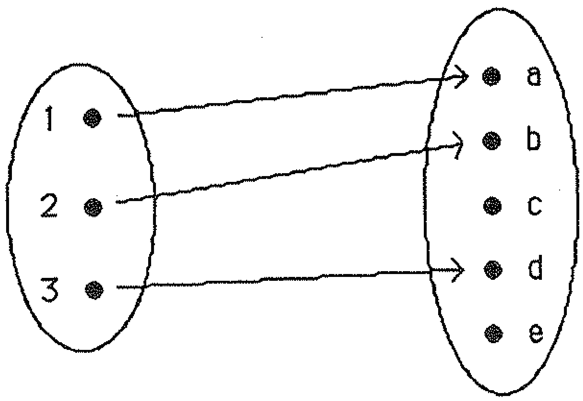
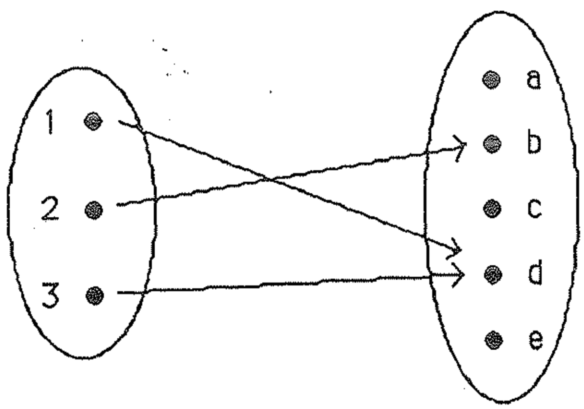
SECTION 3.2 Functions in the Abstract

1. (a) The following picture represents the function $\{(1,a), (2,b), (3,d)\}$. Each element in the codomain is used at most once.



(b) The following picture represents the function $\{(1,d), (2,b), (3,d)\}$. The element d in the codomain is used twice, so the function is not injective.



3. (a) $f(x) = x$
   (b) $f(x) = x + 1$ (not onto since 0 is not in the range)
   (c) $f(x) = \lfloor x/2 \rfloor$ (not one-to-one since, for example, $f(3) = f(2)$)
   (d) $f(x) = 3$

5. Define f by setting $f(0) = 1$, $f(1) = 0$, and $f(x) = x$ for all $x \geq 2$.

7. (a) This function is bijective; its inverse is $f^{-1}(y) = (y-1)/2$.

(b) This function is neither injective ($f(1) = f(-1)$) nor surjective (0 is not in the range).

(c) This function is bijective; its inverse is $f^{-1}(y) = \sqrt[3]{(y-1)/2}$.

(d) This function is injective but not surjective (only numbers between $-\pi/2$ and $\pi/2$ are in the range).

(e) This function is surjective but not injective ($f(1) = f(-1)$).

9. If A is a finite set, and $f: A \to A$, then f is injective if and only if f is surjective.

11. (a) We need x to be either the father of the father or the father of the mother of y, so we can write $x = f(f(y)) \vee x = f(m(y))$.

(b) This time there are four possibilities: $y = m(f(x)) \vee y = m(m(x)) \vee y = f(f(x)) \vee y = f(m(x))$.

(c) We make the simplifying assumption that there are no half-brothers or half-sisters. To say that x is the aunt or uncle of y is to say that x's mother is y's grandmother, but that x is not the mother or father of y. Thus we write $(x \neq m(y) \wedge x \neq f(y)) \wedge \exists u\colon (u = m(x) \wedge (u = m(f(y)) \vee u = m(m(y))))$.

(d) We just need to take this last expression and put it into a set description: $\{\, y \mid (x \neq m(y) \wedge x \neq f(y)) \wedge \exists u\colon (u = m(x) \wedge (u = m(f(y)) \vee u = m(m(y))))\,\}$.

