(b) Just as in part (a),

$$\begin{aligned}
f(x) + x^4 f(x) &= \sum_{k=0}^{\infty} a_k x^k + \sum_{k=4}^{\infty} a_{k-4} x^k \\
&= \sum_{k=4}^{\infty} (a_k + a_{k-4}) x^k + a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\
&= 1 + 2x + 3x^2 + x^3 .
\end{aligned}$$

Therefore

$$\begin{aligned}
f(x) &= \frac{1 + 2x + 3x^2 + x^3}{1 + x^4} \\
&= (1 + 2x + 3x^2 + x^3)(1 - x^4 + x^8 - x^{12} + \cdots).
\end{aligned}$$

From this we read off the answers. If $k \equiv 0 \pmod 4$, then $a_k = (-1)^{k/4}$; if $k \equiv 1 \pmod 4$, then $a_k = 2(-1)^{(k-1)/4}$; if $k \equiv 2 \pmod 4$, then $a_k = 3(-1)^{(k-2)/4}$; and if $k \equiv 3 \pmod 4$, then $a_k = (-1)^{(k-3)/4}$.

23. In Exercise 12 we found that the generating function for the number of partitions of k into parts of unequal size is $\prod_{i=1}^{\infty}(1+x^i)$, and that the generating function for the number of partitions of k into odd parts is $\prod_{j=1}^{\infty}[1/(1-x^{2j-1})]$. Thus we must show that

$$(1+x)(1+x^2)(1+x^3)\cdots = \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdots .$$

We can write the right-hand side as

$$\frac{1}{1-x} \cdot \frac{1-x^2}{1-x^2} \cdot \frac{1}{1-x^3} \cdot \frac{1-x^4}{1-x^4} \cdot \frac{1}{1-x^5} \cdot \frac{1-x^6}{1-x^6} \cdots ,$$

which factors as

$$\frac{1}{1-x} \cdot \frac{(1-x)(1+x)}{1-x^2} \cdot \frac{1}{1-x^3} \cdot \frac{(1-x^2)(1+x^2)}{1-x^4} \cdot \frac{1}{1-x^5} \cdot \frac{(1-x^3)(1+x^3)}{1-x^6} \cdots .$$

We then cancel common factors to obtain the left-hand side.

25. The number of ways to get a sum of k rolling a die t times is the coefficient of x^k in $(x + x^2 + x^3 + x^4 + x^5 + x^6)^t$. Thus for the generating function we need to sum over all t from 0 to ∞, which yields the given formula.

27. We will use the result of Exercise 26 after we first obtain a generating function for $a_k = k^2$. We have

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k .$$

Differentiating both sides, we obtain

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} kx^{k-1},$$

so

$$\frac{x}{(1-x)^2} = \sum_{k=0}^{\infty} kx^k.$$

Differentiating again, we have

$$\frac{(1-x)^2 + 2x(1-x)}{(1-x)^4} = \sum_{k=0}^{\infty} k^2x^{k-1}$$

$$\frac{1+x}{(1-x)^3} = \sum_{k=0}^{\infty} k^2x^{k-1},$$

so

$$\frac{x+x^2}{(1-x)^3} = \sum_{k=0}^{\infty} k^2x^k.$$

Thus by Exercise 26, the generating function for $1^2 + 2^2 + \cdots + k^2$ is

$$\frac{x+x^2}{(1-x)^4} = (x+x^2)\sum_{k=0}^{\infty} C(k+3,3)x^k.$$

Therefore $a_k = C(k+2,3) + C(k+1,3)$, which can be written explicitly as $a_k = k(k+1)(2k+1)/6$.

