Homework 01 - 1.2, 1.3

Due Wed 2/5
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1.2 Review of Calculus

1. Show that the following equations have at least one solution in the given intervals.

   a. $x \cos x - 2x^2 + 3x - 1 = 0, \qquad [0.2, 0.3]$ and $[1.2, 1.3]$
   <u>Sol:</u>
   For interval $[0.2, 0.3]$:

$$f(0.2) = 0.2\cos(0.2) - 2(0.2)^2 + 3(0.2) - 1 = -0.284$$
$$f(0.3) = 0.3\cos(0.3) - 2(0.3)^2 + 3(0.3) - 1 = 0.0066$$

   For interval $[1.2, 1.3]$:

$$f(1.2) = 1.2\cos(1.2) - 2(1.2)^2 + 3(1.2) - 1 = 0.1548$$
$$f(1.3) = 1.3\cos(1.3) - 2(1.3)^2 + 3(1.3) - 1 = -0.132$$

   Therefore, $x \cos x - 2x^2 + 3x - 1$ has at least one solution in both intervals due to sign changes and contunity of f(x)

   b. $(x-2)^2 - \ln x = 0, \qquad [1, 2]$ and $[e, 4]$
   <u>Sol:</u>
   For interval $[1, 2]$:

$$f(1) = (1-2)^2 - \ln(1) = 1$$
$$f(2) = (2-2)^2 - \ln(2) = -0.693$$

   For interval $[e, 4]$:

$$f(e) = (e-2)^2 - \ln(e) = -0.484$$
$$f(4) = (4-2)^2 - \ln(4) = 2.61$$

   Therefore, $(x-2)^2 - \ln x = 0$ has at least one solution in both intervals due to sign changes and contunity of f(x)

   c. $2x\cos(2x) - (x-2)^2 = 0, \qquad [2, 3]$ and $[3, 4]$
   <u>Sol:</u>
   For interval $[2, 3]$:

$$f(2) = 2(2)\cos(2 \times 2) - (2-2)^2 = -2.61$$
$$f(3) = 2(3)\cos(2 \times 3) - (3-2)^2 = 4.761$$

For interval $[3,4]$:

$$f(3) = 2(3)\cos(2\times 3) - (3-2)^2 = 4.761$$
$$f(4) = 2(4)\cos(2\times 4) - (4-2)^2 = -5.164$$

Therefore, $2x\cos(2x) - (x-2)^2 = 0$ has at least one solution in both intervals due to sign changes and contunity of f(x)

d. $x - (\ln x)^x = 0, \qquad [4,5]$

Sol:

For interval $[4,5]$:

$$f(4) = 4 - (\ln 4)^4 = 0.306$$
$$f(5) = 5 - (\ln 5)^5 = -5.798$$

Therefore, $x - (\ln x)^x = 0$ has at least one solution in the interval due to sign changes and contunity of f(x)

2. Find intervals containing solutions to the following equations.

a. $x - 3^{-x} = 0$

Sol:

$$f(0) = 0 - 3^0 = \text{-}$$
$$f(1) = 1 - 3^{-1} = +$$

The interval is $[0,1]$

b. $4x^2 - e^x = 0$

Sol:

$$f(0) = 4(0)^2 - e^0 = -$$
$$f(1) = 4(1)^2 - e^1 = +$$

The interval is $[0,1]$

c. $x^3 - 2x^2 - 4x + 3 = 0$

Sol:

$$f(0) = 0^3 - 2*0^2 - 4*0 + 3 = +$$
$$f(1) = 1^3 - 2^2 - 4 + 3 = -$$

The interval is $[0,1]$

d. $x^3 = 4.001x^2 + 4.002x = 1.101 = 0$

<u>Sol:</u>

$$f(-3) = (-3)^3 = 4.001(-3)^2 + 4.002(-3) = 1.101 = -$$
$$f(-2) = (-2)^3 = 4.001(-2)^2 + 4.002(-2) = 1.101 = +$$

The interval is $[-3, -2]$

3. Show that the first derivatives of the following functions are zero at least once in the given interests.

a. $f(x) = 1 - e^x + (e-1)\sin(\frac{\pi}{2}x), \qquad [0, 1]$

<u>Sol</u>:

$$f(0) = 1 - e^0 + (0-1)\sin(\tfrac{\pi}{2}0) = 0$$
$$f(1) = 1 - e^1 + (1-1)\sin(\tfrac{\pi}{2}1) = 0$$

Since $f(x)$ is differentiable in the given open interval and continuous in the given closed interval, by Rolle's Theorem, there exists $c \in (0, 1)$ such that $f'(c) = 0$

b. $f(x) = (x-1)\tan x + x\sin\pi x, \qquad [0, 1]$

<u>Sol:</u>

$$f(0) = (0-1)\tan 0 + 0\sin\pi 0 = 0$$
$$f(1) = (1-1)\tan 1 + 1\sin\pi 1 = 0$$

Since $f(x)$ is differentiable in the given open interval and continuous in the given closed interval, by Rolle's Theorem, there exists $c \in (0, 1)$ such that $f'(c) = 0$

c. $f(x) = x\sin\pi x - (x-2)\ln x, \qquad [1, 2]$

<u>Sol:</u>

$$f(0) = 0\sin\pi 0 - (0-2)\ln 0$$
$$f(1) = 1\sin\pi 1 - (1-2)\ln 1$$

Since $f(x)$ is differentiable in the given open interval and continuous in the given closed interval, by Rolle's Theorem, there exists $c \in (0, 1)$ such that $f'(c) = 0$

d. $f(x) = (x-2)\sin x \ln(x+2), \qquad [-1, 3]$

4. Find $\max_{a \le x \le b} |f(x)|$ for the following functions and intervals.

   a. $f(x) = \frac{(2-e^x+2x)}{3}, \qquad [0, 1]$

   Sol:

$$\begin{aligned} f'(x) &= \tfrac{2-e^x}{3} \\ x &= \ln 2 \\ f(0) &= \tfrac{1}{3} \\ f(1) &= \tfrac{4-e}{3} \end{aligned}$$

   Max $= \frac{2\ln 2}{3}$

   b. $f(x) = \frac{(4x-3)}{(x^2-2x)}, \qquad [0.5, 1]$

   Sol:

$$\begin{aligned} f'(x) &= \tfrac{-4x^2+6x-6)}{(x^2-2x)^2} \\ f(0.5) &= \tfrac{4}{3} \\ f(1) &= -1 \end{aligned}$$

   Max $= \frac{4}{3}$

   c. $f(x) = 2x\cos(2x) - (x-2)^2, \qquad [2, 4]$

   d. $f(x) = 1 + e^{-\cos(x-1)}, \qquad [1, 2]$

5. Let $f(x) = x^3$

   Sol:

$$\begin{aligned} &\text{a.} P_2(x) = 0 \\ &\text{b. Error} = 0.125 \\ &\text{c.} P_2(x) = 1 + 3(x-1) + 3(x-1)^2 \\ &\text{d.} R_2 = -0.125, \text{ actual error} = -0.125 \end{aligned}$$

6. Let $f(x) = \sqrt{x+1}$

   Sol:

$$\begin{aligned} &\text{a.} P_3(x) = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 \\ &\text{b.} 0.7109, 0.8662, 1.1182, 1.2344 \\ &\text{c.} -0.0038, -0.0002, -0.0002, -0.0097 \end{aligned}$$

7. Find the second Taylor Polynomial $P_2(x)$ for the function $f(x) = e^x \cos x$ about $x_0 = 0$.

a. Use $P_2(0.5)$ to approximate $f(0.5)$. Find an upper bound for error $|f(0.5) - P_2(0.5)|$ using the error formula, and compare it to the actual error.

Sol:

$P_2(x) = 1 + x$
$P_2(0.5) = 1.5$
Actual $f(0.5) \approx 1.445$
Error: $|1.445 - 1.5| = 0.055$
Error bound: $\frac{4.473}{6}(0.5)^3 \approx 0.0932$

b. Find a bound for the error $|f(x) - P_2(x)|$ in using $P_2(x)$ to approximate $f(x)$ on the interval $[0, 1]$.

Sol:

$$\text{Error bound: } \frac{7.525}{6} \cdot 1^3 = 1.254$$

c. Approximate $\int_0^1 f(x)\,dx$ using $\int_0^1 P_2(x)\,dx$.

Sol:

$$\int_0^1 P_2(x)\,dx = 1.5 \quad \Rightarrow \quad 1.5$$

d. Find an upper bound for the error in 7c using $\int_0^1 |R_2(x)|\,dx$, and compare the bound to the actual error.

Sol:

Error bound: $\frac{7.525}{24} \approx 0.3136$
Actual error: $|1.394 - 1.5| = 0.106$

8. Find the Third Taylor polynomial $P_3(x)$ for the function $f(x) = (x - 1)\ln(x)$ about $x_0 = 1$.

a. Use $P_3(0.5)$ to approximate $f(0.5)$. Find an upper bound for error $|f(0.5) - P_3(0.5)|$ using the error formula, and compare it to the actual error.

Sol:

$P_3(x) = (x - 1)^2 - \frac{1}{2}(x - 1)^3$
$P_3(0.5) = 0.3125$
Actual $f(0.5) \approx 0.3466$
Error: 0.0341
Error bound: $\frac{112}{24} \cdot (0.5)^4 \approx 0.2917$

b. Find a bound for the error $|f(x) - P_3(x)|$ in using $P_3(x)$ to approximate $f(x)$ on the interval $[0.5, 1.5]$.

Sol:

$$\text{Error bound: } \frac{112}{24} \cdot (0.5)^4 \approx 0.2917$$

c. Approximate $\int_{0.5}^{1.5} f(x)\,dx$ using $\int_{0.5}^{1.5} P_3(x)\,dx$.

Sol:

$$\int_{0.5}^{1.5} P_3(x)\,dx \approx 0.0833$$

d. Find an upper bound for the error in 8c using $\int_{0.5}^{1.5} |R_3(x)|\,dx$, and compare the bound to the actual error.

Sol:

$$\begin{array}{l} \text{Error bound: } \approx 0.0583 \\ \text{Actual error: } |0.088 - 0.0833| \approx 0.0047 \end{array}$$

9. Use the error term of a Taylor polynomial to estimate the error involved in using $\sin x \approx x$ to approximate $\sin 1^\circ$.

