(11) Estimate your grade in this test. Let x be your guess. If your grade is in the interval $[x-5, x+5]$, you will receive 2 bonus marks.

93, ninety-three

(10) There are $5n$ students in a class and the teacher would like to divide them into n groups of students, each with 2 students (Type A groups), and n groups of students, each with 3 students (Type B groups). Moreover, the teacher has n tags labelled 1 to n and these tags will be distributed to the Type A groups, so that each group has exactly one tag. However, the Type B groups have no tags. [15 marks]

We can choose all Type A groups first, then Type B.
Type A order will matter, due to labeled/numbered tags.
We use multiplication principle at every step.

B n groups — no ordering, doesn't matter

$$\# \text{ways} = \underbrace{\binom{5n}{2}\binom{5n-2}{2}\cdots\binom{3n+2}{2}}_{A \; n \text{ groups}} \cdot \binom{3n}{3}\binom{3n-3}{3}\cdots\binom{6}{3}\binom{3}{3}\Big/ n!$$

$$= \frac{(5n)!}{\cancel{(5n-2)!}\cdot 2!}\cdot\frac{\cancel{(5n-2)!}}{(5n-4)!\cdot 2!}\cdots\frac{\cancel{(3n+4)!}}{\cancel{(3n+2)!}\cdot 2!}\cdot\frac{\cancel{(3n+2)!}}{\cancel{(3n)!}\cdot 2!}\cdot\cancel{n!}\cdot\frac{\cancel{(3n)!}}{\cancel{(3n-3)!}\cdot 3!}\cdot\frac{\cancel{(3n-3)!}}{\cancel{(3n-6)!}\cdot 3!}\cdots\frac{\cancel{6!}}{\cancel{(6-3)!}\cdot 3!}\cdot\frac{\cancel{3!}}{\cancel{(3-3)!}\cdot 3!}$$

$$= \frac{(5n)!}{2!^n}\cdot\frac{1}{3!^n\cdot n!} = \boxed{\frac{(5n)!}{2^n\cdot 6^n\cdot n!}}$$

should be equivalent to unsimplified answer, I hope ".

15

(9) We have $10n$ distinct beads. The task is to make 6 regular necklaces, each consisting of n beads, and two long necklaces, each consisting of $2n$ beads. In addition, we have two special (identical) asymmetrical clasps, two regular (identical) clasps that are symmetrical, one asymmetrical clasp with label 1 written on it (on both sides), and one asymmetrical clasp with label 2 written on it (on both sides). How many ways can we make these necklaces subject to the following conditions?

- Two regular necklaces, each with an asymmetrical clasp (without a label).
- Two regular necklaces, each with a regular clasp.
- Two regular necklaces, each with an asymmetrical clasp with a label.
- Two long necklaces with no clasps.

[15 marks]

Choose beads, then arrange/assemble. Use multiplication principle at each step.

$$\# \text{ ways} = \binom{10n}{n} n! \binom{4n}{n} n! / 2! \qquad [\text{regular, asymm clasps}]$$

$$\cdot \binom{8n}{n} \frac{n!}{2} \binom{7n}{n} \frac{n!}{2} / 2! \qquad [\text{regular, symm clasps}]$$

$$\cdot \binom{6n}{n} n! \binom{5n}{n} n! \qquad [\text{asymm clasps w/ labels}]$$

$$\cdot \binom{4n}{2n} \frac{(2n-1)!}{2} \binom{2n}{2n} \frac{(2n-1)!}{2} / 2! \qquad [\text{long necklaces w/o clasps}]$$

15

Shane J

Test 2

8. $$\sum_{k=2}^{n} k(k-1)^2 \binom{n}{k} 5^{k-2}.$$

Binomial Thm

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$$

$\frac{d}{dx} \rightarrow$ $$\sum_{k=1}^{n} k \binom{n}{k} x^{k-1} = n(1+x)^{n-1}$$

$\frac{d}{dx} \rightarrow$ $$\sum_{k=2}^{n} k(k-1)\binom{n}{k} x^{k-2} = n(n-1)(1+x)^{n-2}$$

$*x \rightarrow$ $$\sum_{k=2}^{n} k(k-1)\binom{n}{k} x^{k-1} = n(n-1)x(1+x)^{n-2}$$

$\frac{d}{dx} \rightarrow$ $$\sum_{k=2}^{n} k(k-1)^2\binom{n}{k} x^{k-2} = n(n-1)(1+x)^{n-2} + n(n-1)(n-2)x\cdot(1+x)^{n-3}.$$

Let $x=5$,

$\Rightarrow$ $$\sum_{k=2}^{n} k(k-1)^2\binom{n}{k} 5^{k-2} = n(n-1)\cdot 6^{n-2} + 5n(n-1)(n-2)\cdot 6^{n-3}.$$

20

Once you have done this, it is no longer pure (combinatorial

(8) Evaluate

$$\sum_{k=2}^{n} k(k-1)^2 \binom{n}{k} 5^{k-2}$$

combinatorially. [20 marks].
(You may evaluate it algebraically for 12 marks.)

