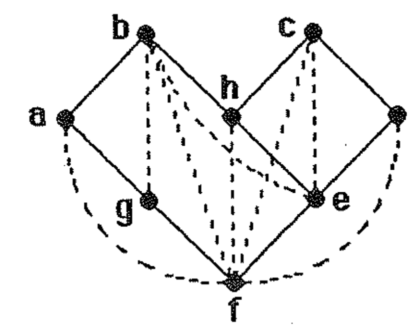
SECTION 3.4 Order Relations and Equivalence Relations

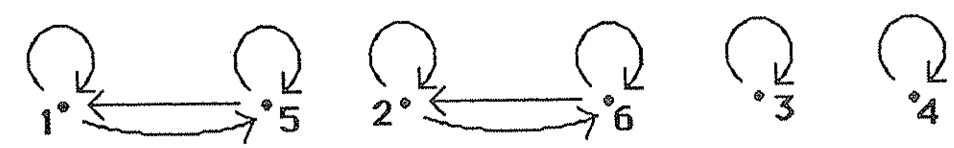
1. (a) We need to list all pairs (x,y) for which there is a rising path from x to y. This gives us $\{(a,a), (a,b), (b,b), (c,c), (d,c), (d,d), (e,b), (e,c), (e,d), (e,e), (e,h), (f,a), (f,b), (f,c), (f,d), (f,e), (f,f), (f,g), (f,h), (g,a), (g,b), (g,g), (h,b), (h,c), (h,h)\}$.

(b) We add a line between two points if there is a rising path between them.



3. All the pairs with a as their first coordinate come first, then the pairs with b as their first coordinates, and so on. The second coordinate determines the order within each group. The list is therefore (a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,a), (c,b), (c,c).

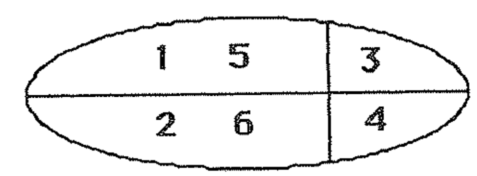
5. (a) We draw an arrow for each pair in the relation.



(b) It is easy to see from the digraph that the relation is reflexive, symmetric, and transitive.

(c) The connected pieces of the digraph give us the sets (equivalence classes) in the partition: $\{\{1,5\}, \{2,6\}, \{3\}, \{4\}\}$.

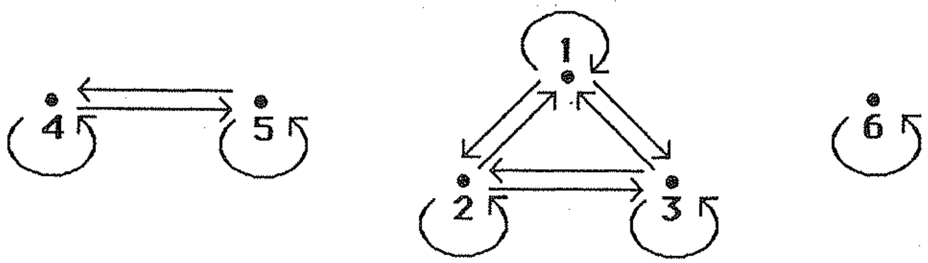
(d) Each set in the partition is represented by one region in this picture.



(e) $[1] = \{1, 5\} = [5]$; $[2] = \{2, 6\} = [6]$; $[3] = \{3\}$; $[4] = \{4\}$

7. **(a)** We list all pairs of numbers that are in the same set in the partition: $\{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (4,4), (4,5), (5,4), (5,5), (6,6)\}$.

(b) An arrow is drawn between two numbers if the corresponding pair is in our list from part (a).

