39. This problem is the "logarithm" of Exercise 34.

(a) The recurrence relation is $a_n = (a_{n-1}a_{n-2})^{1/2}$, with initial conditions $a_0 = x$ and $a_1 = y$. Let $b_n = \log a_n$, and our problem becomes $b_n = (b_{n-1}+b_{n-2})/2$, with $b_0 = \log x$ and $b_1 = \log y$.

(b) We solved this problem for b_n in Exercise 34, obtaining (translated into this context)

$$b_n = \frac{1}{3}(\log x + 2\log y) + \frac{2}{3}(\log x - \log y)\left(-\frac{1}{2}\right)^n .$$

Thus we have

$$\begin{aligned} a_n = 2^{b_n} &= 2^{\frac{1}{3}(\log x+2\log y)+\frac{2}{3}(\log x-\log y)(-\frac{1}{2})^n} \\ &= x^{1/3}y^{2/3}\left(x^{2/3}y^{-2/3}\right)^{(-1/2)^n} . \end{aligned}$$

(c) As $n \to \infty$, $(-1/2)^n \to 0$, so $a_n \to x^{1/3}y^{2/3} = \sqrt[3]{xy^2}$.

SECTION 7.4 The Inclusion–Exclusion Principle

1. Let C, R, and A be the sets of students who like to play chess, bridge, and backgammon, respectively. Then by the inclusion–exclusion principle we have $60 = |C \cup R \cup A| = |C|+|R|+|A|-|C\cap R|-|C\cap A|-|R\cap A|+|C\cap R\cap A| = 37+31+19-11-16-5+|C\cap R\cap A|$. Thus $60 = 55 + |C \cap R \cap A|$, so $|C \cap R \cap A| = 5$.

3. Let B, R, and G stand for the sets of competitors who won blue, red, and green ribbons, respectively. We are told that $|B| = 13$, $|R| = 25$, and $|G| = 23$. We are also told that $|B \cap R| + |B \cap G| + |R \cap G| = 17$ and that $|B \cap R \cap G| = 0$. Therefore by the inclusion–exclusion principle we know that $|B \cup R \cup G| = 13 + 25 + 23 - 17 + 0 = 44$ people won ribbons. Since there are 100 competitors in all, the remaining 56 won no ribbons.

5. $\displaystyle\sum_{i=0}^{n-1} C(n,i)(-1)^i(n-i)^k$

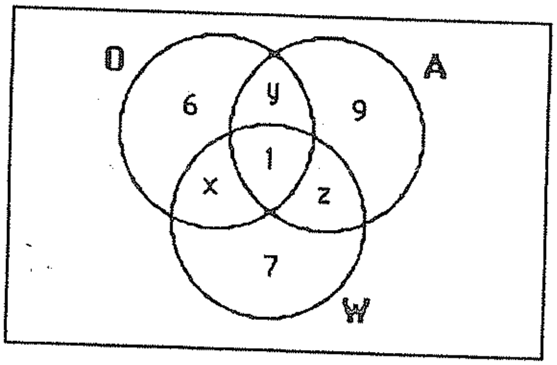
7. The prime numbers less than or equal to 40 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, and 37, for a total of 12. For the calculation based on Theorem 3, we note that since $\sqrt{40} < 7$, we need only consider the primes 2, 3, and 5. Then the number of prime numbers not exceeding 40 is

$$\begin{aligned} 39 - \left(\left\lfloor\frac{40}{2}\right\rfloor - 1\right) - \left(\left\lfloor\frac{40}{3}\right\rfloor - 1\right) - \left(\left\lfloor\frac{40}{5}\right\rfloor - 1\right) \\ + \left\lfloor\frac{40}{2\cdot 3}\right\rfloor + \left\lfloor\frac{40}{2\cdot 5}\right\rfloor + \left\lfloor\frac{40}{3\cdot 5}\right\rfloor - \left\lfloor\frac{40}{2\cdot 3\cdot 5}\right\rfloor \\ = 39 - 19 - 12 - 7 + 6 + 4 + 2 - 1 = 12 . \end{aligned}$$

9. We are told that $|G| = 12$, $|F| = 9$, $|R| = 12$, $|F \cap R| = 1$, $|G \cap R| = 3$, and $|F \cap G| = 2$. Therefore $|G \cup F \cup R| = |G| + |F| + |R| - |F \cap R| - |G \cap R| - |F \cap G| + |F \cap G \cap R| = 12 + 9 + 12 - 1 - 3 - 2 + |F \cap G \cap R| = 27 + |F \cap G \cap R|$, which contradicts the fact that only 26 people were present.