13. (a) $(f+g)(x) = 1+2 = 3$ for all x

(b) $(f \cdot g)(x) = 1 \cdot 2 = 2$ for all x

(c) We need to see what happens to odd and even arguments. The definition of the function is given by

$$(f-g)(x) = f(x) - g(x) = \begin{cases} 1-2=-1 & \text{if } x \text{ is odd} \\ 2-1=1 & \text{if } x \text{ is even.} \end{cases}$$

(d) Again we look at the odd and even arguments separately:

$$(f \circ g)(x) = f(g(x)) = \begin{cases} 2 & \text{if } x \text{ is odd} \\ 1 & \text{if } x \text{ is even} \end{cases} = g(x).$$

Thus $f \circ g = g$.

(e) This time

$$(f \circ f)(x) = f(f(x)) = \begin{cases} 1 & \text{if } x \text{ is odd} \\ 2 & \text{if } x \text{ is even} \end{cases} = f(x).$$

Thus $f \circ f = f$.

(f) Here

$$(g \circ f)(x) = g(f(x)) = \begin{cases} 2 & \text{if } x \text{ is odd} \\ 1 & \text{if } x \text{ is even} \end{cases} = g(x).$$

Thus $g \circ f = g$.

(g) In this case

$$(g \circ g)(x) = g(g(x)) = \begin{cases} 1 & \text{if } x \text{ is odd} \\ 2 & \text{if } x \text{ is even} \end{cases} = f(x).$$

Thus $g \circ g = f$.

15. **(a)** We want to say that C consists of the images of the elements of A; in symbols, $\forall y\colon (y \in C \leftrightarrow \exists x{\in}A\colon f(x) = y)$.

(b) We need to say two things about every y in the codomain—that there are two distinct elements of the domain whose image is y, and that every element of the domain whose image is y is one of these two elements (which we call x_1 and x_2 in the following expression). Thus we have $\forall y{\in}B\colon \exists x_1{\in}A\colon \exists x_2{\in}A\colon \big(x_1 \neq x_2 \wedge f(x_1) = y \wedge f(x_2) = y \wedge \forall x{\in}A\colon [f(x) = y \rightarrow (x = x_1 \vee x = x_2)]\big)$.

17. **(a)** This is true. Suppose that $f(x_1) = f(x_2)$. Then $g(f(x_1)) = g(f(x_2))$. Since $g \circ f$ is one-to-one, this means that $x_1 = x_2$. Therefore f is one-to-one.

(b) This is false. Let $A = \{a\}$, $B = \{b, d\}$, and $C = \{c\}$. Let $f = \{(a, b)\}$ and $g = \{(b, c), (d, c)\}$. Then $g \circ f = \{(a, c)\}$. Clearly $g \circ f$ is one-to-one, but g is not.

19. We will prove the (equivalent) contrapositive: that such a function is not injective if and only if some horizontal line intersects the graph in more than one point. Let $f\colon A \rightarrow \mathbf{R}$ be the given function. If some horizontal line $y = b$ intersects the graph of f in two distinct points (x_1, b) and (x_2, b), then f is not injective, since $f(x_1) = f(x_2)$. Conversely, if f is not injective, then there exist distinct real numbers x_1 and x_2 such that $f(x_1) = f(x_2)$. If b is their common value, then the line $y = b$ intersects the graph in more than one point.

21. Take the graph of f and project it onto the vertical axis by moving each point (x, b) on the graph horizontally over to the point $(0, b)$. The resulting subset of the vertical axis represents the range.

23. **(a)** If $f\colon A \rightarrow \mathbf{R}$ and $g\colon A \rightarrow \mathbf{R}$, then

$$f + g = \{\, (x, u + v) \mid (x, u) \in f \wedge (x, v) \in g \,\}.$$

(b) $i_A = \{\, (a, a) \mid a \in A \,\}$

25. To prove that $f \circ i_A = f$, we simply note that $f \circ i_A(a) = f(i_A(a)) = f(a)$ for all $a \in A$. Similarly, $i_B \circ f(a) = i_B(f(a)) = f(a)$ for all $a \in A$, so $i_B \circ f = f$.