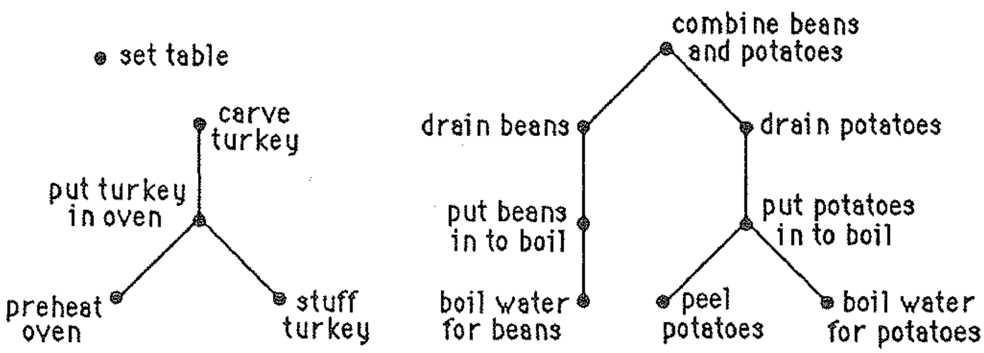


9. **(a)** m is the minimum element of $(A, \preceq)$ if and only if $\forall x \in A\colon m \preceq x$

(b) m is a minimal element of $(A, \preceq)$ if and only if $\forall x \in A\colon (x \preceq m \rightarrow x = m)$

11. The empty relation on $\{1\}$ is antisymmetric and transitive (vacuously), but it is not reflexive. The relation that always holds on $\{1, 2\}$ is reflexive and transitive (trivially), but it is not antisymmetric. The relation $\{(1,1), (1,2), (2,2), (2,3), (3,3)\}$ on $\{1, 2, 3\}$ is reflexive and antisymmetric, but it is not transitive (since $(1,3)$ is missing).

13. A line from one point upward to another point means that the first activity must be completed before the second activity can be started. (We are assuming that the vegetables are cooked in boiling water, rather than boiling the water with the vegetables already in the pot.)



15. (a) If $y = px$ for some prime p, then $x \mid y$ but there can be no intermediate numbers z such that $x \mid z$ and $z \mid y$, since each proper multiple of a number must have at least one more prime factor. This characterizes the notion of being an immediate predecessor for this relation.

(b) For each prime divisor of y, the integer y/p is an immediate predecessor of y; there are no others. Hence the number of immediate predecessors is equal to the numbers of distinct prime divisors of y. (In particular, 1 has no immediate predecessors.)

(c) For each x and each prime p, the integer px is an immediate successor of x. Since there are an infinite number of primes, each x has an infinite number of immediate successors.

17. (a) The proof of Theorem 1 in Section 3.3 showed that the intersection of transitive relations in transitive. Exercise 20d and 20e in Section 3.3 showed the same for reflexivity and antisymmetry.

(b) This is false. Consider the $\leq$ and $\geq$ relations on $\mathbf{N}$. Their union is the relation that always holds—it is not antisymmetric.

(c) This is always false. The complement is definitely not reflexive.

19. (a) Every element $A \in \mathcal{P}(\mathbf{N})$ satisfies $\emptyset \subseteq A \subseteq \mathbf{N}$. Therefore the minimum element (and hence the only minimal element) is $\emptyset$, and the maximum element (and hence the only maximal element) is $\mathbf{N}$.

(b) The minimum element (and hence the only minimal element) is $(1,1)$. The maximum element (and hence the only maximal element) is $(3,3)$.

(c) The minimal elements are a and b. There is no minimum element. The maximum element (and hence the only maximal element) is e.

(d) The minimal elements are a, b, c, h, and i. The maximal elements are f, g, h, and j. Note that h is both minimal and maximal. There are no minimum or maximum elements.

(e) The minimum element (and hence the only minimal element) is f. The maximal elements are b and c. There is no maximum element.

(f) The element 1 is both the minimum and the maximum (and hence also minimal and maximal).

(g) Both elements 1 and 2 are minimal and maximal. There is no minimum or maximum.

21. We could copy the proof of Theorem 2 word for word, replacing "dominate" by "are dominated by," $u \preceq v$ by $v \preceq u$, "maximal" by "minimal," and so on (the word "smallest" is not changed, however). Here is another way to look at it. Let x_1 be any element of A. If x_1 is minimal, then we are done. Otherwise there is an element $x_2 \in A$ such that $x_2 \prec x_1$. If x_2 is minimal, then we are done; otherwise there is an element $x_3 \in A$ such that $x_3 \prec x_2$. We continue in this manner. We can never choose an element previously chosen, for then we would have $x_i \prec x_{i-1} \prec \cdots \prec x_j$, with $x_i = x_j$, whence

by transitivity, $x_i \prec x_j$, a contradiction. Since A has only a finite number of elements, the process must therefore terminate at a minimal element.

