13. The sequence can start with a b or a c, after which any string of length $n-1$ without two consecutive a's gives a string of length n without two consecutive a's. This contributes $2a_{n-1}$ to the sum. Similarly, the sequence can start ab or ac and be followed by a string of length $n-2$ without two consecutive a's, contributing $2a_{n-2}$ to the sum. These are all the possibilities. Therefore the recurrence relation is $a_n = 2a_{n-1} + 2a_{n-2}$. The initial conditions are $a_0 = 1$ (the empty string), and $a_1 = 3$ (all three strings of length 1).

15. **(a)** There is one way for the child to walk 0 inches (take no steps), and there is one way for the child to walk 10 inches (take one little step), so we have $a_0 = a_1 = 1$. In order to cover $10n$ inches, the child can either take a little step and then walk $10(n-1)$ inches (and this can be done in a_{n-1} ways), or take a big step and then walk $10(n-2)$ inches (and this can be done in a_{n-2} ways). Therefore $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 2$. Note that this is the same recurrence relation and initial conditions that defined the Fibonacci sequence, so we conclude that $a_n = f_n$ for all n.

(b) Since 10 feet is $120 = 10 \cdot 12$ inches, we seek a_{12}. Iterating the recurrence relation, we easily compute that $a_{12} = 233$.

17. **(a)** Reasoning as we did above Example 8, we have

$$q(n) = \sum_{k=0}^{M} C(n-1,k)q(n-k-1),$$

where M is the smaller of $n-1$ and 3, since at most three other elements can be in the same set of the partition as n. The initial condition is $q(0) = 1$ (the empty partition).

(b) Clearly $q(n) = p(n)$ for $n \leq 4$. There is only one nonallowed partition of a five-element set (the one with only one class), so $q(5) = p(5) - 1 = 51$. Finally we use the recurrence relation to compute $q(6) = C(5,0)q(5) + C(5,1)q(4) + C(5,2)q(3) + C(5,3)q(2) = 1 \cdot 51 + 5 \cdot 15 + 10 \cdot 5 + 10 \cdot 2 = 196$.

19. We know that $p(n) = \sum_{k=0}^{n-1} C(n-1,k)p(n-k-1)$. Replace the dummy variable k by $n-k-1$. The new range is then from $k = n-0-1 = n-1$ to $n-(n-1)-1 = 0$, which is the same as before, and $C(n-1, n-k-1) = C(n-1, n-1-(n-k-1)) = C(n-1,k)$.

21. Note that we need $0 \leq k \leq n$ in order for $L(n,k)$ to have a nonzero value. The initial conditions are $L(0,0) = 1$ (the empty string) and $L(n,0) = L(n,n) = 1$ (the string of all 1's and the string of all 0's). To form a string of length n with exactly k 0's, either we need to take a string of length $n-1$ with k 0's and append a 1 (and there are $L(n-1,k)$ of these), or else we need to take a string of length $n-1$ with $k-1$ 0's and append a 0 (and there are $L(n-1,k-1)$ of these). Thus for all $n \geq 1$ we have $L(n,k) = L(n-1,k) + L(n-1,k-1)$ as long as $1 \leq k \leq n-1$. These are exactly the same conditions satisfied by $C(n,k)$; see Pascal's triangle. Thus $L(n,k) = C(n,k)$.

23. (a) We see that $a_1 = 1$ (since $1 = 1$ is the only partition), and $a_2 = 2$ (since $2 = 2 = 1+1$ are the only partitions). For $n \geq 3$ we see that $a_n = a_{n-1} + a_{n-2}$, because an ordered partition of n into 1's and 2's can either begin with a 1 and be followed by an ordered partition of $n-1$, or begin with a 2 and be followed by an ordered partition of $n-2$. Note that $\{a_n\}$ is the Fibonacci sequence.

(b) $a_5 = f_5 = 8$

25. (a) There is clearly only one way to get to any lattice point on the coordinate axes, so $T(a,0) = T(0,b) = 1$ for all natural numbers a and b. To get to any other lattice point (a,b), we can either travel along a path from the origin to $(a-1,b)$ (and there are $T(a-1,b)$ such paths) and then move to the right, or else travel along a path from the origin to $(a,b-1)$ (and there are $T(a,b-1)$ such paths) and then move up. Therefore $T(a,b) = T(a-1,b) + T(a,b-1)$.

