MATH-241 Linear Algebra Fall 2015

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Read the instructions below:

- Solutions should be supported with reasons. Just writing the numerical answers is not enough and NO credit will be given for that.
- Please submit both the LaTeXand PDF files. Refer to the instructions on the HW page about submitting the homework .

Assignment # 1

For Grading Only					
Completion Points	5	4	3	2	0
#1	5	4	3	2	0
#2	5	4	3	2	0
#3	5	4	3	2	0
#4	5	4	3	2	0
#5	5	4	3	2	0
#6	5	4	3	2	0
#7	5	4	3	2	0
#8	5	4	3	2	0
#9	5	4	3	2	0
Total					

MATH-241 Homework 1 **Fall 2015**

Matthew Gambogi

DUE DATE Sep 03, 2015 by 11:00PM (in Dropbox)

Problem 1 (POOLE 1.1.18). Solve for the vector $\mathbf{x}$ in terms of the vector $\mathbf{a}$ and $\mathbf{c}$.

$$\mathbf{x} + 2\mathbf{a} - \mathbf{b} = 3(\mathbf{x} + \mathbf{a}) - 2(2\mathbf{a} - \mathbf{b})$$

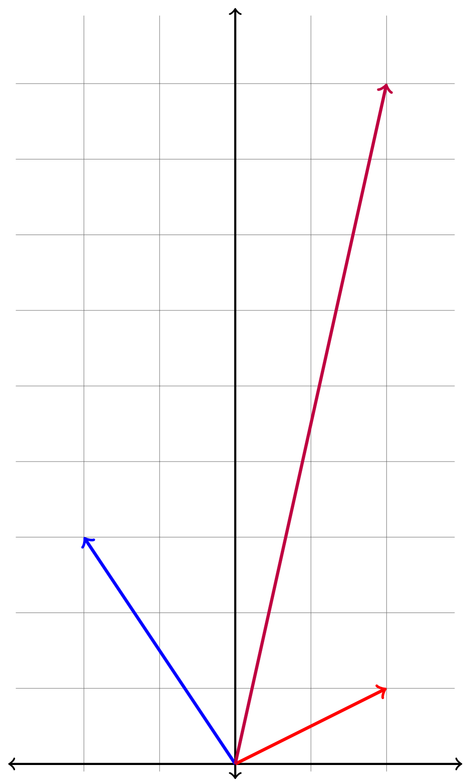
Solution:

$$\begin{aligned}
\mathbf{x} + 2\mathbf{a} - \mathbf{b} &= 3(\mathbf{x} + \mathbf{a}) - 2(2\mathbf{a} - \mathbf{b}) \\
\mathbf{x} + 2\mathbf{a} - \mathbf{b} &= 3\mathbf{x} + 3\mathbf{a} - 4\mathbf{a} + 2\mathbf{b} \\
-2\mathbf{x} &= -3\mathbf{a} - 3\mathbf{b} \\
\mathbf{x} &= \frac{-3(\mathbf{b} + \mathbf{a})}{2}
\end{aligned}$$

Problem 2 (POOLE 1.1.22). Draw the standard coordinate axes on the same diagram as the axes relative to $\mathbf{u}$ and $\mathbf{v}$. Use these to find $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

$$\mathbf{u} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$$

Solution:



$$2\mathbf{u} + 3\mathbf{v} = \mathbf{w}$$

Problem 3 (POOLE 1.2.48). Find all values of the scalar k for which the two vectors are orthagonal

$$\mathbf{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} k+1 \\ k-1 \end{pmatrix}$$

Solution:

In $\mathbb{R}^2$:

$$\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \cos(\pi/2) = 0 \implies \mathbf{u} \text{ is perpendicular to } \mathbf{v}$$

Reifiying, we have:

$$\frac{\begin{pmatrix} 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} k+1 \\ k-1 \end{pmatrix}}{\left\| \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\| \left\| \begin{pmatrix} k+1 \\ k-1 \end{pmatrix} \right\|} = 0$$

$$\frac{2k+2+3k-3}{\sqrt{13}\sqrt{(k+1)^2+(k-1)^2}} = 0$$

$$\frac{5k-1}{\sqrt{13}\sqrt{2k^2+2}} = 0$$

$$\frac{5k-1}{\sqrt{2}\sqrt{13}\sqrt{k^2+1}} = 0$$

$$\frac{5k-1}{\sqrt{k^2+1}} = 0$$

$$\frac{5k}{\sqrt{k^2+1}} = \frac{1}{\sqrt{k^2+1}}$$

$$5k = 1$$

$$k = \frac{1}{5}$$

Problem 4 (POOLE 1.2.62)(a).
Prove that $\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{u}\|^2$ $\forall$ vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$

Solution:

Proof.

