Affine.m – *Mathematica* package for computations in representation theory of finite-dimensional and affine Lie algebras

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Abstract

In this paper we present **Affine.m** — a program for computations in representation theory of finite-dimensional and affine Lie algebras and describe implemented algorithms. The algorithms are based on the properties of weights and Weyl symmetry. Computation of weight multiplicities in irreducible and Verma modules, branching of representations and tensor product decomposition are the most important problems for us. These problems have numerous applications in physics and we provide some examples of these applications. The program is implemented in the popular computer algebra system *Mathematica* and works with finite-dimensional and affine Lie algebras.

Keywords:
Mathematica; Lie algebra; affine Lie algebra; Kac-Moody algebra; root system; weights; irreducible modules, CFT, Integrable systems

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PROGRAM SUMMARY

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No. of lines in distributed program, including test data, etc.: 24 844
No. of bytes in distributed program, including test data, etc.: 1 045 908
Distribution format: tar.gz

*Programming language:*Mathematica
*Computer:*i386-i686, x86_64
Operating system: Linux, Windows, Mac OS, Solaris
RAM: 5-500 Mb
Keywords: Mathematica; Lie algebra; affine Lie algebra; Kac-Moody algebra; root system; weights; irreducible modules, CFT, Integrable systems
Classification: 5 Computer Algebra, 4.2 Other algebras and groups
Nature of problem:
Representation theory of finite-dimensional Lie algebras has many applications in different branches of physics, including elementary particle physics, molecular physics, nuclear physics. Representations of affine Lie algebras appear in string theories and two-dimensional conformal field theory used for the description of critical phenomena in two-dimensional systems. Also Lie symmetries play major role in a study of quantum integrable systems.
Solution method:
We work with weights and roots of finite-dimensional and affine Lie algebras and use Weyl symmetry extensively. Central problems which are the computations of weight multiplicities, branching and fusion coefficients are solved using one general recurrent algorithm based on generalization of Weyl character formula. We also offer alternative implementation based on the Freudenthal multiplicity formula which can be faster in some cases.
Restrictions:
Computational complexity grows fast with the rank of an algebra, so computations for algebras of ranks greater than 8 are not practical.
Unusual features:
We offer the possibility to use a traditional mathematical notation for the objects in representation theory of Lie algebras in computations if **Affine.m** is used in the *Mathematica* notebook interface.
Running time:
From seconds to days depending on the rank of the algebra and the complexity of the representation.

1. Introduction

Representation theory of Lie algebras is of central importance for different areas of physics and mathematics. In physics Lie algebras are used to describe symmetries of quantum and classical systems. Computational methods in representation theory have a long history [1], there exist numerous software packages for computations related to Lie algebras [2], [3], [4, 5], [6], [7].

Most popular programs [2], [3], [6], [5] are created to study representation theory of simple finite-dimensional Lie algebras. The main computational problems are the following:

1. Construction of a root system which determines the properties of Lie algebra including its commutation relations.
2. Weyl group traversal which is important due to Weyl symmetry of root system and characters of representations.
3. Calculation of weight multiplicities, branching and fusion coefficients, which are essential for construction and study of representations.

There are well-known algorithms for these tasks [8], [9], [1], [10]. The third problem is the most computation intensive. There are two different recurrent algorithms which are based on the Weyl character formula and the Freudenthal multiplicity formula. In this paper we analyze them both.

Infinite-dimensional Lie algebras also have a growing number of applications in physics for example in conformal field theory and in a study of integrable systems. But infinite-dimensional algebras are much harder to investigate and the number of available computer programs is much smaller in this case.

Affine Lie algebras [11] constitute an important and tractable class of infinite-dimensional Lie algebras. They are constructed as central extensions of loop algebras of (semi-simple) finite-dimensional Lie algebras and appear naturally in a study of Wess-Zumino-Witten and coset models in conformal field theory [12], [13], [14], [15].

The structure of affine Lie algebras allows us to adapt for them the computational algorithms created for finite-dimensional Lie algebras [7], [16], [17]. The book [17] with the tables of multiplicities and other computed characteristics of affine Lie algebras and representations was published in 1990. But we are not aware of software packages for popular computer algebra systems which can be used to extend these results. We address this issue and present **Affine.m** – a *Mathematica* package for computations in representation theory of affine and finite-dimensional Lie algebras. We describe the features and limitations of the package in the present paper. We also provide a representation-theoretical background of the implemented algorithms and present the examples of computations relevant to physical applications.

The paper starts with an overview of Lie algebras and their representation theory (Sec. 2). Then we describe the datastructures of **Affine.m** used to present different objects related to Lie algebras and representations (Sec. 3) and discuss the implemented algorithms (Sec. 4). The next section consists of physically interesting examples (Sec. 5). The paper is concluded with the discussion of possible extensions and refinements (Sec. 6).

2. Theoretical background

In this section we recall necessary definitions and present formulae used in computations.

2.1. Lie algebras of finite and affine types

A *Lie algebra* $\mathfrak{g}$ is a vector space with a bilinear operation $[\cdot,\cdot]:\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$, which is called a *Lie bracket.* If we choose a basis X_i in $\mathfrak{g}$ we can specify the commutation relations by the *structure constants* C_{ijk}:

$$[X^i, X^j] = \sum_k C^{ij}_k X^k \tag{1}$$

A Lie algebra is *simple* if it contains no non-trivial ideals with respect to a commutator. A *semisimple* Lie algebra is a direct sum of simple Lie algebras. In the present paper we treat simple and semisimple Lie algebras.

A *Cartan subalgebra* $\mathfrak{h}_{\mathfrak{g}}$ is a nilpotent subalgebra of $\mathfrak{g}$ that coincides with its normalizer. We denote the elements of a basis of $\mathfrak{h}_{\mathfrak{g}}$ by H^i.

The Killing form on $\mathfrak{g}$ gives a non-degenerate bilinear form $(\cdot,\cdot)$ on $\mathfrak{h}_{\mathfrak{g}}$ which can be used to identify $\mathfrak{h}_{\mathfrak{g}}$ with the subspace of the dual space $\mathfrak{h}^*_{\mathfrak{g}}$ of linear functionals on $\mathfrak{h}_{\mathfrak{g}}$. *Weights* are the elements of $\mathfrak{h}^*_{\mathfrak{g}}$ and are denoted by Greek letters $\mu,\nu,\omega,\lambda\dots$

A special choice of a basis gives a compact description of the commutation relations (1). This basis can be encoded by the root system which is discussed in Section 2.2 (See also [18, 19]).

The *loop algebra* $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, corresponding to semisimple Lie algebra $\mathfrak{g}$, has commutation relations

$$[X^i t^n, X^j t^m] = t^{n+m} \sum_k C_k^{ij} X^k \tag{2}$$

The central extension leads to the appearance of an additional term

$$[X^i t^n + \alpha c, X^j t^m + \beta c] = t^{n+m} \sum_k C_k^{ij} X^k + (X^i, X^j) n \delta_{n+m,0} c \tag{3}$$

This algebra $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ is called a non-twisted *affine Lie algebra* [11], [20, 21], [17].

We do not treat twisted affine Lie algebras in the present paper.

2.2. Modules, weights and roots

Let $\mathfrak{g}$ be a finite-dimensional or an affine Lie algebra.

Then the $\mathfrak{g}$-module is a vector space V together with a bilinear map $\mathfrak{g} \times V \to V$ such that

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v), \quad \text{for } x, y \in \mathfrak{g}, v \in V \tag{4}$$

A representation of an algebra $\mathfrak{g}$ on a vector space V is a homomorphism $\mathfrak{g} \to gl(V)$ from $\mathfrak{g}$ to a Lie algebra of endomorphisms of the vector space V with the commutator as the bracket.

For an arbitrary representation it is possible to diagonalize the operators corresponding to Cartan generators H^i simultaneously by a special choice of basis $\{v_j\}$ in V:

$$H^i \cdot v_j = \nu_j^i v_j \tag{5}$$

The eigenvalues ν_j^i of Cartan generators on an element of basis v_j determine a weight $\nu_j \in \mathfrak{h}_{\mathfrak{g}}^*$ such that $\nu_j(H^i) = \nu_j^i$. A vector $v \in V$ is called the weight vector of the weight λ if $Hv = \lambda_j(H)v, \ \forall H \in \mathfrak{h}$. The weight subspace consists of all weight vectors $V_\lambda = \{v \in V : Hv = \lambda_j(H)v, \ \forall H \in \mathfrak{h}\}$. The weight multiplicity $m_\lambda = \mathrm{mult}(\lambda) = \mathrm{dim} V_\lambda$ is the dimension of the weight subspace.

The structure of a module is determined by the set of weights since the action of generators E^α on weight vectors is

$$E^\alpha \cdot v_\lambda \propto v_{\lambda+\alpha} \tag{6}$$

The module structure can be encoded by the formal character of the module

$$\mathrm{ch}V = \sum_{\lambda} m_{\lambda} e^{\lambda} \tag{7}$$

The character $\mathrm{ch}V \in \mathcal{E}$ is an element of algebra $\mathcal{E}$ generated by formal exponents of weights. The character can be specialized by taking its value on some element ξ of $\mathfrak{h}$.