27. The range of a function is the empty set if and only if the domain is the empty set.

29. (a) $f(\emptyset) = \{ f(x) \mid x \in \emptyset \} = \{ f(x) \mid \mathrm{F} \} = \emptyset$

(b) $f^{-1}(\emptyset) = \{ x \in A \mid f(x) \in \emptyset \} = \{ x \in A \mid \mathrm{F} \} = \emptyset$

(c) $f^{-1}(B) = \{ x \in A \mid f(x) \in B \} = A$ since $f(x) \in B$ for all $x \in A$

(d) $f(A) = \{ f(x) \mid x \in A \}$, which is the range of f

31. (a) $\{ x \in \mathbf{Z} \mid x \bmod 7 = 0 \} = \{ 7k \mid k \in \mathbf{Z} \}$, which is the set of all multiples of 7

(b) $\{ x \in \mathbf{Z} \mid x \bmod 7 = 1 \} = \{ 7k + 1 \mid k \in \mathbf{Z} \}$, which is the set of all integers that are one greater than a multiple of 7

(c) $\{ x \in \mathbf{Z} \mid x \bmod 7 = 8 \bmod 7 \} = \{ x \in \mathbf{Z} \mid x \bmod 7 = 1 \}$, the same set as in part (b)

(d) $\{ x \in \mathbf{Z} \mid x \bmod 7 = 0 \text{ or } 1 \}$, the union of the two sets obtained in parts (a) and (b)

(e) $\{0, 1, 2, 3, 4, 5, 6\}$, since $r(7) = 0$, $r(29) = 1$, $r(23) = 2$, $r(3) = 3$, $r(11) = 4$, $r(19) = 5$, and $r(13) = 6$

(f) $\{1\}$

(g) $\{0, 1\}$

33. Let $R = \{a\}$ and $S = \{b, c\}$. Then $R \cap S = \emptyset$, so $f(R \cap S) = \emptyset$. But $f(R) \cap f(S) = \{2\} \cap \{1, 2\} = \{2\} \neq \emptyset$.

35. Let $B = \{1\}$ and $A = \{a, b\}$. Let $g(1) = a$ and $f(a) = f(b) = 1$. Then f^{-1} does not exist, so $g \neq f^{-1}$, but $f \circ g : B \to B$ is i_B.

37. There are 14 surjective functions from $\{1, 2, 3, 4\}$ to $\{a, b\}$. The easy way to see this is to observe that all functions from $\{1, 2, 3, 4\}$ to $\{a, b\}$ are surjective except the two that send all elements of the domain to the same element of the codomain. Since there are $|\{a, b\}|^{|\{1,2,3,4\}|} = 2^4 = 16$ functions in all, there must be $16 - 2 = 14$ surjective ones. A more straightforward way to do this calculation is to consider three cases. There are six functions in which exactly two elements of the domain are sent to a, since there are six subsets of $\{1, 2, 3, 4\}$ containing two elements (namely $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$, and $\{3, 4\}$). There are four functions in which exactly one element of the domain is sent to a (there being four ways to pick that element), and similarly four functions in which exactly three elements of the domain are sent to a (there being four ways to pick the element that is sent to b). Thus there are $6 + 4 + 4 = 14$ surjective functions in all.

39. (a) yes (the sum of two continuous functions is continuous)

(b) yes (this follows from Theorem 2)

(c) no (f^{-1} is not a function from A to B unless $A = B$; it is a function from B to A)

41. Let f be defined as follows. Given a rational number a/b in lowest terms, with a an integer and b a positive integer, let $f(a/b)$ be $2^a 3^b$ if $a \geq 0$, and let $f(a/b) = 5^{-a}3^b$ if $a < 0$. It is clear that this function is injective, since a/b can be found uniquely by looking at the prime factorization of $f(a/b)$.

