Geometric Brownian Motion - A Practical Handbook

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Welcome to this comprehensive guide dedicated to Geometric Brownian Motion (GBM), a pivotal concept in stochastic calculus and a foundational element for students pursuing a Master's degree in Financial Engineering. This handbook aims to provide an in-depth, practical understanding of GBM, emphasizing its crucial role in the financial engineering and derivative pricing landscape. As future quantitative finance practitioners, mastering GBM is not just an academic stepping-stone but a necessity for navigating the sophisticated applications in the world of financial modeling and risk management.

Much of the material presented in this handbook is based on *Stochastic Calculus for Finance II* by Steven E. Shreve, coupled with extensive insights and examples drawn from the MTH9821 class notes. These sources have been meticulously synthesized to offer a short yet practical guide to GBM. Through this focused exploration, students will gain a robust understanding of how GBM underpins various financial instruments and risk assessment models. We endeavor to make this complex topic accessible and relevant, providing a valuable resource that will accompany students throughout their academic and professional journey in financial engineering.

1 Origin

1.1 Derivation of Brownian Motion

The widely used model which we will ultimately want to discuss is called the Black-Scholes Model (BSM). It is a fundamental model used in industry to price the value of derivative products like European options, Barrier options, and more. As we will see, the process used to describe the behavior of the underlying stock in BSM is geometric Brownian motion. It is obtained from the n-period binomial model by taking the limit as n goes to infinity.

Let us divide the continuous time interval $[0, t]$ into n sub-intervals of length t/n. We will assume that in each of the sub-intervals, the stock behaves the same way - the stock can either go up by factor u or down by factor d. The interest accumulated over one of these intervals is $e^{rt/n}$.

If we denote by k the total number of up movements, then the total number of down movements is $n-k$ and the stock price can be represented as

$$S(t) = S(0)k^n d^{n-k} = S(0)e^{k \ln u + (n-k) \ln d}. \tag{1}$$

Attempting to connect this to the theory of random walks, let $i \in \{1, 2, ..., n\}$ and let $\omega_i \in \{-1, 1\}$ represent whether the stock went up or down in step i. Let $W_n = \omega_1 + ... + \omega_n$. Thus, we can express W_n in terms of the number of up and down movements $W_n = k \cdot (+1) + (n-k) \cdot (-1) = 2k - n$, hence $k = \frac{W_n + n}{2}$. Plugging this back and after massaging the terms in the exponent in Eq. 1 we obtain

$$S(t) = S(0) \exp\left(\frac{\ln (ud)\, n}{2t} t + \frac{\ln (u/d) \sqrt{n}}{2\sqrt{t}} \sqrt{t} \frac{W_n}{\sqrt{n}} \right). \tag{2}$$

If we assume that the movements are independent of one another and the probability of each up-movement is the same as the probability of each down-movement, then according to the Central Limit Theorem (CLT) we know that $\sqrt{t}\frac{W_n}{\sqrt{n}} \xrightarrow{d} \sqrt{t}Z \sim N(0,t)$. Setting

$$\mu = \frac{\ln(ud)\, n}{2t} \text{ and } \sigma = \frac{\ln(u/d)\sqrt{n}}{2\sqrt{t}}, \tag{3}$$

we can write Eq. 2 as

$$S(t) = S(0)e^{\mu t + \sigma B(t)}, \tag{4}$$

where $B(t) \sim N(0,t)$ is called *Brownian motion*.

1.2 Arriving at GBM Using the Binomial Model Approximation

We can approximate the model with an n-period binomial model where in each period the stock moves up by factor u or down by factor d, where u and d are obtained by solving the system in Eq. 3. The solutions are

$$u = e^{\mu\frac{t}{n} + \sigma\sqrt{\frac{t}{n}}} \text{ and } d = e^{\mu\frac{t}{n} - \sigma\sqrt{\frac{t}{n}}}.$$

We can price the $t = 0$ value of an option with payoff function $\phi(S(t))$ in the n-period binomial model using

$$V(0) = e^{-rt}\tilde{E}\left[\phi(S(t))\right] = e^{-rt}\tilde{E}\left[\phi(S(0)e^{\mu t+\sigma B(t)})\right], \tag{5}$$

where $\tilde{E}[\,\cdot\,]$ denotes the expectation with respect to the probability space in which the steps $\omega_1, ..., \omega_n$ are iid and satisfy

$$\tilde{P}(\omega_i = 1) = \tilde{p} = \frac{e^{rt/n} - d}{u - d} \text{ and } \tilde{P}(\omega_i = -1) = 1 - \tilde{p} = \frac{u - e^{rt/n}}{u - d}, \tag{6}$$

taking into account the no-arbitrage condition constraint. The expectation cannot be calculated in a straightforward way since, in short, $B(t)$ is not normally distributed under the probability $\tilde{P}$. Thus, we express $B(t)$ in the following way:

$$B(t) = \sqrt{t}\frac{W_n}{\sqrt{n}} = \sqrt{t}\frac{\sum_{i=1}^n \omega_i}{\sqrt{n}}.$$

Since $E[\omega_1] = 2\tilde{p} - 1$ and $Var(\omega_1) = 1 - (2\tilde{p} - 1)^2$, by the CLT,

$$\frac{\sum_{i=1}^n \omega_i - n(2\tilde{p} - 1)}{\sqrt{n}} \sim N(0, 1 - (2\tilde{p} - 1)^2). \tag{7}$$

We will now attempt to express $\tilde{p}$ in Eq. 6 in terms of μ, σ, and r. Using the Taylor expansion of e^x and some algebra, we eventually obtain

$$\tilde{p} = \frac{1}{2}\left(1 + \frac{r - \mu - \sigma^2/2}{\sigma}\sqrt{\frac{t}{n}} + O\left(\frac{t}{n}\right)\right)$$
$$1 - \tilde{p} = \frac{1}{2}\left(1 - \frac{r - \mu - \sigma^2/2}{\sigma}\sqrt{\frac{t}{n}} + O\left(\frac{t}{n}\right)\right).$$

