Polyhedral combinatorics – Lecture 11

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1 Non-negative rank (propositions from the tutorial)

Definition (Non-negative rank)**.** *The* non-negative rank $rk_+(M)$ *for* $M_{m\times n} \geq 0$ *is defined as*

$$rk_+(M) = \min\{r \mid \exists T_{m\times r} \geq 0, U_{r\times n} \geq 0 \ s.t. \ M = TU\}$$

Proposition. *$rk_+(M)$ is equal to the minimum number of rank-1 non-negative matrices that sum to M.*

Observation.

- *The non-negative rank of a matrix is at least as large as its rank*
- *The non-negative rank of a matrix is at most as large as the minimum number of rows and columns of the matrix.*
- *The non-negative rank of a matrix is equal to the non-negative rank of its transpose.*

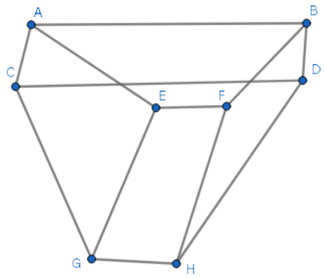
Proposition. *The non-negative rank of the product of two matrices A and B is at most as large as the minimum of the non-negative rank of A and the non-negative rank of B.*

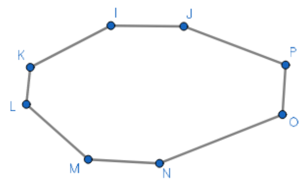
Proposition. *The non-negative rank of the sum of two matrices A and B is at most as large as the sum of the non-negative rank of A and the non-negative rank of B.*

2 Yannakakis theorem

Say we have a problem of size n. We search for a polyhedron representing the problem and its description using linear equalities and inequalities. We might end up with exponentially many inequalities [1] We might search for other descriptions of the polyhedron in $\mathbb{R}^d$ with polynomially many inequalities. If that doesn't work out, another approach is to search for polyhedrons of higher dimension whose projection to $\mathbb{R}^d$ is equivalent to the original polyhedron.

[1] I'd like to stress that we only count the number of inequalities. The equalities don't bother us since we can use linear algebra tools to efficiently solve for the affine space they represent and move into that space. Then we are only left with inequalities.





Definition (Extended formulation)**.** *Let $P \subseteq \mathbb{R}^d, P = \{x| \dots\}$ and $Q \subseteq \mathbb{R}^{d+r}, Q = \{(x,y)| \dots\}$ be polytopes.*

Q is an extended formulation of P if $P = \Pi_x(Q) := \{x|\exists y : (x,y) \in Q\}$

Definition (Extension complexity)**.** *The* extension complexity $xc(P) =$ *min number of inequalities describing any extended formulation of P.*

Definition (Slack matrix)**.** *Let $P = \{x|A_{m\times d}x \leq b\} = conv(V_{n\times d})$.*

The non-negative slack matrix $S(P)$ *is an $m \times n$ matrix s.t.*

$$S_{ij} = b_i - a_i^T v_j$$

The following theorem is important because it enables us to go from bounds on the non-negative rank to bounds on xc.

Theorem (Yannakakis)**.** *Let P be a polytope. Then $xc(P) = rk_+(S(P))$.*

Proof.

reference: Mihalis Yannakakis: Expressing Combinatorial Optimization Problems by Linear Programs, Theorem 3

"$\leq$"

Let's show that for a given slack matrix $S(P)$ of non-negative rank r we can construct an extended formulation Q of the polytope P. Let $P = \{x|A_{m\times n}x \leq b\}$ (WLOG there are no equalities). Let $S(P) = T_{m\times r}U_{r\times n}$ be a non-negative factorization of the slack matrix $S(P)$. We define the extended formulation as follows

$$Q := \{(x,y)|Ax + Ty = b, y \geq 0\}$$

So Q only has r inequalities ($y \geq 0$).

Now let's show that Q is really an extended formulation of P – show that $\Pi_x(Q) = P$.

