Q38 on Pg 355

Prove that $\underbrace{\frac{n^3}{3}+\frac{n^5}{5}+\frac{7n}{15}}_{\frac{5n^3+3n^5+7n}{15}} \in \mathbb{Z} \quad \forall n \geq 0$

Prove $15 \mid 5n^3+3n^5+7n \quad \forall n \geq 0$

Use induction

Base case $n=0 \Rightarrow 5n^3+3n^5+7n = 0$

We know $15 \mid 0$

Assume that the claim is true for $n=k \geq 0$ where k is fixed but arb.

i.e. Assume that $15 \mid 5k^3+3k^5+7k$ I.H.

WTS that the claim is true for $n=k+1$

i.e. WTS $15 \mid 5(k+1)^3+3(k+1)^5+7(k+1)$

Now $5(k+1)^3 + 3(k+1)^5 + 7(k+1)$

$= 5\left[k^3 + 3k^2 + 3k + 1\right]$

$+3\left[k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1\right]$

$+7(k+1)$

$= 5k^3 + 15k^2 + 15k + 5$

$3k^5 + 15k^4 + 30k^3 + 30k^2 + 15k + 3$

$+7k + 7$

$= 5k^3 + 3k^5 + 7k + 15$ (integer)

Since $15 \mid 5k^3 + 3k^5 + 7k$ by I.H.,

$15 \mid 5(k+1)^3 + 3(k+1)^5 + 7(k+1)$

$\therefore$ The claim is true by PMI.