Using $\tilde{E}[\omega_1] = \frac{r-\mu-\sigma^2/2}{\sigma}\sqrt{\frac{t}{n}}$, $\tilde{Var}(\omega_1) = 1 - \left(\frac{r-\mu-\sigma^2/2}{\sigma}\right)^2\frac{t}{n} \approx 1$ and $Z \sim N(0,1)$, we can obtain

$$B(t) = \sqrt{t}\frac{\sum_{i=1}^n \omega_i}{\sqrt{n}} \overset{d}{=} \frac{\sqrt{t}}{\sqrt{n}}\left(\frac{n(r - \mu - \sigma^2/2)}{\sigma}\sqrt{\frac{t}{n}} + \sqrt{n}Z\right),$$

and after some cancellations we get

$$B(t) \overset{d}{=} t\frac{t(r - \mu - \sigma^2/2)}{\sigma} + \sqrt{t}Z. \tag{8}$$

Our stock movement becomes

$$S(t) = S(0)e^{\mu t + \sigma B(t)} = S(0)e^{(r-\sigma^2/2)t+\sigma \tilde{B}(t)}, \tag{9}$$

and Eq. 5 becomes,

$$V(0) = e^{-rt}\tilde{E}\left[\phi\left(S(0)e^{(r-\sigma^2/2)t+\sigma\tilde{B}(t)}\right)\right], \tag{10}$$

where $\tilde{B}(t)$ is a standard Brownian motion with respect to the *risk-neutral* probability measure $\tilde{P}$. In the next section we will show that Eq. 9 is the solution to the *stochastic differential equation* (SDE) of a geometric Brownian motion with drift parameter r and volatility parameter σ. As it turns out, the derivation we just followed uses the fact that we performed a *change of measure* with respect to the no-arbitrage conditions and will expand on this concept further in section 3.

2 Geometric Brownian Motion

Now that we are familiar with the origin of geometric Brownian motion (GBM), we can continue to analyze it further. We must gain a solid understanding of GBM since it is a fundamental stochastic process used in the field of finance, particularly in the modeling of financial markets and instruments. Its significance lies in its ability to capture the essential features of asset price dynamics, including randomness and the continuous nature of market prices. As we have shown in the previous section, it is used to model price movements of underlying assets in the Black-Scholes model.

Definition 2.1 (Geometric Brownian Motion)**.** *A stochastic process $S(t)$ is said to follow a Geometric Brownian Motion (GBM) if it satisfies the following stochastic differential equation (SDE):*

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW(t) \tag{11}$$

where $W(t)$ is a Brownian motion. The constants $\mu \in \mathbb{R}$ (the percentage drift) and $\sigma > 0$ (the percentage volatility) characterize the drift and volatility of the process, respectively.

The drift term ($\mu S(t)dt$) is used to model deterministic trends, while the volatility term ($\sigma S(t)dW(t)$) models unpredictable events occurring during the motion.

2.1 Solving the SDE

Before we can solve the SDE, we will first need to define an *Itô process* and use two useful theorems.

Definition 2.2. *Let $(B(t))_{t\geq 0}$ be a standard Brownian motion and $(\mathcal{F}(t))_{t\geq 0}$ be an associated filtration. An Itô process is a stochastic process of the form*

$$dX(t) = \Delta(t)\,dB(t) + \Theta(t)\,dt, \tag{12}$$

with initial condition $X(0)$ (non-random) and $(\Delta(t))_{t\geq 0}, (\Theta(t))_{t\geq 0}$ are adapted stochastic processes such that for all $t \geq 0$

$$E\left(\int_0^t \Delta^2(s)\,ds\right) < \infty \quad \textit{and} \quad \int_0^t |\Theta(s)|\,ds < \infty \quad \textit{a.s.}.$$

Geometric Brownian motion is an Itô process where $\Delta(t) = \mu S(t)$ and $\Theta(t) = \sigma S(t)$.

Theorem 2.1 (Itô-Doeblin formula for Itô processes)**.** *Let $(X(t))_{t\geq 0}$ be an Itô process and $f \in C^{1,2}([0,\infty) \times \mathbb{R})$. Then for every $t \geq 0$*

$$df(t, X(t)) = f_t(t, X(t))dt + f_x(t, X(t))dX(t) + \frac{1}{2}f_{xx}(t, X(t))d[X]_t \tag{13}$$

almost surely.

Theorem 2.2. *Let $(X(t))_{t\geq 0}$ be an Itô process. Then for every $t \geq 0$*

$$d[X]_t = \Delta^2(t)dt. \tag{14}$$

Exercise 2.1. Assuming the right integrability conditions hold, solve the Geometric Brownian Motion SDE $dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW(t)$ using Theorems 2.1 and 2.2.

Solution: To solve the SDE for GBM we use a trick wise mathematicians discovered where we set f as the natural logarithm function. So in our case, $f(t, X(t)) = f(S(t)) = \ln S(t)$. Using both theorems:

$$\begin{aligned} d\ln S(t) &= 0 + \frac{1}{S(t)}dS(t) + \frac{1}{2}\frac{-1}{S^2(t)}d[S]_t \\ &= \mu dt + \sigma dW(t) - \frac{1}{2S^2(t)}\sigma^2 S^2(t)dt \\ &= (\mu - \sigma^2/2)dt + \sigma dW(t) \end{aligned}$$

Integrating both sides yields

$$\ln S(t) - \ln S(0) = (\mu - \sigma^2/2)t + \sigma W(t)$$

and taking the exponential of both sides and rearranging the terms we finally get

$$S(t) = S(0)e^{(\mu-\sigma^2/2)t+\sigma W(t)} \tag{15}$$

2.2 Properties of GBM

We wish to find the expectation and variance of GBM. To do that, we need to use the fact that $B(t) \sim N(0,t)$ and that the moment-generating function (MGF) of a standard normal is $M_Z(u) = E\left(e^{uZ}\right) = e^{u^2/2}$.

