CHAPTER 6
ELEMENTARY COUNTING TECHNIQUES

SECTION 6.1 Fundamental Principles of Counting

1. **(a)** $2 \cdot 3 = 6$ **(b)** $\{(a,a),\ (a,b),\ (a,c),\ (b,a),\ (b,b),\ (b,c)\}$

3. **(a)** $2^5 = 32$ **(b)** $32 - 1 - 5 = 26$ (exclude no toppings, one topping)

5. If $A_1, A_2, \ldots, A_n$ are finite sets, with $n \geq 1$, then $|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdots |A_n|$.

7. **(a)** Each of the sets $\{(a,a),\ (b,b),\ (c,c),\ (d,d)\}$, $\{(a,a),\ (b,a),\ (c,a),\ (d,a)\}$, and $\{(a,a),$ $(b,c),\ (c,c),\ (d,a)\}$ is a function from X to Y.

(b) There are $5^4 = 625$ possible functions by the multiplication principle, because for each of the four elements in X we must choose one of the five elements of Y as its image under the function.

(c) There are n^k possible functions by the multiplication principle, because for each of the k elements in X we must choose one of the n elements of Y as its image under the function. (If $n = 0$, then there are no functions unless $k = 0$, in which case there is $0^0 = 1$ function, namely the empty set; if $k = 0$, then there is always one function, namely the empty set. These statements are consistent with our definitions that $0^k = 0$ for $k > 0$ and $n^0 = 0$ for all n.)

9. **(a)** $4^7 = 16{,}384$

(b) $4^6 = 4096$ (no choice on Friday)

(c) $4^7 - 3^7 = 14{,}197$ (exclude the number of ways in which beef is not served)

(d) $4^7 - 3^7 - 7 \cdot 3^6 = 9094$ (exclude the number of ways in which beef is served no times or just once; for the latter, we need to choose the day on which beef is to be served, then choose the main courses for the other six days)

(e) $4 \cdot 3^6 = 2916$ (for each day after the first, there are just three choices; if there had been a meal the day before this week began and the restriction were in effect the first day of the week, then the answer would be $3^7 = 2187$, and similar comments apply to parts (f) and (g))

(f) $4 \cdot 3 \cdot 2^5 = 384$ (there are four choices the first day, three the next day, and two each day after that)

(g) $(4 \cdot 3^6)/4 = 729$ (by symmetry, exactly 1/4 of the choices for part (e) have fish on Friday; alternatively, there are three choices for each of Saturday, Thursday, Wednesday, Tuesday, Monday, and Sunday, in that order, so the answer is 3^6)

(h) $4 \cdot (3 \cdot 2 \cdot 1)^2 = 144$ (choose Wednesday's main course first, and then work toward the ends of the week)

11. **(a)** $\aleph_0$ (there are an infinite number of such propositions—they can be arbitrarily long; but there are only countably many by Exercise 34b in Section 2.3, since there are only a finite number of propositions of each finite length)

(b) $2^{32} = 4{,}294{,}967{,}296$ (each of the 32 lines of the truth table can have either a T or an F)

(c) $2^{32} = 4{,}294{,}967{,}296$ (as in part (b))

13. To specify a second place ticket, we must first choose one of the state's numbers to omit from the ticket (this can be done in six ways), and then choose a replacement for it from among the unused numbers (this can be done in 34 ways). Therefore the answer is $6 \cdot 34 = 204$.

