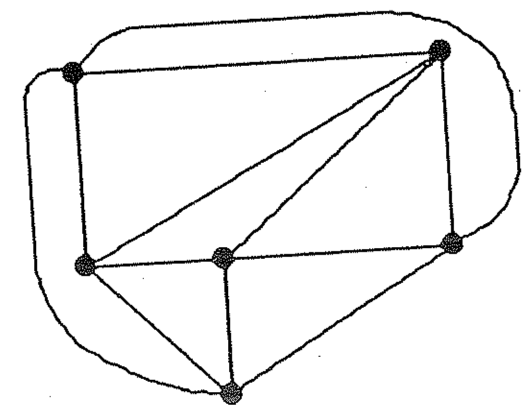
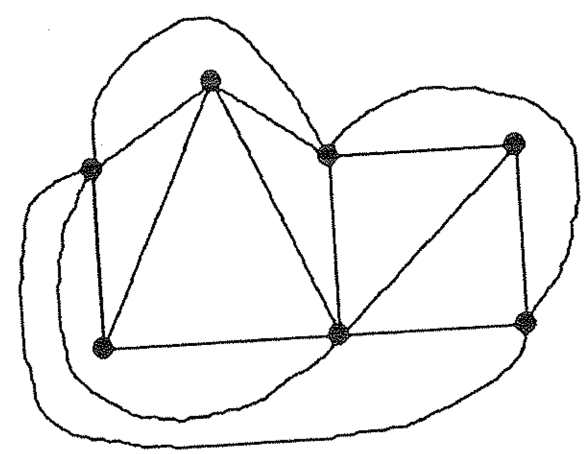
25. **(a)** This is K_5 with one edge missing. By Exercise 2, it is planar. (We can simply reroute two of the nonintersecting diagonals to the outside of the pentagon to obtain a planar embedding.)

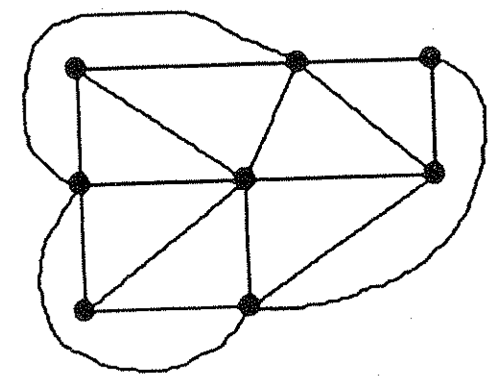
(b) This graph is planar. If we move the point inside the rectangle to the outside and also move the diagonal from upper left to lower right to the outside, we can obtain the following planar embedding.



(c) This graph is planar. We can reroute some of the lines to the outside of the figure to obtain the following embedding.



(d) This graph is planar. We can reroute some of the lines to the outside of the figure to obtain the following embedding.

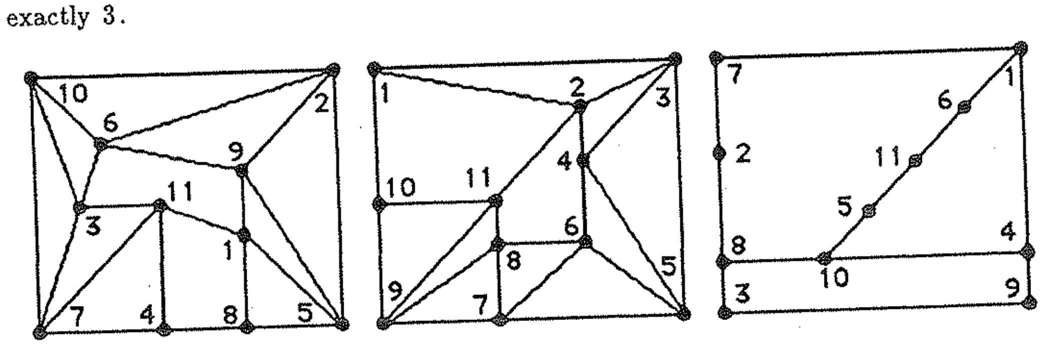


(e) This graph is not planar. Indeed, it is a subdivision of $K_{3,3}$; the points around the outside are the six vertices of $K_{3,3}$, in alternating parts.

(f) This graph is not planar by Theorem 2. It has seven vertices and 17 edges, and $17 \not\leq 3 \cdot 7 - 6$.

