7. Let $x_1, x_2, \ldots, x_n$ be the real numbers, and let A be their average. This means that $A = (x_1 + x_2 + \cdots + x_n)/n$. If the statement is not true, then $x_i > A$ for all i. If we add the inequalities $x_i > A$ for $i = 1, 2, \ldots, n$, we obtain $x_1 + x_2 + \cdots + x_n > nA$, or $(x_1 + x_2 + \cdots + x_n)/n > A$, a contradiction to the definition of A. Thus the statement is true.

9. **(a)** This "proof" starts by assuming what we are trying to prove. We wanted to prove $P \to Q$, but the given argument shows $Q \to P$. In fact the proposition is false; $n = 4$ provides a counterexample.

   (b) One case has been omitted: n might be an even number not divisible by 4, such as 6. In this case, $n^2 - 1$ will be odd (for example, $6^2 - 1 = 35$), not a multiple of 4. Thus the proposition is false.

   (c) This "proof" has things backward. Looking at $p = 17$ would only tell us something about the twinliness of 76 if 17 were a factor of 76, but $17 + 2$ were not. Nevertheless the proposition is true. For a valid proof, we can consider $p = 19$. Then p is a prime factor of 76, but $p + 2 = 21$ is not. Hence 76 is not twinly. (Numbers such as 64 and $5 \cdot 7 \cdot 9 = 315$ are twinly, though.)

   (d) The "proof" erroneously assumes that there is some b such that $P(a, b)$ holds. In fact the proposition is false. Let $P(x, y)$ be the proposition $x \neq x \wedge y \neq y$, which is always false (the quantifiers range over the natural numbers, say). Then the two axioms are both vacuously true, but $\forall a\colon P(a, a)$ is false.

11. **(a)** By the hypothesis we have $a = sb$ and $b = tc$ for some integers s and t. Thus $a = s(tc) = (st)c$, which shows that a is a multiple of c.

    (b) Since $6083824773 = 13 \cdot 467986521$, the statement is true.

    (c) We can write $n^2 + n$ as $n(n + 1)$. There are two cases to consider. If n is even, then by Exercise 3, $n(n+1)$ is even. Otherwise n is odd, so $n + 1$ is even, and again by Exercise 3, $n(n + 1)$ is even.

13. **(a)** Let $2n+1$ and $2m+1$ be the given odd numbers. Then the difference of their squares is $(2n + 1)^2 - (2m + 1)^2 = 4n^2 + 4n + 1 - (4m^2 + 4m + 1) = 4(n^2 + n - m^2 - m)$, a multiple of 4.

    (b) Again, let $2n + 1$ and $2m + 1$ be the given odd numbers. Then the sum of their squares is $(2n+1)^2 + (2m+1)^2 = 4n^2+4n+1+(4m^2+4m+1) = 4(n^2+n+m^2+m)+2$. Clearly the remainder when dividing this by 4 is 2; therefore it is not a multiple of 4.

15. Let $x < y$ be the two distinct real numbers. Let $z = (x+y)/2$. We claim that $x < z < y$. (For the first inequality, add x to both sides of $x < y$ and divide by 2; for the second, add y to both sides of $x < y$ and divide by 2.) Thus z is the desired number strictly between x and y.

17. Suppose that $5\sqrt{2}$ were rational. Since $1/5$ is also rational, the product $(1/5)\cdot(5\sqrt{2}) = \sqrt{2}$ would be rational, contradicting Theorem 5. Therefore $5\sqrt{2}$ is not rational. (To see that the product of two rational numbers is rational, recall first that a rational number is any number of the form x/y, where x is an integer and y is a positive integer. Let a/b and c/d be two rational numbers, where a and b are integers and c and d are positive integers. Their product is $(ab)/(cd)$. Since ab is an integer, and cd is a positive integer, the product is again a rational number.)