- "$\subseteq$" Let $x, y \in Q$. By definition of Q it holds that $Ax + Ty = b$. Because T is non-negative it holds that $Ty \geq 0$. That means that $Ax \leq b$ which by definition means that $x \in P$.
- "$\supseteq$" Let's show that for each vertex v_i of P exists y s.t. $(v_i, y) \in Q$. This will suffice since points of P are convex combinations of vertices and those will then translate to convex combinations of points of Q.

  Let's use $y := U_{*i}$. For each i, (v_i, U_{*i}) is a feasible solution of $Ax + Ty = b, y \geq 0$ because $TU_{*i} = S_{*i}$, which is exactly the slack of vertex v_i.

"$\geq$"

Let $P = \{x | Ax \leq b\} = conv(V)$ and let $Q = \{(x, y) | Ex + Fy \leq g\}$ be its extended formulation with r inequalities. Note that $\forall i : a_i^T x \leq b_i$ is a valid inequality for Q. For each of these inequalities it should be possible to express them as non-negative linear combinations of the inequalities of Q [2]. Let's denote the coefficients of these linear combinations as $\lambda_1^i, ..., \lambda_r^i, \lambda_k^i \geq 0$.

$$(a_i, 0) = \sum_k \lambda_k^i (E_{k*}, F_{k*})$$

$$b_i = \sum_k \lambda_k^i g_k$$

Notice that for a vertex v of P there is a point (v, u) of Q s.t. the inequality $a_i x \leq b_i$ has the same slack w.r.t. v and (v, u). In the lecture we assumed that (v, u) is a vertex of Q. Let me give an explanation of why we can assume this. We can observe that (v, u) lies on a facet – v is an extreme point of P w.r.t. some direction and optimizing along this direction gives us a facet of Q. All of the points of the facet will have the form (v, y) for some y. This facet will contain some vertices of Q. Let (v, u) be one of these vertices. Let's now continue with the proof.

Slack of the inequality of P $a_i^T x \leq b_i$ with respect to vertex v_j of P is the same as the slack of the inequality $a_i^T x \leq b_i$ that we constructed from inequalities of Q w.r.t. a vertex (v_j, u_j) of Q and that can be expressed as the slack of $\sum_{k=1}^r \lambda_k^i \cdot (E_i x + F_i y \leq g_i)$ w.r.t. (v_j, u_j). Since when you combine inequalities you also combine slack, we finaly get this: $\sum_{k=1}^r \lambda_k^i \cdot ($ slack of $E_i x + F_i y \leq g_i$ w.r.t. $(v_j, u_j))$.

Now let's define the non-negative matrices that form the rank r factorization of $S(P)$. One of them will be Λ whose rows are the coefficient vectors λ^i. The second will be submatrix S of the slack matrix of Q consisting of those columns of Q corresponding to vertices (v_j, u_j). Both matrices are non-negative, have the right number of rows and columns and due to what we have shown in the previous paragraph, $\Lambda S = S(P)$. □

In particular, this means that all possible slack matrices of P will have the same non-negative rank.

3 Communication complexity

We now sidestep into the theory of communication complexity. We will show that a specific type of a communication complexity problem can be used to bound the non-negative rank of matrix. That

[2] This intuitively makes sense to me but I wouldn't be able to prove it. In the lecture this was handwaved as a consequence of duality.

will in turn be useful for us when we try to bound the extension complexity of some combinatorial problems.

Communication complexity scenario. Let $M_{m\times n}$ be a non-negative matrix. There are two parties: Alice and Bob. Alice gets a row index i and Bob gets a column index j. They communicate and then one of them outputs a number X_{ij}. Their task is to match M_{ij}.

We count the number of bits exchanged. For a given matrix M the *communication complexity* of a communication protocol is the maximum number of bits exchanged over all i, j. The *communication complexity* of the matrix M ($cc(M)$) is the minimum number of bits exchanged over all possible communication protocols.

We will concern ourselves with a variant of this problem where the communication protocol can make decisions base on chance, X_{ij} is a random variable and the goal is $\mathbb{E}[X_{ij}] = M_{ij}\ \forall i, j$. We will denote the communication complexity of this problem as $cc_+(M)$.

