SAINT-PETERSBURG STATE UNIVERSITY

SPbU-IP-09-08

Recursive algorithms, branching coefficients and applications

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SAINT-PETERSBURG
2009

Abstract

Recurrent relations for branching coefficients in affine Lie algebras integrable highest weight modules are studied. The decomposition algorithm based on the injection fan technique is adopted to the situation where the Weyl denominator becomes singular with respect to a reductive subalgebra. We study some modifications of the injection fan technique and demonstrate that it is possible to define the "subtracted fans" that play the role similar to the original ones. Possible applications of subtracted fans in CFT models are considered.

1 Introduction

The branching problem for affine Lie algebras emerges in conformal field theory, for example, in the construction of modular-invariant partition functions [1]. Recently the problem of the conformal embeddings was considered in the paper [2].

There exist several approaches to deal with the branching coefficients. Some of them use the BGG resolution [3] (for Kac-Moody algebras the algorithm is described in [4],[5]), the Schure function series [6], the BRST cohomology [7], Kac-Peterson formulas [4, 8] or the combinatorial methods applied in [9].

Usually only the maximal reductive subalgebras are considered since the case of non-maximal subalgebra can be obtained using the chain of maximal injections. In this paper we find the recurrent properties for branching coefficients that generalise the relations obtained earlier (see the paper [10] and the references therein) to the case of non-maximal reductive subalgebra. The result is formulated in terms of new injection fan called "the subtracted fan". Using this new tools we formulate simple and explicit algorithm for computations of branching coefficients which is applicable to the non-maximal subalgebras of finite-dimensional and affine Lie algebras.

We demonstrate that our algorithm can be used in the study of conformal embeddings where the central charge of the conformal field theory is preserved, and the computations are simplified by taking into account some physical limitations. In the study of the coset construction of rational conformal field theory the algorithm is useful for the computation of the characters and states.

The paper is organised as follows. In the subsection 1.1 we fix the notations used throughout the paper. In the Section 2 we derive the subtracted recurrent formula for anomalous branching coefficients and describe the decomposition algorithm for integrable highest weight modules of algebra $\mathfrak{g}$

with respect to a reductive subalgebra $\mathfrak{a}$ (subsection 2.2). In the Section 3 we present several simple examples for finite-dimensional Lie algebras to illustrate the algorithm. Then we consider the affine Lie algebras with the applications in CFT models (Section 4). We conclude the paper with a review of obtained results. Possible future developments are discussed (Section 5).

1.1 Notation

Consider affine Lie algebras $\mathfrak{g}$ and $\mathfrak{a}$ with the underlying finite-dimensional subalgebras $\overset{\circ}{\mathfrak{g}}$ and $\overset{\circ}{\mathfrak{a}}$ and an injection $\mathfrak{a} \longrightarrow \mathfrak{g}$ such that $\mathfrak{a}$ is a reductive subalgebra $\mathfrak{a} \subset \mathfrak{g}$ with correlated root spaces: $\mathfrak{h}^*_{\mathfrak{a}} \subset \mathfrak{h}^*_{\mathfrak{g}}$ and $\mathfrak{h}^*_{\overset{\circ}{\mathfrak{a}}} \subset \mathfrak{h}^*_{\overset{\circ}{\mathfrak{g}}}$.

We use the following notations adopted from the paper [10].

L^{μ} $(L^{\nu}_{\mathfrak{a}})$ — the integrable module of $\mathfrak{g}$ with the highest weight μ ; (resp. integrable $\mathfrak{a}$ -module with the highest weight ν);

r , $(r_{\mathfrak{a}})$ — the rank of the algebra $\mathfrak{g}$ (resp. $\mathfrak{a}$) ;

Δ $(\Delta_{\mathfrak{a}})$— the root system; Δ^+ (resp. $\Delta^+_{\mathfrak{a}}$)— the positive root system (of $\mathfrak{g}$ and $\mathfrak{a}$ respectively);

$\mathrm{mult}\,(\alpha)$ $(\mathrm{mult}_{\mathfrak{a}}\,(\alpha))$ — the multiplicity of the root α in Δ (resp. in $(\Delta_{\mathfrak{a}})$);

$\overset{\circ}{\Delta}$, $\left(\overset{\circ}{\Delta}_{\mathfrak{a}}\right)$ — the finite root system of the subalgebra $\overset{\circ}{\mathfrak{g}}$ (resp. $\overset{\circ}{\mathfrak{a}}$); Θ, $(\Theta_{\mathfrak{a}})$ — the highest root of the algebra $\mathfrak{g}$ (resp. subalgebra $\mathfrak{a}$);

$\mathcal{N}^{\mu}$, $(\mathcal{N}^{\nu}_{\mathfrak{a}})$ — the weight diagram of L^{μ} (resp. $L^{\nu}_{\mathfrak{a}}$) ;

W , $(W_{\mathfrak{a}})$— the corresponding Weyl group;

C , $(C_{\mathfrak{a}})$— the fundamental Weyl chamber;

$\bar{C}$, $\left(\bar{C}_{\mathfrak{a}}\right)$ — the closure of the fundamental Weyl chamber;

ρ , $(\rho_{\mathfrak{a}})$ — the Weyl vector;

$\epsilon\,(w) := \det\,(w)$;

α_i , $\left(\alpha_{(\mathfrak{a})j}\right)$ — the i-th (resp. j-th) basic root for $\mathfrak{g}$ (resp. $\mathfrak{a}$); $i = 0, \ldots, r$, $(j = 0, \ldots, r_{\mathfrak{a}})$;

δ — the imaginary root of $\mathfrak{g}$ (and of $\mathfrak{a}$ if any);

$\alpha^{\vee}_i$, $\left(\alpha^{\vee}_{(\mathfrak{a})j}\right)$— the basic coroot for $\mathfrak{g}$ (resp. $\mathfrak{a}$) , $i = 0, \ldots, r$; $(j = 0, \ldots, r_{\mathfrak{a}})$;

$\overset{\circ}{\xi}$, $\overset{\circ}{\xi}_{(\mathfrak{a})}$ — the finite (classical) part of the weight $\xi \in P$, $\left(\text{resp. } \xi_{(\mathfrak{a})} \in P_{\mathfrak{a}}\right)$;

$\lambda = \left(\overset{\circ}{\lambda}; k; n\right)$ — the decomposition of an affine weight indicating the finite part $\overset{\circ}{\lambda}$, level k and grade n .

P (resp. $P_{\mathfrak{a}}$) — the weight lattice;

M (resp. $M_{\mathfrak{a}}$) :=

$$= \left\{ \begin{array}{c} \sum_{i=1}^{r} \mathbf{Z}\alpha_i^{\vee} \ \left(\text{resp. } \sum_{i=1}^{r} \mathbf{Z}\alpha_{(\mathfrak{a})i}^{\vee}\right) \text{ for untwisted algebras or } A_{2r}^{(2)}, \\ \sum_{i=1}^{r} \mathbf{Z}\alpha_i \ \left(\text{resp. } \sum_{i=1}^{r} \mathbf{Z}\alpha_{(\mathfrak{a})i}\right) \text{ for } A_r^{(u\geq 2)} \text{ and } A \neq A_{2r}^{(2)}, \end{array} \right\};$$

$\Psi^{(\mu)} := \sum\limits_{w\in W} \epsilon(w) e^{w\circ(\mu+\rho)-\rho}$ — the singular weight element for the $\mathfrak{g}$-module L^{μ}; $\Psi^{(\nu)}_{(\mathfrak{a})} := \sum\limits_{w\in W_{\mathfrak{a}}} \epsilon(w) e^{w\circ(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}$ — the corresponding singular weight element for the $\mathfrak{a}$-module $L^{\nu}_{\mathfrak{a}}$;

$\widehat{\Psi^{(\mu)}}$ $\left(\widehat{\Psi^{(\nu)}_{(\mathfrak{a})}}\right)$ — the set of singular weights $\xi \in P$ (resp. $\in P_{\mathfrak{a}}$) for the module L^{μ} (resp. $L^{\nu}_{\mathfrak{a}}$) with the coordinates $\left(\overset{\circ}{\xi}, k, n, \epsilon\left(w\left(\xi\right)\right)\right) |_{\xi=w(\xi)\circ(\mu+\rho)-\rho}$, (resp. $\left(\overset{\circ}{\xi}, k, n, \epsilon\left(w_a\left(\xi\right)\right)\right) |_{\xi=w_a(\xi)\circ(\nu+\rho_a)-\rho_a}$), (this set is similar to $P'_{\text{nice}}\left(\mu\right)$ in [5])

$m^{(\mu)}_{\xi}$, $\left(m^{(\nu)}_{\xi}\right)$ — the multiplicity of the weight $\xi \in P$ (resp. $\in P_{\mathfrak{a}}$) in the module L^{μ} , (resp. $\xi \in L^{\nu}_{\mathfrak{a}}$);

$ch\left(L^{\mu}\right)$ (resp. $ch\left(L^{\nu}_{\mathfrak{a}}\right)$)— the formal character of L^{μ} (resp. $L^{\nu}_{\mathfrak{a}}$);

$ch\left(L^{\mu}\right) = \frac{\sum_{w\in W} \epsilon(w) e^{w\circ(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})^{\text{mult}(\alpha)}} = \frac{\Psi^{(\mu)}}{\Psi^{(0)}}$ — the Weyl-Kac formula.

$R := \prod_{\alpha\in\Delta^+}\left(1-e^{-\alpha}\right)^{\text{mult}(\alpha)} = \Psi^{(0)}$ $\left(\text{resp. } R_{\mathfrak{a}} := \prod_{\alpha\in\Delta^+_{\mathfrak{a}}}\left(1-e^{-\alpha}\right)^{\text{mult}_{\mathfrak{a}}(\alpha)} = \Psi^{(0)}_{\mathfrak{a}}\right)$— the denominator.

$L^{\mu}_{\mathfrak{g}\downarrow\mathfrak{a}} = \bigoplus\limits_{\nu\in P^+_{\mathfrak{a}}} b^{(\mu)}_{\nu} L^{\nu}_{\mathfrak{a}}$ — the module decomposition with respect to $\mathfrak{a} \longrightarrow \mathfrak{g}$;

$b^{(\mu)}_{\nu}$ — the branching coefficients;

$$\sum_{\nu\in\bar{C}_{\mathfrak{a}}} b^{(\mu)}_{\nu} \Psi^{(\nu)}_{(\mathfrak{a})} = \sum_{\lambda\in P_{\mathfrak{a}}} k^{(\mu)}_{\lambda} e^{\lambda} \tag{1}$$

k_{λ} — the anomalous branching coefficients, notice that

$$b^{(\mu)}_{\nu} = k^{(\mu)}_{\nu} \text{ for } \nu \in \bar{C}_{\mathfrak{a}} \tag{2}$$

$x_e = \frac{|\pi_{\mathfrak{a}}\Theta|^2}{|\Theta_{\mathfrak{a}}|^2}$ — the embedding index.

2 Recurrent relation for branching coefficients. Singularities and subtractions

Our aim is to demonstrate that despite the zeros arriving in the Weyl denominator (when it is projected to the subalgebra root space) the injection fan technique can be properly modified. The result of this modification is that the generalized recurrent relations for anomalous branching coefficients (1) is to be formulated in the following form:

$$k_{\xi}^{(\mu)} = -\frac{1}{s(\gamma_0)} \left(\sum_{\omega \in W_{\perp} \backslash W} \epsilon(\omega) \dim \left(L_{\mathfrak{a}_{\perp}}^{\pi_{\mathfrak{a}_{\perp}}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_{\perp}}} \right) \delta_{\xi-\gamma_0, \pi_{\mathfrak{a}}(\omega(\mu+\rho)-\rho)} + \right.$$
$$\left. + \sum_{\gamma \in \Gamma_{\mathfrak{a} \subset \mathfrak{g}}} s(\gamma + \gamma_0)\, k_{\xi+\gamma}^{(\mu)} \right) \tag{3}$$

Here $\mathfrak{a}_{\perp}$ is the subalgebra fixed by the roots of $\mathfrak{g}$ orthogonal to the root subsystem of $\mathfrak{a}$, $W_{\perp}$ is the corresponding Weyl group, $\Gamma_{\mathfrak{a} \subset \mathfrak{g}}$ is the set of weights in the expansion of the denominator $\prod_{\alpha \in \Delta^+ \backslash \Delta_{\mathfrak{a}_{\perp}}^+} (1-e^{-\alpha})^{\mathrm{mult}(\alpha)-\mathrm{mult}_{\mathfrak{a}}(\alpha)}$ and $s(\gamma)$ is the coefficient of e^{γ} in this expansion. In the next subsection we study the situation in details and prove the validity of this relation.

In the subsection 2.2 we shall describe the computational algorithm for branching coefficients based on this formula and present some examples.

