5. **(a)** There are three 1's in the row labeled c. Therefore there are three edges incident to c, so $d(c) = 3$.

(b) There are three 1's in the row labeled b. Therefore there are three edges outgoing from b, so $d^{+}(b) = 3$.

(c) There are two 1's in the column labeled b. Therefore there are three edges incoming to b, so $d^{-}(b) = 2$.

7. We simply list the vertices of the graph, and for each vertex list the adjacent vertices.

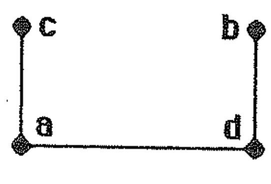
(a)
a: c, d, g
b:
c: a, d
d: a, c, e, f
e: d, f
f: d, e, g
g: a, f

(b)
a: b, c, e
b: a, d
c: a, d, e
d: b, c, e
e: a, c, d

9. The adjacency matrix is easily constructed by putting a 1 in the (i, j)th entry to indicate an edge from vertex i to vertex j, and putting a 0 to indicate no edge.

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

The graph is as shown here.



11. For the incidence matrix, we label the rows with the vertex names, and represent each edge by a column (which we have labeled e_1 through e_8). The first column represents the edge joining vertices a and c, for example.

$$\begin{array}{c c} & \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ g \end{array} & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

13. We count the trails by listing them explicitly. Note that matrix multiplication techniques are not helpful here, since they cannot detect whether walks are trails (do not repeat any edges).

(a) The trails of length 3 are a, e, d, b and a, c, d, b. Thus the answer is 2. We do not count walks that are not trails, such as a, b, d, b.

(b) In addition to the two trails of length 3 found in part (a), there is the trail a, b of length 1. (There are no trails of length 2.) Thus the answer is 3.

(c) The trails of length 4 are a, a, c, b, b; a, a, c, c, b; a, a, d, c, b; a, c, c, b, b; a, d, a, c, b; a, d, c, b, b; and a, d, c, c, b. Thus the answer is 7.

(d) In addition to the seven trails of length 4 found in part (a), there are the trails a, c, b; a, a, c, b; a, c, b, b; a, c, c, b; and a, d, c, b. Thus the answer is 12.

15. **(a)** In the complete graph, there is an edge from every vertex to every other vertex. Thus the adjacency matrix contains only 1's, except that there are 0's on the diagonal, since the graph has no loops.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 0 & 1 & \dots & 1 & 1 \\ 1 & 1 & 1 & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 0 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 0 \end{bmatrix}$$

(b) Let us suppose that the vertices are labeled $1, 2, 3, \ldots, n$, and the edges join vertices whose numbers differ by 1 (there is also an edge between 1 and n to complete the cycle). Then the matrix looks like this.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

(c) We can suppose that the first part of this complete bipartite graph contains vertices $1, 2, \ldots, m$, and the second part contains vertices $m+1, m+2, \ldots, n$. In the adjacency matrix we must put a 1 to represent each edge between a vertex in the first part and a vertex in the second part, but there must be 0's to indicate that no two vertices in the same part are adjacent. Thus the matrix is as shown here.

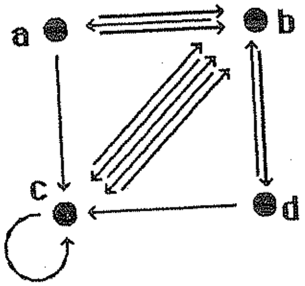
$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

17. (a) Since each 1 in the ith row represents an edge from vertex i to some vertex, the sum of the entries in this row gives the out-degree of the ith vertex.

(b) Since each 1 in the jth column represents an edge from some vertex to vertex j, the sum of the entries in this column gives the in-degree of the jth vertex.

(c) Since each 1 on the main diagonal represents a loop (an edge from some vertex to itself), the sum of the entries on the main diagonal (known as the trace of the matrix) is the number of loops in the digraph.

19. (a) We assume that the vertices are called a, b, c, and d, in the order in which they are represented in the matrix. In this picture we have put two parallel edges from vertex b to vertex c, for instance, since the $(2,3)$th entry in the matrix is 2.