Any Lie algebra is its own module with respect to a special kind of representation. The action that defines this representation is called *adjoint* and is given by the bracket $ad_X Y = [X, Y]$. *Roots* are weights of the adjoint representation of $\mathfrak{g}$. They encode the commutation relations of algebra in the following way. Denote by Δ the set of roots. For each $\alpha \in \Delta$ there exist a root $-\alpha \in \Delta$ and the generators $E^{\alpha}, E^{-\alpha}$ such that

$$[H^i, E^{\alpha}] = \alpha^i E^{\alpha} \tag{8}$$

$$\left[E^{\alpha}, E^{\beta}\right] = \begin{cases} N_{\alpha,\beta} E^{\alpha+\beta}, & \text{if } \alpha+\beta \in \Delta \\ \frac{2}{(\alpha,\alpha)} \sum_i \alpha^i H^i, & \text{if } \alpha = -\beta \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

Given the root system Δ we can choose the set of positive roots. This is a subset $\Delta^+ \subset \Delta$ such that for each root $\alpha \in \Delta$ exactly one of the roots $\alpha, -\alpha$ is contained in Δ^+ and for any two distinct positive roots $\alpha, \beta \in \Delta^+$ such that $\alpha + \beta \in \Delta$ their sum is also positive $\alpha + \beta \in \Delta^+$. The elements of $-\Delta^+$ are called negative roots.

A positive root is *simple* if it cannot be written as a sum of positive roots. The set of simple roots $\Phi = \{\alpha_i\}$ is a basis in $\mathfrak{h}^*_{\mathfrak{g}}$ and each root can be written as $\alpha = \sum_i n_i \alpha_i$ with all n_i non-negative or non-positive simultaneously. In the case of a finite-dimensional Lie algebra $\mathfrak{g}$ simple roots are numbered from 1 to the rank of the algebra $i = 1, \ldots, r, \quad r = \mathrm{rank}(\mathfrak{g})$. Enumerating simple roots with an index i we introduce lexicographic ordering in the root system Δ. The highest root with respect to this ordering is denoted by $\theta = \sum_{i=1,\ldots,r} a_i \alpha_i$, the coefficients a_i are called *marks*. θ is also the highest weight of the adjoint module (See section 2.3). *Comarcs* are the numbers equal to $a_i^{\vee} = \frac{(\alpha_i,\alpha_i)}{2} a_i$.

Although for an affine Lie algebra $\hat{\mathfrak{g}}$ the set of roots Δ is infinite the set of simple roots Φ is finite and its elements are denoted by $\alpha_0, \ldots \alpha_r$ where $r = \mathrm{rank}(\mathfrak{g})$. The roots $\alpha_1, \ldots, \alpha_r$ are the roots of the underlying finite-dimensional Lie algebra $\overset{\circ}{\mathfrak{g}}$. The root $\alpha_0 = \delta - \theta$ is the difference of the *imaginary root* δ and the highest root θ of the algebra $\overset{\circ}{\mathfrak{g}}$. Note that root multiplicity $\mathrm{mult}(\alpha)$ for an affine Lie algebra root can be greater than one.

A subalgebra $\mathfrak{b}_+ \subset \mathfrak{g}$ spanned by the generators H^i, E^α for positive roots $\alpha \in \Delta^+$ is called a *Borel subalgebra*.

A *parabolic subalgebra* $\mathfrak{p}_I \supset \mathfrak{b}_+$ contains a Borel subalgebra and is generated by some subset of simple roots $\{\alpha_j : j \in I, I \subset \{1 \ldots r\}\}$. It is spanned by the subset of generators $\{H^i\} \cup \{E^\alpha : \alpha \in \Delta^+\} \cup \{E^{-\alpha} : \alpha \in \Delta^+, \alpha = \sum_{j\in I} n_j \alpha_j\}$. A *regular subalgebra* $\mathfrak{a} \subset \mathfrak{g}$ is determined by the root system $\Delta_\mathfrak{a}$ with the set of simple roots $\{\beta_i, i = 1, \ldots, r_\mathfrak{a}\}$ being a subset of the set of roots $\{\alpha_1, \ldots, \alpha_r\} \cup \{-\theta\}$.

The *Weyl group* $W_\mathfrak{g}$ is generated by reflections $\{s_i : \mathfrak{h}^*_\mathfrak{g} \to \mathfrak{h}^*_\mathfrak{g}\}$ corresponding to simple roots $\{\alpha_i\}$:

$$s_i \cdot \lambda = \lambda - \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \alpha_i \tag{10}$$

The root system and characters of representation are invariant with respect to the Weyl group action. A root system can be reconstructed from the set of simple roots by the Weyl group transformations.

Weyl groups are finite for finite-dimensional Lie algebras and finitely generated for affine Lie algebras.

Consider an action of the element $s_\alpha s_{\alpha+\delta}$ of the Weyl group of affine Lie algebra $\hat{\mathfrak{g}}$ for α being a simple root of the underlying finite-dimensional Lie algebra $\mathfrak{g}$. Using definition (10) it is easy to see that $s_\alpha s_{\alpha+\delta} \cdot \lambda = \lambda + \frac{2}{(\alpha,\alpha)}\alpha + \left(\frac{(\alpha,\alpha)}{2(\lambda,\delta)} + (\lambda, \alpha)\right)\delta$. So the Weyl group $W_{\hat{\mathfrak{g}}}$ can be presented as a semidirect product of the Weyl group $W_\mathfrak{g}$ of $\mathfrak{g}$ and the set of translations corresponding to the roots of $\mathfrak{g}$.

A Weyl group element can be presented as a product of elementary reflections in multiple ways. The number of elementary reflections in the shortest sequence representing an element $w \in W_\mathfrak{g}$ is called the *length* of w and is denoted by $l(w)$. We also use the notation $\epsilon(w) = (-1)^{l(w)}$ for the parity of the number of Weyl reflections generating w.

A fundamental domain $\bar{C}$ for the Weyl group $W_\mathfrak{g}$ action on $\mathfrak{h}^*_\mathfrak{g}$ is determined by the requirement $\xi \in \bar{C} \Leftrightarrow (\xi, \alpha_i) \geq 0$ for all simple roots α_i. It is called the *main Weyl chamber*.

A *Cartan matrix* A is defined by products of simple roots

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \tag{11}$$

and can be used for a compact description of Lie algebra commutation relations in the Chevalley basis [18], [22], [23].

The form (11) induces a basis dual to the simple roots basis. It is called the *fundamental weights basis.* We denote its elements by ω_i:

$$\langle \omega_i, \alpha_j \rangle = \frac{2(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij} \tag{12}$$

For a finite-dimensional Lie algebra there are r fundamental weights, $i = 1, \dots, r$. For an affine Lie algebra we have an additional fundamental weight $\omega_0 = \lambda$, $(\lambda, \delta) = 1$, $(\lambda, \lambda) = (\delta, \delta) = 0$. Other fundamental weights are equal to $\omega_i = a_i^v \omega_0 + \overset{\circ}{\omega}_i$, where $\overset{\circ}{\omega}_i$ is a fundamental weight of the finite-dimensional Lie algebra $\mathfrak{g}$.

The sum of fundamental weights $\rho = \sum_i \omega_i$ is called a *Weyl vector.* It is an important tool in representation theory.

2.3. Highest weight modules

We consider finitely generated $\mathfrak{g}$-modules V such that $V = \bigoplus_{\xi \in \mathfrak{h}_{\mathfrak{g}}^*} V_\xi$, where each V_ξ is finite dimensional and there exists a finite set of weights $\lambda_1, \dots \lambda_s$ which generates the weight system of V, i.e. if $\dim V_\xi \neq 0$ then $\xi = \lambda_i - \sum_{k=1,\dots,r} n_k \alpha_k$ where $n_k \in \mathbb{Z}_+$ (See [24], [25]).

A highest weight module V^μ contains a single highest weight μ, the other weights are obtained by subtractions of linear combinations of simple roots $\lambda = \mu - n_1\alpha_1 - \dots - n_r\alpha_r$, $n_k \in \mathbb{Z}_+$.

The simplest type of highest weight modules is the Verma module M^μ. Its space can be defined as a module

$$M^\mu = U(\mathfrak{g}) \underset{U(\mathfrak{b}_+)}{\otimes} D^\mu(\mathfrak{b}_+), \tag{13}$$

with respect to a multiplication in $U(\mathfrak{g})$ and $\underset{U(\mathfrak{b}_+)}{\otimes}$ means that the action of elements of $U(\mathfrak{b}_+)$ "falls through" the left part of the tensor product onto the right part. Here $\mathfrak{b}_+$ is the Borel subalgebra, $D^\mu(\mathfrak{b}_+)$ is a representation of $\mathfrak{b}_+$ such that $D(E^\alpha) = 0$, $D(H) = \mu(H)$ for any positive root α. Elements of $\mathfrak{g}$ act from the left and we should commute all the elements of $\mathfrak{b}_+$ to the right, so that they can act on the space $D^\lambda(\mathfrak{b}_+)$.

