which works out to 234,662,231.

27. (a) First consider the colors of the four pegs as fixed. If the four colors are all different, then there are $3! = 6$ arrangements. If two of the colors are the same (and the other two different), then there are three arrangements. If there is a pair of pegs of each of two colors, then there are two arrangements. If three or four of the pegs are of the same color, there is only one arrangement. Since there are six colors to choose from, we can compute the number of ways to choose colors satisfying each of these possibilities (e.g., there are $6 \cdot C(5,2)$ ways to choose one color for a pair of pegs and two other colors for a single peg each). Thus the answer is $C(6,4) \cdot 6 + 6 \cdot C(5,2) \cdot 3 + C(6,2) \cdot 2 + 6 \cdot 5 \cdot 1 + 6 \cdot 1 = 336$.

(b) The reasoning is pretty much the same as in part (a), yielding the answer $C(6,4) \cdot 3 + 6 \cdot C(5,2) \cdot 2 + C(6,2) \cdot 2 + 6 \cdot 5 \cdot 1 + 6 \cdot 1 = 231$.

(c) Again the calculation goes along the same lines, yielding the answer $C(6,4) \cdot 12 + 6 \cdot C(5,2) \cdot 6 + C(6,2) \cdot 4 + 6 \cdot 5 \cdot 2 + 6 \cdot 1 = 666$.

SECTION 6.4 The Pigeonhole Principle

1. In each case the pigeons are the socks, and the pigeonholes are the colors.

(a) There are 10 colors. We need to choose 11 socks to be sure of getting two in the same color class.

(b) There are two colors. We need to choose three socks to be sure of getting two in the same color class.

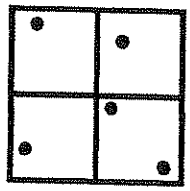
3. The average raise is $\$10000/9 \approx \$1111.11$. If every raise were at least $\$1200$, then the average would not be this low. Hence someone has to get a raise less than $\$1200$.

5. In Theorem 6, take $p = k$, $q = l$, $r = 2$, S equal to the set of people at the party, and T equal to the "is acquainted with" relation.

7. The number of subsets of S is $C(10,4) = 210$. The largest possible sum is $50 + 51 + 52 + 53 = 206$. Since the sums are all natural numbers, by the pigeonhole principle at least two of the subsets have the same sum.

9. Consider the partition $\{\{1,2\}, \{3,4\}, \{5,6\}, \ldots, \{2n-1, 2n\}\}$. By the pigeonhole principle, any subset of cardinality $n+1$ must contain two elements in the same set of this partition, say $2k-1$ and $2k$. Clearly $\gcd(2k-1, 2k) = 1$.

11. Divide the square into four smaller squares as shown below. By the pigeonhole principle, two of the five points are in the same small square. The diameter of each small square is $\sqrt{2}$ by the Pythagorean Theorem. Therefore two of the points are within $\sqrt{2}$ inches of each other.



13. If each hole contained at least m pigeons, then there would be at least mn pigeons altogether, contradicting the hypothesis. Therefore some hole contains fewer than m pigeons.

15. Let S be a sequence of length n. Let $m = \lceil\sqrt{n}\rceil - 1$. Then $m < \sqrt{n}$, so $m^2 < n$, i.e., $m^2 + 1 \le n$. Look at the subsequence of S consisting of the first $m^2 + 1$ terms. By Theorem 3, this subsequence (and hence S, as well) has a monotone subsequence of length $m + 1$. But $m + 1 = \lceil\sqrt{n}\rceil \ge \sqrt{n}$, so there is a monotone subsequence of length at least $\sqrt{n}$.

17. We can show that $R(4) > 9$ with the following situation. Let the people by A, B, C, P, Q, R, X, Y, and Z. Each of A, B, and C knows each other; each of P, Q, and R knows each other; and each of X, Y, and Z knows each other; but there are no other acquaintances among these people. There can clearly be no subset of four mutual acquaintances. On the other hand, among any four people there are at least two in one of the sets $\{A, B, C\}$, $\{P, Q, R\}$, and $\{X, Y, Z\}$. Thus these four people cannot be mutual strangers.