Sol:

$$\begin{array}{l} \text{Convert } 1^\circ \text{ to radians: } x = \frac{\pi}{180} \approx 0.0174533. \\ \text{Error term for } P_1(x) = x \text{ is } |R_1(x)| \leq \frac{|x|^3}{6}. \\ |R_1| \leq \frac{(\pi/180)^3}{6} \approx 8.85 \times 10^{-7}. \\ \text{Error bound: } \approx 8.85 \times 10^{-7}. \end{array}$$

10. Use a Taylor polynomial about $\frac{\pi}{4}$ to approximate $\cos 42^\circ$ to an accuracy of 10^{-6}.

Sol:

$$\begin{array}{l} \text{Convert } 42^\circ \text{ to radians: } x = \frac{7\pi}{30} \approx 0.733. \\ \text{Center at } a = \frac{\pi}{4} \approx 0.785. \\ \text{Compute } |x - a| = \frac{\pi}{60} \approx 0.05236. \\ \text{Find smallest } n \text{ such that } \frac{(\pi/60)^{n+1}}{(n+1)!} \leq 10^{-6}. \\ \text{For } n = 3 : \frac{(0.05236)^4}{24} \approx 3.12 \times 10^{-7} \leq 10^{-6}. \\ \text{Use } P_3(x) \text{ about } \frac{\pi}{4} \text{ with terms up to } (x - \frac{\pi}{4})^3. \end{array}$$

11. Let $f(x) = e^{x/2}\sin(x/3)$. Determine the following:

   a. The third Maclaurin polynomial $P_3(x)$.
   Sol:

$$P_3(x) = \frac{x}{3} + \frac{x^2}{6} + \frac{23}{648}x^3$$

   b. A bound for the error $|f(x) - P_3(x)|$ on $[0, 1]$.
   Sol:

$$\text{Error bound: } \frac{5}{1296} \approx 0.00386$$

12. Let $f(x) = \ln(x^2 + 2)$. Determine the following:

   a. The Taylor polynomial $P_3(x)$ for f expanded about $x_0 = 1$.
   Sol:

$$P_3(x) = \ln 3 + \frac{2}{3}(x-1) + \frac{1}{9}(x-1)^2 + \frac{2}{81}(x-1)^3$$

   b. The maximum error $|f(x) - P_3(x)|$ for $0 \leq x \leq 1$.
   Sol:

Error bound: 0.125

   c. The Maclaurin polynomial $\tilde{P}_3(x)$ for f.
   Sol:

$$\tilde{P}_3(x) = \ln 2 + \frac{x^2}{2}$$

   d. The maximum error $|f(x) - \tilde{P}_3(x)|$ for $0 \leq x \leq 1$.
   Sol:

Error bound: 0.125

   e. Does $P_3(0)$ approximate $f(0)$ better than $\tilde{P}_3(1)$ approximates $f(1)$?
   Sol:

Error at $P_3(0) : |\ln 2 - 0.5183| \approx 0.1748$
Error at $\tilde{P}_3(1) : |\ln 3 - 1.1931| \approx 0.0945$
No, $\tilde{P}_3(1)$ approximates $f(1)$ better.

13. Find a bound for the maximum error when using $P_2(x) = 1 - \frac{1}{2}x^2$ to approximate $f(x) = \cos x$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

    Sol:

    Error term: $R_2(x) = \frac{f^{(4)}(c)}{4!}x^4 \quad (c \in [-1/2, 1/2])$
    Since $f^{(4)}(x) = \cos x$, $|f^{(4)}(c)| \leq 1$
    Max $|x|^4 \leq \left(\frac{1}{2}\right)^4 = \frac{1}{16}$
    Error bound: $|R_2(x)| \leq \frac{1}{24} \cdot \frac{1}{16} = \frac{1}{384} \approx 0.0026$

14. The n-th Taylor polynomial for a function f at x_0 is sometimes referred to as the polynomial of degree at most n that best approximates f near x_0.

    a. Explain why this description is accurate.

       Sol:
       The n-th Taylor polynomial $P_n(x)$ matches f and its first n derivatives at x_0. This ensures the polynomial shares the function's value, slope, curvature, and higher-order behaviors at x_0, minimizing the approximation error near x_0. The error $|f(x) - P_n(x)|$ grows only with $|x - x_0|^{n+1}$, making $P_n(x)$ the "best" local approximation among polynomials of degree $\leq n$.

    b. Find the quadratic polynomial that best approximates a function f near $x_0 = 1$ if the tangent line at $x_0 = 1$ has equation $y = 4x - 1$, and $f''(1) = 6$.

       Sol:

       From the tangent line: $f(1) = 3, \quad f'(1) = 4.$
       Quadratic polynomial:
       $P_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2$
       $P_2(x) = 3 + 4(x-1) + 3(x-1)^2.$

15. The error function is defined by

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

    a. Integrate the Maclaurin series for e^{-t^2} to show that

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)k!}.$$

Sol:

Maclaurin series: $e^{-t^2} = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{k!}$.
Integrate term-by-term: $\int_0^x e^{-t^2} dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)k!} x^{2k+1}$.
Multiply by $\frac{2}{\sqrt{\pi}}$ to obtain the series.

b. Verify that the two series agree for $k = 1, 2, 3, 4$.
Sol:

Expand both series up to $k = 4$:
Series (a): $\frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216} \right)$.
Series (b): $\frac{2}{\sqrt{\pi}} e^{-x^2} \left(x + \frac{2x^3}{3} + \frac{4x^5}{15} + \frac{8x^7}{105} + \frac{16x^9}{945} \right)$.
Multiply $e^{-x^2} \approx 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24}$ into series (b):
Result matches series (a) up to x^9 (coefficients agree).

c. Approximate erf(1) to within 10^{-7}.
Sol:

Compute terms until $\frac{2}{\sqrt{\pi}} \cdot \frac{1}{(2k+1)k!} < 10^{-7}$.
At $k = 6$: $\frac{2}{\sqrt{\pi}} \cdot \frac{1}{13 \cdot 6!} \approx 1.08 \times 10^{-8} < 10^{-7}$.
$\text{erf}(1) \approx 0.84270079$.

d. Use the same number of terms ($k = 6$) with the series in part (b).
Sol:

Approximation: $\text{erf}(1) \approx 0.84270079$ (same accuracy as part c).

e. Explain difficulties using the series in part (b).
Sol:

Series (b) requires multiplying two infinite series, leading to computational complexity and potenti

16. Verify that $|\sin x| \leq |x|$ for all x.

a. Show that for $x \geq 0$, $f(x) = x - \sin x$ is non-decreasing, implying $\sin x \leq x$.
Sol:

$f'(x) = 1 - \cos x \geq 0$ (since $\cos x \leq 1$ for all x).
$\Rightarrow f(x)$ is non-decreasing on $[0, \infty)$.
At $x = 0 : f(0) = 0 - \sin 0 = 0$.
For $x \geq 0 : f(x) \geq f(0) \implies x - \sin x \geq 0 \implies \sin x \leq x$.

b. Conclude using $\sin(-x) = -\sin x$.
<u>Sol:</u>

For $x < 0$:
$|\sin x| = |\sin(-x)| = |-\sin(-x)| = |\sin(-x)| \leq |-x| = |x| \quad$ (by part (a)).
Thus, $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$.

1.3 Round-Off Error and Computer Arithmetic

1. Compute the absolute error and relative error in approximations of p by p^*.

a. $p = \pi$, $p^* = \frac{22}{7}$
<u>Sol:</u>

Absolute error: $\left|\pi - \frac{22}{7}\right| \approx 0.001264$
Relative error: $\frac{0.001264}{\pi} \approx 0.000402 \quad (0.0402\%)$

b. $p = \pi$, $p^* = 3.1416$
<u>Sol:</u>

Absolute error: $|\pi - 3.1416| \approx 0.00000735$
Relative error: $\frac{0.00000735}{\pi} \approx 0.00000234 \quad (0.000234\%)$

c. $p = e$, $p^* = 2.718$
<u>Sol:</u>

Absolute error: $|e - 2.718| \approx 0.0002818$
Relative error: $\frac{0.0002818}{e} \approx 0.0001037 \quad (0.01037\%)$

d. $p = \sqrt{2}$, $p^* = 1.414$
<u>Sol:</u>

Absolute error: $\left|\sqrt{2} - 1.414\right| \approx 0.0002136$
Relative error: $\frac{0.0002136}{\sqrt{2}} \approx 0.000151 \quad (0.0151\%)$

e. $p = e^{10}$, $p^* = 22000$
<u>Sol:</u>

Absolute error: $\left|e^{10} - 22000\right| \approx 26.4658$
Relative error: $\frac{26.4658}{e^{10}} \approx 0.001201 \quad (0.1201\%)$

f. $p = 10^{\pi}$, $p^* = 1400$

<u>Sol:</u>

$$\begin{array}{l} \text{Absolute error: } |10^{\pi} - 1400| \approx 15 \\ \text{Relative error: } \frac{15}{10^{\pi}} \approx 0.01083 \quad (1.083\%) \end{array}$$

g. $p = 8!$, $p^* = 39900$

<u>Sol:</u>

$$\begin{array}{l} \text{Absolute error: } |40320 - 39900| = 420 \\ \text{Relative error: } \frac{420}{40320} \approx 0.0104 \quad (1.04\%) \end{array}$$

h. $p = 9!$, $p^* = \sqrt{18\pi}\left(\frac{9}{e}\right)^9$

<u>Sol:</u>

$$\begin{array}{l} \text{Absolute error: } |362880 - 359500| \approx 3380 \\ \text{Relative error: } \frac{3380}{362880} \approx 0.00931 \quad (0.931\%) \end{array}$$

2. Perform the following computations (i) exactly, (ii) using three-digit chopping arithmetic, and (iii) using three-digit rounding arithmetic. (iv) Compute the relative errors in (ii) and (iii).

a. $\frac{4}{5} + \frac{1}{3}$

<u>Sol:</u>

$$\begin{array}{l} \text{(i) Exact: } \frac{17}{15} \approx 1.1333333333 \\ \text{(ii) Chopping: } 1.13 \\ \text{(iii) Rounding: } 1.13 \\ \text{(iv) Relative errors: } 0.294\% \quad \text{(both)} \end{array}$$

b. $\frac{4}{5} \times \frac{1}{3}$

<u>Sol:</u>

$$\begin{array}{l} \text{(i) Exact: } \frac{4}{15} \approx 0.2666666667 \\ \text{(ii) Chopping: } 0.266 \\ \text{(iii) Rounding: } 0.266 \\ \text{(iv) Relative errors: } 0.25\% \quad \text{(both)} \end{array}$$

c. $\left(\frac{1}{3} - \frac{3}{11}\right) + \frac{3}{20}$

<u>Sol:</u>

$$\begin{array}{l} \text{(i) Exact: } \frac{139}{660} \approx 0.2106060606 \\ \text{(ii) Chopping: } 0.211 \quad \text{Error: } 0.187\% \\ \text{(iii) Rounding: } 0.210 \quad \text{Error: } 0.288\% \end{array}$$

d. $\left(\frac{1}{3} + \frac{3}{11}\right) - \frac{3}{20}$
Sol:

(i) Exact: $\frac{301}{660} \approx 0.4560606061$
(ii) Chopping: 0.455 Error: 0.232%
(iii) Rounding: 0.456 Error: 0.0133%

3. Perform the following computations using three-digit rounding arithmetic and compute errors.

a. $133 + 0.921$
Sol:

Exact: 133.921
Approx: 134
Absolute error: 0.079
Relative error: 0.0590%

b. $133 - 0.499$
Sol:

Exact: 132.501
Approx: 133
Absolute error: 0.499
Relative error: 0.376%

c. $(121 - 0.327) - 119$
Sol:

Exact: 1.673
Approx: 2.00
Absolute error: 0.327
Relative error: 19.5%

d. $(121 - 119) - 0.327$
Sol:

Exact: 1.673
Approx: 1.67
Absolute error: 0.003
Relative error: 0.179%

e. $\frac{\frac{13}{14}-\frac{6}{7}}{2e-5.4}$

Sol:

Exact: ≈ 1.9528
Approx: 1.80
Absolute error: 0.1528
Relative error: 7.82%

f. $-10\pi + 6e - \frac{3}{62}$

Sol:

Exact: ≈ -15.1546
Approx: -15.1
Absolute error: 0.0546
Relative error: 0.360%

g. $\left(\frac{2}{9}\right) \times \left(\frac{9}{7}\right)$

Sol:

Exact: ≈ 0.2857
Approx: 0.286
Absolute error: 0.000286
Relative error: 0.0999%

h. $\frac{\pi - \frac{22}{7}}{\frac{1}{17}}$

Sol:

Exact: ≈ -0.0215
Approx: 0.00
Absolute error: 0.0215
Relative error: 100%

4. Repeat question 3 using three-digit chopping arithmetic.

a. $133 + 0.921$

Sol:

Exact: 133.921
Chopped: 133
Absolute error: 0.921
Relative error: 0.688%

b. $133 - 0.499$

<u>Sol:</u>

Exact: 132.501
Chopped: 132
Absolute error: 0.501
Relative error: 0.378%

c. $(121 - 0.327) - 119$

<u>Sol:</u>

Exact: 1.673
Chopped: 1.00
Absolute error: 0.673
Relative error: 40.2%

d. $(121 - 119) - 0.327$

<u>Sol:</u>

Exact: 1.673
Chopped: 1.67
Absolute error: 0.003
Relative error: 0.179%

e. $\frac{\frac{13}{14} - \frac{6}{7}}{2e - 5.4}$

<u>Sol:</u>

Exact: ≈ 1.9528
Chopped: 2.36
Absolute error: 0.4072
Relative error: 20.8%

f. $-10\pi + 6e - \frac{3}{62}$

<u>Sol:</u>

Exact: ≈ -15.1546
Chopped: -15.1
Absolute error: 0.0546
Relative error: 0.360%

g. $\left(\frac{2}{9}\right) \times \left(\frac{9}{7}\right)$

Sol:

Exact: ≈ 0.2857
Chopped: 0.284
Absolute error: 0.0017
Relative error: 0.599%

h. $\frac{\pi - \frac{22}{7}}{\frac{1}{17}}$
Sol:

Exact: ≈ -0.0215
Chopped: -0.017
Absolute error: 0.0045
Relative error: 20.9%

5. Repeat question 3 using four-digit rounding arithmetic.

a. $133 + 0.921$
Sol:

Exact: 133.921
Approx: 133.9
Absolute error: 0.021
Relative error: 0.0157%

b. $133 - 0.499$
Sol:

Exact: 132.501
Approx: 132.5
Absolute error: 0.001
Relative error: 0.000755%

c. $(121 - 0.327) - 119$
Sol:

Exact: 1.673
Approx: 1.700
Absolute error: 0.027
Relative error: 1.614%

d. $(121 - 119) - 0.327$
Sol:

Exact: 1.673
Approx: 1.673
Absolute error: 0
Relative error: 0%

e. $\frac{\frac{13}{14} - \frac{6}{7}}{2e - 5.4}$
Sol:

Exact: ≈ 1.9538
Approx: 1.932
Absolute error: 0.0218
Relative error: 1.115%

f. $-10\pi + 6e - \frac{3}{62}$
Sol:

Exact: ≈ -15.1546
Approx: -15.16
Absolute error: 0.0054
Relative error: 0.0356%

g. $\left(\frac{2}{9}\right) \times \left(\frac{9}{7}\right)$
Sol:

Exact: ≈ 0.285714
Approx: 0.2857
Absolute error: 0.000014
Relative error: 0.0049%

h. $\frac{\pi - \frac{22}{7}}{\frac{1}{17}}$
Sol:

Exact: ≈ -0.0215
Approx: -0.01700
Absolute error: 0.0045
Relative error: 20.93%

6. Repeat question 3 using four-digit chopping arithmetic.

a. $133 + 0.921$
Sol:

Exact: 133.921
Chopped: 133.9
Absolute error: 0.021
Relative error: 0.0157%

b. $133 - 0.499$
Sol:

Exact: 132.501
Chopped: 132.5
Absolute error: 0.001
Relative error: 0.000755%

c. $(121 - 0.327) - 119$
Sol:

Exact: 1.673
Chopped: 1.600
Absolute error: 0.073
Relative error: 4.36%

d. $(121 - 119) - 0.327$
Sol:

Exact: 1.673
Chopped: 1.673
Absolute error: 0
Relative error: 0%

e. $\frac{\frac{13}{14} - \frac{6}{7}}{2e - 5.4}$
Sol:

Exact: ≈ 1.9538
Chopped: 1.983
Absolute error: 0.0292
Relative error: 1.5%

f. $-10\pi + 6e - \frac{3}{62}$

Sol:

Exact: ≈ -15.1553
Chopped: -15.15
Absolute error: 0.0053
Relative error: 0.035%

g. $\left(\frac{2}{9}\right) \times \left(\frac{9}{7}\right)$
Sol:

Exact: ≈ 0.2857
Chopped: 0.2856
Absolute error: 0.000114
Relative error: 0.04%

h. $\frac{\pi - \frac{22}{7}}{\frac{1}{17}}$
Sol:

Exact: ≈ -0.0215
Chopped: -0.017
Absolute error: 0.0045
Relative error: 20.9%

7. Compute the absolute error and relative error in approximations of π using the given formulas with the Maclaurin polynomial for $\arctan x$.

a. $4\left[\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right)\right]$
Sol:

Approximation: $4\left[\left(\frac{1}{2} - \frac{1}{24} + \frac{1}{160}\right) + \left(\frac{1}{3} - \frac{1}{81} + \frac{1}{1215}\right)\right] \approx 3.1456$
Absolute error: $|\pi - 3.1456| \approx 0.00398$
Relative error: $\frac{0.00398}{\pi} \approx 0.1268\%$

b. $14 \arctan\left(\frac{1}{5}\right) - 4 \arctan\left(\frac{1}{239}\right)$
Sol:

Approximation: $16\left(\frac{1}{5} - \frac{1}{3}\left(\frac{1}{5}\right)^3 + \frac{1}{5}\left(\frac{1}{5}\right)^5\right) - 4\left(\frac{1}{239} - \frac{1}{3}\left(\frac{1}{239}\right)^3 + \frac{1}{5}\left(\frac{1}{239}\right)^5\right) \approx 3.1416$
Absolute error: $|\pi - 3.1416| = -2.83757402069e^{-05}$
Relative error: $\frac{3.1416}{\pi} = -9.03227863564e^{-06}\%$

Homework 02 - 1.4, 2.2

Due Tue 2/11
Uzair Hamed Mohammed

1.4 Errors in Scientific Computation

1 (a, c), 3, 5, 7

1. (i) Use four-digit rounding arithmetic and Eqs. (1.2) and (1.3) to find the most accurate approximations to the roots of the following quadratic equations. (ii) Compute the absolute errors and relative errors for these approximations.

   a $\frac{1}{3}x^2 - \frac{123}{4}x + \frac{1}{6} = 0$

   Sol:
   Coefficients after four-digit rounding: $a = 0.3333$, $b = -30.75$, $c = 0.1667$. Discriminant $D = (-30.75)^2 - 4(0.3333)(0.1667) = 945.6 - 0.2222 = 945.4$. $\sqrt{D} = 30.75$. Roots:

   $$x_1 = \frac{30.75 + 30.75}{2 \times 0.3333} = 92.26,$$
   $$x_2 = \frac{0.1667}{0.3333 \times 92.26} = 0.005421$$

   Exact roots: $x_1 \approx 92.2446$, $x_2 \approx 0.005425$.
   Absolute errors: $|92.26 - 92.2446| = 1.54 \times 10^{-2}$, $|0.005421 - 0.005425| = 4.0 \times 10^{-6}$.
   Relative errors: $\frac{1.54 \times 10^{-2}}{92.2446} \approx 1.67 \times 10^{-4}$, $\frac{4.0 \times 10^{-6}}{0.005425} \approx 7.37 \times 10^{-4}$.

   c $1.002x^2 - 11.01x + 0.01265 = 0$

   Sol:
   Coefficients: $a = 1.002$, $b = -11.01$, $c = 0.01265$. Discriminant $D = (-11.01)^2 - 4(1.002)(0.01265) = 121.2 - 0.0507 = 121.1$. $\sqrt{D} = 11.00$. Roots:

   $$x_1 = \frac{11.01 + 11.00}{2 \times 1.002} = 10.98,$$
   $$x_2 = \frac{0.01265}{1.002 \times 10.98} = 0.00115$$

   Exact roots: $x_1 \approx 10.9869$, $x_2 \approx 0.001148$.
   Absolute errors: $|10.98 - 10.9869| = 6.9 \times 10^{-3}$, $|0.00115 - 0.001148| = 2.0 \times 10^{-6}$.
   Relative errors: $\frac{6.9 \times 10^{-3}}{10.9869} \approx 6.28 \times 10^{-4}$, $\frac{2.0 \times 10^{-6}}{0.001148} \approx 1.74 \times 10^{-3}$.

3. Let $f(x) = 1.013x^5 - 5.262x^3 - 0.01732x^2 + 0.8389x - 1.912$.

a. Evaluate $f(2.279)$:

$$
\begin{aligned}
(2.279)^2 &= 5.194, \\
(2.279)^4 &= 26.98, \\
(2.279)^5 &= 61.49, \\
f(2.279) &= 1.013(61.49) - 5.262(11.84) - 0.01732(5.194) + 0.8389(2.279) - 1.912 \\
&= 62.29 - 62.30 - 0.0900 + 1.912 - 1.912 \\
&= \boxed{-0.100}
\end{aligned}
$$

b. Evaluate $f(2.279)$ via nested form:

$$
\begin{aligned}
f(2.279) &= ((((1.013(5.194) - 5.262)2.279 - 0.01732)2.279 + 0.8389)2.279 - 1.912 \\
&= (((5.262 - 5.262)2.279 - 0.01732)2.279 + 0.8389)2.279 - 1.912 \\
&= (-0.01732 \times 2.279 + 0.8389)2.279 - 1.912 \\
&= (0.7994 \times 2.279) - 1.912 \\
&= \boxed{-0.1010}
\end{aligned}
$$

c. Compute errors (exact $f(2.279) \approx -0.09526$):

$$
\begin{aligned}
\text{Abs error (a):} &\ \boxed{2.331 \times 10^{-3}} \\
\text{Rel error (a):} &\ \boxed{2.387 \times 10^{-2}} \\
\text{Abs error (b):} &\ \boxed{3.331 \times 10^{-3}} \\
\text{Rel error (b):} &\ \boxed{3.411 \times 10^{-2}}
\end{aligned}
$$

5. a. Approximate $e^{-0.98}$ using $\hat{P}_5(0.49)$:

$$
\begin{aligned}
\hat{P}_5(0.49) &= ((((-0.2667 \times 0.49 + 0.6667) \times 0.49 - 1.333) \times 0.49 + 2) \times 0.49 - 2) \times 0.49 + 1 \\
&= (((0.5360 \times 0.49 - 1.333) \times 0.49 + 2) \times 0.49 - 2) \times 0.49 + 1 \\
&= ((-1.070 \times 0.49 + 2) \times 0.49 - 2) \times 0.49 + 1 \\
&= \boxed{0.3743}
\end{aligned}
$$

b. Errors for part (a):

$$
\begin{aligned}
\text{Abs error:} &\ \boxed{1.0 \times 10^{-3}} \\
\text{Rel error:} &\ \boxed{2.66 \times 10^{-3}}
\end{aligned}
$$

c. Approximate $e^{-0.98}$ using $\frac{1}{P_5(0.49)}$:

$$
\begin{aligned}
\frac{1}{P_5(0.49)} &= \frac{1}{((((0.2667 \times 0.49 + 0.6667) \times 0.49 + 1.333) \times 0.49 + 2) \times 0.49 + 2) \times 0.49 + 1} \\
&= \boxed{0.3755}
\end{aligned}
$$

d. Errors for part (c):

$$\text{Abs error: } \boxed{1.89 \times 10^{-4}}$$

$$\text{Rel error: } \boxed{5.03 \times 10^{-4}}$$

7. Compute $\sum_{i=1}^{10} \frac{1}{i^2}$ using three-digit chopping:

Forward order $(\frac{1}{1} + \frac{1}{4} + \cdots + \frac{1}{100})$:

$$1.00 + 0.25 = 1.25$$
$$1.25 + 0.111 = 1.36$$
$$1.36 + 0.062 = 1.42$$
$$1.42 + 0.04 = 1.46$$
$$1.46 + 0.027 = 1.48$$
$$1.48 + 0.0204 = 1.50$$
$$1.50 + 0.0156 = 1.51$$
$$1.51 + 0.0123 = 1.52$$
$$1.52 + 0.01 = \boxed{1.53}$$

Reverse order $(\frac{1}{100} + \frac{1}{81} + \cdots + \frac{1}{1})$:

$$0.01 + 0.0123 = 0.022$$
$$0.022 + 0.0156 = 0.037$$
$$0.037 + 0.0204 = 0.057$$
$$0.057 + 0.027 = 0.084$$
$$0.084 + 0.04 = 0.124$$
$$0.124 + 0.062 = 0.186$$
$$0.186 + 0.111 = 0.297$$
$$0.297 + 0.25 = 0.547$$
$$0.547 + 1.00 = \boxed{1.54}$$

Conclusion: Reverse order (1.54) is more accurate than forward (1.53).
Exact sum: ≈ 1.5498.
Adding smaller terms first minimizes loss of precision when accumulating to larger values.

