MHD

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1 Introduction

Magnetohydrodynamics or MHD is the motion of magnetized fluids. It is also the study of electrically conducting fluids and how the combination of fluid dynamics and electromagnetism govern it's motion. Practical uses for MHD can vary largely. Two specific examples are the confinement of nuclear fusion, and priming rocket fuel in zero gravity. The magnetized fluid we will discuss today is a ferrofluid: a carrier fluid that contains ferromagnetic particles. It is very easy to say that a ferrofluid in a magnetic fluid that will align and move with a magnetic field. Conceptually speaking, we are done (See Figure 1). However, in order to have a coherent understanding of MHD, we need to discuss how properties of fluid dynamics and electricity & magnetism (E&M) fit together to describe the motion of ferrofluids. In this paper we shall explore fluid dynamics, E&M, how fluid dynamics and E&M come together to model MHD, and specific properties of ferrofluids and how they are made. In essence we will describe a fluid motion equation that looks like $F = ma$, an E&M equation that looks like $F = ma$, and then combine them to get an equation of motion for our ferrofluid.

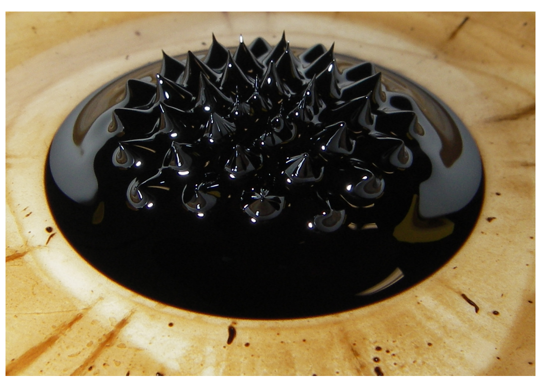

Figure 1: Ferromagnetic fluid (ferrofluid) in the presence of a magnetic field. The ferrofluid aligns with the magnetic field lines.

2 Fluid Dynamics

Before we get into our equations, it will be helpful to define a few things.

1. ρ = mass density
2. $\vec{u}$ = flow velocity
3. P = pressure
4. σ = charge density
5. $\vec{J}$ = current density
6. $\vec{E}$ = electric field
7. $\vec{B}$ = magnetic field

One way of approaching this problem is by considering conservation of mass. We use the equation $d\rho/dt = -\nabla\dot{(}\rho\vec{u})$ to show that the changing mass density is dependent upon the divergence of the mass density and the flow velocity. fluids can have a multitude of forces acting on them like friction, pressure, gravity, EM, etc. We need to unpack each force term and then superpose them. Let's look at pressure. Force due to pressure is pressure times area. the acceleration of our fluid multiplied by our mass will give us force. We can then use conservation of momentum to get an equation of the form

$$d\vec{u}/dt + \vec{u}\dot{\nabla}\vec{u} = -1/\rho\nabla p - \nabla\phi + \vec{F}.$$

This momentum depends on pressure, viscous stress, and any other forces like gravity. This is the Navier-Stokes equation: the main equation used to calculate fluid motion. The left side of the equation is the velocity of the fluid. ∇p is the force pressure term, $\nabla\phi$ is the potential term, and $\vec{F}$ is any force term that we happened to miss. Viscous stress or sheer stress is a frictional force inside the fluid from certain parts of the fluid flowing opposite directions or remaining stationary. Figure 2 shows us what viscous stress is. We now have a working equation of motion for fluids.

3 Electricity and Magnetism

We will now move on the E&M portion. The evolution of of a magnetic field can be shown by $d\vec{B}/dt = -\nabla \times \vec{E}$ This is one of Maxwell's equations. Usually we have $\nabla \times \vec{E} = 0$. But this is only the case with static charges. When the charges start moving, there is some sort of current density and so we use $d\vec{B}/dt = -\nabla \times \vec{E}$.