19. In this proposition the quantifiers on p, q, r, and N range over natural numbers, and the quantifiers on S, P, and Q range over sets: $\forall r{\ge}2{:}\,\forall p{\ge}r{:}\,\forall q{\ge}r{:}\,\exists N{:}\,\forall S{:}\,\forall T{:}\,\big[\big(|S| = N \wedge T \text{ is a symmetric } r\text{-ary relation on } S\big) \rightarrow \big([\exists P{:}\,(P \subseteq S \wedge |P| = p \wedge T \text{ holds on } P)] \vee [\exists Q{:}\,(Q \subseteq S \ \wedge\ |Q| = q \ \wedge\ \overline{T} \text{ holds on } Q)]\big)\big]$.

21. Look at the 128 numbers 7, 77, 777, ..., 77...7. By the pigeonhole principle, two of these numbers must be congruent modulo 127. Hence their difference is congruent to 0 modulo 127, i.e., is a multiple of 127. But their difference is of the form 77...700...0, as desired.

23. (a) If there were 27 or fewer training sessions, then some robot (say Robbie) would have received three or fewer sessions, since $\lceil 27/7 \rceil = 3$. Thus there would be at least $10-3=7$ workers who could not use Robbie. If these seven workers needed a robot at the same time, only six robots would be available to serve them. Therefore there have to be at least 28 training sessions.

(b) We train the robots as follows, where we have numbered the robots and the workers. Robot 1 is trained for workers 1, 2, 3, and 4; robot 2 is trained for workers 2, 3, 4, and 5; robot 3 is trained for workers 3, 4, 5, and 6; ...; robot 7 is trained for workers 7, 8, 9, and 10. To see that this training is sufficient to meet the demand, suppose that we are given seven workers who need to use the robots simultaneously, say with numbers $w_1, w_2, \ldots, w_7$, where $w_1 < w_2 < \cdots < w_7$. Because of these inequalities, it must be the case that $1 \le w_1 \le 4$ (otherwise $w_7 > 10$), $2 \le w_2 \le 5$ (otherwise $w_1 < 1$ or $w_7 > 10$), ..., $7 \le w_7 \le 10$ (otherwise $w_1 < 1$). Thus worker w_i can use robot i.

(c) Assume that we have n workers and k robots. We need $k(n-k+1)$ training sessions (otherwise some robot will serve at most $n-k$ masters, and the other k workers will have only $k-1$ robots available to them). Furthermore, $k(n-k+1)$ sessions are sufficient, for we can train robot i for workers $i, i+1, \ldots, i+n-k$. By the same argument as given in part (b), if we are given any collection of k workers, say with numbers $w_1 < w_2 < \cdots < w_k$, then the ith robot will necessarily have been trained for worker w_i.

25. If $n = 3$, then five such bit strings are necessary to guarantee that two of them agree in at least two places, since the strings 000, 011, 101, and 110 do not so agree. Conversely, given five such strings, by the generalized pigeonhole principle at least three of them agree in the first position. Among these, at least two agree in the second position as well. For $n \ge 4$, three such strings are necessary: Look at $00\ldots0$ and $11\ldots1$. Conversely, suppose that we have three such strings. By the pigeonhole principle at least two of them start with the same bit. Assume that they do not agree in any other bit, and consider the third string. In each of bits 2, 3, and 4, it must agree with one or the other of these first two strings. Therefore again by the pigeonhole principle, it must agree with one of them in at least two of these three bits.

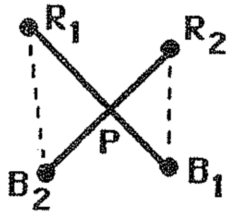
27. The following algorithm follows the hint: l_i will be the length of the longest increasing subsequence starting at x_i, and w_i will be the index of the next term in such a subsequence.

