Prove that

$$8 \mid 3^{2n} - 1 \quad \forall\, n \geq 1$$

Proof: Base case $n = 1$

$$3^{2n} - 1 = 3^{2(1)} - 1 = 3^2 - 1 = 8$$

if $n = 1$

We know $8 \mid 8$.

Now assume that the statement is true for $n = k \geq 1$, where k is fixed but arb.

i.e.

assume $8 \mid 3^{2k} - 1$ I.H.

WTS that the statement is true for $n = k+1$

i.e.

WTS $8 \mid 3^{2(k+1)} - 1$

Now $3^{2(k+1)} - 1$ $\leftarrow$ Goal: WTS this is a multiple of 3

$= 3^{2k+2} - 1$

$= 3^{2k} \cdot 3^2 - 1$ $\quad 9 = 8+1$

$= 9 \cdot 3^{2k} - 1$

$= \underbrace{1 \cdot 3^{2k}} + 8 \cdot 3^{2k} \underbrace{-1}$

$= 3^{2k} - 1 + 8 \cdot 3^{2k}$

By I.H., $8 \mid 3^{2k} - 1$ and

clearly $8 \mid 8 \cdot 3^{2k}$

$\therefore 8 \mid 3^{2k} - 1 + 8 \cdot 3^{2k}$

$\Rightarrow 8 \mid 3^{2(k+1)} - 1$

$\therefore$ By PMI, the claim is true $\forall n \geq 1$

Q7 on Pg 350

$$\frac{2(k+1)-1}{2(k+1)} = \frac{2k+1}{2k+2}$$

Prove that

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} \geq \frac{1}{2n}$$

$$\forall\, n \geq 1$$

Proof: Base case $n = 1$

L.H.S. $= \frac{1}{2}$ R.H.S $= \frac{1}{2(1)} = \frac{1}{2}$

$\therefore$ L.H.S. $\geq$ R.H.S.

Assume that the claim is true for $n = k \geq 1$, where k is fixed by arb.

i.e. Assume that

$$\underbrace{\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2k-1}{2k}} \geq \frac{1}{2k} \quad \text{I.H.}$$

WTS the claim is true for $n = k+1$

WTS

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2k+1}{2k+2} \geq \frac{1}{2k+2}$$

$$\text{L.H.S.} = \underbrace{\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2k-1}{2k}}\ \frac{2k+1}{2k+2}$$

$$\geq \frac{1}{2k} \frac{2k+1}{2k+2} \qquad \text{by I.H.}$$

$$= \frac{2k+1}{2k} \cdot \frac{1}{2k+2} \qquad \geq 1$$

$$\geq 1 \cdot \frac{1}{2k+2}$$

$$= \frac{1}{2k+2} = \text{R.H.S}$$

$\therefore$ By PMI, the claim is true $\forall n \geq 1$

We can prove this without induction.

$$\text{L.H.S.} = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n}$$

$$= \underbrace{\frac{3}{2}}_{\geq 1} \underbrace{\frac{5}{4}}_{\geq 1} \underbrace{\frac{7}{6}}_{\geq 1} \underbrace{\cdots}_{\geq 1} \frac{1}{2n}$$

$$\geq \frac{1}{2n}$$

16b on Pg 351

Prove that

$$7 \mid 11^n - 4^n \quad \forall\, n \geq 1$$

Proof: Base case $n=1$

If $n=1$, then $11^n - 4^n = 11^1 - 4^1 = 7.$

We know $7 \mid 7.$

Assume that the claim is true for $n = k \geq 1$, where k is fixed but arb.

i.e. Assume $7 \mid 11^k - 4^k$ I. H.

WTS the claim is true for $n = k+1$

i.e.

WTS $7 \mid 11^{k+1} - 4^{k+1}$

Now $11^{k+1} - 4^{k+1}$

$$= 11 \cdot 11^k - 4 \cdot 4^k$$

$$= \underbrace{4 \cdot 11^k} + 7 \cdot 11^k - \underbrace{4 \cdot 4^k}$$

$$= 4 \cdot 11^k - 4 \cdot 4^k + 7 \cdot 11^k$$

$$= 4(11^k - 4^k) + 7 \cdot 11^k$$

Now $7 \mid 11^k - 4^k$ by I.H.

and $7 \mid 7 \cdot 11^k$

$\therefore 7 \mid 4(11^k - 4^k) + 7 \cdot 11^k$

$\Rightarrow 7 \mid 11^{k+1} - 4^{k+1}$

$\therefore$ By PMI, the claim is true $\forall n \geq 1$