√(7) Let p be a prime number. Let m and n be two integers. Show that if p divides mn, then p divides m or p divides n, without using the Fundamental Theorem of Arithmetic. [10 marks]

Suppose $p \mid mn$. WTS $p \mid m$ or $p \mid n$.

complementary cases

Case 1: $p \mid m$

We're done, since

T or F = T, and

T or T = T.

Case 2: $p \nmid m$

$\therefore \gcd(m, p) = 1$ (Lemma for primes)

$\Rightarrow \exists x, y \in \mathbb{Z} : 1 = mx + py$ (Bezout's identity)

$\Rightarrow n = nmx + npy$ (for some $x, y \in \mathbb{Z}$)

We supposed $p \mid mn$ and clearly $p \mid p$.

$\therefore p \mid nmx + npy$ (E's Lemma)

$\Rightarrow p \mid n$ (since $n = nmx + npy$)

In either case, it's true that $p \mid m \vee p \mid n$. $\square$

APM 2663

Fall 2024

Instructor: Eddie Cheng

Date: December 10, 2024

Important:

- Recall that the word *if* in a definition means *if and only if.*
- **To receive full credit for a question, you should provide all logical steps. All answers must be justified unless the questions stating otherwise.**
- Recall that $\mathbb{N}$ is the set of positive integers. The definition in the book includes 0.
- Recall that $\mathbb{Z}$ is the set of integers.
- Recall that $\mathbb{Q}$ is the set of rational numbers.
- Recall that $\mathbb{R}$ is the set of real numbers.
- This is a closed book examination. No external aids are allowed, except a calculator.
- Cheating is a serious academic misconduct. Oakland University policy requires that all suspected instances of cheating be reported to the Office of the Dean of Students/Academic Conduct Committee for adjudication. I have forwarded cases to the Office of the Dean of Students/Academic Conduct Committee before and I will not hesitate to do this again if I suspect academic misconduct has occurred. Anyone found responsible of cheating in this assessment will receive a course grade of F, in addition to any penalty assigned by the Academic Conduct Committee.
- I may ask for a meeting for you to explain your solutions.
- This test is worth 110 marks. If you receive x marks, your grade will be $\min\{x, 100\}\%$.

(1) Read the instructions and sign your name (in the space provided below) indicating that you have read the instructions. [1 mark]

Shane Jarach

(2) Write down your name and student number. [1 mark]

Shane Jarach 600158472

(3) Find the gcd of 5271 and 231 using the Euclidean Algorithm. Write the gcd as $5271x+231y$ for some $x, y \in \mathbb{Z}$. [10 marks]

$5271 = 22 \cdot 231 + 189 \longrightarrow 189 = 5271 - 22 \cdot 231$

$231 = 189 + 42 \longrightarrow 42 = 231 - 189$

$189 = 4 \cdot 42 + \boxed{21} \longrightarrow 21 = 189 - 4 \cdot 42$

$42 = 2 \cdot 21 + 0$

We can write $\boxed{\gcd(5271, 231) = 21}$ as,

$$\begin{aligned}
21 &= 189 - 4 \cdot 42 \\
&= 189 - 4(231 - 189) \\
&= 5 \cdot 189 - 4 \cdot 231 \\
&= 5(5271 - 22 \cdot 231) - 4 \cdot 231
\end{aligned}$$

$\Rightarrow \boxed{21 = 5 \cdot 5271 - 114 \cdot 231}$

√(4) Use mathematical induction to prove that for every integer $n \geq 1$,

$$1^2 + 4^2 + 7^2 + \cdots + (3n-2)^2 = \frac{n(6n^2 - 3n - 1)}{2}. \text{ [10 marks]}$$

Base case $n=1$. $1^2 = 1$

$$\frac{1(6-3-1)}{2} = \frac{2}{2} = 1$$

Inductive step Suppose claim true for fixed but arbitrary k, i.e.,

$$1^2+4^2+\cdots+(3k-2)^2 = \frac{k(6k^2-3k-1)}{2}. \quad (\text{some } k \geq 1)$$

$\boxed{IH}$

WTS This implies claim holds for $k+1$ case, i.e.,

$$1^2+4^2+\cdots+(3k-2)^2+[3(k+1)-2]^2 = \frac{(k+1)[6(k+1)^2-3(k+1)-1]}{2}$$

(WTS IH holds for $n=k+1$)

$$= \frac{(k+1)[6k^2+12k+6-3k-4]}{2}$$

$$= \frac{(k+1)[6k^2+9k+2]}{2}$$

$$= \frac{6k^3+9k^2+2k+6k^2+9k+2}{2}$$

$$= \frac{6k^3+15k^2+11k+2}{2}$$

(continued on loose leaf paper)