15. **(a)** 3^n

(b) $[n(n-1)/2] \cdot 2^{n-2}$, assuming that $n \geq 2$ (choose the positions for the a's, then choose the remaining letters from left to right)

(c) $3^n - n \cdot 2^{n-1} - 2^n$, assuming that $n \geq 2$ (exclude strings with one a or no a's)

(d) $[n(n-1)/2] \cdot (2^{n-2} - 2)$, assuming that $n \geq 4$ (choose the positions for the a's, then fill the remaining positions, but exclude the possibility of all b's or all c's)

17. **(a)** $8^5 = 32{,}768$

(b) $6^5 = 7776$ (only six letters are available)

(c) $8^5 - 6^5 = 24{,}992$ (exclude the strings that contain neither)

(d) $7^5 - 6^5 = 9031$ (only seven letters are available; exclude the strings that do not use a)

(e) $6^5 + (7^5 - 6^5) + (7^5 - 6^5) = 25{,}838$ (using parts (b) and (d), since such a string either must contain neither a nor b, or else must contain exactly one of them)

(f) $8^5 - 25838 = 6930$ (exclude the strings counted in part (e))

(g) $8^4 + 8^4 - 8^3 = 7680$ (the other four positions are free; the subtraction is because of having counted twice the strings that both begin and end with e)

(h) $5 \cdot 7^4 = 12{,}005$ (choose the position for the c, then fill in the remaining positions with letters other than c)

19. **(a)** $8 \cdot 9^4 = 52{,}488$ (choose digits from left to right)

(b) $9 \cdot 10^4 - 8 \cdot 9^4 = 37{,}512$ (exclude those that do not contain 5)

(c) $5 \cdot 8 \cdot 8 \cdot 7 \cdot 6 = 13{,}440$ (choose the ones' digit, then the ten thousands', then the remaining digits)

(d) $9^5 = 59{,}049$ (choose the digits from left to right)

21. **(a)** $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 10! = 3{,}628{,}800$

(b) $(6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \cdot (4 \cdot 3 \cdot 2 \cdot 1) = 6!4! = 17{,}280$

(c) $(6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \cdot (7 \cdot 6 \cdot 5 \cdot 4) = 604{,}800$ (after the men have lined up relative to themselves, choose the places for the women among the seven gaps created by the men)

(d) 0

23. **(a)** 2 (101010... or 010101...)

(b) $2^n - 2^{n-2}$ (there are 2^{n-2} strings in which the first and last bits *are* both 1's)

(c) n (choose the location for the 1)

(d) $n(n-1)/2$ (choose the locations for the 1's)

(e) $2^n/2 = 2^{n-1}$ (by symmetry)

25. **(a)** $2^{|A|}$ (for each element of A, either include it in the subset or not)

(b) $2^{|B|+|C|}$ (since $|B \cup C| = |B| + |C|$)

(c) $2^{|B|} \cdot 2^{|C|}$ (choose a subset of B, then a subset of C; note that this is the same as in part (b))

(d) $2^{|A|} - |A| - 1$ (exclude singleton sets and the empty set)

27. First we note that $1500/3 = 500$ numbers have the factor 3 in common with 15. Similarly $1500/5 = 300$ numbers have the factor 5 in common with 15. Now we have counted twice the numbers that are multiples of 15, so we subtract the overcount, $1500/15 = 100$, from the sum, obtaining the answer 700.

29. First we will count the number of such strings in which there are at least two 1's and all the 1's are consecutive. To specify such a string we need to choose the starting and ending positions for the substring of 1's. We can do this in $20 \cdot 19/2 = 190$ ways. Similarly, there are 190 allowable strings in which there are at least two 0's and all the 0's are consecutive. Both of these counts include strings of the form 0^r1^{20-r} and 1^r0^{20-r}, with $2 \le r \le 18$, of which there are $2 \cdot 17 = 34$. This gives us $190 + 190 - 34 = 346$ strings so far. In addition, there are 18 strings containing one 1 in which the 0's are not consecutive, and 18 strings containing one 0 in which the 1's are not consecutive. This gives us a total of $346 + 18 + 18 = 382$.

31. We can count these most easily if we imagine building the sequences from the back, rather than from the front. There are nine decisions to make as we make our way from the back to the front of the sequence: As we add a new number, should it be the largest of the remaining numbers or the smallest? Thus there are $2^9 = 512$ possible sequences.