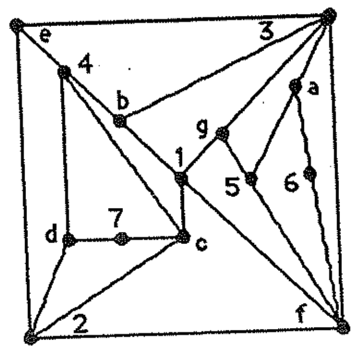
27. Suppose that a graph is embedded on the sphere. Pick a point in the interior of some region, puncture the sphere there, and stretch it open to form a plane. The region that was punctured becomes the unbounded region of the plane. All the other regions, as well as vertices, edges, and components, remain unchanged. Thus Euler's formula for the sphere follows from Euler's formula for the plane.

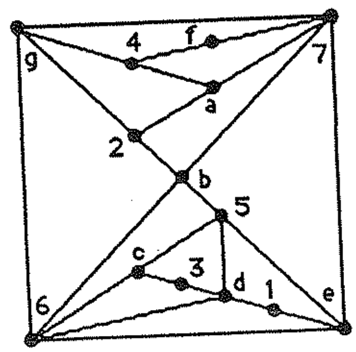
29. **(a)** Suppose that K_{11} is embedded in t planes. Each plane can contain at most $3 \cdot 11 - 6 = 27$ edges. Thus two planes can contain at most 54 edges, one short of the $C(11,2) = 55$ needed for K_{11}. Therefore the thickness t is greater than 2. The following picture, an embedding of K_{11} in three planes, shows that it at most 3. Therefore the thickness is exactly 3.



(b) Suppose that $K_{7,7}$ is embedded in t planes. Each plane can contain at most $2 \cdot 14 - 4 = 24$ edges. Thus two planes can contain at most 48 edges, one short of the 49 needed for $K_{7,7}$. Therefore the thickness is greater than 2. The picture for part (c) shows that it is at most 3 (put the missing edge $a1$ in a third plane). Therefore the thickness is exactly 3.

(c) Every edge of $K_{7,7}$ except edge $a1$ is shown in one or the other of these two planes. Clearly this graph has thickness greater than 1, so we have proved that its thickness is 2.

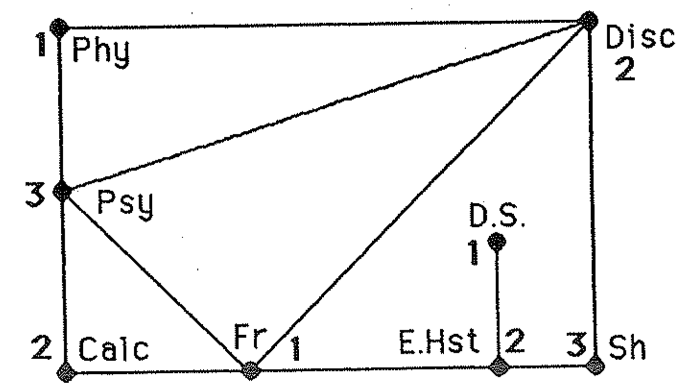




31. We can characterize the arrangement as consisting of d points forming a convex polygon, with $n-d$ points in its interior, where $3 \le d \le n$. When play is finished, the picture must consist of this d-gon with its interior completely triangulated. (Since every polygon of more than three sides contains at least one diagonal in its interior, play cannot stop until only triangles are left inside the d-gon.) Thus the number of moves in the game is e, the number of edges in this graph. It is fixed in advance and does not depend on the players' strategies. To determine e, suppose that we add $d-3$ edges in the exterior of the polygon, thereby obtaining a triangulation. In any triangulation (see Exercise 20) the inequality in Theorem 2 holds as an equality. Therefore in this case we have $e + d - 3 = 3n - 6$, so $e = 3n - d - 3 = 2n - 3 + (n - d)$. The parity of this number depends only on the parity of $n - d$. The first player wins if e is odd, which means that $n - d$ is even; and the second player wins if $n - d$ is odd. Thus the first player would like to see an even number of points inside the convex polygon, and the second player would like to see an odd number of points inside.

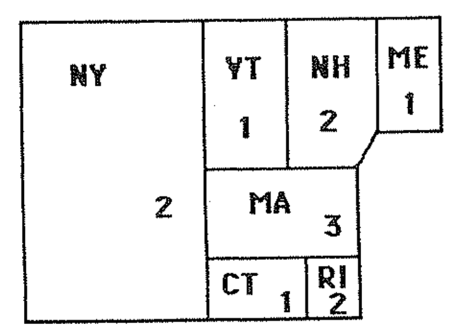
SECTION 8.5 Coloring of Graphs

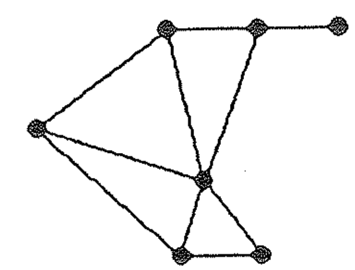
1. (a) Vertices represent courses. Two vertices are adjacent if the courses they represent have a student in common. We want a minimum coloring. The graph is shown below.