Shane J

APM 2663 - Final

4.

$$LHS = \underbrace{1^2+4^2+\cdots+(3k-2)^2}_{IH}+(3k+1)^2 \qquad (\text{Sum for } n=k+1)$$

$$= \frac{k(6k^2-3k-1)}{2} + \frac{2(9k^2+6k+1)}{2}$$

$$= \frac{6k^3-3k^2-k+2(9k^2+6k+1)}{2}$$

$$= \frac{6k^3-3k^2-k+18k^2+12k+2}{2}$$

$$= \frac{6k^3+15k^2+11k+2}{2}$$

$= RHS$, as desired

So, the claim is true $\forall n \geq 1$ by PMI. $\square$

✓(5) Prove the following combinatorially (where $n \geq 3$ is fixed):

$$n(n-1)(n-2)6^{n-3} = \sum_{k=2}^{n-1} k(k-1)(n-k)\binom{n}{k}2^{k-2}4^{n-k-1}. \text{ [15 marks]}$$

among n−k who

Choose 1 student who stayed home to study to get a perfect 100% in the class

of k going to DW, everyone except two who won prizes, have two choices: Motel 6 or Best Western

Among those staying home to study, except the one getting a perfect 100%, everyone has 4 choices: cry, study with Shane, study alone, or don't study (watch TV instead).

Rewrite:
$$\sum_{k=2}^{n-1} \binom{n}{k}\binom{n-k}{1}\binom{k}{1}\binom{k-1}{1} \cdot 2^{k-2} \cdot 4^{n-k-1}$$

Sum over all possible cases, $2 \leq k \leq n-1$.

Any number of students can go to DW.
Sum these disjoint cases together.

Choose k of n students to go to DW, rest stay home to study

of the k going to DW, choose 1 to win free lodging & food (all expense paid)

excluding the 1 student traveling for free, choose one other student to get to take a private tour of DW behind the scenes.

On the other hand we can count like this:

Choose a different student to go behind the scenes at DW

Everyone else, not among 3 special/lucky students, has 6 choices:
- Go to DW & choose between 2 hotels.
- Stay home & either study w/ Shane, study alone, cry, or watch TV. (4 options)

$$n(n-1)(n-2)\,6^{n-3}$$

Choose 1 lucky student to get free lodging, at DW.

Choose 1 different student to stay home, study, and get a perfect 100%

Since we counted both ways correctly, the original formula relating the sum and closed/simplified form is correct. ∎

(6) Give an example of a relation that is symmetric, antisymmetric and transitive but not reflexive, or show that such a relation does not exist. [8 marks]

b

Define Relation R over A. — "ground set"

$\boxed{\text{Let } A = \{a,b,c\} \quad \text{And } R = \phi}$ (R = empty set)

a

c

Defs

× Reflexive: $\forall a \in A: (a,a) \in R$. R not reflexive since $(a,a) \notin R$ (also (b,b) and (c,c) not in R)

✓ Symmetric: $aRb \Rightarrow bRa$. This is vacuously true, ie, aRb always false.

✓ Antisymm: $aRb \wedge bRa \Rightarrow a=b$. Again, vacuously true, since aRb is always false.

✓ Transitive: $aRb \wedge bRc \Rightarrow aRc$ Again, vacuously true.

The hypothesis, LHS, is always true since aRb is never true when $R=\phi$. So, the implication as a whole is always vacuously true. □

(8) There are $10n$ students in a class and the teacher would like to divide them into n groups of students, each with 2 students (Type A groups), n groups of students, each with 3 students (Type B groups), and n groups of students, each with 5 students (Type C groups). Moreover, the teacher has n tags labelled 1 to n and these tags will be distributed to the Type A groups, so that each group has exactly one tag. However, the Type B groups have no tags and the Type C groups have no tags. How many ways can this be done? [10 marks]

Method 1

Total permutations $= (10n)!$

n Groups, type B & C order unimportant $\rightarrow \frac{1}{n! \cdot n!}$

$\Rightarrow \# \text{ways} = \boxed{\frac{(10n)!}{(n!)^2 (2!)^n (3!)^n (5!)^n}}$