(b) By the multiplication principle, the number of walks from vertex i to vertex j that have vertex k as their penultimate vertex is the number of walks from i to k times the number of walks from k to j. The latter quantity is just the number of edges from k to j, and that is precisely the entry in the adjacency matrix. Therefore Theorem 1 is still valid.

(c) The out-degree of vertex i is the number of edges with i as their tail. The entry a_{ij} in the matrix tells us the number of edges from i to j. Therefore the out-degree of vertex i is $\sum_{j=1}^{n} a_{ij}$. Similarly, the in-degree of vertex j is the number of edges with j as their head. The entry a_{ij} in the matrix tells us the number of edges from i to j. Therefore the in-degree of vertex j is $\sum_{i=1}^{n} a_{ij}$.

21. Our algorithm is straightforward. We initialize the incoming adjacency lists L' to be empty. Then we scan the outgoing adjacency lists L in order, and for each entry v_j in the list for v_i (which represents the edge from v_i to v_j), we adjoin v_i to the incoming list for v_j.

```
procedure in_list(L : outgoing adjacency lists)
  {assume that the vertices are labeled v1, v2, ..., vn;
   L and L' are indexed by {v1, v2, ..., vn}}
   initialize L' to have all its entries empty
   for i ← 1 to n do
      for each entry vj in L(vi) do
         adjoin vi to L'(vj)
   return(L') {incoming adjacency lists}
```

23. We represent each edge by a triple (v, w, c), where v and w are the endpoints (in either order, since the graph is undirected) and c is the capacity in hundreds of cars per minute. Thus we obtain the list $(D,U,6)$, $(D,N,5)$, $(D,P,5)$, $(B,U,4)$, $(D,S,4)$, $(N,U,3)$, $(P,U,3)$, $(B,N,2)$, $(B,P,2)$.

25. The identity function from V to itself is tautologically an isomorphism from $G = (V, E)$ to itself, so the relation "is isomorphic to" is reflexive. Next suppose that $\varphi: V \to V'$ is an isomorphism from $G = (V, E)$ to $G' = (V', E')$. Thus

$$\forall u \in V : \forall v \in V : (uv \in E \leftrightarrow \varphi(u)\varphi(v) \in E').$$

Now since φ is bijective it has an inverse φ^{-1}, and the condition is equivalent to

$$\forall u' \in V' : \forall v' \in V' : (\varphi^{-1}(u')\varphi^{-1}(v') \in E \leftrightarrow u'v' \in E').$$

Thus φ^{-1} is an isomorphism from G' to G. This shows that the relation is symmetric. Finally to show transitivity, let $\varphi: V_1 \to V_2$ and $\psi: V_2 \to V_3$ be isomorphisms from $G_1 = (V_1, E_1)$ to $G_2 = (V_2, E_2)$, and from $G_2 = (V_2, E_2)$ to $G_3 = (V_3, E_3)$, respectively. Thus

$$\forall u \in V_1 : \forall v \in V_1 : (uv \in E_1 \leftrightarrow \varphi(u)\varphi(v) \in E_2)$$

and

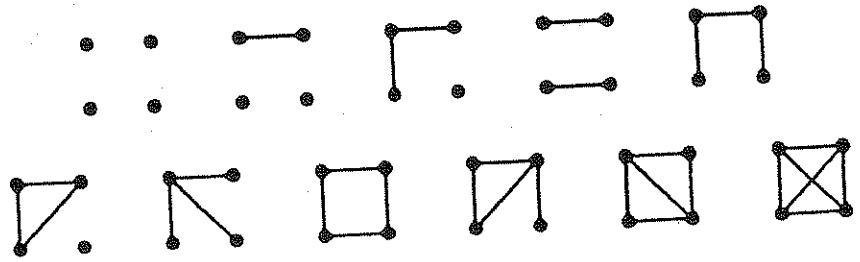
$$\forall u' \in V_2 : \forall v' \in V_2 : (u'v' \in E_2 \leftrightarrow \psi(u')\psi(v') \in E_3).$$

This implies that

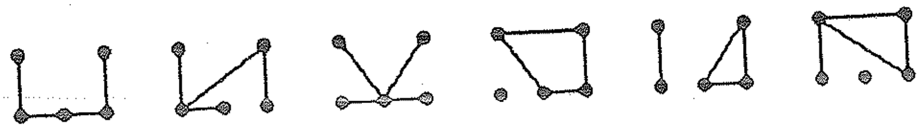
$$\forall u \in V_1 : \forall v \in V_1 : (uv \in E_1 \leftrightarrow \psi(\varphi(u))\psi(\varphi(v)) \in E_3),$$

so $\psi \circ \varphi: V_1 \to V_3$ is an isomorphism from G_1 to G_3.

