Рекурсивные свойства ветвления и БГГ резольвента

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Аннотация

Рекуррентные соотношения для коэффициентов ветвления основываются на определенном разложении сингулярного элемента. Мы показываем, что такое разложение может использоваться для построения параболических модулей Верма и получения обобщенных формул Вейля-Верма для характеров. Также мы демонстрируем, что коэффициенты ветвления определяют обобщенную резольвенту Бернштейна-Гельфанда-Гельфанда.

1 Введение

Свойства ветвления (аффинных) алгебр Ли важны для приложений в квантовой теории поля (смотри, например, модели конформной теории поля [1],[2]). В данной работе мы показываем, что ветвление для произвольной редуктивной подалгебры связано с БГГ резольвентой и проявляет свойства резольвенты в категории $\mathcal{O}^p$ [3] (параболического обобщения категории $\mathcal{O}$ [4]).

Резольвента для неприводимых модулей в терминах бесконечномерных модулей важна для теории интегрируемых спиновых цепочек [5].

В подходе $\mathcal{Q}$-оператора Бакстера [6] общие трансфер-матрицы, соответствующие (обобщенным) модулям Верма, факторизуются в произведение операторов Бакстера. Резольвента позволяет вычислить трансфер-матрицы для конечномерных вспомогательных пространств.

Чтобы продемонстрировать связь БГГ резольвенты с ветвлением мы используем рекурсивный подход, представленный в работе [7] (похожий подход для максимальных вложений использовался в работе [8]). Мы рассматриваем подалгебру $\mathfrak{a} \hookrightarrow \mathfrak{g}$ вместе с $\mathfrak{a}_\perp$ – "ортогональным партнером" $\mathfrak{a}$ по отношению к форме Киллинга, а также $\widetilde{\mathfrak{a}_\perp} := \mathfrak{a}_\perp \oplus \mathfrak{h}_\perp$, где $\mathfrak{h} = \mathfrak{h}_\mathfrak{a} \oplus \mathfrak{h}_{\mathfrak{a}_\perp} \oplus \mathfrak{h}_\perp$. Для любой редуктивной подалгебры $\mathfrak{a}$ подалгебра $\mathfrak{a}_\perp \hookrightarrow \mathfrak{g}$ регулярна и редуктивна. Для интегрируемого модуля старшего веса $L^{(\mu)}$ и ортогональной подалгебры $\mathfrak{a}_\perp$ мы рассматриваем сингулярный элемент $\Psi^{(\mu)}$ (числитель в формуле Вейля для характеров $ch\left(L^\mu\right) = \frac{\Psi^{(\mu)}}{\Psi^{(0)}}$, см., например, [9]) и знаменатель Вейля $\Psi^{(0)}_{\mathfrak{a}_\perp}$ для ортогонального партнера. В работе показано, что элемент $\Psi^{(\mu)}_\mathfrak{g}$ может быть разложен в комбинацию числителей Вейля $\Psi^{(\nu)}_{\mathfrak{a}_\perp}$, где $\nu \in P^+_{\mathfrak{a}_\perp}$. Это разложение дает возможность построить множество модулей старшего веса $L^{\mu_{\mathfrak{a}_\perp}}_{\mathfrak{a}_\perp}$. В том случае, если вложение $\mathfrak{a}_\perp \hookrightarrow \mathfrak{g}$ удовлетворяет "стандартным параболическим" условиям, эти модули порождают параболические модули Верма $M^{\mu_{\mathfrak{a}_\perp}}_{(\mathfrak{a}_\perp \hookrightarrow \mathfrak{g})}$, так что исходный характер $ch\left(L^\mu\right)$ в итоге раскладывается в чередующуюся сумму таких модулей. С другой стороны, если параболическое условие нарушено, конструкция сохраняется и порождает разложение по отношению к набору обобщенных модулей Верма $M^{\mu_{\widetilde{\mathfrak{a}_\perp}}}_{\left(\widetilde{\mathfrak{b}_\perp},\mathfrak{g}\right)}$, где $\widetilde{\mathfrak{b}_\perp}$ уже не является подалгеброй в $\mathfrak{g}$, а оказывается сжатием $\widetilde{\mathfrak{a}_\perp}$.

Некоторые общие свойства предложенного разложения формулируются в терминах определенного формального элемента $\Gamma_{\mathfrak{a}\to\mathfrak{g}}$, называемого "веером вложения". Использование этого инструмента позволило сформулировать простой и явный алгоритм для вычисления правил ветвления, подходящий для произвольной (максимальной или не максимальной) подалгебры в аффинной алгебре Ли [7].

Возможные обобщения полученных результатов обсуждаются в Разделе 4.

1.1 Обозначения

Consider Lie algebras (affine Lie algebras) $\mathfrak{g}$ and $\mathfrak{a}$ and an injection $\mathfrak{a} \hookrightarrow \mathfrak{g}$ such that $\mathfrak{a}$ is a reductive subalgebra $\mathfrak{a} \subset \mathfrak{g}$ with correlated root spaces: $\mathfrak{h}_{\mathfrak{a}}^{*} \subset \mathfrak{h}_{\mathfrak{g}}^{*}$. We use the following notations:

$\mathfrak{g} = \mathfrak{n}^{-} + \mathfrak{h} + \mathfrak{n}^{+}$ — the Cartan decomposition;

r , $(r_{\mathfrak{a}})$ — the rank of the algebra $\mathfrak{g}$ (resp.$\mathfrak{a}$) ;

Δ $(\Delta_{\mathfrak{a}})$— the root system; Δ^{+} (resp.$\Delta_{\mathfrak{a}}^{+}$)— the positive root system (of $\mathfrak{g}$ and $\mathfrak{a}$ respectively);

$\mathrm{mult}\,(\alpha)$ $(\mathrm{mult}_{\mathfrak{a}}\,(\alpha))$ — the multiplicity of the root α in Δ (resp. in $(\Delta_{\mathfrak{a}})$);

S $(S_{\mathfrak{a}})$ — the set of simple roots (for $\mathfrak{g}$ and $\mathfrak{a}$ respectively);

α_i , $\left(\alpha_{(\mathfrak{a})j}\right)$ — the i-th (resp. j-th) simple root for $\mathfrak{g}$ (resp.$\mathfrak{a}$); $i = 0, \ldots, r$, $(j = 0, \ldots, r_{\mathfrak{a}})$;