(b) To reach (a,b) we need a sequence of a moves to the right and b moves up, i.e., a string of length $a+b$ consisting of a R's and b U's. There are clearly $C(a+b,a)$ such strings. Thus $T(a,b) = C(a+b,a)$. Note that the recurrence relation in part (a) is really Pascal's identity.

27. (a) $T(4) = 7T(2)+4.5\cdot4^2 = 7\cdot12+72 = 156$; $T(8) = 7T(4)+4.5\cdot8^2 = 7\cdot156+288 = 1380$

(b) We see from the following table that the reorganized calculation first becomes more efficient for $n = 128$.

k	$n = 2^k$	$T(n) = 8T(n/2) + n^2$	$T(n) = 7T(n/2) + 4.5n^2$
0	1	1	1
1	2	12	12
2	4	112	156
3	8	960	1380
4	16	7936	10,812
5	32	64,512	80,292
6	64	520,192	580,476
7	128	4,177,920	4,137,060

29. (a) For the initial condition we have $T(1) = 1$, since one comparison is needed to see if the element in the list is the one we are looking for. For the recurrence, note that with one comparison the list is cut nearly in half, so $T(n) = 1 + T(\lceil n/2 \rceil)$.

(b) $T(20) = 1 + T(10) = 1 + 1 + T(5) = 1 + 1 + 1 + T(3) = 1 + 1 + 1 + 1 + T(2) = 1 + 1 + 1 + 1 + 1 + T(1) = 1 + 1 + 1 + 1 + 1 + 1 = 6$ (this agrees with the formula $T_B(n) = \lfloor \log(n-1) \rfloor + 2$ presented in Section 4.3, since $\lfloor \log(n-1) \rfloor + 2 = 4 + 2 = 6$)

31. (a) If the bit string of length $n \geq 3$ starts with a 0, then in order not to contain 101 it must continue as a string of length $n-1$ with no 101 as a substring; and there are a_{n-1} such strings. Otherwise the string starts with k 1's, for some $k \geq 1$, followed by 00, unless it ends before the 00 appears. The remaining $n-k-2$ bits must form a bit string without the substring 101. For each k from 1 to $n-2$, then, there are a_{n-k-2} such strings. Finally, the string can be $11\ldots110$ or $11\ldots111$. Putting this all together, we have

$$a_n = a_{n-1} + a_{n-3} + a_{n-4} + a_{n-5} + \cdots + a_1 + a_0 + 2\,.$$

The initial conditions are $a_0 = 1$, $a_1 = 2$, and $a_2 = 4$, since no short strings can possibly contain 101.

(b) We apply the recurrence to obtain $a_3 = a_2 + a_0 + 2 = 7$, $a_4 = a_3 + a_1 + a_0 + 2 = 12$, $a_5 = a_4 + a_2 + a_1 + a_0 + 2 = 21$, $a_6 = a_5 + a_3 + a_2 + a_1 + a_0 + 2 = 37$, $a_7 = a_6 + a_4 + a_3 + a_2 + a_1 + a_0 + 2 = 65$, $a_8 = a_7 + a_5 + a_4 + a_3 + a_2 + a_1 + a_0 + 2 = 114$, $a_9 = a_8 + a_6 + a_5 + a_4 + a_3 + a_2 + a_1 + a_0 + 2 = 200$, and $a_{10} = a_9 + a_7 + a_6 + a_5 + a_4 + a_3 + a_2 + a_1 + a_0 + 2 = 351$.

