### Proof by Induction

The basis of proof by induction is an axiom, i.e., an unprovable basic fact. It serves to prove a statement for all natural numbers—that is, for an infinite number of elements. It consists of two steps: the base case and the induction step. The idea behind it is comparable to the domino effect. In the first step—the base case—one shows that the statement to be proved is valid for the first element considered. After this has been shown, the induction step then successively proves that the statement must also always apply to a subsequent element if it applies to the preceding element. From the correctness of the statement for the first element follows the correctness of the statement for the second element, from the correctness of the statement for the second element follows the correctness of the statement for the third element, and so on. This results in an (infinitely long) chain of proof steps and one proves the statement indirectly for an infinite number of elements without explicitly considering each one.

Mathematically, proof by induction can be described more precisely as follows:

with Let m ∈ ℕn ≥ m0. The principle of proof by induction says if: and be well-defined and chosen, and let A(m)A(n) A(n)n ≥ m be a statement for all A(n + 1) n ∈ ℕ0

Base case : The statement is correct and

Induction step: If the statement is correct for , then is also cor-

then it follows that A(n) is correct for all n ≥ m. A(n) A(m) rect A(m+1). From the correctness of A(m + 1) we can then conclude the correctness of A(n + 1)n ≥ m n ≥ m A(m cludes the correctness of the statement for . From the correctness of A(m) The induction base case ensures that the statement for the very first element is correct. In the induction step one assumes that the statement for is correct and con-

2), and so on. We can continue this chain as we wish (namely for all A(n) ) and there(which we proved at the beginning of the induction) we can conclude the correctness of

fore the statement must be correct for all elements .

#### Example: Proof by induction

Proof:

• Let ⋅ 2 + 21) + 1 > 2(n + 1) + n ∈ ℕ44(. For all n + 1 > 2n + n = 2n ≥ 22 it applies that 2 4n + 1 > 2n + 4n +1 > 2n + 2 42n + 1 + 4 > 2n + 2 + . 4 ⋅ 2 + 1 = 9 > 6 = 4(4n +2

Base case: For the statement is obviously correct because

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Induction step: We assume that is correct and must show that

n + 1) + 1 = 4n + 4 + 1 = 4n + 1 + 4 > 2n + 2 + 4 > 2n + 2 + 2 is also correct. • . Thus, the assertion follows n2 □ . = 4 ≥ 4 = 2 ⋅ 2 ≥ 2(n=

then it is also true that

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1) 1 ≥ 2n 2 2 .

2n + 1 = 4n + □ , it follows that

it follows in general that fore valid.