procedure *long_increase*($x_1, x_2, \ldots, x_n$: distinct numbers)
    $l_n \leftarrow 1$ {x_n is the longest known subsequence starting at x_n}
    $w_n \leftarrow 0$ {there is no next term}
    for $i \leftarrow n-1$ **down to** 1 **do**
        begin
            $l_i \leftarrow 1$ {no subsequence longer than x_i is known yet}
            $w_i \leftarrow 0$ {there is no next term}
            for $j \leftarrow i+1$ **to** n **do** {find a subsequence to stick x_i in front of}
                if $x_i < x_j \land l_i < l_j + 1$ **then**
                    begin {found a longer subsequence starting at x_i}
                        $l_i \leftarrow l_j + 1$
                        $w_i \leftarrow j$
                    end
        end {l_i and w_i are now known}
    $max \leftarrow -\infty$
    for $i \leftarrow 1$ **to** n **do** {find maximum l_i}
        if $l_i > max$ **then**
            begin
                $max \leftarrow l_i$
                $start \leftarrow i$ {starting point for subsequence of length l_i}
            end
    return($max, start, (w_1, w_2, \ldots, w_n)$)
{we return the starting point and length of the longest increasing subsequence, as well as the information needed to reconstruct the subsequence}

29. Fix person A. By the generalized pigeonhole principle, he must have either 10 acquaintances or 10 nonacquaintances at the party. By symmetry we can without loss of generality assume that he has 10 acquaintances. By Exercise 28, among these 10 we can find either three mutual acquaintances or four mutual strangers. In the latter case we are done. In the former case, these three together with A form the desired set of four mutual acquaintances. (Note that the symmetric analogue of Exercise 28 is needed if A has 10 nonacquaintances.)

31. Look at person A. He must have either six nonacquaintances, six friends, or six enemies among the other party-goers (otherwise there would be only $5+5+5=15$ other people). Without loss of generality, assume that there are six nonacquaintances. If any two of these are strangers to each other, then we have our three mutual strangers. Otherwise we apply Theorem 4 to these six people (using the categories of friend and enemy) to get three mutual friends or three mutual enemies.

33. An inductive proof can be given. Here is a cuter direct proof. Among the $n!$ ways to pair red and blue points, one (or more) results in a minimum total length of the line segments used. We claim that the drawing that achieves this minimum has no intersections. If there were an intersection, then the situation shown below must happen for at least one pair of pairs of red and blue points. By the triangle inequality, $R_1B_1 + R_2B_2 = (R_1P+PB_1)+(R_2P+PB_2) = (R_1P+PB_2)+(R_2P+PB_1) > R_1B_2+R_2B_1$, contradicting the choice of the pairing as the one with the smallest total length (if we pair R_1 with B_2 and R_2 with B_1, then we reduce the total length). Thus there must have been no intersections.



CHAPTER 7

ADDITIONAL TOPICS IN COMBINATORICS

SECTION 7.1 Combinatorial Identities

1. (a) $x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

   (b) $a^4 + 4(-3)a^3 + 6(-3)^2a^2 + 4(-3)^3a + (-3)^4 = a^4 - 12a^3 + 54a^2 - 108a + 81$

   (c) $(2x)^5 + 5(2x)^4(3y) + 10(2x)^3(3y)^2 + 10(2x)^2(3y)^3 + 5(2x)(3y)^4 + (3y)^5 = 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5$

   (d) $x^8 + 8x^6 + 28x^4 + 56x^2 + 70 + 56x^{-2} + 28x^{-4} + 8x^{-6} + x^{-8}$

3. The next rows read as follows.

$$\begin{array}{ccccccccccccccc}
 & 1 & 12 & 66 & 220 & 495 & 792 & 924 & 792 & 495 & 220 & 66 & 12 & 1 & \\
 & 1 & 13 & 78 & 286 & 715 & 1287 & 1716 & 1716 & 1287 & 715 & 286 & 78 & 13 & 1 \\
1 & 14 & 91 & 364 & 1001 & 2002 & 3003 & 3432 & 3003 & 2002 & 1001 & 364 & 91 & 14 & 1
\end{array}$$

The sum of the first of these is, as expected $2^{12} = 4096$, the sum of the second is $2^{13} = 8192$, and the sum of the third is $2^{14} = 16384$. In each case the alternating sum is 0.

5. $C(6,6) + C(7,6) + C(8,6) + C(9,6) + C(10,6) = 1 + 7 + 28 + 84 + 210 = 330 = C(11,7)$

7. Each term xy^3z^2 in the expansion can be thought of as an arrangement of one x, three y's, and two z's, where the order is determined by the order in which we draw these symbols from the successive factors, so we need to count the number of ways to form such an arrangement. By Theorem 1 in Section 6.3, the number of such arrangements is $6!/(1!3!2!) = 60$. Hence the coefficient of xy^3z^2 when $(x+y+z)^6$ is multiplied out is 60.

