A necklace problem

We have $8n$ distinct beads. The task is to make 6 regular necklaces, each consisting of n beads, and one long necklace with $2n$ beads. In addition, we have two regular (identical) clasps, two special (identical) clasps that are asymmetrical, one symmetrical clasp with label 1 written on it (on both sides), and one symmetrical clasp with label 2 written on it (on both sides). How many ways can we make these necklaces subject to the following conditions?

- Two regular necklaces, each with an asymmetrical clasp.
- Two regular necklaces, each with a regular clasp.
- Two regular necklaces, each with a symmetrical clasp that has a label.
- One long necklace with no clasp.

See solution on the next page.

Solution

We follow several steps.

(1) We first choose n beads from $8n$ beads for the "first" regular necklace with an asymmetrical clasp in $\binom{8n}{n}$ ways. Then the necklace can be formed in $n!$ ways. We note that there is no need to divide $n!$ by 2 as the clasp is asymmetrical.

(2) Now we have $7n$ beads left. We choose n beads from the $7n$ remaining beads for the "second" regular necklace with an asymmetrical clasp in $\binom{7n}{n}$ ways. Then the necklace can be formed in $n!$ ways.

(3) Now we have $6n$ beads left. We choose n beads from the $6n$ remaining beads for the "first" regular necklace with a regular clasp in $\binom{6n}{n}$ ways. Then the necklace can be formed in $n!/2$ ways.

(4) We choose n beads from the $5n$ remaining beads for the "first" regular necklace with a regular clasp in $\binom{5n}{n}$ ways. Then the necklace can be formed in $n!/2$ ways.

(5) We choose n beads from the $4n$ remaining beads for the regular necklace with a symmetrical clasp with label 1 in $\binom{4n}{n}$ ways. Then the necklace can be formed in $n!/2$ ways.

(6) We choose n beads from the $3n$ remaining beads for the regular necklace with a symmetrical clasp with label 2 in $\binom{3n}{n}$ ways. Then the necklace can be formed in $n!/2$ ways.

(7) We choose $2n$ beads from the $2n$ remaining beads for the long necklace with no clasp in $\binom{2n}{2n} = 1$ ways. Then the necklace can be formed in $(2n-1)!/2$ ways.

So it seems that the answer is

$$\binom{8n}{n}\binom{7n}{n}\binom{6n}{n}\binom{5n}{n}\binom{4n}{n}\binom{3n}{n}(n!)^2\left(\frac{n!}{2}\right)^4\frac{(2n-1)!}{2}.$$

However, we have overcounted. For the two regular necklaces, each with an asymmetrical clasp, we cannot distinguish which one is "first" and which one is "second," so we overcounted by a factor of $2! = 2$. Similarly for the two regular necklaces, each with a regular clasp, we overcounted by a factor of $2! = 2$. Note that there is no such issue with the necklaces with a labelled clasp as we can tell them apart as the clasps are labelled. So the correct answer is

$$\frac{1}{2\cdot 2}\binom{8n}{n}\binom{7n}{n}\binom{6n}{n}\binom{5n}{n}\binom{4n}{n}\binom{3n}{n}(n!)^2\left(\frac{n!}{2}\right)^4\frac{(2n-1)!}{2}.$$

Compressed Solution For Tests/Exams

We follow several steps.

(1) We first choose n beads from $8n$ beads for the "first" regular necklace with an asymmetrical clasp in $\binom{8n}{n}$ ways. Then the necklace can be formed in $n!$ ways. We note that there is no need to divide $n!$ by 2 as the clasp is asymmetrical.

(2) Now we have $7n$ beads left. We choose n beads from the $7n$ remaining beads for the "second" regular necklace with an asymmetrical clasp in $\binom{7n}{n}$ ways. Then the necklace can be formed in $n!$ ways.

(3) We continue in this manner noting that there are $n!/2$ ways to form a necklace with a clasp using n given beads, and there are $(2n-1)!/2$ ways to form the long necklace with no clasp.

So it seems that the answer is

$$\binom{8n}{n}\binom{7n}{n}\binom{6n}{n}\binom{5n}{n}\binom{4n}{n}\binom{3n}{n}(n!)^2\left(\frac{n!}{2}\right)^4\frac{(2n-1)!}{2}.$$

However, we have overcounted. For the two regular necklaces, each with an asymmetrical clasp, we cannot distinguish which one is "first" and which one is "second," so we overcounted by a factor of $2! = 2$. Similarly for the two regular necklaces, each with a regular clasp, we overcounted by a factor of $2! = 2$. So the correct answer is

$$\frac{1}{2\cdot 2}\binom{8n}{n}\binom{7n}{n}\binom{6n}{n}\binom{5n}{n}\binom{4n}{n}\binom{3n}{n}(n!)^2\left(\frac{n!}{2}\right)^4\frac{(2n-1)!}{2}.$$