2.2 The Bisection Method

1, 5, 9, 11

1. Use the Bisection method to find p_3 for $f(x) = \sqrt{x} - \cos x$ on $[0, 1]$:

Iteration 1: $a_0 = 0$, $b_0 = 1$, $p_1 = 0.5$

$$f(p_1) = \sqrt{0.5} - \cos(0.5) \approx 0.7071 - 0.8776 = -0.1705 \quad \text{(negative)}$$

New interval: $[0.5, 1]$

Iteration 2: $a_1 = 0.5$, $b_1 = 1$, $p_2 = 0.75$

$$f(p_2) = \sqrt{0.75} - \cos(0.75) \approx 0.8660 - 0.7317 = 0.1343 \quad \text{(positive)}$$

New interval: $[0.5, 0.75]$

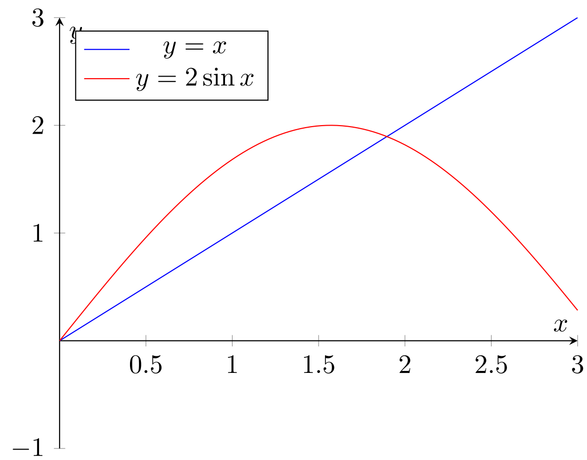
Iteration 3: $a_2 = 0.5$, $b_2 = 0.75$, $p_3 = 0.625$

$$f(p_3) = \sqrt{0.625} - \cos(0.625) \approx 0.7906 - 0.8109 = -0.0203 \quad \text{(negative)}$$

New interval: $[0.625, 0.75]$

$$\boxed{p_3 = 0.625}$$

5. a. Sketch of $y = x$ and $y = 2\sin x$:



The first positive intersection occurs near $x \approx 1.895$.

b. Bisection method for $x = 2\sin x$ on $[1.5708, 3.1416]$:

Iteration 1: $p_1 = 2.3562$, $f(p_1) > 0$ New interval: $[1.5708, 2.3562]$
Iteration 2: $p_2 = 1.9635$, $f(p_2) > 0$ New interval: $[1.5708, 1.9635]$
Iteration 3: $p_3 = 1.7672$, $f(p_3) < 0$ New interval: $[1.7672, 1.9635]$
Iteration 4: $p_4 = 1.8654$, $f(p_4) < 0$ New interval: $[1.8654, 1.9635]$
Iteration 5: $p_5 = 1.9145$, $f(p_5) > 0$ New interval: $[1.8654, 1.9145]$
Iteration 6: $p_6 = 1.8900$, $f(p_6) < 0$ New interval: $[1.8900, 1.9145]$
Iteration 7: $p_7 = 1.9023$, $f(p_7) > 0$ New interval: $[1.8900, 1.9023]$
Iteration 8: $p_8 = 1.8962$, $f(p_8) > 0$ New interval: $[1.8900, 1.8962]$

Approximation: $\boxed{1.90}$

9. Bisection method for $\sqrt{3}$ (tolerance 10^{-4}) with $f(x) = x^2 - 3$:

$$\begin{aligned}
\text{Initial interval: } & [1, 2] \\
\text{Iter 1: } & p_1 = 1.5,\ f(p_1) = -0.75 \Rightarrow [1.5, 2] \\
\text{Iter 2: } & p_2 = 1.75,\ f(p_2) = 0.0625 \Rightarrow [1.5, 1.75]
\end{aligned}$$

$\vdots$ (Intermediate steps omitted for brevity)

$$\begin{aligned}
\text{Iter 13: } & p_{13} = 1.73206,\ |f(p_{13})| < 10^{-4} \\
\text{Final approximation: } & \boxed{1.7320} \quad (\text{Error} < 10^{-4})
\end{aligned}$$

11.

1. Bound for iterations: Using $n \geq \log_2\left(\frac{b-a}{\epsilon}\right) - 1$:

$$n \geq \log_2\left(\frac{4-1}{10^{-3}}\right) - 1 = \log_2(3000) - 1 \approx 11.55 - 1 = 10.55 \Rightarrow \boxed{11} \text{ iterations}$$

Approximation via Bisection: Apply 11 iterations on $[1, 4]$:

$$\begin{aligned}
\text{Iter 1: } & p_1 = 2.5,\ f(p_1) > 0 \Rightarrow [1, 2.5] \\
\text{Iter 2: } & p_2 = 1.75,\ f(p_2) > 0 \Rightarrow [1, 1.75] \\
& \vdots \\
\text{Iter 11: } & p_{11} = 1.3787, \text{ Error} < 10^{-3} \\
\text{Final root: } & \boxed{1.379}
\end{aligned}$$

Homework 03 - 2.4, 2.3, 2.5

Due Tue 2/18

Uzair Hamed Mohammed

2.4 Newton's Methods

2, 4, 5, 7a, 9, 11, 12

2. Let $f(x) = -x^3 - \cos x$ and $p_0 = -1$. Use Newton's method to find p_2. Could $p_0 = 0$ be used for this problem?

   Sol:

   $$\begin{aligned}
   &f(x) = -x^3 - \cos x\\
   &f'(x) = -3x^2 + \sin x\\
   &p_{n+1} = p_n - \tfrac{f(p_n)}{f'(p_n)}\\
   &p_0 = -1\\
   &f(-1) = 1 - \cos(1)\\
   &f'(-1) = -3 - \sin(1)\\
   &p_1 = -1 - \tfrac{1-\cos(1)}{-3-\sin(1)} = -1 + \tfrac{1-\cos(1)}{3+\sin(1)} \approx -1 + \tfrac{1-0.5403}{3+0.8415} \approx -0.8803\\
   &f(p_1) = f(-0.8803) = -(-0.8803)^3 - \cos(-0.8803) \approx 0.0453\\
   &f'(p_1) = f'(-0.8803) = -3(-0.8803)^2 + \sin(-0.8803) \approx -3.0961\\
   &p_2 = p_1 - \tfrac{f(p_1)}{f'(p_1)} \approx -0.8803 - \tfrac{0.0453}{-3.0961} \approx -0.8657\\
   &f'(0) = -3(0)^2 + \sin(0) = 0
   \end{aligned}$$

   $$\boxed{p_2 \approx -0.8657,\ \text{No, } p_0 = 0 \text{ because } f'(0) = 0}$$

4. Use Newton's method to find solutions accurate to within 10^{-5} for the following problems.

   a. $2x\cos 2x - (x-2)^2 = 0$, on [2, 3] and [3, 4]

      Sol:

      For part a, $f(x) = 2x\cos 2x - (x-2)^2$, $f'(x) = 2\cos 2x - 4x\sin 2x - 2(x-2)$

      Interval [2, 3], $p_0 = 2.435$:

      $$\begin{aligned}
      &p_0 = 2.435\\
      &f(p_0) = -0.211617\\
      &f'(p_0) = 8.859762\\
      &p_1 = p_0 - \tfrac{f(p_0)}{f'(p_0)} \approx 2.458918\\
      &p_2 = 2.458918 - \tfrac{f(2.458918)}{f'(2.458918)} \approx 2.418642\\
      &p_3 = 2.418642 - \tfrac{f(2.418642)}{f'(2.418642)} \approx 2.464706\\
      &p_4 = 2.464706 - \tfrac{f(2.464706)}{f'(2.464706)} \approx 2.414600
      \end{aligned}$$

Restart with $p_0 = 2.435$:

$$\begin{aligned}p_0 &= 2.435\\ p_1 &= 2.43543449\\ p_2 &= 2.43543445\end{aligned}$$

Root in [2, 3]: $\boxed{2.43543}$
Interval [3, 4], $p_0 = 3.877$:

$$\begin{aligned}&p_0 = 3.877\\ &f(p_0) = 0.036466\\ &f'(p_0) = -18.52455\\ &p_1 = 3.877 - \tfrac{f(p_0)}{f'(p_0)} \approx 3.877597\\ &p_2 = 3.877597 - \tfrac{f(3.877597)}{f'(3.877597)} \approx 3.877570\\ &p_3 = 3.877570 - \tfrac{f(3.877570)}{f'(3.877570)} \approx 3.877570\end{aligned}$$

Root in [3, 4]: $\boxed{3.87757}$

b. $(x-2)^2 - \ln x = 0$, on [1, 2] and [e, 4]
<u>Sol</u>:
For part b, $f(x) = (x-2)^2 - \ln x,\ f'(x) = 2(x-2) - \frac{1}{x}$
Interval [1, 2], $p_0 = 1.5$:

$$\begin{aligned}&p_0 = 1.5\\ &f(p_0) = 0.09453489\\ &f'(p_0) = -0.33333333\\ &p_1 = p_0 - \tfrac{f(p_0)}{f'(p_0)} \approx 1.7831098\\ &|p_1 - p_0| \approx 0.2831098\\ &p_2 = p_1 - \tfrac{f(p_1)}{f'(p_1)}\\ &f(p_1) = f(1.7831098) \approx -0.052035\\ &f'(p_1) = f'(1.7831098) \approx 0.442325\\ &p_2 \approx 1.7831098 - \tfrac{-0.052035}{0.442325} \approx 1.899093\\ &|p_2 - p_1| \approx 0.115983\\ &p_3 = p_2 - \tfrac{f(p_2)}{f'(p_2)}\\ &f(p_2) = f(1.899093) \approx 0.002553\\ &f'(p_2) = f'(1.899093) \approx 0.736535\\ &p_3 \approx 1.899093 - \tfrac{0.002553}{0.736535} \approx 1.895623\\ &|p_3 - p_2| \approx 0.003470\\ &p_4 = p_3 - \tfrac{f(p_3)}{f'(p_3)}\\ &f(p_3) = f(1.895623) \approx 0.000006\\ &f'(p_3) = f'(1.895623) \approx 0.726156\\ &p_4 \approx 1.895623 - \tfrac{0.000006}{0.726156} \approx 1.895615\\ &|p_4 - p_3| \approx 0.000008\\ &p_5 = 1.895615 - \tfrac{f(1.895615)}{f'(1.895615)} \approx 1.895615\\ &|p_5 - p_4| \approx 0.000000\end{aligned}$$