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{u}\|^2$$

By the definition of the norm:

$$\sqrt{(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})}^2 + \sqrt{(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})}^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{u}\|^2$$

By the distributive property:

$$\sqrt{\mathbf{u}\cdot\mathbf{u} + 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v}}^2 + \sqrt{\mathbf{u}\cdot\mathbf{u} - 2(\mathbf{u}\cdot\mathbf{v}) + \mathbf{v}\cdot\mathbf{v}}^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{u}\|^2$$

Combining terms:

$$2(\mathbf{u}\cdot\mathbf{u}) + 2(\mathbf{v}\cdot\mathbf{v}) = 2\|\mathbf{u}\|^2 + 2\|\mathbf{u}\|^2$$

By raising each term to the $\frac{2}{2}^{\text{th}}$:

$$2\sqrt{\mathbf{u}\cdot\mathbf{u}}^2 + 2\sqrt{\mathbf{v}\cdot\mathbf{v}}^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{u}\|^2$$

Again, using the definition of the norm:

$$2\|\mathbf{u}\|^2 + 2\|\mathbf{u}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{u}\|^2$$

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Problem 5 (POOLE 1.3.14). Give the vector equation of the plane passing through P, Q, and R.

$$P = (1, 0, 0), Q = (0, 1, 0), R = (0, 0, 1)$$

$\boxed{\text{Solution:}}$

The vector form of the equation of a plane in $\mathbb{R}^3$ is:

$$\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$$

Reifying:

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Problem 6. Let $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^n$. If $\|\mathbf{v}\| = 5$ and $\|\mathbf{w}\| = 3$, what are the largest and smallest values of $\|\mathbf{v} - \mathbf{w}\|$? What are the largest and smallest values of $\mathbf{v} \cdot \mathbf{w}$?

Solution: By Cauchy-Sharz-Bunyakowski:

$$|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\|\|\mathbf{w}\|$$

Substituting:

$$0 \leq |\mathbf{v} \cdot \mathbf{w}| \leq 15$$

Therefore:

$$-15 \leq \mathbf{v} \cdot \mathbf{w} \leq 15$$

Now:

$$\begin{aligned}
\|\mathbf{v} - \mathbf{w}\| &= \sqrt{(\mathbf{v} - \mathbf{w})(\mathbf{v} - \mathbf{w})} \\
&= \sqrt{\mathbf{v} \cdot \mathbf{v} - 2(\mathbf{v} \cdot \mathbf{w}) + \mathbf{w} \cdot \mathbf{w}} \\
&= \sqrt{25 - 2(\mathbf{v} \cdot \mathbf{w}) + 9} \\
&= \sqrt{34 - 2(\mathbf{v} \cdot \mathbf{w})} \\
&= \sqrt{2}\sqrt{17 - \mathbf{v} \cdot \mathbf{w}}
\end{aligned}$$

It follows:

$$\sqrt{2}\sqrt{17 - \min(\mathbf{v} \cdot \mathbf{w})} \leq \|\mathbf{v} - \mathbf{w}\| \leq \sqrt{2}\sqrt{17 - \max(\mathbf{v} \cdot \mathbf{w})}$$

Substituting:

$$\sqrt{2}\sqrt{17 - 15} \leq \|\mathbf{v} - \mathbf{w}\| \leq \sqrt{2}\sqrt{17 + 15}$$

Therefore:

$$2\sqrt{2} \leq \|\mathbf{v} - \mathbf{w}\| \leq 5\sqrt{2}$$

Problem 7. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors such that $\|\mathbf{u}\| = 1, \|\mathbf{v}\| = 2$ and $\|\mathbf{w}\| = 3$, $\mathbf{u}$ is orthogonal to $\mathbf{v}$, and that the angle between $\mathbf{u}$ and $\mathbf{w}$ is $\pi/3$ and that between $\mathbf{v}$ and $\mathbf{w}$ is $\pi/6$. Find $\|\mathbf{u} + \mathbf{v} + \mathbf{w}\|$.