Weight multiplicities in Verma modules can be found using the Weyl character formula

$$\mathrm{ch} M^\mu = \frac{e^\mu}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)}} = \frac{e^\mu}{\sum_{w \in W} \epsilon(w) e^{w\rho - \rho}} \tag{14}$$

Here we have used the Weyl denominator identity

$$R := \prod_{\alpha\in\Delta^+} \left(1 - e^{-\alpha}\right)^{\mathrm{mult}(\alpha)} = \sum_{w\in W} \epsilon(w) e^{w\rho-\rho}, \tag{15}$$

and $\epsilon(w) := \det(w)$ is equal to the parity of the sequence of Weyl reflections generating w.

A Verma module M^μ has the unique maximal submodule and the unique nontrivial simple quotient L^μ which is an *irreducible highest weight module.*

Irreducible highest weight modules have no non-trivial submodules. The Weyl character formula for an irreducible highest weight module L^μ is

$$\mathrm{ch}L^\mu = \frac{\sum_{w\in W} \epsilon(w) e^{w(\mu+\rho)-\rho}}{\sum_{w\in W} \epsilon(w) e^{w\rho-\rho}} = \sum_{w\in W} \epsilon(w)\ \mathrm{ch}M^{w(\mu+\rho)-\rho} \tag{16}$$

Thus the character of an irreducible highest weight module can be seen as a combination of characters of Verma modules. (This fact is a consequence of the Bernstein-Gelfand-Gelfand resolution ([26, 27], see also [24]).)

Construction of a generalized Verma module is analogous to (13), but the representation of the Borel subalgebra is substituted by a representation of a parabolic subalgebra $\mathfrak{p}_I \supset \mathfrak{b}_+$ generated by some subset $\{\alpha_I\}$ of simple roots $I \subset \{1, \ldots, r\}$:

$$M_I^\mu = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_I)} L_{\mathfrak{p}_I}^\mu .$$

Introduce a formal element $R_I := \prod_{\alpha\in\Delta^+\setminus\Delta^+_{\mathfrak{p}_I}} \left(1 - e^{-\alpha}\right)^{\mathrm{mult}(\alpha)}$. Then the character of a generalized Verma module can be written as

$$\mathrm{ch}M_I^\mu = \frac{1}{R_I}\mathrm{ch}L_{\mathfrak{p}_I}^\mu . \tag{17}$$

We can use the Weyl character formula to obtain recurrent relations for weight multiplicities – important tools for calculations [28, 29].

For irreducible highest-weight modules the recurrent relation has the following form

$$m_\xi = -\sum_{w\in W\setminus e} \epsilon(w) m_{\xi-(w(\rho)-\rho)} + \sum_{w\in W} \epsilon(w) \delta_{(w(\mu+\rho)-\rho),\xi} . \tag{18}$$

Formulae for Verma and generalized Verma modules differ only in the second term on the right-hand side. In the case of Verma module it is just $\delta_{\xi,\mu}$. For a generalized Verma module the summation in the second term on the right-hand side of (18) is over the Weyl subgroup generated by the reflections corresponding to the roots $\{\alpha_I\}$.

Another recurrent formula can be obtained from a study of Casimir element action on irreducible highest weight modules [18]:

$$m_\lambda = \frac{2}{(\mu+\rho)^2 - (\lambda+\rho)^2} \sum_{\alpha\in\Delta^+} \sum_{k\geq 1} (\lambda + k\alpha, \alpha) m_{\lambda+k\alpha}. \tag{19}$$

It is called the Freudenthal multiplicity formula. Note that it is applicable only to irreducible modules.

We discuss the use of formulae (18) and (19) for the computations in section 4.

Now consider an algebra $\mathfrak{g}$ and a reductive subalgebra $\mathfrak{a} \subset \mathfrak{g}$. Simple roots β_i of the subalgebra $\mathfrak{a}$ can be presented as linear combinations of $\mathfrak{g}$-algebra roots α_j: $\beta_i = \sum_{j=1,\ldots,r_\mathfrak{g}} k_j \alpha_j,\ j = 1,\ldots,r_\mathfrak{a}$.

Each irreducible $\mathfrak{g}$-module is also an $\mathfrak{a}$-module, although $L^\mu_\mathfrak{g}$ is in general not irreducible as an $\mathfrak{a}$-module. It can be decomposed into a direct sum of irreducible $\mathfrak{a}$-modules:

$$L^\mu_\mathfrak{g} = \bigoplus_\nu b^\mu_\nu L^\nu_\mathfrak{a} \tag{20}$$

The coefficients this decomposition are called branching coefficients.

It is possible to calculate branching coefficients by constructing and successively subtracting the submodules $L^\nu_\mathfrak{a}$. This traditional approach has serious limitations especially in case of affine Lie algebras. We discuss them in the end of section 4.

Now we describe an alternative approach which is based on recurrent properties of branching coefficients. But before we proceed to these recurrent relations we need several additional definitions.

For a subalgebra $\mathfrak{a} \subset \mathfrak{g}$ we introduce the subalgebra $\mathfrak{a}_\perp$. Consider a root subspace $\mathfrak{h}^*_{\perp\mathfrak{a}}$ orthogonal to $\mathfrak{h}_\mathfrak{a}$,

$$\mathfrak{h}^*_{\perp\mathfrak{a}} := \{\eta \in \mathfrak{h}^* | \forall h \in \mathfrak{h}_\mathfrak{a}; \eta(h) = 0\},$$

and the roots (correspondingly – positive roots) of $\mathfrak{g}$ orthogonal to the roots of $\mathfrak{a}$,

$$\begin{aligned} \Delta_{\mathfrak{a}_\perp} &: = \{\beta \in \Delta_\mathfrak{g} | \forall h \in \mathfrak{h}_\mathfrak{a}; \beta(h) = 0\}, \\ \Delta^+_{\mathfrak{a}_\perp} &: = \left\{\beta^+ \in \Delta^+_\mathfrak{g} | \forall h \in \mathfrak{h}_\mathfrak{a}; \beta^+(h) = 0\right\}. \end{aligned} \tag{21}$$

Let $W_{\mathfrak{a}_\perp}$ be a subgroup of W generated by the reflections w_β with the roots $\beta \in \Delta^+_{\mathfrak{a}_\perp}$. The subsystem $\Delta_{\mathfrak{a}_\perp}$ determines a subalgebra $\mathfrak{a}_\perp$ with the Cartan subalgebra $\mathfrak{h}_{\mathfrak{a}_\perp}$.

The Cartan subalgebra $\mathfrak{h}$ can be decomposed in the following way: $\mathfrak{h} = \mathfrak{h}_{\mathfrak{a}} \oplus \mathfrak{h}_{\mathfrak{a}_\perp} \oplus \mathfrak{h}_\perp$

We also introduce the notations

$$\widetilde{\mathfrak{a}_\perp} := \mathfrak{a}_\perp \oplus \mathfrak{h}_\perp \qquad \widetilde{\mathfrak{a}} := \mathfrak{a} \oplus \mathfrak{h}_\perp. \tag{22}$$

For $\mathfrak{a}$ and $\mathfrak{a}_\perp$ we consider the corresponding Weyl vectors, $\rho_{\mathfrak{a}}$ and $\rho_{\mathfrak{a}_\perp}$ and compose the so called "defects" $\mathcal{D}_{\mathfrak{a}}$ and $\mathcal{D}_{\mathfrak{a}_\perp}$ of the injection:

$$\mathcal{D}_{\mathfrak{a}} := \rho_{\mathfrak{a}} - \pi_{\mathfrak{a}}\rho, \qquad \mathcal{D}_{\mathfrak{a}_\perp} := \rho_{\mathfrak{a}_\perp} - \pi_{\mathfrak{a}_\perp}\rho. \tag{23}$$

For $\mu \in P^+$ consider the linked weights $\{(w(\mu+\rho)-\rho)\,|w \in W\}$ and their projections to $h^*_{\mathfrak{a}_\perp}$ additionally shifted by the defect $-\mathcal{D}_{\mathfrak{a}_\perp}$:

$$\mu_{\mathfrak{a}_\perp}(w) := \pi_{\mathfrak{a}_\perp}[w(\mu+\rho)-\rho] - \mathcal{D}_{\mathfrak{a}_\perp}, \quad w \in W.$$

Among the weights $\{\mu_{\mathfrak{a}_\perp}(w)\,|w \in W\}$ one can always choose those located in the fundamental chamber $\overline{C_{\mathfrak{a}_\perp}}$. Let U be the set of representatives u for the classes $W/W_{\mathfrak{a}_\perp}$ such that

$$U := \left\{u \in W| \quad \mu_{\mathfrak{a}_\perp}(u) \in \overline{C_{\mathfrak{a}_\perp}}\right\} \quad . \tag{24}$$

Thus we can form the subsets:

$$\mu_{\widetilde{\mathfrak{a}}}(u) := \pi_{\widetilde{\mathfrak{a}}}[u(\mu+\rho)-\rho] + \mathcal{D}_{\mathfrak{a}_\perp}, \quad u \in U, \tag{25}$$

and

$$\mu_{\mathfrak{a}_\perp}(u) := \pi_{\mathfrak{a}_\perp}[u(\mu+\rho)-\rho] - \mathcal{D}_{\mathfrak{a}_\perp}, \quad u \in U. \tag{26}$$

Notice that the subalgebra $\mathfrak{a}_\perp$ is regular by definition since it is built on a subset of roots of the algebra $\mathfrak{g}$.