SECTION 3.3 Relations

1. **(a)** $\{(1,1), (1,2), (2,2)\}$

(b) $\{(1,1), (1,2), (2,2)\}$

(c) $\{(1,1), (2,1), (3,1), (2,2)\}$

(d) $\{(1,1), (2,2)\}$

(e) $\emptyset$ (if $x = y + 3$ and $y = 1$ or 2, then $x = 4$ or 5, which are not in the domain)

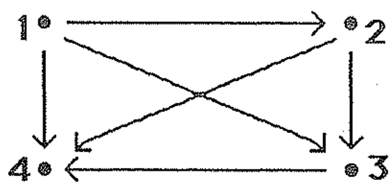
(f) $\{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2)\}$

3. **(a)** $(10,10), (10,11), (4,25) \in R$; $(0,10), (6,6) \notin R$

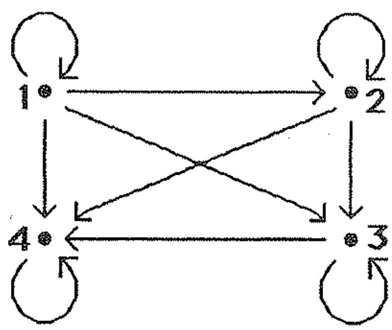
(b) $(3,0), (3,1), (3,2) \in R$; $(2,2), (2,3) \notin R$

(c) There are no elements in R (i.e., $R = \emptyset$), since xy can only be negative if one of x and y is negative. Every element of $\mathbf{N} \times \mathbf{N}$ is not in R; for example, $(2,1), (0,0) \notin R$.

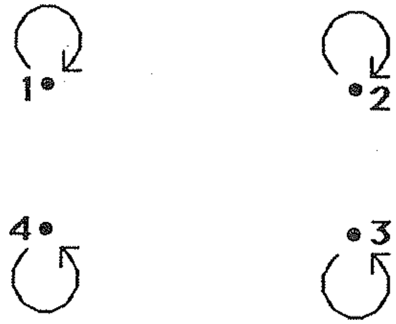
5. **(a)** We draw an arrow from x to y whenever $x < y$.



(b) Note the loops in the digraph, since $x \leq x$ is true for all x.



(c) The relation $\Delta_{\{1,2,3,4\}} = \{(1,1), (2,2), (3,3), (4,4)\}$, so the digraph consists only of loops.



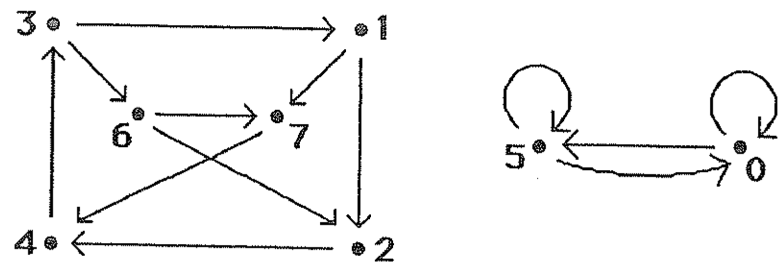
7. (a) C is not a function, since xCy_1 and xCy_2 can both hold for $y_1 \neq y_1$, namely if y_1 is x's mother and y_2 is x's father.

(b) The digraph has no loops since no one is his or her own child.

(c) The relation $C \circ C$ consists of pairs (x, z) such that xCy and yCz for some y. In other words, x is a child of some y, who is a child of z. Thus $C \circ C$ is the "is a grandchild of" relation.

9. (a) The new elements that we need to consider are 6 and 7. We note that $5 \mid 6 + 2 \cdot 2$, $5 \mid 6 + 2 \cdot 7$, $5 \mid 3 + 2 \cdot 6$, $5 \mid 7 + 2 \cdot 4$, and $5 \mid 1 + 2 \cdot 7$. Thus the pairs $(6, 2)$, $(6, 7)$, $(3, 6)$, $(7, 4)$, and $(1, 7)$ must be added to what we had in Example 7. Therefore the relation is $\{(0,0), (0,5), (1,2), (1,7), (2,4), (3,1), (3,6), (4,3), (5,0), (5,5), (6,2), (6,7), (7,4)\}$.