(c) If we subtract from the recurrence relation displayed above the same recurrence relation with $n-1$ in place of n:

$$\begin{aligned} a_n &= a_{n-1} + a_{n-3} + a_{n-4} + a_{n-5} + \cdots + a_1 + a_0 + 2 \\ -(a_{n-1} &= a_{n-2} + a_{n-4} + a_{n-5} + a_{n-6} + \cdots + a_1 + a_0 + 2)\,, \end{aligned}$$

then most of the terms cancel and we obtain $a_n - a_{n-1} = a_{n-1} - a_{n-2} + a_{n-3}$ for all $n \geq 4$. This simplifies to $a_n = 2a_{n-1} - a_{n-2} + a_{n-3}$, with the initial conditions $a_0 = 1$, $a_1 = 2$, $a_2 = 4$, and $a_3 = 7$ (actually the recurrence relation holds for $n = 3$ as well).

33. (a) Let x_0 be a fixed element of the k-set A. There are n ways to choose the image y_0 of x_0 under a surjective function from A to the n-set B. In order for this function to be surjective, we need to have either that it is still surjective when restricted to $A - \{x_0\}$ (and there are $f(k-1,n)$ ways in which this can happen), or that it is surjective to $B - \{y_0\}$ when so restricted (and there are $f(k-1,n-1)$ ways in which this can happen). The recurrence relation follows.

(b) $f(k,1) = 1$ and $f(k,k) = n!$

(c) Working from the bottom up, we find the following values:

$f(3,2) = 2(f(2,2) + f(2,1)) = 2(2+1) = 6;$

$f(4,2) = 2(f(3,2) + f(3,1)) = 2(6+1) = 14;$

$f(4,3) = 3(f(3,3) + f(3,2)) = 3(6+6) = 36;$

$f(5,3) = 3(f(4,3) + f(4,2)) = 3(36+14) = 150;$

$f(5,4) = 4(f(4,4) + f(4,3)) = 4(24+36) = 240;$

$f(6,4) = 4(f(5,4) + f(5,3)) = 4(240+150) = 1560$

35. (a) The paths to $(3,3)$ that do not cross the diagonal are $RRRUUU$, $RRURUU$, $RRUURU$, $RURRUU$, and $RURURU$, where R stands for a move to the right and U stands for a move up.

(b) If $b > a$, then the point is above the diagonal, so there is no way to get there: $T'(a,b) = 0$ in this case. If $b = 0$, then the point is on the x-axis, and the only way to get there is by moving right a times, so $T'(a,0) = 1$ for all a. These are the initial conditions. The recurrence relation is the same as the one found in Exercise 25 (and for the same reason): $T'(a,b) = T'(a-1,b) + T'(a,b-1)$ in all other cases (i.e., $0 < b \le a$).

(c) We construct the following picture from the lower left (representing the number of ways to get to $(0,0)$), working toward the upper right (representing the number of ways to get to $(0,0)$), by using the recurrence relation and boundary (initial) conditions.

0	0	0	0	0	42
0	0	0	0	14	42
0	0	0	5	14	28
0	0	2	5	9	14
0	1	2	3	4	5
1	1	1	1	1	1

Thus we see that $T'(5,5) = 42$.

37. (a) The partitions of 7 into at most three parts are 7, $6+1$, $5+2$, $5+1+1$, $4+3$, $4+2+1$, $3+3+1$, and $3+2+2$, a total of 8. The partitions of 7 into parts no larger than 3 are $3+3+1$, $3+2+2$, $3+2+1+1$, $3+1+1+1+1$, $2+2+2+1$, $2+2+1+1+1$, $2+1+1+1+1+1$, and $1+1+1+1+1+1+1$, again a total of 8.

(b) To each Ferrers diagram representing a partition of k into at most m parts (i.e., k dots in at most m rows), there corresponds the Ferrers diagram representing a partition of k into parts no bigger than m (i.e., k dots in at most m columns), and vice versa: namely the one obtained by flipping the diagram around the line through the upper left corner having slope -1. It is clear that the flip of a Ferrers diagram is again a Ferrers diagram, and since flipping twice restores the original diagram, this is a one-to-one correspondence.

SECTION 7.3 Solving Recurrence Relations

1. **(a)** Plugging $a_n = 3$ into the right-hand side, we have

$$\begin{aligned}\frac{n-1}{n}\cdot a_{n-1}+\frac{2n-4}{n}\cdot a_{n-2} &= \frac{n-1}{n}\cdot 3+\frac{2n-4}{n}\cdot 3\\ &= \frac{3n-3+6n-12}{n}=\frac{9n-15}{n}.\end{aligned}$$

Since this does not equal a_n, we know that the constant function $a_n = 3$ is not a solution.

