An example

Example: Evaluate
$$1^2\binom{n}{1}+2^2\binom{n}{2}+3^2\binom{n}{3}+\cdots+n^2\binom{n}{n}.$$
We will give two solutions.

Boring algebraic proof:
Start with the Binomial Theorem:
$$(1+x)^n=\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\binom{n}{3}x^3+\cdots+\binom{n}{n}x^n.$$
Differentiate both sides with respect to x and we get
$$n(1+x)^{n-1}=\binom{n}{1}+2\binom{n}{2}x+3\binom{n}{3}x^2+\cdots+n\binom{n}{n}x^{n-1}.$$
At this point, we see that the general term is $i\binom{n}{i}x^{i-1}$. This is good as the general term in the sum is $i^2\binom{n}{i}$. One may be tempted to differentiate with respect to x on both sides again. But this will bring down the $i-1$ to get $i(i-1)\binom{n}{i}x^{i-2}$. But we do not want $i(i-1)$, we want i^2. Thus we multiply the above identity by x. This gives
$$nx(1+x)^{n-1}=\binom{n}{1}x+2\binom{n}{2}x^2+3\binom{n}{3}x^3+\cdots+n\binom{n}{n}x^n.$$
Now differentiate both sides with respect to x and we get
$$n(n-1)x(1+x)^{n-2}+n(1+x)^{n-1}=\binom{n}{1}+2^2\binom{n}{2}x+3^2\binom{n}{3}x^2+\cdots+n^2\binom{n}{n}x^{n-1}.$$
(We note that we have used the product rule on the left.) Now let $x=1$ and we have
$$n(n-1)2^{n-2}+n2^{n-1}=1^2\binom{n}{1}+2^2\binom{n}{2}+3^2\binom{n}{3}+\cdots+n^2\binom{n}{n}.$$

Interesting combinatorial proof (G version):
Little Mermaid Elementary School has n pupils and the headmaster wants to sent at least one pupil to Disney World. In addition, among the pupils who are going to Disney World, one would get a Mikey Mouse watch and one would get a Donald Duck hat. It is possible that the same pupil would get both the watch and the hat. We will count the number of ways that this can be done via two different methods. On the one hand, the headmaster can first decide on the number of pupils who would be going to Disney World. Let this number be i. There are $\binom{n}{i}$ ways to select these i pupils. Now among these i pupils, pick one to receive the watch, and there are i ways to do it. Now among these i pupils, pick one to receive the hat, and

there are i ways to do it. This gives $i^2\binom{n}{i}$ ways. Since i can be any number in $\{1, 2, 3, \ldots, n\}$. The correct number is

$$\sum_{i=1}^{n} i^2\binom{n}{i} = 1^2\binom{n}{1} + 2^2\binom{n}{2} + 3^2\binom{n}{3} + \cdots + n^2\binom{n}{n}.$$

On the other hand, the headmaster can first decide who would be getting the watch and who would be getting the hat (and going to the trip) first and then deciding on the rest of the party. There are two cases.

Case 1: The same pupil receives both the watch and the hat. There are n choices. Now we have $n-1$ pupils left and we have two choices for each in deciding whether this pupil is going to Disney World or not. Hence the total number of ways is $n2^{n-1}$.

Case 2: Two different pupils receive the watch and the hat. There are n choices for the pupil to receive the watch and then there are $n-1$ choices to pick a different pupil to receive the hat. Now we have $n-2$ pupils left and we have two choices for each in deciding whether this pupil is going to Disney World or not. Hence the total number of ways is $n(n-1)2^{n-2}$.

Combining the two cases, the total number of ways is $n(n-1)2^{n-2} + n2^{n-1}$. Therefore,

$$n(n-1)2^{n-2} + n2^{n-1} = 1^2\binom{n}{1} + 2^2\binom{n}{2} + 3^2\binom{n}{3} + \cdots + n^2\binom{n}{n}.$$

Exercises:

(1) Evaluate

$$2 \cdot 1\binom{n}{2} + 3 \cdot 2\binom{n}{3} + \cdots + n(n-1)\binom{n}{n}.$$

(2) Evaluate

$$1^2\binom{n}{1}2 + 2^2\binom{n}{2}2^2 + 3^2\binom{n}{3}2^3 + \cdots + n^2\binom{n}{n}2^n.$$

Yes. You should do these exercises.