$\alpha_i^{\vee}$, $\left(\alpha_{(\mathfrak{a})j}^{\vee}\right)$– the simple coroot for $\mathfrak{g}$ (resp.$\mathfrak{a}$) , $i = 0, \ldots, r$; $(j = 0, \ldots, r_{\mathfrak{a}})$;

W , $(W_{\mathfrak{a}})$— the Weyl group;

C , $(C_{\mathfrak{a}})$— the fundamental Weyl chamber;

$\bar{C}$, $\left(\bar{C}_{\mathfrak{a}}\right)$ — the closure of the fundamental Weyl chamber;

$\epsilon\,(w) := (-1)^{\mathrm{length}(w)}$;

ρ , $(\rho_{\mathfrak{a}})$ — the Weyl vector;

L^{μ} $(L_{\mathfrak{a}}^{\nu})$ — the integrable module of $\mathfrak{g}$ with the highest weight μ ; (resp. integrable $\mathfrak{a}$-module with the highest weight ν);

$\mathcal{N}^{\mu}$, $(\mathcal{N}_{\mathfrak{a}}^{\nu})$ — the weight diagram of L^{μ} (resp. $L_{\mathfrak{a}}^{\nu}$);

P (resp. $P_{\mathfrak{a}}$) — the weight lattice;

P^{+} (resp. $P_{\mathfrak{a}}^{+}$) — the dominant weight lattice;

$m_{\xi}^{(\mu)}$, $\left(m_{\zeta}^{(\nu)}\right)$ — the multiplicity of the weight $\xi \in P$ (resp. $\in \mathrm{P}_{\mathfrak{a}}$) in L^{μ}, (resp. in $\zeta \in L_{\mathfrak{a}}^{\nu}$);

$ch\,(L^{\mu})$ (resp. $\mathrm{ch}\,(L_{\mathfrak{a}}^{\nu})$)— the formal character of L^{μ} (resp. of $L_{\mathfrak{a}}^{\nu}$);

$ch\,(L^{\mu}) = \frac{\sum_{w \in W} \epsilon(w) e^{w \circ (\mu+\rho)-\rho}}{\prod_{\alpha \in \Delta^{+}} (1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}$ — the Weyl-Kac formula;

$R := \prod_{\alpha \in \Delta^{+}} (1-e^{-\alpha})^{\mathrm{mult}(\alpha)}$ (resp. $R_{\mathfrak{a}} := \prod_{\alpha \in \Delta_{\mathfrak{a}}^{+}} (1-e^{-\alpha})^{\mathrm{mult}_{\mathfrak{a}}(\alpha)}$)— the Weyl denominator.

2 Orthogonal subalgebra and singular elements

In this section we shall show how the recurrent approach to branching problem leads naturally to a presentation of a formal character of $\mathfrak{g}$-module in terms characters corresponding to a set of parabolic (generalized) Verma modules. Consider a reductive Lie algebra $\mathfrak{g}$ and its reductive subalgebra $\mathfrak{a} \subset \mathfrak{g}$. Let L^{μ} be the highest weight integrable module of $\mathfrak{g}$, $\mu \in P^{+}$. Let L^{μ} be completely reducible with respect to $\mathfrak{a}$,

$$L^{\mu}_{\mathfrak{g}\downarrow\mathfrak{a}} = \bigoplus_{\nu \in P^{+}_{\mathfrak{a}}} b^{(\mu)}_{\nu} L^{\nu}_{\mathfrak{a}}.$$

Using the projection operator $\pi_{\mathfrak{a}}$ (to the weight space ${\mathfrak{h}_{\mathfrak{a}}}^{*}$) one can write this decomposition in terms of formal characters:

$$\pi_{\mathfrak{a}} ch\left(L^{\mu}\right) = \sum_{\nu \in P^{+}_{\mathfrak{a}}} b^{(\mu)}_{\nu} ch\left(L^{\nu}_{\mathfrak{a}}\right). \quad (1)$$

The module L^{μ} has the BGG resolution (see [4, 10, 11] and [12]). All the members of the filtration sequence are the direct sums of Verma modules and all their highest weights ν are strongly linked to μ:

$$\{\nu\} = \left\{w\left(\mu+\rho\right) - \rho | w \in W\right\}.$$

2.1 Orthogonal subalgebra

Let $\mathfrak{h}_{\mathfrak{a}}$ be a Cartan subalgebra of $\mathfrak{g}$. For $\mathfrak{a} \hookrightarrow \mathfrak{g}$ introduce the "orthogonal partner" $\mathfrak{a}_{\perp} \hookrightarrow \mathfrak{g}$.

Consider the root subspace $\mathfrak{h}^{*}_{\perp\mathfrak{a}}$ orthogonal to $\mathfrak{a}$,

$$\mathfrak{h}^{*}_{\perp\mathfrak{a}} := \left\{\eta \in \mathfrak{h}^{*} | \forall h \in \mathfrak{h}_{\mathfrak{a}}; \eta\left(h\right) = 0\right\},$$

and the roots (correspondingly – positive roots) of $\mathfrak{g}$ orthogonal to $\mathfrak{a}$,

$$\begin{array}{lcl} \Delta_{\mathfrak{a}_{\perp}} & : & = \left\{\beta \in \Delta_{\mathfrak{g}} | \forall h \in \mathfrak{h}_{\mathfrak{a}}; \beta\left(h\right) = 0\right\}, \\ \Delta^{+}_{\mathfrak{a}_{\perp}} & : & = \left\{\beta^{+} \in \Delta^{+}_{\mathfrak{g}} | \forall h \in \mathfrak{h}_{\mathfrak{a}}; \beta^{+}\left(h\right) = 0\right\}. \end{array} \quad (2)$$

Let $W_{\mathfrak{a}_\perp}$ be the subgroup of W generated by the reflections w_β with the roots $\beta \in \Delta^+_{\mathfrak{a}_\perp}$. The subsystem $\Delta_{\mathfrak{a}_\perp}$ determines the subalgebra $\mathfrak{a}_\perp$ with the Cartan subalgebra $\mathfrak{h}_{\mathfrak{a}_\perp}$. Let

$$\mathfrak{h}^*_\perp := \{\eta \in \mathfrak{h}^*_{\perp\mathfrak{a}} | \forall h \in \mathfrak{h}_{\mathfrak{a}\oplus\mathfrak{a}_\perp}; \eta\,(h) = 0\}$$

so that $\mathfrak{g}$ has the subalgebras

$$\widetilde{\mathfrak{a}_\perp} := \mathfrak{a}_\perp \oplus \mathfrak{h}_\perp \qquad \widetilde{\mathfrak{a}} := \mathfrak{a} \oplus \mathfrak{h}_\perp. \tag{3}$$

Notice that $\mathfrak{a} \oplus \mathfrak{a}_\perp$ in general is not a subalgebra in $\mathfrak{g}$.