23. For a relation that is symmetric and transitive, but not reflexive, take $\emptyset$ on $\{1\}$. For a relation that is reflexive and transitive, but not symmetric, take $\{(1,1),(1,2),(2,2)\}$ on $\{1,2\}$. For a relation that is reflexive and symmetric, but not transitive, take $\{(1,1),(1,2),(2,1),(2,2),(2,3),(3,2),(3,3)\}$ on $\{1, 2, 3\}$. (See also Exercise 22 in Section 3.3.)

25. **(a)** This is an equivalence relation by Theorem 3.

(b) This relation is not transitive: $1R2 \wedge 2R3$, but $1 \not R 3$.

(c) This is an equivalence relation. One way to see this fact is by applying Theorem 3 using "measures of the angles" as the function.

(d) This is an equivalence relation by Theorem 3, where $f(x)$ is the set of all prime divisors of x.

(e) This is an equivalence relation by Theorem 3, where $f(x) = (x\text{'s father}, x\text{'s mother})$.

(f) This relation is not transitive.

27. the equivalence classes of R

29. Since $x - x = 0 = 0 \cdot m$ is a multiple of m, we know that the relation is reflexive. If $x - y = km$, then $y - x = (-k)m$; this tells us that the relation is symmetric. If $x - y = km$ and $y - z = lm$, then $x - z = (x - y) + (y - z) = km + lm = (k + l)m$; this tells us that the relation is transitive.

31. **(a)** one, namely $\emptyset$

(b) one, namely $\{\{a\}\}$

(c) two, namely $\{\{a,b\}\}$ and $\{\{a\},\{b\}\}$

33. **(a)** Rather than list all the partitions, we will just figure out how many there are with each pattern of cardinalities. There is one partition consisting of just one set. There are four partitions consisting of a set with three elements and a set with one element, since there are four ways to decide on the singleton. There are three partitions with two sets of cardinality 2, since A can be in the same class as B, as C, or as D. There are six partitions in which there is one set of cardinality 2 and two singletons, since there are six ways to pick the doubleton. Finally, there is the one partition in which all the elements are in sets by themselves. Hence the answer is $1 + 4 + 3 + 6 + 1 = 15$.

(b) By part (a) and Theorem 4, the answer is 15.

35. **(a)** This follows from the proof of Theorem 1 in Section 3.3.

(b) This is false—transitivity may fail. For example, let $R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$, and let $R_2 = \{(1,1), a(2,2), (2,3), (3,2), (3,3)\}$. Then $R_1 \cup R_2$ is not transitive.

(c) Let R be an equivalence relation. We must show that R^{-1} is reflexive, symmetric, and transitive. For each $a \in A$, $(a,a) \in R$ since R is reflexive, so $(a,a) \in R^{-1}$ as well. Thus R^{-1} is reflexive. The inverse of any symmetric relation is itself (see Exercise 31 in Section 3.3), so R^{-1} is symmetric. Finally, suppose that $(x,y) \in R^{-1}$ and $(y,z) \in R^{-1}$. Then $(y,x) \in R$ and $(z,y) \in R$, so by the transitivity of R, $(z,x) \in R$. This means that $(x,z) \in R^{-1}$; therefore R^{-1} is transitive.

(d) This is always false; the complement is never reflexive.

37. Every positive integer x can be written as the product of a power of 2 and its "odd part": $x = 2^m \cdot y$, where y is odd and $m \geq 0$.

(a) Here the equivalence classes are the sets of numbers with the same power of 2 in this decomposition (which is the smallest element in each class): $\{\{1,3,5,7,\ldots\}, \{2,6,10,14,\ldots\}, \{4,12,20,28,\ldots\}, \{8,24,40,56,\ldots\}, \ldots\}$.