Denote by $k^{(\mu)}_\xi$ signed branching coefficients defined as follows. If $\xi \in \bar{C}_{\mathfrak{a}}$ is in the main Weyl chamber $k^{(\mu)}_\xi = b^{(\mu)}_\xi$, otherwise $k^{(\mu)}_\xi = \epsilon(w) b^{(\mu)}_{w(\xi+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}$ where $w \in W_{\mathfrak{a}}$ is such that $w(\xi+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}} \in \bar{C}_{\mathfrak{a}}$.

Now we can use the Weyl character formula to write a recurrent relation [30] for the signed branching coefficients $k^{(\mu)}_\xi$ corresponding to an injection $\mathfrak{a} \hookrightarrow \mathfrak{g}$:

$$\begin{aligned} k^{(\mu)}_\xi = -\tfrac{1}{s(\gamma_0)} \Big(& \textstyle\sum_{u\in U} \epsilon(u)\ \dim\left(L^{\mu_{\mathfrak{a}_\perp}(u)}_{\mathfrak{a}_\perp}\right) \delta_{\xi-\gamma_0,\pi_{\widetilde{\mathfrak{a}}}(u(\mu+\rho)-\rho)} + \\ & + \textstyle\sum_{\gamma\in\Gamma_{\widetilde{\mathfrak{a}}\to\mathfrak{g}}} s(\gamma+\gamma_0)\, k^{(\mu)}_{\xi+\gamma} \Big). \end{aligned} \tag{27}$$

The recursion is governed by the set $\Gamma_{\mathfrak{a}\to\mathfrak{g}}$ called the injection fan. The latter is defined by the carrier set $\{\xi\}_{\mathfrak{a}\to\mathfrak{g}}$ for the coefficient function $s(\xi)$

$$\{\xi\}_{\widetilde{\mathfrak{a}}\to\mathfrak{g}} := \{\xi \in P_{\widetilde{\mathfrak{a}}} | s(\xi) \neq 0\}$$

appearing in the expansion

$$\prod_{\alpha\in\Delta^+\setminus\Delta^+_\perp} \left(1 - e^{-\pi_{\widetilde{\mathfrak{a}}}\alpha}\right)^{\mathrm{mult}(\alpha)-\mathrm{mult}_{\mathfrak{a}}(\pi_{\widetilde{\mathfrak{a}}}\alpha)} = -\sum_{\gamma\in P_{\widetilde{\mathfrak{a}}}} s(\gamma)e^{-\gamma}; \qquad (28)$$

The weights in $\{\xi\}_{\widetilde{\mathfrak{a}}\to\mathfrak{g}}$ are to be shifted by γ_0 – the lowest vector in $\{\xi\}$ – and the zero element is to be eliminated:

$$\Gamma_{\mathfrak{a}\to\mathfrak{g}} = \{\xi - \gamma_0 | \xi \in \{\xi\}\} \setminus \{0\}\,. \qquad (29)$$

The formula (18) is a particular case of the recurrent relation for branching coefficients (27) in the case of a Cartan subalgebra $\mathfrak{a} = \mathfrak{h}_{\mathfrak{g}}$.

If the root system of $\mathfrak{a}_\perp$ is generated by some subset of $\mathfrak{g}$ simple roots $\alpha_1, \ldots, \alpha_r$ then the recurrent relation (27) is connected with the generalized Bernstein-Gelfand-Gelfand resolution for parabolic Verma modules [31].

Another particular case of this formula is connected with tensor product decompositions. Consider the tensor product of two irreducible $\mathfrak{g}$-modules $L^\mu \otimes L^\nu$. It is also a $\mathfrak{g}$-module but not irreducible in general. So

$$L^\mu \otimes L^\nu = \bigoplus_\gamma f^{\mu\nu}_\gamma L^\gamma \qquad (30)$$

The coefficients $f^{\mu\nu}_\gamma$ are called the fusion coefficients. The problem of computation of fusion coefficients is equivalent to a branching problem for the diagonal subalgebra $\mathfrak{g} \subset \mathfrak{g} \oplus \mathfrak{g}$ (see [32]). So our implementation of a recurrent algorithm can be used to decompose tensor products (See Section 5.1).

In the case of affine Lie algebras $\mathfrak{g}, \mathfrak{a}$ the multiplicities m_ν and the branching coefficients $b^{(\mu)}_\nu$ can be regarded as the coefficients in the power series decomposition of string and branching functions correspondingly:

$$\sigma_\nu(q) = \sum_{n=0}^{\infty} m_{\nu-n\delta} q^n, \quad \nu = \sum_j c_j\omega_j, \quad c_j \geq 0 \qquad (31)$$

$$b_\nu(q) = \sum_{n=0}^{\infty} b_{\nu-n\delta} q^n, \quad \nu = \sum_j c_j\omega_j, \quad c_j \geq 0 \qquad (32)$$

String and branching functions have modular and analytic properties which are important for conformal field theory, especially in coset models and the study of CFT on higher genus surfaces [33], [13], [12], [34].

Affine.m calculates weight multiplicities and branching coefficients for affine Lie algebras up to some finite grade. We present examples of computations in Sections 5.3, 5.4. Now we proceed to the description of the datastructures and the algorithms implemented in **Affine.m**.

3. Core datastructures

Having introduced necessary mathematical objects, problems and relations we now describe the related datastructures of **Affine.m**. Although *Mathematica* is untyped language it is possible to create structured objects and do the type checks with patterns [35], [36].

3.1. Weights

Weights are represented by two datastructures: `finiteWeight` for finite-dimensional Lie algebras and `affineWeight` for affine.

Internally the finite weight is a List with the Head `finiteWeight`, its components are the coordinates of the weight in the orthogonal Bourbaki basis [23].

An affine weight is an extension of a finite weight by supplying it with the level and grade coordinates. There is a set of functions defined for finite and affine weights. The complete list can be found in the online help of the package. The most important are the definitions of an addition, a multiplication by a number and a scalar product (bilinear form) for weights. These definitions allow us to use a traditional notation with **Affine.m**:

```
w=makeFiniteWeight[{1,0,3}];
v=makeFiniteWeight[{3,2,1}];
2*w+v==makeFiniteWeight[{5,2,7}]
w.v==6
```

The use of an orthogonal basis in the internal structure of weights allows us to work with weights without a complete specification of a root system which is useful for a study of branching, since the roots of the subalgebra can be specified by hand.

3.2. Root systems

To specify an algebra of finite or affine type it is enough to fix its root system. Root systems are represented by two datatypes `finiteRootSystem` and `affineRootSystem`. The latter is an extension of the former. We offer several different constructors for these datastructures. It is possible to specify the set of simple roots explicitly, for example to study the subalgebra $B_2 \subset B_4$ we can use the definition

```
b2b4=makeFiniteRootSystem[ { {1,-1,0,0}, {0,1,0,0} } ]
```

There are constructors for the root systems of simple finite-dimensional Lie algebras:

```
b2=makeSimpleRootSystem[B,2]
```

We use the typographic features of the *Mathematica* frontend to offer a mathematical notation for simple Lie algebras:

```
B2 == makeFiniteRootSystem[ { {1, -1}, {0, 1} }]
```

Non-twisted affine root systems can be created as the affine extensions of finite root systems, e.g.

```
b2affine = makeAffineExtension[b2]
```

In the notebook interface this can be written simply as $\hat{B}_2$.

Semisimple Lie algebras can be created as the sums of simple ones:

```
A1 ⊕ A1 == finiteRootSystem[2, 2, {finiteWeight[2, {1, 0}], finiteWeight[2, {0, 1}]}]
```

The predicate `rootSystemQ` checks if the object is a root system of finite or affine type.

The list of simple roots is a property of the root system so it is accessed as the field `rs[simpleRoots]`.

We have implemented several functions to get major properties of root systems. The Weyl vector is computed by the function `rho[rs_?rootSystemQ]`:

```
In[1]  =  rho[b2]
Out[1]  =  finiteWeight[2, {3/2, 1/2}]
```

The list of positive roots can be constructed with the function `positiveRoots[rs_?rootSystemQ]`. For an affine Lie algebra this and related functions return the list up to some fixed grade. This grade limit is set as the value of the field `rs[gradeLimit]` which is equal to 10 by default. The list of roots (up to `gradeLimit`) is returned by the function `roots[rs]`. The Cartan matrix and the fundamental weights are calculated by the functions `cartanMatrix` and `fundamentalWeights` correspondingly.