29. In each case we want a_{30} for the generating function that models the problem.

(a) The generating function is

$$(1+x^2+x^4+\cdots)(1+x+x^2+\cdots)^2 = \frac{1}{1-x^2}\cdot\frac{1}{(1-x)^2}.$$

We can factor the denominator on the right-hand side and find a partial fraction decomposition:

$$\frac{1}{1+x}\cdot\frac{1}{(1-x)^3} = \frac{A}{1+x} + \frac{B}{1-x} + \frac{C}{(1-x)^2} + \frac{D}{(1-x)^3}.$$

This leads to $A+B+C+D=1$, $-3A-B+D=0$, $3A-B-C=0$, and $-A+B=0$, whence $A=B=1/8$, $C=1/4$, and $D=1/2$. Thus the generating function is

$$f(x) = \frac{1}{8}\sum_{k=0}^{\infty}(-1)^kx^k + \frac{1}{8}\sum_{k=0}^{\infty}x^k + \frac{1}{4}\sum_{k=0}^{\infty}(k+1)x^k + \frac{1}{2}\sum_{k=0}^{\infty}C(k+2,2)x^k.$$

Therefore

$$a_{30} = \frac{1}{8} + \frac{1}{8} + \frac{1}{4}\cdot 31 + \frac{1}{2}\cdot\frac{32\cdot 31}{2} = 256.$$

(b) We can use the result of part (a) directly to solve this problem. If there must be an odd number of white balls, we can select one white ball and then find the number of ways to choose 29 balls such that there are an even number of white balls. This was exactly the problem solved in part (a), except for the meaningless change of color and the fact that we want a_{29} instead of a_{30}. Thus the answer is

$$a_{29} = -\frac{1}{8} + \frac{1}{8} + \frac{1}{4} \cdot 30 + \frac{1}{2} \cdot \frac{31 \cdot 30}{2} = 240 .$$

(c) This is similar to part (a), but the generating function is

$$(1 + x^2 + x^4 + \cdots)(x + x^3 + x^5 + \cdots)(1 + x + x^2 + \cdots) .$$

We write this in closed form, factor the denominator, and look for a partial fraction decomposition.

$$\begin{aligned} \frac{x}{(1-x^2)^2} \cdot \frac{1}{1-x} &= \frac{x}{(1-x)^3(1+x)^2} \\ &= \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3} + \frac{D}{1+x} + \frac{E}{(1+x)^2} . \end{aligned}$$

This leads to $A+B+C+D+E=1$, $B+2C-2D-3E=1$, $-2A-B+C+3E=0$, $-B+2D-E=0$, and $A-D=0$, whence $A=-1/16$, $B=0$, $C=1/4$, $D=-1/16$ and $E=-1/8$. Thus the generating function is

$$f(x) = -\frac{1}{16}\sum_{k=0}^{\infty} x^k + \frac{1}{4}\sum_{k=0}^{\infty} C(k+2,2)x^k - \frac{1}{16}\sum_{k=0}^{\infty}(-1)^k x^k - \frac{1}{8}\sum_{k=0}^{\infty}(-1)^k(k+1)x^k .$$

Therefore

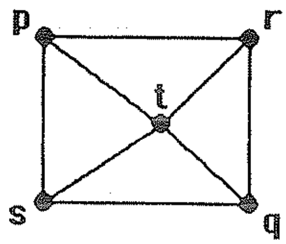
$$a_{30} = -\frac{1}{16} + \frac{1}{4} \cdot 496 - \frac{1}{16} - \frac{1}{8} \cdot 31 = 120 .$$

(d) We use the inclusion–exclusion principle applied to the results of the first three part to obtain the answer of $256 + 240 - 120 = 376$.