Theorem 2.3. *Let $(S(t))_{t\geq 0}$ be geometric Brownian motion. Then the following properties hold:*

- *Expectation:*
$$E(S(t)) = S(0)e^{\mu t} \tag{16}$$
- *Variance:*
$$Var(S(t)) = S^2(0)e^{2\mu t}\left(e^{\sigma^2 t} - 1\right) \tag{17}$$
- *Markov Property*
$$E(f(S(t))|S(s)) = E(f(S(t))|\mathcal{F}(s)) \quad \forall s \leq t$$

Proof. (Expectation and Variance) For the expectation we simply plug in the solution we found in Eq. 15 and take out what is not a random variable

$$E(S(t)) = E\left(S(0)e^{(\mu-\sigma^2/2)t+\sigma W(t)}\right) = S(0)e^{(\mu-\sigma^2/2)t}E\left(e^{\sigma W(t)}\right).$$

Since

$$E\left(e^{\sigma W(t)}\right) \stackrel{d}{=} E\left(e^{\sigma\sqrt{t}Z}\right) = e^{\sigma^2 t/2},$$

we get

$$E(S(t)) = S(0)e^{(\mu-\sigma^2/2)t}e^{\sigma^2 t/2} = S(0)e^{\mu t}.$$

For the variance, we follow a similar procedure. Using the definition of variance, $Var(S(t)) = E(S^2(t)) - E(S(t))^2$, we have already found $E(S(t))$ so we need to find $E(S^2(t))$.

$$\begin{aligned} E(S^2(t)) &= E\left(S^2(0)e^{2(\mu-\sigma^2/2)t+2\sigma W(t)}\right) = S^2(0)e^{2(\mu-\sigma^2/2)t}E\left(e^{2\sigma W(t)}\right) \\ &= S^2(0)e^{2(\mu-\sigma^2/2)t}e^{2\sigma^2 t} = S^2(0)e^{(2\mu+\sigma^2)t}. \end{aligned}$$

Plugging it back to the definition of variance we finally get

$$Var(S(t)) = S^2(0)e^{(2\mu+\sigma^2)t} - S^2(0)e^{2\mu t} = S^2(0)e^{2\mu t}\left(e^{\sigma^2 t} - 1\right).$$

$\square$

Exercise 2.2. Prove the Markov Property of GBM

Solution: Let us start with the RHS.

$$\begin{aligned} E\big(f(S(t))|\mathcal{F}(s)\big) &= E\left(f\left(S(0)e^{(\mu-\sigma^2/2)t+\sigma W(t)}\right)|\mathcal{F}(s)\right) \\ &= E\left(f\left(S(0)e^{(\mu-\sigma^2/2)t}e^{\sigma W(s)}e^{\sigma(W(t)-W(s))}\right)|\mathcal{F}(s)\right) \end{aligned}$$

Since $W(t) - W(s)$ is independent from $\mathcal{F}(s)$ and $W(s)$ is $\mathcal{F}(s)$-measurable, by the Independence Lemma we get

$$E\left(f(S(t))|\mathcal{F}(s)\right) = g\left(e^{(\mu-\sigma^2/2)t+\sigma W(s)}\right)$$

Where

$$g(y) = E\left(f\left(y\, e^{\sigma(W(t)-W(s))}\right)\right).$$

Since it is now $S(s)$-measurable, the tower property yields

$$E\left(f(S(t))|S(s)\right) = E\big[E\big(f(S(t))|\mathcal{F}(s)\big)\big|S(s)\big] = g\left(e^{(\mu-\sigma^2/2)t+\sigma W(s)}\right).$$

Thus

$$E(f(S(t))|S(s)) = E(f(S(t))|\mathcal{F}(s)) \quad \forall s \le t$$

Exercise 2.3. Derive a formula for the moments of geometric Brownian motion, i.e. $E\left(S^\alpha(t)\right)$.

Solution: Given $S^\alpha(t) = S^\alpha(0)e^{\alpha(\mu-\sigma^2/2)t+\alpha\sigma W(t)}$,

$$\begin{aligned} E\left(S^\alpha(t)\right) &= S^\alpha(0)e^{\alpha(\mu-\sigma^2/2)t}E\left(e^{\alpha\sigma W(t)}\right) \stackrel{d}{=} S^\alpha(0)e^{\alpha(\mu-\sigma^2/2)t}E\left(e^{\alpha\sigma\sqrt{t}Z}\right) \\ &= S^\alpha(0)e^{\alpha(\mu-\sigma^2/2)t}e^{\alpha^2\sigma^2 t/2}. \end{aligned}$$

Exercise 2.4. Is geometric Brownian motion a martingale?

Solution: Geometric Brownian motion is not a martingale unless the drift parameter μ is exactly zero. If $\mu > 0$, GBM is a sub-martingale and if $\mu < 0$ then it is a super-martingale. To see this we perform the following:

$$\begin{aligned} E(S(t)|\mathcal{F}(s)) &= E\left(S(0)e^{(\mu-\sigma^2/2)t+\sigma W(t)}\big|\mathcal{F}(s)\right) \\ &= S(s)e^{(\mu-\sigma^2/2)(t-s)}E\left(e^{\sigma(W(t)-W(s))}\big|\mathcal{F}(s)\right) \\ &\stackrel{d}{=} S(s)e^{(\mu-\sigma^2/2)(t-s)}E\left(e^{\sigma\sqrt{t-s}Z)}\right) \\ &= S(s)e^{(\mu-\sigma^2/2)(t-s)}e^{(t-s)\sigma^2/2} = S(s)e^{\mu(t-s)} \end{aligned}$$

So $E(S(t)|\mathcal{F}(s)) > S(s)$ if $e^{\mu(t-s)} > 1$ and $E(S(t)|\mathcal{F}(s)) < S(s)$ if $e^{\mu(t-s)} < 1$.