11. There are four sets, A, B, C, and D, each with 250 elements; $C(4,2) = 6$ pairs (such as B and D), each with 20 elements, and $C(4,3) = 4$ triples (such as A, C, and D), each with three elements. Thus we have by the inclusion–exclusion principle $|A \cup B \cup C \cup D| = 4 \cdot 250 - 6 \cdot 20 + 4 \cdot 3 - 1 = 891$.

13. (a) Let x be the number of artists who use only oil and water color, y the number of artists who use only oil and acrylic, and z the number of artists who use only acrylic and water color. The Venn diagram for this problem is as shown here.



The given information tells us that $x + y = 7$, $x + z = 9$, and $y + z = 8$. We want to find $x + y + z$. If we add the three equations just presented, we find that $2(x + y + z) = 24$, so $x + y + z = 12$.

(b) The sum of all the numbers in the Venn diagram is $x + y + z + 23$, which by the answer to part (a) is $12 + 23 = 35$.

15. Note that unless otherwise stated, we need not assign a task to every employee.

(a) We model this problem by counting functions from the set of tasks to the set of employees (the value of the function at a given task is the employee assigned to do that task). We need to count the number of functions from the set of tasks to the set of employees, which is $10^6 = 1{,}000{,}000$.

(b) This time we need to count the number of surjective functions from the set of tasks to the set of employees, since each employee must be in the range of the assignment function. Since there are more employees than tasks, there are no such functions.

(c) This is really the resource allocation problem from Section 6.3; all that matters is how many of the six tasks get assigned to each employee. Thus we want to count the number of solutions to $x_1 + x_2 + \cdots + x_{10} = 6$, where x_i is the number of tasks assigned to employee i (each $x_i \geq 0$). The answer is therefore $C(10+6-1, 6) = C(15, 6) = 5005$.

(d) Again this is the resource allocation problem, except that now we want the number of solutions to $x_1 + x_2 + \cdots + x_{10} = 6$, where each $x_i \geq 1$. Obviously if each $x_i \geq 1$, then the sum is greater than 6, so there are again no solutions.

17. The formula for the number of derangements is

$$d_n = n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \cdots \pm \frac{n!}{(n-1)!} \mp \frac{n!}{n!}.$$

All of the terms on the right-hand side are even except that $n!/n! = 1$ is odd and $n!/(n-1)! = n$ is either even or odd, depending on the parity of n. Hence if n is even, then the sum is odd; and if n is odd, then the sum is even.

19. If $n-1$ objects are in their original positions, then there is no place for the other object to go except its original position.

21. Note that there are 15! orders in which the hats can be returned. Thus these probabilities will have 15! as their denominators.

(a) By symmetry the answer should be $1/15$; he is just as likely to get his own hat back as any of the 15 hats. More formally, of the 15! ways to return the hats, he can receive his own hat back in 14! ways (the other 14 hats can be permuted arbitrarily); therefore the answer is $14!/15! = 1/15$.

(b) There are d_{14} ways to derange the remaining 14 hats, so the answer is $d_{14}/15! \approx 1/41$.

(c) After we have given these two gentlemen their own hats back, there are 13! ways to permute the remaining hats. Therefore the answer is $13!/15! = 1/(15 \cdot 14) = 1/210$.

(d) After we have given these two gentlemen their own hats back, there are d_{13} ways to derange the remaining hats. Therefore the answer is $d_{13}/15! \approx 1/571$.

(e) There are 5! ways to permute the Zetas' hats and 10! ways to permute the hats of the other fellows, so the answer is $5!10!/15! = 1/3003$.

(f) This can only happen in one way, so the answer is $1/15! = 1/1{,}307{,}674{,}368{,}000$.

(g) There are 15 ways to choose the lucky man to get his own hat back, and then there are d_{14} ways to derange the remaining hats, so the answer is $15d_{14}/15! = d_{14}/14! \approx 1/3$.