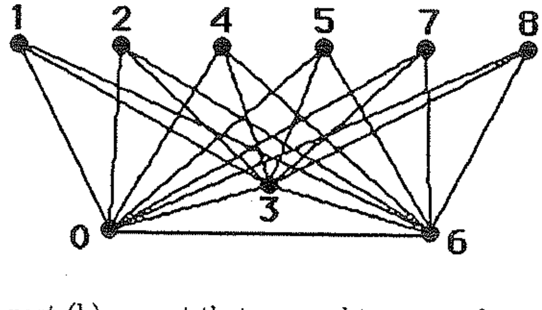
CHAPTER 8
GRAPHS

SECTION 8.1 Basic Definitions in Graph Theory

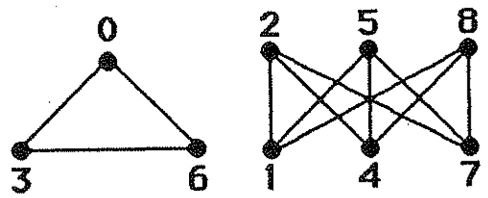
1. (a) Our picture consists of one point for each vertex, labeled p through t, and a line segment joining each pair of distinct vertices other than pq and rs. Note that we were able to draw the picture in such a way that all the lines were straight and no lines crossed, but this is not required; many other pictures represent the same graph.



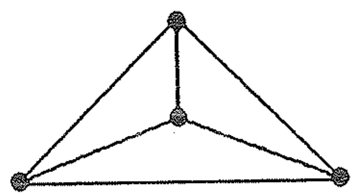
(b) The edge set consists of all uv such that $u \cdot v$ is a multiple of 3, where u and v are distinct natural numbers less than 9. Now $u \cdot v$ is a multiple of 3 if and only if either u or v is a multiple of 3. Thus the vertices 0, 3, and 6 are joined by an edge to every other vertex, but no two vertices among $\{1, 2, 4, 5, 7, 8\}$ are joined by an edge. In our picture, we have put the former three vertices on the bottom and the latter six on the top. All the required edges have been drawn.



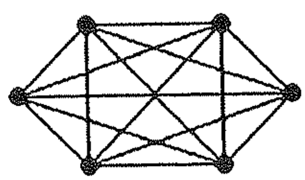
(c) This is similar to part (b), except that we need to worry about the reduction of each number modulo 3. Thus 0, 3, and 6 form one class, and there is an edge from each of these to the others, since the sum of two numbers congruent to 0 modulo 3 is congruent to 0 modulo 3. This portion of the graph is a K_3. The vertices 1, 4, and 7 form a second class (the numbers congruent to 1 modulo 3), and 2, 5, and 8 form a third class (the numbers congruent to 2 modulo 3). Since the sum of any member of this second class and any member of this third class (such as $5+4$) is congruent to 0 modulo 3, we need to have an edge between them. Note that this portion of the graph is a $K_{3,3}$. There are no other edges, since in no other cases do we have the sum congruent to 0 modulo 3.



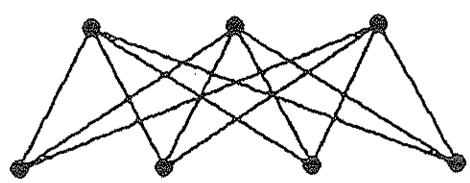
(d) Our picture consists of four vertices and all the edges joining pairs of them. Note that it was possible to avoid edges crossing by arranging the vertices as shown; it would be equally valid to draw K_4 with the four vertices in a square, in which case two of the edges could cross in the middle of the square.



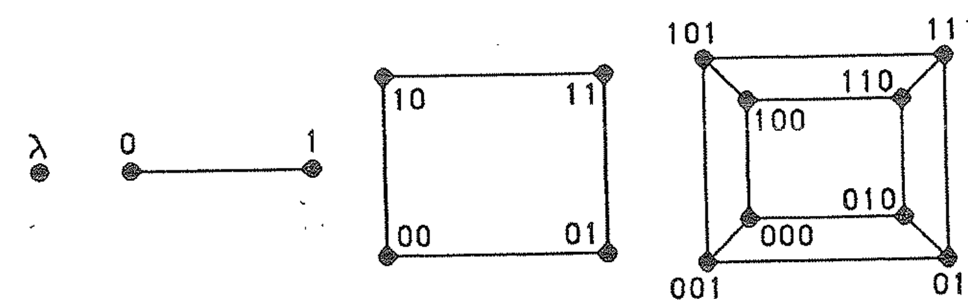
(e) There is no way to avoid edge crossings in drawing the complete graph on six vertices. We have just arranged the six vertices more or less in a circle and drawn all $C(6,2) = 15$ edges joining pairs of vertices.



