9. The symmetric difference of A and B, according to Exercise 15, equals the difference $(A \cup B) - (A \cap B)$. By Exercise 5, $A \cap B \subseteq A \cup B$. Therefore $A \oplus B = A \cup B$ if and only if $A \cap B = \emptyset$. In other words, the symmetric difference equals the difference if and only if the sets are disjoint.

11. The proof is dual to the proof given in the text for the first half of Theorem 2. We can give it more succinctly in symbols.

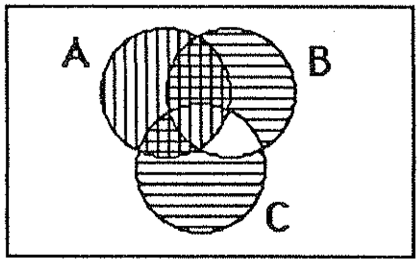
$$\begin{aligned}
\overline{\bigcup_{i\in I} A_i} &= \{\, x \mid x \notin \bigcup_{i\in I} A_i \,\} \\
&= \{\, x \mid \neg(x \in \bigcup_{i\in I} A_i) \,\} \\
&= \{\, x \mid \neg(\exists i{\in}I\colon x \in A_i) \,\} \\
&= \{\, x \mid \forall i{\in}I\colon x \notin A_i \,\} \\
&= \{\, x \mid \forall i{\in}I\colon x \in \overline{A_i} \,\} \\
&= \bigcap_{i\in I} \overline{A_i}
\end{aligned}$$

13. $\{\{0,1\}, \{2,3\}, \{4,5\}, \ldots\}$

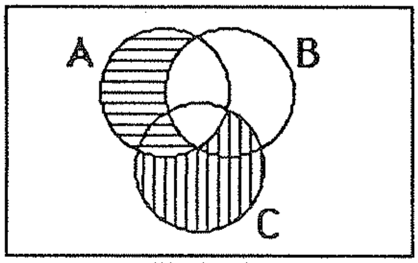
15. The idea in this proof is to tear the definition apart to obtain a complete logical description of when an element is in $A \oplus B$, then to use logic to rewrite that description so that it matches the definition of $(A \cup B) - (A \cap B)$.

$$\begin{aligned}
A \oplus B &= \{\, x \mid (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A) \,\} \\
&= \{\, x \mid (x \in A \vee x \in B) \wedge (x \in A \vee x \notin A) \\
&\qquad\qquad \wedge (x \notin B \vee x \in B) \wedge (x \notin B \vee x \notin A) \,\} \\
&\qquad \text{(this follows from part (c) of Theorem 1 in Section 1.1)} \\
&= \{\, x \mid (x \in A \vee x \in B) \wedge \mathrm{T} \wedge \mathrm{T} \wedge (\overline{x \in B} \vee \overline{x \in A}) \,\} \\
&= \{\, x \mid (x \in A \vee x \in B) \wedge \overline{x \in B \wedge x \in A} \,\} \\
&= \{\, x \mid x \in A \cup B \wedge x \notin A \cap B \,\} \\
&= (A \cup B) - (A \cap B)
\end{aligned}$$

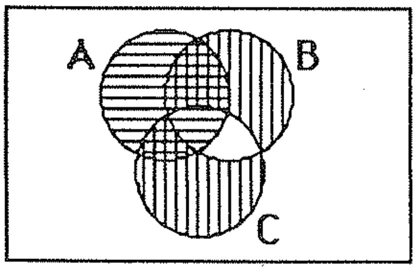
17. **(a)** The double-hatched region is $A \cap (B \oplus C)$.

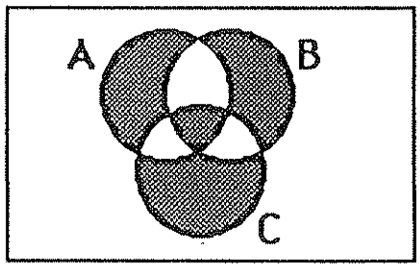


(b) The entire hatched region is $(A - B) \cup (C - A)$.



(c) In the picture on the left we have shaded A with horizontal hatching and $B \oplus C$ with vertical hatching. The symmetric difference of these two sets is therefore the region which is hatched but not double-hatched. It is redrawn, simply shaded, in the Venn diagram on the right.