2.1 Proof of the recurrent relation

Consider the branching of a module $L_{\mathfrak{g}}^{\mu}$ in terms of formal characters and projection operators $\pi_{\mathfrak{a}}$ that bring the weights of $\mathfrak{g}$ to the weight subspace of $\mathfrak{a}$:

$$L_{\mathfrak{g}\downarrow\mathfrak{a}}^{\mu} = \bigoplus_{\nu \in P_{\mathfrak{a}}^+} b_{\nu}^{(\mu)} L_{\mathfrak{a}}^{\nu} \quad \Longrightarrow \quad \pi_{\mathfrak{a}}(chL_{\mathfrak{g}}^{\mu}) = \sum_{\nu \in P_{\mathfrak{a}}^+} b_{\nu}^{(\mu)} chL_{\mathfrak{a}}^{\nu} \tag{4}$$

The Weyl-Kac character formula leads to the equality

$$\pi_{\mathfrak{a}} \left(\frac{\sum_{\omega \in W} \epsilon(\omega) e^{\omega(\mu+\rho)-\rho}}{\prod_{\alpha \in \Delta^+} (1-e^{-\alpha})^{\mathrm{mult}(\alpha)}} \right) = \sum_{\nu \in P_{\mathfrak{a}}^+} b_{\nu}^{(\mu)} \frac{\sum_{\omega \in W_{\mathfrak{a}}} \epsilon(\omega) e^{\omega(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}}{\prod_{\beta \in \Delta_{\mathfrak{a}}^+} (1-e^{-\beta})^{\mathrm{mult}_{\mathfrak{a}}(\beta)}} \tag{5}$$

It is important that the ion of some of positive roots of the algebra $\mathfrak{g}$ can be equal to zero. These roots are orthogonal to the root space of the subalgebra $\mathfrak{a}$ embedded into the root space of the algebra $\mathfrak{g}$. Let's denote the subset of such roots by $\Delta_{\perp}^+ = \left\{ \alpha \in \Delta_{\mathfrak{g}}^+ : \forall \beta \in \Delta_{\mathfrak{a}}^+,\ \alpha \perp \beta \right\}$.

Notice that if the set $\Delta_{\perp}^+$ is non-empty the Weyl reflections corresponding to the positive roots of $\Delta_{\perp}^+$ generate a subgroup $W_{\perp}$ of the Weyl group W. Consider any two positive roots $\alpha,\ \beta \in \Delta_{\perp}^+$ and the corresponding Weyl reflections $\omega_{\alpha},\ \omega_{\beta} \in W_{\perp}$. Since roots of the subalgebra $\mathfrak{a}$ are invariant under $\omega_{\alpha},\ \omega_{\beta}$ they are also invariant under the action of $\omega_{\gamma} = \omega_{\alpha} \cdot \omega_{\beta}$. So the subgroup $W_{\perp}$ preserves the root system of the subalgebra $\mathfrak{a}$.

Thus we have obtained the root system $\Delta_{\perp}$ which is orthogonal to the root system $\Delta_{\mathfrak{a}}$ and invariant with respect to $W_{\perp}$. This root system can be considered as the root system of a subalgebra $\mathfrak{a}_{\perp} \subset \mathfrak{g}$.

Now we are to find out when the subset $\Delta_{\perp}^{+}$ is non-empty and the subgroup $W_{\perp}$ and subalgebra $\mathfrak{a}_{\perp}$ are non-trivial.

If $\mathfrak{a}$ is a maximal regular subalgebra of $\mathfrak{g}$ then the rank of $\mathfrak{a}$ is equal to the rank of $\mathfrak{g}$ and it is clear that $\Delta_{\perp}^{+}$ is empty. On the other hand non-maximal regular embedding of $\mathfrak{a}$ into $\mathfrak{g}$ can be obtained through the chain of maximal embeddings $\mathfrak{a} \subset \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \cdots \subset \mathfrak{g}$. The maximal regular embeddings are constructed by the exclusion of one or two roots from the extended Dynkin diagram of the algebra. Since this process can give us non-connected Dynkin diagrams we can see which roots are orthogonal to the root space of non-maximal regular subalgebra $\mathfrak{a}$. The complete classification of the regular subalgebras of affine Lie algebras can be found in the recent paper [**?**].

Consider for instance the regular embedding of $A_1 \subset B_2$. The extended Dynkin diagram of B_2 is presented in the Figure 1. Drop the central node

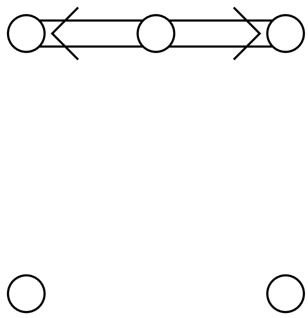


Figure 1: Extended Dynkin diagram of B_2 and embedding of A_1

to describe the embedding $A_1 \oplus A_1 \subset B_2$. In this case we have: $\mathfrak{a} = A_1$ and $\mathfrak{a}_{\perp} = A_1$.

The simple criterion of $\Delta_{\perp}^{+}$'s non-emptiness for a regular embedding $\mathfrak{a} \subset \mathfrak{g}$ when both $\mathfrak{a}$ and $\mathfrak{g}$ are simple can be formulated as follows: if the Dynkin diagram of $\mathfrak{g}$ can be split into the disconnected diagrams of $\mathfrak{a}$ and of some subalgebras $\{\bar{\mathfrak{a}}_j\}$ then the subset $\Delta_{\perp}$ is non-empty, subalgebra $\mathfrak{a}_{\perp}$ is non-trivial and all the $\bar{\mathfrak{a}}_j$ are the subalgebras of $\mathfrak{a}_{\perp}$.

Notice that when we study the regular embedding obtained by dropping the nodes of the extended Dynkin diagram of the algebra $\mathfrak{g}$ and the subalgebra $\mathfrak{a}$ is one of the connected components, the subalgebra $\mathfrak{a}_{\perp}$ may be larger than the algebra generated by the remaining connected components. Consider for example the embedding of $B_2 \subset B_4$ (the Figure 5). In this case by eliminating the simple root $\alpha_2 = e_2 - e_3$ one splits the extended Dynkin diagram of B_4 into the diagrams of the subalgebra $\mathfrak{a} = B_2$ and that of the direct sum $A_1 \oplus A_1$. But the subalgebra $\mathfrak{a}_{\perp}$ is equal not to $A_1 \oplus A_1$ but to B_2 (the root system of B_4 contains not only $\alpha_2 = e_2 - e_3$ but also e_2).

Such effects are due to the fact that the subalgebras $\mathfrak{a}$ and $\mathfrak{a}_\perp$ must not form a direct sum in $\mathfrak{g}$. Consider the case of such a regular embedding $\mathfrak{a} \subset \mathfrak{g}$ where both algebras are simple and the diagram of the subalgebra $\mathfrak{a}_\perp$ is not a subdiagram of the extended Dynkin diagram $\mathfrak{g}$. Drop the subdiagram of $\mathfrak{a}$ and the node α' that connects it with all the remaining nodes of the diagram of $\mathfrak{g}$. Consider the remaining diagram. This diagram is the diagram of the algebra $\bar{\mathfrak{a}}$ of $\mathrm{rank}(\bar{\mathfrak{a}}) = \mathrm{rank}(\mathfrak{g}) - \mathrm{rank}(\mathfrak{a})$. It is clear that $\bar{\mathfrak{a}} \subset \mathfrak{a}_\perp$. So the question is whether $\mathfrak{a}_\perp$ has additional roots, which are not the roots of $\bar{\mathfrak{a}}$ but are the linear combinations of them. The answer is positive when the set of angles between the roots of $\bar{\mathfrak{a}}$ does not contain all the angles between the roots of $\mathfrak{a}$. Then by reflecting the roots of $\mathfrak{a}$ by $s_{\alpha'}$ we get the additional roots of $\mathfrak{a}_\perp$.

All the cases are listed in the table 2.1.

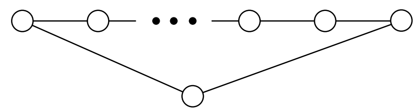
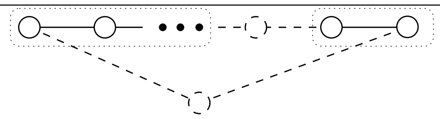
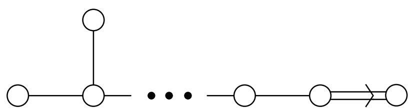
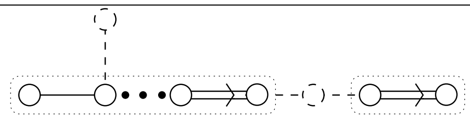
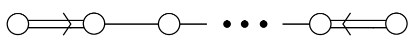
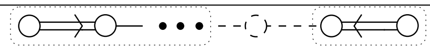
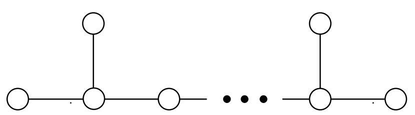
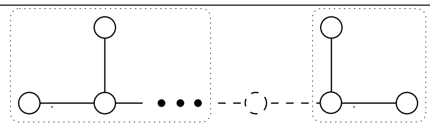
$\mathfrak{g}$	Extended diagram of $\mathfrak{g}$	Diagrams of the subalgebras $\mathfrak{a}$, $\mathfrak{a}_\perp$
A_n		
B_n		
C_n		
D_n		

Table 1: Subalgebras $\mathfrak{a}$, $\mathfrak{a}_\perp$ for the classical series

For the algebra $\mathfrak{g}$ from the series A_r the roots in the orthogonal basis $\{e_i,\ 1 \leq i \leq r+1\}$ are $\Delta = \{\alpha_{ij} = e_i - e_j,\ 1 \leq i,j \leq r+1\}$, $\Delta^+ = \{\alpha_{ij},\ i < j\}$ and the set of simple roots consists of $\alpha_{1,2}, \alpha_{2,3}, \ldots, \alpha_{r,r+1}$. So for the regular subalgebra $\mathfrak{a} = A_{r_\mathfrak{a}}$ and its simple root system consisting of first $r_\mathfrak{a}$ simple roots we get $\Delta_{\mathfrak{a}_\perp} = \{\alpha_{ij},\ r_\mathfrak{a} + 1 < i,j \leq r+1\}$ and $\mathfrak{a}_\perp = A_{r-r_\mathfrak{a}-1}$.

For the algebra $\mathfrak{g}$ from the series B_r the roots in the orthogonal basis $\{e_i,\ 1 \leq i \leq r\}$ are $\Delta = \{\alpha_{\pm i,\pm j} = \pm e_i \pm e_j,\ i < j;\ \alpha_{\pm j} = \pm e_j,\ 1 \leq j \leq r\}$, $\Delta^+ = \{\alpha_{i,-j},\ \alpha_{ij},\ \alpha_j;\ i < j,\ 1 \leq j \leq r\}$ and the set of simple roots consists of $\alpha_{1,-2}, \alpha_{2,-3}, \ldots, \alpha_{r-1,-r}, \alpha_r$. So if the regular subalgebra $\mathfrak{a} = A_{r_\mathfrak{a}}$ and its simple root system consists of first $r_\mathfrak{a}$ simple roots, then $\Delta_{\mathfrak{a}_\perp} = \{\alpha_{\pm i,\pm j},\ \alpha_j,\ r_\mathfrak{a} + 1 < i < j \leq r\}$ and $\mathfrak{a}_\perp = B_{r-r_\mathfrak{a}-1}$. Otherwise if $\mathfrak{a} = B_{r_\mathfrak{a}}$ and its simple roots are $\alpha_{r-r_\mathfrak{a}+1,-r+r_\mathfrak{a}-2}, \ldots, \alpha_{r-1,r}, \alpha_r$ we see that

$\Delta_{\mathfrak{a}_\perp} = \{\alpha_{\pm i,\pm j},\ \alpha_j,\ 1 < i < j \leq r - r_{\mathfrak{a}}\}$ and $\mathfrak{a}_\perp = B_{r-r_{\mathfrak{a}}}$. This is the only case where the simple roots of $\mathfrak{a}_\perp$ can not be obtained from the extended Dynkin diagram, as can be seen in the Table 2.1. There exists the third possibility to get the pair of subalgebras $\mathfrak{a}, \mathfrak{a}_\perp$ with the regular subalgebra $\mathfrak{a}$ by dropping the single node from the extended Dynkin diagram of B_r. This can be done by choosing the set of simple roots of $\mathfrak{a}$ in the form $\{\alpha_{1,-2}, \alpha_{1,2}, \alpha_{2,-3}, \ldots, \alpha_{r_{\mathfrak{a}}-1,-r_{\mathfrak{a}}}\}$. Then $\mathfrak{a} = D_{r_{\mathfrak{a}}}$, $\Delta_{\mathfrak{a}_\perp} = \{\alpha_{\pm i,\pm j},\ \alpha_j,\ r_{\mathfrak{a}} < i < j \leq r\}$ and $\mathfrak{a}_\perp = B_{r-r_{\mathfrak{a}}}$.

For the algebra $\mathfrak{g}$ from the series C_r the roots in the orthogonal basis $\{e_i,\ 1 \leq i \leq r\}$ are $\Delta = \{\alpha_{\pm i,\pm j} = \pm e_i \pm e_j,\ i < j;\ \alpha_{\pm j} = \pm 2e_j,\ 1 \leq j \leq r\}$, $\Delta^+ = \{\alpha_{i,-j},\ \alpha_{ij},\ \alpha_j;\ i < j,\ 1 \leq j \leq r\}$ and the set of simple roots consists of $\alpha_{1,-2}, \alpha_{2,-3}, \ldots, \alpha_{r-1,-r}, \alpha_r$. So if the regular subalgebra $\mathfrak{a} = A_{r_{\mathfrak{a}}}$ and its simple root system consists of first $r_{\mathfrak{a}}$ simple roots, then $\Delta_{\mathfrak{a}_\perp} = \{\alpha_{\pm i,\pm j},\ \alpha_j,\ r_{\mathfrak{a}} + 1 < i < j \leq r\}$ and $\mathfrak{a}_\perp = C_{r-r_{\mathfrak{a}}-1}$. Otherwise if $\mathfrak{a} = C_{r_{\mathfrak{a}}}$ and its simple roots are $\alpha_{r-r_{\mathfrak{a}}+1,-r+r_{\mathfrak{a}}-2}, \ldots, \alpha_{r-1,r}, \alpha_r$ we see that $\Delta_{\mathfrak{a}_\perp} = \{\alpha_{\pm i,\pm j},\ \alpha_j,\ 1 < i < j \leq r - r_{\mathfrak{a}}\}$ and $\mathfrak{a}_\perp = C_{r-r_{\mathfrak{a}}}$.