(f) This complete bipartite graph consists of three vertices in one part, which we have drawn on top, four vertices in the second part (drawn on the bottom), and all $3 \cdot 4 = 12$ edges joining vertices in different parts. There is no way to avoid edges crossings in this case.



3. One set of correct labelings is shown below. Both Q_0 and Q_1 are trivial. The empty string (the only string of length 0) labels the only vertex in Q_0, and the two strings of length 1 label the vertices in Q_1 (they differ in exactly one bit, so an edge joins them). The picture of Q_2 has the four bit strings of length 2 as the vertex labels; two vertices are adjacent if their labels differ in one bit. (This labeling was easy to achieve, since once we labeled 00, we knew that 01 and 10 had to be its neighbors.) Finally, the labeling of Q_3 can be arrived at as follows. First we label the four vertices at the top with bit strings of length 3 beginning with a 1; the second and third bits of these labels are chosen so that adjacent vertices differ in one bit. Then each of the four vertices on the bottom is labeled with the same label as the vertex at the top to which it is joined by an edge, except that the first bit is changed to a 0.



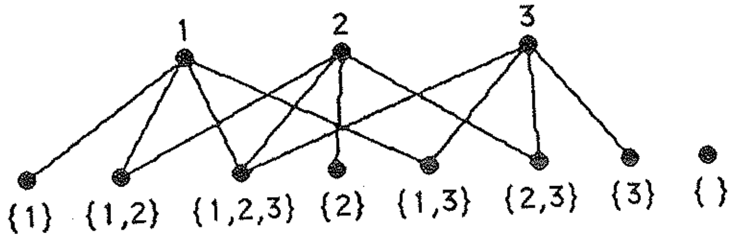
5. (a) Vertex a has two edges outgoing from it (the one to vertex b and the loop), so the out-degree of vertex a is 2; in symbols, $d^+(a) = 2$. Only the loop is an incoming edge to a, so the in-degree of a is 1; in symbols, $d^-(a) = 1$. Similarly, $d^+(b) = 0$, $d^+(c) = 1$, $d^+(d) = 1$, $d^+(e) = 1$, $d^+(f) = 0$, $d^-(b) = 1$, $d^-(c) = 2$, $d^-(d) = 1$, $d^-(e) = 0$, and $d^-(f) = 0$.

(b) There are six flights leaving Detroit, so $d^+(\mathit{Detroit}) = 6$. Similarly, $d^+(\mathit{Boston}) = 3$, $d^+(\mathit{Washington}) = 3$, and $d^+(\mathit{Miami}) = 4$ (one of the edges leaving Miami is the loop). There are six flights coming into Detroit, so $d^-(\mathit{Detroit}) = 6$. Similarly, $d^-(\mathit{Boston}) = 3$, $d^-(\mathit{Washington}) = 3$, $d^-(\mathit{Miami}) = 4$.

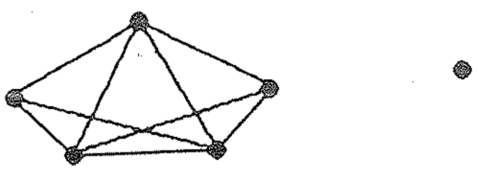
(c) The vertices in this graph are $\{1, 2, 3, 4, 5, 6, 7\}$. There is an edge from x to y whenever $x < y$. Since $1 < y$ for each of the six numbers from 2 through 7, we know that the out-degree of 1 is 6; in symbols, $d^+(1) = 6$. Similarly, $d^+(2) = 5$, $d^+(3) = 4$, $d^+(4) = 3$, $d^+(5) = 2$, $d^+(6) = 1$, and $d^+(7) = 0$ (none of these numbers is greater than 7). On the other hand, $d^-(1) = 0$, $d^-(2) = 1$, $d^-(3) = 2$, $d^-(4) = 3$, $d^-(5) = 4$, $d^-(6) = 5$, and $d^-(7) = 6$.