19. We mimic the proof of Theorem 5. First we need an analogue of Theorem 4: If n^2 is a multiple of 3, then n is a multiple of 3. (*Proof*: If n were not a multiple of 3, then we could write n as $3k+1$ or $3k+2$ for some integer k. In either case, n^2 would be of the form $3m+1$ and hence not a multiple of 3.) Now suppose that $\sqrt{3} = a/b$ in lowest terms. Then $a^2 = 3b^2$. Therefore a^2 is a multiple of 3, so a is as well, say $a = 3n$. Then we have $9n^2 = 3b^2$, or $3n^2 = b^2$. This implies that b^2, and hence b, is a multiple of 3. This has now contradicted our assumption that a/b was in lowest terms. Hence $\sqrt{3}$ is irrational.

21. There exist irrational numbers r and s such that r^s is rational. Our proof is nonconstructive. Look at $\sqrt{2}^{\sqrt{2}}$. If this is rational, then we are finished—take $r = s = \sqrt{2}$. Otherwise, let $r = \sqrt{2}^{\sqrt{2}}$ and $s = \sqrt{2}$. Then $r^s = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = 2$ is rational.

23. Let x and y be positive real numbers. We want to show that $\sqrt{xy} \le (x+y)/2$, with equality holding if and only if $x = y$. Now clearly $(\sqrt{x} - \sqrt{y})^2 \ge 0$, with equality if and only if $x = y$. But this is equivalent to $\sqrt{x}^2 - 2\sqrt{x}\sqrt{y} + \sqrt{y}^2 \ge 0$, i.e., $x + y \ge 2\sqrt{x}\sqrt{y}$, or, as desired, $\sqrt{xy} \le (x+y)/2$.

25. Among five consecutive positive integers, exactly one will be divisible by 5. Hence the product N is a multiple of 5. Similarly, the product is a multiple of 3, since at least one of any three consecutive integers must be divisible by 3. So far, then, we have established that the prime factorization of N must contain a 3 and a 5. Now among the five consecutive numbers there are at least 2 consecutive even ones, say $2a$ and $2a+2$. Their product is $2a(2a+2) = 4a(a+1)$. By Exercise 11c, $a(a+1)$ is even, so $4a(a+1)$ is divisible by $4\cdot 2 = 8$. Thus the prime factorization of N contains at least 2^3. Putting this all together, we see that N is divisible by $2^3\cdot 3\cdot 5 = 120$. Note that we cannot do better than this, since $1\cdot 2\cdot 3\cdot 4\cdot 5 = 120$.

SECTION 1.4 Boolean Functions

1. (a) $f(1,1,1) = 1\cdot\overline{1}\cdot 1 + 1\cdot(1+1) + \overline{1+1} = 1\cdot 0\cdot 1 + 1\cdot 1 + \overline{1} = 0 + 1 + 0 = 1$

(b) $f(0,1,0) = 0\cdot\overline{1}\cdot 0 + 0\cdot(1+0) + \overline{0+1} = 0\cdot 0\cdot 0 + 0\cdot 1 + \overline{1} = 0 + 0 + 0 = 0$

3. The following table shows that equality holds in all four cases: The first and fourth columns are identical.

x	y	$x+y$	$x(x+y)$
0	0	0	0
0	1	1	0
1	0	1	1
1	1	1	1

5. This circuit is the sum (the last *OR* gate) of two products (the two *AND* gates), whose inputs are the original inputs and their complements (because of the inverters). Thus the Boolean expression is $x\,\overline{y}+\overline{x}\,y$.

7. **(a)** The motion will pass if all three people vote for it or if two of them vote for it and one votes against. We write down a minterm for each of these cases and take their sum: $x\,y\,z + x\,y\,\overline{z} + x\,\overline{y}\,z + \overline{x}\,y\,z$.

(b) A motion passes if and only if (at least) two of the members vote in the affirmative. Thus we can represent the condition as $x\,y + x\,z + y\,z$.