(h) This is clearly $d_{15}/15! \approx 1/3$. Note that for all practical purposes, this is the same probability as in part (g), namely almost exactly $1/e \approx 0.3679$.

(i) At least one person receives his hat back if it is not the case that no person receives his hat back. Therefore the answer is $1 - (d_{15}/15!)$, or about 2 out of 3.

23. (a) If we want to leave all the odd numbers in their original positions, then we simply permute the five even numbers. Therefore the answer is $5! = 120$.

(b) We must derange the even numbers, so the answer is $d_5 = 44$.

(c) There are $C(10,5)$ ways to choose the five numbers to leave fixed, and then there are d_5 ways to derange the numbers not chosen. Therefore the answer is $C(10,5)d_5 = 252 \cdot 44 = 11088$.

(d) A permutation fails to meet this condition if it is a derangement (and there are $d_{10} = 1334961$ of these), or if it leaves one number fixed and deranges the rest (and there are $10 \cdot d_9 = 1334960$ of these). Therefore the answer is $10! - 1334961 - 1334960 = 958{,}879$.

25. There are $300/2 = 150$ odd positive integers not exceeding 300. There are $\lfloor\sqrt{300}\rfloor = 17$ perfect squares in this range. Furthermore, $\lceil 17/2 \rceil = 9$ of these perfect squares are odd. By the inclusion–exclusion principle there are $150 + 17 - 9 = 158$ positive integers in this range that are either odd or perfect squares. This includes the nine numbers that were both odd and perfect squares, and we are asked to omit these, so the answer is 149.

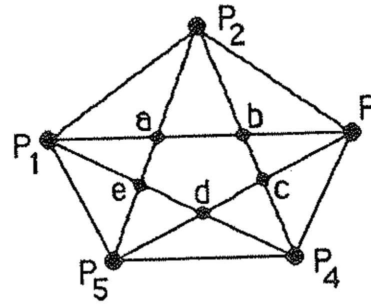
27. Let A_i be the set of 7-bit strings that have 00000 starting in position i, for $i = 1, 2, 3$. Then we want to compute as follows.

$$\begin{aligned}|\overline{A_1 \cup A_2 \cup A_3}| &= 2^7 - |A_1| - |A_2| - |A_3| + |A_1 \cap A_2| + |A_1 \cap A_3| \\ &\quad + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3| \\ &= 2^7 - 3 \cdot 2^2 + 2 + 1 + 2 - 1 = 2^7 - 8 = 120\end{aligned}$$

29. A number in this range is divisible by some perfect square if and only if it is divisible by a number from the set $\{4, 9, 25, 49, 121, 169, 289, 361\}$ (we need only worry about squares of prime numbers). We count (using the inclusion–exclusion principle) in much the same way that we counted to determine the number of prime numbers (the terms left out are all 0).

$$\begin{aligned}&\left\lfloor\frac{500}{4}\right\rfloor + \left\lfloor\frac{500}{9}\right\rfloor + \left\lfloor\frac{500}{25}\right\rfloor + \left\lfloor\frac{500}{49}\right\rfloor + \left\lfloor\frac{500}{121}\right\rfloor \\ &\quad + \left\lfloor\frac{500}{169}\right\rfloor + \left\lfloor\frac{500}{289}\right\rfloor + \left\lfloor\frac{500}{361}\right\rfloor - \left\lfloor\frac{500}{4\cdot 9}\right\rfloor - \left\lfloor\frac{500}{4\cdot 25}\right\rfloor \\ &\quad - \left\lfloor\frac{500}{4\cdot 49}\right\rfloor - \left\lfloor\frac{500}{4\cdot 121}\right\rfloor - \left\lfloor\frac{500}{9\cdot 25}\right\rfloor - \left\lfloor\frac{500}{9\cdot 49}\right\rfloor \\ &= 125 + 55 + 20 + 10 + 4 + 2 + 1 + 1 - 13 - 5 - 2 - 1 - 2 - 1 = 194\end{aligned}$$