19. **(a)** $(A \cap \overline{B \cup C}) \cup (B \cap C \cap \overline{A}) = (A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap C)$

(b) $(A \cap B) \cup (A \cap C) \cup (B \cap C)$

(c) $(A \cup B \cup C) - (A \cap B) - (A \cap C) - (B \cap C) = (A - B - C) \cup (B - A - C) \cup (C - A - B)$

(d) $(B - A) \cup (A \cap C \cap \overline{B})$

21. **(a)** Suppose that $x \in A \cup B$. Then $x \in A$ or $x \in B$. In either case, $x \in B \cup A$. Hence $A \cup B \subseteq B \cup A$. Similarly $B \cup A \subseteq A \cup B$. Hence $A \cup B = B \cup A$.

(b) We need to prove, for an arbitrary x, that $x \in A \cap (B \cup C)$ if and only if $x \in (A \cap B) \cup (A \cap C)$. Suppose that $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. The latter condition means that $x \in B$ or $x \in C$. If $x \in B$, then $x \in A \cap B$; and if $x \in C$, then $x \in A \cap C$. In either case $x \in (A \cap B) \cup (A \cap C)$. Conversely, suppose that $x \in (A \cap B) \cup (A \cap C)$. First, assume that $x \in A \cap B$. Then $x \in A$ and $x \in B$. Hence $x \in B \cup C$ as well, so $x \in A \cap (B \cup C)$. On the other hand, if $x \notin A \cap B$, then $x \in A \cap C$. Thus $x \in A$ and $x \in C$. Hence $x \in B \cup C$ as well, so again $x \in A \cap (B \cup C)$.

(c) Suppose that $x \in A \cap U$. Then in particular $x \in A$. Conversely, if $x \in A$, then $x \in A \cap U$, since by convention x is always an element of U.

(d) Suppose that $x \in A \cup A$. Then $x \in A$ or $x \in A$, i.e., $x \in A$. Conversely, if $x \in A$, then $x \in A \cup A$.

(e) Suppose that $x \in A \cap \overline{A}$. Then $x \in A$ and $x \in \overline{A}$, i.e., $x \in A$ and $x \notin A$. This is impossible, so no such elements x exist. Therefore $A \cap \overline{A} = \emptyset$.

(f) Suppose that $x \in \overline{\overline{A}}$. Then it is not the case that $x \in \overline{A}$, i.e., it is not the case that $x \notin A$. Therefore $x \in A$. Conversely, if $x \in A$, then $x \notin \overline{A}$, whence $x \in \overline{\overline{A}}$.

(g) Suppose that $x \in \overline{A \cup B}$. Then $x \notin A \cup B$. This means that it is not the case that $x \in A$ or $x \in B$, which implies that x is in neither A nor B. Hence $x \in \overline{A}$ and $x \in \overline{B}$, so $x \in \overline{A} \cap \overline{B}$. Conversely, if $x \in \overline{A} \cap \overline{B}$, then $x \notin A$ and $x \notin B$, so it is not the case that $x \in A \cup B$. Thus $x \in \overline{A \cup B}$.

23. **(a)** To see that $A - B$ is not equal to $B - A$ in general, take $A = \emptyset$ and $B = \{1\}$. Then $A - B = \emptyset$, but $B - A = \{1\}$. It is also easy to see from the Venn diagrams that $A - B \neq B - A$.

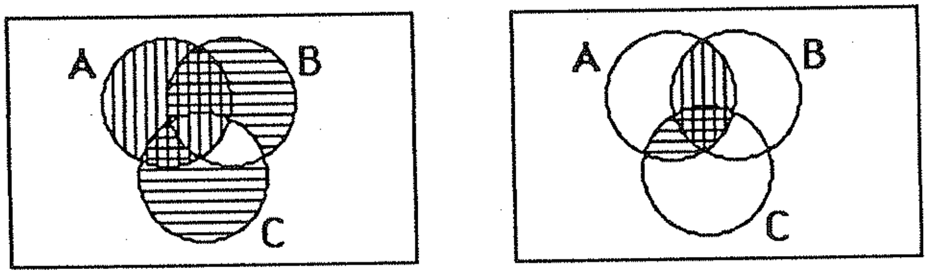
(b) It is true that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$. Each side consists of all those objects that are elements of an odd number of the sets A, B, and C. Indeed,

$$\begin{aligned}(A \oplus B) \oplus C &= \{\, x \mid (x \in A \oplus B \wedge x \notin C) \vee (x \in C \wedge x \notin A \oplus B) \,\} \\ &= \{\, x \mid (x \in A \wedge x \notin B \wedge x \notin C) \vee (x \in B \wedge x \notin A \wedge x \notin C) \\ &\qquad \vee (x \in C \wedge x \in A \wedge x \in B) \vee (x \in C \wedge x \notin A \wedge x \notin B) \,\}.\end{aligned}$$

The calculation for $A \oplus (B \oplus C)$ is similar. See the Venn diagram for the solution to Exercise 17c.

