APM 2663 Test 1

Fall 2024

Instructor: Eddie Cheng
Date: October 31, 2024

100%

I have to work very hard to take off 1 point.

Instructions and Important Information:

- Recall that the word *if* in a definition means *if and only if*.
- **To receive full credit for a question, you should provide all logical steps. All answers must be justified unless the questions stating otherwise.**
- Recall that $\mathbb{N}$ is the set of positive integers. The definition in the book includes 0.
- Recall that $\mathbb{Z}$ is the set of integers.
- Recall that $\mathbb{Q}$ is the set of rational numbers.
- Recall that $\mathbb{R}$ is the set of real numbers.
- This is a closed book examination. No external aids are allowed, except a calculator.
- Cheating is a serious academic misconduct. Oakland University policy requires that all suspected instances of cheating be reported to the Office of the Dean of Students/Academic Conduct Committee for adjudication. I have forwarded cases to the Office of the Dean of Students/Academic Conduct Committee before and I will not hesitate to do this again if I suspect academic misconduct has occurred. Anyone found responsible of cheating in this assessment will receive a course grade of F, in addition to any penalty assigned by the Academic Conduct Committee.
- I may ask for a meeting for you to explain your solutions.
- Until the solution to this test is posted/discussed by me, you may not discuss this test with others.
- This test is worth 110 marks. If you receive x marks, your grade will be $\min\{x, 100\}$%.
- Solutions must be uploaded to Moodle unless otherwise arranged.

This is a remark that I don't usually give to students.

(1) Read the instructions and sign your name indicating that you have read the instructions. [1 mark]

Shane Jaroch

(2) Write down your name. [1 mark]

Shane Jaroch

your proofs are too long.

(3) Let $A = \{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$ and $B = \{\longrightarrow, \emptyset, \Omega\}$.

(a) What is the cardinality of the power set of $(B - (A \cup \{\heartsuit\}))$? [4 marks]

(b) What is the cardinality of $A \times A - B \times B$? [4 marks]

a) $B - (A \cup \{\heartsuit\}) = B \Rightarrow |B - (A \cup \{\heartsuit\})| = |B| = 3$

Since $|P(B)| = 2^{|B|}$, $\boxed{|P(B)| = 2^3 = 8.}$

b) $A \times A - B \times B = A \times A$ (since $A \cap B = \phi$)

$\Rightarrow |A \times A - B \times B| = |A \times A| = 4 \cdot 4 = \boxed{16}$

8

(4) Define a surjective function. (You may assume the definition of a function.) [5 marks]

A surjective function is one which, for every element in the co-domain (B), there exists at least one value in the domain (A) "pointing to" it (s.t. $f(a) = b$).

Define: $f : A \to B$.

Then, surjective means

$$\forall b \in B : \exists a \in A : f(a) = b.$$

5

(5) Answer each of the following:

(a) Disprove: If n^2 is a multiple of 49, then n is a multiple of 49 [5 marks]

$\boxed{\text{False. Let } n=7,\text{ then } n^2=49.}$

Since $49|49$; $49|n^2$, i.e., $n^2 = 1\cdot(49)$.

But $49\nmid 7$; $49 \nmid n$, i.e. $n=7$ cannot be written as a multiple of 49.

(b) Prove: If n^2 is a multiple of 7, then n is a multiple of 7, without using the Fundamental Theorem of Arithmetics. [5 marks]

(If you want to use the general result: "let p be a prime number; if p divides n^2, then p divides n," then you need to prove it.)

We want to show $7|n^2 \Rightarrow 7|n$, for all $n \in \mathbb{Z}$.

This is the form $P \Rightarrow Q$, where $P: 7|n^2$ and $Q: 7|n$.

Let's show the (logically equivalent) contrapositive holds true.

We want to show $\overline{Q} \Rightarrow \overline{P}$, or $7\nmid n \Rightarrow 7 \nmid n^2$.

If $\overline{Q}$, then $n = 7k+i$ where $k \in \mathbb{Z}$ and $i \in \{1,2,3,4,5,6\}$.

Let's look at the first case. (The remaining 5 cases are similar).

<u>Case 1:</u> $n = 7k+1$.

$\therefore n^2 = 49k^2+14k+1 = 7(7k^2+2k)+1 = 7j+1$. $(j \in \mathbb{Z})$

It's clear that, in case 1, $7 \nmid n^2$. (It will have remainder of 1)

The other cases are similar, since the k^2 and k terms will have a coefficient which is a multiple of 7, but the constant term will not be divisible by 7.