33. **(a)** There are two ways: $B{-}SS{-}SSS$ or $BS{-}SS{-}SS$.

(b) There are six ways to label the boxes in the first way of distributing the marbles given in the solution to part (a), and there are three ways to label the boxes in the second. Thus the answer is $6 + 3 = 9$.

(c) In the first of the solutions for part (a), we can choose the two small marbles that go into a box by themselves in $5 \cdot 4/2 = 10$ ways. In the second, we can choose the small marble to go with the big marble in five ways, after which there are three ways to pair up the remaining small marbles. Thus the answer is $10 + 5 \cdot 3 = 25$.

(d) Once we have made the decisions in part (c), we need to label the boxes, and there are six ways to do so. Therefore the answer is $25 \cdot 6 = 150$.

35. There are 29 ways to choose a partner for the oldest student. There are then 27 ways to choose a partner for the oldest unpaired student, etc. Thus the answer is $29 \cdot 27 \cdot 25 \cdots 3 \cdot 1 \approx 6.2 \times 10^{15}$. (See also Exercise 24 in Section 7.1.)

SECTION 6.2 Permutations and Combinations

1. **(a)** $7!/4! = 7 \cdot 6 \cdot 5 = 210$ **(b)** $7!/6! = 7$ **(c)** $7!/0! = 7! = 5040$
(d) $7!/1! = 7! = 5040$ **(e)** $10!/2! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 1{,}814{,}400$

3. **(a)** $P(5,2) = 5 \cdot 4 = 20$

(b) 13, 15, 17, 19, 31, 35, 37, 39, 51, 53, 57, 59, 71, 73, 75, 79, 91, 93, 95, 97

(c) $C(5,3) = 5 \cdot 4/2 = 10$

(d) 135, 137, 139, 157, 159, 179, 357, 359, 379, 579

5. **(a)** $C(10,2) = 45$ **(b)** $C(10,6) = 210$ **(c)** $C(10,0) = 1$ **(d)** $C(10,1) = 10$

7. $C(5,1) + C(5,2) = 5 + 10 = 15$

9. $C(9,2) + C(9,3) + C(9,5) + C(9,7) = 36 + 84 + 126 + 36 = 282$

11. $P(21,3) + C(21,3) = 7980 + 1330 = 9310$ (there were two cases to consider)

13. **(a)** $C(39,9) = 211{,}915{,}132$

(b) $C(10,3) \cdot C(12,3) \cdot C(17,3) = 120 \cdot 220 \cdot 680 = 17{,}952{,}000$

(c) $C(10,9) + C(12,9) + C(17,9) = 10 + 220 + 24310 = 24{,}540$

(d) $C(29,9) + C(27,9) - C(17,9) = 10015005 + 4686825 + 24310 = 14{,}677{,}520$ (the subset either contains only adult women and children, or it contains only adult men and children; those choices in which there are only children have been counted twice)

15. There are two ways to look at this. We can note that there are $C(80,11) = 10{,}477{,}677{,}064{,}400$ possible ways for the state's drawing to turn out, and there are $C(73,4) = 1{,}088{,}430$ of these in which the state has picked the player's seven numbers and four others. Therefore the answer is $1088430/10477677064400$, or about 1 in 9,626,413. Alternatively, we can view the state as having picked its numbers first; then the player can pick his numbers in $C(80,7) = 3{,}176{,}716{,}400$ ways, of which $C(11,7) = 330$ will result in a winning ticket. Therefore the chances of winning are $330/3176716400$, which has the same value as that previously obtained.