(c) To see that $A \cup (B \oplus C) \neq (A \cup B) \oplus (A \cup C)$ in general, take $A = \{1\}$ and $B = C = \emptyset$. Then $A \cup (B \oplus C) = \{1\}$, but $(A \cup B) \oplus (A \cup C) = \{1\} \oplus \{1\} = \emptyset$. One can also look at the Venn diagrams for these two expressions.

(d) The double-hatched region on the left is $A \cap (B \oplus C)$. The shaded but not double-hatched region on the right is $(A \cap B) \oplus (A \cap C)$. These two sets are equal.



(e)
$$
\begin{aligned}
A \times (B \cap C) &= \{ (x,y) \mid x \in A \wedge y \in B \cap C \} \\
&= \{ (x,y) \mid x \in A \wedge y \in B \wedge y \in C \} \\
&= \{ (x,y) \mid (x \in A \wedge y \in B) \wedge (x \in A \wedge y \in C) \} \\
&= \{ (x,y) \mid x \in A \wedge y \in B \} \cap \{ (x,y) \mid x \in A \wedge y \in C \} \\
&= (A \times B) \cap (A \times C)
\end{aligned}
$$

(f)
$$
\begin{aligned}
A \times (B \cup C) &= \{ (x,y) \mid x \in A \wedge y \in B \cup C \} \\
&= \{ (x,y) \mid x \in A \wedge (y \in B \vee y \in C) \} \\
&= \{ (x,y) \mid (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C) \} \\
&= \{ (x,y) \mid x \in A \wedge y \in B \} \cup \{ (x,y) \mid x \in A \wedge y \in C \} \\
&= (A \times B) \cup (A \times C)
\end{aligned}
$$

25. These operations correspond in the sense that $x \in A - B$ if and only if it is not the case that $x \in A$ implies $x \in B$. (This follows from the definition of the difference of sets and part (k) of Theorem 1 in Section 1.1.)

27. **(a)** The $\subseteq$ relation for sets is analogous to the logical implication relation for propositions. Just as $A = B$ means that $A \subseteq B$ and $B \subseteq A$, so $P \Longleftrightarrow Q$ means that $P \Longrightarrow Q$ and $Q \Longrightarrow P$.

(b) Analogous to $P \Longrightarrow P \vee Q$ is $A \subseteq A \cup B$. This is clear: If $x \in A$, then $x \in A \cup B$. Analogous to $P \wedge Q \Longrightarrow P$ is $A \cap B \subseteq A$. Again this is clear: If $x \in A \cap B$, then $x \in A$.

29. **(a)** This is true. If $A \in F$, then A is finite, so certainly $A \cap B$, which is a subset of A, is finite.

(b) This is false. Let A be the set of even natural numbers, and let B be the set of odd natural numbers together with 0 and 2. Then $A \cap B = \{0, 2\} \notin I$.

(c) This is true since $|A \cup B| \leq |A| + |B|$.

(d) This is true. If either A or B is infinite, then certainly $A \cup B$ is infinite. On the other hand, $\emptyset \cup \emptyset = \emptyset$.

31. **(a)** $\mathbf{Q}$, since $\bigcap_{\epsilon>0}(x-\epsilon, x+\epsilon) = \{x\}$

(b) $\mathbf{R}$, since $\bigcup_{\epsilon>0}(x-\epsilon, x+\epsilon) = \mathbf{R}$ for each x

(c) $\mathbf{R}$, since $\bigcup_{x\in\mathbf{Q}}(x-\epsilon, x+\epsilon) = \mathbf{R}$ for each $\epsilon > 0$

(d) $\emptyset$, since $\bigcap_{x\in\mathbf{Q}}(x-\epsilon, x+\epsilon) = \emptyset$ for each $\epsilon > 0$