For the Cartan subalgebras we have the decomposition

$$\mathfrak{h} = \mathfrak{h}_{\mathfrak{a}} \oplus \mathfrak{h}_{\mathfrak{a}_\perp} \oplus \mathfrak{h}_\perp = \mathfrak{h}_{\widetilde{\mathfrak{a}}} \oplus \mathfrak{h}_{\mathfrak{a}_\perp} = \mathfrak{h}_{\widetilde{\mathfrak{a}_\perp}} \oplus \mathfrak{h}_{\mathfrak{a}}. \tag{4}$$

For $\mathfrak{a}$ and $\mathfrak{a}_\perp$ consider the corresponding Weyl vectors, $\rho_{\mathfrak{a}}$ and $\rho_{\mathfrak{a}_\perp}$. Form the so called "defects" $\mathcal{D}_{\mathfrak{a}}$ and $\mathcal{D}_{\mathfrak{a}_\perp}$ of the injection:

$$\mathcal{D}_{\mathfrak{a}} := \rho_{\mathfrak{a}} - \pi_{\mathfrak{a}}\rho, \qquad \mathcal{D}_{\mathfrak{a}_\perp} := \rho_{\mathfrak{a}_\perp} - \pi_{\mathfrak{a}_\perp}\rho. \tag{5}$$

For $\mu \in P^+$ consider the linked weights $\{(w(\mu+\rho) - \rho)\,|w \in W\}$. Consider the projections to $h^*_{\mathfrak{a}_\perp}$ additionally shifted by the defect $-\mathcal{D}_{\mathfrak{a}_\perp}$:

$$\mu_{\mathfrak{a}_\perp}(w) := \pi_{\mathfrak{a}_\perp}[w(\mu+\rho) - \rho] - \mathcal{D}_{\mathfrak{a}_\perp}, \quad w \in W.$$

Among the weights $\{\mu_{\mathfrak{a}_\perp}(w)\,|w \in W\}$ one can always choose those located in the fundamental chamber $\overline{C_{\mathfrak{a}_\perp}}$. Let U be the set of representatives u for the classes $W/W_{\mathfrak{a}_\perp}$ such that

$$U := \left\{u \in W | \quad \mu_{\mathfrak{a}_\perp}(u) \in \overline{C_{\mathfrak{a}_\perp}}\right\} \quad . \tag{6}$$

Thus we can form the subsets:

$$\mu_{\widetilde{\mathfrak{a}}}(u) := \pi_{\widetilde{\mathfrak{a}}}[u(\mu+\rho) - \rho] + \mathcal{D}_{\mathfrak{a}_\perp}, \quad u \in U, \tag{7}$$

and

$$\mu_{\mathfrak{a}_\perp}(u) := \pi_{\mathfrak{a}_\perp}[u(\mu+\rho) - \rho] - \mathcal{D}_{\mathfrak{a}_\perp}, \quad u \in U. \tag{8}$$

Notice that the subalgebra $\mathfrak{a}_\perp$ is regular by definition since it is built on a subset of roots of the algebra $\mathfrak{g}$.

For the modules we are interested in the Weyl-Kac formula for $\mathrm{ch}\,(L^\mu)$ can be written in terms of singular elements [9],

$$\Psi^{(\mu)} := \sum_{w\in W} \epsilon(w) e^{w(\mu+\rho)-\rho},$$

namely,

$$\mathrm{ch}\,(L^\mu) = \frac{\Psi^{(\mu)}}{\Psi^{(0)}} = \frac{\Psi^{(\mu)}}{R}. \tag{9}$$

The same is true for the submodules $\mathrm{ch}\,(L^\nu_{\mathfrak{a}})$ in (1)

$$\mathrm{ch}\,(L^\nu_{\mathfrak{a}}) = \frac{\Psi^{(\nu)}_{\mathfrak{a}}}{\Psi^{(0)}_{\mathfrak{a}}} = \frac{\Psi^{(\nu)}_{\mathfrak{a}}}{R_{\mathfrak{a}}},$$

with

$$\Psi^{(\nu)}_{\mathfrak{a}} := \sum_{w\in W_{\mathfrak{a}}} \epsilon(w) e^{w(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}.$$

Applying formula (9) to the branching rule (1) we get the relation connecting the singular elements $\Psi^{(\mu)}$ and $\Psi^{(\nu)}_{\mathfrak{a}}$:

$$\begin{aligned}
\pi_{\mathfrak{a}}\left(\frac{\sum_{w\in W}\epsilon(w)e^{w(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}\right) &= \sum_{\nu\in P^+_{\mathfrak{a}}} b^{(\mu)}_\nu \frac{\sum_{w\in W_{\mathfrak{a}}}\epsilon(w)e^{w(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}}{\prod_{\beta\in\Delta^+_{\mathfrak{a}}}(1-e^{-\beta})^{\mathrm{mult}_{\mathfrak{a}}(\beta)}},\\
\pi_{\mathfrak{a}}\left(\frac{\Psi^{(\mu)}}{R}\right) &= \sum_{\nu\in P^+_{\mathfrak{a}}} b^{(\mu)}_\nu \frac{\Psi^{(\nu)}_{\mathfrak{a}}}{R_{\mathfrak{a}}}.
\end{aligned} \tag{10}$$

2.2 Decomposing the singular element.

Now we shall perform a decomposition of the singular element $\Psi^{(\mu)}$ in terms of singular elements of the orthogonal partner modules:

Lemma 1. *Let $\mathfrak{a}_\perp$ be the orthogonal partner of a reductive subalgebra $\mathfrak{a}\hookrightarrow\mathfrak{g}$ with $\mathfrak{h}=\mathfrak{h}_{\mathfrak{a}}\oplus\mathfrak{h}_{\mathfrak{a}_\perp}\oplus\mathfrak{h}_\perp$, $\widetilde{\mathfrak{a}_\perp}=\mathfrak{a}_\perp\oplus\mathfrak{h}_\perp$ and $\widetilde{\mathfrak{a}}=\mathfrak{a}\oplus\mathfrak{h}_\perp$.*

L^μ be the highest weight integrable module with $\mu\in P^+$ and

$\Psi^{(\mu)}$ – the singular element of L^μ.