17. **(a)** $C(12,5) \cdot 3^7 = 1{,}732{,}104$ (choose the positions for the a's, then choose the symbols for the remaining positions, from left to right)

(b) $C(12,3) \cdot C(9,3) \cdot C(6,3) \cdot C(3,3) = 369{,}600$ (choose the positions for the a's, then choose the positions for the b's, and so on)

(c) $C(12,5) \cdot C(7,4) \cdot 2^3 = 221{,}760$ (choose the positions for the a's, then choose the positions for the b's, then fill the remaining positions from left to right)

(d) $C(12,5) \cdot 3^7 + C(12,4) \cdot 3^8 - 221760 = 4{,}758{,}039$ (subtract the overcount of those strings that have exactly five a's and exactly four b's, computed in part (c))

19. 1 (there is only one outcome—k of the boxes will have a ball and the rest will not)

21. Think of gluing a pair of 1's to the right of each 0. Then there are four 011's and 10 remaining 1's to form a string with. We just need to choose the four locations in this string of 14 things to receive the 011's, so the answer is $C(14,4) = 1001$.

23. $12!/12 = 11! = 39{,}916{,}800$ (by the circular symmetry)

25. **(a)** $12!/2 = 239{,}500{,}800$ (the rectangular table has two-fold rotational symmetry)

(b) $12!/4 = 119{,}750{,}400$ (the square table has four-fold rotational symmetry)

27. Since we just have to choose the positions for the n left parentheses among the $2n$ positions available, the answer is $C(2n,n)$. (The question is much harder if the parentheses have to make sense in order as grouping symbols, so that, for example, ())(is not allowed; see Exercise 42 in Section 7.2.)

29. Once we know which elements are in $A \cup B$, we automatically know which are in each of the two subsets. Therefore the answer is $C(n, k+l)$.

31. Since we need to choose a 0 or a 1 for each entry in a matrix with n^2 entries, the answer is 2^{n^2}.

33. Of the 2^{16} bit strings, $C(16,8)$ have an equal number of 0's and 1's. By symmetry, half of the rest have more 0's than 1's. Therefore the answer is $(2^{16} - C(16,8))/2 = 26{,}333$.

35. **(a)** $C(52,13) = 635{,}013{,}559{,}600$

(b) $C(13,4) \cdot C(13,4) \cdot C(13,3) \cdot C(13,2) = 11{,}404{,}407{,}300$ (choose the card from the suits one suit at a time)

(c) $C(4,2) \cdot C(13,4)^2 \cdot 2C(13,3) \cdot C(13,2) = 136{,}852{,}887{,}600$ (choose the two suits from which the four cards are to come, choose the cards from each of these suits, choose the suit from which three cards are to come, choose the cards from this suit, choose the cards from the last suit)

(d) none (the hand needs 13 cards in all)

(e) There are 32 cards no better than a 9. Thus there are $C(32,13) = 347{,}373{,}600$ hands that qualify as a Yarborough. The chances of holding a Yarborough are therefore $C(32,13)/C(52,13)$, or about 1 in 1828.

37. The number of such tickets is $C(6,3) \cdot C(34,3) = 119680$. The total number of tickets is $C(40,6) = 3838380$. Therefore the chance of holding such a ticket is $119680/3838380$, or about 1 in 32.

39. **(a)** $C(n-k, k)$ (arrange $n-k$ things consisting of the k couples and $n-2k$ blank seats; the only choice is in which positions to put the couples)

(b) $C(n-k, k) \cdot k! = P(n-k, k)$ (once it has been determined which seats will be occupied, which is what we counted in part (a), assign an order to the couples)

(c) $C(n-k, k) \cdot k! \cdot 2^k$ (once it has been determined which seats will be occupied by which couples, which is what we counted in part (b), decide for each couple whether the younger person will sit on the older person's left or right)

41. **(a)** There are $C(4,2)$ ways to choose the two suits. There are $C(26,13)$ ways to choose cards from just these two suits. However, this includes the four cases in which all the cards came from just one suit (which do we not wish to include), and each such case is counted three times. Thus we have overcounted by 12. Therefore the answer is $C(4,2) \cdot C(26,13) - 12 = 62{,}403{,}588$.