31. We label the points of the figure as shown in the following diagram.



Let A_i be the set of triangles in this figure that have the point P_i as one of their vertices. Note that every triangle involves at least one of these five points. Now each $|A_i| = 15$; for example the triangles involving P_1 are P_1P_2a, P_1P_2e, $P_1P_2P_5$, P_1P_2b, $P_1P_2P_4$, $P_1P_2P_3$, P_1ae, P_1aP_5, P_1bP_4, P_1P_3d, $P_1P_3P_5$, $P_1P_3P_4$, P_1eP_5, P_1dP_5, and $P_1P_4P_5$. Next to compute $|A_i \cap A_j|$, where $i \neq j$, we note that there are two cases. If P_i and P_j are adjacent, then (as we see from the list above for the case of P_1 and P_2) there are six triangles in $A_i \cap A_j$. If P_i and P_j are not adjacent, then (as we see from the list above for the case of P_1 and P_3) there are four triangles in $A_i \cap A_j$. Finally, we see that $|A_i \cap A_j \cap A_k| = 1$ for all triples of distinct i, j, and k. Now we apply the inclusion–exclusion principle, noting that there are five A_i's, five adjacent A_i–A_j pairs, five nonadjacent A_i–A_j pairs, and 10 triples. Thus the answer is $5\cdot15-5\cdot6-5\cdot4+10\cdot1 = 35$. (An alternative approach is to base the count on the number of triangles using each of the segments P_iP_{i+1} and P_5P_1. We find $5\cdot6-5=25$ triangles involving one or more of these segments, together with (by inspection) 10 triangles involving none of them.)

SECTION 7.5 Generating Functions

1. (a) If we expand this binomial, we get $4+12x+9x^2$. Therefore the sequence that this function represents is 4, 12, 9, or 4, 12, 9, 0, 0, 0, ...

(b) Since $1/(1+x) = 1-x+x^2-x^3+\cdots$, the sequence is 1, −1, 1, −1, ... Explicitly, $a_k = (-1)^k$.

(c) Since $3/(1-4x) = 3(1+4x+(4x)^2+(4x)^3+\cdots)$, the sequence is 3, $3\cdot4$, $3\cdot4^2$, $3\cdot4^3$, ..., i.e., 3, 12, 48, 192, ... Explicitly, $a_k = 3\cdot4^k$

(d) Since $3/(1-x^4) = 3(1+x^4+(x^4)^2+(x^4)^3+\cdots) = 3(1+x^4+x^8+x^{12}+\cdots)$, the sequence is 3, 0, 0, 0, 3, 0, 0, 0, 3, 0, 0, 0, ... Explicitly, $a_k = 3$ if $k \equiv 0 \pmod 4$ and $a_k = 0$ otherwise.

(e) Since $2x/(1+x^2) = 2x(1-x^2+x^4-x^6+\cdots) = 2(x-x^3+x^5-x^7+\cdots)$, the sequence is 0, 2, 0, −2, 0, 2, 0, −2, ... Explicitly, $a_k = 2$ if $k \equiv 1 \pmod 4$; $a_k = -2$ if $k \equiv 3 \pmod 4$; and $a_k = 0$ otherwise.

3. In each case we use Theorem 3.

(a) The general term is $C(k+1,1)=k+1$; thus the sequence begins 1, 2, 3, 4, 5, ...

(b) The general term is $C(k+2,2)=(k+2)(k+1)/2$; thus the sequence begins 1, 3, 6, 10, 15, ...

(c) The general term is $C(k+3,3)=(k+3)(k+2)(k+1)/6$; thus the sequence begins 1, 4, 10, 20, 35, ...

(d) The general term is $C(k+4,4)=(k+4)(k+3)(k+2)(k+1)/24$; thus the sequence begins 1, 5, 15, 35, 70, ...

5. (a) We found that the generating function is $1+2x+3x^2+3x^3+3x^4+3x^5+2x^6+x^7$. Since the coefficient of x^4 in this expression is 3, there are three ways to choose four balls.

(b) We found that the generating function is $x+x^2+2x^3+x^4+2x^5+x^6+x^7$. Since the coefficient of x^4 in this expression is 1, there is one way to choose four balls.

7. (a) This sequence is just 2 times the sequence $1, -1, 1, -1, \ldots$. Therefore its generating function is $f(x)=2/(1+x)$.

(b) This sequence is 4 times the sequence 1, 2, 4, 8, ... Therefore by Theorem 1 with $a=2$ its generating function is $f(x)=4/(1-2x)$.