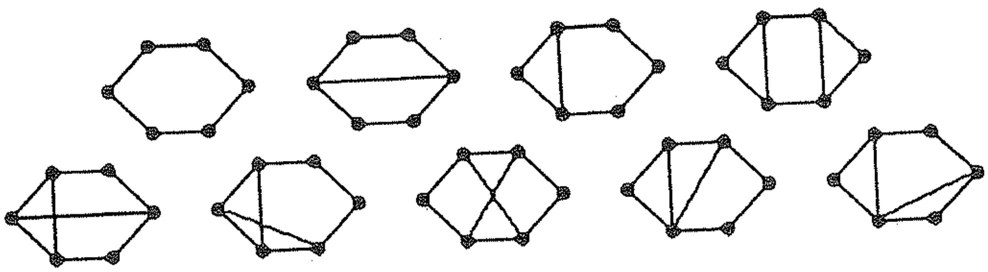
27. **(a)** We organize our catalog by the number of edges. There is clearly only one graph on four vertices with no edges, and only one with one edge. There are two graphs with two edges, depending on whether or not the edges are adjacent. If the graph has three edges, then the edges can form a path, a triangle, or a star. Finally, all the graphs with four, five, or six edges are just the complements of the graphs with two, one, or zero edges, respectively. The entire collection of 11 graphs is shown below.



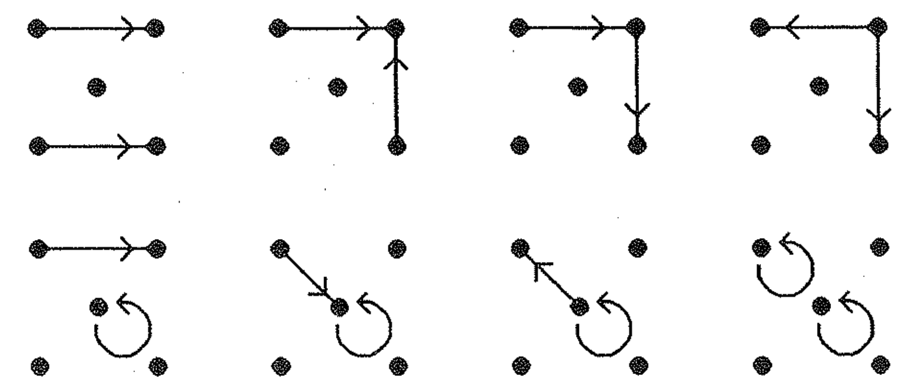
(b) First we consider graphs in which there are no cycles. The first three pictures below exhaust the possibilities (there is one graph each in which the longest path has length 4, 3, and 2). The edges could form a 4-cycle, or they could form a 3-cycle. In the latter case there are two possibilities for the fourth edge. Thus we find a total of six such graphs.



(c) We draw the Hamilton cycle first (which requires six edges) and then consider where the additional edges, if any, might go. The first picture below shows the only such graph with only six edges. If there are seven edges, then the seventh edge can either join vertices diametrically opposite on the cycle or only removed by one intermediate vertex. These two possibilities are not isomorphic, since the latter graph contains a triangle. Finally we consider where we can place two additional edges. Some careful thought shows that the six possibilities shown here are a complete catalog. This gives us nine graphs in all.



(d) We need to decide where to place the two edges. First suppose that the edges are not loops. The first picture below shows the only case in which they are not adjacent. In the next three pictures we find all the cases in which they are adjacent nonloops. If the digraph has a loop, then the other edge can either be a nonloop related to the loop as shown in the next three pictures, or it can be another loop. Thus there are eight possibilities in all.

