Recall: Let R be an relation on the ground set A.

If R is reflexive, antisymmetric and transitive, then R is a <u>POSET</u> (partially ordered set)

We have seen examples of a poset before.

Basic examples.

Let $☺ = \{1, 2, 3\}$.

Let $A = P(☺)$. Recall $|A| = 2^3 = 8$

Let $X, Y \in A$. Define $X R Y$ iff $X \subseteq Y$. [Note: X and Y are elements of $P(☺)$, so subsets of ☺]

Claim: R is a POSET on A.

(1) reflexive?

Let $X \in A = \mathcal{P}(☺)$.

Is $X \subseteq X$? Yes. $\therefore XRX$.

(2) antisym?

Let $X, Y \in A$. Suppose

XRY and YRX. WTS $X = Y$.

Now XRY and YRX

$\Rightarrow X \subseteq Y$ and $Y \subseteq X$

$\therefore X = Y$ (from what we know about sets)

(3) transitive?

Let $X, Y, Z \in A$

Suppose XRY and YRZ.

WTS XRZ.

i.e. WTS.

$X \subseteq Y$ and $Y \subseteq Z \implies X \subseteq Z$.

[We know this is true from earlier chapter on sets]

$\therefore$ R is a POSET of A

Wait! Something is fishy. We did not explicitly use the fact that ☺ $= \{1, 2, 3\}$

This is because the proof works for every set ☺.

Let's continue working on

$$☺ = \{1, 2, 3\}.$$

Then $A = \mathcal{P}(X)$ consist of the following elements

$$\{1, 2, 3\}$$

$$\{1, 2\} \qquad \{1, 3\} \qquad \{2, 3\}$$

$$\{1\} \qquad \{2\} \qquad \{3\}$$

$$\emptyset$$

Next we want to put an arrow from X to Y if $X \subseteq Y$.

Since R is reflexive, we will always have

$$\begin{tikzcd} X \arrow[loop right] \end{tikzcd}$$

Stupid to have this arrow explicitly when we know it is there. So skip it.

Since R is transitive, if we have $X \subseteq Y$ and $Y \subseteq Z$, then $X \subseteq Z$.

So

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend right] & Y \arrow[r] & Z \end{tikzcd}$$

stupid to have the blue arrow explicitly when we knows it is there.

So skip it.

∴ Now we can draw such a diagram

I am lazy, I don't want to draw the arrow heads.

If all the arrows are pointing upwards in the diagram as we have done, then I don't need the arrow heads.

Yes. this can be done.

Note that we cannot have something like

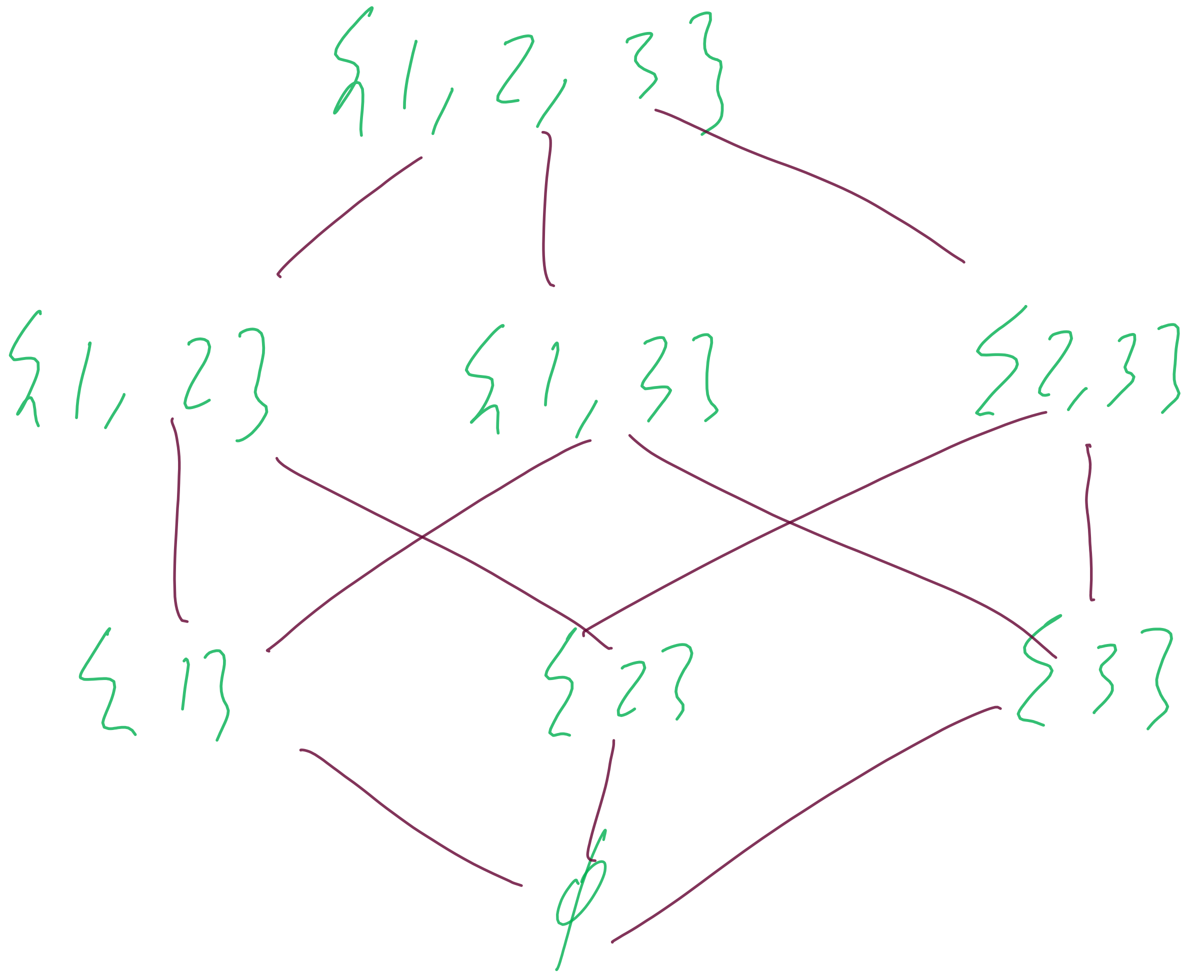
$$\underbrace{X_1 \subseteq X_2 \subseteq X_3}_{X_1 \subseteq X_3} \subseteq X_1 .$$