7. The complement of the complement of G contains all the edges that the complement of G does not contain. The complement of G does not contain precisely the edges that G does contain. Therefore the complement of the complement of G contains exactly the same edges as G does, so $\overline{\overline{G}} = G$.

9. We put the elements of $\{1, 2, 3\}$ on top and the (eight) elements of $\mathcal{P}(\{1, 2, 3\})$ (i.e., the subsets of $\{1, 2, 3\}$) on the bottom, and we draw a line from each number to each subset of which it is an element.



11. **(a)** Aside from the one isolated vertex, our graph will be a subgraph of K_5. There are 10 edges in K_5, so we need to draw all but one of these edges in order to have nine edges in our graph. Thus the picture is as shown here.



(b) The statement is true. As we saw in part (a), if one vertex is isolated, then there can be at most $C(5, 2) = 10$ edges. Therefore if there are 11 edges, no vertex can be isolated.

(c) The statement is false, essentially because there is no limit to the number of edges in a pseudograph on a fixed number of vertices. For a counterexample, we can take a multigraph with six vertices, one pair of which has 11 parallel edges between them, and the other four of which are isolated.

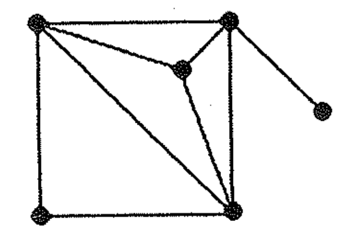
13. **(a)** This is true. We can see C_5 as the perimeter of the picture of the Petersen graph as shown in Figure 8.10.

(b) This is false. After looking at the Petersen graph for a few minutes, we can clearly see that there are no cycles of length less than 5.

(c) This is false. The Petersen graph does not even have a triangle (K_3) as a subgraph, so it certainly cannot contain K_5 (which itself contains K_3).

(d) This is false. A bipartite graph cannot contain K_3, since by the pigeonhole principle at least two of the three vertices in any purported K_3 must be in the same part and hence cannot be adjacent.

15. (a) With a little trial and error we find that the following graph has the desired degree sequence.

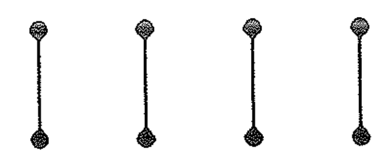


(b) There is no such graph, since the sum of the terms in this sequence is odd (such a graph would violate the corollary to Theorem 1).

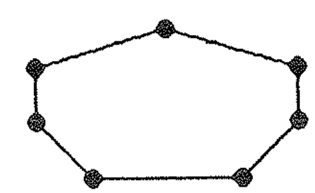
(c) There is no such graph. Suppose that there were, say with vertices v_1, v_2, v_3, and v_4 of degree 4, and vertices u_1 and u_2 of degree 1. Since at most three edges from any v_i can go to vertices v_j, there must be at least one edge from each v_i to u_1 and/or u_2. This makes a total of at least four edges involving u_1 and u_2, so u_1 and u_2 could not both have degree 1.

(d) There is no such graph, since the largest possible degree in a graph with 5 vertices is 4.

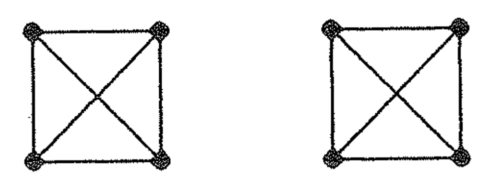
(e) Clearly the following graph has the desired degree sequence.



(f) Clearly the following graph has the desired degree sequence.



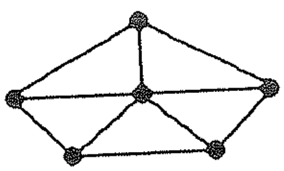
(g) Clearly the following graph has the desired degree sequence.