3 GBM and Applications

3.1 Black-Scholes-Merton PDE

Geometric Brownian motion is used in the Black-Scholes model to describe the dynamics of stock prices (which agree with empirical observations). This model implies that stock prices follow a log-normal distribution and adhere to random walk theory, suggesting that price changes are random and unpredictable. As we will shortly see, the Black-Scholes Partial Differential Equation (PDE) is derived by applying Itô's Lemma to the option pricing problem, assuming no arbitrage opportunities and the ability to continuously hedge the option. This leads to the Black-Scholes formula, which is crucial for determining the fair price of derivatives like call or put options. The ability to assess the price of derivative products through the framework provided by the Black-Scholes PDE is key for risk management and the formulation of hedging strategies in financial portfolios. Furthermore, the PDE has been a foundational element for further research and development of more complex models in financial mathematics.

3.1.1 Derivation

The derivation of the Black-Scholes PDE involves the following setting. We consider a simple market consisting of one risky asset, a stock, with price $S(t)$ following a GBM

$$dS(t) = \alpha S(t)\,dt + \sigma S(t)\,dB(t) \tag{18}$$

a risk-less asset, a money market account (MMA), which has a constant interest rate r with continuous compounding. We denote the overall portfolio value at time t by $X(t)$ and assume that at time t the investor holds $\Delta(t)$ shares of the stock. The remainder cash $X(t) - \Delta(t)S(t)$ is invested is invested in the MMA. Thus, the evolution of the portfolio value is described by

$$dX(t) = \Delta(t)dS(t) + r\left(X(t) - \Delta(t)S(t)\right)dt. \tag{19}$$

Substituting $dS(t)$ into Eq. 19 we get

$$dX(t) = \sigma\Delta(t)S(t)dB(t) + rX(t)dt + (\alpha - r)\Delta(t)S(t)dt. \tag{20}$$

The three terms in Eq. 20 are the volatility proportional to the size of the stock investment; the average underlying rate of return r of the portfolio; and the risk premium for investing in stock.

Consider now a *European call option* on the stock with payoff $(S(T) - K)_+$ at time T. Black, Scholes, and Merton argued that the value of this call at time t should depend only on $T - t$ and the price of the stock at time t (other parameters r, σ, K being constant). If we let $c(t,x)$ denote the price of the option at time t when $S(t) = x$ then $c(t,x)$ is a non-random function. The stochastic process $c(t, S(t))$ is the price of the option at time t. Assuming that $c(t,x) \in C^{1,2}([0,T] \times [0,\infty))$ we get for all $0 \leq t < T$ that

$$dc(t,S(t)) = c_t(t,S(t))dt + c_x(t,S(t))dS(t) + \frac{1}{2}c_{xx}(t,S(t))d[S]_t \tag{21}$$

$$= \left(c_t(t,S(t)) + aS(t)c_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))\right)dt + \sigma\, c_x(t,S(t))\,S(t)dB(t).$$

The above equation represents the evolution of the option value.

A hedging portfolio for a short position on an option has to satisfy the equation X(t) = c(t, S(t)) for all t$\in [0,T]$. If we attempt to simply equate Eqs. 19 and 21 then the equation will contain a $rX(t)dt$ term and others that we cannot work with so let us attempt something else. To get rid of the $rX(t)dt$ term, we can equate the present values instead,

$$e^{-rt}X(t) = e^{-rt}c(t,S(t)), \quad t \in [0,T], \tag{22}$$

and the corresponding evolutions are given by

$$d\left(e^{-rt}X(t)\right) = e^{-rt}\Delta(t)(\alpha - r)S(t)dt + \sigma e^{-rt}\Delta(t)S(t)dB(t) \tag{23}$$

and

$$d\left(e^{-rt}c(t,S(t))\right) = e^{-rt}\left(-r\,c(t,S(t)) + c_t(t,S(t)) + \alpha S(t)\,c_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)\,c_{xx}(t,S(t))\right)dt \tag{24}$$

$$+e^{-rt}\sigma\,c_x(t,S(t))\,S(t)dB(t).$$

Equating the dt-part and the $dB(t)$-part of Eqs. 23 and 24 we get

$$r\,c(t,S(t)) = c_t(t,S(t)) + rS(t)c_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))\,, \quad t \in [0,T),\ \text{a.s.} \tag{25}$$

and

$$\Delta(t) = c_x(t,S(t))\,, \quad t \in [0,T),\ \text{a.s.} \tag{26}$$

respectively. Eq. 25 tells us that we need to find a smooth function $c(t,x)$ which satisfies the following Black-Scholes-Merton partial differential equation (BSM PDE)

$$r\,c(t,x) = c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x)\,, \quad t \in [0,T),\ \text{a.s.} \tag{27}$$

and the terminal condition $c(T,x) = (x-K)_+$. For the solution to be unique we need additional boundary conditions at $x = 0$ and as $x \to \infty$. The first one is obtained by plugging $x = 0$ in the the BSM PDE to get $c_t(t,0) = rc(t,0)$. Solving this ODE with $c(0,0) = 0$ we get the condition at $x = 0$: $c(t,0) = 0$ for all $t \in [0,T]$. The condition at infinity can be stated in the following form:

$$\lim_{x\to\infty}\left(c(t,x) - \left(x - e^{-(T-t)}K\right)\right) = 0, \quad t \in [0,T].$$

As x grows large, the call option will be deep in the money, and it will very likely finish in the money. In this case, the price of the call at time t is almost as much as the time t price of the forward contract with delivery price K and expiration T, i.e. $S(t) - e^{-(T-t)}K$. This explains the condition at infinity.