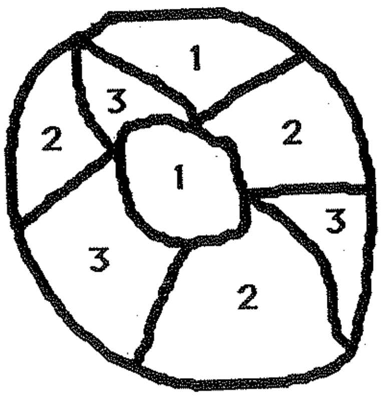
(b) One minimum coloring gives rise to the scheduling of Physics, Data Structures, and French exams on Day 1; Calculus, Discrete Math, and European History exams on Day 2; and Psychology and Shakespeare exams on Day 3. There is no 2-coloring because of the triangles such as Calculus–French–Psychology.

3. Our stylized map is shown on the left. The numbers given there provide a 3-coloring. Certainly no 2-coloring is possible, since Massachusetts, Connecticut, and New York, for example, are mutually adjacent. The associated multigraph is obtained by putting a vertex inside each state and connecting two vertices by an edge if the states share a common border.

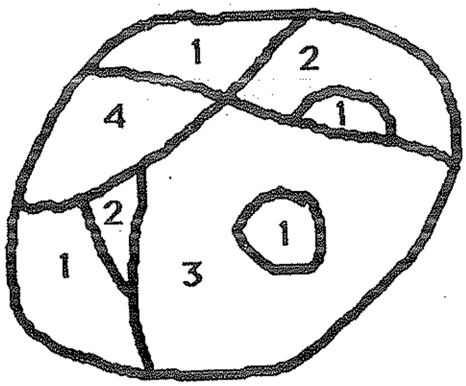




5. (a) A 3-coloring is shown below. Because the three countries in the lower left part of the map are mutually adjacent, no 2-coloring is possible.



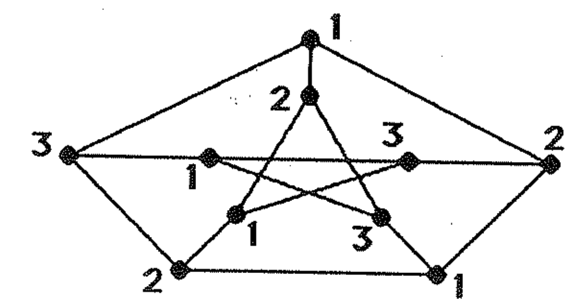
(b) A 4-coloring is shown below. Because the four countries in the lower left part of the map are mutually adjacent, no 3-coloring is possible.



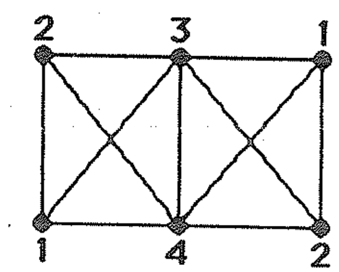
7. (a) First we order the vertices by decreasing degree (breaking ties with alphabetical order). The order is C, P, D, F, M, R. First we assign color 1 to vertex C. Since vertex C is adjacent to all other vertices, no other vertex receives color 1. Next we assign color 2 to vertex P; again no other vertices can receive color 2. Color 3 is assigned to vertices D and F; then color 4 is assigned to vertices M and R, and the coloring is complete. Note that a minimum coloring was achieved here (although the algorithm will not find a minimum coloring for *all* graphs).

(b) The decreasing degree order of the vertices is H, F, G, I, J, K, D, E, L, M, C, B, O, A, N. We begin by assigning color 1 to the nonadjacent vertices in this list, in order: H, C, O, A, and N. Then we assign color 2 to F and K; color 3 to G, L, and B; color 4 to I and D; color 5 to J and E; and finally color 6 to M. As was pointed out in Example 3, six colors are in fact necessary.