(e) $\emptyset$, as in part (d)

(f) $\mathbf{R}$, as in part (c)

33. For each prime number p, let $A_p = \{pq \mid q \text{ is prime}\}$. For example, $A_2 = \{4, 6, 10, 14, \ldots\}$, and $A_3 = \{4, 9, 15, 21, \ldots\}$. The collection of all such sets A_p is an example with the desired property. Indeed, for every distinct p and q, $A_p \cap A_q = \{pq\} \neq \emptyset$. On the other hand, it is impossible for any number to have three distinct prime factors if it has only two prime factors, so $A_p \cap A_q \cap A_r = \emptyset$ whenever p, q, and r are distinct primes.

35. **(a)** C is closed under complementation, since if A is finite, then $\overline{A}$ has a finite complement (namely A), and if $\overline{A}$ is finite, then A has a finite complement.

(b) Suppose that A and B are both in C. If either is finite, then $A \cap B$ is finite, hence in C. Otherwise both $\overline{A}$ and $\overline{B}$ are finite, so $\overline{A} \cup \overline{B}$ is finite. But $\overline{A} \cup \overline{B} = \overline{A \cap B}$, so $A \cap B$ has a finite complement and hence is an element of C.

(c) Since $A \cup B = \overline{\overline{A} \cap \overline{B}}$, the fact that C is closed under complementation and intersection implies that C is closed under union.

(d) Similar to part (c), since $A - B = A \cap \overline{B}$.

29. **(a)** This is true. If $A \in F$, then A is finite, so certainly $A \cap B$, which is a subset of A, is finite.

(b) This is false. Let A be the set of even natural numbers, and let B be the set of odd natural numbers together with 0 and 2. Then $A \cap B = \{0, 2\} \notin I$.

(c) This is true since $|A \cup B| \leq |A| + |B|$.

(d) This is true. If either A or B is infinite, then certainly $A \cup B$ is infinite. On the other hand, $\emptyset \cup \emptyset = \emptyset$.

31. **(a)** $\mathbf{Q}$, since $\bigcap_{\epsilon>0}(x-\epsilon, x+\epsilon) = \{x\}$

(b) $\mathbf{R}$, since $\bigcup_{\epsilon>0}(x-\epsilon, x+\epsilon) = \mathbf{R}$ for each x

(c) $\mathbf{R}$, since $\bigcup_{x\in\mathbf{Q}}(x-\epsilon, x+\epsilon) = \mathbf{R}$ for each $\epsilon > 0$

(d) $\emptyset$, since $\bigcap_{x\in\mathbf{Q}}(x-\epsilon, x+\epsilon) = \emptyset$ for each $\epsilon > 0$

(e) $\emptyset$, as in part (d)

(f) $\mathbf{R}$, as in part (c)

33. For each prime number p, let $A_p = \{pq \mid q \text{ is prime}\}$. For example, $A_2 = \{4, 6, 10, 14, \ldots\}$, and $A_3 = \{4, 9, 15, 21, \ldots\}$. The collection of all such sets A_p is an example with the desired property. Indeed, for every distinct p and q, $A_p \cap A_q = \{pq\} \neq \emptyset$. On the other hand, it is impossible for any number to have three distinct prime factors if it has only two prime factors, so $A_p \cap A_q \cap A_r = \emptyset$ whenever p, q, and r are distinct primes.

35. **(a)** C is closed under complementation, since if A is finite, then $\overline{A}$ has a finite complement (namely A), and if $\overline{A}$ is finite, then A has a finite complement.

(b) Suppose that A and B are both in C. If either is finite, then $A \cap B$ is finite, hence in C. Otherwise both $\overline{A}$ and $\overline{B}$ are finite, so $\overline{A} \cup \overline{B}$ is finite. But $\overline{A} \cup \overline{B} = \overline{A \cap B}$, so $A \cap B$ has a finite complement and hence is an element of C.

(c) Since $A \cup B = \overline{\overline{A} \cap \overline{B}}$, the fact that C is closed under complementation and intersection implies that C is closed under union.

(d) Similar to part (c), since $A - B = A \cap \overline{B}$.