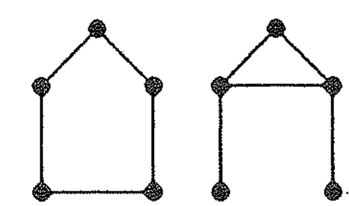


(e) A regular graph must consist of disjoint cycles. We organize our catalog in terms of the length of the longest cycle or cycles. It should be clear that the following list of 17 graphs is complete: C_{15}, $C_{12} \cup C_3$, $C_{11} \cup C_4$, $C_{10} \cup C_5$, $C_9 \cup C_6$, $C_8 \cup C_7$, $C_9 \cup C_3 \cup C_3$, $C_8 \cup C_4 \cup C_3$, $C_7 \cup C_5 \cup C_3$, $C_7 \cup C_4 \cup C_4$, $C_6 \cup C_6 \cup C_3$, $C_6 \cup C_5 \cup C_4$, $C_5 \cup C_5 \cup C_5$, $C_6 \cup C_3 \cup C_3 \cup C_3$, $C_5 \cup C_4 \cup C_3 \cup C_3$, $C_4 \cup C_4 \cup C_4 \cup C_3$, $C_3 \cup C_3 \cup C_3 \cup C_3 \cup C_3$.

(f) Since a graph must have an even number of vertices of odd degree, there are *no* such graphs.

29. (a) Since the complete graph on four vertices has six edges, a self-complementary graph on four vertices must have three edges. It is not hard to see that the graph consisting of three edges in a path is self-complementary.

(b) Since the complete graph on five vertices has ten edges, a self-complementary graph on five vertices must have five edges. It is not hard to see that the two graphs shown here are both self-complementary.



(c) There is no way for G and $\overline{G}$ to each have half of the edges, since K_3 has 3 edges.

31. Let us begin by trying to find an isomorphism. In order for out-degrees to correspond, vertex e must be paired with vertex x (the only vertices with out-degree 1), and vertex c must be paired with vertex z. By looking at in-degrees, we see that vertex b must be paired with vertex w (the only vertices with in-degree 1), and vertex f must be paired with vertex y. Now vertex d is the only remaining vertex in the digraph whose matrix is A_1 with out-degree 2, so it must be paired with the similarly describable vertex u in the other digraph. This leaves vertex a to be paired with vertex v. Thus *if* there is an isomorphism it must be given by the following correspondence: $a \leftrightarrow v$, $b \leftrightarrow w$, $c \leftrightarrow z$,

$d \leftrightarrow u$, $e \leftrightarrow x$, $f \leftrightarrow y$. It remains to check that all the adjacencies are preserved. For instance, there are edges from a to a, d, and f, so there must be (and are) edges from v to v, u, and y. We check the remaining five vertices and conclude that the digraphs are isomorphic.

33. The easiest way to prove this is to observe that the graph on the right is bipartite, but the graph on the left is not.

35. Given a graph G, we can consider *all* the possible 0–1 matrices that can be the adjacency matrix for G. We can order these lexicographically, where we consider the n^2 entries of the matrix ordered row by row, from left to right in each row. Let $M(G)$ be the matrix in this collection that comes first in this order. For example, the following matrices all represent the graph containing three vertices and one edge, but the one on the right is the one that we would pick for our invariant, since $000001010 < 001000100 < 010100000$.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Clearly this matrix is well-defined by our description, so it is an invariant. Given two matrices obtained in this way, it is trivial (in $O(n^2)$ steps) to see whether they represent the same graph: They will represent the same graph if and only if they are the same matrix. The difficulty is that given G, there is no good way known to compute $M(G)$—trying all possible $n!$ orders of the vertices has more than polynomial time complexity.

37. In time $O(v^2)$ we first construct an adjacency matrix for G by initializing it to be a v by v matrix of 0's and then going through the adjacency lists and putting a 1 in each position corresponding to an edge of G. Now for each edge uu' of G (obtained from the adjacency lists), and for each vertex w, we determine whether both u and u' are adjacent to w. It takes $O(\max(e,v))$ steps to peruse the adjacency lists for all the edges, and $O(v)$ steps to check out all possible w for each edge (using the adjacency matrix). Thus the efficiency of this algorithm is $O(\max(ev, v^2))$.