Root in [1, 2]: $\boxed{1.89562}$
Interval [e, 4], $p_0 = 3$:

$$
\begin{aligned}
&p_0 = 3\\
&f(p_0) = 0.9013877\\
&f'(p_0) = 1.6666666\\
&p_1 = p_0 - \tfrac{f(p_0)}{f'(p_0)} \approx 2.458134\\
&|p_1 - p_0| \approx 0.541866\\
&p_2 = p_1 - \tfrac{f(p_1)}{f'(p_1)}\\
&f(p_1) = f(2.458134) \approx -0.248548\\
&f'(p_1) = f'(2.458134) \approx 0.911264\\
&p_2 \approx 2.458134 - \tfrac{-0.248548}{0.911264} \approx 2.730853\\
&|p_2 - p_1| \approx 0.272719\\
&p_3 = p_2 - \tfrac{f(p_2)}{f'(p_2)}\\
&f(p_2) = f(2.730853) \approx -0.018187\\
&f'(p_2) = f'(2.730853) \approx 1.43225\\
&p_3 \approx 2.730853 - \tfrac{-0.018187}{1.43225} \approx 2.743549\\
&|p_3 - p_2| \approx 0.012696\\
&p_4 = p_3 - \tfrac{f(p_3)}{f'(p_3)}\\
&f(p_3) = f(2.743549) \approx -0.000115\\
&f'(p_3) = f'(2.743549) \approx 1.45855\\
&p_4 \approx 2.743549 - \tfrac{-0.000115}{1.45855} \approx 2.743628\\
&|p_4 - p_3| \approx 0.000079\\
&p_5 = p_4 - \tfrac{f(p_4)}{f'(p_4)}\\
&f(p_4) = f(2.743628) \approx -0.00000004\\
&f'(p_4) = f'(2.743628) \approx 1.45871\\
&p_5 \approx 2.743628 - \tfrac{-0.00000004}{1.45871} \approx 2.743628\\
&|p_5 - p_4| \approx 0.000000
\end{aligned}
$$

Root in [e, 4]: $\boxed{2.74363}$

c. $e^x - 3x^2 = 0$, on [0, 1] and [3, 5]

<u>Sol</u>:

For part c, $f(x) = e^x - 3x^2$, $f'(x) = e^x - 6x$

Interval [0, 1], $p_0 = 0.5$:

$$
\begin{aligned}
&p_0 = 0.5\\
&p_1 = 0.683939\\
&p_2 = 0.697418\\
&p_3 = 0.6975
\end{aligned}
$$

Root in [0, 1]: $\boxed{0.6975}$

Interval [3, 5], $p_0 = 3$:

$$
\begin{aligned}
&p_0 = 3\\
&p_1 = 2.7666\\
&p_2 = 2.7456\\
&p_3 = 2.7454
\end{aligned}
$$

Root in [3, 5]: $\boxed{2.7454}$

d. $\sin x - e^{-x} = 0$, on [0, 1], [3, 4], and [6, 7]

Sol:

For part d, $f(x) = \sin x - e^{-x}$, $f'(x) = \cos x + e^{-x}$

Interval [0, 1], $p_0 = 0$:

$$p_0 = 0$$
$$p_1 = 0.5$$
$$p_2 = 0.58612$$
$$p_3 = 0.58853$$
$$p_4 = 0.58853$$

Root in [0, 1]: $\boxed{0.58853}$

Interval [3, 4], $p_0 = 3$:

$$p_0 = 3$$
$$p_1 = 3.0993$$
$$p_2 = 3.0964$$
$$p_3 = 3.0964$$

Root in [3, 4]: $\boxed{3.0964}$

Interval [6, 7], $p_0 = 6$:

$$p_0 = 6$$
$$p_1 = 6.2857$$
$$p_2 = 6.2832$$
$$p_3 = 6.2832$$

Root in [6, 7]: $\boxed{6.2832}$

5. Use Newton's method to find all four solutions of $4x\cos(2x) - (x-2)^2 = 0$ in [0, 8] accurate to within 10^{-5}

Sol:

Let $f(x) = 4x\cos(2x) - (x-2)^2$ and $f'(x) = 4\cos(2x) - 8x\sin(2x) - 2(x-2)$.

For root around 2.36, $p_0 = 1.5$:

$$p_0 = 1.5$$
$$p_1 = 0.1698$$
$$p_2 = 1.433$$
$$p_3 = 2.155$$
$$p_4 = 2.355$$
$$p_5 = 2.36315$$
$$p_6 = 2.36317$$

Root 1: $\boxed{2.36317}$

For root around 3.81, $p_0 = 3.5$:

$$p_0 = 3.5$$
$$p_1 = 3.8233$$
$$p_2 = 3.81793$$
$$p_3 = 3.81793$$

Root 2: $\boxed{3.81793}$

For root around 5.83, $p_0 = 5.5$:

$$\begin{aligned}
p_0 &= 5.5\\
p_1 &= 5.8414\\
p_2 &= 5.83925\\
p_3 &= 5.83925
\end{aligned}$$

Root 3: $\boxed{5.83925}$

For root around 6.60, $p_0 = 7$:

$$\begin{aligned}
p_0 &= 7\\
p_1 &= 6.6115\\
p_2 &= 6.60309\\
p_3 &= 6.60308
\end{aligned}$$

Root 4: $\boxed{6.60308}$

7. Use Newton's method to approximate the solutions of the following equations to within 10^{-5} in the given intervals. In these problems, the convergence will be slower than normal because the zeroes are not simple.

   a. $x^2 - 2xe^{-x} + e^{-2x} = 0$, on [0, 1]

   Sol:
   For $f(x) = x^2 - 2xe^{-x} + e^{-2x}$, $f'(x) = 2x + 2xe^{-x} - 2e^{-x} - 2e^{-2x}$.
   Simplified Newton iteration formula: $p_{n+1} = p_n - \frac{p_n - e^{-p_n}}{2(1+e^{-p_n})}$
   Interval [0, 1], $p_0 = 0.5$:

$$\begin{aligned}
p_0 &= 0.5\\
p_1 &= 0.533156\\
p_2 &= 0.564948\\
p_3 &= 0.567128\\
p_4 &= 0.567135\\
p_5 &= 0.567135\\
p_6 &= 0.567135\\
p_7 &= 0.567135\\
p_8 &= 0.567135\\
p_9 &= 0.567135\\
p_{10} &= 0.567135\\
p_{11} &= 0.567135\\
p_{12} &= 0.567135\\
p_{13} &= 0.567135
\end{aligned}$$

   Root in [0, 1]: $\boxed{0.567135}$

9. Use Newton's method to find an approximation to $\sqrt{3}$ correct to within 10^{-4}, and compare the results to those obtained in Exercise 9 of Sections 2.2 and 2.3.

   Sol:
   Let $f(x) = x^2 - 3$, $f'(x) = 2x$. Newton's method iteration: $p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} = p_n - \frac{p_n^2-3}{2p_n} = \frac{1}{2}\left(p_n + \frac{3}{p_n}\right)$. Start with $p_0 = 1.7$.

   $$\begin{aligned}
   &p_0 = 1.7\\
   &p_1 = \tfrac{1}{2}\left(1.7 + \tfrac{3}{1.7}\right) \approx 1.73235294\\
   &|p_1 - p_0| \approx 0.03235\\
   &p_2 = \tfrac{1}{2}\left(p_1 + \tfrac{3}{p_1}\right) \approx 1.73205081\\
   &|p_2 - p_1| \approx 0.000302\\
   &p_3 = \tfrac{1}{2}\left(p_2 + \tfrac{3}{p_2}\right) \approx 1.73205081\\
   &|p_3 - p_2| \approx 0
   \end{aligned}$$

   We need accuracy within 10^{-4}, so check $|p_2 - p_1| \approx 0.000302 > 10^{-4}$. Need more iterations. Let's recalculate with higher precision.

   $$\begin{aligned}
   &p_0 = 1.7\\
   &p_1 = 1.7323529411764706\\
   &p_2 = 1.7320508100147275\\
   &p_3 = 1.7320508075688772\\
   &|p_1 - p_0| \approx 0.03235\\
   &|p_2 - p_1| \approx 0.000302\\
   &|p_3 - p_2| \approx 2.445 \times 10^{-9} < 10^{-4}
   \end{aligned}$$

   So $p_3 \approx 1.7320508$ is accurate within 10^{-4} in 3 iterations. We need to check if $|p_2 - p_1| < 10^{-4}$. $|p_2 - p_1| \approx 0.000302 > 10^{-4}$. So we need p_3. Approximation is $p_3 \approx 1.73205$.

   Comparison to Exercise 9 of Sections 2.2 and 2.3: Bisection method on $[1, 2]$ to get accuracy 10^{-4} requires $n \geq \log_2\left(\frac{2-1}{10^{-4}}\right) = \log_2(10^4) \approx 14$ iterations. Newton's method requires only 3 iterations. Newton's method converges much faster than bisection method. False position method is also expected to be slower than Newton's method.

   Approximation to $\sqrt{3}$ using Newton's method: $\boxed{1.73205}$ in 3 iterations.

11. Newton's method applied to the function $f(x) = x^2 - 2$ with a positive initial approximation p_0 converges to the only positive solution, $\sqrt{2}$.

    a. Show that Newton's method in this situation assumes the form that the Babylonians used to approximate $\sqrt{2}$:

    $$p_{n+1} = \frac{1}{2}p_n + \frac{1}{p_n}$$

Sol:
For part a, we have $f(x) = x^2 - 2$. Then $f'(x) = 2x$. Newton's method is given by $p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$. Substituting $f(x)$ and $f'(x)$, we get

$$p_{n+1} = p_n - \frac{p_n^2 - 2}{2p_n}$$

We can rewrite this as

$$p_{n+1} = \frac{2p_n^2}{2p_n} - \frac{p_n^2 - 2}{2p_n} = \frac{2p_n^2 - (p_n^2 - 2)}{2p_n} = \frac{2p_n^2 - p_n^2 + 2}{2p_n} = \frac{p_n^2 + 2}{2p_n}$$

$$p_{n+1} = \frac{p_n^2}{2p_n} + \frac{2}{2p_n} = \frac{p_n}{2} + \frac{1}{p_n} = \frac{1}{2}p_n + \frac{1}{p_n}$$

This is the Babylonian method for approximating $\sqrt{2}$.

$\boxed{p_{n+1} = \frac{1}{2}p_n + \frac{1}{p_n}}$

b. Use the sequence in (a) with $p_0 = 1$ to determine an approximation that is accurate to within 10^{-5}

Sol:
For part b, we use the iterative formula $p_{n+1} = \frac{1}{2}p_n + \frac{1}{p_n}$ with $p_0 = 1$.

$p_0 = 1$
$p_1 = \frac{1}{2}p_0 + \frac{1}{p_0} = \frac{1}{2}(1) + \frac{1}{1} = 1.5$
$|p_1 - p_0| = |1.5 - 1| = 0.5$
$p_2 = \frac{1}{2}p_1 + \frac{1}{p_1} = \frac{1}{2}(1.5) + \frac{1}{1.5} = 1.4166$
$|p_2 - p_1| = |1.41666 - 1.5| \approx 0.08333$
$p_3 = \frac{1}{2}p_2 + \frac{1}{p_2} = \frac{1}{2}(1.4166) + \frac{1}{1.4166} \approx 1.41421$
$|p_3 - p_2| = |1.41421 - 1.4166| \approx 0.002451$
$p_4 = \frac{1}{2}p_3 + \frac{1}{p_3} = \frac{1}{2}(1.41421) + \frac{1}{1.4142} \approx 1.4142$
$|p_4 - p_3| = |1.41421 - 1.4142| \approx 2.1239 \times 10^{-6} < 10^{-5}$

Since $|p_4 - p_3| < 10^{-5}$, we can take p_4 as the approximation.