For the algebra $\mathfrak{g}$ from the series D_r the roots in the orthogonal basis $\{e_i,\ 1 \leq i \leq r\}$ are $\Delta = \{\alpha_{\pm i,\pm j} = \pm e_i \pm e_j,\ 1 \leq i < j \leq r\}$, $\Delta^+ = \{\alpha_{i,-j},\ \alpha_{ij},\ i < j,\ 1 \leq j \leq r\}$ and the set of simple roots consists of $\alpha_{1,-2}, \alpha_{2,-3}, \ldots, \alpha_{r-1,-r}, \alpha_{r-1,r}$. So if the regular subalgebra $\mathfrak{a} = A_{r_{\mathfrak{a}}}$ and its simple root system consists of first $r_{\mathfrak{a}}$ simple roots, then $\Delta_{\mathfrak{a}_\perp} = \{\alpha_{\pm i,\pm j},\ r_{\mathfrak{a}} + 1 < i < j \leq r\}$ and $\mathfrak{a}_\perp = D_{r-r_{\mathfrak{a}}-1}$. Otherwise if $\mathfrak{a} = D_{r_{\mathfrak{a}}}$ and its simple roots are $\alpha_{r-r_{\mathfrak{a}}+1,-r+r_{\mathfrak{a}}-2}, \ldots, \alpha_{r-1,r}, \alpha_{r-1,r}$ we see that $\Delta_{\mathfrak{a}_\perp} = \{\alpha_{\pm i,\pm j},\ 1 < i < j \leq r - r_{\mathfrak{a}}\}$ and $\mathfrak{a}_\perp = D_{r-r_{\mathfrak{a}}}$.

In the case of special embeddings the set $\Delta_\perp^+$ can be empty as for the special embedding of $A_1 \subset A_2$ with the embedding index equal to 4, or non-empty for example for the embedding $A_1 \subset A_2 \subset A_3$ which is depicted at the Figure 2.1.

Using the existing classification of maximal special subalgebras [11] we immediately have the following pairs of the orthogonal subalgebras $\mathfrak{a}$, $\mathfrak{a}_\perp$

$$\begin{array}{lll} su(p) \oplus su(q) & \subset su(pq) & \\ so(p) \oplus so(q) & \subset so(pq) & \\ sp(2p) \oplus sp(2q) & \subset so(4pq) & \\ sp(2p) \oplus so(q) & \subset sp(2pq) & \\ so(p) \oplus so(q) & \subset so(p+q) & \text{for } p \text{ and } q \text{ odd} \end{array} \tag{6}$$

Exceptional Lie algebras and other non-maximal subalgebras will be considered elsewhere.

Up to this point we considered the problem of constructing the subalgebra $\mathfrak{a}_\perp$ for a given regular injection $\mathfrak{a} \in \mathfrak{g}$ in terms of Dynkin diagrams. When the root systems Δ and $\Delta_{\mathfrak{a}}$ are known explicitly all that we need is to select the

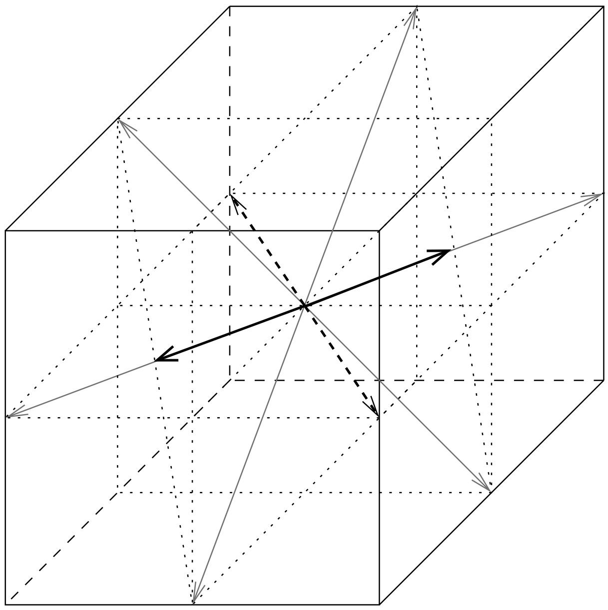


Figure 2: Special embedding $A_1 \subset A_2 \subset A_3$. Grey vectors are the roots of A_2, thick black - of $\mathfrak{a} = A_1$, dashed black are the orthogonal roots of A_1 which is contained in $\mathfrak{a}_\perp$

roots $\Delta_\perp = \{\alpha \in \Delta : \alpha \perp \Delta_\mathfrak{a}\}$ and correspondingly the positive roots $\Delta_\perp^+ = \{\alpha \in \Delta^+ : \alpha \perp \Delta_\mathfrak{a}\}$.

Now consider the direct sum $\mathfrak{a}_\perp \oplus \mathfrak{h}_\mathfrak{a}$ and its module $L^\mu_{\mathfrak{a}_\perp \oplus \mathfrak{h}}$ with the highest weight μ. The character $chL^\mu_{\mathfrak{a}_\perp \oplus \mathfrak{h}}$ can be written as

$$chL^\mu_{\mathfrak{a}_\perp \oplus \mathfrak{h}} = \frac{\sum_{\omega \in W_\perp} \epsilon(\omega) e^{\omega(\mu + \rho_{\mathfrak{a}_\perp}) - \rho_{\mathfrak{a}_\perp}}}{\prod_{\alpha \in \Delta_\perp^+} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)}}. \tag{7}$$

Its projection $\pi_\mathfrak{a}(ch\, L^\mu_{\mathfrak{a}_\perp \oplus \mathfrak{h}})$ is the single element $e^{\pi_\mathfrak{a} \cdot \mu}$ of the formal algebra $\mathcal{E}(\mathfrak{a})$ with the multiplicity equal to the dimension of the module $L^\mu_{\mathfrak{a}_\perp \oplus \mathfrak{h}}$, since all the roots of $\mathfrak{a}_\perp$ are orthogonal to that of $\Delta_\mathfrak{a}$.

Using this property we can consider the restriction $ch\, L^\mu_{\mathfrak{g} \downarrow \mathfrak{a}_\perp \oplus \mathfrak{h}}$, that is the character of the direct sum of $(\mathfrak{a}_\perp \oplus \mathfrak{h})$-modules. Multiply the equation (5) by the element

$$\pi_\mathfrak{a} \left(\prod_{\alpha \in \Delta^+ \setminus \Delta_\perp^+} (1 - e^{-\alpha})^{\mathrm{mult}_\mathfrak{g}(\alpha)} \right) \tag{8}$$

Taking into account that the projection commutes with the multiplication,

$$\pi_\mathfrak{a}(Q) \pi_\mathfrak{a}(1 - e^{-\alpha}) = \pi_\mathfrak{a}\left(Q \cdot (1 - e^{-\alpha})\right), \tag{9}$$

we can rewrite the product of (5) and (8):

$$\pi_{\mathfrak{a}}\left(\frac{\sum_{\omega\in W}\epsilon(\omega)e^{\omega(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^+_\perp}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}\right)=$$
$$\pi_{\mathfrak{a}}\left(\prod_{\alpha\in\Delta^+\setminus\Delta^+_\perp}(1-e^{-\alpha})^{\mathrm{mult}_{\mathfrak{g}}(\alpha)}\right)\sum_{\nu\in P^+_{\mathfrak{a}}}b^{(\mu)}_\nu\frac{\sum_{\omega\in W_{\mathfrak{a}}}\epsilon(\omega)e^{\omega(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}}{\prod_{\beta\in\Delta^+_{\mathfrak{a}}}(1-e^{-\beta})^{\mathrm{mult}_{\mathfrak{a}}(\beta)}}. \quad (10)$$

The right-hand side of this equation can be reorganised similarly to what was performed in the paper [10], by introducing the anomalous branching coefficients k_λ,

$$\sum_{\nu\in P_{\mathfrak{a}}}b^{(\mu)}_\nu\Psi^{(\nu)}_{(\mathfrak{a})}=\sum_{\lambda\in P_{\mathfrak{a}}}k^{(\mu)}_\lambda e^\lambda \quad (11)$$

and simplifying the multiplier:

$$\pi_{\mathfrak{a}}\left(\frac{\sum_{\omega\in W}\epsilon(\omega)e^{\omega(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^+_\perp}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}\right)=$$
$$\left(\prod_{\alpha\in\pi_{\mathfrak{a}}(\Delta^+\setminus\Delta^+_\perp)}(1-e^{-\alpha})^{\mathrm{mult}_{\mathfrak{g}}(\alpha)-\mathrm{mult}_{\mathfrak{a}}(\alpha)}\right)\sum_{\lambda\in P_{\mathfrak{a}}}k^{(\mu)}_\lambda e^\lambda \quad (12)$$

If the set $\Delta^+_\perp$ is non-empty then the Weyl reflections corresponding to the positive roots of $\Delta^+_\perp$ generate a subgroup $W_\perp$ of the Weyl group W. Let us reorganise the summation in the left-hand side of (12). Consider the factor-space $W_\perp\setminus W$. For the class $\tilde{\omega}\in W_\perp\setminus W$ choose the representative $\omega\in\tilde{\omega}$ such that $\pi_{\mathfrak{a}_\perp}\omega(\mu+\rho)\in\bar{C}_{\mathfrak{a}_\perp}$,

$$\pi_{\mathfrak{a}}\left(\frac{\sum_{\omega\in W}\epsilon(\omega)e^{\omega(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^+_\perp}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}\right)=$$
$$\pi_{\mathfrak{a}}\left(\sum_{\omega\in W_\perp\setminus W}\epsilon(\omega)\frac{\sum_{\nu\in W_\perp}\epsilon(\nu)e^{\nu\cdot\omega(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^+_\perp}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}\right) \quad (13)$$

The fraction in the right-hand side of the equation is similar to the character of some $\mathfrak{a}_\perp$-module. Let us rewrite the shifted weights

$$\nu\cdot\omega(\mu+\rho)-\rho=\nu\cdot\big(\omega(\mu+\rho)-\pi_{\mathfrak{a}}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}+\rho_{\mathfrak{a}_\perp}+\pi_{\mathfrak{a}}(\omega(\mu+\rho))\big)-\rho \quad (14)$$

Since $\nu \cdot \pi_{\mathfrak{a}}(\omega(\mu+\rho)) = \pi_{\mathfrak{a}}(\omega(\mu+\rho))$ and $\omega(\mu+\rho) - \pi_{\mathfrak{a}}(\omega(\mu+\rho)) = \pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))$, we get

$$\sum_{\omega \in W_\perp \backslash W} \epsilon(\omega) \frac{\sum_{\nu \in W_\perp} \epsilon(\nu) e^{\nu\cdot\omega(\mu+\rho)-\rho}}{\prod_{\alpha \in \Delta_\perp^+} (1-e^{-\alpha})^{\mathrm{mult}(\alpha)}} =$$

$$\sum_{\omega \in W_\perp \backslash W} \epsilon(\omega) e^{\pi_{\mathfrak{a}}(\omega(\mu+\rho))-\rho} \frac{e^{\rho_{\mathfrak{a}_\perp}} \sum_{\nu \in W_\perp} \epsilon(\nu) e^{\nu\cdot(\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}+\rho_{\mathfrak{a}_\perp})-\rho_{\mathfrak{a}_\perp}}}{\prod_{\alpha \in \Delta_\perp^+} (1-e^{-\alpha})^{\mathrm{mult}(\alpha)}} =$$

$$\sum_{\omega \in W_\perp \backslash W} \epsilon(\omega) e^{\pi_{\mathfrak{a}}(\omega(\mu+\rho))-\rho} e^{\rho_{\mathfrak{a}_\perp}} \mathrm{ch} L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}} \tag{15}$$

The projector $\pi_{\mathfrak{a}}$ transforms the character $\mathrm{ch} L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}$ into the unit element of $\mathcal{E}$ multiplied by $\dim L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}$:

$$\pi_{\mathfrak{a}} \left(\sum_{\omega \in W_\perp \backslash W} \epsilon(\omega) e^{\pi_{\mathfrak{a}}(\omega(\mu+\rho))-\rho} e^{\rho_{\mathfrak{a}_\perp}} \mathrm{ch} L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}} \right) =$$

$$\sum_{\omega \in W_\perp \backslash W} \epsilon(\omega) \dim \left(L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}} \right) e^{\pi_{\mathfrak{a}}(\omega(\mu+\rho)-\rho)} \tag{16}$$

Thus we obtained the relation

$$\sum_{\omega \in W_\perp \backslash W} \epsilon(\omega) \dim \left(L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}} \right) e^{\pi_{\mathfrak{a}}(\omega(\mu+\rho)-\rho)} =$$

$$\left(\prod_{\alpha \in \pi_{\mathfrak{a}}\left(\Delta^+ \backslash \Delta_\perp^+\right)} (1-e^{-\alpha})^{\mathrm{mult}_{\mathfrak{g}}(\alpha)-\mathrm{mult}_{\alpha}} \right) \sum_{\lambda \in P_{\mathfrak{a}}} k_\lambda^{(\mu)} e^{\lambda}. \tag{17}$$

Let us rewrite the multiplier in the right-hand side:

$$\prod_{\alpha \in \pi_{\mathfrak{a}} \circ (\Delta^+ \backslash \Delta_\perp^+)} \left(1-e^{-\alpha}\right)^{\mathrm{mult}(\alpha)-\mathrm{mult}_{\mathfrak{a}}(\alpha)} = -\sum_{\gamma \in P_{\mathfrak{a}}} s(\gamma) e^{-\gamma} \tag{18}$$