(b) Here the equivalence classes are the sets of numbers with the same odd part (which is again the smallest element in each class): $\{\{1,2,4,8,16,\ldots\}, \{3,6,12,24,\ldots\}, \{5,10,20,40,\ldots\}, \{7,14,28,56,\ldots\}, \ldots\}$.

39. Since the equivalence relation must be reflexive, symmetric, and transitive, we need to form its reflexive, symmetric, transitive closure. This is discussed at length in Section 3.3.

41. **(a)** Since the identity function on A is bijective, we have ARA. If ARB, then there is a bijective function $f: A \to B$. Then $f^{-1}: B \to A$ is a bijective function as well, so BRA. If ARB and BRC, then there are bijective functions $f: A \to B$ and $g: B \to C$. Then $g \circ f$ is a bijective function from A to C (Theorem 2 in Section 3.2), so ARC. Alternatively, we can apply Theorem 3 with $f(A) = |A|$, recalling that all infinite subsets of $\mathbf{N}$ have cardinality $\aleph_0$.

(b) For $i = 0,1,2,\ldots$, let C_i be the set of all finite subsets of $\mathbf{N}$ with cardinality i; and let C_∞ be the set of all infinite subsets of $\mathbf{N}$. For example, $C_1 = \{\{1\}, \{2\}, \{3\}, \ldots\}$. The C_i's and C_∞ are the equivalence classes.

43. **(a)** The $\leq$ relation on $\mathbf{R}$ is dense. If $x \neq y$, then $(x+y)/2$ lies between them.

(b) The relation $\preceq$ on $\mathbf{R} \times \mathbf{R}$ given by $(a,b) \preceq (c,d) \leftrightarrow (a \leq c \land b \leq d)$ is dense but not total. The pairs $(1,3)$ and $(2,2)$, for example, are not comparable, but if $(a,b) \prec (c,d)$, then $a \leq c$, $b \leq d$, and either $a < c$ or $b < d$, and in either case $((a+c)/2, (b+d)/2)$ lies between them.

(c) The $\leq$ relation on $\mathbf{Z}$ is not dense, since, for example, nothing lies between 3 and 4.

(d) Let $a \prec b$. Then there is an element x_1 such that $a \prec x_1 \prec b$. Using the definition of denseness again, we obtain an element x_2 such that $x_1 \prec x_2 \prec b$. Continuing in this way we obtain an infinite sequence of distinct elements $x_1 \prec x_2 \prec x_3 \prec \cdots$.

45. **(a)** Let $\pi_1 = \{\{1\}, \{2,3\}, \{4,5\}\}$ and $\pi_2 = \{\{1,2,3\}, \{4,5\}\}$. Every set in π_1 is contained in some set in π_2 (for example, $\{2,3\} \subseteq \{1,2,3\}$).

(b) Let $\pi_1 = \{\{1,2,3\}, \{4,5\}\}$ and $\pi_2 = \{\{1,2\}, \{3,4\}, \{5\}\}$. The set $\{1, 2, 3\}$ in π_1 is contained in no set in π_2, and the set $\{3, 4\}$ in π_2 is contained in no set in π_1.

(c) We always have $\pi \preceq \pi$, since we can take $Y = X$ in the definition. Thus the refinement relation is reflexive. Next suppose that $\pi_1 \preceq \pi_2$ and $\pi_2 \preceq \pi_1$. Let X be an arbitrary element of π_1. Then $X \subseteq Y$ for some $Y \in \pi_2$. Furthermore, $Y \subseteq X'$ for some $X' \in \pi_1$. Hence $X \subseteq X'$. Since the elements of π_1 are nonempty and pairwise disjoint, this can only happen if $X = X'$. Thus we have $X \subseteq Y \subseteq X$, whence $X = Y$. This shows that every element of π_1 is also in π_2. By similar reasoning every element of π_2 is in π_1, so $\pi_1 = \pi_2$. Thus we have established that $\preceq$ is antisymmetric. Finally, suppose that $\pi_1 \preceq \pi_2$ and $\pi_2 \preceq \pi_3$. We must show that $\pi_1 \preceq \pi_3$. Let $X \in \pi_1$. Then there is a $Y \in \pi_2$ such that $X \subseteq Y$, and hence also a $Z \in \pi_3$ such that $Y \subseteq Z$. But then $X \subseteq Z$, as desired.