Rewrite: $\frac{1}{25}\sum_{k=2}^{n} \binom{n}{k}\binom{k}{1}\binom{k-1}{1}\binom{k-1}{1}\cdot 5^{k}$

All possible cases, from all pass to all fail and everything between. Sum cases

choose k of n students to pass class

choose 1 student who can skip the final

choose 2 more students (who must take final), but can win one of two vacation prizes. Same student can win both vacations

Among those passing, they can wear 5 different colors of outfits.

On the other hand, use mult & addition principles (add distinct cases) on all possible ways,

- n ways to choose student exempt from final
- n-1 ways to choose student for 1st vacation prize
- n-1 ways to choose student for 2nd vacation prize (2 cases, same n-1 or different n-2)
- Everyone else (n-2 remain or n-3), 6 choices: fail, or pass → 1 of 5 color outfits

Case 1: same student — Case 2: diff students for vacations

$$\rightarrow n\cdot 5\cdot(n-1)\cdot 5\cdot 6^{n-2} + n\cdot 5\cdot(n-1)\cdot 5\cdot(n-2)\cdot 5\cdot 6^{n-3}$$

5 color choices

n-2 other students, 6 choices

3 prizes

n-3 remain

Then, multiply by $\frac{1}{25}$ coefficient

$$\Rightarrow \boxed{\#\text{ ways} = n(n-1)\cdot 6^{n-2} + 5n(n-1)(n-2)\,6^{n-3}.}$$

Since we counted correctly, the LHS & RHS are equivalent

(7) How many integers between 1 and 2024 (inclusive) are divisible by at least one of 3, 11 and 17? [10 marks]

$$|M_3 \cup M_{11} \cup M_{17}| = |M_3| + |M_{11}| + |M_{17}|$$
$$- |M_3 \cap M_{11}| - |M_3 \cap M_{17}| - |M_{11} \cap M_{17}|$$
$$+ |M_3 \cap M_{11} \cap M_{17}|.$$

$$|M_3| = \left\lfloor \frac{2024}{3} \right\rfloor, \quad |M_{11}| = \left\lfloor \frac{2024}{11} \right\rfloor, \quad |M_{17}| = \left\lfloor \frac{2024}{17} \right\rfloor.$$

$|M_3 \cap M_{11}| = \left\lfloor \frac{2024}{3 \cdot 11} \right\rfloor$, and so on...

What are M_3, M_{11} and M_{17}

So, the answer is;

$$\boxed{\begin{aligned} |M_3 \cup M_{11} \cup M_{17}| &= \left\lfloor \frac{2024}{3} \right\rfloor + \left\lfloor \frac{2024}{11} \right\rfloor + \left\lfloor \frac{2024}{17} \right\rfloor \\ &\quad - \left\lfloor \frac{2024}{3 \cdot 11} \right\rfloor - \left\lfloor \frac{2024}{3 \cdot 17} \right\rfloor - \left\lfloor \frac{2024}{11 \cdot 17} \right\rfloor \\ &\quad + \left\lfloor \frac{2024}{3 \cdot 11 \cdot 17} \right\rfloor \end{aligned}}$$

8

(6) Let p be a prime number and n be an integer. Show that if p divides n^2, then p divides n, without using the Fundamental Theorem of Arithmetic. [10 marks]

Prove: $p \mid n^2 \Rightarrow p \mid n$ $\quad (p \in \text{primes}, n \in \mathbb{N})$

Suppose not, i.e., $p \mid n^2$ but $p \nmid n$.

Since p is prime, $p \nmid n \Rightarrow \gcd(n,p) = 1$. (lemma for primes)

$\therefore \exists x, y \in \mathbb{Z} : 1 = nx + py$. (Bezout's identity)

$\Rightarrow n = n^2x + npy = n^2(x) + p(ny)$.

We are supposing the hypothesis $p \mid n^2$, and clearly also $p \mid p$.

$\therefore p \mid n^2(x) + p(ny)$ (E's lemma)

$\Rightarrow p \mid n$. (Since $n = n^2 \cdot x + p \cdot ny$)

Recap: $p \mid n^2$, $p \nmid n$, $p \mid n$. ※

10

So, we cannot have $p \mid n^2$ and $p \nmid n$. It leads to contradiction.