Then the element $\Psi^{(\mu)}$ can be decomposed into the sum over $u\in U$ (see (6)) of singular elements $\Psi^{\mu_{\mathfrak{a}_\perp}(u)}_{\mathfrak{a}_\perp}$ with the coefficients $\epsilon(u)e^{\mu_{\widetilde{\mathfrak{a}}}(u)}$:

$$\Psi^{(\mu)} = \sum_{u\in U}\epsilon(u)e^{\mu_{\widetilde{\mathfrak{a}}}(u)}\Psi^{\mu_{\mathfrak{a}_\perp}(u)}_{\mathfrak{a}_\perp}. \tag{11}$$

Доказательство. Let

$$u(\mu+\rho)=\pi_{(\widetilde{\mathfrak{a}})}u(\mu+\rho)+\pi_{(\mathfrak{a}_\perp)}u(\mu+\rho)$$

with $u\in U$. For any $v\in W_{\mathfrak{a}_\perp}$ consider the singular weight $vu(\mu+\rho)-\rho$ and perform the decomposition:

$$\begin{array}{rcl} vu(\mu+\rho)-\rho & = & \pi_{(\mathfrak{a})}\left(u(\mu+\rho)\right)-\rho+\rho_{\mathfrak{a}_\perp} \\ & & +\,v\left(\pi_{(\widetilde{\mathfrak{a}}_\perp)}u(\mu+\rho)-\rho_{\mathfrak{a}_\perp}+\rho_{\mathfrak{a}_\perp}\right)-\rho_{\mathfrak{a}_\perp}. \end{array} \tag{12}$$

Use the defect $\mathcal{D}_{\mathfrak{a}_\perp}$ (5) to simplify the first line in (12):

$$\begin{array}{r} \pi_{(\widetilde{\mathfrak{a}})}\left(u(\mu+\rho)\right)-\rho+\rho_{\mathfrak{a}_\perp}= \\ \pi_{(\widetilde{\mathfrak{a}})}\left(u(\mu+\rho)\right)-\pi_{\widetilde{\mathfrak{a}}}\rho-\pi_{\mathfrak{a}_\perp}\rho+\rho_{\mathfrak{a}_\perp}= \\ =\pi_{(\widetilde{\mathfrak{a}})}\left(u(\mu+\rho)-\rho\right)+\mathcal{D}_{\mathfrak{a}_\perp}, \end{array}$$

and the second one:

$$\begin{array}{r} v\left(\pi_{(\mathfrak{a}_\perp)}u(\mu+\rho)-\rho_{\mathfrak{a}_\perp}+\rho_{\mathfrak{a}_\perp}\right)-\rho_{\mathfrak{a}_\perp}= \\ v\left(\pi_{(\mathfrak{a}_\perp)}u(\mu+\rho)-\mathcal{D}_{\mathfrak{a}_\perp}-\pi_{(\mathfrak{a}_\perp)}\rho+\rho_{\mathfrak{a}_\perp}\right)-\rho_{\mathfrak{a}_\perp}= \\ =v\left(\pi_{(\mathfrak{a}_\perp)}\left[u(\mu+\rho)-\rho\right]-\mathcal{D}_{\mathfrak{a}_\perp}+\rho_{\mathfrak{a}_\perp}\right)-\rho_{\mathfrak{a}_\perp}. \end{array}$$

This provides the desired decomposition of the singular element Ψ^μ in terms of singular elements $\Psi^\eta_{\mathfrak{a}_\perp}$ of the $\mathfrak{a}_\perp$-modules $L^\eta_{\mathfrak{a}_\perp}$:

$$\begin{array}{l} \Psi^\mu=\sum_{u\in U}\sum_{v\in W_{\mathfrak{a}_\perp}}\epsilon(v)\epsilon(u)e^{vu(\mu+\rho)-\rho}= \\ =\sum_{u\in U}\epsilon(u)e^{\pi_{\widetilde{\mathfrak{a}}}[u(\mu+\rho)-\rho]+\mathcal{D}_{\mathfrak{a}_\perp}}\sum_{v\in W_{\mathfrak{a}_\perp}}\epsilon(v)e^{v\left(\pi_{(\mathfrak{a}_\perp)}[u(\mu+\rho)-\rho]-\mathcal{D}_{\mathfrak{a}_\perp}+\rho_{\mathfrak{a}_\perp}\right)-\rho_{\mathfrak{a}_\perp}}= \\ =\sum_{u\in U}\ \epsilon(u)\Psi_{\mathfrak{a}_\perp}^{\pi_{(\mathfrak{a}_\perp)}[u(\mu+\rho)-\rho]-\mathcal{D}_{\mathfrak{a}_\perp}}e^{\pi_{(\widetilde{\mathfrak{a}})}[u(\mu+\rho)-\rho]+\mathcal{D}_{\mathfrak{a}_\perp}}. \end{array} \tag{13}$$

□

Remark 1. This relation can be considered as a generalized form of the Weyl formula for the singular element $\Psi^\mu_{\mathfrak{g}}$: the vectors $\mu_{\widetilde{\mathfrak{a}}}(u)$ play the role of singular weights while the alternating factors $\epsilon(u)$ are extended to $\epsilon(u)\Psi_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}$. In fact when $\mathfrak{a}=\mathfrak{g}$ both $\mathfrak{a}_\perp$and $\mathfrak{h}_\perp$are zeros, $U=W$, and the original Weyl formula is reobtained so far as the singular elements $\epsilon(u)\Psi_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}=\epsilon(u)$ become trivial. In the opposite limit when $\mathfrak{a}=\mathfrak{o}$, $\Delta_{\mathfrak{a}_\perp}=\Delta_{\mathfrak{g}}$, $\mathfrak{h}^*_\perp=0$, $\mathfrak{a}_\perp=\mathfrak{g}$, $\mathcal{D}_{\mathfrak{a}_\perp}=0$ and $U=W/W_{\mathfrak{a}_\perp}=e$ the singular element Ψ^μ is again reobtained, now via the trivilization of the set of vectors $\mu_{\mathfrak{a}}(e)=0$.

