1.23

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$\Delta u = 0, \quad 0 \leq r < 3$

$u|_{r=3} = -\varphi^2 + 2\pi\varphi$

$$A_n = \frac{1}{\pi}\int_0^{2\pi} u(\varphi)\cdot\cos(n\varphi)\,d\varphi$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} u(\varphi)\cdot\sin(n\varphi)\cdot d\varphi$$

$$u(\varphi) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n\cos n\varphi + B_n\sin n\varphi\right)\left(\frac{r}{a}\right)^n$$

$$A_0 = \frac{1}{\pi}\int_0^{2\pi}\left(-\varphi^2 + 2\pi\varphi\right)d\varphi = \frac{4\pi^2}{3}$$

$$A_n = \frac{1}{\pi}\int_0^{2\pi}\left(-\varphi^2 + 2\pi\varphi\right)\cdot\cos(n\varphi)\,d\varphi =$$

$$= \frac{4\cos^2(\pi n)(tg(\pi n) - \pi n)}{\pi n^3} = -\frac{4}{n^2}$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi}\left(-\varphi^2 + 2\pi\varphi\right)\sin(n\varphi)\,d\varphi =$$

$$= -\frac{2(\pi n\sin(2\pi n) + \cos(2\pi n) - 1)}{\pi n^3} = 0$$

Подставляем коэффициенты:

$$u = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{r}{3}\right)^n\left(-\frac{4}{n^2}\cdot\cos(n\varphi)\right)$$

5.23

$\Delta u = xz, \quad 2 < r < 3$

$u|_{r=2} = 0, \quad u|_{r=3} = 4$

Используя сфер. коорд., представим решение в виде суммы:

$$u(r, \theta, \varphi) = v(r, \theta, \varphi) + \omega(r),$$

где $v(r, \theta, \varphi)$ служит решением задачи:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right) + \frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2 v}{\partial\varphi^2} =$$

$$= \frac{r^2}{2}\cos\varphi\sin 2\theta,$$

~~и $\omega(r)$ задачи~~

1) $\omega(r)$

$\omega(2) = 0, \quad \omega(3) = 4$

$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \omega}{\partial r}\right) = 0$

$\frac{1}{r^2}\left(2r\frac{\partial\omega}{\partial r} + r^2\frac{\partial^2\omega}{\partial r^2}\right) = 0$

$\frac{2t}{r} + t' = 0$, где $t = \omega'$

$\frac{dt}{dr} = -\frac{2t}{r}$

$\frac{1}{2}\frac{dt}{t} = -\frac{dr}{r}$

$\frac{1}{2}\ln t = -\ln r + C$

$t = \frac{C}{r^2} \Rightarrow \omega = -\frac{C_1}{r} + C_2$

$\begin{cases} -\frac{C_1}{2} + C_2 = 0 \\ -\frac{C_1}{3} + C_2 = 4 \end{cases} \Rightarrow \begin{matrix} C_1 = 24 \\ C_2 = 12 \end{matrix}$

$\omega = -\frac{24}{r} + 12$

2) $u(r, \theta, \varphi)$

$$u(2, \theta, \varphi) = 0, \quad u(3, \theta, \varphi) = 0$$

$$u(r, \theta, \varphi) = R(r) P_2^{(1)}(\cos\theta) \cos\varphi,$$

где $P_2^{(1)}(x)$ – присоединённая функция Лежандра

Обозначим $P_2^{(1)}(\cos\theta)\cos\varphi = Y_2^{(1)}(\theta, \varphi)$, тогда:

$$Y_2^{(1)} \frac{d}{dr}(r^2 R') + R\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial Y_2^{(1)}}{\partial\theta}\right) + R\frac{1}{\sin^2\theta}\frac{\partial^2 Y_2^{(1)}}{\partial\varphi^2} = \frac{r^4}{6} Y_2^{(1)}(\theta, \varphi)$$

По определению сферической функции $Y_2^{(1)}(\theta, \varphi)$ имеет место равенство:

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_2^{(1)}}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y_2^{(1)}}{\partial\varphi^2} + 6Y_2^{(1)} = 0$$

Поэтому:

$$Y_2^{(1)}\frac{d}{dr}(r^2R') - 6R\, Y_2^{(1)} = \frac{r^4}{6} Y_2^{(1)}$$

откуда получим:

$$\frac{d}{dr}(r^2R') - 6R = \frac{r^4}{6}$$

$$\Downarrow$$

$$r^2R'' + 2rR' - 6R = \frac{r^4}{6}$$

$$R(2) = R(3) = 0$$

Решим уравнение:

$$r^2R'' + 2rR' - 6R = 0$$

$$R = r^k$$

$$R' = kr^{k-1}$$

$$R'' = k(k-1)r^{k-2}$$

$$r^2 k(k-1)r^{k-2} + 2rkr^{k-1} - 6r^k = 0$$

$$k^2 + k - 6 = 0$$

$k_1 = 2$, $k_2 = -3$ $\Rightarrow$ $R_1 = r^2$, $R_2 = r^{-3}$ $\Rightarrow R_{однородное} = C_1 r^2 + C_2 r^{-3}$

Частное решение:

$$R = Ar^4 + Br^3 + Cr^2 + Dr + E$$

$$R' = 4Ar^3 + 3Br^2 + 2Cr + D$$

$$R'' = 12Ar^2 + 6Br + 2C$$

$$r^2R'' + 2rR' - 6R = \frac{r^4}{6} \Rightarrow r^2(12Ar^2 + 6Br + 2C) +$$

$$+ 2r(4Ar^3 + 3Br^2 + 2Cr + D) - 6(Ar^4 + Br^3 + Cr^2 + Dr + E) = \frac{r^4}{6} \Rightarrow$$

$$\Rightarrow 14Ar^4 + 6Br^3 + 0\cdot Cr^2 - 4Dr - 6E = \frac{r^4}{6}$$

$$\begin{cases} A = \frac{1}{84} \\ B = 0 \\ C \in \mathbb{R} \\ D = 0 \\ E = 0 \end{cases} \Rightarrow R_{\text{частное}} = \frac{r^4}{84} + Cr^2$$

Общее решение:

$$R(r) = \frac{r^4}{84} + C_1 r^2 + C_2 r^{-3}$$

$$R(2) = R(3) = 0$$

$$\begin{cases} \frac{16}{84} + C_1 4 + \frac{C_2}{8} = 0 \\ \frac{81}{84} + C_1 9 + \frac{C_2}{27} = 0 \end{cases} \Rightarrow \begin{matrix} C_1 = -\frac{2059}{17724} \\ C_2 = \frac{3240}{1477} \end{matrix}$$

$$R(r) = \frac{r^4}{84} - \frac{2059}{17724} r^2 + \frac{3240}{1477} r^{-3}$$

В итоге получаем:

$$u(r, \theta, \varphi) = \left(\frac{r^4}{84} - \frac{2059}{17724} r^2 + \frac{3240}{1477} r^{-3}\right) P_2^{(1)}(\cos\theta)\cos\varphi - \frac{24}{r} + 12$$