(b) We need to add new points for 6 and 7 and the lines corresponding to the pairs we have added to the relation.



(c) As usual, we put a 1 in location (i, j) to indicate that $5 \mid i + 2 \cdot j$, and a 0 to indicate that $5 \nmid i + 2 \cdot j$.

$$
\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}
\begin{array}{c}
\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

11. (a) For each $(x, y) \in S$, we need to look for what y is related to under R. For example, since $(a, 3) \in S$ and $(3, a) \in R$, we know that $(a, a) \in R \circ S$. Since $(3, b) \in R$ as well, we also have $(a, b) \in R \circ S$. Doing this for all pairs in S, we obtain $R \circ S = \{(a,a), (a,b), (b,a), (b,c), (c,c)\}$.

(b) This is the same as part (a), except that we reverse the roles of R and S: For each $(x, y) \in R$, we need to look for what y is related to under S. Here we obtain $S \circ R = \{(1,2), (1,3), (1,4), (3,1), (3,2), (3,3), (3,4), (4,2), (4,4)\}$.

(c) We need to list all pairs in $\{1,2,3,4\} \times \{a,b,c\}$ that are *not* in R. This gives us $\overline{R} = \{(1,b), (2,a), (2,b), (2,c), (3,c), (4,a), (4,b)\}$.

(d) We reverse all the pairs in R; this gives us a relation from $\{a, b, c\}$ to $\{1, 2, , 3, 4\}$: $R^{-1} = \{(a,1), (c,1), (a,3), (b,3), (c,4)\}$.

(e) First we need to compute $S^{-1} = \{(1,b), (2,b), (2,c), (3,a), (4,b), (4,c)\}$. Then we need to compose this with R^{-1} using the method explained in part (a). We obtain $R^{-1} \circ S^{-1} = \{(1,3), (2,1), (2,3), (2,4), (3,1), (3,3), (4,1), (4,3), (4,4)\}$.

(f) This is just like part (e), but the composition is in the opposite order: $S^{-1} \circ R^{-1} = \{(a,a), (a,b), (b,a), (c,b), (c,c)\}$.

(g) We take the reverses of all the pairs we obtained in part (a): $(R \circ S)^{-1} = \{(a,a), (a,b), (b,a), (c,b), (c,c)\}$. Note that this is the same as the answer to part (f); that is, $S^{-1} \circ R^{-1} = (R \circ S)^{-1}$.

13. (a) By definition, $R \circ \Delta_A = \{(a,b) \mid \exists a' \in A : (a,a') \in \Delta_A \wedge (a',b) \in R\}$. Now for each $a \in A$ there is a unique $a' \in A$ such that $(a,a') \in \Delta_A$, namely $a' = a$. Therefore $(a,b) \in R \circ \Delta_A$ if and only if $(a,b) \in R$. In other words, $R \circ \Delta_A = R$.

(b) By definition, $\Delta_B \circ R = \{(a,b) \mid \exists b' \in B : (a,b') \in R \wedge (b',b) \in \Delta_B\}$. Now for each $b \in B$ there is a unique $b' \in B$ such that $(b',b) \in \Delta_B$, namely $b' = b$. Therefore $(a,b) \in \Delta_B \circ R$ if and only if $(a,b) \in R$. In other words, $\Delta_B \circ R = R$.

(c) This is false. Let $A = B = \{1, 2\}$, and let $R = A \times A$. Then $R^{-1} = A \times A$ and $R \circ R^{-1} = A \times A \neq \Delta_A$.