Remark 2. In [7] the decomposition analogous to (13) was used to construct the recurrent relations for branching coefficients $k_{\xi}^{(\mu)}$ corresponding to the injection $\mathfrak{a} \hookrightarrow \mathfrak{g}$:

$$\begin{gathered} k_{\xi}^{(\mu)} = -\frac{1}{s(\gamma_0)} \left(\sum_{u \in U} \epsilon(u) \ \dim \left(L_{\mathfrak{a}_{\perp}}^{\mu_{\mathfrak{a}_{\perp}}(u)} \right) \delta_{\xi - \gamma_0, \pi_{\widetilde{\mathfrak{a}}}(u(\mu+\rho)-\rho)} + \right. \\ \left. + \sum_{\gamma \in \Gamma_{\widetilde{\mathfrak{a}} \rightarrow \mathfrak{g}}} s\left(\gamma + \gamma_0\right) k_{\xi+\gamma}^{(\mu)} \right). \end{gathered} \tag{14}$$

The recursion is goverened by the set $\Gamma_{\widetilde{\mathfrak{a}} \rightarrow \mathfrak{g}}$ called the injection fan. The latter is defined by the carrier set $\{\xi\}_{\mathfrak{a} \rightarrow \mathfrak{g}}$ for the coefficient function $s(\xi)$

$$\{\xi\}_{\widetilde{\mathfrak{a}} \rightarrow \mathfrak{g}} := \{\xi \in P_{\widetilde{\mathfrak{a}}} | s(\xi) \neq 0\}$$

appearing in the expansion

$$\prod_{\alpha \in \Delta^{+} \setminus \Delta_{\perp}^{+}} \left(1 - e^{-\pi_{\widetilde{\mathfrak{a}}} \alpha}\right)^{\mathrm{mult}(\alpha) - \mathrm{mult}_{\mathfrak{a}}(\pi_{\widetilde{\mathfrak{a}}} \alpha)} = - \sum_{\gamma \in P_{\widetilde{\mathfrak{a}}}} s(\gamma) e^{-\gamma}; \tag{15}$$

The weights in $\{\xi\}_{\widetilde{\mathfrak{a}} \rightarrow \mathfrak{g}}$ are to be shifted by γ_0 – the lowest vector in $\{\xi\}$ – and the zero element is to be eliminated:

$$\Gamma_{\widetilde{\mathfrak{a}} \rightarrow \mathfrak{g}} = \{\xi - \gamma_0 | \xi \in \{\xi\}\} \setminus \{0\}. \tag{16}$$

The recursion relation (14) was originally used to describe branchings for integrable modules. Notice that there exists an important class of modules that also can be reduced with the help of the injection fan – these are Verma modules.

2.3 Weyl-Verma formulas.

Statement 1. *For an orthogonal subalgebra $\mathfrak{a}_{\perp}$ in $\mathfrak{g}$ (an orthogonal partner of a reductive $\mathfrak{a} \hookrightarrow \mathfrak{g}$) the character of an integrable highest weight module L^{μ} can be presented as a combination (with integral coefficients) of parabolic Verma modules distributed by the set of weights $e^{\mu_{\widetilde{\mathfrak{a}}}(u)}$:*

$$\mathrm{ch}\left(L^{\mu}\right) = \sum_{u \in U} \epsilon(u) e^{\mu_{\widetilde{\mathfrak{a}}}(u)} \mathrm{ch} M_{I}^{\mu_{\mathfrak{a}_{\perp}}(u)}, \tag{17}$$

where $U := \left\{u \in W | \quad \mu_{\mathfrak{a}_{\perp}}(u) \in \overline{C_{\mathfrak{a}_{\perp}}}\right\}$ and I is such a subset of S that Δ_{I}^{+} is equivalent to $\Delta_{\mathfrak{a}_{\perp}}^{+}$.

Доказательство. By the definition (2) the subalgebra $\mathfrak{a}_\perp$ is regular and reductive. Consider its Weyl denominator $R_{\mathfrak{a}_\perp} := \prod_{\alpha\in\Delta^+_{\mathfrak{a}_\perp}} \left(1-e^{-\alpha}\right)^{\mathrm{mult}_{\mathfrak{a}}(\alpha)}$ and the element $R_J := \prod_{\alpha\in\Delta^+\setminus\Delta^+_{\mathfrak{a}_\perp}} \left(1-e^{-\alpha}\right)^{\mathrm{mult}(\alpha)}$ as the factors in R:

$$R = R_J R_{\mathfrak{a}_\perp}.$$

According to this factorization and the decomposition (11) the character $\mathrm{ch}\left(L^\mu\right)$ can be written as

$$\begin{aligned}
\mathrm{ch}\left(L^\mu\right) &= \left(R_J\right)^{-1}\left(R_{\mathfrak{a}_\perp}\right)^{-1}\Psi^\mu = \left(R_J\right)^{-1}\sum_{u\in U} e^{\mu_{\widetilde{\mathfrak{a}}}(u)}\epsilon(u)\left(R_{\mathfrak{a}_\perp}\right)^{-1}\Psi^{\mu_{\mathfrak{a}_\perp}(u)}_{\mathfrak{a}_\perp}\\
&= \left(R_J\right)^{-1}\sum_{u\in U} e^{\mu_{\widetilde{\mathfrak{a}}}(u)}\epsilon(u)\mathrm{ch}\left(L^{\mu_{\mathfrak{a}_\perp}(u)}_{\mathfrak{a}_\perp}\right),
\end{aligned}$$

where $\left\{L^{\mu_{\mathfrak{a}_\perp}(u)}_{\mathfrak{a}_\perp}|u\in U\right\}$ is the set of finite-dimensional $\mathfrak{a}_\perp$-modules with the highest weights $\mu_{\mathfrak{a}_\perp}(u)$. We are interested in nontrivial subalgebras $\mathfrak{a}$ and correspondingly in nontrivial $\mathfrak{a}_\perp$ (the case of a trivial orthogonal subalgebra was considered above (see Remark 1)). This means that $r_{\mathfrak{a}}\geq 1$ and $r_{\mathfrak{a}_\perp}<r$. Due to the fact that any maximal regular subalgebra has the Dynkin scheme obtained by one or two node subtractions from the extended Dynkin scheme and the extended scheme has at most one dependent root (the highest root) the set of roots $\Delta^+_{\mathfrak{a}_\perp}$ is always equivalent to the one Δ^+_I generated by some subset $I\subset S$ of simple roots.