9. **(a)** The idea is to apply DeMorgan's law to break up $\overline{xy}$ into pieces that we can then work with by applying the distributive law.

$$\begin{aligned} x \cdot \overline{xy} &= x \cdot (\overline{x} + \overline{y}) \qquad \text{(DeMorgan's law)} \\ &= x \cdot \overline{x} + x \cdot \overline{y} \qquad \text{(distributive law)} \\ &= 0 + x \cdot \overline{y} \qquad \text{(complement law)} \\ &= x\,\overline{y} \qquad \text{(identity law)} \end{aligned}$$

(b) This is exactly dual to part (a). It seems less natural to distributive addition over multiplication, but in Boolean algebra it is equally valid.

$$\begin{aligned} x + \overline{x+y} &= x + (\overline{x}\,\overline{y}) \qquad \text{(DeMorgan's law)} \\ &= (x+\overline{x}) \cdot (x + \overline{y}) \qquad \text{(distributive law)} \\ &= 1 \cdot (x + \overline{y}) \qquad \text{(complement law)} \\ &= x + \overline{y} \qquad \text{(identity law)} \end{aligned}$$

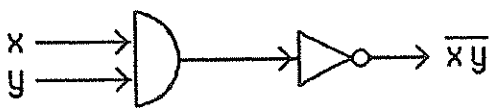
(c) If we think to "factor out" the x, the rest is easy.

$$\begin{aligned} x + x\,y &= x \cdot 1 + x\,y \qquad \text{(identity law)} \\ &= x(1+y) \qquad \text{(distributive law)} \\ &= x(y+1) \qquad \text{(commutative law)} \\ &= x \cdot 1 \qquad \text{(dominance law)} \\ &= x \qquad \text{(identity law)} \end{aligned}$$

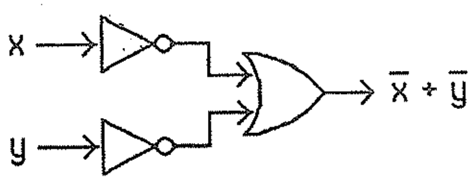
(d) This is dual to part (c). Again, it is less natural, but the proof for part (c) dualizes. An alternative approach is to expand the left-hand side, replace $x \cdot x$ by x, and then use the result of part (c) directly.

$$\begin{aligned} x(x+y) &= (x+0)(x+y) && \text{(identity law)} \\ &= x + 0 \cdot y && \text{(distributive law)} \\ &= x + y \cdot 0 && \text{(commutative law)} \\ &= x + 0 && \text{(dominance law)} \\ &= x && \text{(identity law)} \end{aligned}$$

11. **(a)** We just invert the output of the *AND* gate.



(b) By DeMorgan's law, $\overline{xy} = \overline{x} + \overline{y}$. For the circuit, we feed the two inverted inputs to an *OR* gate.



(c) $\overline{x} + \overline{y} = \overline{x}(y + \overline{y}) + \overline{y}(x + \overline{x}) = \overline{x}\,y + \overline{x}\,\overline{y} + \overline{y}\,x + \overline{y}\,\overline{x} = \overline{x}\,y + x\,\overline{y} + \overline{x}\,\overline{y}$

13. **(a)** $x\,y + x\,\overline{y} = x(y + \overline{y}) = x \cdot 1 = x$

(b) $x\,y + x\,\overline{y} + \overline{x}\,y + \overline{x}\,\overline{y} = x(y + \overline{y}) + \overline{x}(y + \overline{y}) = x \cdot 1 + \overline{x} \cdot 1 = x + \overline{x} = 1$

(c) $x\,y\,\overline{z} + x\,\overline{y}\,z + \overline{x}\,y\,\overline{z} + \overline{x}\,\overline{y}\,z = x(y\,\overline{z} + \overline{y}\,z) + \overline{x}(y\,\overline{z} + \overline{y}\,z) = (x + \overline{x})(y\,\overline{z} + \overline{y}\,z) = 1 \cdot (y\,\overline{z} + \overline{y}\,z) = y\,\overline{z} + \overline{y}\,z$