Hence, the original claim is true by proof by contradiction, and $p \mid n^2 \Rightarrow p \mid n$. □

Proof: $\boxed{p \nmid n \Rightarrow \gcd(n,p) = 1}$ Call $g = \gcd(n,p)$. (lemma for primes)

Since p is prime, and $g \mid p$, either $g = p$ or $g = 1$.
But if $g = p$, this means $p \mid n$ (which would violate our hypothesis that $p \nmid n$),
so $g = 1$. □

$$\sum_{k=1}^{n} (-1)^{k-1} \cdot k^2.$$

(5) Use mathematical induction to prove that

$$1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1}n^2 = (-1)^{n-1}n(n+1)/2 \text{ for } n \geq 1. \text{ [15 marks]}$$

Base case: $n=1$, $LHS = 1^2 = 1$

$RHS = (-1)^{1-1} \cdot 1^2 = 1 \cdot 1 = 1$ ✓

Inductive step: Assume claim is true for a fixed but arbitrary integer, $k \geq 1$, i.e. $\underbrace{1^2 - 2^2 + 3^2 - \ldots + (-1)^{k-1} \cdot k^2 = (-1)^{k-1} \cdot k(k+1)/2}_{IH}$.

WTS: The IH assumption implies the $k+1$ case also holds true, i.e., $1^2 - 2^2 + 3^2 - \ldots + (-1)^{k-1} \cdot k^2 + (-1)^k \cdot (k+1)^2 = (-1)^k \cdot (k+1)(k+2)/2$.

$LHS = \underbrace{1^2 - 2^2 + 3^2 - \ldots + (-1)^{k-1} \cdot k(k+1)}_{IH} + (-1)^k \cdot (k+1)^2$

$= (-1)^{k-1} k(k+1)/2 + (-1)^k (k+1)^2$

$= (-1)^k \left[-\frac{k(k+1)}{2} + \frac{(k+1)^2}{2} \right] = (-1)^k (k+1) \left[\frac{-k + 2(k+1)}{2} \right] = (-1)^k (k+1) \left[\frac{k+2}{2} \right]$

$= (-1)^k (k+1)(k+2)/2$

$= RHS$, as desired.

15

$\therefore$ The claim is proved true $\forall n \geq 1$, by PMI. $\square$

(4) Find the gcd of 7590 and 615. Write the gcd as $7590x + 615y$ for some $x, y \in \mathbb{Z}$. [13 marks]

$7590 = 12 \cdot 615 + 210 \rightarrow 210 = 7590 - 12 \cdot 615$

$615 = 2 \cdot 210 + 195 \rightarrow 195 = 615 - 2 \cdot 210$

$210 = 195 + \boxed{15} \rightarrow 15 = 210 - 195$

$195 = 13 \cdot 15 + 0$

$15 = 210 - 195$

$= 210 - (615 - 2 \cdot 210)$

$= 3 \cdot 210 - 615$

$= 3(7590 - 12 \cdot 615) - 615$

$\Rightarrow \boxed{15 = 3 \cdot 7590 - 37 \cdot 615}$

13

$x = 3$

$y = -37$

$\boxed{\gcd(7590, 615) = 15.}$

(3) Give a polynomial time algorithm, using the arithmetic model, to solve the following problem: Input n numbers, find the square of the product of these numbers, and output this number. (That is, output $(a_1 a_2 \cdots a_n)^2$ if the numbers are $a_1, a_2, \ldots, a_n$.) [10 marks]

(variable declaration of n & num is assumed)

```
read n
prod ← 1
for i, 1 to n
    read num
    prod ← prod • num
prod ← prod • prod
write prod
```

10

APM 2663 Test 2

Fall 2024

Instructor: Eddie Cheng

Date: November 26, 2024

100%

Important:

- Recall that the word *if* in a definition means *if and only if*.
- **To receive full credit for a question, you should provide all logical steps. All answers must be justified unless the questions stating otherwise.**
- Recall that $\mathbb{N}$ is the set of positive integers. The definition in the book includes 0.
- Recall that $\mathbb{Z}$ is the set of integers.
- Recall that $\mathbb{Q}$ is the set of rational numbers.
- Recall that $\mathbb{R}$ is the set of real numbers.
- This is a closed book examination. No external aids are allowed, except a calculator.
- Cheating is a serious academic misconduct. Oakland University policy requires that all suspected instances of cheating be reported to the Office of the Dean of Students/Academic Conduct Committee for adjudication. I have forwarded cases to the Office of the Dean of Students/Academic Conduct Committee before and I will not hesitate to do this again if I suspect academic misconduct has occurred. Anyone found responsible of cheating in this assessment will receive a course grade of F, in addition to any penalty assigned by the Academic Conduct Committee.
- I may ask for a meeting for you to explain your solutions.
- Until the solution to this test is posted/discussed by me, you may not discuss this test with others.
- This test is worth 110 marks. If you receive x marks, your grade will be $\min\{x, 100\}\%$.
- Solutions must be uploaded to Moodle unless otherwise arranged.

(1) Read the instructions and sign your name (in the space provided below) indicating that you have read the instructions. [1 mark]

Shane Jar

(2) Write down your name. [1 mark]

Shane Jaroch