Solution:

Solving for $\mathbf{u} \cdot \mathbf{v}$:

$$\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = 0 = \cos(\frac{\pi}{2})$$
$$\implies \mathbf{u} \cdot \mathbf{v} = 0$$

Solving for $\mathbf{u} \cdot \mathbf{w}$:

$$\frac{\mathbf{u} \cdot \mathbf{w}}{\|\mathbf{u}\|\|\mathbf{w}\|} = \frac{1}{2} = \cos(\frac{\pi}{3})$$
$$\implies \mathbf{u} \cdot \mathbf{w} = \frac{3}{2}$$

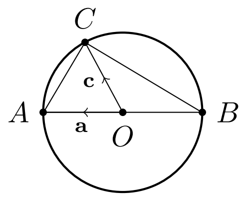
Solving for $\mathbf{u} \cdot \mathbf{w}$:

$$\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{\sqrt{3}}{2} = \cos(\frac{\pi}{6})$$
$$\implies \mathbf{v} \cdot \mathbf{w} = 3\sqrt{3}$$

Rewriting $\|\mathbf{u} + \mathbf{v} + \mathbf{w}\|$

$$\begin{aligned}
&\|\mathbf{u} + \mathbf{v} + \mathbf{w}\| \\
=&\sqrt{(\mathbf{u} + \mathbf{v} + \mathbf{w}) \cdot (\mathbf{u} + \mathbf{v} + \mathbf{w})} \\
=&\sqrt{\mathbf{u} \cdot (\mathbf{u} + \mathbf{v} + \mathbf{w}) + \mathbf{v} \cdot (\mathbf{u} + \mathbf{v} + \mathbf{w}) + \mathbf{w} \cdot (\mathbf{u} + \mathbf{v} + \mathbf{w})} \\
=&\sqrt{\mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w} + 2(\mathbf{u} \cdot \mathbf{v}) + 2(\mathbf{u} \cdot \mathbf{w}) + 2(\mathbf{v} \cdot \mathbf{w})} \\
=&\sqrt{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 + 2(\mathbf{u} \cdot \mathbf{v}) + 2(\mathbf{u} \cdot \mathbf{w}) + 2(\mathbf{v} \cdot \mathbf{w})} \\
=&\sqrt{1^2 + 2^2 + 3^2 + 2(0) + 2(3/2) + 2(3\sqrt{3})} \\
=&\sqrt{17 + 6\sqrt{3}}
\end{aligned}$$

Problem 8. Let A and B be the endpoints of a diameter of a circle and O be the center (without the loss of generality we can place it at the origin). Suppose C is any point on the circle (see the figure below). Let the position vectors of A be $\mathbf{a}$ and that of C be $\mathbf{c}$. Using this prove that $\angle ACB$ is a right angle. Hint: Try to express the vectors $\overrightarrow{AC}$ and $\overrightarrow{CB}$ in terms of $\mathbf{a}$ and $\mathbf{c}$.



Solution:

Proof.

$$\|\mathbf{a}\| = \|\mathbf{b}\| = \|\mathbf{c}\|$$

$$\overrightarrow{AC} = \mathbf{a} - \mathbf{c}$$

$$\overrightarrow{CB} = \mathbf{c} + \mathbf{a}$$

$$\cos(\angle ABC) = \frac{(\mathbf{a} - \mathbf{c}) \cdot (\mathbf{c} + \mathbf{a})}{\|\mathbf{a} - \mathbf{c}\|\|\mathbf{c} + \mathbf{a}\|}$$

$$\cos(\angle ABC) = \frac{(\mathbf{a} \cdot \mathbf{a} - \mathbf{c} \cdot \mathbf{c})}{\|\mathbf{a} - \mathbf{c}\|\|\mathbf{c} + \mathbf{a}\|}$$

$$\cos(\angle ABC) = \frac{\|\mathbf{a}\|^2 - \|\mathbf{c}\|^2}{\|\mathbf{a} - \mathbf{c}\|\|\mathbf{c} + \mathbf{a}\|}$$

$$\cos(\angle ABC) = \frac{0}{\|\mathbf{a} - \mathbf{c}\|\|\mathbf{c} + \mathbf{a}\|}$$

$$\cos(\angle ABC) = 0$$

$$\angle ABC = \frac{\pi}{2}$$

$\angle ABC$ must be perpendicular.