Students within groups unordered, all 3 types $\rightarrow \frac{1}{(2!)^n(3!)^n(5!)^n}$

Method 2

$$\binom{10n}{2}\binom{10n-2}{2}\cdots\binom{8n+2}{2}$$ Type A

$$\cdot\binom{8n}{3}\binom{8n-3}{3}\cdots\binom{5n+3}{3}\Big/n!$$ Type B

$$\cdot\binom{5n}{5}\binom{5n-5}{5}\cdots\binom{10}{5}\binom{5}{5}\Big/n!$$ Type C

$$= \frac{(10n)!}{2!\cancel{(10n-2)!}}\cdot\frac{\cancel{(10n-2)!}}{2!\cancel{(10n-4)!}}\cdots\frac{\cancel{(8n+2)!}}{2!(8n)!}\cdot\frac{(8n)!}{3!(8n-3)!}\cdot\frac{\cancel{(8n-3)!}}{3!(8n-6)!}\cdots\frac{(5n+3)!}{3!(5n)!}\cdot\frac{\cancel{(5n)!}}{5!(5n-5)!}\cdots\frac{5!}{5!\cdot 0!}\cdot\frac{1}{n!\cdot n!}$$

$$= \frac{(10n)!}{n!\cdot n!\cdot(2!)^n(3!)^n(5!)^n} = \frac{(10n)!}{(n!)^2(2!)^n(3!)^n(5!)^n}. \square$$

(Same answer as method 1 :))

(9) Jessie Hurse has n pieces of (identical) kit-kats and m pieces of (identical) Reese's peanut butter cups. He wants to distribute all of them to 6 boys Aaron, Ben, Chad, Don, Eli and Fred, and 6 girls Amy, Beth, Carrie, Diana, Elizabeth and Fiona such that for every letter $X \in \{A, B, C, D, E, F\}$, the boy whose name starts with X has twice as many kit-kats as the girl whose name starts with X, but he has half as many Reese's peanut butter cups as her. How many ways can this be done? [10 marks]

Let's find # ways to distribute each type of candy & use multiplication principle.

$\boxed{\text{Kit Kats}}$

Imagine boy-girl pairs based on first name. Each "pair" gets 3 "units" of candy, i.e,

$$X_1 + X_2 + \cdots + X_6 = \frac{n}{3} \Rightarrow \# \text{ ways} = \begin{cases} 0 & \text{if } 3 \nmid n \\ \binom{\frac{n}{3}+6-1}{6-1} & \text{otherwise} \end{cases}$$

$\boxed{\text{Reese's}}$

Again boy-girl pairs. This time boys get $1i \Rightarrow$ girls get $2i$, so when paired, it's the same, i.e,

$$X_1 + X_2 + \cdots + X_6 = \frac{m}{3} \Rightarrow \# \text{ ways} = \begin{cases} 0 & \text{if } 3 \nmid m \\ \binom{\frac{m}{3}+6-1}{6-1} & \text{otherwise} \end{cases}$$

$$\# \text{ total ways} = \begin{cases} 0 & \text{if } 3 \nmid n \text{ or } 3 \nmid m \\ \binom{\frac{n}{3}+5}{5} \cdot \binom{\frac{m}{3}+5}{5} & \text{otherwise} \end{cases}$$

use multiplication principle

both conditions must hold, otherwise # ways $= 0$

set: $V_{\searrow}$ .2

✓(11) Let G be a simple graph where the vertex set is $\{1, 2, 3, \ldots, 999, 1000\}$ and two vertices are adjacent if they are relatively prime, that is, vertices m and n are adjacent if the greatest common divisor of m and n is 1.

(a) Find a vertex in G with the largest degree. [2 marks]

(b) Is G connected? [2 marks]

(c) Is G Eulerian? [2 marks]

(d) Does G contain a K_5 as a subgraph? [2 marks]

(e) Is G planar? [2 marks]

(a) Since $\gcd(m,1)=1 \ \forall m \in V$, $\boxed{v_1 \text{ is the most connected}}$ vertex (1 has highest degree $=999$)
$\hookrightarrow d(v_1)=999$

(b) Yes, G is connected, since every vertex is connected to 1 (except 1 itself / no loops)
$\hookrightarrow$ So, G has only 1 component.