(b) Since there are an odd number of cards in the hand, it is impossible for there to be an equal number of red cards and black cards. Therefore by symmetry the answer is $C(52,13)/2 = 317{,}506{,}779{,}800$.

(c) First let us count the number of hands with the same number of spades as hearts, and denote this number by S. Since such a hand can have anywhere from zero to six cards in each of these two suits, it is clear that $S = C(13,0)^2 \cdot C(26,13) + C(13,1)^2 \cdot C(26,11) + C(13,2)^2 \cdot C(26,9) + C(13,3)^2 \cdot C(26,7) + C(13,4)^2 \cdot C(26,5) + C(13,5)^2 \cdot C(26,3) + C(13,6)^2 \cdot C(26,1) \approx 1.12 \times 10^{11}$. Half of the remaining hands will have more spades than hearts, so the answer is $(C(52,13) - S)/2 \approx 2.6 \times 10^{11}$.

43. **(a)** $C(n,k)$ (choose the elements to be included in the subsequence)

(b) 2^n (the number of subsets of $\{1, 2, \ldots, n\}$)

(c) If $k = 0$ then there is just one subsequence (the empty sequence). Otherwise, the consecutive subsequence can start in positions $1, 2, \ldots, n-k+1$, a total of $n-k+1$ possibilities.

(d) There are $C(n,2)$ consecutive subsequences of length greater than 1, since to specify such a subsequence is to specify a pair of locations to be its beginning and end. In addition, there are n consecutive subsequences of length 1 and the empty sequence. Therefore the answer is $C(n,2) + n + 1 = (n^2 + n + 2)/2$.

SECTION 6.3 Combinatorial Problems Involving Repetitions

1. **(a)** $\dfrac{8!}{2!} = 20{,}160$ (because of the two e's)

(b) $\dfrac{12!}{2!3!2!} = 19{,}958{,}400$ (there are two m's, three a's, and two f's)

(c) $\dfrac{4!}{2!2!} = 6$ (there are two n's and two o's)

3. **(a)** There are $C(10+7-1,7) = C(16,7) = 11{,}440$ ways to choose a collection of seven dinners from 10 dinners, with repetitions allowed.

(b) There are 10^7 ways to choose an ordered sequence of seven dinners from 10 dinners, with repetitions allowed.

5. **(a)** There are six choices for each of the four positions, so the answer is $6^4 = 1296$.

(b) This time order is irrelevant, so we are asking for a collection of four items from a set of six items, with repetition allowed. There are $C(6+4-1,4) = C(9,4) = 126$ such collections.

7. **(a)** $3^3 = 27$

(b) $C(3+3-1,3) = C(5,3) = 10$

(c) This amounts to arranging the entire bag, so Theorem 1 applies, and the answer is $\frac{10!}{4!3!3!} = 4200$.

(d) 1 (take the whole bag)

(e) There are $3^5 = 243$ sequences of five colors using the colors W, R, and B. Not all of these correspond to actual choices from the bag, since there are not enough balls of each color. Let us count the sequences that are not allowed. There are three sequences using only one color, and these are not allowed, since there are not five balls of any one color. Furthermore, there are $2 \cdot 2 \cdot 5 = 20$ sequences that use either four R's or four B's (choose the color to use four of, choose the other color to be used, choose the position for the other color). Therefore the answer is $243 - 3 - 20 = 220$.

(f) We need to count the number of solutions to $w+r+b=5$, with $0 \le w \le 4$, $0 \le r \le 3$, and $0 \le b \le 3$. Without these restrictions, there are $C(3+5-1,5) = C(7,5) = 21$ solutions. There is one way for the restriction on w to be violated (take $w = 5$); and there are three ways each for the restriction on r or b to be violated (since then we are counting solutions to an equation of the form $x+y+z=1$). Therefore the answer is $21 - 1 - 2 \cdot 3 = 14$.