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Problem 9. Find the vector form of the equation of the plane that contains the point $\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$ and the line

$$\mathbf{r} = \left\{ \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \; : \; t \in \mathbb{R} \right\}.$$

Solution:

$$\mathbf{x} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + s \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} : t, s \in \mathbb{R}$$

Problem 10. Let L be the line that passes through the points $\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 10 \\ -1 \\ 4 \end{pmatrix}$.

1. Find the vector equation of line L.
2. Does the point $\begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$ lie on the line. Give reasons.
3. Find the equation of a line that passes through the origin and is orthogonal to L. How many such distinct lines are possible?

Solution:

1.

$$A = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, B = \begin{pmatrix} 5 \\ 10 \\ -1 \\ 4 \end{pmatrix}, \mathbf{v} = B - A = \begin{pmatrix} 7 \\ 9 \\ -2 \\ 4 \end{pmatrix}$$

$$L = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 7 \\ 9 \\ -2 \\ 4 \end{pmatrix}, \forall t \in \mathbb{R}$$

2. *Proof.*

$$\begin{pmatrix}1\\0\\2\\1\end{pmatrix} = \begin{pmatrix}-2+7t\\1+9t\\1-2t\\4t\end{pmatrix}$$
$$\implies \frac{3}{7} = t = \frac{-1}{9}$$
$$\text{Contradiction.}$$
$$\begin{pmatrix}1\\0\\2\\1\end{pmatrix} \notin L$$

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3.

Let $\mathbf{u}$ be the direction vector in:

$$M = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} + s\begin{pmatrix}x\\y\\z\\w\end{pmatrix}$$

$$\mathbf{u}\cdot\mathbf{v} = 0 \implies 7\mathbf{x} + 9\mathbf{y} - 2\mathbf{z} + 4\mathbf{w}$$

One solution:

$$\begin{pmatrix}x\\y\\z\\w\end{pmatrix} = \begin{pmatrix}1\\-1\\1\\1\end{pmatrix}$$

Only one unique solution exists.

Problem 11. Let $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ n \end{pmatrix}$ be two vectors in $\mathbb{R}^n$. Let θ_n be the angle between $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$.

(a) Find θ_n.

(b) What will be $\lim_{n\to\infty} \theta_n$? Give a mathematical justification.

Solution:

$$\cos(\theta_n) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

$$\mathbf{u} \cdot \mathbf{v} = 1(1) + 2(1) + ... + n(1) = 1 + 2 + ... + n = \frac{n(n+1)}{2}$$

$$\|\mathbf{u}\| = \sqrt{1^2 + ... + 1^2} = \sqrt{n}$$

$$\|\mathbf{v}\| = \sqrt{1^2 + 2^2 + ... + n^2} = \sqrt{\frac{n(n+1)(2n+1}{6}}$$

$$\cos(\theta_n) = \frac{\frac{n(n+1)}{2}}{\sqrt{n}\sqrt{\frac{n(n+1)(2n+1}{6}}}$$

$$\cos(\theta_n) = \frac{\sqrt{6}}{2}\sqrt{\frac{n+1}{2n+1}}$$

$$\theta_n = \cos^{-1}(\frac{\sqrt{6}}{2}\sqrt{\frac{n+1}{2n+1}})$$

$$\begin{aligned}\lim_{n \to \inf} \cos^{-1}(\frac{\sqrt{6}}{2}\sqrt{\frac{n+1}{2n+1}}) &= \cos^{-1} \frac{\sqrt{6}}{2\sqrt{2}} \\ &= \cos^{-1} \frac{\sqrt{3}}{2} \\ &= \frac{\pi}{6}\end{aligned}$$