$\therefore X_1 \subseteq X_3 , \; X_3 \subseteq X_1$

$\Rightarrow X_1 = X_3$

So for the above example,

we have



If A is finite, we can draw it explicitly.

↳ Hasse diagram

↳ This will look familiar when you learn DAG (Directed acyclic digraphs) in APM/CSI 3610

For POSET, we often use $\preceq$ instead of R for the symbol of such a relation.

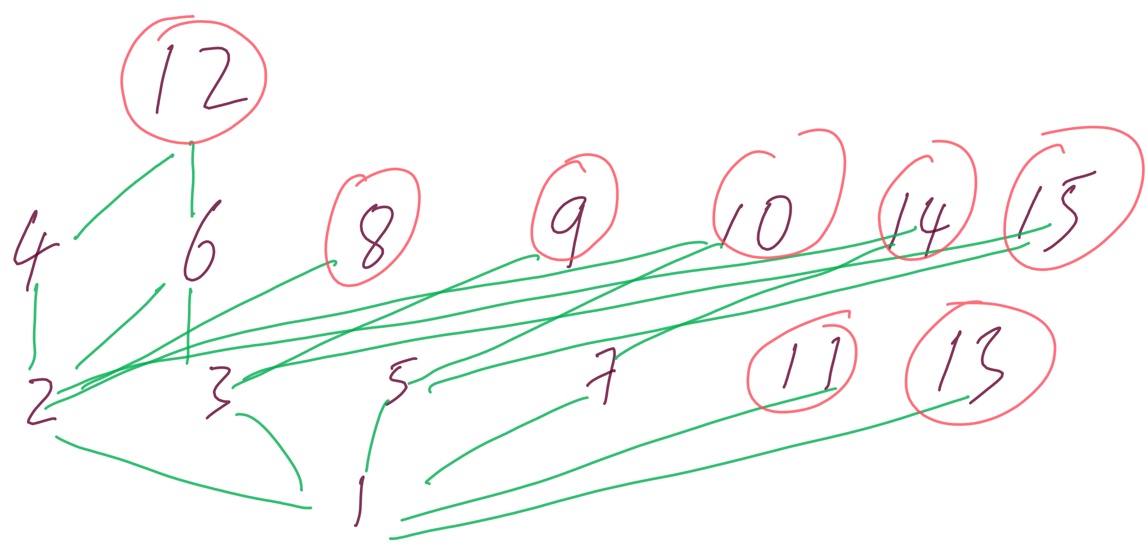
NOTE: $\mathbb{R}$ on $\leq$ is a POSET

Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$

Let $x, y \in A$. Define $x \preceq y$ if

$x \mid y$

Check that this is a POSET



Def: Let $\preceq$ be a POSET on A.

(a maximal element) largest element

x is <u>maximal</u> in $\preceq$ if

$x \preceq y \Rightarrow y = x$

[i.e. there is no element "larger" than x]

maximal elements are circled in red.

Def: Let $\preceq$ be a POSET on A

(a minimal element) smallest element

x is <u>minimal</u> in $\preceq$ if

$y \preceq x \Rightarrow y = x$

[i.e. there is no element "smaller" than x]

Def: Let $\preceq$ be a POSET on A.

x is the <u>maximum</u> in $\preceq$

if $y \neq x \Rightarrow y \preceq x$

Note: Yes. If a maximum exists, then it is unique.

Proof: Suppose x, y both are maximum. Then

$x \preceq y$ since y is maximum

and $y \preceq x$ since x is maximum.

$\therefore x = y$ since $\preceq$ is transitive.

Def: Let $\preceq$ be a POSET on A.

x is the minimum in $\preceq$

if $y \neq x \Rightarrow y \succeq x$

Note: If a minimum exists,

then it is unique.

Defn: Let $\preceq$ be a POSET on A.

x and y are comparable

if $x \preceq y$ or $y \preceq x$; otherwise,

they are incomparable

NOTE: $\preceq$ POSET on A

$+$

x and y are comparable

$\forall\, x, y \in A$

linear order

$\hookrightarrow$ $\preceq$ is a total order on A

NOTE: $\preceq$ is a total order on A

$+$

$\forall\, B \subseteq A$, $B \neq \emptyset$, B has a smallest element

$\hookrightarrow$ $\preceq$ is a well-ordered set.

Learn more about this

in MTH 3002.