9. (a) $\displaystyle\sum_{k=0}^{n} C(n,k)2^k = \sum_{k=0}^{n} C(n,k)2^k1^{n-k} = (2+1)^n = 3^n$

   (b) $\displaystyle\sum_{k=0}^{n} C(n,k)(-2)^k = \sum_{k=0}^{n} C(n,k)(-2)^k1^{n-k} = (-2+1)^n = (-1)^n$

11. A triangular arrangement with one dot in the top row, two dots in the second row, ..., $n-1$ dots in the last row, has a total of $1+2+\cdots+(n-1)=(n-1)n/2=C(n,2)$ dots in all.

13. (a) $C(2n,2)=C(n,2)+C(n,2)+n\cdot n=2C(n,2)+n^2$

(b) $C(2n,2)=\dfrac{2n(2n-1)}{2}=n(2n-1)=2n^2-n$

and $2C(n,2)+n^2=2\dfrac{n(n-1)}{2}+n^2=n^2-n+n^2=2n^2-n$

15. (a) $5C(10,5)=5\cdot 252=1260=10\cdot 126=10C(9,4)$

(b) $kC(n,k)=k\dfrac{n!}{k!(n-k)}!=\dfrac{n!}{(k-1)!(n-k)!}$

and $nC(n-1,k-1)=n\dfrac{(n-1)!}{(k-1)!((n-1)-(k-1))!}=\dfrac{n!}{(k-1)!(n-k)!}$

(c) If $n=1$, then $k=1$ and we have $1\cdot C(1,1)=1=1\cdot C(0,0)$. Assume that the identity is true for $n-1$; we will show it for n. If $k=n$, then both sides equal n. Otherwise, $k\le n-1$, and we have by algebra, the inductive hypothesis and two applications of Pascal's identity

$$\begin{aligned}kC(n,k)&=k(C(n-1,k)+C(n-1,k-1))\\&=kC(n-1,k)+(k-1)C(n-1,k-1)+C(n-1,k-1)\\&=(n-1)C(n-2,k-1)+(n-1)C(n-2,k-2)+C(n-1,k-1)\\&=(n-1)[C(n-2,k-1)+C(n-2,k-2)]+C(n-1,k-1)\\&=(n-1)C(n-1,k-1)+C(n-1,k-1)\\&=nC(n-1,k-1)\,,\end{aligned}$$

as desired.

(d) To choose from a set of n people a committee of k people including a chairperson, we can either choose the committee members (in one of $C(n,k)$ ways) and then choose a chairperson from among them (in one of $C(k,1)=k$ ways), or else we can choose the chairperson first (in one of $C(n,1)=n$ ways), and then choose the other $k-1$ members of the committee from among the other $n-1$ people (in one of $C(n-1,k-1)$ ways).

17. We want to show that $\sum_{k=0}^{n}C(n,k)2^k=3^n$. Let us count the number of ways to paint the elements of an n-set red, white, or blue. We can first decide how many of the elements are to be red or white. Let k be this number, so $0\le k\le n$. (The remaining elements will all be blue.) Now there are $C(n,k)$ ways to select a subset of k elements to be red/white, and then there are 2^k ways to decide on the colors (red or white) for the elements in our chosen set. Thus the left-hand side counts the desired quantity. Clearly the right-hand side does so as well.

19. The right-hand side is the number of ways to choose n people from $2n$ people, who happen to consist of n men and n women. The left-hand side is the number of ways to choose from this collection k men and $n-k$ women (namely, $C(n,k)\cdot C(n,n-k)=C(n,k)^2$), summed over all possible values of k, which is again simply the number of ways to choose n people.

21. Both sides equal $\dfrac{(2m+2n)!}{(m!)^2(n!)^2}$.

23. The number of permutations of k copies of each of n distinct things (kn things in all) is, by Theorem 1 in Section 6.3, $\dfrac{(kn)!}{k!k!\cdots k!}=\dfrac{(kn)!}{(k!)^n}$. Since this is a whole number, $(k!)^n$ must divide $(kn)!$.