(d) This is false. Let $A = \{1\}$, $B = \{2, 3\}$, $C = \{4\}$, $R = \{(1,2)\}$, and $S = \{(2,4)\}$. Then $S \circ R = \{(1,4)\}$, so $\overline{S \circ R} = \emptyset$. On the other hand, $\overline{R} = \{(1,3)\}$ and $\overline{S} = \{(3,4)\}$, so $\overline{S} \circ \overline{R} = \{(1,4)\}$.

(e) We must show that each pair in the left-hand side is in the right-hand side, and vice versa. Suppose that $(a,c) \in S \circ (R \cup R')$. Then for some $b \in B$ we have $(a,b) \in R \cup R'$ and $(b,c) \in S$. If $(a,b) \in R$, then $(a,c) \in S \circ R$, and if $(a,b) \in R'$, then $(a,c) \in S \circ R'$; in either case $(a,c) \in (S \circ R) \cup (S \circ R')$. Conversely, suppose that $(a,c) \in (S \circ R) \cup (S \circ R')$. Without loss of generality, suppose that $(a,c) \in S \circ R$. Then for some $b \in B$ we have $(a,b) \in R$ and $(b,c) \in S$. Thus perforce $\exists b \in B : ((a,b) \in R \cup R' \wedge (b,c) \in S)$, so by definition, $(a,c) \in S \circ (R \cup R')$.

(f) This is false. Let $A = \{1\}$, $B = \{2, 3\}$, $C = \{4\}$, $R = \{(1,2)\}$, $R' = \{(1,3)\}$, and $S = \{(2,4), (3,4)\}$. Then $S \circ (R \cap R') = S \circ \emptyset = \emptyset$, but $(S \circ R) \cap (S \circ R') = \{(1,4)\} \cap \{(1,4)\} = \{(1,4)\}$.

11. **(a)** For each $(x,y) \in S$, we need to look for what y is related to under R. For example, since $(a,3) \in S$ and $(3,a) \in R$, we know that $(a,a) \in R \circ S$. Since $(3,b) \in R$ as well, we also have $(a,b) \in R \circ S$. Doing this for all pairs in S, we obtain $R \circ S = \{(a,a), (a,b), (b,a), (b,c), (c,c)\}$.

(b) This is the same as part (a), except that we reverse the roles of R and S: For each $(x,y) \in R$, we need to look for what y is related to under S. Here we obtain $S \circ R = \{(1,2), (1,3), (1,4), (3,1), (3,2), (3,3), (3,4), (4,2), (4,4)\}$.

(c) We need to list all pairs in $\{1,2,3,4\} \times \{a,b,c\}$ that are *not* in R. This gives us $\overline{R} = \{(1,b), (2,a), (2,b), (2,c), (3,c), (4,a), (4,b)\}$.

(d) We reverse all the pairs in R; this gives us a relation from $\{a, b, c\}$ to $\{1, 2, ,3, 4\}$: $R^{-1} = \{(a,1), (c,1), (a,3), (b,3), (c,4)\}$.

(e) First we need to compute $S^{-1} = \{(1,b), (2,b), (2,c), (3,a), (4,b), (4,c)\}$. Then we need to compose this with R^{-1} using the method explained in part (a). We obtain $R^{-1} \circ S^{-1} = \{(1,3), (2,1), (2,3), (2,4), (3,1), (3,3), (4,1), (4,3), (4,4)\}$.

(f) This is just like part (e), but the composition is in the opposite order: $S^{-1} \circ R^{-1} = \{(a,a), (a,b), (b,a), (c,b), (c,c)\}$.

(g) We take the reverses of all the pairs we obtained in part (a): $(R \circ S)^{-1} = \{(a,a), (a,b), (b,a), (c,b), (c,c)\}$. Note that this is the same as the answer to part (f); that is, $S^{-1} \circ R^{-1} = (R \circ S)^{-1}$.