(c) Consider v_1: v_1 is adjacent to all $v \in V$, except itself, since $\gcd(m,1)=1 \ \forall m \in V$.
$\Rightarrow d(v_1)=999$. (see part (a))
Observe: 999 is an odd number!

$\Rightarrow$ G is not Eulerian, since not all vertices have even degree. □

(d) Yes, G contains K_5 as a subgraph, since all primes are fully connected to each other and there are clearly at least 5 primes between 1 and 1000.

(e) By part (d), G is not planar, since any graph which contains K_5 (or a subdivision of K_5) is non planar. □

(12) State the Four-Color Theorem, and prove it for (simple) triangle-free planar graphs. (Recall that a graph is triangle-free if its girth is at least 4, that is, it has no cycles of length 3.) [15 marks]

4-color thm: Any planar, Δ-free graph G can have all its vertices colored with one of four (chosen) colors such that no two adjacent vertices have the same color.
$\chi(G) \leq 4$.

Lemma 1 $\quad e \leq 2v-4 \quad$ ($e = \#$ edges, $v = \#$ vertices)

Proof: We know $2e = \sum_{r \in R} \deg(r) \geq kr$, girth is $k=4$ in this case. (Assume G has one cycle at least)

$\Rightarrow 2e \geq kr$. By Euler's identity for planar graphs, $v - e + r = c + 1$.
We know/assume $c \geq 1 \Rightarrow r \geq 2 - v + e$

$\therefore 2e \geq k(2-v+e) \Rightarrow 0 \geq 2k - kv + (k-2)e \Rightarrow (k-2)e \leq kv - 2k$

Let $k=4$

$\Rightarrow 2e \leq 4v - 8 \Rightarrow \boxed{e \leq 2v-4.}$ (True for all G, if $e \geq 2$)

Lemma 2 Every planar, Δ-free graph with at least 2 edges has a vertex v, s.t. $\deg(v) \leq 3$.

Proof: Suppose not, i.e., $\forall v \in V: d(v) \geq 4 \Rightarrow 2e = \sum_{v \in V} d(v) \geq 4v \Rightarrow e \geq 2v$.

Recap: $e \leq 2v-4$, $e \geq 2v$ #

Therefore, lemma 2 is true by contradiction.

(continued on loose leaf paper)

Shane J

APM 2663 - Final

(12.) __Prove:__ 4-color theorem

* Proof by induction on # of vertices. *

Let G be a planar, Δ-free graph, with k vertices.

__se case:__ $v \leq 4$. Clearly this is 4-colorable, i.e., $\chi(G) \leq 4$.

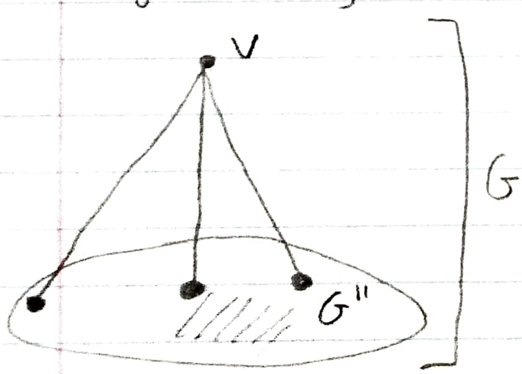
__nductive step:__ Assume claim true for G (Δ-free, planar graph, with k vertices) ($k \geq 4$, fixed arbitrary)

__WTS__ The claim is true for G', a planar, Δ-free graph with $k+1$ vertices.

(if $e \leq 1$, clearly G is 4-colorable (or less))

By lemma 2, we know there's a vertex in G' with degree ≤ 3.

Imagine deleting this vertex, i.e., $G' - v$. Call $G'' = G' - v$.



Since G'' has k vertices, we supposed, by IH it is 4-colorable.

Now, since v has at most 3 neighbors, v can be colored (using the 4th color) in a way which doesn't conflict with its neighbors' colors.

Hence G' ($G''+v$), which has $k+1$ vertices, can __also__ be 4-colored.

So, by induction, __any__ planar, Δ-free graph needs at most 4 colors to color it., i.e., $\chi(G) \leq 4$. $\square$

Estimate your grade in this test. Let x be your guess. If your grade is in the interval $[x-5, x+5]$, you will receive 2 bonus marks.

$(100)0.6 + (x)0.4 = 95 \Rightarrow 0.4x = 35 \Rightarrow x = 87.5$

Guess: $\boxed{x = 87}$