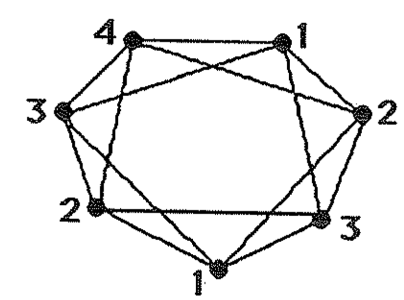
9. (a) A 3-coloring is shown here. Since the graph has an odd-length cycle, no 2-coloring is possible, so the chromatic number is indeed 3.



(b) A 4-coloring is shown here. Since the graph has K_4 as a subgraph, no 3-coloring is possible, so the chromatic number is indeed 4.



(c) A 4-coloring is shown here. To see that no 3-coloring is possible, we can try to 3-color the graph as we proceed clockwise around the figure, starting with the vertex in the upper right. There are essentially no choices involved, and the use of color 4 is forced when we come to the last vertex.



11. Algorithm 2 found a 6-coloring in Exercise 7b.

13. Assume that the colors are red, blue, green, and yellow. The indentation in the following list shows the order in which the vertices are assigned tentative colors. When the algorithm stops, the 3-coloring in which vertex 1 is red, vertices 2 and 5 are blue, and vertices 3 and 4 are green has been found.

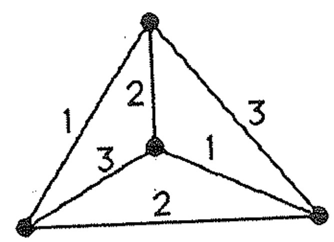
Color 1 red.
    Color 2 blue.
        Color 3 green.
            Color 4 blue.
                Color 5 yellow (4-coloring found).
            Color 4 green.
                Color 5 blue (3-coloring found).
            Color 4 yellow.
                Color 5 blue.

done

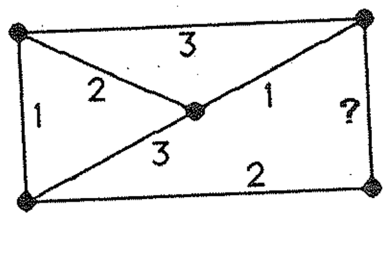
15. A graph with no edges clearly has chromatic number 1, since every vertex can be assigned the same color. On the other hand, any graph with at least one edge must have chromatic number at least 2. Therefore those graphs G for which $\chi(G) = 1$ are precisely the graphs with no edges.

17. (a) There are d edges with a common endpoint, so at least d colors are required in any edge coloring.

(b) The following picture shows that $\chi'(K_4) = 3$, and 3 is the maximum degree of the vertices in K_4.



(c) Suppose that we try to find a coloring of the edges of this graph using only the colors 1, 2, and 3. Without loss of generality, we can assign colors 1, 2, and 3 to the three mutually adjacent edges in the triangle on the left. This forces the other edges to be colored as shown here.

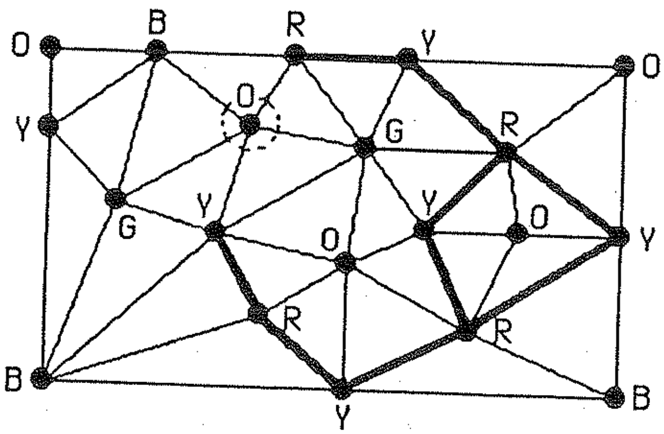


Now there is no way to color the last edge with one of these three colors. Thus the edge-chromatic number of the graph is 4. Note, however, that the maximum vertex degree is 3.