13. **(a)** By definition, $R \circ \Delta_A = \{\, (a,b) \mid \exists a' {\in} A : (a,a') \in \Delta_A \wedge (a',b) \in R \,\}$. Now for each $a \in A$ there is a unique $a' \in A$ such that $(a,a') \in \Delta_A$, namely $a' = a$. Therefore $(a,b) \in R \circ \Delta_A$ if and only if $(a,b) \in R$. In other words, $R \circ \Delta_A = R$.

(b) By definition, $\Delta_B \circ R = \{\, (a,b) \mid \exists b' {\in} B : (a,b') \in R \wedge (b',b) \in \Delta_B \,\}$. Now for each $b \in B$ there is a unique $b' \in B$ such that $(b',b) \in \Delta_B$, namely $b' = b$. Therefore $(a,b) \in \Delta_B \circ R$ if and only if $(a,b) \in R$. In other words, $\Delta_B \circ R = R$.

(c) This is false. Let $A = B = \{1, 2\}$, and let $R = A \times A$. Then $R^{-1} = A \times A$ and $R \circ R^{-1} = A \times A \neq \Delta_A$.

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(e) We must show that each pair in the left-hand side is in the right-hand side, and vice versa. Suppose that $(a,c) \in S \circ (R \cup R')$. Then for some $b \in B$ we have $(a,b) \in R \cup R'$ and $(b,c) \in S$. If $(a,b) \in R$, then $(a,c) \in S \circ R$, and if $(a,b) \in R'$, then $(a,c) \in S \circ R'$; in either case $(a,c) \in (S \circ R) \cup (S \circ R')$. Conversely, suppose that $(a,c) \in (S \circ R) \cup (S \circ R')$. Without loss of generality, suppose that $(a,c) \in S \circ R$. Then for some $b \in B$ we have $(a,b) \in R$ and $(b,c) \in S$. Thus perforce $\exists b {\in} B : ((a,b) \in R \cup R' \wedge (b,c) \in S)$, so by definition, $(a,c) \in S \circ (R \cup R')$.

(f) This is false. Let $A = \{1\}$, $B = \{2, 3\}$, $C = \{4\}$, $R = \{(1,2)\}$, $R' = \{(1,3)\}$, and $S = \{(2,4), (3,4)\}$. Then $S \circ (R \cap R') = S \circ \emptyset = \emptyset$, but $(S \circ R) \cap (S \circ R') = \{(1,4)\} \cap \{(1,4)\} = \{(1,4)\}$.

15. A 1-ary relation is any set of 1-tuples from A, which can be thought of, more simply, as just any subset of A.

17. **(a)** Let a be any element of A. Then $(a,a) \in R$, so $(a,a) \notin \overline{R}$. Therefore $\overline{R}$ is not reflexive.

(b) Suppose that $(x,y) \in \overline{R}$. Then $(x,y) \notin R$. If (y,x) were in R, then (x,y) would be in R as well; therefore $(y,x) \notin R$. In other words, $(y,x) \in \overline{R}$. Thus $\overline{R}$ is symmetric.

(c) This is false. Take $A = \{x, y\}$ and $R = \emptyset$. Then R is antisymmetric, but $\overline{R}$ is not.

(d) This is false. Take $A = \{x\}$ and $R = \emptyset$. Then both R and $\overline{R}$ are transitive.

19. Suppose that $\{R_i\}_{i \in I}$ is the given collection of relations on A, and let $R = \bigcap_{i \in I} R_i$ be their intersection.

(a) Let $a \in A$. Then since each R_i is reflexive, $(a,a) \in R_i$ for each i. Therefore $(a,a) \in R$, so R is reflexive.

(b) Let $(x,y) \in R$, so that $(x,y) \in R_i$ for each i. Then since each R_i is symmetric, $(y,x) \in R_i$ for each i. Therefore $(y,x) \in R$, so R is symmetric.

21. **(a)** This relation is clearly symmetric. It is also clear that it is not reflexive or antisymmetric. Although one would have to research the publishing habits of mathematicians to make sure, it is surely not transitive.