(d) The partition $\{\{1,2,3,4,5\}\}$ is the maximum element, since we can always take $Y = \{1, 2, 3, 4, 5\}$ in the definition.

(e) The partition $\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$ is the minimum element (i.e., $\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\} \preceq \pi_2$ for all π_2), since every element of $\{1, 2, 3, 4, 5\}$ is in some element of π_2.

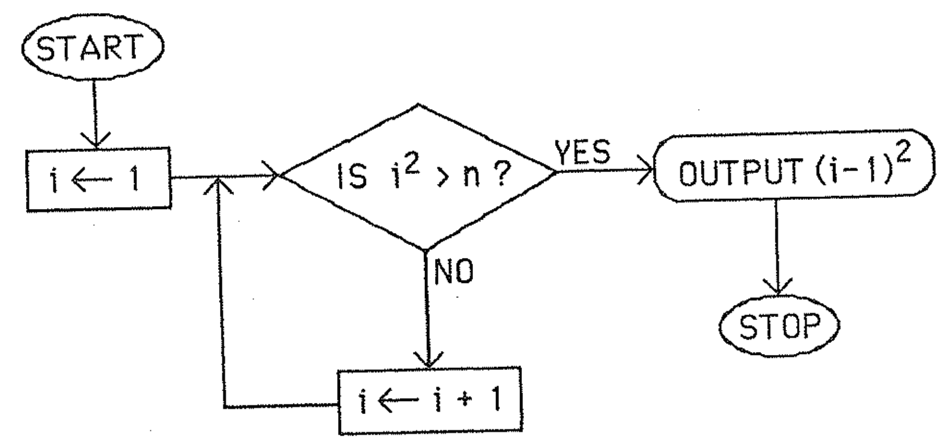
CHAPTER 4
ALGORITHMS

SECTION 4.1 The Idea of an Algorithm

1. Suppose that a, b, and c are the three numbers. Compare a and b. If $a < b$, then compare a and c. If $c < a$, then give a as output; otherwise $(c > a)$ compare b and c, and give b as output if $b < c$ and c as output if $c < b$. On the other hand, if $b < a$, then compare b and c. If $c < b$, then give b as output; otherwise $(c > b)$ compare a and c, and give a as output if $a < c$ and c as output if $c < a$.

3. Set $i = 1$. As long as $i^2 \leq n$, continue to add 1 to i. Stop as soon as $i^2 > n$. Then give $(i-1)^2$ as output.

5. When the counter i first gets so large that $i^2 > n$, then the algorithm takes the branch to the right and outputs the correct answer, namely $(i-1)^2$.



7. (a) Since $900 = 2^2 \cdot 3^2 \cdot 5^2$ and $1750 = 2 \cdot 5^3 \cdot 7$, $\gcd(900, 1750) = 2 \cdot 5^2 = 50$.
   (b) Since $46 = 2 \cdot 23$ and $27 = 3^3$, $\gcd(46, 27) = 1$.
   (c) Since $2323 = 23 \cdot 101$ and $9191 = 7 \cdot 13 \cdot 101$, $\gcd(2323, 9191) = 101$.

9. $d = \gcd(x, y) \leftrightarrow [d \mid x \wedge d \mid y \wedge \forall a\colon ((a \mid x \wedge a \mid y) \rightarrow a \leq d)]$

11. Let the prime factorizations of x and y be $x = p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$ and $y = p_1^{y_1} p_2^{y_2} \cdots p_k^{y_k}$, respectively, where we included all the prime factors of both x and y, so that some of the x_i's or y_i's might be 0. Then $\gcd(x,y) = p_1^{g_1} p_2^{g_2} \cdots p_k^{g_k}$, where $g_i = \min(x_i, y_i)$. Now suppose that $d \mid x \land d \mid y$. Then $d = p_1^{d_1} p_2^{d_2} \cdots p_k^{d_k}$, where each $d_i \leq x_i$ and $d_i \leq y_i$. Therefore $\forall i : d_i \leq g_i$, so $d \mid \gcd(x,y)$.