For the coefficient function $s\left(\gamma\right)$ define the carrier $\Phi_{\mathfrak{a}\subset\mathfrak{g}} \subset P_{\mathfrak{a}}$:

$$\Phi_{\mathfrak{a}\subset\mathfrak{g}} = \left\{ \gamma \in P_{\mathfrak{a}} \mid s\left(\gamma\right) \neq 0 \right\}. \tag{19}$$

From the obtained equation for the formal elements,

$$\sum_{\omega\in W_{\perp}\backslash W}\epsilon(\omega)\mathrm{dim}\left(L_{\mathfrak{a}_{\perp}}^{\pi_{\mathfrak{a}_{\perp}}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_{\perp}}}\right)e^{\pi_{\mathfrak{a}}(\omega(\mu+\rho)-\rho)}=$$
$$=-\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}}s\left(\gamma\right)e^{-\gamma}\sum_{\lambda\in P_{\mathfrak{a}}}k_{\lambda}^{(\mu)}e^{\lambda}$$
$$=-\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}}\sum_{\lambda\in P_{\mathfrak{a}}}s\left(\gamma\right)k_{\lambda}^{(\mu)}e^{\lambda-\gamma}\quad(20)$$

we can deduce the following property of the anomalous branching coefficients,

$$\sum_{\omega\in W_{\perp}\backslash W}\epsilon(\omega)\mathrm{dim}\left(L_{\mathfrak{a}_{\perp}}^{\pi_{\mathfrak{a}_{\perp}}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_{\perp}}}\right)\delta_{\xi,\pi_{\mathfrak{a}}(\omega(\mu+\rho)-\rho)}+\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}}s(\gamma)\;k_{\xi+\gamma}^{(\mu)}=0;$$
$$\xi\in P_{\mathfrak{a}}.\quad(21)$$

To get the recurrent relations for the coefficients $k_{\xi+\gamma}^{(\mu)}$ we use the following procedure (similar to that in [10]). Let γ_0 be the lowest vector with respect to the natural ordering in $\overset{\circ}{\Delta}_{\mathfrak{a}}$ in the lowest grade of $\Phi_{\mathfrak{a}\subset\mathfrak{g}}$ and decompose the defining relation (18),

$$\prod_{\alpha\in\pi_{\mathfrak{a}^{\circ}}(\Delta^{+}\backslash\Delta_{\perp}^{+})}\left(1-e^{-\alpha}\right)^{\mathrm{mult}(\alpha)-\mathrm{mult}_{\mathfrak{a}}(\alpha)}=-s\left(\gamma_{0}\right)e^{-\gamma_{0}}-\sum_{\gamma\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}\backslash\{\gamma_{0}\}}s\left(\gamma\right)e^{-\gamma},\quad(22)$$

then the equality (21) leads to the desired recurrent relation for the anomalous branching coefficients:

$$k_{\xi}^{(\mu)}=-\frac{1}{s\left(\gamma_{0}\right)}\left(\sum_{\omega\in W_{\perp}\backslash W}\epsilon(\omega)\;\mathrm{dim}\left(L_{\mathfrak{a}_{\perp}}^{\pi_{\mathfrak{a}_{\perp}}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_{\perp}}}\right)\delta_{\xi-\gamma_{0},\pi_{\mathfrak{a}}(\omega(\mu+\rho)-\rho)}+\right.$$
$$\left.+\sum_{\gamma\in\Gamma_{\mathfrak{a}\subset\mathfrak{g}}}s\left(\gamma+\gamma_{0}\right)k_{\xi+\gamma}^{(\mu)}\right)\quad(23)$$

where the set

$$\Gamma_{\mathfrak{a}\subset\mathfrak{g}}=\{\xi-\gamma_{0}|\xi\in\Phi_{\mathfrak{a}\subset\mathfrak{g}}\}\setminus\{0\}\quad(24)$$

was introduced that is called the injection fan.

Now consider the case $\Delta_{\perp}^{+}=0$. There are three different reasons for $\Delta_{\perp}^{+}$ to be empty: i) $\mathrm{dim}\mathfrak{h}_{\mathfrak{a}}=\mathrm{dim}\mathfrak{h}_{\mathfrak{g}}$, ii) $\mathfrak{a}_{\perp}=0$ and iii) $\mathfrak{a}_{\perp}\subset\mathfrak{h}_{\mathfrak{g}}$. Both the first and the second cases can be treated as corresponding to the trivial orthogonal subalgebra: $\mathfrak{a}_\perp = 0$. In any of these cases instead of the formal characters in the right-hand side of (13) we obtain the formal element $e^{\pi_{\mathfrak{a}_\perp}\omega(\mu+\rho)}$. In the first two cases (equivalent to $\mathfrak{a}_\perp = 0$) the projection operator retains its purely geometrical meaning: the vector $\omega(\mu+\rho)$ is projected to the subspace orthogonal to the weight space of $\mathfrak{a}$. It is clear that in any of the three variants the final vector $\pi_{\mathfrak{a}}\pi_{\mathfrak{a}_\perp}\omega(\mu+\rho)$ leads to the unit of the formal algebra $\mathcal{E}$. Thus when the set $\Delta_\perp^+$ is empty the recurrent relation is simplified:

$$k_\xi^{(\mu)} = -\frac{1}{s(\gamma_0)}\left(\sum_{w\in W}\epsilon(w)\,\delta_{\xi,\pi_{\mathfrak{a}}\circ(w\circ(\mu+\rho)-\rho)+\gamma_0} + \sum_{\gamma\in\Gamma_{\mathfrak{a}\subset\mathfrak{g}}} s(\gamma+\gamma_0)\,k_{\xi+\gamma}^{(\mu)}\right), \tag{25}$$

the latter coinsides with the one obtained in [10] (formula (16)).

In the next section we describe a computation algorithm for branching coefficients based on the relation (23).

2.2 Algorithm for the recursive computation of the branching coefficients

The recurrent relation (23) allows us to formulate an algorithm for recursive computation of the branching coefficients. In this algorithm there is no need to construct the module $L_{\mathfrak{g}}^{(\mu)}$ or any of the modules $L_{\mathfrak{a}}^{(\nu)}$.

It contains the following steps:

1. Construct the sets Δ^+ and $\Delta_{\mathfrak{a}}^+$ of positive roots for the algebras $\mathfrak{a}\subset\mathfrak{g}$.

2. Select the positive roots $\alpha\in\Delta^+$ which are orthogonal to the root subspace of $\mathfrak{a}$ and form the set $\Delta_\perp^+$.

3. Construct the set Γ (24).

4. Construct the set $\widehat{\Psi^{(\mu)}} = \{\omega(\mu+\rho)-\rho;\ \omega\in W\}$ of the anomalous weights of the $\mathfrak{g}$-module $L^{(\mu)}$.

5. Select the weights $\left\{\lambda = \omega(\mu+\rho) | \pi_{\mathfrak{a}_\perp}\lambda \in \bar{C}_{\mathfrak{a}_\perp}\right\}$ Since we have constructed the set $\Delta_\perp^+$ we can easily check wether the weight $\pi_{\mathfrak{a}_\perp}\lambda$ lies in the main Weyl chamber of $\mathfrak{a}_\perp$ by computing the scalar product of λ with the roots of $\Delta_\perp^+$, it must be non-negative.

6. For $\lambda = \omega(\mu+\rho)$, $\pi_{\mathfrak{a}_\perp}\lambda\in\bar{C}_{\mathfrak{a}_\perp}$ calculate the dimensions of the corresponding modules $\dim\left(L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}\right)$ using the Weyl formula with the set $\Delta_\perp^+$.

7. Calculate the anomalous branching coefficients in the main Weyl chamber $\bar{C}_{\mathfrak{a}}$ of the subalgebra $\mathfrak{a}$ using the recurrent relation (23).

If we are interested in the branching coefficients for the embedding of the finite-dimensional Lie algebra into the affine Lie algebra we can construct the set of the anomalous weights up to the required grade and use the steps 4-7 of the algorithm for each grade. We can also speed up the algorithm by one-time computation of the representatives of the conjugate classes $W_{\perp}\backslash W$.

The next section contains several examples computed using this algorithm.

3 Finite dimensional Lie algebras

3.1 Regular embedding of A_1 into B_2

Consider the regular embedding of A_1 into B_2. Simple roots α_1, α_2 of B_2 are drawn as the dashed vectors at the Figure 3. We denote the corresponding Weyl reflections by ω_1, ω_2. Simple root β of the embedded A_1 is equal to $\alpha_1 + 2\alpha_2$ and is drawn as the grey vector.

Let's describe the reduction of the fundamental representation of B_2 with the highest weight equal to $(1, 0)$ (in the fundamental weight basis), it is drawn as the black vector in the Figure 3. There we have also shown the

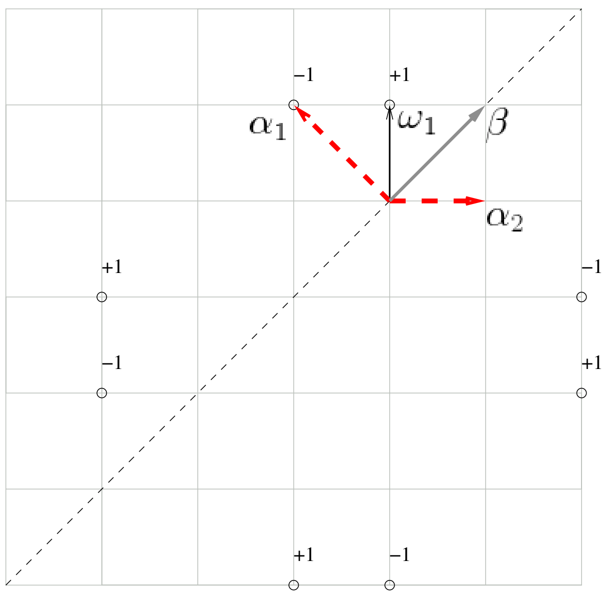


Figure 3: Regular embedding of A_1 into B_2

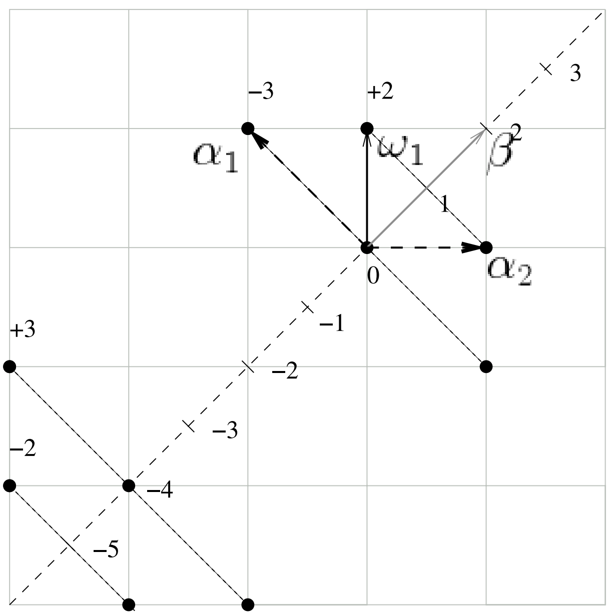


Figure 4: Anomalous weights and the corresponding $\mathfrak{a}_{\perp} = A_1$-modules for the embedding $A_1 \subset B_2$

set of weights $\omega(\mu+\rho)-\rho,\ \omega \in W$ of the fundamental representation of B_2 with the corresponding determinants $\epsilon(\omega)$ of Weyl transformations. Now we have to factorise the Weyl group W by $W_{\perp} = \{\omega_1\}$. We get the following set of anomalous weights $\omega(\mu+\rho)-\rho,\ \omega \in W_{\perp}\backslash W$: We have also depicted the corresponding $\mathfrak{a}_{\perp} = A_1$-modules $L_{\mathfrak{a}_{\perp}}^{\pi_{\mathfrak{a}_{\perp}}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_{\perp}}}$. Then we project these weights and dimensions of modules onto the root space of subalgebra $\mathfrak{a} = A_1$ and get the following anomalous weights in fundamental weights basis with corresponding multiplicities:

$$(1,2),\ (0,-3),\ (-4,3),\ (-5,-2) \tag{26}$$

For the function $s(\gamma)$ and the set Γ (using the definition (19,24)) we have

$$(1,2),\ (2,-1) \tag{27}$$

Here the second component denotes the value of $s(\gamma)$.

As far as the anomalous branching coefficient $k_1^{(1,0)} = 2$ for the coefficient $k_0^{(1,0)}$ the formula (23) gives the value

$$k_0^{(1,0)} = -1 \cdot k_2^{(1,0)} + 2 \cdot k_1^{(1,0)} - 3 \cdot \delta_{0,0} = 1. \tag{28}$$

The recurrence property defines the branching.

3.2 Embedding of B_2 into B_4

Consider the regular embedding $B_2 \longrightarrow B_4$. We calculate the branching coefficients for the fundamental vector representation of B_4. The corresponding Dynkin diagrams are in the Figure 5.

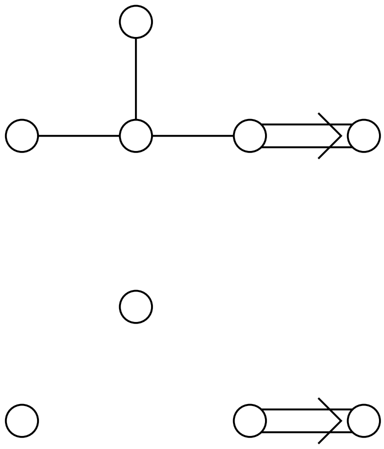


Figure 5: Dynkin diagrams for the regular embedding $B_2 \subset B_4$. Note the dropped node and the roots of $\mathfrak{a}_\perp$.