(b) Plugging in $a_n = 2^n$, we have

$$\begin{aligned}\frac{n-1}{n}\cdot a_{n-1}+\frac{2n-4}{n}\cdot a_{n-2} &= \frac{n-1}{n}\cdot 2^{n-1}+\frac{2n-4}{n}\cdot 2^{n-2}\\ &= \frac{n2^{n-1}-2^{n-1}+2n\cdot 2^{n-2}-4\cdot 2^{n-2}}{n}\\ &= \frac{n2^n-3\cdot 2^{n-1}}{n}=2^n-\frac{3\cdot 2^{n-1}}{n}.\end{aligned}$$

Since this does not equal a_n, we know that the function $a_n = 2^n$ is not a solution.

(c) Plugging in $a_n = 0$, we have

$$\frac{n-1}{n}\cdot a_{n-1}+\frac{2n-4}{n}\cdot a_{n-2} = \frac{n-1}{n}\cdot 0+\frac{2n-4}{n}\cdot 0 = 0 = a_n\,.$$

Therefore the function $a_n = 0$ is a solution.

(d) Plugging in $a_n = 2^n/n$, we have

$$\begin{aligned}\frac{n-1}{n}\cdot a_{n-1}+\frac{2n-4}{n}\cdot a_{n-2} &= \frac{n-1}{n}\cdot\frac{2^{n-1}}{n-1}+\frac{2n-4}{n}\cdot\frac{2^{n-2}}{n-2}\\ &= \frac{2^{n-1}}{n}+\frac{2\cdot 2^{n-2}}{n}=\frac{2^n}{n}=a_n\,.\end{aligned}$$

Therefore this function is a solution.

(e) Plugging in $a_n = 7\cdot 2^n/n$, we have

$$\frac{n-1}{n}\cdot a_{n-1}+\frac{2n-4}{n}\cdot a_{n-2} = \frac{n-1}{n}\cdot 7\cdot\frac{2^{n-1}}{n-1}+\frac{2n-4}{n}\cdot 7\cdot\frac{2^{n-2}}{n-2}=7\cdot\frac{2^n}{n}=a_n\,.$$

Therefore this is a solution; note that it is a linear multiple of the solution checked in part (d).

3. **(a)** yes, order 3 **(b)** no, not linear **(c)** no, not homogeneous
(d) no, does not have constant coefficients

5. **(a)** $a=3$, $b=3$, $c=7$, $d=1$; thus $a=b^d$, so $f(n)\in O(n^d\log n)=O(n\log n)$
(b) $a=4$, $b=2$, $c=1$, $d=2$; thus $a=b^d$, so $f(n)\in O(n^d\log n)=O(n^2\log n)$
(c) $a=1$, $b=2$, $c=1$, $d=1$; thus $a<b^d$, so $f(n)\in O(n^d)=O(n)$
(d) $a=2$, $b=2$, $c=1$, $d=1/2$; thus $a>b^d$, so $f(n)\in O(n^{\log_b a})=O(n)$

7. (a) We take the general solution $a_n = A(-2)^n + B \cdot 3^n$ obtained in Exercise 6a and plug in $a_0 = 2$ and $a_1 = -1$ to obtain the equations $A + B = 2$ and $-2A + 3B = -1$. The solution to these equations is $A = 7/5$, $B = 3/5$. Therefore the solution is $a_n = (7/5) \cdot (-2)^n + (3/5) \cdot 3^n$. As a check, $a_2 = 11$, whether computed from the recurrence or computed from this formula.

(b) We take the general solution $a_n = A(-2)^n + Bn \cdot (-2)^n$ obtained in Exercise 6b and plug in $a_0 = 2$ and $a_1 = -1$ to obtain the equations $A = 2$ and $-2A - 2B = -1$. The solution to these equations is $A = 2$, $B = -3/2$. Therefore the solution is $a_n = 2(-2)^n - (3/2)n \cdot (-2)^n$. As a check, $a_2 = -4$, whether computed from the recurrence or computed from this formula.

