega_0-5\delta}_{\hat{A}_1} \oplus
$$
$$
42L^{\omega_1-5\delta}_{\hat{A}_1} \oplus 78L^{\omega_1-6\delta}_{\hat{A}_1} \oplus 85L^{\omega_0-6\delta}_{\hat{A}_1} \oplus 120L^{\omega_1-7\delta}_{\hat{A}_1} \oplus
$$

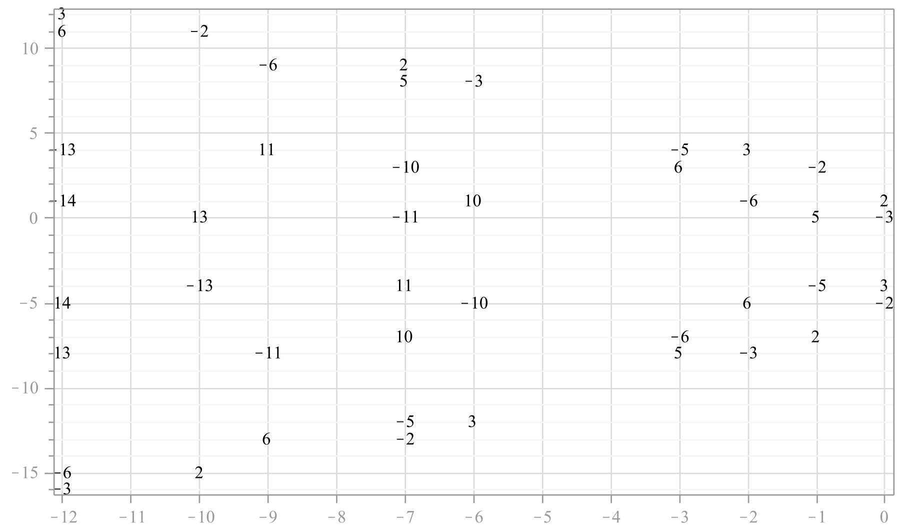


Figure 12. The projected anomalous weights $\pi_{\hat{A}_1}\left(\Psi^{(1,0;1;0)}_{\hat{B}_2}\right)$. The dimensions of the corresponding $\mathfrak{a}_\perp = A_1$-modules with the signs $\epsilon(w)$ are shown.

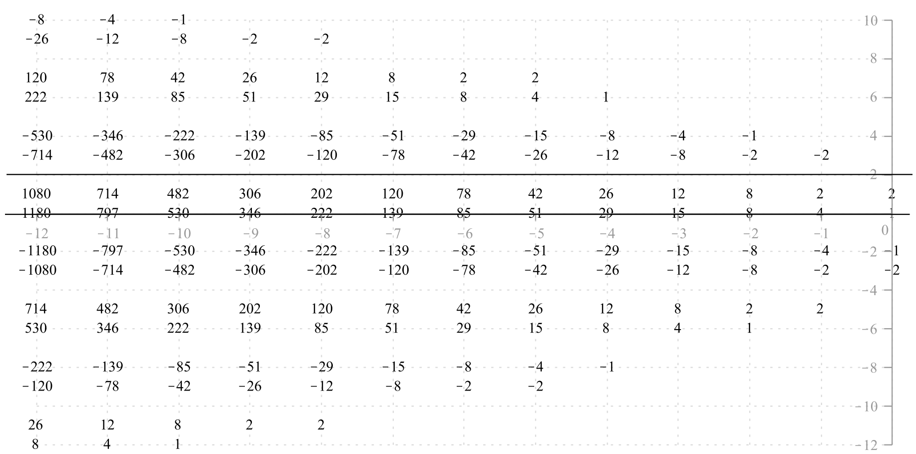


Figure 13. Anomalous branching coefficients for $\hat{A}_1 \subset \hat{B}_2$. The boundaries of the main Weyl chamber $\bar{C}_{\hat{A}_1}$ are indicated by the black lines. The anomalous branching coefficients inside the main Weyl chamber are equal to the branching coefficients of the embedding $\hat{A}_1 \longrightarrow \hat{B}_2$.