13. **(a)** $m = \operatorname{lcm}(x,y) \leftrightarrow [x \mid m \land y \mid m \land \forall z : ((x \mid z \land y \mid z) \to m \leq z)]$

(b) Let the prime factorizations of x and y be $x = p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$ and $y = p_1^{y_1} p_2^{y_2} \cdots p_k^{y_k}$, respectively, where we included all the prime factors of both x and y, so that some of the x_i's or y_i's might be 0. Then $\operatorname{lcm}(x,y) = p_1^{l_1} p_2^{l_2} \cdots p_k^{l_k}$, where $l_i = \max(x_i, y_i)$.

(c) If x and y are positive integers and m is any positive common multiple of x and y, then $\operatorname{lcm}(x,y) \mid m$. *Proof*: (This is very similar to Exercise 11, above.) Write the prime factorizations of x, y, and their least common multiple as in part (b). Now suppose that $x \mid m$ and $y \mid m$. Then $m = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$, where each $m_i \geq x_i$ and $m_i \geq y_i$. Therefore $\forall i : m_i \geq l_i$, so $\operatorname{lcm}(x,y) \mid m$.

(d) Using the notation in part (b) and Exercise 11, we have $x = p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$, $y = p_1^{y_1} p_2^{y_2} \cdots p_k^{y_k}$, $\operatorname{lcm}(x,y) = p_1^{l_1} p_2^{l_2} \cdots p_k^{l_k}$, and $\gcd(x,y) = p_1^{g_1} p_2^{g_2} \cdots p_k^{g_k}$, where $l_i = \max(x_i, y_i)$ and $g_i = \min(x_i, y_i)$. Then the prime p_i appears to the power $x_i + y_i$ in $x \cdot y$, and to the power $l_i + g_i$ in $\operatorname{lcm}(x,y) \cdot \gcd(x,y)$. Since for any two numbers u and v it is always true that $\max(u,v) + \min(u,v) = u + v$, these two exponents agree. The desired conclusion follows.

(e) Compute $\gcd(x,y)$ by the Euclidean Algorithm. Then compute $xy/\gcd(x,y)$, which by part (d) equals $\operatorname{lcm}(x,y)$.

(f) First compute $\gcd(50059, 98789) = \gcd(98789, 50059) = \gcd(50059, 48730) = \gcd(48730, 1329) = \gcd(1329, 886) = \gcd(886, 443) = \gcd(443, 0) = 443$.
Then $\operatorname{lcm}(50059, 98789) = 50059 \cdot 98789/\gcd(50059, 98789) = 50059 \cdot 98789/443 = 11163157$.

15. We want to prove that if $d \mid \gcd(x,y)$, then $d \mid x$ and $d \mid y$. This is just a special case of the transitivity of the "divides" relation (see Exercise 14 in Section 3.4), since $\gcd(x,y) \mid x$ and $\gcd(x,y) \mid y$.

17. Note that the remainder when $x^2 - 1$ is divided by x is $x - 1$, since $x^2 - 1 = x(x-1) + (x-1)$. Thus by Theorem 2 we have $\gcd(x^2 - 1, x) = \gcd(x, x-1) = \gcd(x-1, 1) = \gcd(1, 0) = 1$.

19. **(a)** Since we can compute by the Euclidean Algorithm that $\gcd(238133, 341936) = 7$, we have $\dfrac{238133}{341936} = \dfrac{238133/7}{341936/7} = \dfrac{34019}{48848}$ in lowest terms.

(b) Since we can compute by the Euclidean Algorithm that $\gcd(238134, 341936) = 6106$, we have $\dfrac{238134}{341936} = \dfrac{238134/6106}{341936/6106} = \dfrac{39}{56}$ in lowest terms.

(c) Since we can compute by the Euclidean Algorithm that $\gcd(238135, 341936) = 1$, we know that $\dfrac{238135}{341936}$ is already in lowest terms.