CHAPTER 3
FUNCTIONS AND RELATIONS

SECTION 3.1 Functions

1. (a) no (it is not even a set)
(b) no (it is a function from $\{2\}$ to $\{1, 2, 3, 4\}$, however)
(c) yes
(d) no (3 is missing from the domain)
(e) no (1 has two associated values)
(f) no (5 is not in the given codomain)

3. (a) $\{(1,2), (2,3), (3,4), (4,5)\}$
(b) $\{(1,8), (2,8), (3,8), (4,8)\}$
(c) $\{(1,2), (2,2), (3,4), (4,4)\}$

5. (a) $\ln 5 \div \ln 2 = 2.32193$
(b) $\ln 10 \div \ln 2 = 3.32193$ (note that this is $1 + \log 5$, since $\log(2 \cdot 5) = (\log 2) + (\log 5)$)
(c) $50 \log 10 = 50 \ln 10 \div \ln 2 = 166.09640$

7. (a) $6 + 5 = 11 \equiv 4$ (b) $1 + 1 = 2$ (c) $6 \cdot 4 = 24 \equiv 3$
(d) $0 \cdot 2 = 0$ (e) $5^6 = (5^2)^3 = 25^3 \equiv 4^3 = 64 \equiv 1$

9. Each entry is the sum or product in $\mathbf{N}$, reduced modulo 5.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

X	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

11. (a) Let q be Prince William of Great Britain. Since q has no children, $\neg\exists p\colon f(p) = q$.

(b) Since f is a function, this cardinality is always 1. Everybody has exactly one father.

(c) Since f is a function, this cannot happen. A person cannot have two different fathers.

(d) This can happen. Let q be Prince Charles of Great Britain, and let p and p' be his sons William and Harry.

13. $\forall a{\in}A\colon \exists! b{\in}B\colon (a,b) \in f$

15. $\{(\{1,2,3\},1),\ (\{1,2\},1),\ (\{1,3\},1),\ (\{2,3\},2),\ (\{1\},1),\ (\{2\},2),\ (\{3\},3)\}$

17. $D : \mathbf{R} \times (\mathbf{R} - \{0\}) \to \mathbf{R}$ given by $D(x,y) = x/y$ is a function of two variables.

19. $\odot = \{ ((a,b),c) \mid (c = a \wedge a \geq b) \vee (c = b \wedge a \leq b) \}$; or $x \odot y = (x + y + |x - y|)/2$

21. (a) no $(3 - 7 \notin \mathbf{N})$

(b) yes

(c) yes $((a/b) + (c/d) = (ad + bc)/(bd))$

(d) yes ($\sqrt{a}$ is a positive real number for all positive real numbers a)

(e) yes (see Exercise 3a in Section 1.3)

(f) no $(3 + 5 = 8)$

23. The answer is 1 if $1 \leq n < 10$, 2 if $10 \leq n < 100$, and so on. Taking common (base 10) logarithms, we have

$$0 \leq \log_{10} < 1 \quad \text{if } 1 \leq n < 10,$$
$$1 \leq \log_{10} < 2 \quad \text{if } 10 \leq n < 100,$$

and so on. It is then easy to see that in general the number of digits is $\lfloor \log_{10} n \rfloor + 1$.

25. (a) $\log 2 + \log x = 1 + y$

(b) $2 \log x = 2y$

(c) $(\log x)/(\log 4) = y/2$

(d) x

(e) $4^{\log x} = (2^2)^{\log x} = 2^{2\log x} = 2^{\log(x^2)} = x^2$

(f) Since $\log(x^{1/\log x}) = (1/\log x) \cdot \log x = 1$, we see that $x^{1/\log x} = 2$.

11. (a) Let q be Prince William of Great Britain. Since q has no children, $\neg\exists p{:}\, f(p) = q$.

(b) Since f is a function, this cardinality is always 1. Everybody has exactly one father.

(c) Since f is a function, this cannot happen. A person cannot have two different fathers.

(d) This can happen. Let q be Prince Charles of Great Britain, and let p and p' be his sons William and Harry.

13. $\forall a{\in}A{:}\, \exists! b{\in}B{:}\, (a,b) \in f$

15. $\{(\{1,2,3\},1),\ (\{1,2\},1),\ (\{1,3\},1),\ (\{2,3\},2),\ (\{1\},1),\ (\{2\},2),\ (\{3\},3)\}$

17. $D : \mathbf{R} \times (\mathbf{R} - \{0\}) \to \mathbf{R}$ given by $D(x,y) = x/y$ is a function of two variables.