25. The point of this exercise is the combinatorial identity

$$nC(2n-1,n-1)=\sum_{i=1}^{n} iC(n,i)^2\,.$$

(a) Choose a captain from among the n boys. Then choose the remaining $n-1$ debaters from among the $2n-1$ remaining people.

(b) Let i be the number of boys who will be on the team. Then $n-i$ is the number of girls who will be on the team. The number of ways to choose the team with i boys and $n-i$ girls is therefore $C(n,i)\cdot C(n,n-i)=C(n,i)^2$. The number of ways to choose the captain is i, since he must be one of the male debaters. We sum over all possible values of i to obtain the total number of ways to pick the team and captain.

27. Call the top row row 1. Consider the kth person in row m. If $2\le k\le m-1$, then this person bears half the weight borne by person k in row $m-1$ and half the weight borne by person $k-1$ in row $m-1$, plus w (i.e., half the weight of each of these two people). If $k=m$ or $k=1$, then there is only one person above, contributing half of his weight and burden. Thus we have the following equations recursively defining the weight $B(m,k)$ borne by the kth person in row m.

$$B(1,1)=0$$

$$B(m,1)=\frac{1}{2}(w+B(m-1,1))\quad\text{for } m\ge 2$$

$$B(m,m)=\frac{1}{2}(w+B(m-1,m-1))\quad\text{for } m\ge 2$$

$$B(m,k)=w+\frac{1}{2}(B(m-1,k)+B(m-1,k-1))\quad\text{for } 2\le k\le m-1,\ m\ge 3$$

In our triangle, we take $w=1$ for simplicity.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 0 & & & & & & \\
 & & & & & 0.5 & & 0.5 & & & & & \\
 & & & & 0.75 & & 1.5 & & 0.75 & & & & \\
 & & & 0.875 & & 2.125 & & 2.125 & & 0.875 & & & \\
 & & 0.9375 & & 2.5 & & 3.125 & & 2.5 & & 0.9375 & & \\
 & 0.96875 & & 2.71875 & & 3.8125 & & 3.8125 & & 2.71875 & & 0.96875 & \\
0.984375 & & 2.84375 & & 4.265625 & & 4.8125 & & 4.265625 & & 2.84375 & & 0.984375
\end{array}
$$

29. We first observe the data for small values of n.

$n=0$	$C(0,0)=1$
$n=1$	$C(1,0)=1$
$n=2$	$C(2,0)+C(1,1)=1+1=2$
$n=3$	$C(3,0)+C(2,1)=1+2=3$
$n=4$	$C(4,0)+C(3,1)+C(2,2)=1+3+1=5$
$n=5$	$C(5,0)+C(4,1)+C(3,2)=1+4+3=8$
$n=6$	$C(6,0)+C(5,1)+C(4,2)+C(3,3)=1+5+6+1=13$
$n=7$	$C(7,0)+C(6,1)+C(5,2)+C(4,3)=1+6+10+4=21$

These numbers look like numbers from the Fibonacci sequence, so let us conjecture, as holds up to $n=7$, that $\sum_{i=0}^{\lfloor n/2\rfloor} C(n-i,i)=f_n$, where f_n is the nth term in the Fibonacci sequence defined in Section 5.1. Since we have already established the base case, we only need to show that our sequence satisfies the same recursive property that the Fibonacci sequence satisfies, namely that the nth term plus the $(n+1)$th term equals the $(n+2)$th term. In our case that means showing that

$$\sum_{i=0}^{\lfloor n/2\rfloor} C(n-i,i)+\sum_{j=0}^{\lfloor (n+1)/2\rfloor} C(n+1-j,j)=\sum_{k=0}^{\lfloor (n+2)/2\rfloor} C(n+2-k,k)\,.$$

Assume first that n is even. Then the left-hand side of this equation equals

$$\begin{aligned} &C(n,0)+C(n-1,1)+\cdots+C((n/2)+1,(n/2)-1)+C(n/2,n/2)\\ +\,&C(n+1,0)+C(n,1)+C(n-1,2)+\cdots+C((n/2)+1,n/2)\,,\end{aligned}$$

which by Pascal's identity applied to the terms written together vertically equals

$$1+C(n+1,1)+C(n,2)+C(n-1,3)+\cdots+C((n/2)+2,n/2)+1\,.$$

On the other hand, the right-hand side is the same, since its first and last terms are also both equal to 1. A similar messy calculation applies when n is odd.