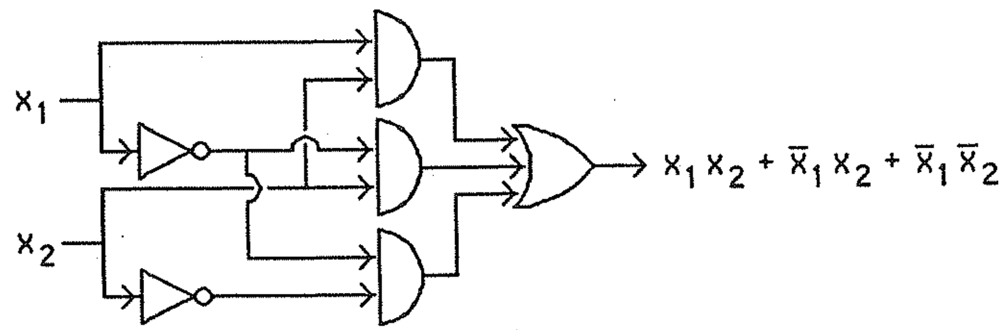
(d) $x\,y\,\overline{z} + x\,\overline{y}\,z + x\,\overline{y}\,\overline{z} = x(y\,\overline{z} + \overline{y}\,z + \overline{y}\,\overline{z})$, which can be written $x(\overline{yz})$ or $x(\overline{y} + \overline{z})$

15. **(a)** $\overline{\overline{x}\,\overline{y}\,\overline{z}} = \overline{\overline{x}} + \overline{\overline{y}} + \overline{\overline{z}} = x + y + z = x(y + \overline{y})(z + \overline{z}) + (x + \overline{x})y(z + \overline{z}) + (x + \overline{x})(y + \overline{y})z =$ $x\,y\,z + x\,y\,\overline{z} + x\,\overline{y}\,z + x\,\overline{y}\,\overline{z} + \overline{x}\,y\,z + \overline{x}\,y\,\overline{z} + \overline{x}\,\overline{y}\,z$ (after canceling repeated terms)

(b) $(\overline{x + y}\,\overline{z})(x + \overline{yz}) = (\overline{x}\,\overline{y}\,\overline{z})(x + \overline{y} + \overline{z}) = \overline{x}\,\overline{y}\,\overline{z}\,x + \overline{x}\,\overline{y}\,\overline{z}\,\overline{y} + \overline{x}\,\overline{y}\,\overline{z}\,\overline{z} = 0 + \overline{x}\,\overline{y}\,\overline{z} + \overline{x}\,\overline{y}\,\overline{z} = \overline{x}\,\overline{y}\,\overline{z}$

(c) $(x + y)(y + z)\overline{x} = (x\,y + x\,z + y + y\,z)\overline{x} = 0 + 0 + \overline{x}\,y + \overline{x}\,y\,z = \overline{x}\,y(z + \overline{z}) + \overline{x}\,y\,z =$ $\overline{x}\,y\,z + \overline{x}\,y\,\overline{z}$

17. Since this expression has three terms, we use an *OR* gate with three inputs.



19. **(a)** If $x=1$ and $y=0$, then $\overline{xy}=1$ but $\overline{x}\,\overline{y}=0$.
(b) If $x=1$ and $y=0$, then $\overline{x+y}=0$ but $\overline{x}+\overline{y}=1$.

21. **(a)** For each case in which $f(a_1,a_2,\ldots,a_n)=0$, we write down a sum $y_1+y_2+\cdots+y_n$ that will be 0 only in this case; we do so by letting $y_i=x_i$ if $a_i=0$, and $y_i=\overline{x}_i$ if $a_i=1$. Then we take the product of all such sums. This expression will have the value 0 if and only if $f(a_1,a_2,\ldots,a_n)=0$.

(b) The motion fails in four cases, for each of which we form a sum. It fails if $x=y=z=0$; this gives the sum $x+y+z$. It fails if $x=y=0$ and $z=1$; this gives the sum $x+y+\overline{z}$. Similarly we obtain the sums $x+\overline{y}+z$ and $\overline{x}+y+z$. We take the product of these sums to obtain the conjunctive normal form for the desired function: $(x+y+z)(x+y+\overline{z})(x+\overline{y}+z)(\overline{x}+y+z)$. This expression has the value 0 if and only if the motion fails. Therefore it has the value 1 if and only if the motion passes.