19. $\odot = \{\, ((a,b),c) \mid (c = a \wedge a \geq b) \ \vee\ (c = b \wedge a \leq b) \,\}$; or $x \odot y = (x + y + |x - y|)/2$

21. (a) no $(3 - 7 \notin \mathbf{N})$

(b) yes

(c) yes $((a/b) + (c/d) = (ad + bc)/(bd))$

(d) yes ($\sqrt{a}$ is a positive real number for all positive real numbers a)

(e) yes (see Exercise 3a in Section 1.3)

(f) no $(3 + 5 = 8)$

23. The answer is 1 if $1 \leq n < 10$, 2 if $10 \leq n < 100$, and so on. Taking common (base 10) logarithms, we have

$$\begin{aligned} 0 \leq \log_{10} < 1 \quad &\text{if } 1 \leq n < 10, \\ 1 \leq \log_{10} < 2 \quad &\text{if } 10 \leq n < 100, \end{aligned}$$

and so on. It is then easy to see that in general the number of digits is $\lfloor \log_{10} n \rfloor + 1$.

25. (a) $\log 2 + \log x = 1 + y$

(b) $2 \log x = 2y$

(c) $(\log x)/(\log 4) = y/2$

(d) x

(e) $4^{\log x} = (2^2)^{\log x} = 2^{2 \log x} = 2^{\log(x^2)} = x^2$

(f) Since $\log(x^{1/\log x}) = (1/\log x) \cdot \log x = 1$, we see that $x^{1/\log x} = 2$.

27. (a) We can write the real number x as $\lfloor x \rfloor + \epsilon$, so that ϵ is a real number satisfying $0 \le \epsilon < 1$. Since $\epsilon = x - \lfloor x \rfloor$, we have $0 \le x - \lfloor x \rfloor < 1$. The desired inequalities, $\lfloor x \rfloor \le x$ and $x - 1 < \lfloor x \rfloor$, follow algebraically.

(b) This is similar to part (a), but this time we write $x = \lceil x \rceil - \epsilon$, where again $0 \le \epsilon < 1$. Then $0 \le \lceil x \rceil - x < 1$, and again the desired inequalities, $x \le \lceil x \rceil$ and $\lceil x \rceil < x + 1$, follow by trivial algebra.

29. The easiest way to accomplish this is to add $1/2$ and then round down. This has the effect of rounding up if the original number was more than half way to the next highest integer. Thus $\text{round}(x) = \lfloor x + 0.5 \rfloor$.

31. If $x \equiv 0 \pmod 6$ or $x \equiv 3 \pmod 6$, then $x = 6k$ or $x = 6k + 3$; in either case, x is divisible by 3, and therefore (since $x > 3$), x is not prime. Similarly, if $x \equiv 2 \pmod 6$ or $x \equiv 4 \pmod 6$, then x is divisible by 2, hence not prime. Thus every prime number must be congruent to either 1 or 5, modulo 6. Numbers of the former form can be written as $6k + 1$, and numbers of the latter form (since $5 \equiv -1 \pmod 6$) can be written as $6k - 1$.

33. This is false. Let $m = 5$, $a = b = 2$, $c = 1$, and $d = 6$. Then $2 \equiv 2 \pmod 5$ and $1 \equiv 6 \pmod 5$, but $2^1 = 2 \not\equiv 64 = 2^6 \pmod 5$.

35. The number 2 has no multiplicative inverse modulo 6, since $2k$ is always congruent to 0, 2, or $4 \pmod 6$, never 1.

37. (a) yes, since multiplication of integers is associative

(b) no, since $a \odot (b \odot c) = a \odot (b + 2c) = a + 2(b + 2c) = a + 2b + 4c$, whereas $(a \odot b) \odot c = (a + 2b) \odot c = (a + 2b) + 2c = a + 2b + 2c$

(c) yes, since $a \odot (b \odot c) = a \odot b = a$ and $(a \odot b) \odot c = a \odot c = a$

39. The domain is $\mathcal{P}(\mathrm{N}) - \{\emptyset\}$. For each S in the domain, the image n of S under this function is an element of S that satisfies $n \le m$ for all $m \in S$. Thus we can describe this function as the following set of ordered pairs: $\{\, (S, n) \mid S \in \mathcal{P}(\mathrm{N}) - \{\emptyset\} \ \wedge \ n \in S \ \wedge \ \forall m{\in}S{:}\, n \le m \,\}$.