APM 263

Combinatorial Identities

All identities have implicit assumptions on the ranges of the given parameters that are implied by the binomial coefficients.

(1) Prove that

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}.$$

Proof. We would like to find the number of ways to choose n students from a group consisting of n boys and n girls. We will find this number in two different ways. Clearly this number is $\binom{2n}{n}$, which is the RHS of the identity. On the other hand, we can first decide on the number of boys and the number of girls in the selected group. Let k be the number of boys in the selected group. Then we must have $n-k$ girls in the selected group. The number of ways to select n students in the prescribed boy-girl composition is $\binom{n}{k}\binom{n}{n-k}$. Since k ranges from 0 to n, summing $\binom{n}{k}\binom{n}{n-k}$ over this range gives the LHS. $\square$

(2) Prove that

$$\binom{n}{k+r}\binom{k+r}{k} = \binom{n}{k}\binom{n-k}{r}.$$

Proof. We would like to find the number of ways to choose $k+r$ students from n students, then select k of them, each receiving a candy bar; the remaining r student will each receive a can of soda. We will find this number in two different ways. On the one hand, we can first pick $k+r$ students from the group of n in $\binom{n}{k+r}$ ways, we then choose k students from the chosen $k+r$ students for the candy bars in $\binom{k+r}{k}$ ways, giving the LHS. (Of course, the remaining r students will receive the sodas.) On the other hand, we can first decide who will receive the candy bars by choosing k students from n students. This can be done in $\binom{n}{k}$ ways. Now we have $n-k$

students remaining and we want to choose r of them to receive the sodas and this can be done in $\binom{n-k}{r}$ ways. This gives the RHS. $\square$

(3) Prove that

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}.$$

Proof. We would like to find the number of ways to choose a subset of $r+1$ elements from the set $\{1, 2, 3, \ldots, n+1\}$. We will find this number in two different ways. Clearly this number is $\binom{n+1}{r+1}$, which is the RHS of the identity. On the other hand, we can first decide on the largest number in the subset. Since we are choosing $r+1$ elements, the largest number can be $r+1, r+2, \ldots, n+1$. If the largest number in the subset is $r+1$, then we need to choose r more elements from $\{1, 2, 3, \ldots, r\}$ and this can be done in $\binom{r}{r}$ ways. If the largest number in the subset is $r+2$, then we need to choose r more elements from $\{1, 2, 3, \ldots, r+1\}$ and this can be done in $\binom{r+1}{r}$ ways. Continue with this argument. If the largest number in the subset is $n+1$, then we need to choose r more elements from $\{1, 2, 3, \ldots, n\}$ and this can be done in $\binom{n}{r}$ ways. Summing these numbers give the LHS. $\square$