(c) Form the disjunction normal form of the complement of the desired function (i.e., write down all the minterms that are missing from the given disjunctive normal form expression), complement it, and then apply DeMorgan's law in two stages (and the double complement law if necessary), pushing the complementation operation inside. The outermost $+$ operations thereby become $\cdot$ operations, and the innermost $\cdot$ operations become $+$ operations. For example, for majority voting in a committee of three we have

$$\begin{aligned}\overline{x\,\overline{y}\,\overline{z}+\overline{x}\,y\,\overline{z}+\overline{x}\,\overline{y}\,z+\overline{x}\,\overline{y}\,\overline{z}} &= (\overline{x\,\overline{y}\,\overline{z}})(\overline{\overline{x}\,y\,\overline{z}})(\overline{\overline{x}\,\overline{y}\,z})(\overline{\overline{x}\,\overline{y}\,\overline{z}})\\ &= (\overline{x}+\overline{\overline{y}}+\overline{\overline{z}})(\overline{\overline{x}}+\overline{y}+\overline{\overline{z}})(\overline{\overline{x}}+\overline{\overline{y}}+\overline{z})(\overline{\overline{x}}+\overline{\overline{y}}+\overline{\overline{z}})\\ &= (\overline{x}+y+z)(x+\overline{y}+z)(x+y+\overline{z})(x+y+z).\end{aligned}$$

CHAPTER 2
SETS

SECTION 2.1 Basic Definitions in Set Theory

1. {Ronald Reagan, Jimmy Carter, $\{25, 4, 48\}$, $\{\{4\}, \emptyset, \{1,2\}, \mathbf{N}, \mathbf{Z}\}$}

3. **(a)** no (not a set) **(b)** yes **(c)** yes
(d) yes **(e)** no (not proper) **(f)** no

5. **(a)** $\{-99, -98, \ldots, -1, 0, 1, 2, \ldots, 99\}$
(b) $\{1, 2, \ldots, 67, 68, 70, 71, 72, \ldots, 100\}$
(c) $\{0, 1, 4, 9, 16, 25, \ldots\}$

7. {red, blue}, {red, green}, {red, yellow}, {blue, green}, {blue, yellow}, {green, yellow}

9. **(a)** F (the left-hand side has two elements; the right-hand side has three elements)
(b) F (the left-hand side has two elements; the right-hand side has one element)
(c) T (both sets contain the three numbers 1, 2, and 3)

11. **(a)** We know that $A \subseteq B$ if and only if $\forall x: (x \in A \to x \in B)$. Thus

$$\begin{aligned} A \not\subseteq B &\iff \neg\forall x: (x \in A \to x \in B) \\ &\iff \exists x: \neg(x \in A \to x \in B) \\ &\iff \exists x: (x \in A \wedge x \notin B)\,. \end{aligned}$$

The last equivalence follows from part (k) of Theorem 1 in Section 1.1.
(b) $\emptyset \not\subseteq A \iff \exists x: (x \in \emptyset \wedge x \notin A)$ This is clearly false, since there is no x such that $x \in \emptyset$.

13. **(a)** $\{\, x \mid x \in \mathrm{N} \wedge x < 100 \wedge \neg\exists m{\in}\mathrm{N}: x = 10m \,\}$
(b) $\{\, 1/x^3 \mid x \in \mathrm{N} \wedge x > 0 \,\}$
(c) $\{\, n(n+1) \mid n \in \mathrm{N} \wedge 1 \le n \le 99 \,\} = \{1 \cdot 2, 2 \cdot 3, 3 \cdot 4, 4 \cdot 5, \ldots, 99 \cdot 100\}$
(d) $\{\, x \mid \exists m{\in}\mathrm{Z}: x = 3m \,\}$
(e) $\{\, 6n + 1 \mid n \in \mathrm{Z} \,\}$

15. $\{\text{Henry VIII}, \{2,15\}, \{25,4\}, \{11,17\}\}$

17. We are given that $A \subseteq B$ and $B \subseteq C$, and we want to show that $A \subseteq C$, i.e., that $\forall x{:}(x \in A \rightarrow x \in C)$. So let x be an arbitrary element of A. Since $A \subseteq B$, it follows that $x \in B$. Then since $B \subseteq C$, we conclude that $x \in C$, as desired.

19. **(a)** Let $A = \{\emptyset\}$. Then $\mathcal{P}(A) = \{\emptyset, \{\emptyset\}\}$, so $A \subseteq \mathcal{P}(A)$ (the one element of A is also in $\mathcal{P}(A)$).