$$
\begin{aligned}
&139L^{\omega_0-7\delta}_{\hat{A}_1} \oplus 202L^{\omega_1-8\delta}_{\hat{A}_1} \oplus 222L^{\omega_0-8\delta}_{\hat{A}_1} \oplus 306L^{\omega_1-9\delta}_{\hat{A}_1} \oplus \\
&346L^{\omega_0-9\delta}_{\hat{A}_1} \oplus 530L^{\omega_0-10\delta}_{\hat{A}_1} \oplus 482L^{\omega_1-10\delta}_{\hat{A}_1} \oplus 714L^{\omega_1-11\delta}_{\hat{A}_1} \oplus \\
&797L^{\omega_0-11\delta}_{\hat{A}_1} \oplus 1080L^{\omega_1-12\delta}_{\hat{A}_1} \oplus 1180L^{\omega_0-12\delta}_{\hat{A}_1} \oplus \dots
\end{aligned}
$$

This result can be presented as the set of branching functions:

$$
\begin{aligned}
b_0^{(\omega_1)} &= 1 + 4\,q^1 + 8\,q^2 + 15\,q^3 + 29\,q^4 + 51\,q^5 + 85\,q^6 + 139\,q^7 + \\
&\quad 222\,q^8 + 346\,q^9 + 530\,q^{10} + 797\,q^{11} + 1180\,q^{12} + \dots \\
b_1^{(\omega_1)} &= 2 + 2\,q^1 + 8\,q^2 + 12\,q^3 + 26\,q^4 + 42\,q^5 + 78\,q^6 + 120\,q^7 + \\
&\quad 202\,q^8 + 306\,q^9 + 482\,q^{10} + 714\,q^{11} + 1080\,q^{12} + \dots
\end{aligned}
$$

Here $q = \exp(2\pi i \tau)$ and the lower index enumerates the branching functions according to their highest weights in $P^+_{\hat{A}_1}$. These are the fundamental weights $\omega_0 = \lambda_0 = (0,1,0)$, $\omega_1 = \alpha/2 = (1,1,0)$.

Now we can use the relation (21) to get the expansion of the B_2/A_1-coset characters:

$$\begin{aligned} \chi_1^{(\omega_1)}(q) = \;& q^{\frac{7}{12}}\,(2 + 2\,q^1 + 8\,q^2 + 12\,q^3 + 26\,q^4 + 42\,q^5 + 78\,q^6 + 120\,q^7 + \\ & 202\,q^8 + 306\,q^9 + 482\,q^{10} + 714\,q^{11} + 1080\,q^{12} + \ldots) \\ \chi_0^{(\omega_1)}(q) = \;& q^{\frac{5}{6}}\,(1 + 4\,q^1 + 8\,q^2 + 15\,q^3 + 29\,q^4 + 51\,q^5 + 85\,q^6 + 139\,q^7 + \\ & 222\,q^8 + 346\,q^9 + 530\,q^{10} + 797\,q^{11} + 1180\,q^{12} + \ldots) \end{aligned}$$

Further amelioration of the algorithm can be achieved by using the folded fan technique [28] to get the explicit expression for the branching functions and the corresponding coset characters in conformal field theory.

5. Conclusion

We have demonstrated that the injection fan technique can be used to deal with the nonmaximal subalgebras. It was found out that for such subalgebras an auxiliary subset $\Delta^+_{\mathfrak{a}_\perp}$ must be extracted from the set of positive roots $\Delta^+_{\mathfrak{g}}$. The role of the subset $\Delta^+_{\mathfrak{a}_\perp}$ is to modify both the injection fan (formed here by the weights $\left(\Delta^+_{\mathfrak{g}} \setminus \Delta^+_{\mathfrak{a}}\right) \setminus \Delta^+_{\mathfrak{a}_\perp}$) and the anomalous weights of the initial module. This modification reduces to a simple procedure: the anomalous weights multiplicities are to be substituted by the dimensions of the corresponding $\mathfrak{a}_\perp$-modules.

The efficiency of the injection fan algorithm was verified. Its possible applications to some physical problems were discussed. In particular we considered the construction of modular-invariant partition functions in the framework of conformal embedding method and the coset construction in the rational conformal field theory. This construction is useful in the study of WZW-models emerging in the context of the AdS/CFT correspondence [21, 22, 23].

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