SECTION 8.4 Planarity of Graphs

1. (a) We find $v = 16$ vertices, $e = 28$ edges, $r = 15$ regions (including the outer region), and $c = 2$ components. Euler's formula checks: $16 - 28 + 15 = 2 + 1$.

(b) Here $v = 14$, $e = 14$, $r = 4$, and $c = 3$; we see that $14 - 14 + 4 = 3 + 1$.

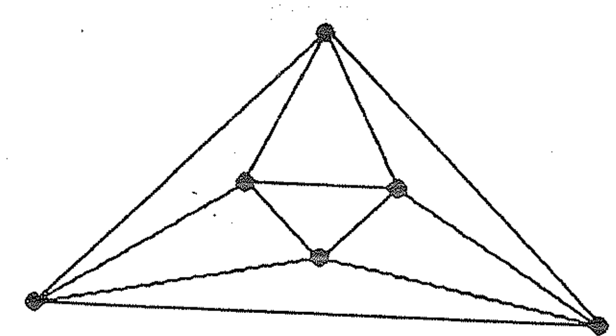
(c) In this pseudograph there are four vertices, 10 edges, eight regions, and one component; sure enough, $4 - 10 + 8 = 1 + 1$.

(d) There are 20 vertices, 22 edges, six regions, and three components: $20-22+6 = 3+1$.

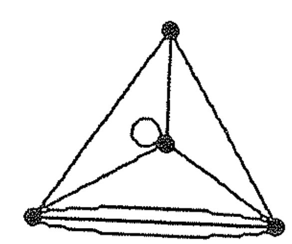
3. Clearly if G is a subgraph of H, and if H is planar, then G is planar, since an embedding of H gives an embedding of G. Equivalently, if G is nonplanar, then so is H. In the situation here, note that K_r is a subgraph of K_s if $r \leq s$. Thus since K_4 is planar, K_n is planar for all $n \leq 4$. On the other hand, no K_n is planar for $n \geq 5$, since K_5 is already nonplanar.

5. We can model the problem with a bipartite graph. The three houses are vertices in one part, and the three utility companies are vertices in the other part. The brain-teaser asks whether we can draw $K_{3,3}$ in the plane without edge crossings. The construction cannot be done because $K_{3,3}$ is not planar.

7. Intuitively, we just push the three vertices of the inner triangle toward the center of the figure, thereby making room for the curved edges to become straight line segments. The result is shown here.

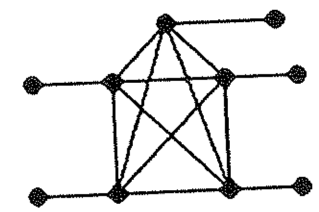


9. In the planar embedding of H, we can slightly fatten each edge and replace it by the required number of parallel edges. We can put any required loops in empty space near their vertices. The following picture, for instance, shows how to embed a pseudograph whose underlying graph is K_4.



11. We can rewrite $2e \geq 3r$ as $r \leq 2e/3$. Then we have from Euler's formula (since a graph has at least one component) $2 \leq 1 + c = v - e + r \leq v - e + 2e/3 = v - e/3$. Multiplying through by 3 we obtain $6 \leq 3v - e$, which is clearly equivalent to $e \leq 3v - 6$.

13. (a) Following the instructions, we obtain the following picture.



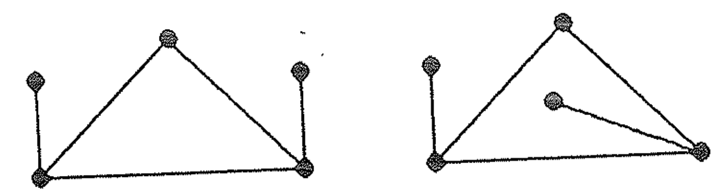
(b) Clearly $v = 10$ and $e = 15$. There is just one component.

(c) Since $e = 15$, and $3v - 6 = 3 \cdot 10 - 6 = 24$, the inequality holds: $15 \leq 24$.