(b) Let $A = \{1\}$. Then $\mathcal{P}(A) = \{\emptyset, \{1\}\}$, so $A \not\subseteq \mathcal{P}(A)$ (the one element of A is not an element of $\mathcal{P}(A)$).

(c) Let $A = \{1\}$ and $B = \{1, \{1\}\}$. Then A is an element of B, and the one element of A is an element of B (so $A \subseteq B$).

21. Given $A \subseteq B$, we must show that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, i.e., that $\forall C{:}(C \subseteq A \rightarrow C \subseteq B)$. Suppose that C is an arbitrary subset of A. We must show that $C \subseteq B$. But this follows directly from Exercise 17.

23. $|\emptyset| = 0$, $|\mathcal{P}(\emptyset)| = 2^0 = 1$, $|\mathcal{P}(\mathcal{P}(\emptyset))| = 2^1 = 2$, $|\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset)))| = 2^2 = 4$, $|\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))))| = 2^4 = 16$, $|\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset)))))| = 2^{16} = 65536$

25. By the observation made before Theorem 1, we know that $A = \emptyset \leftrightarrow (A \subseteq \emptyset \wedge \emptyset \subseteq A)$. By Theorem 1, we know that $\emptyset \subseteq A$ is always true. Since $P \wedge \mathrm{T} \iff P$ for any proposition P, we conclude that $A = \emptyset \leftrightarrow A \subseteq \emptyset$.

27. **(a)** We can list the set of even positive integers as $\{2, 4, 6, 8, \ldots\}$. In other words, by letting $n \in \mathrm{N}$ correspond to $2n+2$ in the set of even positive integers, we have set up a one-to-one correspondence between these two sets.

(b) This set is finite, hence countable. (Indeed, there are, by most estimates, fewer than 10^{100} atoms in the entire universe.)

(c) We can list this set as $\{0, \frac{1}{2}, -\frac{1}{2}, 1, -1, 1\frac{1}{2}, -1\frac{1}{2}, 2, -2, 2\frac{1}{2}, -2\frac{1}{2}, \ldots\}$.

29. **(a)** $\forall A{:}\forall B{:}\exists C{:}\forall x{:}\,(x \in C \leftrightarrow (x = A \vee x = B))$

(b) $\exists S{:}\forall x{:}\overline{x \in S}$

(c) $\forall A{:}\exists P{:}\forall C{:}\,(C \in P \leftrightarrow \forall x{:}(x \in C \rightarrow x \in A))$

31. First we show that it is possible to list all the positive rational numbers expressed in lowest terms as $r_1, r_2, r_3, \ldots$. We order them as follows: first all the positive rational numbers in lowest terms, the sum of whose numerator and denominator is 2; then those for which the sum is 3; then those for which the sum is 4, and so on; and within each group, we order by increasing numerator. Thus the list is $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{1}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}, \frac{1}{5}, \frac{5}{1}, \frac{1}{6}, \frac{2}{5}, \frac{3}{4}, \frac{4}{3}, \frac{5}{2}$, and so on. Finally, we list all the rational numbers in the order $0, r_1, -r_1, r_2, -r_2, \ldots$.

33. **(a)** All real numbers between 0 and 1 can be written as infinite decimals. To avoid ambiguity in notation, since any finite decimal between 0 and 1 can be written in two ways (either ending with a string of all 0's or ending with a string of all 9's), we arbitrarily declare that we will use no representations ending with a string of all 9's.

(b) Assume that the set of real numbers between 0 and 1 is countable, and can therefore be labeled as $r_1, r_2, r_3, \ldots$ (it is clearly infinite). Let d_{ij} be the jth digit in the decimal for r_i. For example, if $r_1 = 1/7 = .142857142857\ldots$, then $d_{14} = 8$.

(c) Define a number $r = 0.d_1d_2d_3\ldots$ as follows. If $d_{ii} \neq 4$, then $d_i = 4$; if $d_{ii} = 4$, then $d_i = 5$. Note that $d_i \neq d_{ii}$ for all i, so that r is not in the list. On the other hand, r represents a real number and does not end in a string of 9's, so r must be in the list. This contradiction shows that our assumption that the set of real numbers between 0 and 1 is countable was wrong. Therefore this set is uncountable.