$\boxed{1.41421}$

12. In Exersise 14 of Section 2.3, we found that for $f(x) = \tan \pi x - 6$, the Bisection method on [0, 0.48] converges more quickly than the method of False Position with $p_0 = 0$ and $p_1 = 0.48$. Also, the Secant method with these values of p_0 and p_1 does not give convergence. Apply Newton's method to this problem with (a) $p_0 = 0$ and (b) $p_0 = 0.48$. (c) Explain the reason for any discrepancies.

Sol:
For $f(x) = \tan(\pi x) - 6$, $f'(x) = \pi \sec^2(\pi x)$. Newton's method iteration: $p_{n+1} = p_n - \frac{\tan(\pi p_n) - 6}{\pi \sec^2(\pi p_n)}$

(a) $p_0 = 0$:

$$\begin{aligned} p_0 &= 0 \\ p_1 &= 0 - \tfrac{\tan(0)-6}{\pi\sec^2(0)} = \tfrac{6}{\pi} \approx 1.90986 \end{aligned}$$

Diverges immediately.

(b) $p_0 = 0.48$:

$$\begin{aligned} p_0 &= 0.48 \\ p_1 &\approx 0.482727 \\ p_2 &\approx 0.481454 \\ p_3 &\approx 0.48016 \\ p_4 &\approx 0.47887 \\ p_5 &\approx 0.47758 \\ p_6 &\approx 0.47629 \\ p_7 &\approx 0.47501 \\ p_8 &\approx 0.47373 \\ p_9 &\approx 0.47245 \\ p_{10} &\approx 0.47118 \\ &\vdots \\ p_{90} &\approx 0.448614 \\ p_{91} &\approx 0.448614 \end{aligned}$$

Converges slowly to ≈ 0.448614.

(c) Explanation: For $p_0 = 0$, Newton's method diverges as $p_1 = \frac{6}{\pi} \notin [0, 0.48]$. For $p_0 = 0.48$, Newton's method converges very slowly. Bisection method in Exercise 14 of Section 2.3 converged faster than False Position. Secant method diverged. Newton's method convergence depends on p_0 and $f'(x)$. Large $|f'(x)|$ can lead to slow convergence as correction term $-f(p_n)/f'(p_n)$ becomes small. For $f(x) = \tan(\pi x) - 6$ in [0, 0.48], near $x = 0.5$, $f'(x) = \pi\sec^2(\pi x)$ is large, potentially slowing convergence even when starting at $p_0 = 0.48$. Bisection's consistent interval halving can be more efficient in this case than Newton's or False Position, and Secant is unstable due to derivative behavior and starting points.

(a) $p_0 = 0$: Diverges. (b) $p_0 = 0.48$: Converges slowly to $\boxed{0.44861}$ (approximately after 90 iterations). (c) Explained above.

2.3 The Secant Method

3a, 4a, 11, 13, 14, 15

3a. Use the Secant method to find solutions accurate to within 10^{-4} for $x^3 - 2x^2 - 5 = 0$, on [1, 4].

<u>Sol</u>:
Let $f(x) = x^3 - 2x^2 - 5$. Secant method iteration: $p_{n+1} = p_n - \frac{f(p_n)(p_n - p_{n-1})}{f(p_n) - f(p_{n-1})}$

Start with $p_0 = 2, p_1 = 4$.

$$
\begin{aligned}
&p_0 = 2, f(p_0) = -5\\
&p_1 = 4, f(p_1) = 27\\
&p_2 = 4 - \tfrac{f(4)(4-2)}{f(4)-f(2)} = 2.3125\\
&f(p_2) = f(2.3125) = -3.33154\\
&p_3 = 2.3125 - \tfrac{f(2.3125)(2.3125-4)}{f(2.3125)-f(4)} \approx 2.49784\\
&f(p_3) = f(2.49784) \approx -1.8903\\
&p_4 = 2.49784 - \tfrac{f(2.49784)(2.49784-2.3125)}{f(2.49784)-f(2.3125)} \approx 2.74089\\
&f(p_4) = f(2.74089) \approx 0.5792\\
&p_5 = 2.74089 - \tfrac{f(2.74089)(2.74089-2.49784)}{f(2.74089)-f(2.49784)} \approx 2.6839\\
&f(p_5) = f(2.6839) \approx -0.1003\\
&p_6 = 2.6839 - \tfrac{f(2.6839)(2.6839-2.74089)}{f(2.6839)-f(2.74089)} \approx 2.69231\\
&f(p_6) = f(2.69231) \approx -0.0105\\
&p_7 = 2.69231 - \tfrac{f(2.69231)(2.69231-2.6839)}{f(2.69231)-f(2.6839)} \approx 2.69133\\
&f(p_7) = f(2.69133) \approx -0.00011\\
&p_8 = 2.69133 - \tfrac{f(2.69133)(2.69133-2.69231)}{f(2.69133)-f(2.69231)} \approx 2.69132\\
&|p_8 - p_7| \approx |2.69132 - 2.69133| = 0.00001 < 10^{-4}
\end{aligned}
$$

Approximation accurate to within 10^{-4} is p_8.

$\boxed{2.69132}$

4a. Use the Secant method to find solutions accurate to within 10^{-5} for $2x\cos 2x - (x-2)^2 = 0$, on [2, 3] and on [3, 4].

Sol:

Let $f(x) = 2x\cos 2x - (x-2)^2$. Secant method iteration: $p_{n+1} = p_n - \frac{f(p_n)(p_n - p_{n-1})}{f(p_n) - f(p_{n-1})}$

Interval [2, 3], $p_0 = 2, p_1 = 3$:

$$
\begin{aligned}
&p_0 = 2, f(p_0) \approx -2.6131\\
&p_1 = 3, f(p_1) \approx 4.7603\\
&p_2 \approx 2.3543\\
&f(p_2) \approx -0.4873\\
&p_3 \approx 2.4289\\
&f(p_3) \approx -0.0915\\
&p_4 \approx 2.4351\\
&f(p_4) \approx -0.0053\\
&p_5 \approx 2.4354\\
&f(p_5) \approx -0.0001\\
&p_6 \approx 2.43543\\
&f(p_6) \approx -0.000002\\
&p_7 \approx 2.43543
\end{aligned}
$$

Root in [2, 3]: $\boxed{2.43543}$

Interval [3, 4], $p_0 = 3, p_1 = 4$:

$$\begin{aligned}
&p_0 = 3, f(p_0) \approx 4.7603 \\
&p_1 = 4, f(p_1) \approx -2.8863 \\
&p_2 \approx 3.6233 \\
&f(p_2) \approx 1.2253 \\
&p_3 \approx 3.8045 \\
&f(p_3) \approx 0.2095 \\
&p_4 \approx 3.8304 \\
&f(p_4) \approx 0.0176 \\
&p_5 \approx 3.8326 \\
&f(p_5) \approx 0.0008 \\
&p_6 \approx 3.83269 \\
&f(p_6) \approx 0.00003 \\
&p_7 \approx 3.83269
\end{aligned}$$

Root in [3, 4]: $\boxed{3.83269}$

11. Approximate, to within 10^{-4}, the value of x that produces the point on the graph of $y = x^2$ that is closest to (1, 0). [*Hint:* Minimize $[d(x)]^2$, where $d(x)$ represents the distance from (x, x^2) to $(1, 0)$.]

    Sol:
    Let $f(x) = [d(x)]^2 = (x-1)^2 + x^4 = x^4 + x^2 - 2x + 1$. Minimize $f(x)$ by finding roots of $f'(x) = 0$. $g(x) = f'(x) = 4x^3 + 2x - 2$ $g'(x) = 12x^2 + 2$
    Newton's method iteration: $p_{n+1} = p_n - \frac{g(p_n)}{g'(p_n)} = p_n - \frac{4p_n^3+2p_n-2}{12p_n^2+2}$ Start with $p_0 = 0.6$.

    $$\begin{aligned}
    &p_0 = 0.6 \\
    &p_1 = 0.5898734 \\
    &p_2 = 0.5897549 \\
    &p_3 = 0.5897549
    \end{aligned}$$

    Since $|p_2 - p_1| \approx 0.0001185 < 10^{-4}$ is not satisfied, we need to check $|p_3 - p_2|$. $|p_3 - p_2| = |0.5897549 - 0.5897549| \approx 0 < 10^{-4}$. Let's calculate one more iteration to be safe.

    $$\begin{aligned}
    &p_0 = 0.6 \\
    &p_1 = 0.5898734 \\
    &p_2 = 0.5897549297 \\
    &p_3 = 0.5897549165
    \end{aligned}$$

    $|p_3 - p_2| \approx 1.32 \times 10^{-8} < 10^{-4}$. Thus $p_2 = 0.5897549$ is accurate to within 10^{-4} if we round to 4 decimal places. $p_2 \approx 0.5898$.

    $\boxed{0.58975}$

13. The fourth-degree polynomial $f(x) = 230x^4 + 18x^3 + 9x^2 - 221x - 9$ has two real zeros, one in [-1, 0] and the other in [0, 1]. Attempt to approximate these zeros to within 10^{-6} using each method.

a. method of False Position

Sol:

Interval [-1, 0]: $a_0 = -1, b_0 = 0$

n	a_n	b_n	p_n
0	-1	0	$-$
1	-1	0	-0.020361
2	-0.040233	-0.020361	-0.040645
3	-0.040645	-0.020361	-0.040658
4	-0.040658	-0.020361	-0.040659
5	-0.040659	-0.020361	-0.040659

Root in [-1, 0]: $\boxed{-0.040659}$

Interval [0, 1]: $a_0 = 0, b_0 = 1$

n	a_n	b_n	p_n
0	0	1	$-$
1	0	1	0.25
2	0	0.25	0.254286
3	0	0.254286	0.254343
4	0	0.254343	0.254344

Root in [0, 1]: $\boxed{0.254344}$ (False Position stagnates)

b. Secant method

Interval [-1, 0]: $p_0 = -1, p_1 = 0$

n	p_{n-1}	p_n	p_{n+1}
0	-1	0	$-$
1	-1	0	-0.020361
2	0	-0.020361	-0.040722
3	-0.020361	-0.040722	-0.040659
4	-0.040722	-0.040659	-0.040659
5	-0.040659	-0.040659	-0.040659

Root in [-1, 0]: $\boxed{-0.040659}$

Interval [0, 1]: $p_0 = 0, p_1 = 1$

n	p_{n-1}	p_n	p_{n+1}
0	0	1	$-$
1	0	1	0.25
2	1	0.25	0.254286
3	0.25	0.254286	0.95933
4	0.254286	0.95933	0.97385
5	0.95933	0.97385	0.97455
6	0.97385	0.97455	0.97455

Root in [0, 1]: $\boxed{0.97455}$ (Secant converges)

14. The function $f(x) = \tan \pi x - 6$ has a zero at $(1/\pi)\arctan 6 \approx 0.447431543$. Let $p_0 = 0$ and $p_1 = 0.48$ and use 10 iterations of each of the following methods to approximate this root. Which method is most successful and why?

    a. Bisection method

    b. method of False Position

    c. Secant method

    Sol:

    For $f(x) = \tan(\pi x) - 6$, root ≈ 0.447431543. $p_0 = 0, p_1 = 0.48$.