In the orthogonal basis $e_1, \dots, e_4$ the simple roots of B_4 are

$$(e_1 - e_2,\ e_2 - e_3,\ e_3 - e_4,\ e_4) \tag{29}$$

The positive roots are

$$(e_1 - e_2,\ e_2 - e_3,\ e_3 - e_4,\ e_4,\ e_1 - e_3,\ e_2 - e_4,\ e_3 + e_4,\ e_3,\ e_1 - e_4,\\ e_2 + e_4,\ e_2,\ e_1 + e_4,\ e_2 + e_3,\ e_1,\ e_1 + e_3,\ e_1 + e_2) \tag{30}$$

The simple roots of the embedded subalgebra $\mathfrak{a} = B_2$ are

$$(e_3 - e_4, e_4) \tag{31}$$

The set $\Delta_\perp^+$ contains the roots

$$\{e_1 - e_2, e_1 + e_2, e_1, e_2\} \tag{32}$$

and is the set of positive roots for the algebra $\mathfrak{a}_\perp = B_2$.

Firstly we construct the fan for this injection. Using the definition (24) we obtain the set Γ with the corresponding values $s(\gamma + \gamma_0)$, depicted at the Figure 7.

To find the branching coefficients we need to compute the anomalous weights of B_4, select the weights belonging to $C_{\mathfrak{a}_\perp}^{(0)}$ and compute the dimensions of the corresponding $\mathfrak{a}_\perp$-modules.

Consider the B_4-module L^μ with the highest weight $\mu = (0,1,0,2) = 2e_1 + 2e_2 + e_3 + e_4$; $dim(L^{(0,1,0,2)}) = 2772$.

The set of the anomalous weights $\{\omega(\mu+\rho) - \rho,\ \omega \in W\}$ contains of 384 vectors.

We need to select the weights $\psi \in \omega(\mu+\rho)$ with the property $\pi_{\mathfrak{a}_\perp}(\psi) \in C_{\mathfrak{a}_\perp}^{(0)}$. It means that the scalar product of these weights with all the roots in $\Delta_\perp^+$ is non-negative.

To compute the dimensions of the corresponding $\mathfrak{a}_\perp$-modules we need to project each selected weight onto the root space $\Delta_\perp^+$, substract $\rho_{\mathfrak{a}_\perp}$ and apply the Weyl dimension formula.

The result of this procedure is shown in the Figure 6.

Applying the recurrent relation (23) we obtain the following branching coefficients:

$$\pi_{\mathfrak{a}}\left(chL_{B_4}^{(0,1,0,2)}\right) = 6\ chL_{B_2}^{(0,0)} + 60\ chL_{B_2}^{(0,2)} + 30\ chL_{B_2}^{(1,0)} + 19\ chL_{B_2}^{(2,0)} + 40\ chL_{B_2}^{(1,2)} + 10\ chL_{B_2}^{(2,2)}. \quad (33)$$

4 Applications to the conformal field theory

4.1 Conformal embeddings

Branching coefficients for an embedding of affine Lie subalgebra into affine Lie algebra can be used to construct modular invariant partition functions for Wess-Zumino-Novikov-Witten models of conformal field theory ([1], [12], [13], [14]). In these models current algebras are affine Lie algebras.

The modular invariant partition function is crucial for the conformal theory to be valid on the torus and higher genus Riemann surfaces. It is important for the applications of CFT to the string theory and to the critical phenomena description.

The simplest modular-invariant partition function has the diagonal form:

$$Z(\tau) = \sum_{\mu \in P_{\mathfrak{g}}^+} \chi_\mu(\tau)\bar{\chi}_\mu(\bar{\tau}) \quad (34)$$

Here the sum is over the set of the highest weights of integrable modules of WZW-model and $\chi_\mu(\tau)$ are the modified characters of these modules.

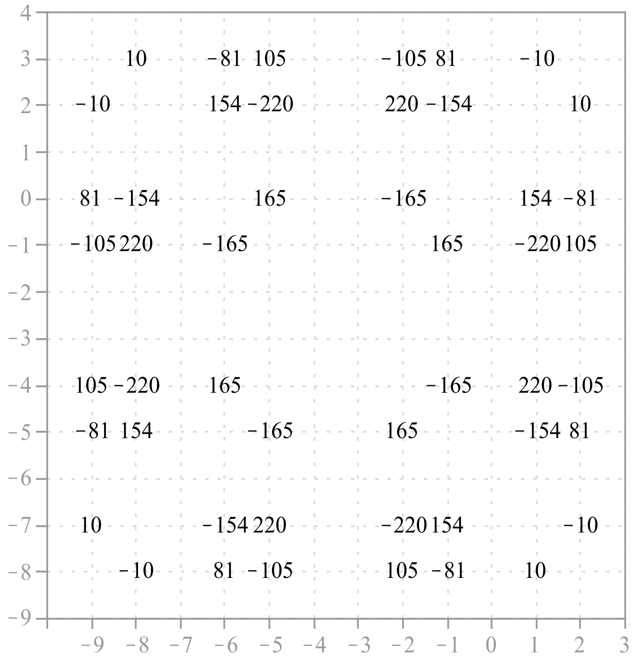


Figure 6: Projected weights of $\pi_{B_2}\left(\Psi_{B_4}^{(0,1,0,2)}\right)$ with the dimensions of the corresponding $\mathfrak{a}_\perp$-modules.

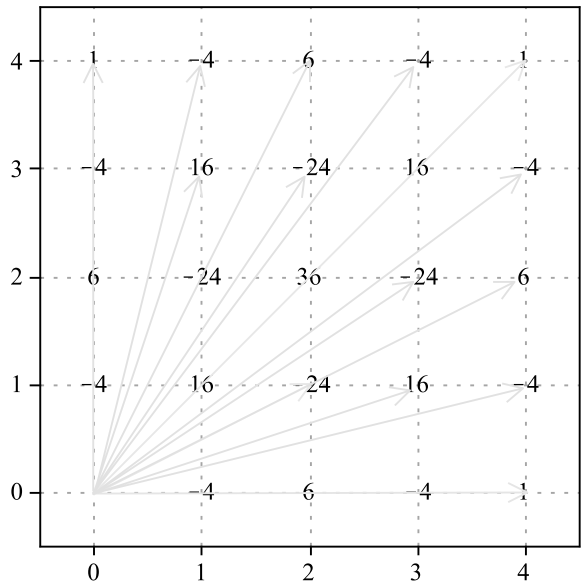


Figure 7: The fan for $B_2 \subset B_4$

The task of the construction of non-diagonal modular invariants is not easy, although the complete classification of modular invariants for some models is known [116, 219, 117, 119, 46] in [15].

Consider the Wess-Zumino-Witten model with the affine Lie algebra $\mathfrak{a}$. Non-diagonal modular invariants for this model can be constructed from the diagonal invariant if there exists affine algebra $\mathfrak{g}$ such that $\mathfrak{a} \subset \mathfrak{g}$. Then we can replace the characters of the $\mathfrak{g}$-modules in the diagonal modular-invariant partition function (36) by the decompositions

$$\sum_{\nu \in P_{\mathfrak{a}}^{+}} b_{\nu}^{(\mu)} \chi_{\nu} \tag{35}$$

containing the modified characters χ_ν of the corresponding $\mathfrak{a}$-modules. Thus we obtain the non-diagonal modular-invariant partition function for the theory with the current algebra $\mathfrak{a}$.

$$Z_{\mathfrak{a}}(\tau) = \sum_{\nu,\lambda \in P_{\mathfrak{a}}^{+}} \chi_{\nu}(\tau) M_{\nu\lambda} \bar{\chi}_{\lambda}(\bar{\tau}) \tag{36}$$

The reduction of the representations is crucial in this construction.

For the construction to be valid the embedding is required to preserve conformal invariance. Let $X_{-n_j}^{a_j}$ and $\tilde{X}_{-n_j}^{a'_j}$ be the lowering generators for $\mathfrak{g}$ and for $\mathfrak{a} \subset \mathfrak{g}$ correspondingly. Let $\pi_{\mathfrak{a}}$ be the projection operator of $\pi_{\mathfrak{a}} : \mathfrak{g} \longrightarrow \mathfrak{a}$. In the theory attributed to $\mathfrak{g}$ with the vacuum $|\lambda\rangle$ the states can be described as

$$X_{-n_1}^{a_1} X_{-n_2}^{a_2} \ldots |\lambda\rangle \quad n_1 \geq n_2 \geq \cdots > 0. \tag{37}$$

And for the sub-algebra $\mathfrak{a}$ the corresponding states are

$$\tilde{X}_{-n_1}^{a'_1} \tilde{X}_{-n_2}^{a'_2} \ldots |\pi_{\mathfrak{a}}(\lambda)\rangle . \tag{38}$$

The $\mathfrak{g}$-invariance of the vacuum entails its $\mathfrak{a}$-invariance, but this is not the case for the energy-momentum tensor. So the energy-momentum tensor of the larger theory should consist only of the generators of $\tilde{X}$. Then

$$T_{\mathfrak{g}}(z) = T_{\mathfrak{a}}(z), \tag{39}$$

which leads to the equality of the central charges

$$c(\mathfrak{g}) = c(\mathfrak{a}). \tag{40}$$

This leads to the equation

$$\frac{k \dim \mathfrak{g}}{k+g} = \frac{x_e k \dim \mathfrak{a}}{x_e k + a} \tag{41}$$

Here x_e is the embedding index and g, a are the dual Coxeter numbers for the corresponding algebras.

It can be demonstrated that the solutions of the equation (41) exist only for the level $k = 1$ [1].

The complete classification of conformal embeddings is given in the paper [14].

The relation (41) and the asymptotics of the branching functions can be used to prove the finite reducibility theorem [16]. It states that for the conformal embedding $\mathfrak{a} \subset \mathfrak{g}$ only finite number of branching coefficients have non-zero values.

Note 4.1. *The orthogonal subalgebra $\mathfrak{a}_\perp$ is always empty for the conformal embeddings $\mathfrak{a} \subset \mathfrak{g}$.*

Proof. It can be seen from the following consideration. In the case of the conformal embedding energy-momentum tensors are to be equal (39). The energy-momentum tensor can be expanded into the modes L_n

$$T(z) = \frac{1}{2(k+h^v)} \sum_n z^{-n-1} L_n \tag{42}$$

The modes are constructed as the combination of normally-ordered products of the generators of $\mathfrak{g}$.

$$L_n = \frac{1}{2(k+h^v)} \sum_a \sum_m : X^a_m X^a_{n-m} : \tag{43}$$

The substitution of the generators of the algebra $\mathfrak{a}$ in terms of the generators of $\mathfrak{g}$ into these combinations of $T_{\mathfrak{a}}$ should give the energy-momentum tensor $T_{\mathfrak{g}}$. But if the set of the generators $\Delta_\perp$ is not empty it is not possible, since $T_{\mathfrak{g}}$ contains the combinations of the generators X^α_n, $\alpha \in \Delta_\perp$, which can not be obtained in the expansion of components of $T_{\mathfrak{a}}$. □

4.1.1 Special embedding $\hat{A}_1 \subset \hat{A}_2$

Consider the embedding of the affine Lie algebra $\hat{A}_1$ into $\hat{A}_2$ constructed as the affine extension of the special embedding $A_1 \subset A_2$ with the embedding index $x_e = 4$. The level of the representations of the algebra $\mathfrak{g} = \hat{A}_2$ is equal to one, so the level of the modules of the subalgebra is equal $\tilde{k} = kx_e = 4$.

There exist three level 1 fundamental weights in the weight space of $\hat{A}_2$. It is easy to see that the set $\Delta_\perp$ is empty and the subalgebra $\mathfrak{a}_\perp = 0$.

Using the definition (24) we construct the fan $\Gamma_{\hat{A}_1 \to \hat{A}_2}$ with the corresponding values of the function $s(\gamma + \gamma_0)$ (see the Figure 8).

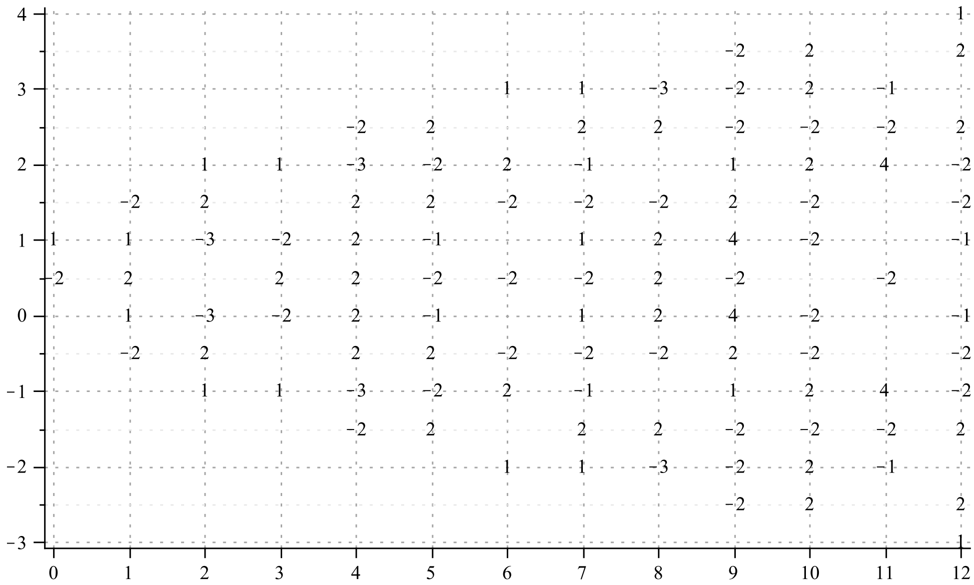


Figure 8: The fan for $\hat{A}_1 \subset \hat{A}_2$

Let us consider the representation with the highest weight $w_0 = (0,0;1;0)$ in details. Here the first two components are the Dynkin indices (coordinates in the fundamental weights basis) of the finite part of the weight.