(b) antisymmetric, transitive (vacuously)

(c) antisymmetric, transitive

(d) reflexive, transitive

(e) None of the properties holds.

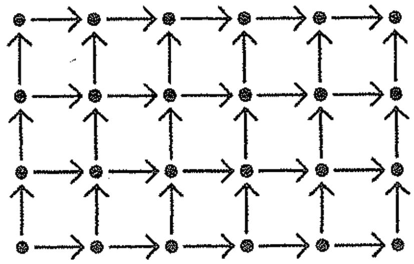
(f) None of the properties holds.

(g) symmetric

(h) antisymmetric, transitive

(i) reflexive, antisymmetric, and transitive (it is the same relation as $\geq$)

23. Two points are related if and only if the first is just below or just to the left of the second.



25. (a) Suppose that $(x,y) \in R \wedge (y,z) \in R$. By symmetry we know that $(y,x) \in R$ and hence by antisymmetry, $x = y$. Therefore $(y,z) = (x,z)$, so $(x,z) \in R$.

(b) If $(x,y) \in R$, then, as above, $x = y$. Hence R can contain no pairs (x,y) with $x \neq y$. In short, $R \subseteq \Delta_A$.

27. (a) Two authors a and b are related if there is a sequence of authors $a = a_0$, a_1, a_2, $\ldots$, $a_n = b$ such that each author in the sequence has written a joint paper with the next one (i.e., a_{i-1} and a_i have written a joint paper, for $i = 1, 2, \ldots, n$).

(b) This is the relation that always holds.

(c) "is an ancestor of"

(d) This is the relation that always holds, as long as you adopt the view that every two people have a common ancestor (Adam and Eve?).

29. (a) There are all 1's along the main diagonal (i.e., each $m_{ii} = 1$).

(b) There are all 0's along the main diagonal (i.e., each $m_{ii} = 0$).

(c) The matrix is symmetric about the main diagonal (i.e., $m_{ij} = m_{ji}$ for every i and j).

(d) There are never two 1's symmetrically located around the main diagonal (i.e., for every $i \neq j$, if $m_{ij} = 1$, then $m_{ji} \neq 1$).

31. (a) Assume that R is symmetric. If $(x,y) \in R$, then $(y,x) \in R$, so $(x,y) \in R^{-1}$. Thus $R \subseteq R^{-1}$. Similarly, if $(x,y) \in R^{-1}$, then $(y,x) \in R$, which implies by symmetry that $(x,y) \in R$. Thus $R^{-1} \subseteq R$. Therefore $R = R^{-1}$. Conversely, assume that $R = R^{-1}$. If $(x,y) \in R$, then $(x,y) \in R^{-1}$, so $(y,x) \in R$. In other words, R is symmetric.

(b) Assume that R is transitive. If $(x,y) \in R^2$, then there exists a $z \in A$ such that $(x,z) \in R$ and $(z,y) \in R$. Thus by transitivity, $(x,y) \in R$. This shows that $R^2 \subseteq R$. Conversely, assume that $R^2 \subseteq R$. Let $(x,y) \in R$ and $(y,z) \in R$. Then by definition, $(x,z) \in R^2$, so $(x,z) \in R$. This shows that R is transitive.

(c) R is reflexive $\Longleftrightarrow \forall a \in A: (a,a) \in R \Longleftrightarrow \Delta_A \subseteq R$

(d) Suppose that R is antisymmetric, and let $(x,y) \in R \cap R^{-1}$. Then $(x,y) \in R \wedge (x,y) \in R^{-1}$. The latter condition means that $(y,x) \in R$, so by antisymmetry, $x = y$. This shows that $R \cap R^{-1} \subseteq \Delta_A$. Conversely, suppose that $R \cap R^{-1} \subseteq \Delta_A$. If both $(x,y) \in R$ and $(y,x) \in R$, then $(x,y) \in R \cap R^{-1}$, so $(x,y) \in \Delta_A$. This means that $x = y$. Therefore R is antisymmetric.