9. Let n be the length of the walk. We need r steps to the right and u steps up, $r-2$ steps left, and $u-3$ steps down, in order to reach $(2,3)$, where $r+u+(r-2)+(u-3) = n$. Thus we need $r \ge 2$, $u \ge 3$, and $2(r+u) = n+5$. This implies that n is odd and at least 5, and that

$$r = \frac{n+5}{2} - u \le \frac{n+5}{2} - 3 = \frac{n-1}{2}.$$

Thus r can have any value from 2 to $(n-1)/2$. We see also that $u = [(n+5)/2] - r$ and $u - 3 = [(n-1)/2] - r$. Reasoning now as in Example 3, we see that the number of walks from $(0,0)$ to $(2,3)$ is given by

$$\sum_{r=2}^{(n-1)/2} \frac{n!}{r!(r-2)!([(n+5)/2]-r)!([(n-1)/2]-r)!}.$$

We apply this formula to obtain the answers below.

(a) There is only one term, with $r = 2$, so the answer is $5!/(2!3!) = 10$.

(b) none (there are not enough steps to reach $(2,3)$; equivalently, our summation would run from 2 to 1, and hence would be empty—the empty sum is 0)

(c) Plugging into our formula with $n = 15$ we have

$$\frac{15!}{2!0!8!5!}+\frac{15!}{3!1!7!4!}+\frac{15!}{4!2!6!3!}+\frac{15!}{5!3!5!2!}+\frac{15!}{6!4!4!1!}+\frac{15!}{7!5!3!0!} = 19{,}324{,}305\,.$$

(d) none (n must be odd)

(e) The general formula for odd n is displayed before our answer to part (a); there are no such walks if n is even.

11. (a) $C(4+40-1,40) = C(43,40) = 12{,}341$

(b) $C(5+39-1,39) = C(43,39) = 123{,}410$ (the number of solutions to $x_1+x_2+x_3+x_4+x_5 = 39$, where $x_5 \geq 0$ is a slack variable)

(c) $C(4+32-1,32) = C(35,32) = 6545$ (the number of solutions to $x_1'+x_2'+x_3+x_4' = 32$, where $x_1' = x_1 - 1$, $x_2' = x_2 - 1$, and $x_4' = x_4 - 6$)

(d) $C(4+40-1,40) = C(43,40) = 12{,}341$ (the number of solutions to $x_1'+x_2+x_3+x_4' = 40$, where $x_1' = x_1 + 2$ and $x_4' = x_4 - 2$)

13. We seek the number of integer solutions to $x_1+x_2+x_3+x_4 = 25$ with $2 \leq x_i \leq 10$ for each i. By letting $x_i = 2 + x_i'$, we transform this into the problem of finding the number of natural number solutions to $x_1'+x_2'+x_3'+x_4' = 17$, with each $x_i' \leq 8$. Now without the restriction the number of solutions is $C(4+17-1,17) = C(20,17)$. There are four ways to violate the restriction: Any of the variables might be greater than or equal to 9 (but it is impossible for two of the conditions to be violated simultaneously, since $9+9 > 17$). The number of solutions with $x_1' \geq 9$ is the same as the number of solutions of $x_1''+x_2'+x_3'+x_4' = 8$, which is $C(4+8-1,8) = C(11,8)$. Therefore the answer to our problem is $C(20,17) - 4C(11,8) = 480$.

15. We can distribute the blue balls in $C(10+8-1,8)$ ways and the red balls in $C(10+9-1,9)$ ways. Therefore the answer is $C(17,8)\cdot C(18,9) = 1{,}181{,}952{,}200$.

17. By the multiplication principle, the number of permutations with repetitions allowed is just n^k.

19. All salaries are in terms of multiples of \$100,000. Let x_1 be the salary of the MVP shortstop, let x_2 be the salary of the ace relief pitcher, let x_3 through x_{15} be the salaries of the other first-string players, and let x_{16} through x_{25} be the salaries of the remaining players. Thus the conditions are $x_1 \geq 10$, $x_2 \geq 15$, $x_i \geq 2$ for $3 \leq i \leq 15$, and $x_i \geq 1$ for $16 \leq i \leq 25$.