(c) This sequence is just 2 times the sequence 1, 2, 3, 4, ... Therefore by Theorem 3 with $r=1$ its generating function is $f(x)=2/(1-x)^2$.

(d) This sequence is the same as 1, 1, 1, 1, ..., shifted over three places. Therefore its generating function is $f(x)=x^3/(1-x)$.

(e) Since $f(x)=1/(1+x^2)$ is the generating function for $1, 0, -1, 0, 1, 0, -1, \ldots$, the answer is $f(x)=x/(1+x^2)$.

(f) Note that

$$\begin{aligned}2+3x+4x^2+\cdots &= \frac{1}{x}(1+2x+3x^2+4x^3+\cdots-1)\\ &= \frac{1}{x}\left(\frac{1}{(1-x)^2}-1\right)=\frac{1-(1-x)^2}{x(1-x)^2}\\ &= \frac{1-1+2x-x^2}{x(1-x)^2}=\frac{2-x}{(1-x)^2}.\end{aligned}$$

Therefore the generating function is $f(x)=(2-x)/(1-x)^2$.

(g) Using our result in part (c),

$$\begin{aligned}&1+3x+5x^2+7x^3+\cdots\\ &\quad=(2+4x+6x^2+8x^3+\cdots)-(1+x+x^2+x^3+\cdots)\\ &\quad=\frac{2}{(1-x)^2}-\frac{1}{1-x}=\frac{1+x}{(1-x)^2}.\end{aligned}$$

Therefore the generating function is $f(x) = (1+x)/(1-x)^2$.

(h) Here we use the results from parts (c) and (g):

$$\begin{aligned} &2 + x + 4x^2 + 3x^3 + 6x^4 + 5x^5 + \cdots \\ &= (2 + 4x^2 + 6x^4 + \cdots) + x(1 + 3x^2 + 5x^4 + \cdots) \\ &= \frac{2}{(1-x^2)^2} + \frac{x(1+x)}{(1-x^2)^2} = \frac{x^2+x+2}{(1-x^2)^2}. \end{aligned}$$

Therefore the generating function is $f(x) = (x^2+x+2)/(1-x^2)^2$.

9. Thinking of the sequence split in this way, we see that the generating function is

$$f(x) = \frac{1}{1-x} - x^2\left(\frac{1}{1-x^3}\right).$$

We find a common denominator by recalling that $1-x^3 = (1-x)(1+x+x^2)$, and after a little algebra this function simplifies to $f(x) = (1+x)/(1-x^3)$.

11. **(a)** Here the coefficient of x^k represents the number of ways to achieve k cents. For the 3-cent stamps the generating function is $1+x^3+x^6+x^9+\cdots = 1/(1-x^3)$. Similarly the generating function for choosing just the 4-cent stamps is $1/(1-x^4)$, and analogously for the 5-cent stamps. Therefore the generating function is

$$\begin{aligned} f(x) &= (1+x^3+x^6+\cdots)(1+x^4+x^8+\cdots)(1+x^5+x^{10}+\cdots) \\ &= \frac{1}{(1-x^3)(1-x^4)(1-x^5)}. \end{aligned}$$

To write this out explicitly we multiply out the left-hand side of the displayed equation, obtaining $f(x) = 1 + x^3 + x^4 + x^5 + x^7 + 2x^8 + 2x^9 + 2x^{10} + \cdots$.

(b) Since the coefficient of x^{10} in this expression is 2, there are two ways to achieve 10 cents (in fact, we can use two 5-cent stamps or two 3-cent stamps and a 4-cent stamp).

13. This question asks for the number of partitions of 6 under various restrictions. It is probably easier to count the partitions than to work with the algebra. We can write 6 as 6, $5+1$, $4+2$, $4+1+1$, $3+3$, $3+2+1$, $3+1+1+1$, $2+2+2$, $2+2+1+1$, $2+1+1+1+1$, and $1+1+1+1+1+1$.

(a) The list above has 11 partitions, so the answer is 11.

(b) Four of the partitions listed above use only odd parts (1, 3, and 5), so the answer is 4.

(c) Four of the partitions listed above use parts of unequal size (the first three and the sixth one), so the answer is 4.