(c) We take the general solution $a_n = A + B \cdot 2^n$ obtained in Exercise 6c and plug in $a_0 = 2$ and $a_1 = -1$ to obtain the equations $A + B = 2$ and $A + 2B = -1$. The solution to these equations is $A = 5$, $B = -3$. Therefore the solution is $a_n = 5 - 3 \cdot 2^n$.

(d) We take the general solution $a_n = A + Bn$ obtained in Exercise 6d and plug in $a_0 = 2$ and $a_1 = -1$ to obtain the equations $A = 2$ and $A + B = -1$. The solution to these equations is $A = 2$, $B = -3$. Therefore the solution is $a_n = 2 - 3n$.

9. (a) We plug $a_n = 1/(1 + Cn)$ and $a_{n-1} = 1/(1 + C(n-1))$ into the right-hand side to obtain

$$\frac{(1-n)a_{n-1}}{a_{n-1} - n} = \frac{(1-n)/(1+C(n-1))}{[1/(1+C(n-1))] - n} = \frac{1-n}{1 - n(1 + C(n-1))}$$
$$= \frac{1-n}{1 - n + Cn(1-n)} = \frac{1}{1 + Cn} = a_n ,$$

which is the left-hand side.

(b) We set $1/(1 + C \cdot 1) = 3$ and solve to obtain $C = -2/3$. Thus the specific solution is $a_n = 1/(1 - (2/3)n) = 3/(3 - 2n)$.

(c) From the recurrence relation we compute $a_2 = (1-2) \cdot 3/(3-2) = -3$, then $a_3 = (1-3)(-3)/((-3)-3) = -1$, and finally $a_4 = (1-4)(-1)/((-1)-4) = -3/5$. In each case this is the same as $3/(3-2n)$.

(d) One solution is $a_n = 1$, which corresponds to letting $C = 0$. Our verification in part (a) is valid for $C = 0$ (in fact it is valid for any C other than the reciprocal of a negative integer). If we really want a solution that is not in this form for any real C, we see that as $C \to \infty$ the fraction approaches 0. This suggests that maybe $a_n = 0$ is a solution. Plugging this into the recurrence relation gives the identity $0 = 0$, so it is indeed a solution. It corresponds to the initial condition $a_1 = 0$.

11. The recurrence relation is $A_{n+1} = 1.05A_n$, with initial condition $A_0 = 1200$. By the iterative method (or applying the formula we obtained in this section, with $i = 1.05$ and $d = 0$), we see that $A_n = 1.05^n A_0 = 1200 \cdot 1.05^n$.

13. (a) The recurrence relation is $A_n = 2000 + 1.05A_n$, with initial condition $A_0 = 0$. By the iterative method (or applying the formula we obtained in this section, with $i = 1.05$ and $d = 2000$), we see that $A_n = 2000[(1.05^n - 1)/(1.05 - 1)] + 1.05^n A_0 = 40000(1.05^n - 1)$. Recall that we had $A_n = 42000(1.05^n - 1)$ when investments were made at the beginning of the year.

(b) We see from the two formulas that the ratio of money in the two accounts is always $40000/42000$. In other words, under the original method, the balance is always 5% higher. After 5 years these balances (to the nearest dollar) are \$11,051 and \$11,604; after 10 years the balances are \$43,157 and \$45,315; and after 15 years they are \$132,878 and \$139,522.

15. We are told that the complexity function satisfies $f(n) = 3f(n/2) + cn$ for some constant c. We invoke Theorem 1, with $a = 3$, $b = 2$, and $d = 1$ (the hypothesis that f is increasing is reasonable to assume). Since $a > b^d$, we see that $f \in O(n^{\log 3}) \approx O(n^{1.585})$.

17. (a) We easily compute that the sequence begins $1, 3, 4, 7, 11, 18, 29, 47, 76$, and so on.