The set $\widehat{\Psi^{(w_0)}}$ is depicted in the Figure 9 up to the sixth grade. The weights $\omega(w_0+\rho)-\rho$ is shown as cross if $\epsilon(\omega)=1$ and by diamond if $\epsilon(\omega)=-1$. Simple roots of classical subalgebra A_2 are shown in grey and grey diagonal plane is the Cartan subalgebra of the embedded algebra $\hat{A}_1$.

The next step is to project the anomalous weights to $P_{\hat{A}_1}$. The result up to the twelfth grade is presented in the Figure 10

Using the recurrent relation for the anomalous branching coefficients we get the result presented in Figure 11.

We see that inside $\bar{C}_{\hat{A}_1}$, which is shown at the Figure 11, there are only two non-zero anomalous weights. These are the branching coefficients. So the finite reducibility theorem holds and we get the decomposition

$$L^{(0,0;1;0)}_{\hat{A}_2\downarrow\hat{A}_1} = L^{(0;4;0)}_{\hat{A}_1} \oplus L^{(4;4;0)}_{\hat{A}_1}. \tag{44}$$

For the other level 1 irreducible modules of $\hat{A}_2$ we get the trivial branching

$$L^{(1,0;1;0)}_{\hat{A}_2\downarrow\hat{A}_1} = L^{(2;4;0)}_{\hat{A}_1}, \tag{45}$$

$$L^{(0,1;1;0)}_{\hat{A}_2\downarrow\hat{A}_1} = L^{(2;4;0)}_{\hat{A}_1}. \tag{46}$$

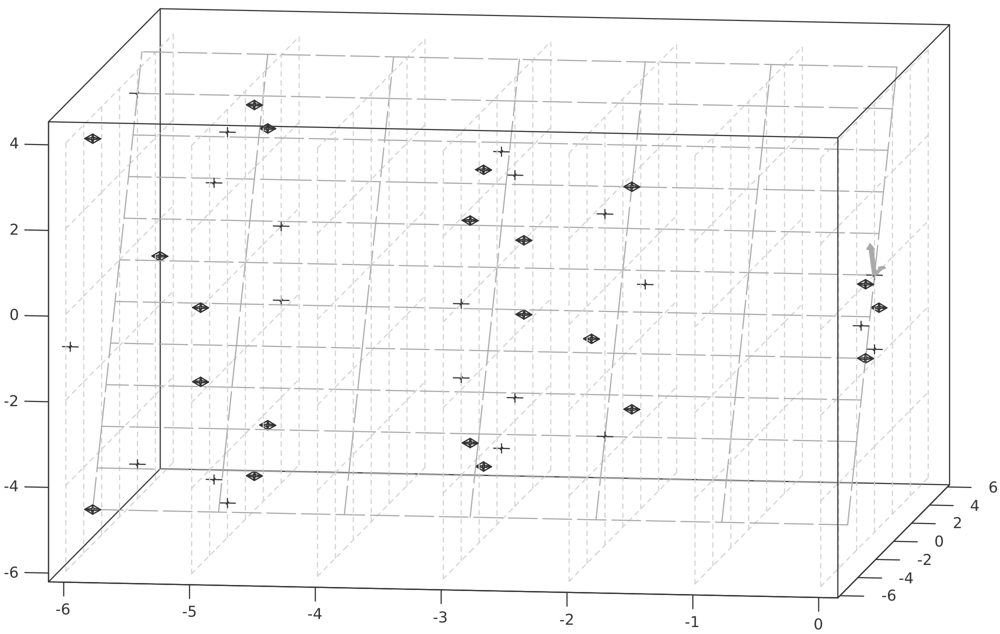


Figure 9: The anomalous weights of the module $L^{(0,0;1;0)}_{\hat{A}_2}$

Using these results the modular-invariant partition function is easily found,

$$Z = \left|\chi_{(4;4;0)} + \chi_{(0;4;0)}\right|^2 + 2\chi^2_{(2;4;0)}. \tag{47}$$

4.1.2 Embedding of semisimple subalgebras

Consider the embedding $\mathfrak{a} \subset \mathfrak{g}$ where $\mathfrak{a} = \hat{A}_1 \oplus \hat{A}_1$ and $\mathfrak{g} = \hat{A}_3$, which is the affine extension of the special embedding $A_1 \oplus A_1 \subset A_3$. Let's construct the special embedding $A_1 \oplus A_1 \subset A_3$ using the method of [17]. We start with 4-dimensional representation of $A_1 \oplus A_1$ with the highest weight $(1,1)$. The weights of this representation are numbered as depicted at the Figure 12 and have the following coordinates in the fundamental weights basis: $\nu_1 = (1,1),\ \nu_2 = (-1,1),\ \nu_3 = (1,-1),\ \nu_4 = (-1,-1)$.

Then for the matrix elements of representation of Cartan subalgebra generators b_1, b_2 in Weyl basis we have $d(b_i) = \mathrm{diag}\left(\frac{2(\nu_1,\alpha_i)}{(\alpha_i,\alpha_i)}, \frac{2(\nu_2,\alpha_i)}{(\alpha_i,\alpha_i)}, \frac{2(\nu_3,\alpha_i)}{(\alpha_i,\alpha_i)}, \frac{2(\nu_4,\alpha_i)}{(\alpha_i,\alpha_i)}\right)$ [17], so $d(b_1) = \mathrm{diag}(1,-1,1,-1),\ d(b_2) = \mathrm{diag}(1,1,-1,-1)$. The embedded roots α_1, α_2 of $A_1 \oplus A_1$ in terms of roots $\tilde{\alpha}_i$ of A_3 are

$$\begin{aligned} \alpha_1 &= \tfrac{1}{2}(\tilde{\alpha}_1 + \tilde{\alpha}_3) \\ \alpha_2 &= \tfrac{1}{2}(\tilde{\alpha}_1 + 2\tilde{\alpha}_2 + \tilde{\alpha}_3) \end{aligned} \tag{48}$$

They are depicted at the Figure 13.

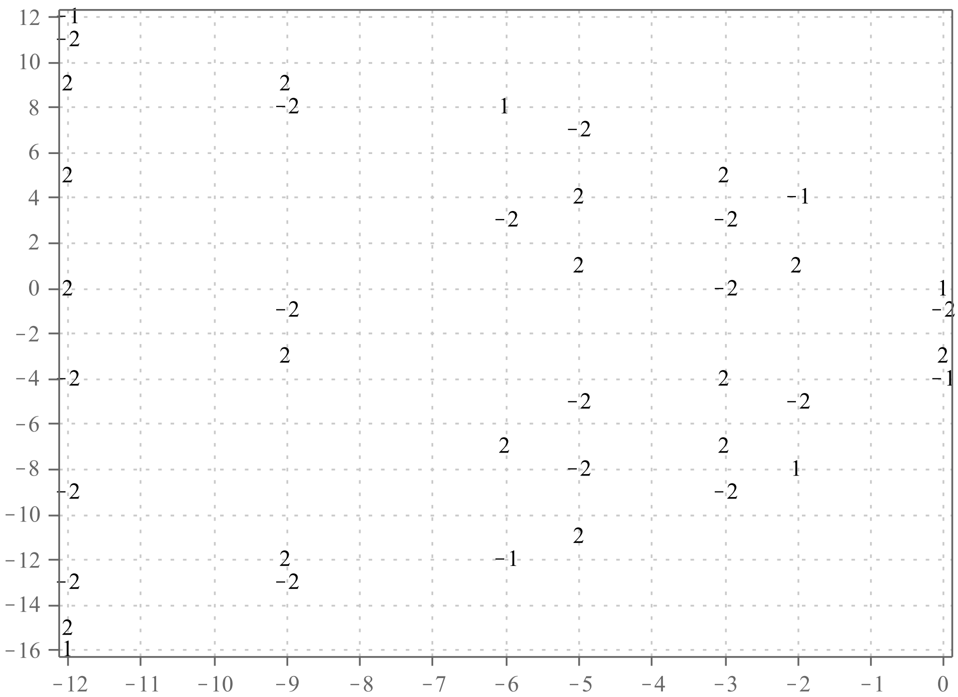


Figure 10: Projected anomalous weights of $L^{(0,0;1;0)}_{\hat{A}_2}$.

This embedding is characterised by the embedding indices $(2,2)$ and is conformal, since $c(A_1 \oplus A_1) = c(A_1) + c(A_1) = 2\frac{x_e \dim(A_1)}{x_e+2} = \frac{\dim A_3}{5} = c(A_3)$.

We are interested in the reduction of the fundamental modules of $\hat{A}_3$. Four fundamental weights of level 1 have the following coordinates in the orthogonal basis:

$$\begin{array}{rcl} w_0 & = & (0,0,0,0;1;0), \\ w_1 & = & (\frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}; 1; 0), \\ w_2 & = & (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}; 1; 0), \\ w_3 & = & (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{3}{4}; 1; 0). \end{array} \tag{49}$$

For the sets of positive roots Δ^+ and $\Delta^+_{\mathfrak{a}}$ we have

$$\begin{array}{rcl} \Delta^+ & = & \Big\{ \overset{\circ}{\Delta}{}^+ = \{\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_1 + \tilde{\alpha}_2, \tilde{\alpha}_2 + \tilde{\alpha}_3, \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3\}; \\ & & \quad \overset{\circ}{\Delta} + n\delta;\ +n\delta \text{ with multiplicity } 3;\ n = 1, 2, \ldots, \Big\} \\ \Delta^+_{\mathfrak{a}} & = & \{\alpha_1, \alpha_2;\ \pm\alpha_1 + n\delta, \pm\alpha_2 + n\delta;\ +n\delta \text{ with multiplicity } 2;\ n = 1, 2, \ldots\} \end{array} \tag{50}$$

The set $\Delta^+_{\perp}$ is empty. The fan $\Gamma_{\mathfrak{a}\subset\mathfrak{g}}$ is shown at the Figure 14. Coordinates of fan element's finite part are given in the basis of fundamental roots of

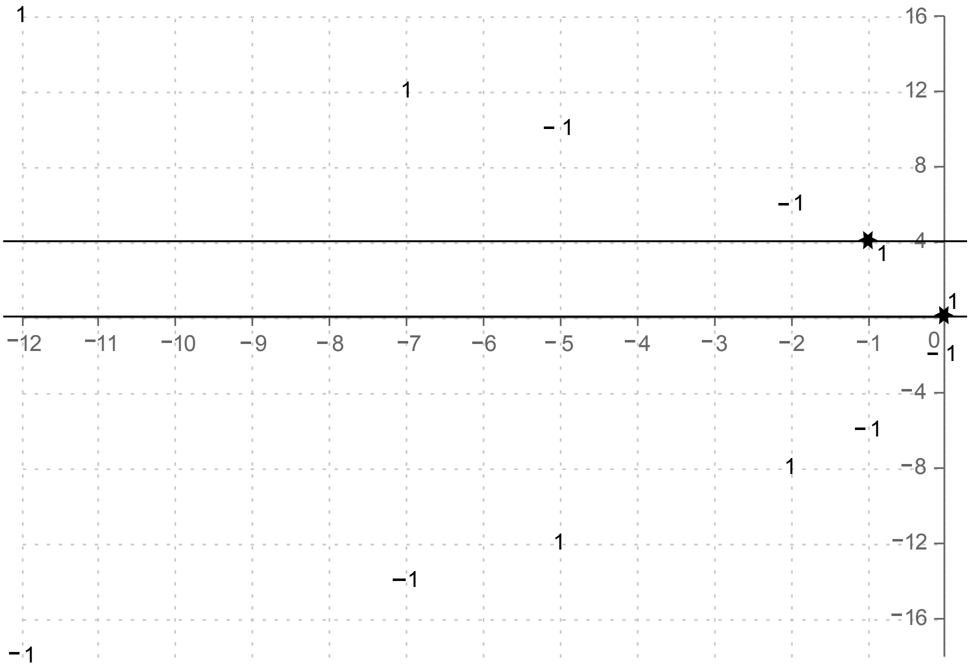


Figure 11: Anomalous branching coefficients for $\hat{A}_1 \subset \hat{A}_2$

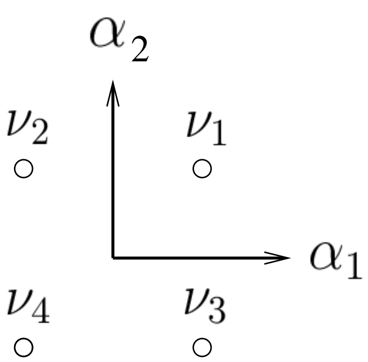


Figure 12: Representation for the special embedding $A_1 \oplus A_1 \subset A_3$

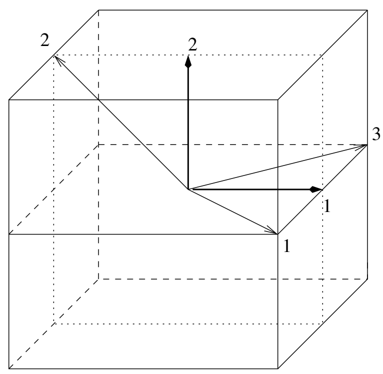


Figure 13: Embedded roots for the special embedding $A_1 \oplus A_1 \subset A_3$

$A_1 \oplus A_1$. Element γ is shown by cross if $s(\gamma) = 1$ and by diamond if $s(\gamma) = -1$.