It follows that we can (by redefining the set Δ^+) identify $\Delta^+_{\mathfrak{a}_\perp}$ with the subset Δ^+_I where $I\subset S$. This allows us to introduce the elements necessary to compose the generalized Verma modules [3, 12]. We have two sets of root vectors $\left\{x_\xi\in\mathfrak{g}_\xi|\xi\in\Delta^+_I\right\}$ and $\left\{x_\eta\in\mathfrak{g}_\eta|\eta\in\Delta^+\setminus\Delta^+_I\right\}$ and the corresponding nilpotent subalgebras in $\mathfrak{n}^+$:

$$\mathfrak{n}^+_I := \sum_{\xi\in\Delta^+_I}\mathfrak{g}_\xi, \quad \mathfrak{u}^+_I := \sum_{\eta\in\Delta^+\setminus\Delta^+_I}\mathfrak{g}_\eta.$$

The first subalgebra together with its negative counterpart $\mathfrak{n}^-_I$ generates a simple subalgebra

$$\mathfrak{s}_I = \mathfrak{n}^-_I + \mathfrak{h}_I + \mathfrak{n}^+_I.$$

We enlarge it with the remaining Cartan generators:

$$\mathfrak{l}_I = \mathfrak{n}^-_I + \mathfrak{h} + \mathfrak{n}^+_I.$$

The semidirect product of $\mathfrak{l}_I$ and $\mathfrak{u}_I^+$ gives a parabolic subalgebra $\mathfrak{p}_I \hookrightarrow \mathfrak{g}$:

$$\mathfrak{p}_I = \mathfrak{l}_I \rhd \mathfrak{u}_I^+. \tag{18}$$

Its universal enveloping $U(\mathfrak{p}_I)$ is a subalgebra in $U(\mathfrak{g})$. The $\mathfrak{l}_I$-modules $L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}$ can be easily lifted to $\mathfrak{p}_I$-modules using the trivial action of the nilradical $\mathfrak{u}_I^+$. The latter induce $U(\mathfrak{g})$-modules in a standard way:

$$M_I^{\mu_{\mathfrak{a}_\perp}(u)} = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_I)} L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}.$$

These are the *generalized Verma modules* [3] generated by the highest weights $\mu_{\mathfrak{a}_\perp}(u)$. As a $U\left(\mathfrak{u}_I^-\right)$-module each $M_I^{\mu_{\mathfrak{a}_\perp}(u)}$ is isomorphic to $U\left(\mathfrak{u}_I^-\right) \otimes L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}$ and thus its character can be written in terms of Kostant-Heckman function [13] corresponding to the injection of the orthogonal partner $\mathfrak{a}_\perp \hookrightarrow \mathfrak{g}$:

$$\mathrm{ch} M_I^{\mu_{\mathfrak{a}_\perp}(u)} = \mathcal{KH}_{\mathfrak{a}_\perp \hookrightarrow \mathfrak{g}} \mathrm{ch} L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}.$$

The function $\mathcal{KH}_{\mathfrak{a}_\perp \hookrightarrow \mathfrak{g}}$ is generated by the denominator R_I thus the last expression can be written in the form

$$\mathrm{ch} M_I^{\mu_{\mathfrak{a}_\perp}(u)} = \frac{1}{R_I} \mathrm{ch} L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}.$$

This means that we have obtained the generalized Weyl-Verma character formula – the decomposition of $\mathrm{ch}\left(L^\mu\right)$ in terms of generalized Verma module characters:

$$\mathrm{ch}\left(L^\mu\right) = \sum_{u \in U} e^{\mu_{\widetilde{\mathfrak{a}}}(u)} \epsilon(u) \mathrm{ch} M_I^{\mu_{\mathfrak{a}_\perp}(u)}. \tag{19}$$

$\square$

Remark 3. Here the generalized Weyl-Verma character formula (called the alternating sum formula in [12]) appears in a special form: the weights $\mu_{\widetilde{\mathfrak{a}}}$ and the generalized Verma module highest weights $\mu_{\mathfrak{a}_\perp}$ are separated. The reason is that the highest weight of M_I-module is not equal to the projection of its maximal weight to $h_{\mathfrak{a}_\perp}^*$ (but must be additionally shifted by the defect).

Example 1. Consider the generalized Verma modules for the embedding $A_1 \hookrightarrow B_2$ with the subalgebra $\mathfrak{a}_\perp$ attributed to the root α_1 of B_2. The generalized Verma module $M_I^{\omega_1}$ with the highest weight $\omega_1 = e_1$ is shown in Figure 1.