17. The possible degrees in a graph with n vertices are the n values $0, 1, 2, \ldots, n-1$. The only way to avoid having two vertices with the same degree is for the degree sequence to be precisely $(n-1, n-2, \ldots, 2, 1, 0)$. But this is impossible, since if one vertex has degree $n-1$, then there can be no vertex with degree 0. The statement need not hold for multigraphs. For example, a multigraph on vertex set $\{a, b, c\}$ in which there is one edge from a to b and two parallel edges from b to c has degree sequence $(3, 2, 1)$.

19. For the base case we just note that Q_0 is a single vertex. Now Q_{n+1} is obtained from two disjoint copies of Q_n, say with corresponding vertices $v_1, v_2, \ldots, v_{2^n}$ and $v'_1, v'_2, \ldots, v'_{2^n}$, by adding the edges $v_i v'_i$ for each i. To see that this is really Q_{n+1}, we can label the vertices as follows. Label both copies of Q_n in the same way (i.e., for each i the label on v_i is the same as the label on v'_i). Then put a 1 in front of each label on v_i and put a 0 in front of each label on v'_i.

21. (a) The graph consists of a 5-cycle, with one additional vertex (which we put in the middle) joined to each vertex on the cycle.



(b) The n-cycle contributes n edges, and the additional vertex contributes n new edges, so there are $2n$ edges in all.

(c) The additional vertex has degree n, and each of the n vertices on the cycle has degree 3, so the degree sequence is $(n, 3, 3, \ldots, 3)$, in which there are n 3's.

23. (a) Since K_n is the graph with n vertices and all possible edges, $\overline{K}_n$ must be the graph with n vertices and no edges.

(b) The graph $K_{m,n}$ consists of a set of m vertices and a set of n vertices, together with all edges joining vertices in one set with vertices in the other. Therefore the complement consists of these same $m+n$ vertices, together with all edges joining vertices in the same part. This is clearly the disjoint union of K_m and K_n.

25. We need to find the degree of each of the 64 vertices in this graph, that is, the number of attacked squares for each square on the chessboard. Consider the following picture, in which we have labeled the squares of the chessboard with the letters A, B, C, or D.

A	A	A	A	A	A	A	A
A	B	B	B	B	B	B	A
A	B	C	C	C	C	B	A
A	B	C	D	D	C	B	A
A	B	C	D	D	C	B	A
A	B	C	C	C	C	B	A
A	B	B	B	B	B	B	A
A	A	A	A	A	A	A	A

Note that there are 28 squares labeled A, 20 squares labeled B, 12 squares labeled C, and four squares labeled D. We can see that a queen on any square labeled A attacks 21 other squares (seven in its own row, seven in its own column, and seven in its diagonals); a queen on any square labeled B attacks 23 other squares (seven in its own row, seven in its own column, and nine in its diagonals); a queen on any square labeled C attacks 25 other squares (seven in its own row, seven in its own column, and 11 in its diagonals); and a queen on any square labeled D attacks 27 other squares (seven in its own row, seven in its own column, and 13 in its diagonals). Therefore the degree sequence is $(4\cdot 27,\ 12\cdot 25,\ 20\cdot 23,\ 28\cdot 21)$, where $k\cdot d$ denotes k copies of the number d. This gives a vertex degree total of $4\cdot 27+12\cdot 25+20\cdot 23+28\cdot 21=1456$, so the number of edges is $1456/2=728$.

27. (a) We represent each task with a vertex. To see which edges are present, we need to determine which tasks must precede which other tasks. Before writing the draft of the history paper, Sam must do the research and rent the word processor. Thus there is an edge from task 1 to task 3 and from task 2 to task 3. Similarly, task 3 precedes task 4 (editing the draft), which precedes task 5 (returning the word processor), which precedes task 12 (flying home). Task 6 (studying for history exam) precedes task 7 (taking the exam), and by the same token task 9 (studying for discrete math exam) precedes task 10 (taking that exam). Now in order to perform task 11 (celebrating), all the work must be done, and taking into account the prerequisites we have already discussed, this is forced by requiring tasks 4, 7, 8, and 10 to precede task 11. Finally, task 11 must also precede task 12. The resulting digraph is shown here.