(d) All but the first two partitions shown above satisfy this condition, so the answer is $11 - 2 = 9$.

15. (a) We want the coefficient of x^{30} in $(1+x+x^2+x^3+\cdots)^4 = 1/(1-x)^4$. By Theorem 3, this is $C(30+3,3) = C(33,3) = 5456$.

(b) We want the coefficient of x^{30} in the generating function

$$\begin{aligned}
&(1+x+x^2+x^3+\cdots+x^9)^4\\
&= \left(\frac{1-x^{10}}{1-x}\right)^4\\
&= \frac{1-4x^{10}+6x^{20}-4x^{30}+x^{40}}{(1-x)^4}\\
&= \frac{1}{(1-x)^4} - 4x^{10}\frac{1}{(1-x)^4} + 6x^{20}\frac{1}{(1-x)^4} - 4x^{30}\frac{1}{(1-x)^4} + x^{40}\frac{1}{(1-x)^4}.
\end{aligned}$$

Now an x^{30} term arises from each of the first four terms in the last line. The coefficient of x^{30} in $1/(1-x)^4$ is 5456, as we found in part (a). The coefficient of x^{30} in $4x^{10}/(1-x)^4$ is 4 times the coefficient of x^{20} in $1/(1-x)^4$, namely $4C(20+3,3)$. A similar analysis is applied to the next two terms. The desired coefficient is therefore $C(30+3,3) - 4C(20+3,3) + 6C(10+3,3) - 4C(0+3,3) = 5456 - 7084 + 1716 - 4 = 84$.

(c) We want the coefficient of x^{30} in

$$(x+x^2+x^3+\cdots)^4 = \frac{x^4}{(1-x)^4},$$

which is the coefficient of x^{26} in $1/(1-x)^4$, namely $C(26+3,3) = 3654$.

(d) We want the coefficient of x^{30} in

$$(x+x^2+x^3+\cdots+x^9)^4 = x^4\left(\frac{1-x^9}{1-x}\right)^4 = \frac{x^4-4x^{13}+6x^{22}-4x^{31}+x^{40}}{(1-x)^4},$$

which is $C(26+3,3) - 4C(17+3,3) + 6C(8+3,3) = 3654 - 4560 + 990 = 84$.

17. For each die the generating function for the number of spots showing is $x+x^2+x^3+x^4+x^5+x^6$. Therefore the generating function for the total number of spots showing on n dice is

$$f(x) = (x+x^2+x^3+x^4+x^5+x^6)^n = \left(\frac{x(1-x^6)}{1-x}\right)^n = \frac{x^n(1-x^6)^n}{(1-x)^n}.$$

19. In each case we let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ be the generating function for the given sequence. Then

$$xf(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1}x^k \quad\text{and}\quad x^2 f(x) = \sum_{k=0}^{\infty} a_k x^{k+2} = \sum_{k=2}^{\infty} a_{k-2}x^k.$$

(a) First we note that $1/(1-4x) = \sum_{k=0}^{\infty} 4^k x^k$. Then in order to take advantage of the recurrence relation, we look at

$$\begin{aligned}
f(x) - 5xf(x) + 6x^2f(x) - \frac{1}{1-4x} \\
&= \sum_{k=0}^{\infty} a_k x^k - 5\sum_{k=1}^{\infty} a_{k-1}x^k + 6\sum_{k=2}^{\infty} a_{k-2}x^k - \sum_{k=0}^{\infty} 4^k x^k \\
&= \sum_{k=2}^{\infty}(a_k - 5a_{k-1} + 6a_{k-2} - 4^k)x^k + a_0 + a_1x - 5a_0x - 1 - 4x \\
&= 1 + 2x - 5x - 1 - 4x = -7x\,.
\end{aligned}$$

We then solve algebraically for $f(x)$ and try to decompose it into partial fractions (noting that $1 - 5x + 6x^2$ factors into $(1-2x)(1-3x)$).

$$\begin{aligned}
f(x) &= \frac{-7x + [1/(1-4x)]}{(1-2x)(1-3x)} = \frac{-7x + 28x^2 + 1}{(1-2x)(1-3x)(1-4x)} \\
&= \frac{A}{1-2x} + \frac{B}{1-3x} + \frac{C}{1-4x}\,.
\end{aligned}$$

