Homework 14 - 8.4, 8.6

Due Thu 5/15
Uzair Hamed Mohammed

8.4 Chebyshev Polynomials

3b, 4b, 7

3b. Use the zeroes of $\tilde{T}_4$ to construct an interpolating polynomial of degree 3 for the function $f(x) = \sin x$.

Sol:
Chebyshev zeros for $\tilde{T}_4$ on $[-1, 1]$:

$$x_k = \cos\left(\frac{(2k+1)\pi}{8}\right) \approx \{0.9238795,\ 0.3826834,\ -0.3826834,\ -0.9238795\}$$

Function values at nodes:

$$\sin(x_k) \approx \{0.7956988,\ 0.3745929,\ -0.3745929,\ -0.7956988\}$$

Newton's divided differences (ordered as $x_0 > x_1 > x_2 > x_3$):

$$\begin{aligned}
f[x_0] &= 0.7956988, \\
f[x_0, x_1] &\approx \frac{0.3745929 - 0.7956988}{0.3826834 - 0.9238795} \approx 0.7778, \\
f[x_0, x_1, x_2] &\approx \frac{0.9789 - 0.7778}{-0.7654} \approx -0.1539, \\
f[x_0, x_1, x_2, x_3] &\approx \frac{-0.1539 - 0.1539}{-1.8478} \approx -0.1667.
\end{aligned}$$

Interpolating polynomial in Newton form:

$$\boxed{\begin{aligned}&P_3(x) = 0.7957 + 0.7778(x - 0.9238795) - 0.1539(x - 0.9238795)(x - 0.3826834) \\ &-0.1667(x - 0.9238795)(x - 0.3826834)(x + 0.3826834)\end{aligned}}$$

4b. Find a bound for the maximum error of the approximation computed in the previous exercise.

Sol:
Error bound for interpolation using Chebyshev nodes:

$$\|f - P_3\|_\infty \leq \frac{\|f^{(4)}\|_\infty}{4!} \cdot \max_{x \in [-1,1]} \left| \prod_{k=0}^{3} (x - x_k) \right|$$

Fourth derivative of $f(x) = \sin x$ is $\sin x$, so $\|f^{(4)}\|_\infty = 1$. For Chebyshev zeros of $\tilde{T}_4$, the product term is minimized:

$$\max_{x \in [-1,1]} \left| \prod_{k=0}^{3} (x - x_k) \right| = \frac{1}{2^{4-1}} = \frac{1}{8}$$

Thus, the error bound:

$$\frac{1}{4!} \cdot \frac{1}{8} = \frac{1}{192}$$

Final bound: $\boxed{\frac{1}{192}}$.

7. Show that for any positive integers i and j with $i > j$, we have

$$T_i(x)T_j(x) = \frac{1}{2}[T_{i+j}(x) + T_{i-j}(x)].$$

<u>Sol</u>:
Using the definition $T_n(x) = \cos(n \arccos x)$, let $\theta = \arccos x$. Then:

$$T_i(x)T_j(x) = \cos(i\theta)\cos(j\theta)$$

Apply trigonometric identity:

$$\cos(i\theta)\cos(j\theta) = \frac{1}{2}\left[\cos((i+j)\theta) + \cos((i-j)\theta)\right]$$

Recognize $\cos((i \pm j)\theta) = T_{i\pm j}(x)$, hence:

$$T_i(x)T_j(x) = \frac{1}{2}\left[T_{i+j}(x) + T_{i-j}(x)\right]$$

Thus, $\boxed{T_i(x)T_j(x) = \frac{1}{2}\left[T_{i+j}(x) + T_{i-j}(x)\right]}$.

8.6 Trigonometric Polynomial Approximation

1, 5, 9, 14

1. Find the continuous least squares trigonometric polynomial $S_2(x)$ for $f(x) = x^2$ on $[-\pi, \pi]$.

<u>Sol</u>:
Compute coefficients for $S_2(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x$. Since $f(x) = x^2$ is even, $b_1 = b_2 = 0$.

$$a_0 = \tfrac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \tfrac{2\pi^2}{3}$$
$$a_1 = \tfrac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos x dx = -4$$
$$a_2 = \tfrac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos 2x dx = 1$$

$$\boxed{S_2(x) = \frac{\pi^2}{3} - 4\cos x + \cos 2x}$$

5. Find the general continuous least squares trigonometric polynomial $S_n(x)$ for

$$f(x) = \begin{cases} 0, & \text{if } -\pi < x \leq 0, \\ 1, & \text{if } 0 < x < \pi. \end{cases}$$

Sol:
Compute Fourier coefficients for $f(x)$:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} 1\, dx = 1 \implies \frac{a_0}{2} = \frac{1}{2}$$

For $k \geq 1$:

$$a_k = \frac{1}{\pi}\int_0^{\pi} \cos(kx)dx = 0, \quad b_k = \frac{1}{\pi}\int_0^{\pi} \sin(kx)dx = \frac{1-(-1)^k}{k\pi}$$

$S_n(x)$ includes only sine terms with odd k:

$$S_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \frac{1-(-1)^k}{k\pi} \sin(kx)$$

Simplifying for odd k:

$$S_n(x) = \frac{1}{2} + \sum_{m=1}^{\lfloor n/2 \rfloor} \frac{2}{(2m-1)\pi} \sin((2m-1)x)$$

Final general form: $\boxed{\frac{1}{2} + \sum_{k=1}^{n} \frac{1-(-1)^k}{k\pi} \sin(kx)}$.

9. Determine the discrete least squares trigonometric polynomial $S_3(x)$, using $m = 4$ for $f(x) = e^x \cos 2x$ on the interval $[-\pi, \pi]$. Compute the error $E(S_3)$.

Sol:

$$a_0 = \frac{1}{4}\sum_{j=0}^{7} f(x_j) \approx -0.9937858 \implies \frac{a_0}{2} = -0.4968929,$$

$$a_1 \approx 0.2391965, \quad a_2 \approx 1.515393, \quad a_3 \approx 0.2391965,$$

$$b_1 \approx -1.150649, \quad b_2 = b_3 = 0.$$

$S_3(x) = -0.4968929 + 0.2391965\cos x + 1.515393\cos 2x + 0.2391965\cos 3x - 1.150649\sin x.$

$$E(S_3) = \sum_{j=0}^{7} |f(x_j) - S_3(x_j)|^2 = \boxed{7.271197}.$$

14. In Example 1, the Fourier series was determined for $f(x) = |x|$. Use this series and the assumption that it represents f at zero to find the value of the convergent infinite series $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$.

Sol:
Fourier series for $f(x) = |x|$ on $[-\pi, \pi]$:

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\cos((2k+1)x)}{(2k+1)^2}.$$

At $x = 0$:

$$0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}.$$

Solving:

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

$\boxed{\frac{\pi^2}{8}}$.