It is possible to specify the weight of a Lie algebra by its Dynkin labels

```
weight[b2][1,2] == makeFiniteWeight[{2, 1}]
```

The function `dynkinLabels[rs_?rootSystemQ][wg_?weightQ]` returns Dynkin labels of a weight `wg` in the root system `rs`.

An element of the Weyl group can be specified by the set of the indices of the reflections, so the element $w = s_1 s_2 s_1$ of the Weyl group of algebra B_2 is constructed with `weylGroupElement[b2][1,2,1]`. Then it can be applied to the weights:

```
w = weylGroupElement[b2][1,2,1];
w @ makeFiniteWeight[{1,0}] == makeFiniteWeight[{-1,0}]
```

A computation of a lexicographically minimal form [10, 37] for the Weyl group elements can be conveniently implemented using pattern-matching in *Mathematica*. In [38] the rewrite rules for simple finite dimensional and affine Lie algebras are presented as the *Mathematica* patterns. Our presentation of the Weyl group elements is compatible with the code of [38]:

```
In[1]  = <<A3reduce;
         reduce[s[1,2,1,2,1,3,2,1,1]]
Out[1] = s[2, 3, 2]

In[2]  = (weylGroupElement[A3] @@ reduce[s[1,2,1,2,1,3,2,1,1]]) @ weight[A3][-1,-2,-1]
Out[2] = finiteWeight[4, {-2, 2, 1, -1}]

In[3]  = dynkinLabels[A3][Out[2]]
Out[3] = {-4, 1, 2}
```

3.3. Formal elements

We represent formal characters of modules by a special structure `formalElement`. The constructor `makeFormalElement`$[\{\gamma_1,\ldots,\gamma_n\},\{m_1,\ldots,m_n\}]$ creates a datastructure which represents the element $\sum_{i=1}^{n} m_i e^{\gamma_i}$ of the formal algebra $\mathcal{E}$. This structure is a hash-table implemented with DownValues. The keys are weights at the exponents and the values are the corresponding multiplicities. The operations in $\mathcal{E}$ are implemented for the `formalElement` data-type: formal elements can be added, multiplied by a number or by an exponent of a weight. There exists also a multiplication of formal elements but no division.

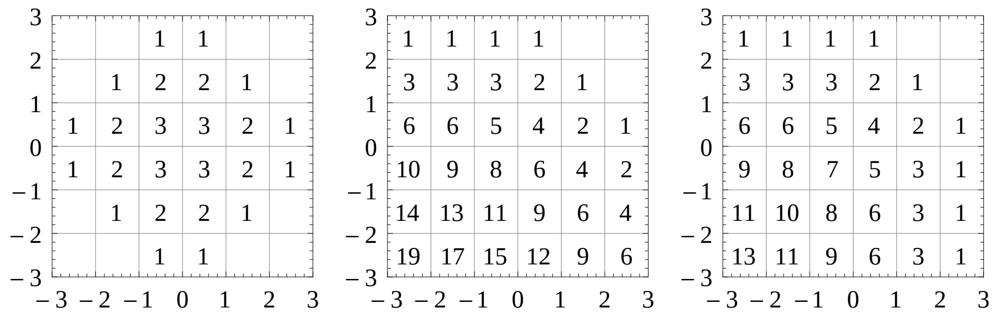
```
In[1]  = makeFormalElement[{makeFiniteWeight[{1,1}],makeFiniteWeight[{0,0}]},{1,2}] *
           (2 * Exp[makeFiniteWeight[{1,0}]] *
           makeFormalElement[{makeFiniteWeight[{1,1}],makeFiniteWeight[{0,0}]},{1,2}]);
In[2]  = In[1][weights]
Out[2] = {finiteWeight[2, {1, 0}], finiteWeight[2, {2, 1}], finiteWeight[2, {3, 2}]}

In[3]  = In[1][multiplicities]
Out[3] = {8, 8, 2}
```

3.4. Modules

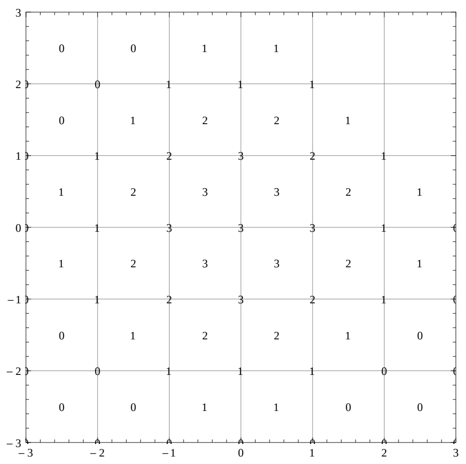
Affine.m can be used to study different kinds of modules, i.e. Verma modules, irreducible modules and parabolic Verma modules. We need the datastructure `module` to represent a generic module of a Lie algebra $\mathfrak{g}$. Module properties can be deduced from its set of singular weights using the Weyl character formulae (14),(15),(17),(16). A set of singular weights can have Weyl symmetry. It can be a symmetry with respect to the Weyl group $W_{\mathfrak{g}}$ or with respect to some subgroup $W_{\mathfrak{a}}$ as in the case of parabolic Verma modules. Then it is possible to study only the main Weyl chamber $C_{\mathfrak{a}}$. To use this symmetry a generic constructor for the `module` datastructure accepts several parameters `makeModule[rs_?rootSystemQ][singWeights_formalElement,subs_?rootSystemQ|emptyRootSystem[],limit:10.` Here `rs` is the root system of a Lie algebra $\mathfrak{g}$, `singWeights` is the set of singular weights, `subs` is the root system corresponding to the Weyl group $W_{\mathfrak{a}}$ which is the (anti-)symmetry of the set of singular weights. The parameter `limit` limits the computation for infinite-dimensional representations such as Verma or parabolic Verma modules. There are several specialized constructors for different types of highest weight modules:

```
vm=makeVermaModule[B_2][{2,1}];
pm=makeParabolicVermaModule[B_2][weight[B_2][2,1],{1}];
im=makeIrreducibleModule[B_2][2,1];
GraphicsRow[textPlot/@{im,vm,pm}]
```



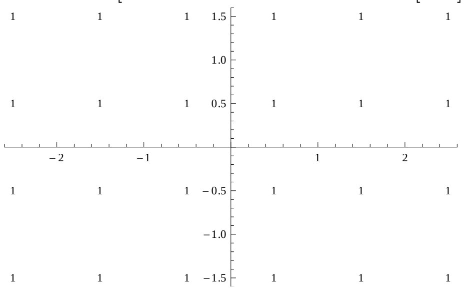
As we have already stated the properties of a module are encoded by its singular element. The function `singularElement[m_module]` returns the singular element of a module as a `formalElement` datastructure. The character (up to `limit` for (parabolic) Verma modules) is returned by the function `character[m_module]`. A direct sum of modules is a module and we use natural notation

```
In[1] := im1=makeIrreducibleModule[B_2][weight[B_2][2,1]];
         im2=makeIrreducibleModule[B_2][weight[B_2][1,2]];
         Head[im1⊕ im2]
Out[1] = module
In[2] := textPlot[im1⊕ im2]
```



The tensor product is also implemented but only for finite-dimensional Lie algebras, since the tensor product of affine Lie algebra modules leads to rich new structures [39, 40, 41] which are out of the scope of the present paper.

```
textPlot[makeIrreducibleModule[A₁][5]⊗ makeIrreducibleModule[A₁][3]];
```



4. Computational algorithms

As we have already stated in the section 2.3 there exist two recurrent relations which can be used to calculate weight multiplicities in irreducible modules. Both algorithms proceed in the following way to calculate weight multiplicities:

1. Create the list of weights in the main Weyl chamber by subtracting all possible combinations of simple roots from the highest weight (e.g. for a finite-dimensional algebra subtract α_1 from μ while inside $\bar{C}$, then subtract α_2 from all the weights already obtained etc).
2. Sort the list of weights by their product with the Weyl vector.

3. Use a recurrent formula. If the weight required for the recurrent computation is outside the main chamber use the Weyl symmetry.

The difference in the performance of algorithms comes from the number of previous values required to compute the multiplicity of a weight under consideration. For the recurrent relation (18) based on the Weyl formula it is constant and equal to the number of elements in the Weyl group (if we are far from the boundary of the representation diagram). When the Freudenthal formula (19) is used the number of previous values grows with the distance from the external border of representation. So the Freudenthal formula is faster if the weight is close to the border or the rank of the algebra and the size of the Weyl group is large [8]. Note that the Freudenthal formula is valid for the irreducible modules only, so it can not be used to study (generalized) Verma modules.

We have made some experiments with our implementations of the Freudenthal formula and formula (19) and prepared the Fig. 1, which depicts the dependence of the computation time on the number of weights in a module.