SECTION 7.2 Modeling Combinatorial Problems with Recurrence Relations

1. **(a)** The recurrence relation is $A_{n+1} = 1.05A_n$, since 5% of the previous balance (the interest) is added to the account. The initial condition is $A_0 = 1200$, representing the initial deposit.

(b) We have $A_1 = 1.05 \cdot 1200 = 1260$, $A_2 = 1.05 \cdot 1260 = 1323$, $A_3 = 1.05 \cdot 1323 = 1389.15$, and $A_4 = 1.05 \cdot 1389.15 \approx 1458.61$. Therefore after 4 years the balance will be \$1458.61.

3. $p(6) = C(5,0)p(5) + C(5,1)p(4) + C(5,2)p(3) + C(5,3)p(2) + C(5,4)p(1) + C(5,5)p(0) = 1 \cdot 52 + 5 \cdot 15 + 10 \cdot 5 + 10 \cdot 2 + 5 \cdot 1 + 1 \cdot 1 = 203$

5. The partitions of 8 into parts no larger than 3 are $3+3+2$, $3+3+1+1$, $3+2+2+1$, $3+2+1+1+1$, $3+1+1+1+1+1$, $2+2+2+2$, $2+2+2+1+1$, $2+2+1+1+1+1$, $2+1+1+1+1+1+1$, and $1+1+1+1+1+1+1+1$. Thus $P_3(8) = 10$. Working with the recurrence relation, we have $P_3(8) = P_2(8) + P_3(5) = P_1(8) + P_2(6) + P_2(5) + P_3(2) = 1 + P_1(6) + P_2(4) + P_1(5) + P_2(3) + P_2(2) = 1 + 1 + P_1(4) + P_2(2) + 1 + P_1(3) + P_2(1) + P_2(2) = 1 + 1 + 1 + 2 + 1 + 1 + 1 + 2 = 10$.

7. In each case we apply the recurrence relation $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + n$, with $T(1) = 0$.

(a) $T(8) = 2T(4) + 8 = 2(2T(2) + 4) + 8 = 2(2(2T(1) + 2) + 4) + 8 = 2(2 \cdot 2 + 4) + 8 = 24$

(b) $T(9) = T(4) + T(5) + 9 = 2T(2) + 4 + T(2) + T(3) + 5 + 9 = 3T(2) + T(3) + 18 = 3T(2) + T(1) + T(2) + 3 + 18 = 4T(2) + 21 = 4(2T(1) + 2) + 21 = 29$

(c) $T(20) = 2T(10) + 20 = 2 \cdot 34 + 20 = 88$ (using the value of $T(10)$ found in Example 11)

(d) $T(50) = 2T(25) + 50 = 2(T(12) + T(13) + 25) + 50 = 2T(12) + 2T(13) + 100 = 2(2T(6) + 12) + 2(T(6) + T(7) + 13) + 100 = 6T(6) + 2T(7) + 150 = 6(2T(3) + 6) + 2(T(3) + T(4) + 7) + 150 = 14T(3) + 2T(4) + 200 = 14(T(1) + T(2) + 3) + 2(2T(2) + 4) + 200 = 18T(2) + 250 = 18(2T(1) + 2) + 250 = 286$

9. $A_n = A_{n-1} + 2000 + 0.05A_{n-1} = 2000 + 1.05A_{n-1}$, with $A_0 = 0$

11. The sequence can start with a 1, after which any string of length $n-1$ without three consecutive 0's gives a string of length n without three consecutive 0's. This contributes a_{n-1} to the sum. Similarly, the sequence can start 01 and be followed by a string of length $n-2$ without three consecutive 0's, or it can start 001 and be followed by a string of length $n-3$ without three consecutive 0's. These are all the possibilities. Therefore the recurrence relation is $a_n = a_{n-1} + a_{n-2} + a_{n-3}$. The initial conditions are $a_0 = 1$ (the empty string), $a_1 = 2$ (both strings of length 1), and $a_2 = 4$ (all four strings of length 2).