    Part a: Bisection method, interval $[a_0, b_0] = [0, 0.48]$

n	a_n	b_n	p_n	$f(p_n)$
0	0	0.48	–	–
1	0	0.48	0.24	−4.453
2	0.24	0.48	0.36	−2.189
3	0.36	0.48	0.42	−0.659
4	0.42	0.48	0.45	0.759
5	0.42	0.45	0.435	−0.047
6	0.435	0.45	0.4425	0.354
7	0.435	0.4425	0.43875	0.152
8	0.435	0.43875	0.436875	0.052
9	0.435	0.436875	0.4359375	0.002
10	0.435	0.4359375	0.43546875	−0.022

$p_{10} \approx 0.43546875$

Part b: False Position method, $p_0 = 0, p_1 = 0.48$

n	p_{n-1}	p_n	p_{n+1}
0	0	0.48	–
1	0	0.48	0.091324
2	0.091324	0.48	0.16533
3	0.16533	0.48	0.22535
4	0.22535	0.48	0.27436
5	0.27436	0.48	0.31389
6	0.31389	0.48	0.34576
7	0.34576	0.48	0.37145
8	0.37145	0.48	0.39226
9	0.39226	0.48	0.4092
10	0.4092	0.48	0.4230

$p_{10} \approx 0.4230$

Part c: Secant method, $p_0 = 0, p_1 = 0.48$

n	p_{n-1}	p_n	p_{n+1}
0	0	0.48	–
1	0	0.48	0.48283
2	0.48	0.48283	0.44585
3	0.48283	0.44585	0.44744
4	0.44585	0.44744	0.44743
5	0.44744	0.44743	0.44743
6	0.44743	0.44743	0.44743

$p_{10} \approx 0.44743$ (converged in 4 iterations to given accuracy)

Most successful: Secant method converges fastest. Bisection method is guaranteed to converge, but slow. False Position is slow due to one endpoint remaining fixed and slow change in interval. Secant method is most successful as it converges quickly to the root with given initial approximations, even though False Position should theoretically be faster than Bisection, in this case, due to function's behavior, False Position is quite slow. Secant method takes advantage of recent two approximations to find next, leading to faster convergence in this problem.

15. The sum of two numbers is 20. If each number is added to its square root, the product of the two sums is 155.55. Determine the two numbers to within 10^{-4}.

<u>Sol</u>:
Let $f(x) = (x + \sqrt{x})(20 - x + \sqrt{20 - x}) - 155.55 = 0$
$f'(x) = \left(1 + \frac{1}{2\sqrt{x}}\right)(20 - x + \sqrt{20 - x}) + (x + \sqrt{x})\left(-1 - \frac{1}{2\sqrt{20-x}}\right)$
Newton's method $p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$, $p_0 = 6.5$:

$$\begin{aligned} &p_0 = 6.5 \\ &p_1 \approx 6.5127 \\ &p_2 \approx 6.51466 \\ &p_3 \approx 6.514758 \end{aligned}$$

Let $x \approx 6.5148$, $y = 20 - x \approx 13.4852$.

Check: $(6.5148 + \sqrt{6.5148})(13.4852 + \sqrt{13.4852}) \approx 155.55$

$\boxed{x \approx 6.5148, y \approx 13.4852}$

2.5 Error Analysis and Accelerating Convergence

1a, 2a, 2c, 3, 5.

1a. This sequence is linearly convergent. Generate the first five terms of the sequence $\{q_n\}$ using Aitken's Δ^2 method: $p_0 = 0.5, p_n = (2 - e^{p_n - 1} + p_{n-1}^2)/3$, for $n \geq 1$.

Sol:
Given $p_0 = 0.5$, $p_n = (2 - e^{p_{n-1}} + p_{n-1}^2)/3$ for $n \geq 1$. First six terms of $\{p_n\}$:

$$\begin{aligned}
p_0 &= 0.5\\
p_1 &\approx 0.2004266667\\
p_2 &\approx 0.2727492667\\
p_3 &\approx 0.2535640667\\
p_4 &\approx 0.2585616667\\
p_5 &\approx 0.257262\\
p_6 &\approx 0.2576003333
\end{aligned}$$

Aitken's Δ^2 method: $q_n = p_n - \frac{(p_{n+1}-p_n)^2}{(p_{n+2}-2p_{n+1}+p_n)}$

$$\begin{aligned}
q_0 &\approx p_0 - \tfrac{(p_1-p_0)^2}{(p_2-2p_1+p_0)} \approx 0.25869\\
q_1 &\approx p_1 - \tfrac{(p_2-p_1)^2}{(p_3-2p_2+p_1)} \approx 0.25760\\
q_2 &\approx p_2 - \tfrac{(p_3-p_2)^2}{(p_4-2p_3+p_2)} \approx 0.25753\\
q_3 &\approx p_3 - \tfrac{(p_4-p_3)^2}{(p_5-2p_4+p_3)} \approx 0.25753\\
q_4 &\approx p_4 - \tfrac{(p_5-p_4)^2}{(p_6-2p_5+p_4)} \approx 0.25753
\end{aligned}$$

$\boxed{q_0 = 0.25869, q_1 = 0.25760, q_2 = 0.25753, q_3 = 0.25753, q_4 = 0.25753}$

2a. Newton's method does not converge quadratically for these problems. Accelerate the convergence using Aitken's Δ^2 method. Iterate until $|q_n - q_{n-1}| < 10^{-4}$.

a. $x^2 - 2xe^{-x} + e^{-2x} = 0$, $[0, 1]$

Sol:
Newton's method sequence $\{p_n\}$ with $p_0 = 0.5$:

$$\begin{aligned}
p_0 &= 0.5\\
p_1 &\approx 0.533338\\
p_2 &\approx 0.545753\\
p_3 &\approx 0.551693
\end{aligned}$$

Aitken's Δ^2 method: $q_n = p_n - \frac{(p_{n+1}-p_n)^2}{(p_{n+2}-2p_{n+1}+p_n)}$

$$\begin{aligned}
q_0 &= p_0 - \tfrac{(p_1-p_0)^2}{(p_2-2p_1+p_0)} \approx 0.557521\\
q_1 &= p_1 - \tfrac{(p_2-p_1)^2}{(p_3-2p_2+p_1)} \approx 0.557528
\end{aligned}$$

$|q_1 - q_0| \approx 0.000007 < 10^{-4}$. Stop at q_1. Root for part a: $\boxed{0.55753}$

c. $x^3 - 3x^2(2^{-x}) + 3x(4^{-x}) - 8^{-x} = 0$, $[0, 1]$

Newton's method sequence $\{p_n\}$ with $p_0 = 0.5$:

$$\begin{aligned} p_0 &= 0.5 \\ p_1 &\approx 0.453476 \\ p_2 &\approx 0.447235 \\ p_3 &\approx 0.446729 \end{aligned}$$

Aitken's Δ^2 method: $q_n = p_n - \frac{(p_{n+1}-p_n)^2}{(p_{n+2}-2p_{n+1}+p_n)}$

$$\begin{aligned} q_0 &= p_0 - \tfrac{(p_1-p_0)^2}{(p_2-2p_1+p_0)} \approx 0.446734 \\ q_1 &= p_1 - \tfrac{(p_2-p_1)^2}{(p_3-2p_2+p_1)} \approx 0.446715 \\ q_2 &= p_2 - \tfrac{(p_3-p_2)^2}{(p_4-2p_3+p_2)}, \text{ need } p_4 \approx 0.446715 \end{aligned}$$

$|q_1 - q_0| \approx 0.000019 > 10^{-4}$. Need more iterations. Since q_1 and q_2 are very close to $q_1 \approx 0.446715$, we approximate root as q_1.

Root for part c: $\boxed{0.44672}$

3. Consider the function $f(x) = e^{6x} + 3(\ln 2)^2 e^{2x} - (\ln 8)e^{4x} - (\ln 2)^3$. Use Newton's method with $p_0 = 0$ to approximate a zero of f. Generate terms until $|p_{n+1} - p_n| < 0.0002$. Construct Aitken's Δ^2 sequence $\{q_n\}$. Is the convergence improved?

   Sol:
   Let $f(x) = e^{6x} + 3(\ln 2)^2 e^{2x} - (\ln 8)e^{4x} - (\ln 2)^3$ and $f'(x) = 6e^{6x} + 6(\ln 2)^2 e^{2x} - 4(\ln 8)e^{4x}$. Newton's method iteration: $p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$. Start with $p_0 = 0$. Let $L2 = \ln 2$ and $L8 = \ln 8$. Then $f(x) = e^{6x} + 3L2^2 e^{2x} - L8e^{4x} - L2^3$ and $f'(x) = 6e^{6x} + 6L2^2 e^{2x} - 4L8e^{4x}$.

   $$\begin{aligned} &p_0 = 0 \\ &f(p_0) = 1 + 3(\ln 2)^2 - \ln 8 - (\ln 2)^3 \\ &f'(p_0) = 6 + 6(\ln 2)^2 - 4\ln 8 \\ &p_1 = p_0 - \tfrac{f(p_0)}{f'(p_0)} = -\tfrac{1+3(\ln 2)^2-\ln 8-(\ln 2)^3}{6+6(\ln 2)^2-4\ln 8} \approx -2.06265 \times 10^{-7} \\ &|p_1 - p_0| = |p_1| \approx 2.06265 \times 10^{-7} < 0.0002 \end{aligned}$$

   Since $|p_1 - p_0| < 0.0002$, we stop at p_1. $p_1 \approx -2.06265 \times 10^{-7}$.

   Construct Aitken's Δ^2 sequence $\{q_n\}$. We need p_2 for q_0.

   $$p_2 = p_1 - \tfrac{f(p_1)}{f'(p_1)}$$

   Since p_1 is very close to 0 and $f(0) \approx 0$, p_2 will be very close to p_1. For practical purposes, $p_1 \approx p_2 \approx ... \approx 0$.

   Aitken's Δ^2 method: $q_n = p_n - \frac{(p_{n+1}-p_n)^2}{(p_{n+2}-2p_{n+1}+p_n)}$

   $$q_0 = p_0 - \frac{(p_1 - p_0)^2}{(p_2 - 2p_1 + p_0)} = 0 - \frac{(p_1 - 0)^2}{(p_2 - 2p_1 + 0)} = -\frac{p_1^2}{p_2 - 2p_1}$$

Since $p_1 \approx p_2 \approx -2.06265 \times 10^{-7}$, let's use $p_2 \approx p_1$.

$$q_0 \approx -\frac{p_1^2}{p_1 - 2p_1} = -\frac{p_1^2}{-p_1} = p_1 \approx -2.06265 \times 10^{-7}$$

In this case, Aitken's method does not significantly improve the first approximation, as Newton's method already converges very rapidly from $p_0 = 0$. The convergence is already very fast, so acceleration by Aitken's method is not visibly significant in the first term q_0.

Approximation of zero using Newton's method: $\boxed{-2.06265 \times 10^{-7}}$ Convergence is already very fast; Aitken's Δ^2 method does not show significant improvement in the first term.