(d) The set of real numbers between 0 and 1 is a subset of the set of all real numbers. If the latter were countable, the former would also have to be countable, and we just showed that this is not the case.

35. **(a)** $\omega + 2 = \{0, 1, 2, \ldots, \omega, \omega + 1\}$; $\omega + 3 = \{0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2\}$

(b) $\omega \cdot 2 = \{0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots\}$

(c) $\omega^2 = \{0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots, \omega \cdot 2, \omega \cdot 2 + 1, \omega \cdot 2 + 2, \ldots, \omega \cdot 3, \omega \cdot 3 + 1, \omega \cdot 3 + 2, \ldots, \ldots\}$; $\omega^2 + 1 = \{0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots, \omega \cdot 2, \omega \cdot 2 + 1, \omega \cdot 2 + 2, \ldots, \omega \cdot 3, \omega \cdot 3 + 1, \omega \cdot 3 + 2, \ldots, \ldots, \omega^2\}$

SECTION 2.2 Sets with Structure

1. **(a)** $\{(1,1), (1,3), (2,1), (2,3)\}$

(b) $\{(a,a,a), (a,a,b), (a,b,a), (a,b,b), (b,a,a), (b,a,b), (b,b,a), (b,b,b)\}$

(c) $\{(1,5), (2,5), (3,5), (4,5)\}$

(d) $\{(1,4,5), (2,4,5), (3,4,5)\}$

(e) $\emptyset$

3. $rolloverrover = r(oll)(ove)rr(ove)r = uwvuuvu$

5. The initial substrings are λ, r, ro, rol, and $roll$. The other substrings are o, ol, oll, l, and ll.

7. It is straightforward to calculate the entries of these matrices. For example, the $(4,2)$th entry for part (b) is 0 because 2 is not a multiple of 4; and the $(2,4)$th entry for part (c) is the substring of *papa* extending from the second letter to the fourth, namely *apa*.

(a) $$\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) $$\begin{bmatrix} p & pa & pap & papa \\ \lambda & a & ap & apa \\ \lambda & \lambda & p & pa \\ \lambda & \lambda & \lambda & a \end{bmatrix}$$

9. **(a)** $5+8+11+14=38$

(b) $2^{2^0}+2^{2^1}+2^{2^2}+2^{2^3}=2^1+2^2+2^4+2^8=2+4+16+256=278$

(c) $7+7+\cdots+7=700$

(d) $2^2+3^2+5^2+7^2+11^2=208$

(e) $1+1+2+2+3+3=12$

11. Let $(a,c)\in A\times C$. Then $a\in A$ and $c\in C$. Since $A\subseteq B$ and $C\subseteq D$, we know that $a\in B$ and $c\in D$. Hence $(a,c)\in B\times D$, as desired.

13. If $A=B$, then we have the identity $A\times A=A\times A$. Conversely, suppose that $A\neq B$. Then there is an element in one of the sets but not in the other; without loss of generality, assume that it is an element a that is in A but not in B. Since $B\neq\emptyset$, we can also find an element $b\in B$. Then $(a,b)\in A\times B$, but $(a,b)\notin B\times A$ since $a\notin B$. Thus $A\times B\neq B\times A$.

15. 00001111, 00010111, 00011011, 00011101, 00100111, 00101011, 00101101, 00110011, 00110101, 01000111, 01001011, 01001101, 01010011, 01010101

17. $\{(0,0,3),(0,3,0),(3,0,0),(0,1,2),(0,2,1),(1,0,2),(1,2,0),(2,0,1),(2,1,0),(1,1,1)\}$

19. (a) $$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 0 & 5 \\ 1 & -1 & -3 & 4 \end{bmatrix}$$

(b) $$\begin{bmatrix} 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 7 \\ 0 & 3 & 0 & 0 \\ 1 & 0 & 4 & 0 \end{bmatrix}$$

(c) $$\begin{bmatrix} 0 & 1 & 1 & \sqrt{8} \\ 1 & 0 & \sqrt{2} & \sqrt{5} \\ 1 & \sqrt{2} & 0 & \sqrt{5} \\ \sqrt{8} & \sqrt{5} & \sqrt{5} & 0 \end{bmatrix}$$

21. **(a)** An open set is determined by its endpoints, which can be any real numbers (as long as the left endpoint is less than the right endpoint). Thus the set is $\{\,\{x\mid a<x<b\}\mid a<b\,\}$, indexed by pairs of real numbers (a,b) with $a<b$.

