### Definition: Divide

Let ⋅ n = a . We say “n divides a” and write n|a for it if there is an x ∈ ℤ so that x

#### Examples: Divide 2. 5|35

1. 3|−15 3 ⋅ 2 = 3 ⋅ 8 = (−5) ⋅ 3 624= −15

3.

4.

5.. Let b) n ∈ ℕ and a, b ∈ ℤ. It applies that n|a and n|b. It follows that n|(a + b) and n|(a − Theorem: Divisor of Sums and Differences

also applies.

Proof:

) ⋅

4 ⋅ 2 = 7 ⋅ 2 = 14 = 6 + 3|9 8 3 ⋅ 3 ⋅ 2+3

m|(b − a)Example: Congruent moduloLet a, b ∈ ℤ applies. and let a ≡ b mod mm ∈ ℕ. We write is read “a ≡ b mod ma is congruent to , if m divides the number b modulo m.” b − a, i.e., if

Definition: Equivalence Relation2.3.4.1. It applies that It applies that It applies that It applies that −85221 ≡ −19 mod ≡ 12 mod ≡ 0 mod ≡ 4 mod 572, because 4, because 572|(12 − |(0 |(4 4|(−19 − − 21)− (−82) because 5)) because 2 ⋅ 5 = −36 ⋅ 2 = ⋅ 7 10= −21.⋅12.

following properties apply to all An equivalence relation is therefore a set whose elements must fulfill certain properties.Let 1.2.3. MReflexivity: Symmetry: If Transitivity: If be a set. A subset (x, x) ∈ R((x, y) ∈ Rx, y) ∈ RR ⊆ M × M, then and x, y, z ∈ M((y, z) ∈ Ry, x) ∈ R is called an equivalence relation to :, then also applies (x, z) ∈ R also applies M if the three

Often,reads this “cates to which quantity the equivalence relation refers.If R is an equivalence relation on instead of x is equivalent to x ∼R y, one simply writes y.” M and (x, y) ∈ Rx ∼ y in abbreviated form, if the context indi-, then we write x ∼R y for it. One a ∼Example: Equivalence relationsn b is an equivalence relation to ℤ:

a) ∼ , then n| relation If Definition: Equivalence Classa ∼∼( n b∼, then n is therefore also called congruent modulo equivalence relation.□n|(a − b) applies. This is equivalent to n|(a − b) (a − b) + (b − c) = a − ca ≡ b mod n. The equivalencen|(a − b) +n|(a − c)n|−(a − b) b)

b − c) then is c)

Let MFrom be a set and x, y ∈ LR be an equivalence relation to Rx ∼ y y ∈ L M. A non-empty subset L M L ⊆ M is called equivalence class with respect to if the following two properties apply:

it follows that .

If x ∈ L and y ∈ M, and if x ∼ y, then .

equivalence class L must be equivalent to each other in pairs. Property 2 states that every An equivalence class thus denotes a non-empty subset of a set , with respect to which

element of M which is equivalent to an element of equivalence class L must also be an we have formed an equivalence relation. Property 1 states that all elements of such an

element of this equivalence class.

#### Example: Equivalence classes

Let’s look at the amount M of students enrolled in the subjects medicine, computerM

science, and history. We have seen that the property of whether two students are in the same course of study forms an equivalence relation on . We assume that each course has at least one student (otherwise the university would not offer it). Each of the students is therefore enrolled in exactly one of these courses. Then we can define three sets:First, we note that SSSMGI: =: =: = xxSx∈∈∈ ≠∅MMMx studies computx studies history, x studies medicineS ≠ ∅ and S ≠ ∅er ⊆. This is important according to the previ-science⊆MM ⊆ M

ous definition, because equivalence classes are always non-empty subsets.M I G

property for equivalence classes is fulfilled for all three sets.same course. Thus If x, y ∈ is SM, then x ∼ yx and . The same applies to y are both studying medicine, i.e., they are enrolled in thex, y ∈ SI and x, y ∈ SG. Thus the first

Let us consider the equivalence relation course of study and in particular are both studying medicine. But that also means SIf sets x ∈ S, because S and M is equivalent to SSS is just the quantity of all medical students. The same applies to the, S and S y ∈ M are equivalence classes with respect to , i.e., if x ∼ y∼n applies, then x and [a] := {a + kn|k ∈y. □ are in the samey ∈ three sets.M I G. Thus the second property for equivalence classes is also fulfilled for allM

M I G on , which is explained as fol-

Because

and . It applies

. Because of it applies that

One can now ask the question whether an element could also be contained in more than−=i.e., m)n a + kn − y. We reformulate the equation and obtain [a] y ∈ [a]n|x = a + kn(a + kn) − y. x = a + kn. Furthermore, let ). Thus there is an y = a + kn − mn = a + (kn|n(k − k')y = a + k'n∼ □ nmn y,

Altogether it follows that is an equivalence class with respect to n. one equivalence class. The following theorem states that this is not possible.

If M is a set and R is an equivalence relation to M R M, then every x ∈ M is in exactly one Theorem: Uniqueness of equivalence classes

equivalence class of with respect to .

Let x ∈ M and be LxL := {y ∈ M |y ∼ x} is an equivalence class containing . x. Then we will prove that L Proof:

unique.First, we show that x x is Let unique, i.e., that Let Because y, y' ∈ L∼ is also transitive, and x. Then L ≠ ∅y' ∈ My ∼ x, because due to the reflexivity of . It applies that and x ∼ y'y ∼ y' and thus continues to follow . Because is symmetrical, it follows that xy' ∈ L∼. It remains to be shown that , x ∼y ∈ L is . x and therefore y ∼ y'. x ∈ Lx ∼ y'x ∼ yL .

It applies that xx.

Because L'y ∈ L∼x is transitive, it follows that Lx is an equivalence class containing is not in any other equivalence class.M R x x it applies that x .

Consequently, x x is

Let x be another equivalence class of with respect to , which contains .

x

class and because L'Let Let and because x. Thus y ∈ Ly' ∈ L'y' ∈ is Lx. Then it applies that xL ⊆ L'. Then it applies that x ∈ L'. Therefore, .x ∈ Lx, it follows with the second property of equivalence classes that L', it follows that with the second property of equivalence classes ⊆ Ly ∼ xy' ∼ x and therefore and also and also y ∈ ML = L'y' ∈ M. Because altogether. . Because L'x is an equivalence class□L is an equivalencey ∈

that x x x x x x x x

With this we have answered the question asked above: Each element of the set lies exactly in one equivalence class. Finally, let us briefly consider how equivalence classes can be related to each other.