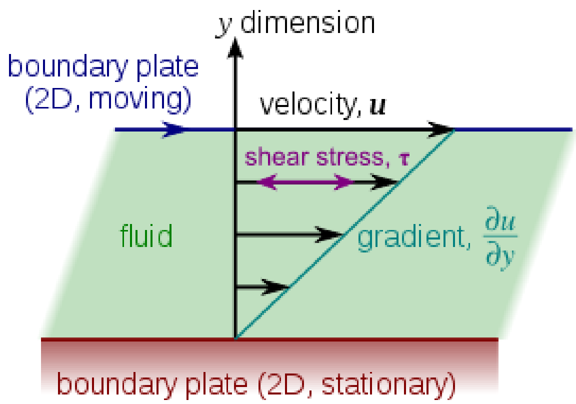

Figure 2: The viscous stress on a "cube" of fluid. The velocity of the fluid is slower near the edge and such causes friction in the fluid.

From Coulomb's law we can see that the force on a stationary charge due to another charge is $F = kqQ/r^2$ where $k = 1/4\pi\epsilon_0$ and $\epsilon_0 = 8.85418782 * 10^{-12}$. This will yield a repulsive force between the two particles if $F > 0$, and an attractive force if $F < 0$. If we add more test charges to our system, we can get the force by using the principle of superposition. We also can calculate the electric field from the force. this equation is $E = F/Q$. These are our initial backbone equations, but these are only useful for stationary discrete charges. We introduce and σ to account for a volume of charges. our electric field can then take the form $E = 1/k \int \sigma d\tau/r^2$. This is important because we can then use Gauss's law to get one of Maxwell's equations.

$\nabla \cdot \vec{E} = \sigma/\epsilon_0$

Along those same lines, for a stationary charge we get that $\nabla \times \vec{E} = 0$. This is another one of Maxwell's equations, but it is useless to us right now. These equations, simply put, state that the divergence of an electric field is equal to the volume charge divided by ϵ_0, which is the vacuum permittivity, and that the curl of an electric field is equal to zero. Now, We know that this is the case for stationary charges, but when we introduce moving charges, we have a different expression.

Before we explain moving charges and their expressions, let us first describe the magnetic field. The force due to a magnetic field can be described as $\vec{F} = Q(\vec{v} \times \vec{B})$ where Q is some test charge, $\vec{v}$ is the velocity of the test charge, and $\vec{B}$ is the magnetic field the charge moves through. You can see that $\vec{F}$ must be perpendicular to both the velocity vector and the magnetic field vector. If the charge is in the presence of both a magnetic field and an electric field, the force it feel would just superpose the two forces together. $\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B})$

We can derive three important equations using a magnetic field and moving charges.

1. $\nabla \times \vec{B} = \mu_0 \vec{J}$
2. $\nabla \cdot \vec{B} = 0$
3. $\nabla \times \vec{E} = -d\vec{B}/dt$

The first one tells us that the curl of a magnetic field is equal to the permeability of free space, which is $\mu_0 = 4\pi * 10^7$ multiplied by $\vec{J}$ which is the current density.

The second one tells us that the divergence of a magnetic field is always zero. This makes sense because if you were to look at one straight magnetic field line due to a magnet and measure its strength at the base compared to its strength some distance away, the magnitude would be the same. You may wonder why the magnet gets stronger closer to the magnet? This is because there are more field lines entering/exiting the magnet near the base of it. Figure 3 shows this.

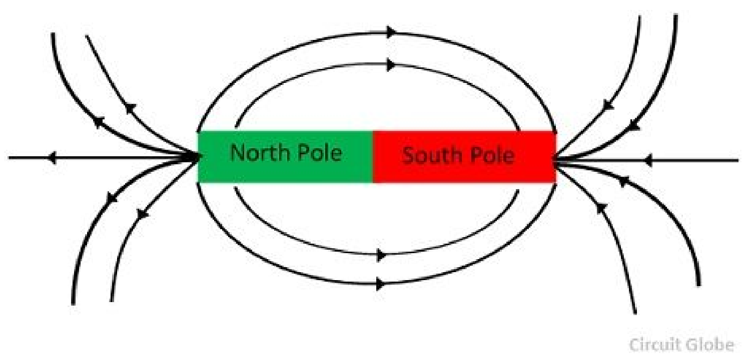

Figure 3: magnetic field lines. You can clearly see that there are more field lines closer to the magnets poles.

The third one is a bit confusing. We just showed you up above that $\nabla \times \vec{E} = 0$ and now we are changing that. It can be confusing, but this new equation holds for moving charges while our previous equation holds for static charges. We want to use this new equation because in a fluid we aren't dealing with static fluids, we are dealing with the motion of these fluids and therefore the motion of the charges within these fluids.