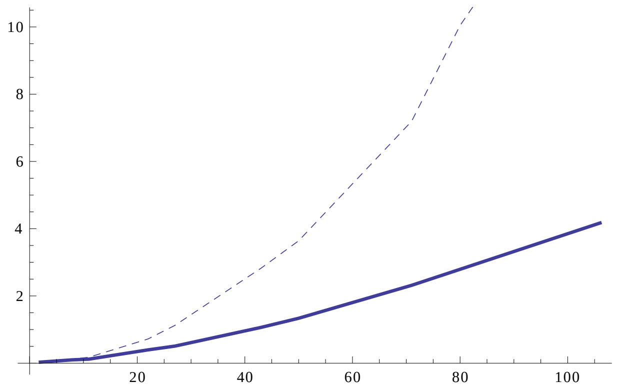


Figure 1: The dependence of the running time of the algorithms based on the Freudenthal formula (19) (dashed) and the recurrent relation (18) (solid) on the number of weights in $\bar{C}$ for calculation of multiplicities in representations of B_2.

In the calculation of branching coefficients the application of the Freudenthal formula requires a complete construction of the formal characters of an algebra module and all the representations of a subalgebra. It is impractical if the ranks of an algebra and a subalgebra are big, for example for the maximal subalgebras.

An alternative algorithm was presented in the paper [30]. It contains the following steps:

1. Construct the root system $\Delta_{\mathfrak{a}}$ for an embedding $\mathfrak{a} \to \mathfrak{g}$.
2. Select all the positive roots $\alpha \in \Delta^+$ orthogonal to $\mathfrak{a}$, i.e. form the set $\Delta^+_{\mathfrak{a}_\perp}$.
3. Construct the set $\Gamma_{\mathfrak{a}\to\mathfrak{g}}$. The relation (2) defines the sign function $s(\gamma)$ and the set $\Phi_{\mathfrak{a}\subset\mathfrak{g}}$. The lowest weight γ_0 is subtracted to get the fan: $\Gamma_{\mathfrak{a}\to\mathfrak{g}} = \{\xi - \gamma_0 | \xi \in \Phi_{\mathfrak{a}\subset\mathfrak{g}}\} \setminus \{0\}$.
4. Construct the set $\widehat{\Psi^{(\mu)}} = \{w(\mu+\rho) - \rho;\ w \in W\}$ of singular weights for the $\mathfrak{g}$-module $L^{(\mu)}$.
5. Select the weights $\left\{\mu_{\widetilde{\mathfrak{a}_\perp}}(w) = \pi_{\widetilde{\mathfrak{a}_\perp}}[w(\mu+\rho)-\rho] - \mathcal{D}_{\mathfrak{a}_\perp} \in \overline{C_{\widetilde{\mathfrak{a}_\perp}}}\right\}$. Since the set $\Delta^+_{\mathfrak{a}_\perp}$ is fixed we can easily check that the weight $\mu_{\widetilde{\mathfrak{a}_\perp}}(w)$ belongs to the main Weyl chamber $\overline{C_{\widetilde{\mathfrak{a}_\perp}}}$ (by computing its scalar product with the fundamental weights of $\mathfrak{a}_\perp$).
6. For the weights $\mu_{\widetilde{\mathfrak{a}_\perp}}(w)$ calculate dimensions of the corresponding modules, $\dim\left(L^{\mu_{\widetilde{\mathfrak{a}_\perp}}(u)}_{\widetilde{\mathfrak{a}_\perp}}\right)$, using the Weyl dimension formula and construct the singular element $\Psi^{(\mu)}_{(\mathfrak{a},\mathfrak{a}_\perp)}$.
7. Calculate the signed branching coefficients using the recurrent relation (27) and select among them those corresponding to the weights in the main Weyl chamber $\overline{C_{\mathfrak{a}}}$.

We can speed up the algorithm by one-time computation of the representatives of conjugate classes $W/W_{\mathfrak{a}_\perp}$.

Consider the regular embedding $B_2 \subset B_4$. In this case the fan consists of 24 elements. In order to decompose B_4 module we need to construct the subset of singular weights of the module which projects to the main Weyl chamber of the subalgebra B_2. The full set of singular weights consists of 384 elements. The required subset contains at most 48 elements. The time for the construction of this required subset is negligible if the number of branching coefficients is greater than that.

We may estimate the total number of required operations for the computation of branching coefficients as the product of the number of elements with non-zero branching coefficients in the main Weyl chamber of a subalgebra and the number of elements in the fan. So we have a linear growth. If we use a direct algorithm we need to compute the multiplicities for each module in the decomposition. So the number of operations grows faster than the square of the number of elements with non-zero branching coefficients in the main Weyl chamber of a subalgebra.

To illustrate this performance issue we present the Figure 2 where we show the time required to compute the branching coefficients for $B_3 \subset B_4$.

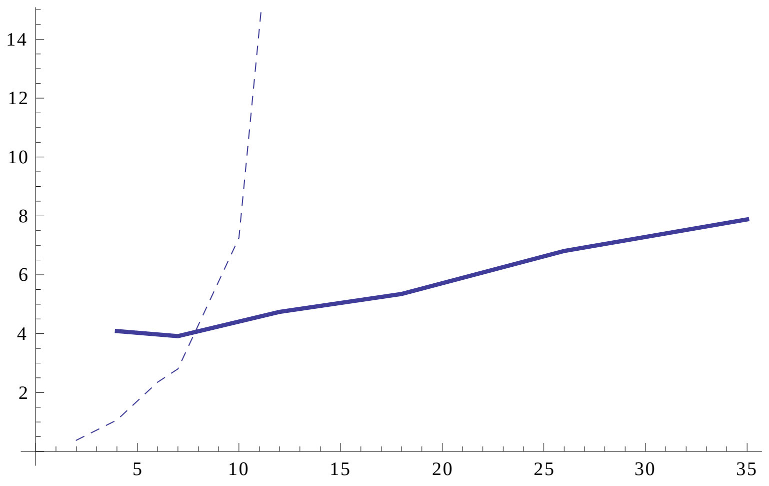


Figure 2: Dependence of the time required to compute branching coefficients for $B_3 \subset B_4$ on the number of weights in $\bar{C}$. The dashed line corresponds to the direct algorithm based on the Freudenthal formula (19), the solid one to the use of recurrent relation (27).

5. Examples

In this section we present some examples of computations available with **Affine.m** and the code required to produce these results.

5.1. Tensor product decompositon for finite-dimensional Lie algebras

Computation of fusion coefficients for decomposition of the tensor product of highest-weight modules to the direct sum of irreducible modules has numerous applications in physics. For example, we can consider the spin of a composite system such as an atom. Another interesting example is the integrable spin chain consisting of N particles with the spins living in some representation L of a Lie algebra $\mathfrak{g}$ with a $\mathfrak{g}$-invariant Hamiltonian H, describing nearest-neighbour spin-spin interaction. In order to solve such a system, i.e. to find eigenstates of the Hamiltonian, we need to decompose $L^{\otimes N}$ into the direct sum of the irreducible $\mathfrak{g}$-modules of lower dimension and diagonalize the Hamiltonian on these modules.

For the fundamental representations of simple Lie algebras it is sometimes possible to get an analytic formula for the dependence of the decomposition coefficients on N (See [32]). Our code provides the numerical values and can be used to check the analytic results.

Consider for example a fourth tensor power of the first fundamental representation $\left(L^{[1,0]}\right)^{\otimes 4}$ of algebra B_2 . Decomposition coefficients are just the branching coefficients for the tensor power module reduced to the diagonal subalgebra $B_2 \subset B_2 \oplus B_2 \oplus B_2 \oplus B_2$. So the following code calculates these coefficients:

```
fm = makeIrreducibleModule[B2][1, 0];
tp = ((fm⊗ ]fm)⊗ fm)⊗]fm;
subs = makeFiniteRootSystem[
  {1/4*{1, −1, 1, −1, 1, −1, 1, −1},
   1/4*{0, 1, 0, 1, 0, 1, 0, 1}}];
bc = branching[tp, subs];
{bc[#], dynkinLabels[subs][#]} & /@ bc[weights]
```

It produces a list of highest weights and tensor product decomposition coefficients:

```
{{1, {4, 0}}, {3, {2, 2}}, {0, {3, 0}},
{2, {0, 4}}, {3, {1, 2}}, {6, {2, 0}},
{6, {0, 2}}, {1, {1, 0}}, {3, {0, 0}}}]
```

Returning to the problem of spin chain Hamiltonian diagonalization we can see that instead of diagonalizing the operator in a space of dimension 625 we can diagonalize the operators in the spaces of dimensions $55, 81, 30, 35, 35, 14, 10, 5, 1$.