We limit our calculations by the first 5 grades.

The set $\widehat{\Psi^{(\mu)}} = \{\omega(\mu+\rho)-\rho;\ \omega \in W\}$ of the anomalous weights of $L^{w_2}_{\hat{A}_3}$ consists of 192 elements and its projection $\pi_{\mathfrak{a}}\left(\widehat{\Psi^{(\mu)}}\right)$ for $\mu = w_2 =$

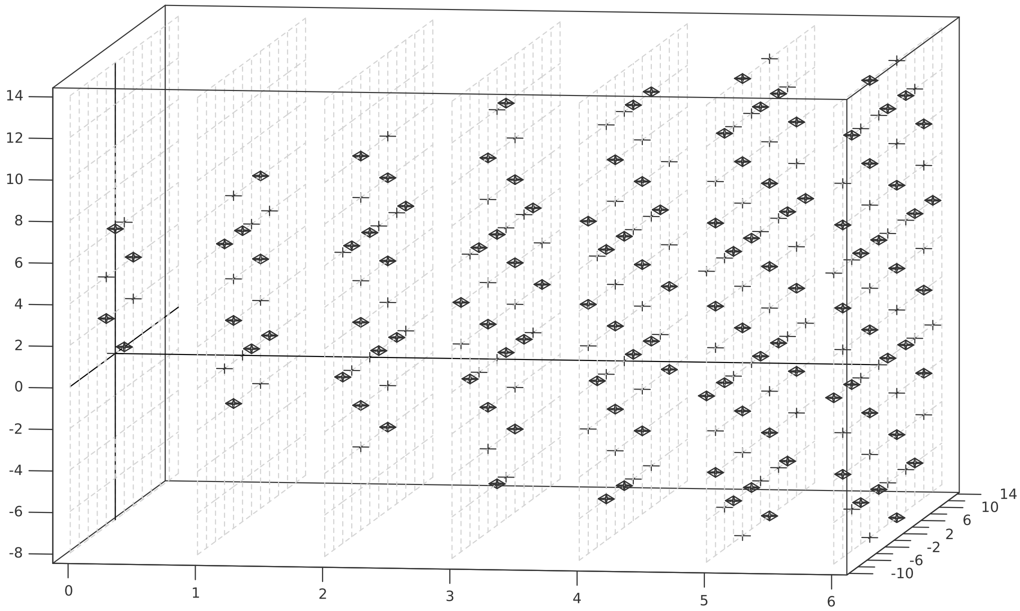


Figure 14: Fan for the special embedding $\hat{A}_1 \oplus \hat{A}_1 \subset \hat{A}_3$

$(0,1,0;1;0)$ is shown at the Figure 15. Coordinates of anomalous weight's finite part are given in the basis of fundamental weights of $A_1 \oplus A_1$. The weights are shown by crosses if the determinant has the value $\epsilon(\omega) = 1$ and by diamonds otherwise.

The diagrams for $L^{w_0}_{\hat{A}_3}, L^{w_1}_{\hat{A}_3}, L^{w_1}_{\hat{A}_3}$ are similar to that of $L^{w_2}_{\hat{A}_3}$.

The anomalous branching coefficients for the module $L^{(w_2)}$ are shown at the Figure 16. The anomalous branching coefficients in the $C_{\mathfrak{a}}$ are equal to the branching coefficients of $L^{w_2}_{\hat{A}_3 \downarrow \hat{A}_1 \oplus \hat{A}_1}$ and are shown by the stars in the Figure 16.

We get the following branching rules

$$\begin{array}{rcl}
L^{(0,0,0;1;0)}_{\hat{A}_3 \downarrow \hat{A}_1 \oplus \hat{A}_1} & = & L^{(0;2;0)}_{\hat{A}_1} \otimes L^{(0;2;0)}_{\hat{A}_1} \\
L^{(1,0,0;1;0)}_{\hat{A}_3 \downarrow \hat{A}_1 \oplus \hat{A}_1} & = & L^{(1;2;0)}_{\hat{A}_1} \otimes L^{(1;2;0)}_{\hat{A}_1} \\
L^{(0,1,0;1;0]}_{\hat{A}_3 \downarrow \hat{A}_1 \oplus \hat{A}_1} & = & \left(L^{(2;2;0)}_{\hat{A}_1} \otimes L^{(0;2;0)}_{\hat{A}_1}\right) \oplus \left(L^{(0;2;0)}_{\hat{A}_1} \otimes L^{(2;2;0)}_{\hat{A}_1}\right) \\
L^{(0,0,1;1;0)}_{\hat{A}_3 \downarrow \hat{A}_1 \oplus \hat{A}_1} & = & L^{(1;2;0)}_{\hat{A}_1} \otimes L^{(1;2;0)}_{\hat{A}_1}
\end{array} \tag{51}$$

Now we can obtain modular invariant partition function for WZW-model with the chiral algebra $\hat{A}_1 \oplus \hat{A}_1$.

$$Z = \left|\chi_{(0;2;0)}\chi_{(0;2;0)}\right|^2 + 2\left|\chi_{(1;2;0)}\chi_{(1;2;0)}\right|^2 + \left|\chi_{(2;2;0)}\chi_{(0;2;0)} + \chi_{(0;2;0)}\chi_{(2;2;0)}\right|^2 = \left|\chi_{(0;2;0)}\right|^4 + 2\left|\chi_{(1;2;0)}\right|^4 + 4\left|\chi_{(2;2;0)}\chi_{(0;2;0)}\right|^2 \tag{52}$$

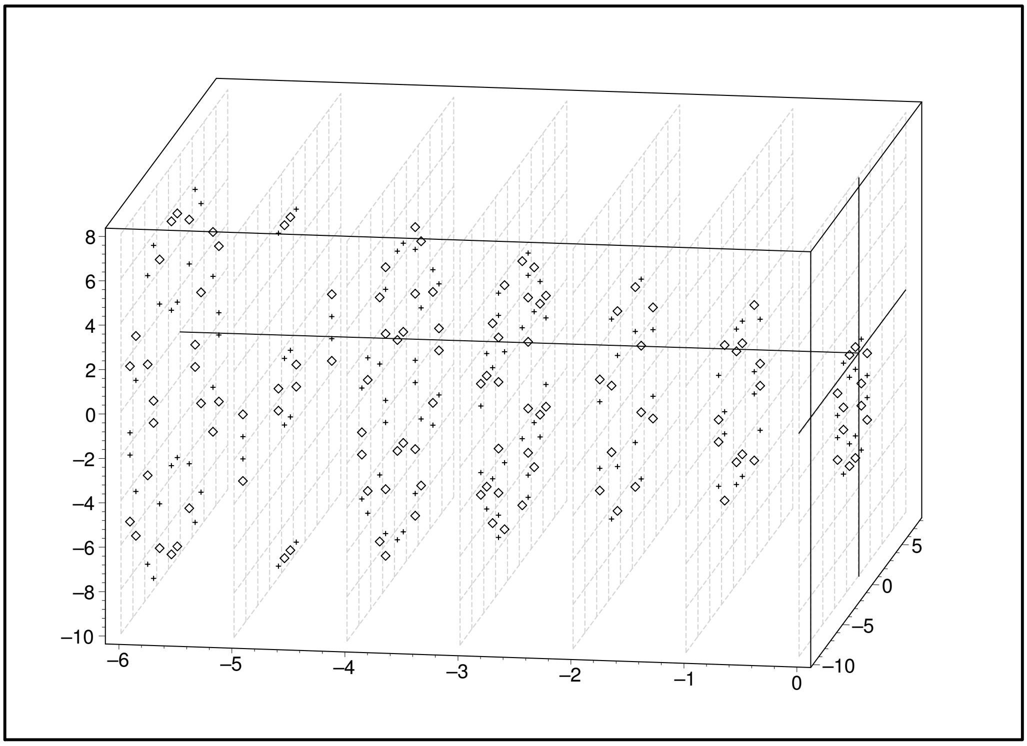


Figure 15: Projected anomalous weights for $L_{\hat{A}_3}^{w_2}$

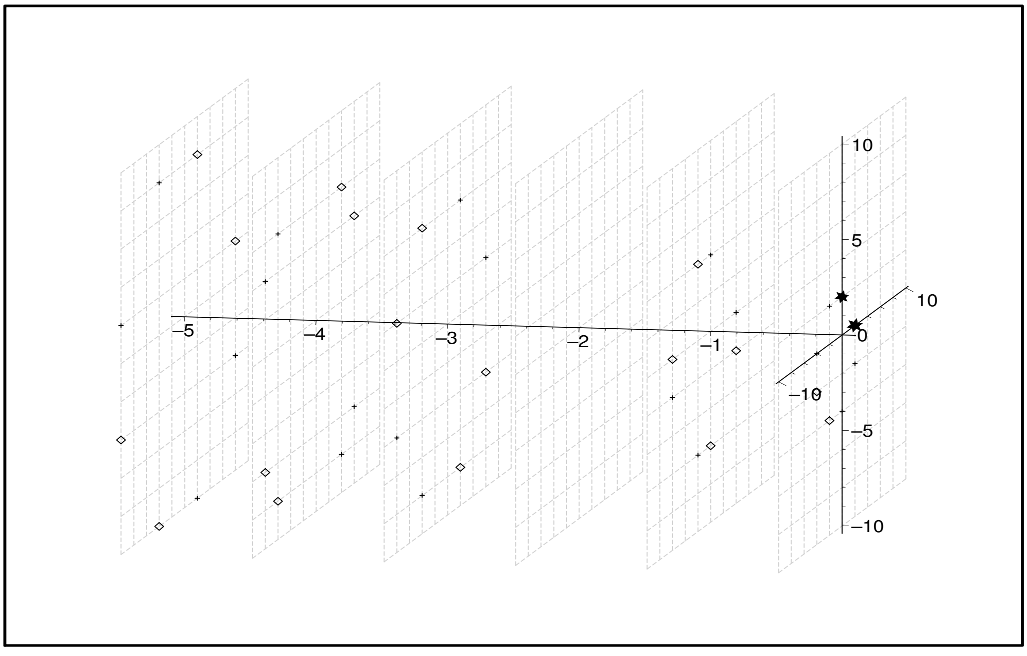


Figure 16: Anomalous branching coefficients for $L_{\hat{A}_3}^{w_2}$

We have chosen this example to illustrate the mechanism of out algorithm. In this case the modules could be reduced by the method described in [13], which is based upon the properties of A_n-series of Lie algebras. We must stress that our method is universal and can be applied for an arbitrary pair of affine Lie algebras $\mathfrak{a} \subset \mathfrak{g}$.

4.2 Coset models

Another natural setting where the reduction problem for affine Lie algebras appears is the coset construction of rational conformal field theories [18]. Coset models are tightly connected with the gauged WZW-models and are actively studied in string theory especially in string models on anti-de-Sitter space [19, 20, 21, 22, 23]. The characters of the coset model are connected with the branching functions in the following way:

$$\chi_{\nu}^{(\mu)}(\tau) = e^{2\pi i \tau (m_\mu - m_\nu)} b_{\nu}^{(\mu)}(\tau), \tag{53}$$

where

$$m_\mu = \frac{|\mu + \rho|^2}{2(k+g)} - \frac{|\rho|^2}{2g}. \tag{54}$$

The problem of the branching functions construction in the coset models was considered in the papers [24], [7], [25]. The results were mostly given in rather cumbersome form and could not be easily used in computations. The method of calculation of the branching coefficients proposed in this text can be used to get power series expansion of the branching functions up to any required degree. In future work we are going to write complete expansion of the branching functions.

In this section we present the example of the computation of branching functions power series expansion for the embedding $A_1 \subset B_2$.

Consider the affine extension of the example 3.1. Since this embedding is regular and $x_e = 1$, the level of the subalgebra modules is equal to that of the initial module.

The set $\Delta_{\perp}^{+}$ of the orthogonal positive roots with the zero projection on the root space of the subalgebra $\hat{A}_1$ is the same as in the finite-dimensional case.

Using the definition (24) we get the fan $\Gamma_{\hat{A}_1 \longrightarrow \hat{B}_2}$ with the corresponding values $s(\gamma + \gamma_0)$(see the Figure 17). Here we restricted the computation to the twelfth grade.

