APM 2663

Sun 20 Oct 2024

Optional Feedback 3

38. Prove: $n^3/3 + n^5/5 + 7n/15$ is integer for all $n \in \mathbb{N}$. $(n \geq 0)$

Proof by induction.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$(n=0)$ base case: $0+0+0=0$. ✓

inductive step: WTS $P(k) \Rightarrow P(k+1)$.

$$(k+1)^3/3 + (k+1)^5/5 + 7(k+1)/15 = \tfrac{1}{3}(k^3+3k^2+3k+1) + \tfrac{1}{5}(k^5+5k^4+10k^3+10k^2+5k+1) + 7(k+1)/15.$$

$$= \underbrace{\left(\frac{k^3}{3} + \frac{k^5}{5} + \frac{7k}{15}\right)}_{①} + \underbrace{\left(\frac{3k^2+3k}{3} + \frac{5k^4+10k^3+10k^2+5k}{5}\right)}_{②} + \underbrace{\left(\frac{1}{3} + \frac{1}{5} + \frac{1}{15}\right)}_{③}$$

① Is the induction hypothesis (with $n=k$). [Presumed to be an integer]
② Is clearly an integer
③ Is equal to 1.

Thus, ①+②+③ is an integer, and the k+1 case holds.
Hence, the original claim is true by induction, and it's always an integer. □

✓