4 Fluid Dynamics and E&M Combined

We have now described all four of Maxwell's equations. We shall use these and their force laws in combination with fluid motion force laws to get our expression for MHD.

We can now describe our F term in our Navier-Stokes equation. We can unpack it to be $F = \nu\nabla^2 u + 1/\rho(J \times B)$ The first term is due to sheer stress and vorticity. The second term is from E&M. Vorticity is a property of fluids. It essentially measures the curl of a fluid. Often times, vorticity arises from friction of the fluid against the fluid container. You could imagine a river that is flowing past you. If there are no impedances, the river will be flowing the fastest in the center and the slowest near the riverbanks. If you were to throw a stick into the river such that one end was in the middle of the river and the other was near the edge, the stick would spin because the vorticity of the river. (See figure 4)

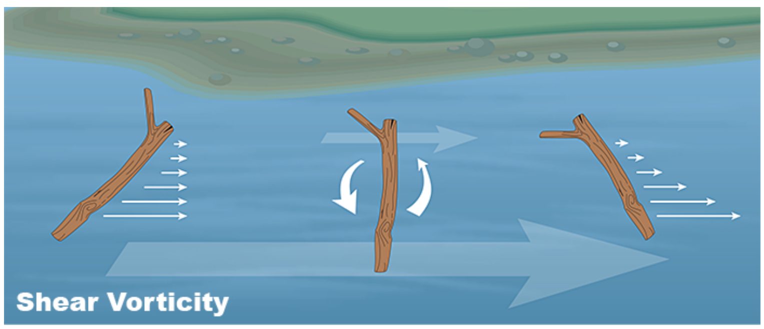

Figure 4: you can see that the stick seems to be almost pivoting from a point. The velocity in the river is different at different spots so the river has vorticity.

Vorticity can be represented as ω and we can define it as $\omega = \nabla \times v$. With this equation and a few manipulations we can rewrite our Navier-Stokes equation as $d\omega/dt + \nabla \times (\omega \times v) = \nu\nabla^2\omega + \nabla \times (J \times B)/\rho$. This gives us our equation of motion including a magnetic force. This is exactly what we were looking for. Now that we have our equation, let's look at specific properties of ferrofluids.

5 Ferrofluids

Temperature is a critical consideration when it comes to MHD. This is because vorticity can be generated by thermomagnetic interaction even in the absence of viscosity (think of convection). Ferrohydrodynamics is the fluid dynamic and heat transfer processes associated with the motion of incompressible, magnetically polarizable fluids in the presence of magnetic field and temperature gradients. Strong coupling occurs when the polarization is field and temperature dependent.

When making ferrofluids, the most common magnetic particles used are magnetite and maghemite. There are many ways to get make these particles, but the most common way is decomposition of organo-metallic compounds. This is done through thermolysis of our substance in a hydrocarbon solution. This is essentially just a reaction of substances that breaks down our substance into the particles we want. These particles should be close in size though. often times performing thermolysis at a higher temperature will yield smaller particles. If the particles are too big then we can get sediment build ups in our fluid and this will ruin our ferrofluid. You can think of it like a bloodstream. You don't want your bloodstream to clot. In order for the colloidal suspension within a ferrofluid to remain stable, the magnetic particles generally have to be approximately 10 nm in diameter. When they are this size they have only one magnetic domain. These particles then feel an attraction to each other. To stop this attraction to each other we must add a repulsive interaction. We can introduce a charge the particles to give them a repulsive electrostatic force. These charged particles are introduced into the fluid. When this fluid is placed within a magnetic field, the particles slip relative to the fluid and transmit drag to it. This is what causes the whole fluid to move.

6 Wrap Up

We now have a more coherent understanding of MHD. The most important fluid dynamic equation is the Navier-Stokes equation. This gives the motion of a fluid given certain forces. The most important equations from E&M are Maxwell's equations. These four equations can give us a force term, which we can plug into the Navier-Stokes equation to give us a full picture. We also account for vorticity and sheer stress within this equation. We also saw that ferrofluids must be made with small ferromagnetic particles injected into a fluid. The particles cannot be too big and must be charged to prevent clumping. There is much more to learn within this area, like other methods of preparing ferrofluid particles or looking at restoring forces in MHD. I hope this has given you a taste of MHD and that your palate entices you to delve deeper into it.

7 References

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