(b) Since the recurrence relation is the same as that for the Fibonacci sequence, the general solution is the same:

$$L_n = A \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n + B \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n .$$

We need to plug in the initial conditions to determine A and B. Note that rather than starting at $L_1 = 1$, we can take $L_0 = 2$ to simplify the algebra (since $L_2 = 1 + 2$). Then our initial conditions tell us that

$$\begin{aligned} 2 &= A + B \\ 1 &= A \cdot \left(\frac{1+\sqrt{5}}{2}\right) + B \cdot \left(\frac{1-\sqrt{5}}{2}\right) . \end{aligned}$$

Solving, we obtain $A = 1$ and $B = 1$. Thus we have

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n .$$

(c) A calculator reports that for $n = 8$ the sum shown here is $46.9787 + 0.0213 = 47$. Thus $L_8 = 47$.

(d) We expand the binomial terms and simplify.

$$\begin{aligned} L_3 &= \left(\frac{1+\sqrt{5}}{2}\right)^3 + \left(\frac{1-\sqrt{5}}{2}\right)^3 \\ &= \frac{1}{8}(1+3\sqrt{5}+3\sqrt{5}^2+\sqrt{5}^3+1-3\sqrt{5}+3\sqrt{5}^2-\sqrt{5}^3) \\ &= \frac{1}{8}(2+6\cdot 5) = 4 \end{aligned}$$

19. (a) The characteristic equation of this homogeneous recurrence relation is $r^2-r-3=0$, and the quadratic formula tells us that the roots are $r=(1\pm\sqrt{13})/2$. Therefore the general solution to the recurrence relation is

$$a_n = A\cdot\left(\frac{1+\sqrt{13}}{2}\right)^n + B\cdot\left(\frac{1-\sqrt{13}}{2}\right)^n .$$

(b) If we plug in the initial conditions, we obtain

$$\begin{aligned} 3 &= A+B \\ 1 &= A\cdot\left(\frac{1+\sqrt{13}}{2}\right) + B\cdot\left(\frac{1-\sqrt{13}}{2}\right) . \end{aligned}$$

To solve this system, we substitute $A=3-B$ (from the first equation) into the second equation, and find that

$$B = \frac{1+3\sqrt{13}}{2\sqrt{13}} = \frac{39+\sqrt{13}}{26} ,$$

and so $A=(39-\sqrt{13})/26$. Thus

$$a_n = \left(\frac{39-\sqrt{13}}{26}\right)\cdot\left(\frac{1+\sqrt{13}}{2}\right)^n + \left(\frac{39+\sqrt{13}}{26}\right)\cdot\left(\frac{1-\sqrt{13}}{2}\right)^n .$$

21. The characteristic equation is $r^4-8r^2-9=0$, which factors as $(r^2-9)(r^2+1) = (r-3)(r+3)(r-i)(r+i)=0$. Therefore the general solution is $a_n = A\cdot 3^n + B\cdot(-3)^n + C\cdot i^n + D\cdot(-i)^n$.

23. (a) 1, 1, 1, 2, 3, 3, 3, 4, 5, 5, 5, 6

(b) The characteristic equation is $r^4-2r^3+2r^2-2r+1=0$. By inspection we see that $r=1$ is a root, so we factor out $(r-1)$ and write $(r-1)(r^3-r^2+r-1)=0$. Again $(r-1)$ is a factor, and after dividing again we obtain the complete factorization: $(r-1)^2(r^2+1)=0$. This yields the roots 1 (twice), i, and $-i$. Therefore the general solution is $a_n = A+Bn+Ci^n+D(-i)^n$.

Next we plug in the initial conditions and obtain the following system of linear equations.

$$\begin{aligned} 1 &= A \qquad\;\; + C + D \\ 1 &= A + B + iC - iD \\ 1 &= A + 2B - C - D \\ 2 &= A + 3B - iC + iD \end{aligned}$$

Solving this system, we obtain $A = B = 1/2$, $C = D = 1/4$. Therefore the specific solution we seek is

$$a_n = \frac{1}{2} + \frac{1}{2}n + \frac{1}{4}i^n + \frac{1}{4}(-i)^n$$

(c) For a_7 (which we saw above to be 4) we compute

$$a_7 = \frac{1}{2} + \frac{1}{2}\cdot 7 + \frac{1}{4}i^7 + \frac{1}{4}(-i)^7 = 4 - \frac{1}{4}i + \frac{1}{4}i = 4\,,$$

and similarly for $a_{10} = 5$.

