Homework 01 - 1.2, 1.3

Due Wed 2/5
Uzair Hamed Mohammed

1.2 Review of Calculus

1. Show that the following equations have at least one solution in the given intervals.

   a. $x\cos x - 2x^2 + 3x - 1 = 0, \qquad [0.2, 0.3]$ and $[1.2, 1.3]$

   Sol:
   For interval $[0.2, 0.3]$:

$$f(0.2) = 0.2\cos(0.2) - 2(0.2)^2 + 3(0.2) - 1 = -0.284$$
$$f(0.3) = 0.3\cos(0.3) - 2(0.3)^2 + 3(0.3) - 1 = 0.0066$$

   For interval $[1.2, 1.3]$:

$$f(1.2) = 1.2\cos(1.2) - 2(1.2)^2 + 3(1.2) - 1 = 0.1548$$
$$f(1.3) = 1.3\cos(1.3) - 2(1.3)^2 + 3(1.3) - 1 = -0.132$$

   Therefore, $x\cos x - 2x^2 + 3x - 1$ has at least one solution in both intervals due to sign changes and contunity of f(x)

   b. $(x-2)^2 - \ln x = 0, \qquad [1, 2]$ and $[e, 4]$

   Sol:
   For interval $[1, 2]$:

$$f(1) = (1-2)^2 - \ln(1) = 1$$
$$f(2) = (2-2)^2 - \ln(2) = -0.693$$

   For interval $[e, 4]$:

$$f(e) = (e-2)^2 - \ln(e) = -0.484$$
$$f(4) = (4-2)^2 - \ln(4) = 2.61$$

   Therefore, $(x-2)^2 - \ln x = 0$ has at least one solution in both intervals due to sign changes and contunity of f(x)

   c. $2x\cos(2x) - (x-2)^2 = 0, \qquad [2, 3]$ and $[3, 4]$

   Sol:
   For interval $[2, 3]$:

$$f(2) = 2(2)\cos(2 \times 2) - (2-2)^2 = -2.61$$
$$f(3) = 2(3)\cos(2 \times 3) - (3-2)^2 = 4.761$$

For interval $[3, 4]$:

$$f(3) = 2(3)\cos(2 \times 3) - (3 - 2)^2 = 4.761$$
$$f(4) = 2(4)\cos(2 \times 4) - (4 - 2)^2 = -5.164$$

Therefore, $2x\cos(2x) - (x - 2)^2 = 0$ has at least one solution in both intervals due to sign changes and contunity of f(x)

d. $x - (\ln x)^x = 0, \qquad [4, 5]$

Sol:

For interval $[4, 5]$:

$$f(4) = 4 - (\ln 4)^4 = 0.306$$
$$f(5) = 5 - (\ln 5)^5 = -5.798$$

Therefore, $x - (\ln x)^x = 0$ has at least one solution in the interval due to sign changes and contunity of f(x)

2. Find intervals containing solutions to the following equations.

a. $x - 3^{-x} = 0$

Sol:

$$f(0) = 0 - 3^0 = \text{-}$$
$$f(1) = 1 - 3^{-1} = +$$

The interval is $[0, 1]$

b. $4x^2 - e^x = 0$

Sol:

$$f(0) = 4(0)^2 - e^0 = -$$
$$f(1) = 4(1)^2 - e^1 = +$$

The interval is $[0, 1]$

c. $x^3 - 2x^2 - 4x + 3 = 0$

Sol:

$$f(0) = 0^3 - 2 * 0^2 - 4 * 0 + 3 = +$$
$$f(1) = 1^3 - 2^2 - 4 + 3 = -$$

The interval is $[0, 1]$

d. $x^3 = 4.001x^2 + 4.002x = 1.101 = 0$

Sol:

$$f(-3) = (-3)^3 = 4.001(-3)^2 + 4.002(-3) = 1.101 = -$$
$$f(-2) = (-2)^3 = 4.001(-2)^2 + 4.002(-2) = 1.101 = +$$