(b) $\{$last name of $x\mid x$ is a person in the United States$\}$

(c) The sets listed here are the multiples of 1, 2, 3, and so on. The set of multiples of n is the set of all numbers of the form mn, where $m\in\mathbf{N}$. Thus the set is $\{\,\{mn\mid m\in\mathbf{N}\}\mid n\in\mathbf{N}\wedge n\neq 0\,\}$, indexed by the positive integers.

23. Let A_n be the set of prime divisors of the natural number n. Then $\{A_n \mid n = 5, 10, 20\}$ is different from $\{A_n \mid n = 5, 10\}$ as indexed sets (the first has an indexing set with three elements, the second has an indexing set with two elements). But as sets, both are simply $\{\{5\}, \{2,5\}\}$.

25. **(a)** $1 \cdot 2 + 1 \cdot 3 + 2 \cdot 3 = 11$

(b) $1 \cdot 1 + 1 \cdot 2 + 1 \cdot 3 + 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + 3 \cdot 1 + 3 \cdot 2 + 3 \cdot 3 = 36$

27. **(a)** $\sigma(n) = \sum_{\substack{d \mid n \\ 1 \le d < n}} d$

(b) $\operatorname{trace}(A) = \sum_{i=1}^{n} a_{ii}$, where $A = (a_{ij})$ is an n by n matrix

29. **(a)** $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 5040$

(b) $46 \cdot 45 \cdot 44 \cdots 2 \cdot 1 \cdot 0 = 0$ (no arithmetic is needed because of the final factor)

(c) $(1 \cdot 1)(2 \cdot 1 + 2 \cdot 2)(3 \cdot 1 + 3 \cdot 2 + 3 \cdot 3) = 108$

(d) $(1 \cdot 1) + ((2 \cdot 1)(2 \cdot 2)) + ((3 \cdot 1)(3 \cdot 2)(3 \cdot 3)) = 171$

31. **(a)** yes (n corresponds to $2n$; the key point is that $2(n+m) = 2n + 2m$)

(b) no ($\mathbf{Z}$ has a multiplicative identity, but E does not)

(c) no (addition is commutative but subtraction is not)

33. A palindrome is any string of the form $u_1u_2 \ldots u_{n-1}u_nu_{n-1} \ldots u_2u_1$, where $n \ge 1$ and each $u_i \in U$, or of the form $u_1u_2 \ldots u_{n-1}u_nu_nu_{n-1} \ldots u_2u_1$, where $n \ge 0$ and each $u_i \in U$.

SECTION 2.3 Operations on Sets

1. **(a)** $\emptyset$ **(b)** $\{3, 9\}$ **(c)** U **(d)** $\{2, 3, 4, 6, 8, 9, 10\}$

(e) $\{1, 2, 4, 5, 7, 8, 10\}$ **(f)** $\{1, 5, 7\}$ **(g)** $\{2, 3, 4, 8, 9, 10\}$

3. **(a)** $M \cap C$ **(b)** $\overline{M} \cap C$ **(c)** $\overline{M} - C = \overline{M} \cap \overline{C} = \overline{M \cup C}$

(d) $A \cap \overline{C} \cap M = (A \cap M) - C$ **(e)** $\overline{A} \cap \overline{M} \cap C = \overline{A \cup M} \cap C = C - (A \cup M)$

5. If $x \in A \cap B$, then $x \in A$ and $x \in B$. In particular, $x \in A$, so $x \in A \vee x \in B$. Thus $x \in A \cup B$.

7. **(a)** $\{1\}$ and $\{2, 3\}$ **(b)** $\{4\}$ and $\{0, 5, 6, 7, \ldots\}$ **(c)** $\{2, 3\}$ and $\{0, 3, 4, 5, \ldots\}$