25. (a) $a_0 = 1$, $a_1 = 2$, $a_2 = 2^3/1^2 = 8$, $a_3 = 8^3/2^2 = 128$, $a_4 = 128^3/8^2 = 32768$
(b) Let $b_n = \log a_n$. Then upon taking logarithms the initial conditions become $b_0 = 0$ and $b_1 = 1$, and the recurrence relation becomes $b_n = 3b_{n-1} - 2b_{n-2}$. The general solution to this is easily found to be $b_n = A + B \cdot 2^n$, and then the initial conditions are easily seen to yield $A = -1$ and $B = 1$. Therefore we have $b_n = 2^n - 1$. Thus the solution to the original problem is $a_n = 2^{2^n - 1}$. For $n = 4$, for example, we have $a_4 = 2^{2^4 - 1} = 2^{15} = 32768$.

27. The recurrence relation we obtained there was $a_n = a_{n-1} + a_{n-2} + a_{n-3}$. The characteristic equation for this problem is therefore $r^3 - r^2 - r - 1 = 0$, which, by the rational root test, is seen to have no rational roots. Thus we cannot easily find the roots. In fact, there is only one real root, approximately 1.84, and two complex roots. In principle we can write down these roots (they will be messy expressions involving the cube root symbol) and (after plugging in the initial conditions) write down the specific solution, but the formula will certainly not be very pretty.

29. The number of bit strings of length n that *do not* contain two consecutive 0's is f_{n+1} (see Example 6 in Section 7.2). Thus the number we seek is $2^n - f_{n+1}$. By Example 5 in the present section, we know a formula for f_{n+1}. Thus the solution is

$$2^n - \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} + \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+2}.$$

31. Suppose that $\{a_n\}$ satisfies the recurrence relation $a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k}$, with $c_k \neq 0$. If the characteristic equation has k distinct roots, $r_1, r_2, \ldots, r_k$, then $a_n = A_1 r_1^n + \cdots + A_k r_k^n$, where the A_i's are arbitrary constants. Conversely, $a_n = A_1 r_1^n + \cdots + A_k r_k^n$ satisfies the recurrence relation.

33. (a) The recurrence relation is $a_n = a_{n-1}+n^2$, with $a_1 = 1$. The solution of the associated homogeneous recurrence relation is $a_n = A \cdot 1^n = A$. The particular solution of the given recurrence relation is of the form $a_n = Bn + Cn^2 + Dn^3$ (a polynomial of degree 2, multiplied by n because of the root $r = 1$ of the associated homogeneous characteristic equation). Plugging this into the relation and simplifying, we obtain $0n^3+(-3D+1)n^2+(-2C+3D)n+(-B+C-D) = 0$. Equating coefficients and solving, we obtain $D = 1/3$, $C = 1/2$, and $B = 1/6$. Therefore the general solution to the nonhomogeneous relation is

$$a_n = A + \frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3}.$$

Finally we plug in the initial condition $a_1 = 1$ and solve to find that $A = 0$. Therefore the formula we desire is $a_n = (n/6) + (n^2/2) + (n^3/3)$.

(b) It is clear that the two formulas are algebraically equivalent.

35. (a) $1 + 2 + \cdots + n = n(n+1)/2$

(b) Using part (a), we see that

$$P_n = P_{n-1} + \frac{n(n+1)}{2} = P_{n-1} + \frac{1}{2}n^2 + \frac{1}{2}n,$$

with initial condition $P_0 = 0$.

(c) The associated homogeneous equation has the solution $P_n = A$. We seek a particular solution of the nonhomogeneous equation by trying $P_n = Bn+Cn^2+Dn^3$ (a polynomial of degree 2, multiplied by n because of the root $r = 1$ of the associated homogeneous characteristic equation). Plugging this into the relation and simplifying, we obtain

$$0n^3 + \left(-3D + \frac{1}{2}\right)n^2 + \left(-2C + 3D + \frac{1}{2}\right)n + (-B + C - D) = 0.$$

Equating coefficients and solving, we obtain $D = 1/6$, $C = 1/2$, and $B = 1/3$. Therefore the general solution to the nonhomogeneous relation is

$$P_n = A + \frac{n}{3} + \frac{n^2}{2} + \frac{n^3}{6}.$$

Plugging in the initial condition tells us that $A = 0$, so (after factoring to improve the presentation) $P_n = n(n+1)(n+2)/6$. In particular, $P_d = d(d+1)(d+2)/6$.