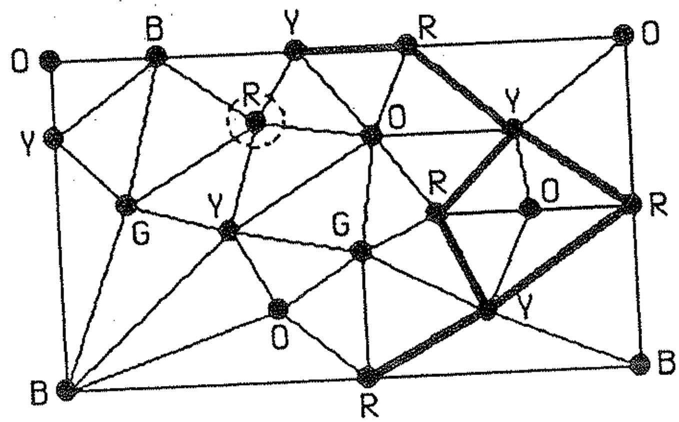
19. The chromatic number of a graph with more than one component is the largest of the chromatic numbers of its components. To see this rigorously, let c be the chromatic number of the component of the graph G with largest chromatic number. Then clearly $\chi(G) \geq c$. On the other hand, since the chromatic number of each component of G is at most c, we can find c-colorings of each component using the colors $\{1, 2, \ldots, c\}$. Putting these colorings together gives a c-coloring of G. Therefore $\chi(G) \leq c$, as well, so $\chi(G) = c$.

21. (a) F (the complete graph on n vertices requires n colors)
(b) T (since every planar graph has chromatic number at most 4)
(c) T (its chromatic number is 5)
(d) F (since $K_{3,3}$ is bipartite, its chromatic number is 2)
(e) F ($K_{3,3}$ is 4-colorable but not planar)
(f) F ($K_{3,3}$ is 2-colorable but not planar)
(g) T (a 1-colorable graph has no edges)

23. (a) We look at the maximal red-yellow connected subgraph containing the red vertex adjacent to the circled vertex (shown in heavy lines in the following figure). It contains the yellow vertex adjacent to the circled vertex, so it contains a path that surrounds the orange vertex adjacent to the circled vertex. Therefore we can interchange the colors orange and green for all vertices within the region formed by this red-yellow fence, thereby freeing up the color orange for the circled vertex, as shown.

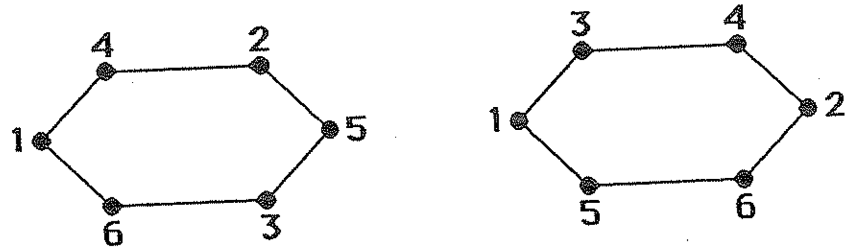


(b) This time the maximal red-yellow subgraph containing the red vertex adjacent to the circled vertex does not contain the yellow vertex adjacent to the circled vertex. Therefore we interchange the colors red and yellow in this subgraph and thereby free up the color red for the circled vertex. The new coloring is shown here.



25. The main loop is performed $\chi(G)$ times. The **for** loop and **if** statement require essentially looking at each vertex and each edge. Thus the algorithm's efficiency is in $O(\chi(G)\cdot(v+e))$, where v is the number of vertices and e is the number of edges of G.

27. The algorithm will produce a 2-coloring of the graph with the numbering shown on the left, but not with the numbering shown on the right. In the graph shown on the right, the algorithm will assign the same color to vertices 1 and 2, after which it cannot recover.



29. **(a)** If we remove edge ab from K_n, then we can $(n-1)$-color the resulting graph by using the same color for a and b.

(b) The 5-spoked wheel is 4-critical. It is clear that this graph has chromatic number 4. If we delete an edge along the rim of the wheel, then we can color the rest of the rim with just two colors, so the entire broken wheel can be colored with three colors. If we delete a spoke, then we can 3-color the rim in such a way that color 3 is used only for the vertex at the end of the missing spoke, and the center of the wheel can receive color 3.

(c) Let $\chi(G)=k$. If G is k-critical, then we are done. If not, then G has a proper subgraph G_1 such that $\chi(G_1)=k$. If G_1 is k-critical, then we are done. If not, we repeat this process, obtaining a proper subgraph G_2 of G_1 with $\chi(G_2)=k$. We continue as long as possible. Since G has only finitely many vertices and edges, the process must eventually halt, giving us a k-critical graph.

(d) By Exercise 19, the chromatic number of a graph equals the chromatic number of one of its components. Thus some proper subgraph of a nonconnected graph has the same chromatic number as the graph, so no nonconnected graph can be k-critical.

(e) The graph G' obtained by deleting one edge uv from G has chromatic number less than k. If its chromatic number were less than $k-1$, then a $(k-2)$-coloring of G', modified to assign color $k-1$ to vertex v, would be a $(k-1)$-coloring of G, a contradiction.