The interval is $[-3, -2]$

3. Show that the first derivatives of the following functions are zero at least once in the given interests.

a. $f(x) = 1 - e^x + (e-1)\sin(\frac{\pi}{2}x), \qquad [0, 1]$

Sol:

$$f(0) = 1 - e^0 + (0-1)\sin(\tfrac{\pi}{2}0) = 0$$
$$f(1) = 1 - e^1 + (1-1)\sin(\tfrac{\pi}{2}1) = 0$$

Since $f(x)$ is differentiable in the given open interval and continuous in the given closed interval, by Rolle's Theorem, there exists $c \in (0, 1)$ such that $f'(c) = 0$

b. $f(x) = (x-1)\tan x + x \sin \pi x, \qquad [0, 1]$

Sol:

$$f(0) = (0-1)\tan 0 + 0 \sin \pi 0 = 0$$
$$f(1) = (1-1)\tan 1 + 1 \sin \pi 1 = 0$$

Since $f(x)$ is differentiable in the given open interval and continuous in the given closed interval, by Rolle's Theorem, there exists $c \in (0, 1)$ such that $f'(c) = 0$

c. $f(x) = x \sin \pi x - (x-2)\ln x, \qquad [1, 2]$

Sol:

$$f(0) = 0 \sin \pi 0 - (0-2)\ln 0$$
$$f(1) = 1 \sin \pi 1 - (1-2)\ln 1$$

Since $f(x)$ is differentiable in the given open interval and continuous in the given closed interval, by Rolle's Theorem, there exists $c \in (0, 1)$ such that $f'(c) = 0$

d. $f(x) = (x-2)\sin x \ln(x+2), \qquad [-1, 3]$

4. Find $\max_{a \le x \le b} |f(x)|$ for the following functions and intervals.

   a. $f(x) = \frac{(2-e^x+2x)}{3}, \qquad [0, 1]$

   <u>Sol:</u>

$$\begin{aligned}
&f'(x) = \tfrac{2-e^x}{3} \\
&x = \ln 2 \\
&f(0) = \tfrac{1}{3} \\
&f(1) = \tfrac{4-e}{3}
\end{aligned}$$

   Max $= \frac{2\ln 2}{3}$

   b. $f(x) = \frac{(4x-3)}{(x^2-2x)}, \qquad [0.5, 1]$

   <u>Sol:</u>

$$\begin{aligned}
&f'(x) = \tfrac{-4x^2+6x-6)}{(x^2-2x)^2} \\
&f(0.5) = \tfrac{4}{3} \\
&f(1) = -1
\end{aligned}$$

   Max $= \frac{4}{3}$

   c. $f(x) = 2x\cos(2x) - (x-2)^2, \qquad [2, 4]$

   d. $f(x) = 1 + e^{-\cos(x-1)}, \qquad [1, 2]$

5. Let $f(x) = x^3$

6. Let $f(x) = \sqrt{x+1}$

7. Find the second Taylor Polynomial $P_2(x)$ for the function $f(x) = e^x \cos x$ about $x_0 = 0$.

   a. Use $P_2(0.5)$ to approximate $f(0.5)$. Find an upper bound for error $|f(0.5) - P_2(0.5)|$ using the error formula, and compare it to the actual error.

   <u>Sol:</u>

$$\begin{aligned}
&P_2(x) = 1 + x \\
&P_2(0.5) = 1.5 \\
&\text{Actual } f(0.5) \approx 1.445 \\
&\text{Error: } |1.445 - 1.5| = 0.055 \\
&\text{Error bound: } \tfrac{4.473}{6}(0.5)^3 \approx 0.0932
\end{aligned}$$

b. Find a bound for the error $|f(x) - P_2(x)|$ in using $P_2(x)$ to approximate $f(x)$ on the interval $[0, 1]$.
Sol:

$$\text{Error bound: } \frac{7.525}{6} \cdot 1^3 = 1.254$$

c. Approximate $\int_0^1 f(x)\,dx$ using $\int_0^1 P_2(x)\,dx$.
Sol:

$$\int_0^1 P_2(x)\,dx = 1.5 \quad \Rightarrow \quad 1.5$$

d. Find an upper bound for the error in 7c using $\int_0^1 |R_2(x)|\,dx$, and compare the bound to the actual error.
Sol:

Error bound: $\frac{7.525}{24} \approx 0.3136$
Actual error: $|1.394 - 1.5| = 0.106$

8. Find the Third Taylor polynomial $P_3(x)$ for the function $f(x) = (x - 1)\ln(x)$ about $x_0 = 1$.

a. Use $P_3(0.5)$ to approximate $f(0.5)$. Find an upper bound for error $|f(0.5) - P_3(0.5)|$ using the error formula, and compare it to the actual error.
Sol:

$P_3(x) = (x-1)^2 - \frac{1}{2}(x-1)^3$
$P_3(0.5) = 0.3125$
Actual $f(0.5) \approx 0.3466$
Error: 0.0341
Error bound: $\frac{112}{24} \cdot (0.5)^4 \approx 0.2917$

b. Find a bound for the error $|f(x) - P_3(x)|$ in using $P_3(x)$ to approximate $f(x)$ on the interval $[0.5, 1.5]$.
Sol:

$$\text{Error bound: } \frac{112}{24} \cdot (0.5)^4 \approx 0.2917$$

c. Approximate $\int_{0.5}^{1.5} f(x)\,dx$ using $\int_{0.5}^{1.5} P_3(x)\,dx$.
Sol:

$$\int_{0.5}^{1.5} P_3(x)\,dx \approx 0.0833$$

d. Find an upper bound for the error in 8c using $\int_{0.5}^{1.5} |R_3(x)|\, dx$, and compare the bound to the actual error.
Sol:

$$\begin{array}{l}\text{Error bound: } \approx 0.0583\\ \text{Actual error: } |0.088 - 0.0833| \approx 0.0047\end{array}$$

9. Use the error term of a Taylor polynomial to estimate the error involved in using $\sin x \approx x$ to approximate $\sin 1^\circ$.

Sol:

$$\begin{array}{l}\text{Convert } 1^\circ \text{ to radians: } x = \frac{\pi}{180} \approx 0.0174533.\\ \text{Error term for } P_1(x) = x \text{ is } |R_1(x)| \leq \frac{|x|^3}{6}.\\ |R_1| \leq \frac{(\pi/180)^3}{6} \approx 8.85 \times 10^{-7}.\\ \text{Error bound: } \approx 8.85 \times 10^{-7}.\end{array}$$

10. Use a Taylor polynomial about $\frac{\pi}{4}$ to approximate $\cos 42^\circ$ to an accuracy of 10^{-6}.

Sol:

$$\begin{array}{l}\text{Convert } 42^\circ \text{ to radians: } x = \frac{7\pi}{30} \approx 0.733.\\ \text{Center at } a = \frac{\pi}{4} \approx 0.785.\\ \text{Compute } |x - a| = \frac{\pi}{60} \approx 0.05236.\\ \text{Find smallest } n \text{ such that } \frac{(\pi/60)^{n+1}}{(n+1)!} \leq 10^{-6}.\\ \text{For } n = 3 : \frac{(0.05236)^4}{24} \approx 3.12 \times 10^{-7} \leq 10^{-6}.\\ \text{Use } P_3(x) \text{ about } \frac{\pi}{4} \text{ with terms up to } (x - \frac{\pi}{4})^3.\end{array}$$

11. Let $f(x) = e^{x/2} \sin(x/3)$. Determine the following:

a. The third Maclaurin polynomial $P_3(x)$.
Sol:

$$P_3(x) = \frac{x}{3} + \frac{x^2}{6} + \frac{23}{648}x^3$$

b. A bound for the error $|f(x) - P_3(x)|$ on $[0, 1]$.
Sol:

$$\text{Error bound: } \frac{5}{1296} \approx 0.00386$$

12. Let $f(x) = \ln(x^2 + 2)$. Determine the following:

a. The Taylor polynomial $P_3(x)$ for f expanded about $x_0 = 1$.
Sol:

$$P_3(x) = \ln 3 + \frac{2}{3}(x-1) + \frac{1}{9}(x-1)^2 + \frac{2}{81}(x-1)^3$$

b. The maximum error $|f(x) - P_3(x)|$ for $0 \le x \le 1$.
Sol:

$$\text{Error bound: } 0.125$$

c. The Maclaurin polynomial $\tilde{P}_3(x)$ for f.
Sol:

$$\tilde{P}_3(x) = \ln 2 + \frac{x^2}{2}$$

d. The maximum error $|f(x) - \tilde{P}_3(x)|$ for $0 \le x \le 1$.
Sol:

$$\text{Error bound: } 0.125$$

e. Does $P_3(0)$ approximate $f(0)$ better than $\tilde{P}_3(1)$ approximates $f(1)$?
Sol:

Error at $P_3(0) : |\ln 2 - 0.5183| \approx 0.1748$
Error at $\tilde{P}_3(1) : |\ln 3 - 1.1931| \approx 0.0945$
No, $\tilde{P}_3(1)$ approximates $f(1)$ better.

13. Find a bound for the maximum error when using $P_2(x) = 1 - \frac{1}{2}x^2$ to approximate $f(x) = \cos x$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

Sol:

Error term: $R_2(x) = \frac{f^{(4)}(c)}{4!}x^4 \quad (c \in [-1/2, 1/2])$
Since $f^{(4)}(x) = \cos x$, $|f^{(4)}(c)| \le 1$
Max $|x|^4 \le \left(\frac{1}{2}\right)^4 = \frac{1}{16}$
Error bound: $|R_2(x)| \le \frac{1}{24} \cdot \frac{1}{16} = \frac{1}{384} \approx 0.0026$

14. The n-th Taylor polynomial for a function f at x_0 is sometimes referred to as the polynomial of degree at most n that best approximates f near x_0.

   a. Explain why this description is accurate.

      Sol:
      The n-th Taylor polynomial $P_n(x)$ matches f and its first n derivatives at x_0. This ensures the polynomial shares the function's value, slope, curvature, and higher-order behaviors at x_0, minimizing the approximation error near x_0. The error $|f(x) - P_n(x)|$ grows only with $|x-x_0|^{n+1}$, making $P_n(x)$ the "best" local approximation among polynomials of degree $\leq n$.

   b. Find the quadratic polynomial that best approximates a function f near $x_0 = 1$ if the tangent line at $x_0 = 1$ has equation $y = 4x - 1$, and $f''(1) = 6$.

      Sol:

      From the tangent line: $f(1) = 3, \quad f'(1) = 4.$
      Quadratic polynomial:
      $P_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2$
      $P_2(x) = 3 + 4(x-1) + 3(x-1)^2.$

15. The error function is defined by

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.$$

   a. Integrate the Maclaurin series for e^{-t^2} to show that

$$erf(x) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)k!}.$$

      Sol:

      Maclaurin series: $e^{-t^2} = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{k!}$.
      Integrate term-by-term: $\int_0^x e^{-t^2} dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)k!} x^{2k+1}$.
      Multiply by $\frac{2}{\sqrt{\pi}}$ to obtain the series.

   b. Verify that the two series agree for $k = 1, 2, 3, 4$.

Sol:

Expand both series up to $k = 4:$
Series (a): $\frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216}\right).$
Series (b): $\frac{2}{\sqrt{\pi}}e^{-x^2}\left(x + \frac{2x^3}{3} + \frac{4x^5}{15} + \frac{8x^7}{105} + \frac{16x^9}{945}\right).$
Multiply $e^{-x^2} \approx 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24}$ into series (b):
Result matches series (a) up to x^9 (coefficients agree).

c. Approximate $erf(1)$ to within 10^{-7}.
Sol:

Compute terms until $\frac{2}{\sqrt{\pi}} \cdot \frac{1}{(2k+1)k!} < 10^{-7}$.
At $k = 6 : \frac{2}{\sqrt{\pi}} \cdot \frac{1}{13 \cdot 6!} \approx 1.08 \times 10^{-8} < 10^{-7}$.
$erf(1) \approx 0.84270079$.

d. Use the same number of terms ($k = 6$) with the series in part (b).
Sol:

Approximation: $erf(1) \approx 0.84270079 \quad$ (same accuracy as part c).

e. Explain difficulties using the series in part (b).
Sol:

Series (b) requires multiplying two infinite series, leading to computational complexity and pot

16. Verify that $|\sin x| \leq |x|$ for all x.

a. Show that for $x \geq 0$, $f(x) = x - \sin x$ is non-decreasing, implying $\sin x \leq x$.
Sol:

$f'(x) = 1 - \cos x \geq 0 \quad (since \cos x \leq 1 forallx).$
$\Rightarrow f(x) isnon-decreasingon [0, \infty).$
$Atx = 0 : f(0) = 0 - \sin 0 = 0.$
$Forx \geq 0 : f(x) \geq f(0) x - \sin x \geq 0 \sin x \leq x.$

b. Conclude using $\sin(-x) = -\sin x$.
Sol:

$Forx < 0 :$
$|\sin x| = |\sin(-x)| = |-\sin(-x)| = |\sin(-x)| \leq |-x| = |x| \quad (bypart(a)).$
$Thus, |\sin x| \leq |x| forallx \in R.$

1.3 Round-Off Error and Computer Arithmetic

1. Compute the absolute error and relative error in approximations of p by p^*.

   a. $p = \pi$, $p^* = \frac{22}{7}$
   <u>Sol:</u>

   $$\begin{aligned}&\text{Absolute error: } \left|\pi - \tfrac{22}{7}\right| \approx 0.001264\\ &\text{Relative error: } \tfrac{0.001264}{\pi} \approx 0.000402 \quad (0.0402\%)\end{aligned}$$

   b. $p = \pi$, $p^* = 3.1416$
   <u>Sol:</u>

   $$\begin{aligned}&\text{Absolute error: } |\pi - 3.1416| \approx 0.00000735\\ &\text{Relative error: } \tfrac{0.00000735}{\pi} \approx 0.00000234 \quad (0.000234\%)\end{aligned}$$

   c. $p = e$, $p^* = 2.718$
   <u>Sol:</u>

   $$\begin{aligned}&\text{Absolute error: } |e - 2.718| \approx 0.0002818\\ &\text{Relative error: } \tfrac{0.0002818}{e} \approx 0.0001037 \quad (0.01037\%)\end{aligned}$$

   d. $p = \sqrt{2}$, $p^* = 1.414$
   <u>Sol:</u>

   $$\begin{aligned}&\text{Absolute error: } \left|\sqrt{2} - 1.414\right| \approx 0.0002136\\ &\text{Relative error: } \tfrac{0.0002136}{\sqrt{2}} \approx 0.000151 \quad (0.0151\%)\end{aligned}$$

   e. $p = e^{10}$, $p^* = 22000$
   <u>Sol:</u>

   $$\begin{aligned}&\text{Absolute error: } \left|e^{10} - 22000\right| \approx 26.4658\\ &\text{Relative error: } \tfrac{26.4658}{e^{10}} \approx 0.001201 \quad (0.1201\%)\end{aligned}$$

   f. $p = 10^{\pi}$, $p^* = 1400$
   <u>Sol:</u>

   $$\begin{aligned}&\text{Absolute error: } |10^{\pi} - 1400| \approx 15\\ &\text{Relative error: } \tfrac{15}{10^{\pi}} \approx 0.01083 \quad (1.083\%)\end{aligned}$$

   g. $p = 8!$, $p^* = 39900$
   <u>Sol:</u>

   $$\begin{aligned}&\text{Absolute error: } |40320 - 39900| = 420\\ &\text{Relative error: } \tfrac{420}{40320} \approx 0.0104 \quad (1.04\%)\end{aligned}$$

h. $p = 9!$, $p^* = \sqrt{18\pi}\left(\frac{9}{e}\right)^9$
<u>Sol:</u>

$$\begin{array}{l} \text{Absolute error: } |362880 - 359500| \approx 3380 \\ \text{Relative error: } \frac{3380}{362880} \approx 0.00931 \quad (0.931\%) \end{array}$$