(d) $P_{12} = 12 \cdot 13 \cdot 14/6 = 364$ (almost enough to last the whole year).

37. By Exercise 36 in Section 7.2, the recurrence relation is $R_n = R_{n-1} + n$, with $R_0 = 1$. The solution to this is easily found by either of the techniques of this section to be $R_n = (n^2 + n + 2)/2$.

39. This problem is the "logarithm" of Exercise 34.

(a) The recurrence relation is $a_n = (a_{n-1}a_{n-2})^{1/2}$, with initial conditions $a_0 = x$ and $a_1 = y$. Let $b_n = \log a_n$, and our problem becomes $b_n = (b_{n-1}+b_{n-2})/2$, with $b_0 = \log x$ and $b_1 = \log y$.

(b) We solved this problem for b_n in Exercise 34, obtaining (translated into this context)

$$b_n = \frac{1}{3}(\log x + 2\log y) + \frac{2}{3}(\log x - \log y)\left(-\frac{1}{2}\right)^n .$$

Thus we have

$$\begin{aligned} a_n = 2^{b_n} &= 2^{\frac{1}{3}(\log x+2\log y)+\frac{2}{3}(\log x-\log y)(-\frac{1}{2})^n} \\ &= x^{1/3}y^{2/3}(x^{2/3}y^{-2/3})^{(-1/2)^n} . \end{aligned}$$

(c) As $n \to \infty$, $(-1/2)^n \to 0$, so $a_n \to x^{1/3}y^{2/3} = \sqrt[3]{xy^2}$.

SECTION 7.4 The Inclusion–Exclusion Principle

1. Let C, R, and A be the sets of students who like to play chess, bridge, and backgammon, respectively. Then by the inclusion–exclusion principle we have $60 = |C \cup R \cup A| = |C|+|R|+|A|-|C\cap R|-|C\cap A|-|R\cap A|+|C\cap R\cap A| = 37+31+19-11-16-5+|C\cap R\cap A|$. Thus $60 = 55 + |C \cap R \cap A|$, so $|C \cap R \cap A| = 5$.

3. Let B, R, and G stand for the sets of competitors who won blue, red, and green ribbons, respectively. We are told that $|B| = 13$, $|R| = 25$, and $|G| = 23$. We are also told that $|B \cap R| + |B \cap G| + |R \cap G| = 17$ and that $|B \cap R \cap G| = 0$. Therefore by the inclusion–exclusion principle we know that $|B \cup R \cup G| = 13 + 25 + 23 - 17 + 0 = 44$ people won ribbons. Since there are 100 competitors in all, the remaining 56 won no ribbons.

5. $\displaystyle\sum_{i=0}^{n-1} C(n,i)(-1)^i(n-i)^k$

7. The prime numbers less than or equal to 40 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, and 37, for a total of 12. For the calculation based on Theorem 3, we note that since $\sqrt{40} < 7$, we need only consider the primes 2, 3, and 5. Then the number of prime numbers not exceeding 40 is

$$\begin{aligned} &39 - \left(\left\lfloor\frac{40}{2}\right\rfloor - 1\right) - \left(\left\lfloor\frac{40}{3}\right\rfloor - 1\right) - \left(\left\lfloor\frac{40}{5}\right\rfloor - 1\right) \\ &\qquad + \left\lfloor\frac{40}{2\cdot 3}\right\rfloor + \left\lfloor\frac{40}{2\cdot 5}\right\rfloor + \left\lfloor\frac{40}{3\cdot 5}\right\rfloor - \left\lfloor\frac{40}{2\cdot 3\cdot 5}\right\rfloor \\ &= 39 - 19 - 12 - 7 + 6 + 4 + 2 - 1 = 12 . \end{aligned}$$