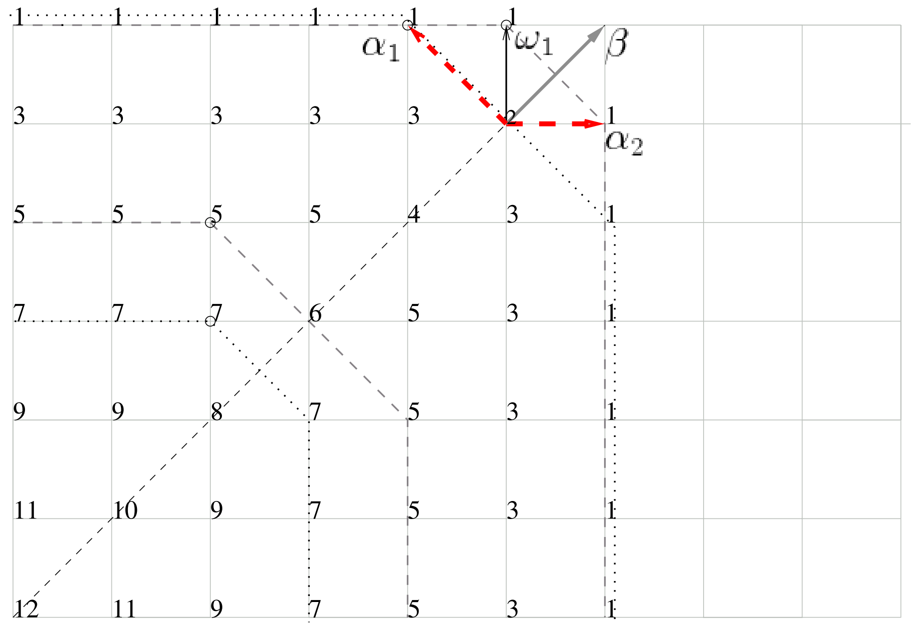

Рис. 1: Generalized Verma modules for the regular embedding of A_1 into B_2. Simple roots α_1, α_2 of B_2 are presented as the dashed vectors. The simple root $\beta = \alpha_1 + 2\alpha_2$ of A_1 is indicated as the grey vector. The decomposition of L^{ω_1} is indicated by the set of contours of the involved generalized Verma modules. Dashed contours correspond to positive $\epsilon(u)$ and dotted to negative.

Remark 4. As it was proved in [12] (see Proposition 9.6) characters of the generalized Verma modules $M_I^{\mu_{\mathfrak{a}_\perp}(u)}$ can be described as linear combinations of ordinary Verma modules of $\mathfrak{g}$:

$$\mathrm{ch} M_I^{\mu_{\mathfrak{a}_\perp}(u)} = \sum_{w \in W_{\mathfrak{a}_\perp}} \epsilon\left(w\right) \mathrm{ch} M^{w\left(\mu_{\mathfrak{a}_\perp}(u)+\rho_{\mathfrak{a}_\perp}\right)-\rho_{\mathfrak{a}_\perp}}$$

Substituting this expression in (19) and using the definitions (7,8) and (5) we reconstruct the standard Weyl-Verma decomposition of the character:

$$\mathrm{ch}\left(L^\mu\right) = \sum_{w \in W} \epsilon(u) \mathrm{ch} M^{w(\mu+\rho)-\rho}.$$

3 BGG resolution and branching

In [3] it was demonstrated that for the highest weight module L^{μ} with $\mu \in P^{+}$ the sequence

$$0 \rightarrow M_r^I \xrightarrow{\delta_r} M_{r-1}^I \xrightarrow{\delta_{r-1}} \dots \xrightarrow{\delta_1} M_0^I \xrightarrow{\varepsilon} L^{\mu} \rightarrow 0, \qquad (20)$$

with

$$M_k^I = \bigoplus_{u \in U,\ \mathrm{length}(u)=k} M_I^{u(\mu+\rho)-\rho}, \quad M_0^I = M_I^{\mu} \qquad (21)$$

(the generalized BGG resolution) is exact and formula (17) is a cosequence of this resolution.

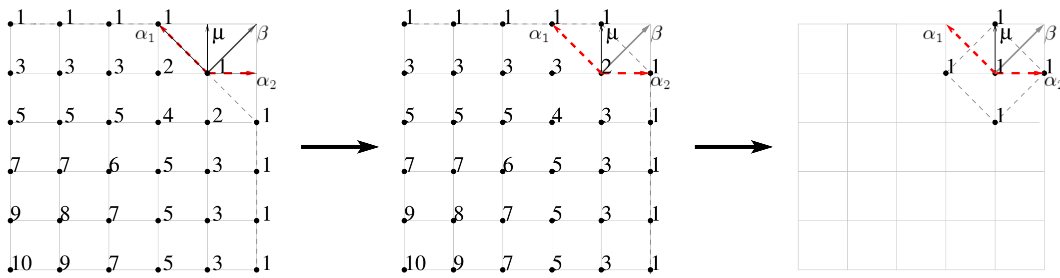

Рис. 2: Injection $A_1 \hookrightarrow B_2$ (see Figure 1). The orthogonal partner is A_1 corresponding to the root α_1. The resolution of the simple module L^{ω_1}. Presented is the central part of the exact sequence $0 \rightarrow Im(\delta_2) \rightarrow \left(e^{\mu_{\widetilde{\mathfrak{a}}}(e)} \mathrm{ch} M_I^{\pi_{\mathfrak{a}_\perp}[\omega_1]-\mathcal{D}_{\mathfrak{a}_\perp}} = M_I^{\omega_1}\right) \rightarrow L^{\omega_1} \rightarrow 0$. Here $\mu_{\widetilde{\mathfrak{a}}}(e) = \pi_{\widetilde{\mathfrak{a}}}[\mu] + \mathcal{D}_{\mathfrak{a}_\perp}$.

Statement 2. *Let L^{μ} be the highest weight $\mathfrak{g}$-module with $\mu \in P^{+}$,let its regular subalgebra $\mathfrak{a}_\perp \hookrightarrow \mathfrak{g}$ be orthogonal to a reductive subalgebra $\mathfrak{a} \hookrightarrow \mathfrak{g}$. Then the decomposition (11) defines both the generalized resolution of L^{μ} with respect to $\mathfrak{a}_\perp$ and the branching rules for L^{μ} with respect to $\mathfrak{a}$.*

Доказательство. Put

$$\mathrm{ch} M_I^{u(\mu+\rho)-\rho} = e^{\mu_{\widetilde{\mathfrak{a}}}(u)} \mathrm{ch} M_I^{\mu_{\mathfrak{a}_\perp}(u)}, \mathrm{ch} M_I^{\mu} = e^{\mu_{\widetilde{\mathfrak{a}}}(e)} \mathrm{ch} M_I^{\pi_{\mathfrak{a}_\perp}[\mu]-\mathcal{D}_{\mathfrak{a}_\perp}}$$

with $\mu_{\widetilde{\mathfrak{a}}}(u), \mu_{\mathfrak{a}_\perp}(u)$ and $\mathcal{D}_{\mathfrak{a}_\perp}$ as in Lemma 1 and $u \in U$ defined by (6). This gives the elements of the filtration sequence (20).