Suppose that we have found such a function $c(t,x)$. Then at each time t we have both the option price $c(t,S(t))$ ($S(t)$ is known at time t) and the replicating portfolio. Indeed, if the investor starts with initial capital $X(0) = c(0,S(0))$ and at time t has $\Delta(t) = c_x(t,S(t))$ shares of the underlying asset in his portfolio, then the $dB(t)$ terms in Eqs. 23 and 24 agree. By the BSM PDE the dt-terms in Eqs. 23 and 24 also agree. This and the equality of the initial values imply that Eq. 22 holds for all $t \in [0,T)$. As $t \uparrow T$ we get by continuity of $X(t)$ and $c(t,S(t))$ that $X(T) = c(T,S(T)) = (S(T)-K)_+$, i.e. the short position is successfully hedged.

3.2 Girsanov's Theorem and Risk-Neutral Measure

In the world of derivative pricing, Girsanov's Theorem stands out as a fundamental tool that facilitates the crucial transition from real-world expected returns to the risk-free rate in a risk-neutral setting. This theorem is not just a mathematical curiosity; it is the very linchpin that makes the shift in measures both possible and practical.

Dealing with real-world expected returns involves grappling with the complexities of varying risk premiums associated with different assets. This variability introduces a level of complexity and uncertainty that can make the pricing of derivatives an extremely challenging task.

Contrast this with the risk-neutral world, where Girsanov's Theorem comes into play. This theorem allows us to transform the probability measure under which financial assets are priced. By applying Girsanov's Theorem, we can effectively change the drift of the asset's return from the uncertain real-world expected return to the stable and known risk-free rate. This shift is not just a simplification; it's a paradigm shift that aligns the pricing of derivatives with the risk-neutral perspective, where all assets are assumed to grow at the risk-free rate.

Furthermore, in a market that respects the no-arbitrage principle, Girsanov's Theorem ensures consistency. By using this theorem to standardize the drift across different assets to the risk-free rate, it prevents the emergence of arbitrage opportunities, thereby maintaining the market's integrity and efficiency.

Let us consider the following setting: Let $(B(t))_{t\geq 0}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, P)$, and let $(\mathcal{F}(t))_{t\geq 0}$ be the filtration for this Brownian motion, where $\mathcal{F}(t) \subset \mathcal{F}$ for all $t \geq 0$. We shall consider a market with

- a Money Market Account (MMA) with an adapted interest rate process $(R(t))_{t\geq 0}$ so that the discounting process $(D(t))_{t\geq 0}$ satisfies

$$\mathrm{d}D(t) = -R(t)D(t)\,\mathrm{d}t, \quad t \geq 0 \tag{28}$$

$$\text{i.e. } D(t) = \exp\left(-\int_0^t R(u)\,\mathrm{d}u\right); \tag{29}$$

- a single stock whose price under some probability measure P satisfies the equation

$$dS(t) = \alpha(t)S(t)\,dt + \sigma(t)S(t)\,dB(t), \tag{30}$$

  where $\alpha(t)$, $\sigma(t)$, $t \geq 0$, are adapted processes, with $\sigma(t) \neq 0$ a.s.. The solution is given by

$$S(t) = S(0)\exp\left(\int_0^t \sigma(u)\,\mathrm{d}B(u) + \int_0^t \left[\alpha(u) - \frac{\sigma^2(u)}{2}\right]\,\mathrm{d}u\right), \tag{31}$$

  which is a generalized geometric Brownian motion (GBM).

We call the above model a *simple market model.* Our objective is to find a probability measure $\tilde{P}$ under which the discounted stock price $D(t)S(t)$ is an $\mathcal{F}(t)$-martingale. More formally:

Definition 3.1. *A risk-neutral measure $\tilde{P}$ is a measure on $(\Omega, \mathcal{F}, P)$ that is equivalent to P and such that under $\tilde{P}$ the discounted stock price $(D(t)S(t))_{t\geq 0}$ is an $\mathcal{F}(t)$-martingale.*

How we go from our initial setting to the risk-neutral setting is via Girsanov's Theorem.

Theorem 3.1 (Girsanov, $d = 1$)**.** *Let B be a standard Brownian motion and $\mathcal{F}(t)$, $t \geq 0$, be a filtration for this Brownian motion. Let $\Theta(t)$, $t \in [0, T]$, be an $\mathcal{F}(t)$-adapted process and define*

$$Z(t) = \exp\left(-\int_0^t \Theta(s)\,dB(s) - \frac{1}{2}\int_0^t \Theta^2(s)\,ds\right), \quad d\tilde{B}(t) = dB(t) + \Theta(t)\,dt, \quad t \in [0, T].$$

Assume that

$$E\left(\int_0^T \Theta^2(t)Z^2(t)\,dt\right) < \infty. \tag{32}$$

Set $Z = Z(T)$. Then $E(Z) = 1$, and under the probability measure

$$\tilde{P}(A) = \int_A Z(\omega)\,dP(\omega) \tag{33}$$

the process $\tilde{B}(t)$, $0 \leq t \leq T$, is a standard Brownian motion.

To find $\tilde{P}$ we calculate $d(D(t)S(t))$ and try and get rid of the dt-term. Using Itô's product rule, and using the fact that $d[S, D]_t = 0$ we get

$$\begin{aligned} d(D(t)S(s)) &= D(t)dS(t) + S(t)dD(t) = D(t)\big(\alpha(t)S(t)\,dt + \sigma(t)S(t)\,dB(t)\big) + S(t)\big(-R(t)D(t)dt\big) \\ &= D(t)S(t)\big[\big(\alpha(t) - R(t)\big)dt + \sigma(t)dB(t)\big] = \sigma(t)D(t)S(t)\left(\frac{\alpha(t) - R(t)}{\sigma(t)}dt + dB(t)\right). \end{aligned}$$

Let $\Theta(t) = \sigma(t)^{-1}(\alpha(t) - R(t))$ be the *market price of risk*. (It is the excess instantaneous rate of return of the stock (over MMA) per unit volatility.) We set

$$\tilde{P}(A) = \int_A Z dP, \quad \text{where} \quad Z = Z(T) = \exp\left(-\int_0^T \Theta(t)\, dB(t) - \frac{1}{2}\int_0^T \Theta^2(t)\, dt\right).$$

Then, by Girsanov's Theorem,

- $\tilde{B}(t) = B(t) + \int_0^t \Theta(s) ds \quad 0 \le t \le T$, is a standard Brownian motion.
- $d(D(t)S(t)) = \sigma(t)D(t)S(t)d\tilde{B}(t) \quad 0 \le t \le T$, is an $\mathcal{F}(t)$-martingale under $\tilde{P}$.