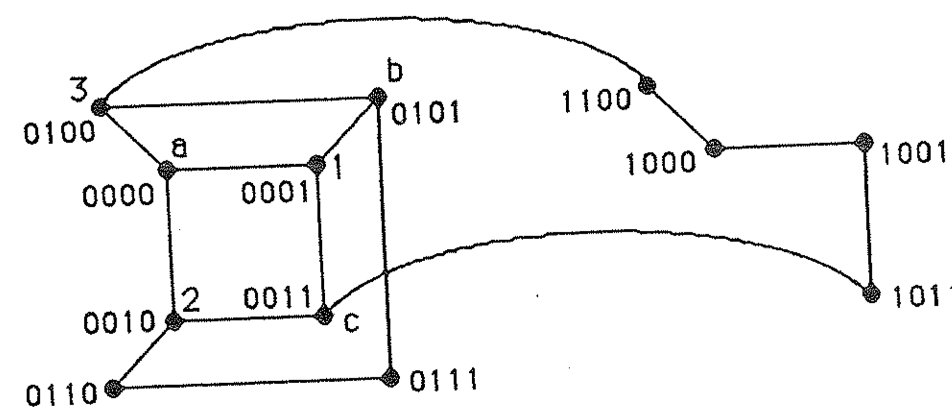
(d) This graph is not planar because it contains the nonplanar graph K_5 as a subgraph.

(e) Theorem 2 said that *if* G is planar, then $e \leq 3v - 6$, not the converse. If we have a graph that is not planar, it may or may not be the case that $e \leq 3v - 6$. Note that this example points out that you can never prove a graph to be planar just by counting edges and vertices and checking that $e \leq 3v - 6$.

15. The following two embeddings of the same graph differ in that in one case the two vertices of degree 1 are in the same region, whereas in the other they are not.

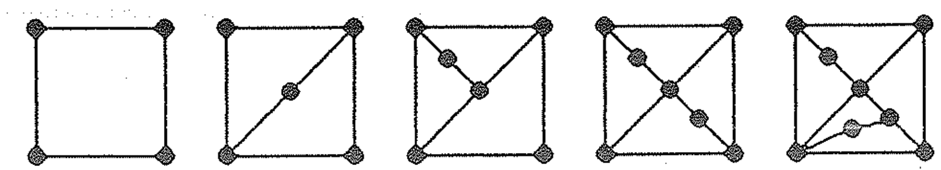


17. Recall that the vertices of Q_4 are bit strings of length 4, with two vertices adjacent if the strings differ in exactly one bit. The following picture shows a subdivision of $K_{3,3}$ in Q_4, where the parts are $\{1, 2, 3\}$ and $\{a, b, c\}$. Therefore Q_4 is not planar.



Alternatively we can give a proof by contradiction. In Q_4 there are $v = 16$ vertices, $e = 32$ edges, $c = 1$ component, and no triangles. If Q_4 were planar, Theorem 3 would tell us that $32 \leq 2 \cdot 16 - 4$, a falsehood. Therefore Q_4 is not planar.

19. We give a construction that amounts to a proof by mathematical induction. For $v = 4$, the graph C_4 embedded in the plane satisfies $v = 2e - 4$. Note that all (i.e., both) regions are four-sided. Given a planar embedding of a graph for which the equality holds, we can form a planar embedding of a graph with one more vertex for which the equality holds by putting a new vertex u inside a region $vwxy$ and adding edges uv and ux. Having added one more vertex and two more edges, we have not disturbed the equality; and the resulting planar embedding still has only four-sided regions. The following pictures show the idea for $v = 4$, 5, 6 and 7.



21. Consider the graph K_2, consisting of two vertices and one edge. It satisfies neither inequality: $e \not\leq 3v - 6$ and $e \not\leq 2v - 4$. Otherwise, however, the hypotheses of these theorems are met.

23. (a) Since every vertex has degree 4, the sum of the degrees must be $4v$, whence the number of edges must be $2v$. If there are 10% more regions than vertices, we also know that $r = 1.10v$. Plugging these facts into Euler's formula tells us that $v - 2v + 1.1v = 2$, and so $0.1v = 2$, or $v = 20$.

(b) The following picture shows such a graph.