Consider the set $\{\mu_{\mathfrak{a}_\perp}(u)\,|u\in U\}$ as the highest weights for the simple modules $L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}$ and evaluate their dimensions. Together with $\{\mu_{\widetilde{\mathfrak{a}}}(u)\,|u\in U\}$ this gives the set of singular weights

$$\left\{\epsilon(u)\, e^{\mu_{\widetilde{\mathfrak{a}}}(u)}\dim\left(L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)}\right)\right\}.$$

The branching $L^\mu_{\mathfrak{g}\downarrow\mathfrak{a}}=\bigoplus\limits_{\nu\in P^+_{\mathfrak{a}}} b_\nu^{(\mu)}L^\nu_{\mathfrak{a}}$ is then fixed by the injection fan $\Gamma_{\mathfrak{a}\rightarrow\mathfrak{g}}$ and the relation (14). The latter gives us the coefficients $k_\xi^{(\mu)}$ and thus defines $b_\nu^{(\mu)}$ due to the property $b_\nu^{(\mu)}=k_\nu^{(\mu)}$ for $\nu\in\overline{C_{\mathfrak{a}}}$. □

Corollary 0.1. *Let L^μ be the highest weight $\mathfrak{g}$-module with $\mu\in P^+$ and $\mathfrak{a}\hookrightarrow\mathfrak{g}$ – a reductive subalgebra in $\mathfrak{g}$. Let $\mathfrak{a}_\perp$, the orthogonal partner for $\mathfrak{a}$, be equivalent to A_1, $\mathfrak{a}_\perp\approx A_1$, and $\widetilde{\mathfrak{a}}=\mathfrak{a}\oplus\mathfrak{h}_\perp$ with $\mathfrak{h}=\mathfrak{h}_{\mathfrak{a}}\oplus\mathfrak{h}_{\mathfrak{a}_\perp}\oplus\mathfrak{h}_\perp$. Let $L^\mu_{\mathfrak{g}\downarrow\widetilde{\mathfrak{a}}}=\bigoplus\limits_{\nu\in P^+_{\widetilde{\mathfrak{a}}}} b_\nu^{(\mu)}L^\nu_{\widetilde{\mathfrak{a}}}$ be the branching of L^μ with respect to $\widetilde{\mathfrak{a}}$. Then the branching coefficients $b_\nu^{(\mu)}$ define the generalized resolution (20) of L^μ with respect to $\mathfrak{a}_\perp$.*

Доказательство. Let α be the simple root of A_1. Use the Weyl transformations to identify it with some simple root of $\mathfrak{g}$, say α_1. Construct the singular element for the module $L^\mu_{\mathfrak{g}\downarrow\widetilde{\mathfrak{a}}}$, i.e. the $\Psi_{\widetilde{\mathfrak{a}}}^{\left(L^\mu_{\mathfrak{g}\downarrow\widetilde{\mathfrak{a}}}\right)}=\sum_{\nu\in P^+_{\widetilde{\mathfrak{a}}},b_\nu^{(\mu)}>0} b_\nu^{(\mu)}\Psi_{\widetilde{\mathfrak{a}}}^{(\nu)}$, and decompose it $\Psi_{\widetilde{\mathfrak{a}}}^{\left(L^\mu_{\mathfrak{g}\downarrow\widetilde{\mathfrak{a}}}\right)}=k_\xi^{(\mu)}e^\xi$. In our case the representatives u in the recurrent relation (14) are uniquely determined by the weight ξ:

$$\epsilon(u\left(\xi\right))\ \dim\left(L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u(\xi))}\right)=-s\left(\gamma_0\right)k_\xi^{(\mu)}-\sum_{\gamma\in\Gamma_{\widetilde{\mathfrak{a}}\rightarrow\mathfrak{g}}}s\left(\gamma+\gamma_0\right)k_{\xi+\gamma}^{(\mu)}.$$

We have

$$\dim\left(L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u(\xi))}\right)=\left|s\left(\gamma_0\right)k_\xi^{(\mu)}+\sum_{\gamma\in\Gamma_{\widetilde{\mathfrak{a}}\rightarrow\mathfrak{g}}}s\left(\gamma+\gamma_0\right)k_{\xi+\gamma}^{(\mu)}\right|$$

and

$$\mu_{\mathfrak{a}_\perp}\left(u\left(\xi\right)\right)=\frac{1}{2}\left(\dim\left(L_{A_1}^{\mu(\xi)}\right)-1\right)\alpha_1$$

The set of generalized Verma modules $e^{\xi+\mathcal{D}_{\mathfrak{a}_\perp}}\mathrm{ch}M_I^{\mu_{\mathfrak{a}_\perp}(u(\xi))}$ is thus fixed:

$$\left\{ e^{\mu_{\tilde{\mathfrak{a}}}(u)}\mathrm{ch}M_I^{\mu_{\mathfrak{a}_\perp}(u)} | u \in U \right\}.$$

Classifying these modules according to the length of u we get the components (21) of the resolution (20). $\square$

4 Conclusions

In [7] it was demonstrated that the injection fan recursive mechanism works also for special injections. It must be mentioned that in this case the Weyl-Verma decompositions can also be obtained. The resolutions corresponding to special subalgebras describe the relations between the projections of characters of the initial module and the generalized Verma modules with highest weights in the subspace of h^*.

Consider the situation where the simple roots are prescribed by some external factors (originating in physical applications conditions, for example). In this case the orthogonal partner cannot be generated by simple root vectors only. The elements $\mathfrak{u}_I^+ := \sum_{\eta \in \Delta^+ \setminus \Delta_I^+} \mathfrak{g}_\eta$ do not form a subalgebra in $\mathfrak{g}$ because some nonsimple roots are lost in $\Delta^+ \setminus \Delta_I^+$. It is important to indicate that in this case the Weyl-Verma formula still exists. In it the generalized Verma modules correspond to the contractions [14] of the algebra $\mathfrak{n}^+$ and the Weyl-Verma relations describe the decomposition of the representation space of L^μ into the set of generalized Verma modules of contracted algebra $U(\mathfrak{n}_c^+)$. The weight vectors are formed by the PBW-basis of $U(\mathfrak{n}_c^+)$ and of $U(\mathfrak{a}_\perp)$. To consider such space as a $\mathfrak{g}$-module we must perform the deformation [15] of the algebra $\mathfrak{n}_c^+$ (and thus restore the initial composition law). The space survives and after such a deformation the initial algebra generators will act properly on it.

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