Using the fact that $dB(t) = d\tilde{B}(t) - \sigma(t)^{-1}(\alpha(t) - R(t))$, we substitute $dB(t)$ in Eq. 30 to obtain

$$dS(t) = R(t)S(t)dt + \sigma(t)S(t)d\tilde{B}(t). \tag{34}$$

This shows that the instantaneous rate of return of the stock under $\tilde{P}$ is the same as for the MMA,

$$M(t) := D(t)^{-1} = \exp\left(\int_0^t R(u)\, \mathrm{d}u\right); \quad dM(t) = R(t)M(t)dt.$$

This explains the name *risk-neutral* measure for $\tilde{P}$.

3.3 Applications

In this section we will provide examples of providing the time-zero price of derivative products; a European call option and a Down-and-Out Barrier option. The examples will use tools we acquired from the previous two sections.

3.3.1 Pricing a European Call Option

The objective is to compute the price of a European Call option with payoff $V(T) = f(S(T)) = (S(T) - K)_+$ where T is the maturity of the contract and K is the strike price of the option. We assume that $R(t) = r \ge 0$ and $\sigma(t) = \sigma \ge 0$. Then we know that under the risk-neutral measure $\tilde{P}$

$$dS(t) = rS(t)dt + \sigma S(t)d\tilde{B}(t); \quad S(t) = S(0)e^{(r-\sigma^2/2)t + \sigma\tilde{B}(t)}.$$

We wish to compute

$$V(t) = \tilde{E}\left(e^{-r(T-t)}f(S(T)) \mid \mathcal{F}(t)\right) = e^{-r(T-t)}\tilde{E}\left(f(S(T)) \mid \mathcal{F}(t)\right), \quad \forall t \in [0, T].$$

Writing $S(T) = S(t)e^{(r-\sigma^2/2)(T-t)+\sigma(\tilde{B}(T)-\tilde{B}(t))}$ we get

$$\begin{aligned}
V(t) &= e^{-r(T-t)}\tilde{E}\left(f(S(T)) \mid \mathcal{F}(t)\right) \\
&= e^{-r(T-t)}\tilde{E}\left(f\left(S(t)e^{(r-\sigma^2/2)(T-t)+\sigma(\tilde{B}(T)-\tilde{B}(t))}\right) \mid \mathcal{F}(t)\right) \\
&\overset{d}{=} e^{-r(T-t)}\tilde{E}\left(f\left(S(t)e^{(r-\sigma^2/2)(T-t)-\sigma\sqrt{T-t}Z}\right) \mid \mathcal{F}(t)\right),
\end{aligned}$$

where $Z = -(\tilde{B}(T) - \tilde{B}(t))/\sqrt{T-t}$ is standard normal (negative for convenience later). Since $\sqrt{T-t}Z$ is independent of $\mathcal{F}(t)$ and $S(t)$ is $\mathcal{F}(t)$-measurable, by the independence lemma we get

$$\tilde{E}\left(f\left(S(t)e^{(r-\sigma^2/2)(T-t)-\sigma\sqrt{T-t}Z}\right) \mid \mathcal{F}(t)\right) = g(t, S(t)),$$

where

$$\begin{aligned}
g(t, x) &= \frac{1}{2\pi}\int_{\mathbb{R}} f\left(x\, e^{(r-\sigma^2/2)(T-t)-\sigma\sqrt{T-t}z}\right) e^{-z^2/2}dz \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \left(x\, e^{(r-\sigma^2/2)(T-t)-\sigma\sqrt{T-t}z} - K\right)_+ e^{-z^2/2}dz.
\end{aligned}$$

We have accomplished two things: (1) we have found a closed-form formula for the price

$$V(t) = e^{-r(T-t)} g(t, S(t)), \quad \forall t \in [0, T],$$

and (2) we have shown that a GBM with constant drift and volatility parameters is a Markov process. While we will go on to derive the complete closed-form solution for the price of the European call option, it is important to note that $g(t, S(t))$ could also be obtained probabilistically via Monte-Carlo simulations (which we will perform in this handbook).

Exercise 3.1. Compute the integral

$$g(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}} f\left(x\, e^{(r-\sigma^2/2)(T-t)-\sigma\sqrt{T-t}z}\right) e^{-z^2/2} dz$$

for $f(x) = (x-K)_+$ and show that the result is the standard BSM formula

$$C_{BS}(\tau, x;\, K, r, \sigma) = x\, N(d_+(\tau, x)) - e^{-r\tau}\, K\, N(d_-(\tau, x)) \tag{35}$$

where $\tau = T - t$, $N(\cdot)$ is the standard normal CDF and

$$d_-(\tau, x) = \frac{1}{\sigma\sqrt{\tau}} \left(\log \frac{x}{K} + \left(r - \frac{1}{2}\sigma^2\right)\tau\right) \tag{36}$$

$$d_+(\tau, x) = d_-(\tau, x) + \sigma\sqrt{\tau}. \tag{37}$$

Solution: Let $\tau = T - t$. The payoff for a call option is positive if an only if

$$x\, e^{(r-\sigma^2/2)\tau - \sigma\sqrt{\tau}z} - K > 0$$

which is equivalent to

$$z < \frac{1}{\sigma\sqrt{\tau}} \left(\log \frac{x}{K} + \left(r - \frac{1}{2}\sigma^2\right)\tau\right) =: d_-(\tau, x).$$