Thus, we have shown $\overline{Q} \Rightarrow \overline{P}$, or $7\nmid n \Rightarrow 7\nmid n^2$, is true.

Hence, by contrapositive, $P \Rightarrow Q$, or $7|n^2 \Rightarrow 7|n$. $\square$

10

(6) Prove that $\sqrt{7}$ is irrational without using the Fundamental Theorem of Arithmetics. [15 marks]

Suppose not. Then $\sqrt{7} = \frac{m}{n}$, $m, n \in \mathbb{N}$.

WLOG, we may assume $\frac{m}{n}$ is in lowest terms, i.e., m and n have no common factors.

$\therefore 7 = \frac{m^2}{n^2}$. Thus

(1) $7n^2 = m^2$. $\Rightarrow 7 | m^2$.

We know from question 5(b), that since $7|m^2$, $7|m$. Thus,

(2) $m = 7i$ (for some $i \in \mathbb{Z}$).

Plug eq (2) into eq (1).

$\therefore 7n^2 = (7i)^2$.

$\therefore 7n^2 = 49i^2 \Rightarrow n^2 = 7i^2 \Rightarrow 7|n^2 \Rightarrow 7|n$.

Recap: m & n have no common factors, but $7|m \wedge 7|n$. #

Hence, $\sqrt{7}$ is irrational by proof by contradiction. □

15

(7) Let A, B be sets. Let A and B be sets in some universal set. Prove that $(A \cup B) \cap (A \cup \overline{B}) = A$ without using distributive laws and without using Venn diagrams. [15 marks]

Prove: $(A \cup B) \cap (A \cup \overline{B}) = A$.

(I) $\boxed{\text{Let } x \in A.}$

Case 1: $x \in B$

$\therefore x \in A \cup B$. (def of $\cup$)

But also $x \in A \cup \overline{B}$. (def of $\cup$)

Hence, $x \in (A \cup B) \cap (A \cup \overline{B})$. (def of $\cap$)

Case 2: $x \notin B$

$\therefore x \in A \cup \overline{B}$ (def of $\cup$)

Also, $x \in A \cup B$ (def of $\cup$)

Hence, $x \in (A \cup B) \cap (A \cup \overline{B})$. (def of $\cap$)

In either case, $x \in A \Rightarrow x \in (A \cup B) \cap (A \cup \overline{B})$.

$\therefore A \subseteq (A \cup B) \cap (A \cup \overline{B})$.

(II) $\boxed{\text{Let } x \in (A \cup B) \cap (A \cup \overline{B}).}$

$\therefore x \in A \cup B$ and $x \in A \cup \overline{B}$. $\Rightarrow (x \in A \lor x \in B) \land (x \in A \lor x \notin B)$.

Since, if x were not an element of A, both conditions would fail, we can conclude $x \in A$ (x cannot be both in B and $\overline{B}$, so it must at least be in A).

$\therefore (A \cup B) \cap (A \cup \overline{B}) \subseteq A$.

15

Combining I and II with the def of set equality gives,

$A = (A \cup B) \cap (A \cup \overline{B})$, as desired. $\square$

(8) Define $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ by $f(x) = 2024x^3 - 2663x$. Determine whether or not f is one-to-one and/or onto. [20 marks]

$f(x) = x(2024x^2 - 2663)$

onto : Is there any $x \in \mathbb{Z}$ s.t. $f(x) = 1$?

That means $1 = x(2024x^2 - 2663)$. We have two possible solutions:

$x = -1$ and $(2024x^2 - 2663) = -1$,

or

$x = 1$ and $(2024x^2 - 2663) = 1$. (only integer factors matter, since $x \in \mathbb{Z}$).

If we show $2024x^2 \neq 2663 \pm 1$ $\forall x \in \mathbb{Z}$, we're done (and f is not onto).

Since neither 2664 nor 2662 are divisible by 2024, we clearly see no x-value exists satisfying this equation.

Hence, 1 is not in the image of f, and f is not onto. ☐

1-to-1 : Does $2024a^3 - 2663a = 2024b^3 - 2663b$ for any $a, b \in \mathbb{Z}$ $(a \neq b)$?

Let's look for solutions. Set $2024(a^3 - b^3) = 2663(a-b)$

Since $a = b$ is not desired or allowed, we can divide by $(a-b)$.