5.2. Branching and parabolic Verma modules

We illustrate the generalized BGG-resolution by the diagrams of G_2 parabolic Verma modules which appear in the decomposition of the irreducible module $L^{[1,1]}_{G_2}$:

$$\mathrm{ch}\left(L^{\mu}\right) = \sum_{u \in U} e^{\mu_{\tilde{\mathfrak{a}}}(u)} \epsilon(u) \mathrm{ch} M_I^{\mu_{\mathfrak{a}_\perp}(u)}. \tag{33}$$

The character of $L^{[1,1]}$ is presented in Figure 3, the characters of the generalized Verma modules in the decomposition (33) are shown in Figure 4. The characters in the upper row appear in (33) with a positive sign and in the lower row with a negative.

5.3. String functions of affine Lie algebras and CFT models

String functions can be used to present a formal character of an affine Lie algebra highest weight representation. They have interesting analytic and modular properties [11, 33, 42].

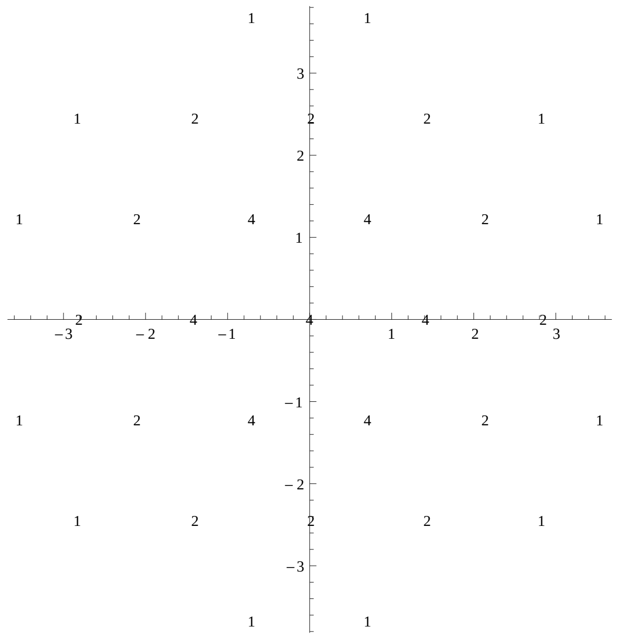


Figure 3: Character of the irreducible G_2-module $L^{[1,1]}$

Affine.m produces power series decomposition for string functions. Consider an affine Lie algebra $\hat{sl(3)} = \hat{A}_2$ and its highest weight module $L^{(1,0,0)}$. To get the string functions we can use the code:

```
stringFunctions[Â₂,{1,1,2}]
{{0, 0, 4},
  2q + 10q^2 + 40q^3 + 133q^4 + 398q^5 + 1084q^6 + 2760q^7 + 6632q^8 + 15214q^9 + 33508q^10},
{{0, 3, 1},
  2q + 12q^2 + 49q^3 + 166q^4 + 494q^5 + 1340q^6 + 3387q^7 + 8086q^8 + 18415q^9 + 40302q^10},
{{1, 1, 2},
  1 + 6q + 27q^2 + 96q^3 + 298q^4 + 836q^5 + 2173q^6 + 5310q^7 + 12341q^8 + 27486q^9 + 59029q^10},
{{2, 2, 0},
  1 + 8q + 35q^2 + 124q^3 + 379q^4 + 1052q^5 + 2700q^6 + 6536q^7 + 15047q^8 + 33248q^9 + 70877q^10},
{{3, 0, 1},
  2 + 12q + 49q^2 + 166q^3 + 494q^4 + 1340q^5 + 3387q^6 + 8086q^7 + 18415q^8 + 40302q^9 + 85226q^10}
```

Similarly for the affine Lie algebra $\hat{G}_2$ we get

```
stringFunctions[Ĝ₂,{1,1,0}]
{{2, 0, 0},
  1 + 8q + 37q^2 + 138q^3 + 431q^4 + 1227q^5 + 3208q^6 + 7901q^7},
{{0, 0, 1},
  3q + 18q^2 + 73q^3 + 247q^4 + 736q^5 + 2000q^6 + 5070q^7},
{{1, 1, 0},
  1 + 7q + 32q^2 + 117q^3 + 370q^4 + 1055q^5 + 2780q^6 + 6880q^7},
{{0, 2, 0},
```

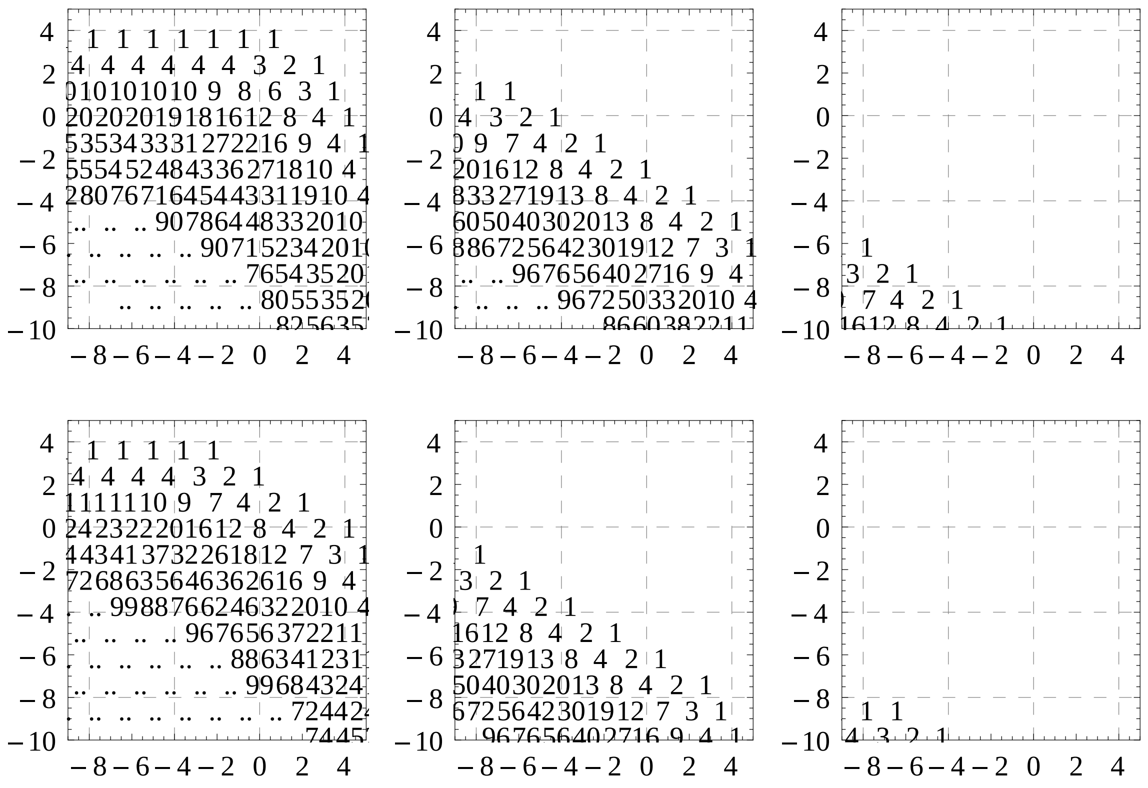


Figure 4: The characters of generalized Verma modules of G_2 appearing in the decomposition of $L^{[1,1]}$. The parabolic Verma modules in the upper row appear in the decomposition with a positive sign, in lower row with a negative.

$3q + 15q^2 + 63q^3 + 210q^4 + 633q^5 + 1725q^6 + 4407q^7\}$

5.4. Branching functions and coset models of conformal field theory

It is believed that most rational models of CFT can be obtained from the cosets G/A corresponding to the embedding $\mathfrak{a} \subset \mathfrak{g}$. These models can be studied as gauge theories [43, 44].

Branching functions for an embedding $\mathfrak{a} \subset \mathfrak{g}$ are the partition functions of CFT on the torus (see [13]).

As a first example we show how to construct branching functions for the embedding $\hat{A}_1 \to \hat{B}_2$ up to the tenth grade:

```
branchingFunctions[B̂2,makeAffineExtension[makeFiniteRootSystem[{{1, 1}}]], {1, 1, 1}]
```

```
{{3, 0},
  2 + 14q + 52q^2 + 154q^3 + 410q^4 + 994q^5 + 2248q^6 + 4832q^7 + 9934q^8 + 19680q^9 + 37802q^10},
 {{2, 1},
  4 + 20q + 72q^2 + 220q^3 + 584q^4 + 1424q^5 + 3248q^6 + 7012q^7 + 14488q^8 + 28844q^9 + 55616q^10},
 {{0, 3},
  4q + 20q^2 + 68q^3 + 200q^4 + 516q^5 + 1224q^6 + 2736q^7 + 5808q^8 + 11820q^9 + 23236q^10},
 {{1, 2},
  2 + 14q + 54q^2 + 168q^3 + 462q^4 + 1148q^5 + 2656q^6 + 5812q^7 + 12130q^8 + 24358q^9 + 47328q^10}}
```

Another example demonstrates the computation of branching functions for the regular embedding $\hat{B}_2 \subset \hat{C}_3$:

```
sub=makeAffineExtension[parabolicSubalgebra[C3][2,3]];
branchingFunctions[Ĉ3,sub, {2, 0, 0, 0}]
```

```
{{0, 1, 0},
  2q - 20q^3 + 24q^4 + 82q^5 - 320q^6 + 108q^7},
{{1, 0, 0},
  1 - q - 8q^2 + 19q^3 + 16q^4 - 156q^5 + 205q^6 + 640q^7},
{{0, 0, 1},
  q - 5q^3 + 7q^5}
```

6. Conclusion

We have presented the package **Affine.m** for computations in representation theory of finite-dimensional and affine Lie algebras. It can be used to study Weyl symmetry, root systems, irreducible, Verma and parabolic Verma modules of finite-dimensional and affine Lie algebras. In the present paper we have also discussed main ideas used for the implementation of the package and described the most important notions of representation theory required to use **Affine.m**.