Thus,

$$\begin{aligned}
C_{BS}(\tau, x;\, K, r, \sigma) &= e^{-r\tau} \frac{1}{2\pi} \int_{\mathbb{R}} \left(x\, e^{(r-\sigma^2/2)\tau - \sigma\sqrt{\tau}z} - K\right)_+ e^{-z^2/2} dz \\
&= \frac{1}{2\pi} \int_{-\infty}^{d_-} e^{-r\tau} \left(x\, e^{(r-\sigma^2/2)\tau - \sigma\sqrt{\tau}z} - K\right) e^{-z^2/2} dz \\
&= \frac{1}{2\pi} \int_{-\infty}^{d_-} x\, e^{-z^2/2 - \sigma\sqrt{\tau}z - \sigma^2\tau/2} dz - e^{-r\tau} \frac{1}{2\pi} \int_{-\infty}^{d_-} K e^{-z^2/2} dz \\
&= \frac{x}{2\pi} \int_{-\infty}^{d_-} e^{-\frac{1}{2}(z+\sigma\sqrt{\tau})^2} dz - e^{-r\tau} K N(d_-(\tau, x)) \\
&= \frac{x}{2\pi} \int_{-\infty}^{d_- + \sigma\sqrt{\tau}} e^{-z^2/2} dz - e^{-r\tau} K N(d_-(\tau, x)) \\
&= x\, N(d_+(\tau, x)) - e^{-r\tau}\, K\, N(d_-(\tau, x))
\end{aligned}$$

where $d_+(\tau, x) := d_-(\tau, x) + \sigma\sqrt{\tau}$. We have found a closed-form formula for the price of a European call option.

3.3.2 Pricing a Down-and-Out Barrier Option

A barrier option is a type of financial derivative that's similar to standard options, but with a key difference: its payoff depends on whether the underlying asset's price reaches a certain level (the barrier) during the option's life. A Down-and-Out option falls under the category of barrier options and is a type of knock-out option, which means its validity is contingent upon the behavior of the underlying asset's price relative to a predetermined barrier level. In the case of a down-and-out option, this barrier is set below the current price of the underlying asset at the time the option is issued. The defining characteristic of this option is that it starts off active, just like a standard option. However, if at any point during the option's lifespan the price of the underlying asset drops to or below this barrier level, the option is immediately rendered inactive or "knocked out," and becomes worthless. This deactivation remains in effect irrespective of any future price movements of the asset.

In order for us to develop the mathematical formulation that models the price of the down-and-out option, we will need to know a useful tool, namely the joint distribution of Brownian motion and its running maximum.

Theorem 3.2. *Let $(B(t))_{t\geq 0}$ be a Brownian motion and $B^*(t) := \max_{0\leq s\leq t} B(s)$. For all $a > 0$, $x \leq a$ and all $t \geq 0$*

$$P(B^*(t) \geq a\, B(t) \leq x) = 1 - N\left(\frac{2a-x}{\sqrt{t}}\right) \tag{38}$$

where $N(x) = (2\pi)^{-1/2}\int_{-\infty}^{x} e^{-y^2/2}dy$.

We will now find the time zero price of a down-and-out call option with strike K, barrier $\mathcal{L} < K$, and expiration T. Again, we assume that $R(t) = r \geq 0$ and $\sigma(t) = \sigma \geq 0$. Then we know that under the risk-neutral measure $\tilde{P}$

$$dS(t) = rS(t)dt + \sigma S(t)d\tilde{B}(t); \quad S(t) = S(0)e^{(r-\sigma^2/2)t+\sigma\tilde{B}(t)}.$$

Let $\hat{B}(t) = -\tilde{B}(t) - \alpha t$ where $\alpha = (r - \sigma^2/2)/\sigma$, and define $\hat{B}^*(t) = \max_{0\leq s\leq t}\hat{B}(s)$. Then we get that

$$\min_{0\leq s\leq t} S(t) = S(0)e^{-\hat{B}^*(T)}$$

and the payoff of the down-and-out option is

$$\begin{aligned} V(T) &= \left(S(0)e^{-\sigma\hat{B}(T)} - K\right)_+ \mathbb{1}_{\{S(0)e^{-\sigma\hat{B}^*(T)}>L\}} \\ &= \left(S(0)e^{-\sigma\hat{B}(T)} - K\right) \mathbb{1}_{\{S(0)e^{-\sigma\hat{B}(T)}\geq K;\, S(0)e^{-\sigma\hat{B}^*(T)}>L\}} \\ &= \left(S(0)e^{-\sigma\hat{B}(T)} - K\right) \mathbb{1}_{\{\hat{B}\leq k;\hat{B}^*(T)<b\}} \end{aligned}$$

where we have defined

$$k := \frac{1}{\sigma}\log\frac{S(0)}{K}, \quad b := \frac{1}{\sigma}\log\frac{S(0)}{L},$$

such that $k < b$. According to risk-neutral pricing formula, the time zero price of the down-and-out call option considered here is given by

$$V(0) = e^{-rT}\tilde{E}\left[\left(S(0)e^{-\sigma\hat{B}(T)} - K\right)\mathbb{1}_{\{\hat{B}(T)\leq k;\hat{B}^*(T)<b\}}\right].$$

The joint distribution of $(\hat{B}(T), \hat{B}^*(T))$ under the probability measure $\tilde{P}$ can be derived using Thm. 3.2

$$\hat{f}(x,a) = \frac{2(2a-x)}{T\sqrt{2\pi T}}e^{-\frac{(2a-x)^2}{2T}-\alpha x-\frac{1}{2}\alpha^2 T}\mathbb{1}_{\{x\leq a,\ a\geq 0\}}.$$

Therefore,

$$V(0) = e^{-rT}\int_{-\infty}^{k}\int_{x}^{b}\left(S(0)e^{-\sigma x} - K\right)\frac{2(2a-x)}{T\sqrt{2\pi T}}e^{-\frac{(2a-x)^2}{2T}-\alpha x-\frac{1}{2}\alpha^2 T}\,da\,dx,$$

where, again, where $\alpha = (r - \sigma^2/2)/\sigma$.