Definition (Communication protocol (formally))**.** A communication protocol *is a binary tree with internal nodes labeled Alice/Bob. The leaves represent output. The left downwards edge represents sending the other party a 0 bit, the right downwards edge represents sending the other party a 1 bit.*

For the probabilistic version of the problem each internal node labeled Alice has a probability $p(i) \in [0,1]$ *of sending the 0 bit dependent on the row index i asociated with it and each internal node labeled Bob has a probability* $p(j) \in [0,1]$ *of sending the 0 bit dependent on the column index j asociated with it.*

Theorem. $\log(rk_+(M)) \sim cc_+(M)$

We leave this theorem without a proof for now. The proof will be presented in the next lecture. Instead, we present an example of usage of this theorem.

4 The spanning tree polytope

Let $G = (V, E)$ be a graph. We call the polytope $P_{ST}(G) = \{\chi^{E'} \in \{0,1\}^{|E|} | E' \text{ is a spanning tree of } G\}$ the *spanning tree polytope* of G. Here is a description of the polytope using linear inequalities:

$$\left\{ x \;\middle|\; \begin{array}{cc} \sum_{e\in E} x_e = n-1 & \\ x_e \geq 0 & \forall e \in E \\ \sum_{e\in E[U]} \leq |U|-1 & \forall U \subseteq V \end{array} \right\}$$

The third system of inequalities basically says that each subset of vertices should induce a forest in the spanning tree E'.

How does the slack matrix of this polytope look like? It has one column for each possible spanning tree of G. The rows of the slack matrix corresponding to the first two systems of inequalities are trivial. For the first system we get all zeros. For the second system we get a row for each edge e where positions represent whether e is contained in the spanning tree corresponding to the column.

The part of the matrix corresponding to the third set of inequalities is more interesting. Rows correspond to subsets $U \subseteq V$. Here is a formula for elements of this part of the matrix:

$$S(U,T) = \left(|U| - 1 - \sum_{e\in E[U]} [e \in T]\right) - 1$$

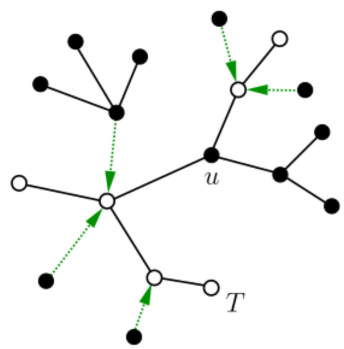
This can be interpreted as (# components in $T[U]$) $- 1$.

4.1 Communication protocol

reference: "basics" paper, section 5.2
Let's bound the extension complexity of the spanning tree polytope by constructing a communication protocol for its slack matrix.

Let $G = (V, E)$ be a graph. $P_{ST}(G)$ is its spanning tree polytope. In terms of the corresponding communication problem, Alice has a proper nonempty set $U \subsetneq V$ and Bob a spanning tree T. Together, they wish to compute $S(U, T)$.

Alice sends the name of some (arbitralily chosen) vertex u of U. Then Bob picks an edge e of T uniformly at random and sends to Alice the endpoints v and w of e as an ordered pair of vertices (v, w), where the order is chosen in such a way that w is on the unique path from v to u in the tree. That is, she makes sure that the directed edge (v, w) "points" towards the root u. Then Alice checks that $v \in U$ and $w \notin U$, in which case she outputs $n - 1$; otherwise she outputs 0.



The resulting randomized protocol is clearly of complexity $\log |V| + \log |E| + O(1)$. Moreover, it computes the slack matrix in expectation because for each connected component of $T[U]$ distinct from that which contains u, there is exactly one directed edge (v, w) that will lead Alice to output a non-zero value. Since she outputs $(n - 1)$ in this case, the expected value of the protocol on pair (U, T) is $(n - 1) \cdot (k - 1)/(n - 1) = k - 1$, where k is the number of connected components of $T[U]$. Therefore we obtain the following result.

Proposition. *For every graph G with n vertices and m edges, $xc(P_{ST}(G)) \in O(mn)$*