First, let's factor the difference of cubes. $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.

Thus if we find no solutions to this (below) it's 1-to-1, else f is not 1-1.

$2024(a^2 + ab + b^2) = 2663$.

20

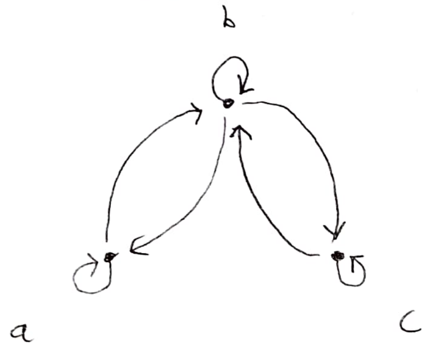
Again, since $2024 \nmid 2663$, we clearly see, since $a^2 + ab + b^2$ is an integer, there are no integer solutions for a and b.

Hence, f is one-to-one. ☐

(9) Give an example of a binary relation that is reflexive, symmetric, not transitive and not anti-symmetric, or show that it does not exist. [10 marks]

$\boxed{r/s}$

R on A. $A = \{a, b, c\}$.



Not transitive, since $aRb \wedge bRc$, but $a \not R c$.

also say something about the other properties

9

$$\boxed{\begin{aligned} R = \{&(a,a), (b,b), (c,c), \\ &(a,b), (b,a), \\ &(b,c), (c,b)\}. \end{aligned}}$$

(10) Let R be a relation on A. Then R is *irreflexive* if $(a, a) \notin R$ for all $a \in A$. Let $A \neq \emptyset$ and R be a relation on A.

(a) Is it possible for R to be both reflexive and irreflexive? [5 marks]

(b) Is it possible for R to be both not reflexive and not irreflexive? [5 marks]

(a) R can only be reflexive and irreflexive if $A = \emptyset$. Let's prove this.

(1) Suppose R is reflexive, then $\exists a \in A : (a,a) \in R$. But, by definition of irreflexivity, R is not irreflexive.

~~(2) Conversely, suppose R is irreflexive, then $\forall a \in A : (a,a) \notin R$. This is logically equivalent to $\neg \exists a \in A : (a,a) \in R$ (by the law of negated quantifiers).~~

not needed

~~But since A is non-empty, this means the definition of reflexive fails to hold on R over A. Hence, in this case, R over A cannot be reflexive.~~

~~In both cases, which was a bit redundant, we have shown R (over A) cannot be both reflexive and irreflexive. $\square$~~

(b) Yes, it's possible for R to be both not reflexive & not irreflexive.

Example:

$$A = \{1, 2\}, \qquad R = \{(1,1)\}$$

R not reflexive because $(2,2) \notin R$.

R not irreflexive because $(1,1) \in R$. $\square$

10

(11) Let $f : A \longrightarrow B$, $h : A \longrightarrow B$ and $g : B \longrightarrow C$.

(a) Show that the statement, $g \circ f = g \circ h$ implies $f = h$, is not true. [5 marks]

(b) Prove that if $g \circ f = g \circ h$ and g is injective, then $f = h$. (*Hint: To show that* $f = h$, *let* $a \in A$ *and show that* $f(a) = h(a)$.) [10 marks]

(a) Let $g(b) = b^2$, $f(a) = \lfloor a \rfloor$, and $h(a) = -\lfloor a \rfloor$ $\quad A = \{-1, 1\}$

Then $\forall a \in A: g(f(a)) = g(h(a))$, $\quad B = \{-1, 1\}$

But $f \neq h$. (So $f = h$ is false by counterexample). $\quad C = \{1\}$

(b) Let $a \in A$.

Then $g(f(a)) = g(h(a)) = c$ only if $f(a) = h(a)$. [since g is one-to-one]

Since g is one-to-one, only one value exists (call it b) such that $g(b) = c$. $\neg \exists d \in B : g(b) = g(d) \wedge b \neq d$.

Since we know $g \circ f = g \circ h$, as we have above, $g(f(a)) = g(h(a))$, but since g is 1-to-1, it's required that $f(a) = h(a)$.

$\therefore$ $f = h$ is true $\forall a \in A$. $\square$

10

(12) Estimate your grade in this test. Let x be your guess. If your grade is in the interval $[x-5, x+5]$, you will receive 2 bonus marks.

92 (ninety-two)