21. We exploit here the facts that (1) we can assume that $i \le j \le k \le l$ and (2) there is always a solution.

1. Set i equal to 0.
2. Set j equal to i.
3. If $i^2 + 3j^2 > N$, then replace i by $i+1$, return to step 2, and continue from there. Otherwise, continue with step 4.
4. Set k equal to j.
5. If $i^2 + j^2 + 2k^2 > N$, then replace j by $j+1$, return to step 3, and continue from there. Otherwise, continue with step 6.
6. Set l equal to k.
7. If $i^2 + j^2 + k^2 + l^2 = N$, then output (i, j, k, l) and stop. If $i^2 + j^2 + k^2 + l^2 > N$, then replace k by $k+1$, return to step 5, and continue from there. Otherwise (i.e., when $i^2 + j^2 + k^2 + l^2 < N$), replace l by $l+1$ and repeat this step.

23. Set *largest* equal to s_1, and set *second* equal to $-\infty$. [This is not the only way to initialize the algorithm; we could also start by setting *largest* equal to the larger of s_1 and s_2, and *second* equal to the smaller.] Now for each i from 2 to n, repeat the following steps: compare s_i to *second*, and if $s_i >$ *second*, then replace *second* by s_i, compare the new value of *second* to *largest*, and if *second* $>$ *largest*, interchange *second* and *largest*. At the end, output *second*.

25.

1. Set i equal to 1.
2. If $i > N$, then output 0 and stop. Otherwise, continue with step 3.
3. If $w = w_i$, then output i and stop. Otherwise continue with step 4.
4. Replace i by $i+1$, return to step 2, and continue from there.

27. In each case, let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$ be the input sets, with $a_1 < a_2 < \cdots < a_n$ and $b_1 < b_2 < \cdots < b_m$. Set a_{n+1} and b_{m+1} equal to ∞. The output is the set $C = \{c_1, c_2, \ldots, c_k\}$.

(a) 1. Set i equal to 1; set j equal to 1; and set k equal to 0.

2. If $i = n+1$ and $j = m+1$, then output $\{c_1, c_2, \ldots, c_k\}$ and stop. Otherwise, continue with step 3.

3. If $a_i = b_j$, then continue with step 4. If $a_i < b_j$, then replace i by $i+1$; otherwise (when $a_i > b_j$) replace j by $j+1$. In either of these cases, return to step 2 and continue from there.

4. Replace k by $k+1$; set c_k equal to a_i; replace i by $i+1$; replace j by $j+1$; and return to step 2 and continue from there.

(b) 1. Set i equal to 1; set j equal to 1; and set k equal to 0.

2. If $i = n+1$ and $j = m+1$, then output $\{c_1, c_2, \ldots, c_k\}$ and stop. Otherwise, replace k by $k+1$ and continue with step 3.

3. If $a_i = b_j$, then continue with step 4. If $a_i < b_j$, then skip to step 5 and continue from there. If $a_i > b_j$, then skip to step 6 and continue from there.

4. Set c_k equal to a_i; replace i by $i+1$; replace j by $j+1$; and return to step 2 and continue from there.

5. Set c_k equal to a_i; replace i by $i+1$; and return to step 2 and continue from there.

6. Set c_k equal to b_j; replace j by $j+1$; and return to step 2 and continue from there.

(c) 1. Set i equal to 1; set j equal to 1; and set k equal to 0.

2. If $i = n+1$, then output $\{c_1, c_2, \ldots, c_k\}$ and stop. Otherwise, continue with step 3.

3. If $a_i = b_j$, then replace i by $i+1$, replace j by $j+1$, return to step 2 and continue from there. Otherwise, if $a_i > b_j$, then replace j by $j+1$ and repeat this step from its beginning. Otherwise (i.e., when $a_i < b_j$) continue with step 4.

4. Replace k by $k+1$; set c_k equal to a_i; replace i by $i+1$; and return to step 2 and continue from there.

29. If and when we obtain a square, we are done—we have found the common measure. Given any nonsquare rectangle $B'CDE$, we fold along AC, to bring corner B' to point B on the opposite longer side, as shown below.