To find A, B, and C, we multiply through by the common denominator and collect terms, obtaining

$$28x^2 - 7x + 1 = (12A + 8B + 6C)x^2 + (-7A - 6B - 5C)x + (A + B + C)\,.$$

Now we equate like coefficients and obtain three linear equations, which are solved to yield $A = 9$, $B = -16$, and $C = 8$. Therefore

$$f(x) = \frac{9}{1-2x} - \frac{16}{1-3x} + \frac{8}{1-4x}\,.$$

Since we know the sequences generated by these three functions, we immediately write down the answer: $a_k = 9\cdot 2^k - 16\cdot 3^k + 8\cdot 4^k$.

(b) Working as in part (a), we have

$$\begin{aligned}
f(x) - xf(x) - x^2f(x) &= \sum_{k=0}^{\infty} a_k x^k - \sum_{k=1}^{\infty} a_{k-1}x^k - \sum_{k=2}^{\infty} a_{k-2}x^k \\
&= \sum_{k=2}^{\infty}(a_k - a_{k-1} - a_{k-2})x^k + a_0 + a_1x - a_0x \\
&= 1 + x - x = 1\,.
\end{aligned}$$

We then solve algebraically for $f(x)$ and, in preparation for the partial fraction decomposition, factor the denominator (which, unfortunately, has irrational roots):

$$f(x) = \frac{1}{1-x-x^2} = \frac{1}{(1-\alpha x)(1-\beta x)} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}\,,$$

where by the quadratic formula

$$\frac{1}{\alpha} = \frac{1+\sqrt{5}}{-2} \quad \text{and} \quad \frac{1}{\beta} = \frac{1-\sqrt{5}}{-2}.$$

Clearing fractions and equating powers of x leads to the equations $A + B = 1$ and $\beta A + \alpha B = 0$, whence $A = \alpha/(\alpha - \beta)$ and $B = \beta/(\beta - \alpha)$. (We will use shortly the fact that $\beta - \alpha = \sqrt{5}$, which follows from the definitions of α and β.) Thus we have

$$f(x) = \frac{\alpha}{\alpha-\beta}(1 + \alpha x + \alpha^2 x^2 + \cdots) + \frac{\beta}{\beta-\alpha}(1 + \beta x + \beta^2 x^2 + \cdots)$$

and so

$$\begin{aligned} a_k &= \frac{1}{\alpha-\beta}\alpha^{k+1} + \frac{1}{\beta-\alpha}\beta^{k+1} \\ &= -\frac{1}{\sqrt{5}}\left(\frac{-2}{1+\sqrt{5}}\right)^{k+1} + \frac{1}{\sqrt{5}}\left(\frac{-2}{1-\sqrt{5}}\right)^{k+1} \\ &= -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{k+1} + \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}. \end{aligned}$$

Note that this agrees with the formula for the Fibonacci sequence that we obtained in Section 7.3.

21. The basic technique is the same as in Exercises 18 and 19; we let $f(x) = \sum_{k=0}^{\infty} a_k x^k$.

(a) First we see that

$$\begin{aligned} f(x) + 4x^2 f(x) &= \sum_{k=0}^{\infty} a_k x^k + 4\sum_{k=2}^{\infty} a_{k-2}x^k \\ &= \sum_{k=2}^{\infty}(a_k + 4a_{k-2})x^k + a_0 + a_1 x \\ &= 1 + 2x. \end{aligned}$$

Therefore

$$\begin{aligned} f(x) &= \frac{1+2x}{1+4x^2} = \frac{1+2x}{1+(2x)^2} \\ &= (1+2x)(1 - 2^2x^2 + 2^4x^4 - 2^6x^6 + \cdots) \\ &= (1 - 2^2x^2 + 2^4x^4 - \cdots) + (2x - 2^3x^3 + 2^5x^5 - \cdots). \end{aligned}$$

Thus if k is even, then $a_k = (-1)^{k/2} \cdot 2^k$, whereas if k is odd, then $a_k = (-1)^{(k-1)/2} \cdot 2^k$. We can express this in one formula by $a_k = (-1)^{\lfloor k/2 \rfloor} \cdot 2^k$.