If we let $v(t,x)$ be the price of the option at time t, $S(t) = x$, and assuming that the call has not been knocked out before t, we can show that $v(t,x)$ solves the BSM PDE in $[0,T)\times[L,\infty)$ and has to satisfy the following boundary conditions

$$\begin{aligned}&v(T,x)=(x-K)_+, \quad x\geq L;\\&v(t,L)=0, \quad t\in[0,T];\\&\frac{v(t,x)}{x}\to 1, \quad \text{as } x\to\infty.\end{aligned}$$

Using the tower property, $e^{-rt}V(t,S(t))$ is a $\tilde{P}$-martingale. Define $\tau=\inf\{t\geq 0 \;:\; S(t)=L\}$ as the knock-out time, which is a stopping time. Using Optional Stopping Theorem, $e^{-r(t\wedge\tau)}V(t\wedge\tau, S(t\wedge\tau))$ for all $t\in[0,\tau]$ is a $\tilde{P}$-martingale. Using Thm. 2.1,

$$\begin{aligned}d\left(e^{-rt}v(t,S(t))\right)&=e^{-rt}\left(-rvdt+v_tdt+v_xdS(t)+\frac{1}{2}v_{xx}d[S](t)\right)\\&=e^{-rt}\left(-rvdt+v_tdt+v_x\left(rS(t)dt+\sigma S(t)d\tilde{B}(t)\right)+\frac{1}{2}v_{xx}\sigma^2S^2(t)dt\right)\\&=e^{-rt}\left(-rv+v_t+rS(t)v_x+\frac{1}{2}\sigma^2S^2(t)v_{xx}\right)dt+e^{-rt}\sigma S(t)v_xd\tilde{B}(t)\end{aligned}$$

Since it is a $\tilde{P}$-martingale, we need to set the dt-term to zero

$$-rv+v_t+rS(t)v_x+\frac{1}{2}\sigma^2S^2(t)v_{xx}=0$$

and since The process $(S(t))_{0\leq t\leq\tau}$ can reach a neighborhood of any point in $[0,T)\times[L,\infty)$ with positive probability. This means that the BSM PDE should hold for all $(t,x)\in[0,T)\times[L,\infty)$:

$$-rv+v_t+rxv_x+\frac{1}{2}\sigma^2x^2v_{xx}=0$$

As for the boundary conditions:

- At time T, if $x\geq B$, the down-and-out option is never knocked-out, and it will have the same payoff at maturity as a plain vanilla option. Thus, $v(T,x)=(x-K)_+, x\geq B$.
- For $S(t)=B$ at any time t, the down-and-out option is knocked out ad expires worthless. Thus, $v(t,B)=0, \quad t\in[0,T]$.
- For $S(t)\to\infty$, it is highly unlikely the stock price will hit the barrier and the down-and-out option gets knocked-out. In other words, the down-and-out option will very likely become a plain vanilla option *deep in the money* and end up with being in the money. In this case, the price of the down-and-out option and also the call is almost as much as the price of a forward contract:

$$\lim_{x\to\infty}\frac{v(t,x)}{x}=\lim_{x\to\infty}\frac{x-e^{-r(T-t)}K}{x}=1.$$

4 Numerical Simulations

In this section, we will only provide the pseudocode for the algorithms to simulate both the European call option and the down-and-out barrier option we worked with in the previous section. The full simulations and Python codes can be found in the simulations.ipynb in the following GitHub repository: `https://github.com/galweitz/MTH9831_GBM_handbook`.

Algorithm 1: Monte Carlo Simulation for European Call Option

function MonteCarloEuropeanCall ($S0, K, T, r, \sigma, N, M$);

Input : Initial stock price $S0$, Strike price K, Time to maturity T, Risk-free interest rate r, Volatility σ, Number of time steps N, Number of simulations M

Output: Estimated option price

Initialize stock prices array $S[M][N+1]$ with all values as $S0$;
for $m \leftarrow 1$ **to** M **do**
  for $t \leftarrow 1$ **to** N **do**
    Generate a random standard normal value z;
    Update $S[m][t]$ based on Geometric Brownian Motion formula;
Initialize payoffs array;
for $m \leftarrow 1$ **to** M **do**
  Calculate the payoff at maturity for $S[m]$;
  `// For a European call option, payoff is max(S - K, 0) at maturity`
  Calculate payoff as $\max(S[m][N] - K, 0)$;
Calculate the average of payoffs;
Discount the average payoff to present value using r;
return *discounted average payoff as option price*;

Algorithm 2: Monte Carlo Simulation for Down-and-Out Barrier Option

function MonteCarloBarrierOption ($S0, K, T, r, \sigma, B, N, M$);

Input : Initial stock price $S0$, Strike price K, Time to maturity T, Risk-free interest rate r, Volatility σ, Barrier level B, Number of time steps N, Number of simulations M

Output: Estimated option price

Initialize stock prices array $S[M][N+1]$ with all values as $S0$;
for $m \leftarrow 1$ **to** M **do**
  for $t \leftarrow 1$ **to** N **do**
    Generate a random standard normal value z;
    Update $S[m][t]$ based on Geometric Brownian Motion formula;
Initialize payoffs array;
for $m \leftarrow 1$ **to** M **do**
  Calculate the payoff at maturity for $S[m]$;
  if *path in* $S[m]$ *hits the barrier* **then**
    Set payoff to 0;
Calculate the average of payoffs;
Discount the average payoff to present value using r;
return *discounted average payoff as option price*;