We have demonstrated that the recurrent approach based on the Weyl character formula is not only useful for calculations but also allows us to establish connections with the (generalized) Bernstein-Gelfand-Gelfand resolution.

Also we have presented examples of computations connected with problems of physics and mathematics.

In future versions of our software we are going to treat twisted affine Lie algebras, extended affine Lie algebras and provide more direct support for tensor product decompositions.

Acknowledgements

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Appendix A. Software package

The package can be freely downloaded from `http://github.com/naa/Affine`. To get the development code use the command

```
git clone git://github.com/naa/Affine.git
```

Contents of the package:

`Affine/`	root folder
`demo/`	demonstrations
`demo.nb`	demo notebook
`paper.nb`	code for the paper
`doc/`	documentation folder
`figures/`	figures in paper
`timing.pdf`	diagram showing performance
`branching-timing.pdf`	... for branching coefficients
`irrep-sum.pdf`	sum of B2 irreps
`irrep-verma-pverma.pdf`	irrep, Verma, (p)Verma for B2
`G2-irrep.pdf`	irrep for G2
`G2-pverma.pdf`	parabolic Verma for G2
`tensor-product.pdf`	tensor product of A1-modules
`bibliography.bib`	bibliographic database
`paper.pdf`	present paper
`paper.tex`	paper source
`TODO.org`	list of issues
`src/`	source folder
`affine.m`	main software package
`tests/`	unit tests folder
`tests.m`	unit tests
`README.markdown`	installation and usage notes

References

[1] J. Belinfante, B. Kolman, A survey of Lie groups and Lie algebras with applications and computational methods, Society for Industrial Mathematics, 1989.

[2] T. Nutma, Simplie (2011), `http://code.google.com/p/simplie/`.

[3] M. Van Leeuwen, LiE, a software package for Lie group computations, Euromath Bull 1 (1994) 83–94, `http://www-math.univ-poitiers.fr/~maavl/LiE/`.

[4] J. Stembridge, A Maple package for symmetric functions, Journal of Symbolic Computation 20 (1995) 755–758.

[5] J. Stembridge, Coxeter/weyl packages for maple (2011), `http://www.math.lsa.umich.edu/~jrs/maple.html`.

[6] T. Fischbacher, Introducing LambdaTensor1.0 - A package for explicit symbolic and numeric Lie algebra and Lie group calculations (2002), `arXiv:hep-th/0208218`.

[7] J. Fuchs, A. N. Schellekens, C. Schweigert, A matrix S for all simple current extensions, Nucl. Phys. B473 (1996) 323–366, `arXiv:hep-th/9601078`, `http://www.nikhef.nl/~t58/kac.html`.

[8] R. Moody, J. Patera, Fast recursion formula for weight multiplicities, Bulletin (New Series) of the American Mathematical Society 7 (1982) 237–242.

[9] J. Stembridge, Computational aspects of root systems, Coxeter groups, and Weyl characters, Interaction of combinatorics and representation theory, MSJ Mem., vol. 11, Math. Soc. Japan, Tokyo (2001) 1–38.

[10] W. Casselman, Machine calculations in Weyl groups, Inventiones mathematicae 116 (1994) 95–108.

[11] V. Kac, Infinite dimensional Lie algebras, Cambridge University Press, 1990.

[12] M. Walton, Affine Kac-Moody algebras and the Wess-Zumino-Witten model (1999), `arXiv:hep-th/9911187`.

[13] P. Di Francesco, P. Mathieu, D. Senechal, Conformal field theory, Springer, 1997.

[14] P. Goddard, A. Kent, D. Olive, Virasoro algebras and coset space models, Physics Letters B 152 (1985) 88 – 92.

[15] D. C. Dunbar, K. G. Joshi, Characters for coset conformal field theories, Int. J. Mod. Phys. A8 (1993) 4103–4122, `arXiv:hep-th/9210122`.

[16] T. Gannon, Algorithms for affine Kac-Moody algebras (2001), `arXiv:hep-th/0106123`.

[17] S. Kass, R. Moody, J. Patera, R. Slansky, Affine Lie algebras, weight multiplicities, and branching rules, Sl, 1990.

[18] J. Humphreys, Introduction to Lie algebras and representation theory, Springer, 1997.

[19] J. Humphreys, Reflection groups and Coxeter groups, Cambridge Univ Pr, 1992.

[20] M. Wakimoto, Infinite-dimensional Lie algebras, American Mathematical Society, 2001.

[21] M. Wakimoto, Lectures on infinite-dimensional Lie algebra, World scientific, 2001.

[22] W. Fulton, J. Harris, Representation theory: a first course, volume 129, Springer Verlag, 1991.

[23] N. Bourbaki, Lie groups and Lie algebras, Springer Verlag, 2002.

[24] J. Humphreys, Representations of semisimple Lie algebras in the BGG category O, Amer Mathematical Society, 2008.

[25] R. Carter, Lie algebras of finite and affine type, Cambridge University Press, 2005.

[26] J. Bernstein, I. Gel'fand, S. Gel'fand, Category of g-modules, Funktsional'nyi Analiz i ego prilozheniya 10 (1976) 1–8.

[27] I. Bernstein, I. Gel'fand, S. Gel'fand, Structure of representations generated by vectors of highest weight, Functional Analysis and Its Applications 5 (1971) 1–8.

[28] M. Il'in, P. Kulish, V. Lyakhovsky, Folded fans and string functions, Zapiski Nauchnykh Seminarov POMI 374 (2010) 197–212.

[29] P. Kulish, V. Lyakhovsky, String Functions for Affine Lie Algebras Integrable Modules, Symmetry, Integrability and Geometry: Methods and Applications 4 (2008), `arXiv:0812.2381 [math.RT]`.

[30] V. Lyakhovsky, A. Nazarov, Recursive algorithm and branching for nonmaximal embeddings, Journal of Physics A: Mathematical and Theoretical 44 (2011) 075205, `arXiv:1007.0318 [math.RT]`.

[31] V. Lyakhovsky, A. Nazarov, Recursive properties of branching and BGG resolution, Theor. Math. Phys. 169 (2011) 1551–1560, `arXiv:1102.1702 [math.RT]`.

[32] P. Kulish, V. Lyakhovsky, O. Postnova, Tensor powers for non-simply laced lie algebras b2-case, Journal of Physics: Conference Series 346 (2012) 012012.

[33] V. Kac, M. Wakimoto, Modular and conformal invariance constraints in representation theory of affine algebras, Advances in mathematics(New York, NY. 1965) 70 (1988) 156–236.

[34] M. Walton, Conformal branching rules and modular invariants, Nuclear Physics B 322 (1989) 775–790.

[35] L. Shifrin, Mathematica programming: an advanced introduction, 2009, `http://mathprogramming-intro.org/`.

[36] R. Maeder, Computer science with Mathematica, Cambridge University Press, 2000.

[37] W. Casselman, Automata to perform basic calculations in coxeter groups, in: Representations of groups: Canadian Mathematical Society annual seminar, June 15-24, 1994, Banff, Alberta, Canada, Canadian Mathematical Society, 1995, volume 16, p. 35.

[38] W. van der Kallen, Computing shortlex rewite rules with mathematica (2011), `http://www.staff.science.uu.nl/~kalle101/ickl/shortlex.html`.

[39] D. Kazhdan, G. Lusztig, Tensor structures arising from affine lie algebras. iii, Journal of the American Mathematical Society 7 (1994).

[40] D. Kazhdan, G. Lusztig, Tensor structures arising from affine lie algebras. i, Journal of the American Mathematical Society 6 (1993) 905–947.

[41] D. Kazhdan, G. Lusztig, Tensor structures arising from affine lie algebras. ii, Journal of the American Mathematical Society 6 (1993).

[42] V. Kac, D. Peterson, Infinite-dimensional Lie algebras, theta functions and modular forms, Adv. in Math 53 (1984) 125–264.

[43] S. Hwang, H. Rhedin, General branching functions of affine Lie algebras, Mod. Phys. Lett. A10 (1995) 823–830, `arXiv:hep-th/9408087`.

[44] S. Hwang, H. Rhedin, The brst formulation of g/h wznw models, Nuclear Physics B 406 (1993) 165–184.