APM 2663 Final Exam

Fall 2021
Instructor: Eddie Cheng

Important:

- Recall that the word *if* in a definition means *if and only if*.
- Recall that $\mathbb{N}$ is the set of positive integers.
- Recall that $\mathbb{Z}$ is the set of integers.
- Recall that $\mathbb{Q}$ is the set of rational numbers.
- Recall that $\mathbb{R}$ is the set of real numbers.
- Recall that $\emptyset$ is the empty set.
- This is a closed book examination. No external aids are allowed, except a calculator.
- Cheating is a serious academic misconduct. Oakland University policy requires that all suspected instances of cheating be reported to the Office of the Dean of Students/Academic Conduct Committee for adjudication. I have forwarded cases to the Office of the Dean of Students/Academic Conduct Committee before and I will not hesitate to do this again if I suspect academic misconduct has occurred. Anyone found responsible of cheating in this assessment will receive a course grade of F, in addition to any penalty assigned by the Academic Conduct Committee.
- I may ask for a meeting for you to explain your solutions.
- You may not give a "cheap" proof. For example, if I ask you to prove the 6-Color Theorem, you are not allowed to use the 4-Color Theorem.
- Until the solution to this exam is posted/discussed by me, you may not discuss this exam with others.
- This test is worth 110 marks. If you receive x marks, your grade will be $\min\{x, 100\}\%$.

(1) Read the instructions and sign your name (in the space provided below) indicating that you have read the instructions. [1 mark]

(2) Write down your name and student number. [1 mark]

(3) Find the gcd of 8205 and 615. Write the gcd as $8205x + 615y$ for some $x, y \in \mathbb{Z}$. [8 marks]

(4) Michael Wadsworth makes a special blend of hot sauce. In his store, only 3-packs (each containing 3 bottles) and 7-packs (each containing 7 bottles) are available. Use mathematical induction to show that he can fulfill an order of size n for every integer $n \geq 12$. [15 marks]

(5) Let p be a prime number. Let m and n be two integers. Show that if p divides mn, then p divides m or p divides n without using the Fundamental Theorem of Arithmetic. [10 marks]

(6) Consider the following sum.

$$\sum_{k=2}^{n} k(k-1)^2 \binom{n}{k} 5^k.$$

(a) Evaluate it by using the Binomial Theorem [5 marks]

(b) Evaluate it combinatorially. [15 marks]

Extra space

(7) Define $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ by $f(x) = 2021x^3 - 2663x + 115$. Determine whether or not f is one-to-one and/or onto. [10 marks]

(8) Aura Cazares is holding a concert and she is giving 12 families front row seats. Each family consists of a couple and three kids. The task is to sit the 60 guests in the first row with 60 seats such that (a) each family must sit together and (b) the two seats at the ends of the row (the first seat and the last seat) must be occupied by adults. How many ways can this be done? [10 marks]

(9) Jessie Hurse is distributing m candies to 25 boys, with name tags 1 to 25, and 25 girls, with name tags 1 to 25. How many ways can this be done if girl i must have twice as many candies as boy i for every $i = 1, 2, \ldots, 25$. [15 marks]

(10) Let $G(V, E)$ be a simple graph. We define a relation $\sim$ on V by $u \sim v$ (where $u, v \in V$) if there is a path from u to v. Prove that $\sim$ is an equivalence relation on V. What can you say about equivalence classes with respect to $\sim$? [10 marks]

(11) State the 4-Color Theorem and prove it for triangle-free planar graphs. [10 marks]

Estimate your grade in this test. Let x be your guess. If your grade is in the interval $[x-5, x+5]$, you will receive 2 bonus marks.