Q28 on Page 203

$\mathbb{Z}$ ground set

$x \sim y$ if $3 \mid x^2 - y^2$

Show that $\sim$ is an equiv rel on $\mathbb{Z}$

Moreover, find all equiv classes.

(I) Reflexive?

Let $x \in \mathbb{Z}$ WTS $x \sim x$

Need to check $3 \mid \underbrace{x^2 - x^2}_{0}$

Yes we know $3 \mid 0$.

$\therefore$ $\sim$ is reflexive.

(II) Symmetric?

Let $a, b \in \mathbb{Z}$

Suppose $a \sim b$

$\Rightarrow 3 \mid a^2 - b^2$

$\Rightarrow 3 \mid -(a^2 - b^2)$

$\Rightarrow 3 \mid -a^2 + b^2$

$\Rightarrow 3 \mid b^2 - a^2$

$\therefore b \sim a$

$\therefore \sim$ is sym.

Recall

$3 \mid$ (1)

$\Rightarrow 3 \mid -$ (1)

(III) transitive?

Suppose $a \sim b$ and $b \sim c$

[$a, b, c \in \mathbb{Z}$]

$\therefore 3 \mid a^2 - b^2$ and $3 \mid b^2 - c^2$

$\Rightarrow 3 \mid 1(a^2 - b^2) + 1(b^2 - c^2)$ by E's Lemma

$\Rightarrow 3 \mid a^2 - b^2 + b^2 - c^2$

$\Rightarrow 3 \mid a^2 - c^2$

$\therefore a \sim c$

$\therefore \sim$ is transitive.

$\therefore \sim$ is an equiv rel on $\mathbb{Z}$

Find all the equiv classes wrt to $\sim$ $3 | b^2 - 1$

$\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, \ldots$

$$[0]_\sim = \{ b \in \mathbb{Z} \mid 0 \sim b \}$$

$$= \{ b \in \mathbb{Z} \mid 3 | b^2 - b^2 \}$$

$$= \{ b \in \mathbb{Z} \mid 3 | -b^2 \}$$

$$= \{ b \in \mathbb{Z} \mid 3 | b^2 \}$$

$$= \{ \ldots, -3, 0, 3, 6, \ldots \}$$

guess $= \{b \in \mathbb{Z} \mid 3 \mid b\}$

$[1]_N = \{b \in \mathbb{Z} \mid 1 \sim b\}$

$= \{b \in \mathbb{Z} \mid 3 \mid 1 - b^2\}$

$= \{b \in \mathbb{Z} \mid 3 \mid b^2 - 1\}$

guess $= \{b \in \mathbb{Z} \mid 3 \nmid b\}$

Only two equiv classes mod N if our guesses are correct.

Every integer can be written in one of these forms:

$$3k,\ 3k+1,\ 3k+2$$

Now

Claim $3k \in [0]_n$

Check: $3k \in [0]_n \iff 0 \sim 3k$

$\iff 3 \mid 0^2 - (3k)^2$

Now $0^2 - (3k)^2 = -(3k)^2$

$= -9k^2$ ← This is divisible by 3.

Claim: $3k+1,\ 3k+2 \in [1]_n$

Want to check

$$3 \mid \underbrace{(3k+1)}^{2} - 1$$

$$(3k)^2 + 6k + \cancel{1} - \cancel{1}$$

$$9k^2 + 6k \leftarrow \text{multiple of } 3$$

and $3 \mid \underbrace{(3k+2)}^{2} - 1$

$$(3k)^2 + 12k + 4 - 1$$

$$9k^2 + 12k + 3 \leftarrow \text{multiple of } 3.$$