APM 263 Sample Test 1
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Important:

- Recall that the word *if* in a definition means *if and only if*.
- To receive full credit for a question, you should provide all logical steps.
- Recall that $\mathbb{N}$ is the set of natural numbers, that is, the set of positive integers.
- Recall that $\mathbb{Z}$ is the set of integers.
- Recall that $\mathbb{Q}$ is the set of rational numbers.
- Recall that $\mathbb{R}$ is the set of real numbers.
- This test is worth 120 marks. If you receive x marks, your grade will be $\min\{x, 100\}\%$.

(1) Let $X = \{1, 2, 3\}$.
 (a) What is the cardinality of X? [5 marks]
 (b) Is $\{1, 2\}$ a subset of X? [5 marks]

(2) Define an injective function. (You may assume the definition of a function.) [10 marks]

(3) Answer each of the following:
 (a) Disprove: If n^2 is a multiple of 9, then n is a multiple of 9. [5 marks]
 (b) Prove: If n^2 is a multiple of 3, then n is a multiple of 3. [5 marks]
 (*Hint: Recall that if $a \in \mathbb{Z}$ is not a multiple of 3, then $a = 3b + 1$ or $a = 3b + 2$ for some $b \in \mathbb{Z}$.*)

(4) Prove that $\sqrt{3}$ is irrational. [10 marks]

(5) Explain why the following statement is not true:
 For every positive integer x,

$$f(x) = 2001x^5 + 7801x^4 + 19197x^3 + 2011917x^2 + 287777x + 2567811111111$$

 produces a prime number.
 (Recall the following definition: An integer $p > 1$ is *prime* if the only divisors of p are $1, -1, p, -p$.) [5 marks]

(6) Let $A = \{(-1, 2), (4, 5), (0, 0), (6, -5), (5, 1), (4, 3)\}$. List the elements in each of the following sets.
 (a) $\{a + b \mid (a, b) \in A\}$ [3 marks]
 (b) $\{a \mid a > 0 \text{ and } (a, b) \in A \text{ for some } b\}$ [3 marks]
 (c) $\{b \mid b = k^2 \text{ for some } k \in \mathbb{Z} \text{ and } (a, b) \in A \text{ for some } a\}$ [4 marks]

(7) Let A, B, C be subsets of some universal set U.
 (a) Prove that

$$A \cap B \subseteq A \cap C \text{ and } \overline{A} \cap B \subseteq \overline{A} \cap C \Longrightarrow B \subseteq C. \text{ [10 marks]}$$

 (b) Is the following true? Explain.

$$A \cap B \subseteq A \cap C \text{ and } \overline{A} \cap B \subseteq \overline{A} \cap C \Longrightarrow B = C. \text{ [5 marks]}$$

(8) Define $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ by $f(x) = x^2 - 5x + 5$. Determine whether or not f is one-to-one and/or onto. [10 marks]

(9) Suppose $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are functions. Prove that if $g \circ f$ is one-to-one and f is onto, then g is one-to-one. [10 marks]

(10) For (x, y) and (u, v) in $\mathbb{R}^2$, define $(x, y) \sim (u, v)$ if $x^2 + y^2 = u^2 + v^2$. Prove that $\sim$ defines an equivalence relation on $\mathbb{R}^2$ and interpret the equivalence classes geometrically. [15 marks]

(11) Prove that the product of any five consecutive positive integers (such as $17 \cdot 18 \cdot 19 \cdot 20 \cdot 21$) must be divisible by 120. [15 marks]

Solutions

(1) (a) 3.
(b) Yes.

(2) Let $f : A \longrightarrow B$ be a function. It is *injective* if $x_1 = x_2$ whenever $f(x_1) = f(x_2)$.

(3) (a) Let $n = 3$. Then $n^2 = 9$ is a multiple of 9 but $n = 3$ is not.
(b) Suppose the result is not true. Then $n = 3b+1$ or $n = 3b+2$ for some $b \in \mathbb{Z}$. We consider two cases.
(i) $n = 3b+1$: Then $n^2 = 9b^2+6b+1 = 3(3b^2+2b)+1$. So n^2 is not a multiple of 3, contradiction.
(ii) $n = 3b+2$: Then $n^2 = 9b^2+12b+4 = 3(3b^2+4b+1)+1$. So n^2 is not a multiple of 3, contradiction.

(4) Suppose $\sqrt{3}$ is rational. Let $\sqrt{3} = m/n$. ($m \neq 0$ and $n \neq 0$.) Without loss of generality, we may assume m and n has no common multiple, in particular, we may assume at most one of m and n is a multiple of 3. Now, $\sqrt{3} = m/n$ implies $n\sqrt{3} = m$ which implies $n^2 3 = m^2$. Since $n^2 3$ is a multiple of 3, m^2 is a multiple of 3. Therefore, m is a multiple of 3. (We did this in the previous question.) Since m is a multiple of 3, $m = 3a$ for some $a \in \mathbb{Z}$. So $n^2 3 = 9a^2$ which implies $n^2 = 3a^2$. Since $3a^2$ is a multiple of 3, n^2 is a multiple of 3. Therefore n is a multiple of 3. So both m and n are multiples of 3, a contradiction.

(5) $f(2567811111111)$ is a multiple of 2567811111111. Since $2567811111111 \neq 1$ and $2567811111111 \neq f(2567811111111)$, $f(2567811111111)$ is not prime.