Consider the level 1 module $L_{\hat{B}_2}^{(1,0;1;0)}$ with the highest weight $w_1 = (1, 0; 1; 0)$, where the first two components are the coordinates of the classical part in

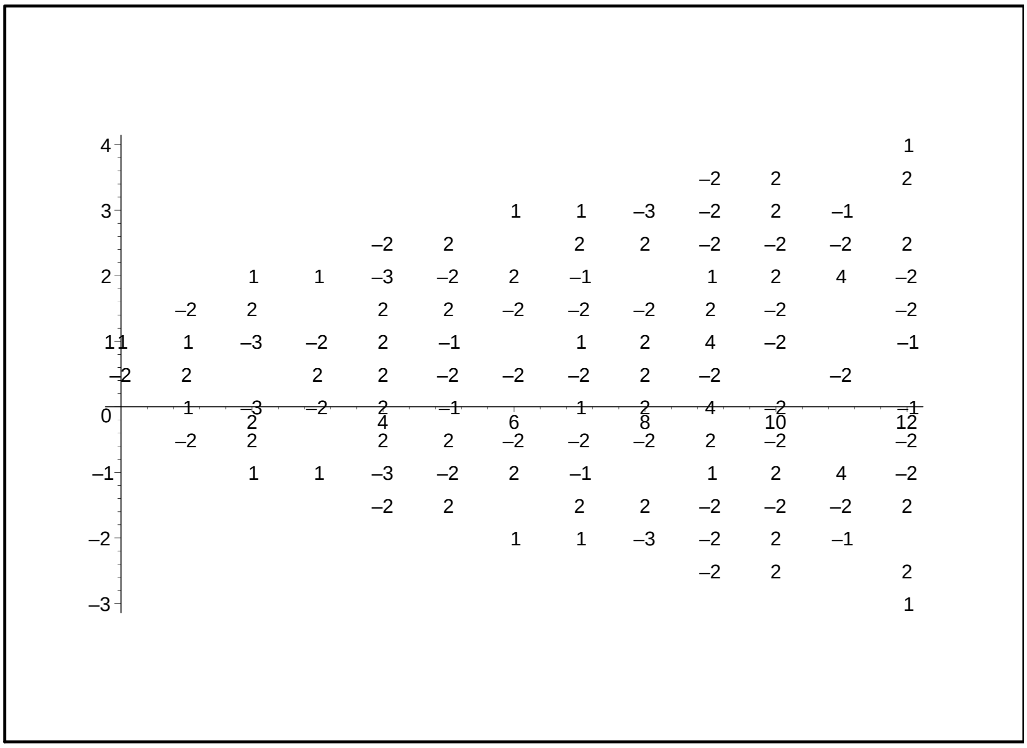


Figure 17: The fan for $\hat{A}_1 \subset \hat{B}_2$

the orthogonal basis e_1, e_2, the third is the level of the weight and the fourth is the grade.

The set of the anomalous weights for this module up to the sixth grade is depicted in the Figure 18. In the grade 0 it is exactly the set of the anomalous weights for the embedding of the classical Lie algebras $A_1 \subset B_2$ depicted in the Figure 3. The weights $\omega(w_1+\rho)-\rho$ is shown as cross if $\epsilon(\omega)=1$ and by circle otherwise. Simple roots of classical subalgebra B_2 are shown in grey and grey diagonal plane is the Cartan subalgebra of the embedded algebra $\hat{A}_1$.

Performing the next step of the algorithm 2.2 we project the anomalous weights to the weight space of the subalgebra $\hat{A}_1$ and find the dimensions of the corresponding $\mathfrak{a}_\perp$-modules $L_{\mathfrak{a}_\perp}^{\pi_{\mathfrak{a}_\perp}(\omega(\mu+\rho))-\rho_{\mathfrak{a}_\perp}}$. The result of this computation up to the twelfth grade is presented in the Figure 19.

Notice that here the lowest weight γ_0 of the fan is zero, since we have excluded all the roots of $\Delta_\perp^+$ from the defining relation (24).

Selecting the elements inside the Weyl chamber $\bar{C}_{\hat{A}_1}^{(0)}$ we get the following

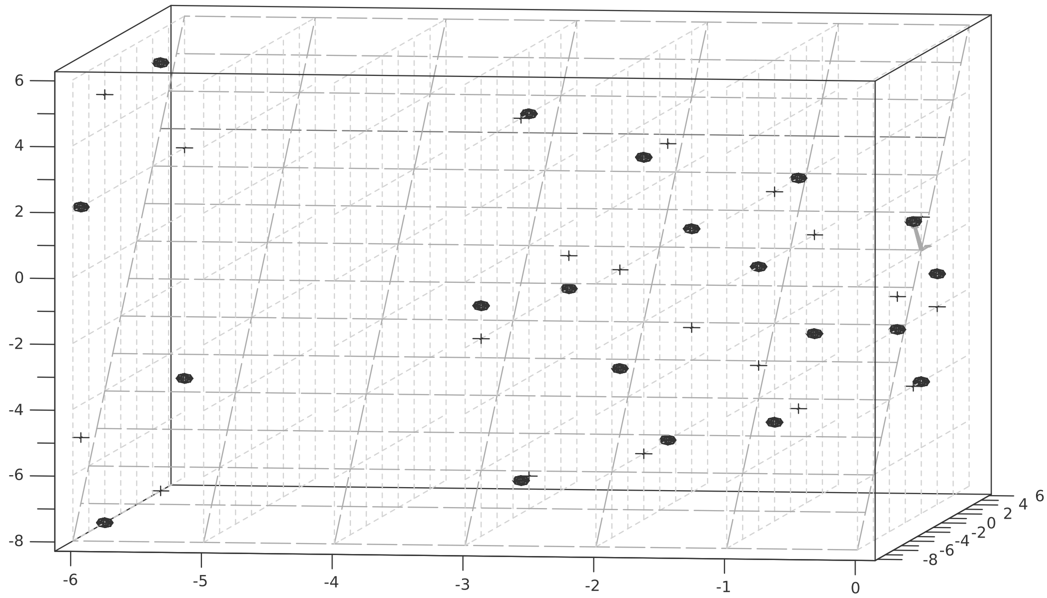


Figure 18: The anomalous weights of the $L_{\hat{B}_2}^{(1,0;1;0)}$. The weights of the grade 0 are depicted at the Figure 3
.

results for the branching coefficients up to twelfth grade

$$\begin{aligned} L^{w_1}_{\hat{B}_2\downarrow\hat{A}_1} = 2L^{w_1}_{\hat{A}_1}(0) \oplus 1L^{w_0}_{\hat{A}_1}(0) \oplus 4L^{w_0}_{\hat{A}_1}(-1)\oplus \\ 2L^{w_1}_{\hat{A}_1}(-1) \oplus 8L^{w_0}_{\hat{A}_1}(-2) \oplus 8L^{w_1}_{\hat{A}_1}(-2) \oplus 15L^{w_0}_{\hat{A}_1}(-3)\oplus \\ 12L^{w_1}_{\hat{A}_1}(-3) \oplus 26L^{w_1}_{\hat{A}_1}(-4) \oplus 29L^{w_0}_{\hat{A}_1}(-4) \oplus 51L^{w_0}_{\hat{A}_1}(-5)\oplus \\ 42L^{w_1}_{\hat{A}_1}(-5) \oplus 78L^{w_1}_{\hat{A}_1}(-6) \oplus 85L^{w_0}_{\hat{A}_1}(-6) \oplus 120L^{w_1}_{\hat{A}_1}(-7)\oplus \\ 139L^{w_0}_{\hat{A}_1}(-7) \oplus 202L^{w_1}_{\hat{A}_1}(-8) \oplus 222L^{w_0}_{\hat{A}_1}(-8) \oplus 306L^{w_1}_{\hat{A}_1}(-9)\oplus \\ 346L^{w_0}_{\hat{A}_1}(-9) \oplus 530L^{w_0}_{\hat{A}_1}(-10) \oplus 482L^{w_1}_{\hat{A}_1}(-10) \oplus 714L^{w_1}_{\hat{A}_1}(-11)\oplus \\ 797L^{w_0}_{\hat{A}_1}(-11) \oplus 1080L^{w_1}_{\hat{A}_1}(-12) \oplus 1180L^{w_0}_{\hat{A}_1}(-12) \end{aligned} \quad (55)$$

This result can be presented as the set of branching functions [4].

$$\begin{aligned} b_0^{(w_1)} = \; & 1 + 4\,q^1 + 8\,q^2 + 15\,q^3 + 29\,q^4 + 51\,q^5 + 85\,q^6 + 139\,q^7 + \\ & 222\,q^8 + 346\,q^9 + 530\,q^{10} + 797\,q^{11} + 1180\,q^{12} + \dots \end{aligned} \quad (56)$$

$$\begin{aligned} b_1^{(w_1)} = \; & 2 + 2\,q^1 + 8\,q^2 + 12\,q^3 + 26\,q^4 + 42\,q^5 + 78\,q^6 + 120\,q^7 + \\ & 202\,q^8 + 306\,q^9 + 482\,q^{10} + 714\,q^{11} + 1080\,q^{12} + \dots \end{aligned} \quad (57)$$

Here the lower index enumerates the branching function according to their

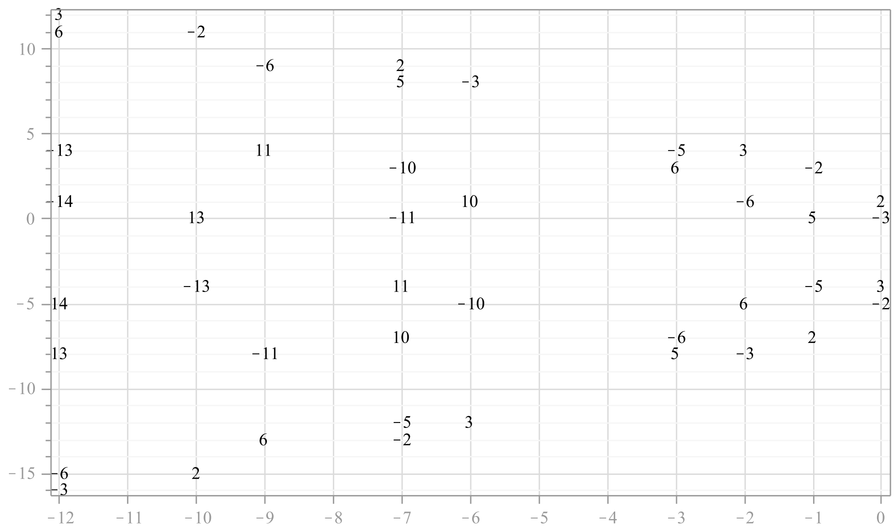


Figure 19: The projected anomalous weights $\pi_{\hat{A}_1}\left(\Psi_{\hat{B}_2}^{(1,0;1;0)}\right)$ and the dimensions of $\mathfrak{a}_\perp$-modules.

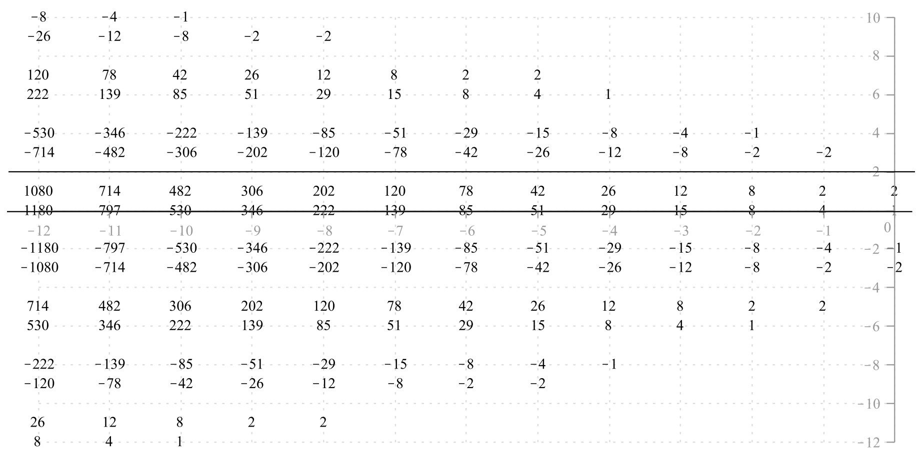


Figure 20: Anomalous branching coefficients for $\hat{A}_1 \subset \hat{B}_2$

highest weights in $P^+_{\hat{A}_1}$, these are the fundamental weights $w_0 = \lambda_0 = (0,1,0)$, $w_1 = \alpha/2 = (1,1,0)$, and $q = \exp(2\pi i\tau)$.

Now we can use the equation (53) to get the expansion of the B_2/A_1-coset theory characters:

$$\begin{aligned}
\chi_1^{(w_1)}(q) = &\quad q^{\frac{7}{12}}\,(2 + 2\,q^1 + 8\,q^2 + 12\,q^3 + 26\,q^4 + 42\,q^5 + 78\,q^6 + 120\,q^7 + \\
& 202\,q^8 + 306\,q^9 + 482\,q^{10} + 714\,q^{11} + 1080\,q^{12} + \dots) \\
\chi_0^{(w_1)}(q) = &\quad q^{\frac{5}{6}}\,(1 + 4\,q^1 + 8\,q^2 + 15\,q^3 + 29\,q^4 + 51\,q^5 + 85\,q^6 + 139\,q^7 + \\
& 222\,q^8 + 346\,q^9 + 530\,q^{10} + 797\,q^{11} + 1180\,q^{12} + \dots)
\end{aligned} \tag{58}$$

The folded fan technique [**?**] can be used to get the explicit expression for the branching functions and the corresponding coset conformal field theory characters. This subject will be discussed in the future works.

5 Conclusion

We have shown that the injection fan technique can be used to deal with the nonmaximal subalgebras. It was demonstrated that in such cases in the set of positive roots $\Delta^+_{\mathfrak{g}}$ it is necessary to separate an additional subset $\Delta^+_{\perp}$. The injection fan is formed by the weights $\left(\Delta^+_{\mathfrak{g}} \setminus \Delta^+_{\mathfrak{a}}\right) \setminus \Delta^+_{\perp}$ and the additional role of the subset $\Delta^+_{\perp}$ is to modify the anomalous weights of the initial module. This modification reduces to a simple procedure: the anomalous weights are to be substituted by the dimensions of the corresponding $\mathfrak{a}_{\perp}$-modules.

We have demonstrated the effectiveness of the proposed generalizations of the injection fan algorithm and discussed its possible application to some physical problems. In particular we considered the construction of modular-invariant partition functions in the conformal field theory in the framework of conformal embedding method and the coset construction of the rational conformal field theory. This construction is useful in the study of WZW-models emerging in the context of the AdS/CFT correspondence [19, 20, 21].

The proposed general approach to the reduction problem has the additional applications in the study of integrable chains.

6 Acknowledgements

The work was supported in part by RFFI grant N 09-01-00504 and the National Project RNP.2.1.1./1575.

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