(a) The number of solutions to the equation $x_1+x_2+\cdots+x_{25} = 200$ subject to the constraints is the number of natural number solutions to $x_1'+x_2'+\cdots+x_{25}' = 139$. This equals $C(25+139-1,139) = C(163,139) \approx 3.4\times 10^{28}$.

(b) As we saw in part (a), there are 139 units of salary free to be distributed. If all of it goes to the ace relief pitcher (who already has the highest minimum), then he will have a salary of $15 + 139 = 154$ units, i.e., \$15,400,000.

(c) As we saw in part (a), there are 139 units of salary free to be distributed. If all of it goes to one of the second string players, then he will have a salary of $1 + 139 = 140$ units, i.e., \$14,000,000.

(d) We may as well assume that players 17 through 25 each get their minimum one unit of salary and ignore them for the rest of the problem. This leaves 191 units for the remaining players, and we want to arrange for x_{16} to be as large as possible, under the requirement that $x_1 \geq 10$, $x_2 \geq 15$, and $x_i \geq x_{16}$ for all remaining i. Clearly x_{16} can be at most $\lfloor 191/16 \rfloor = 11$, for otherwise there will not be enough money to go around. Furthermore, we can achieve $x_{16} = 11$ by setting $x_i = 11$ for $i \neq 2$, and $x_2 = 26$. Thus the answer is \$1,100,000.

21. **(a)** The largest part can have size $1, 2, \ldots, \lceil k/2 \rceil$. Thus the answer is $\lceil k/2 \rceil$.

(b) Choose a subset S other than the k-set A itself or $\emptyset$. There are $2^k - 2$ ways to do this. The parts of our partition are S and $A - S$. This overcounts the number of partitions by a factor of 2, since each partition can arise by choosing either S or $A - S$. Therefore the answer is $2^{k-1} - 1$. (We assume that $k \geq 2$; otherwise the answer is clearly 0.)

23. We will first count the number of subsets of six numbers from $\{1, 2, \ldots, 40\}$, such that there are no adjacent numbers in the subset. Let $a_1, a_2, \ldots, a_6$ be the six numbers. Let $x_1 = a_1 + 1$, $x_2 = a_2 - a_1$, $x_3 = a_3 - a_2$, $x_4 = a_4 - a_3$, $x_5 = a_5 - a_4$, $x_6 = a_6 - a_5$, and $x_7 = 42 - a_6$. Then the sum $x_1 + x_2 + \cdots + x_7$ telescopes to equal 43, and the given conditions on the a_i's translate exactly into the condition that each $x_i \geq 2$. By the usual trick, the number of solutions to this equation is same as the number of natural number solutions to $x_1' + x_2' + \cdots + x_7' = 29$, which is $C(7 + 29 - 1, 29) = C(35, 29) = 1623160$. Now since there are $C(40, 6) = 3838380$ subsets in all, we see that the remaining 2215220 subsets have at least one pair of adjacent elements. Therefore the answer to the question is $2215220/3838380 \approx 58\%$.

25. **(a)** Any subset of the 10 numbered balls gives us one choice (we choose enough unnumbered balls to bring the total to 10). Thus the answer is 2^{10}.

(b) $20!/10! = 670{,}442{,}572{,}800$ by Theorem 1

(c) There are $C(10, 10 - i) \cdot (10!/i!)$ ways to make a selection that uses i unnumbered balls (choose the $10 - i$ numbered balls to include, then choose a permutation of the 10 chosen balls, of which i are identical). Since i can take any value from 0 to 10, the answer is

$$\sum_{i=0}^{10} \frac{10!^2}{i!^2(10-i)!},$$