(6) (a) $\{-1+2, 4+5, 0+0, 6-5, 5+1, 4+3\} = \{1, 9, 0, 6, 7\}$
(b) $\{4, 6, 5\}$
(c) $\{0, 1\}$

(7) (a) Let $b \in B$. We want to prove $b \in C$. We consider two cases.
(i) $b \in A$: Then $b \in A \cap B$. Since $A \cap B \subseteq A \cap C$, $b \in A \cap C$. Therefore $b \in C$.
(ii) $b \in \overline{A}$: Then $b \in \overline{A} \cap B$. Since $\overline{A} \cap B \subseteq \overline{A} \cap C$, $b \in \overline{A} \cap C$. Therefore $b \in C$.
Hence $b \in C$.
(b) Not true since if we let $U = \{a\}$, $A = \{a\}$, $C = \{a\}$ and $B = \emptyset$, then the hypothesis are satisfied but $B \neq C$.

(8) We claim that f is neither one-to-one nor onto. By completing the square, we have $f(x) = (x - \frac{5}{2})^2 - \frac{5}{4}$. (Drawing the graph may help.) By picking $x_1 = 0 \in \mathbb{Z}$ and $x_2 = 5 \in \mathbb{Z}$, we have $f(x_1) = f(x_2) = 5 \in \mathbb{Z}$. Hence f is not one-to-one.
We claim that it is not onto. Since $f(x) = (x - \frac{5}{2})^2 - \frac{5}{4}$, $f(x) > -2$ for all $x \in \mathbb{Z}$. Hence there is no $x \in \mathbb{Z}$ that will give $f(x) = -2 \in \mathbb{Z}$. So f is not onto.

(9) Suppose $b_1, b_2 \in B$ and $g(b_1) = g(b_2)$. We want to prove $b_1 = b_2$. Since f is onto and $b_1, b_2 \in B$, there is $a_1, a_2 \in A$ such that $f(a_1) = b_1$ and $f(a_2) = b_2$. Hence $g(f(a_1)) = g(b_1)$ and $g(f(a_2)) = g(b_2)$. Since $g(b_1) = g(b_2)$, $g(f(a_1)) = g(f(a_2))$. Now $g \circ f(a_1) = g \circ f(a_2)$ implies $a_1 = a_2$ since $g \circ f$ is one-to-one. Since f is a function, we have $f(a_1) = f(a_2)$. Hence $b_1 = b_2$.

(10) (a) Let $(x, y) \in \mathbb{R}^2$. Since $x^2 + y^2 = x^2 + y^2$, $(x, y) \sim (x, y)$. Hence $\sim$ is reflexive.
(b) Let $(x, y), (u, v) \in \mathbb{R}^2$ such that $(x, y) \sim (u, v)$. Then $x^2 + y^2 = u^2 + v^2$. Hence $u^2 + v^2 = x^2 + y^2$. Therefore $(u, v) \sim (x, y)$. Hence $\sim$ is symmetric.
(c) Let $(x, y), (u, v), (w, z) \in \mathbb{R}^2$ such that $(x, y) \sim (u, v)$ and $(u, v) \sim (w, z)$. Then $x^2 + y^2 = u^2 + v^2$ and $u^2 + v^2 = w^2 + z^2$. Therefore $x^2 + y^2 = w^2 + z^2$. Hence $\sim$ is transitive.

Hence $\sim$ is an equivalence relation on $\mathbb{R}^2$.
Let $(x, y) \in \mathbb{R}^2$. Then

$$[(x, y)] = \{(u, v) \mid x^2 + y^2 = u^2 + v^2\}.$$

Since $(x, y) \in \mathbb{R}^2$ is fixed, $x^2 + y^2$ is just a nonnegative real number. So $[(x, y)]$ is a circle centered at $(0, 0)$. So the equivalence classes are circles centered at $(0, 0)$.

(11) Let $a \in \mathbb{N}$. We want to prove $a(a + 1)(a + 2)(a + 3)(a + 4)$ is divisible 120. Since $120 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5$, it is enough to show $a(a+1)(a+2)(a+3)(a+4)$ has three factors of 2, one factor of 3 and one factor of 5. Since exactly one of five consecutive integers has a factor of 5 and at least one of five consecutive integers has a factor of 3, it is enough to show $a(a+1)(a+2)(a+3)(a+4)$ has a factor of 8. We consider two cases.
(a) $a = 2b$ for some $b \in \mathbb{N}$. Then

$$\begin{aligned} a(a+1)(a+2)(a+3)(a+4) &= 2b(2b+1)(2b+2)(2b+3)(2b+4) \\ &= 8b(2b+1)(b+1)(2b+3)(b+2) \end{aligned}$$

which has a factor of 8.

(b) $a = 2b + 1$ for some $b \in \mathbb{N} \cup \{0\}$. Then

$$\begin{aligned} a(a+1)(a+2)(a+3)(a+4) &= (2b+1)(2b+2)(2b+3)(2b+4)(2b+5) \\ &= 4(2b+1)(b+1)(2b+3)(b+2)(2b+5). \end{aligned}$$

Now either $b+1$ is even or $b+2$ is even. Hence $a(a+1)(a+2)(a+3)(a+4)$ has a factor of 8.