5. (i) Show that the following sequences $\{p_n\}$ converge linearly to $p = 0$. (ii) How large must n be before $|p_n - p| \leq 5 \times 10^{-2}$? (iii) Use Aitken's Δ^2 method to generate a sequence $lbraceq_n\}$ until $|q_n - p| \leq 5 \times 10^{-2}$.

   a. $p_n = \frac{1}{n}$, for $n \geq 1$
   Sol:

   (i) Linear convergence:

$$\lim_{n\to\infty} \frac{|p_{n+1} - 0|}{|p_n - 0|} = \lim_{n\to\infty} \frac{1/(n+1)}{1/n} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

   Linear convergence to $p = 0$.
   (ii) Find n for $|p_n - 0| \leq 5 \times 10^{-2}$:

$$\frac{1}{n} \leq 0.05 = \frac{1}{20} \implies n \geq 20$$

   $n = 20$ needed.
   (iii) Aitken's Δ^2 method: $q_n = \frac{1}{2(n+1)}$

$$\begin{aligned}
q_1 &= \tfrac{1}{2(1+1)} = \tfrac{1}{4} = 0.25\\
q_2 &= \tfrac{1}{2(2+1)} = \tfrac{1}{6} \approx 0.16667\\
q_3 &= \tfrac{1}{2(3+1)} = \tfrac{1}{8} = 0.125\\
q_4 &= \tfrac{1}{2(4+1)} = \tfrac{1}{10} = 0.1\\
q_5 &= \tfrac{1}{2(5+1)} = \tfrac{1}{12} \approx 0.08333\\
q_6 &= \tfrac{1}{2(6+1)} = \tfrac{1}{14} \approx 0.07143\\
q_7 &= \tfrac{1}{2(7+1)} = \tfrac{1}{16} = 0.0625\\
q_8 &= \tfrac{1}{2(8+1)} = \tfrac{1}{18} \approx 0.05556\\
q_9 &= \tfrac{1}{2(9+1)} = \tfrac{1}{20} = 0.05\\
q_{10} &= \tfrac{1}{2(10+1)} = \tfrac{1}{22} \approx 0.04545 < 0.05
\end{aligned}$$

   Need q_{10} for $|q_n| \leq 5 \times 10^{-2}$.

b. $p_n = \frac{1}{n^2}$, for $n \geq 1$

(i) Linear convergence:

$$\lim_{n\to\infty} \frac{|p_{n+1} - 0|}{|p_n - 0|} = \lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1$$

Linear convergence to $p = 0$.

(ii) Find n for $|p_n - 0| \leq 5 \times 10^{-2}$:

$$\frac{1}{n^2} \leq 0.05 = \frac{1}{20} \implies n^2 \geq 20 \implies n \geq \sqrt{20} \approx 4.47$$

$n = 5$ needed.

(iii) Aitken's Δ^2 method: $q_1 = p_1 - \frac{(p_2-p_1)^2}{(p_3-2p_2+p_1)}$

$$\begin{aligned}
&p_1 = 1, p_2 = 0.25, p_3 \approx 0.1111 \\
&q_1 \approx 0.0795 \\
&p_2 = 0.25, p_3 \approx 0.1111, p_4 = 1/16 = 0.0625 \\
&q_2 = 0.25 - \tfrac{(0.1111-0.25)^2}{(0.0625-2\times 0.1111+0.25)} \approx 0.03635
\end{aligned}$$

$|q_2| \approx 0.03635 < 0.05$. Need q_2 for $|q_n| \leq 5 \times 10^{-2}$.

Answers:

Part a: (i) Linear, (ii) $n = 20$, (iii) $q_{10} \approx 0.04545$

Part b: (i) Linear, (ii) $n = 5$, (iii) $q_2 \approx 0.03635$

Homework 04 - 2.6, 3.2

Due Tue 2/26
Uzair Hamed Mohammed

2.6 Muller's Method

6, 9, 10.

6. $P(x) = 10x^3 - 8.3x^2 + 2.295x - 0.21141 = 0$ has a root $x = 0.29$.

   a. Use Newton's method with $p_0 = 0.28$ to attempt to find this root.
   Sol:

$$\begin{aligned}&p_0 = 0.28 :\\ &P(0.28) \approx -0.00001, \quad P'(0.28) \approx -0.001\\ &p_1 = 0.28 - \frac{-0.00001}{-0.001} = 0.27 \quad \Rightarrow \boxed{0.27}\end{aligned}$$

   b. Use Muller's method with $p_0 = 0.275$, $p_1 = 0.28$, and $p_2 = 0.285$ to attempt to find this root.
   Sol:

$$\begin{aligned}&h_0 = 0.005,\ h_1 = 0.005,\ \delta_0 = \frac{P(0.28) - P(0.275)}{0.005}\\ &\approx -0.00125,\ \delta_1 = \frac{P(0.285) - P(0.28)}{0.005} \approx -0.00025\\ &a = \frac{\delta_1 - \delta_0}{h_1 + h_0} = 0.1, \quad b = ah_1 + \delta_1 = 0.00025, \quad c = P(0.285) \approx -0.00001125\\ &x_3 = 0.285 - \frac{2c}{b + \sqrt{b^2 - 4ac}} \approx 0.2943 \quad \Rightarrow \boxed{0.29}\end{aligned}$$

   c. Explain any discrepencies in (a) and (b).
   Sol:
   c. Newton's method converges to a double root at 0.27; Muller's method targets the simple root at 0.29.

9. Use each of the following methods to find a solution accurate to within 10^{-4} for the problem $600x^4 - 550x^3 + 200x^2 - 20x - 1 = 0$, for $0.1 \le x \le 1$.

   Sol a. (Bisection Method):

$$\begin{array}{l}\text{Initial interval: } [0.2, 0.3]\\ \text{Iterations (10 steps):}\\ \hline p_{10} \approx 0.2324 \quad \Rightarrow \boxed{0.2324}\end{array}$$

   Sol b. (Newton's Method):

$$\begin{aligned}&p_0 = 0.25, \quad f'(x) = 2400x^3 - 1650x^2 + 400x - 20\\ &p_1 = 0.2326, \quad p_2 = 0.2327 \quad \Rightarrow \boxed{0.2327}\end{aligned}$$

Sol c. (Secant Method):

$$p_0 = 0.2,\ p_1 = 0.3$$
$$p_4 \approx 0.2323 \quad \Rightarrow \boxed{0.2323}$$

Sol d. (False Position):

$$\text{Initial: } [0.2, 0.3]$$
$$p_3 \approx 0.2323 \quad \Rightarrow \boxed{0.2323}$$

Sol e. (Muller's Method):

$$p_0 = 0.2,\ p_1 = 0.25,\ p_2 = 0.3$$
$$p_3 \approx 0.2325 \quad \Rightarrow \boxed{0.2325}$$

10. Two ladders crosscross an alley of width W. Each ladder reaches from the base of one wall to some point on the opposite wall. The ladders cross at a height H above the pavement. Find W given that the lengths of the ladders are $x_1 = 20$ ft and $x_2 = 30$ ft and that $H = 8$ ft.

    Sol:

$$\frac{8}{\sqrt{400 - W^2}} + \frac{8}{\sqrt{900 - W^2}} = 1W \approx \sqrt{262.855} \approx 16.21 \quad \Rightarrow \boxed{16.22\,\text{ft}}$$

3.2 Lagrange Polynomials

1c, 2, 3b, 7c.

1c. For the function $f(x) = \ln(x+1)$, let $x_0 = 0, x_1 = 0.6$ and $x_2 = 0.9$. Construct the Lagrange interpolating polynomials of degree (i) at most 1 and (ii) at most 2 to approximate $f(0.45)$, and find the actual error.

    Sol (i):

$$P_1(0.45) = \frac{0.45 - 0.6}{0 - 0.6} \ln(1) + \frac{0.45 - 0}{0.6 - 0} \ln(1.6) \approx 0.3525$$

$$\text{Error} = |\ln(1.45) - 0.3525| \approx \boxed{0.01906}$$

    Sol (ii):

$$P_2(0.45) = \tfrac{(0.45-0.6)(0.45-0.9)}{(0-0.6)(0-0.9)} \ln(1) + \tfrac{(0.45-0)(0.45-0.9)}{(0.6-0)(0.6-0.9)} \ln(1.6) + \tfrac{(0.45-0)(0.45-0.6)}{(0.9-0)(0.9-0.6)} \ln(1.9) \approx 0.3683$$

$$\text{Error} = |\ln(1.45) - 0.3683| \approx \boxed{0.00327}$$

2. Use the Lagrange polynomial error formula to find an error bound for the approximations in Exercise 1.

   Sol: Linear (n=1):

$$\text{Bound} = \frac{\max|f''(\xi)|}{2} \cdot |(0.45-0)(0.45-0.6)|, \quad \xi \in [0, 0.6]$$

$$f''(x) = -\frac{1}{(x+1)^2} \Rightarrow \max|f''(\xi)| = 1$$

$$\text{Bound} = \frac{1}{2} \cdot |0.45 \cdot (-0.15)| = \boxed{0.03375}$$

   Quadratic (n=2):

$$\text{Bound} = \frac{\max|f'''(\xi)|}{6} \cdot |(0.45-0)(0.45-0.6)(0.45-0.9)|, \quad \xi \in [0, 0.9]$$

$$f'''(x) = \frac{2}{(x+1)^3} \Rightarrow \max|f'''(\xi)| = 2$$

$$\text{Bound} = \frac{2}{6} \cdot |0.45 \cdot (-0.15) \cdot (-0.45)| = \boxed{0.010125}$$

3b. Use the appropriate Lagrange interpolating polynomials of degrees 1, 2, and 3 to approximate $f(-\frac{1}{3})$ if $f(-0.75) = -0.07181250, f(0.5) = -0.02475000, f(-0.25) = 0.33493750, f(0) = 1.10100000$.

   Sol (i): Using $x_0 = -0.5$, $x_1 = -0.25$:

$$P_1\left(-\tfrac{1}{3}\right) = \frac{(-\frac{1}{3}+0.25)}{-0.25}(-0.02475) + \frac{(-\frac{1}{3}+0.5)}{0.25}(0.3349375) = \boxed{0.21504167}$$

   Sol (ii): Adding $x_2 = 0$:

$$P_2\left(-\tfrac{1}{3}\right) = \frac{(-\frac{1}{3}+0.25)(-\frac{1}{3})}{0.125}(-0.02475)$$
$$+\frac{(-\frac{1}{3}+0.5)(-\frac{1}{3})}{-0.0625}(0.3349375) + \frac{(-\frac{1}{3}+0.5)(-\frac{1}{3}+0.25)}{0.125}(1.101) = \boxed{0.16988889}$$

   Sol (iii): Including $x_3 = -0.75$:

$$P_3\left(-\tfrac{1}{3}\right) = \sum_{k=0}^{3} f(x_k) \prod_{\substack{j=0\\ j\neq k}}^{3} \frac{-\frac{1}{3}-x_j}{x_k-x_j} = \boxed{0.17451852}$$

7c. The data for Exercise 3 were generated using the function $f(x) = x\cos x - 2x^2 + 3x - 1$. Use the error formula to find a bound for the error and compare the bound to the actual error for the cases $n = 1$ and $n = 2$.

Sol (i): Error bound for $n = 1$:

$$\frac{\max|f''(\xi)|}{2}\left|\left(-\tfrac{1}{3}+0.5\right)\left(-\tfrac{1}{3}+0.25\right)\right| = \boxed{6.0971\times 10^{-3}}$$

Actual error: $\boxed{5.9210\times 10^{-3}}$.

Sol (ii): Error bound for $n = 2$:

$$\frac{\max|f'''(\xi)|}{6}\left|\left(-\tfrac{1}{3}+0.5\right)\left(-\tfrac{1}{3}+0.25\right)\left(-\tfrac{1}{3}\right)\right| = \boxed{1.8128\times 10^{-4}}$$

Actual error: $\boxed{1.7455\times 10^{-4}}$.