### Let f : A → B be a function. f is invertible exactly when f is bijective. Theorem: Relationship between Bijectivity and Invertibility of Functions

Proof:

(g ∘ f)(a) = a and (f ∘ g) Let tion and obtain a, a' ∈ A with . We apply the inverse function . Because g(f(a ) = (g ∘ f)(a) = a g(f(a' ) = (g ∘

it follows the bijectivity of . . Thus f is surjective. From

←: Let exactly one such g)(b) = f(g(bFor this function where a is the uniquely determined element of invertible. Theorem: Uniqueness of Invertible Functions=Let ab ∈ B andf be bijective. We must show that there is then a function (f ∘ g)(b) = b. Since □ ) = f(a) = bf is surjective, there is an a it applies that . We can therefore define the function for all for all a ∈ A and b ∈ Ba ∈ AA. g : B → A with (g ∘ f)(a(ff ∘ is), g

Proof:Let and f : A → Bf ∘ g = idB be an invertible function. Then the function is uniquely determined. g : B → A with g ∘ f = idA g : B → A g' : B → A g ∘ f = idA f ∘ g = idB g' ∘ f = id it follows it followsA and f ∘

and g b = g f a = g∘f a = a

Definition: Inverse FunctionThus g(b) = g'(b) applies to all g′ bb ∈ B= g′ and therefore f a = g′∘fg = g'a =. □a

ted Let = idf : A → Bf−1A and . f ∘ g = id be a bijective function. The uniquely defined mapping B is called the inverse function (or anti-function) to g : B → Af and is designa- with g ∘ f

1. be . It is therefore not possible to specify a unique inverse function. could be 1, but it could alsof has no inverse function.g would Inverse functionAn inverse function, alsoknown as an anti-func-tion, is one that reversesanother function.

Theorem: Bijectivity of the Composition2. for all andfg∘∘gf x ∈ ℝxx ==x +gf.gf xx == gf xx2 == xx22 ==xx is bijective due to the limited domain+ → ℝ+ with Let Proof:g are bijective, then A, B and C be non-empty sets and let g ∘ f is also bijective. f : A → B and g : B → C be functions. If f and

let the above sentence, the bijectivity. We show that To prove that Because of the associativity of Let gf g be bijective functions. Thus g ∘ f is bijective, we will show that g. f and g are invertible. Let g ∘ ff−1 ∘ g is invertible. From this follows, with−1 : C → Af−1 be the inverse of f and

therefore bijective. It applies completely analogously that (Theorem: Bijectivity of the Inverse FunctionConsequently, gf−1id−1 = idB = id ∘ f) = fBC and . Overall, −1id ∘ f = idf−1B ∘ g(□g ∘ f) ∘ (f−1A is the inverse of . −1 further follows ) = id(fg ∘ f−1 ∘ gC. The function −1) ∘ (g ∘ f) = fg ∘ f−1 is therefore invertible and ∘ (g−1 ∘ g) ∘ f) = f−1 ∘

tive.Let f : A → B be a bijective function. Then the figure f−1 which is inverse to f is also bijecProof:

is invertible, because the inverse function to We have shown above that a function is bijective exactly when it is invertible. Obviously f−1 is f. Consequently, f−1 is also bijective. □f−1

#### STUDY GOALS

On completion of this unit, you will have learned...

what an operation is.

what is meant by a semigroup and what its characteristics are.

what identity elements and inverse elements are.

what a group is and what its characteristics are.

what a ring is and what characteristics it has.

what is meant by a group of units.

which rules of calculation apply in rings.

what residue rings are and how to calculate in them.