6. (a) Define $L : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ by $L(A) = A + A^T$.

(a) Show that L is a linear operator.

We need to check

$$L(\alpha A + \beta B) = \alpha L(A) + \beta L(B)$$

for any scalars α, β and $A, B \in \mathbb{R}^{2\times 2}$

$$\underline{LHS} = \alpha A + \beta B + (\alpha A + \beta B)^T$$
$$= \alpha A + \beta B + \alpha A^T + \beta B^T$$

$$RHS = \alpha(A + A^T) + \beta(B + B^T)$$
$$= \alpha A + \alpha A^T + \beta B + \beta B^T$$

the same

So $LHS = RHS$. L is a linear transformation.

(b) Find ker L and its dimension.

Solve $\underbrace{L(A) = 0}_{A + A^T = 0}$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}$

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & c \\ b & d \end{pmatrix}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2a & b+c \\ c+b & 2d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2a & b+c \\ c+b & 2d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\left.\begin{array}{l} 2a = 0 \to a = 0 \\ \boxed{\begin{array}{l} b+c=0 \\ c+b=0 \end{array}} \to b = -c \quad c = -b \\ 2d = 0 \to d = 0 \end{array}\right\} \to A = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$$

$$\operatorname{Ker} L = \left\{ \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \right\} = \operatorname{span} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

$$\to \dim(\operatorname{Ker} L) = 1.$$

(c) Find the matrix representation of L.

$$L(A) = \begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$E = \{\text{standard basis of } \mathbb{R}^{2\times 2}\}$

$$\to [A]_E = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}; \quad [L(A)]_E = \begin{pmatrix} 2a \\ b+c \\ c+b \\ 2d \end{pmatrix}$$

$$\begin{pmatrix} 2a \\ b+c \\ c+b \\ 2d \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$E = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

5. Determine whether the following are linear transformation in $C^1[-1,1]$, the set of all differentiable functions in [-1,1]:

(a) $L(f(x)) = x^2 + f(x)$ for $f \in C^1[-1,1]$.

Check $L(\alpha f + \beta g) = \alpha L(f) + \beta L(g)$

for any $\alpha, \beta \in \mathbb{R}$, f, g are functions.

$\underline{LHS} = x^2 + (\alpha f + \beta g)(x)$

$= x^2 + \alpha f(x) + \beta g(x)$

$\underline{RHS} = \alpha(x^2 + f(x)) + \beta(x^2 + g(x))$

$= \alpha x^2 + \alpha f(x) + \beta x^2 + \beta g(x)$.

B/c $\alpha x^2 + \beta x^2 = (\alpha+\beta)x^2 \neq x^2$, $LHS \neq RHS$.

* $E = \{v_1, v_2, \dots, v_n\}$ is a basis of V.

Any vector $v \in V$ has a coordinate $[v]_E = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$

if $v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$

1b) $E = \{\underbrace{(1,1)^T}_{v_1}, \underbrace{(1,2)^T}_{v_2}\}$

$X = (2,3)^T$.

Whether $[X]_E = (1,1)^T$ $\longrightarrow$ $x = 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (true)

(d) $L(x_1, x_2) = (x_1 x_2, x_2)$ is a linear operator from $\mathbb{R}^2$ to $\mathbb{R}^2$.

$L(0,1) = (0,1)$

$$\left.\begin{aligned} L(\underbrace{0,1}_{x}) &= (0,1) \\ L(\underbrace{1,2}_{y}) &= (2,2) \\ L(\underbrace{1,3}_{x+y}) &= (3,3) \end{aligned}\right\} \rightarrow \underbrace{L(x+y)}_{(3,3)} \neq \underbrace{L(x)+L(y)}_{\underbrace{(0,1)+(2,2)}_{(2,3)}}$$

not a linear transformation.

8. (5 pts for each) Given an 5×4 matrix A with $\text{rank}(A) = 4$.

(a) How many solutions are there for equation $Ax = 0$? Explain your answer.

$$\underbrace{\text{rank}(A)}_{\#\text{ leading variables}} + \underbrace{\text{nullity}(A_{m \times n})}_{\#\text{ free variable}} = n$$

$$4 + 0 = 4$$

A unique solution of $Ax = 0$.

(b) How many solutions are there for equation $A^T y = 0$? Explain your answer.

$$\underbrace{\text{rank}(A^T)}_{\underbrace{\text{rank}(A)}_{4}} + \text{nullity}(A^T_{n \times m}) = m$$

$$4 + 1 = 5$$

$\text{nullity}(A^T) = 1$

$\rightarrow$ one free variable $\rightarrow$ infinitely many solutions

7, A is an $6 \times n$ matrix of rank r.

7/3.6

A is an $6 \times n$ matrix of rank r.

(a) $n = 7,\ r = 5$. How many solutions of $Ax = b$?

$$\text{rank } A + \text{nullity}(A) = n :$$

$$\underbrace{\text{rank } A}_{5} + \underbrace{\text{nullity}(A)}_{2} = 7$$

$\longrightarrow$ <u>2 free variables</u>.

no solution

or

infinitely many solutions.

(c) $n = 5, \quad r = 5$

$$\text{rank}(A) + \underbrace{\text{nullity}(A)}_{0} = n$$

$$5 + 0 = 5$$

$\longrightarrow$ $\boxed{\text{no free variable}}$

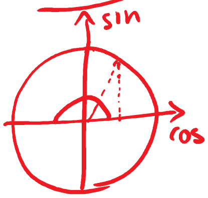
$$(A \mid b) \xrightarrow{\text{Gaussian}} \cdots\cdots \left(\begin{array}{ccccc|c} \ddots & & & & & \\ & & & & & \\ 0 & 0 & 0 & \cdots & 0 & \end{array}\right)_{6\times 5} \rightarrow \text{zero or nonzero.}$$

No solution or a unique solution.

$\sqrt{4} = 2$

$x^2 = 4$

$\to x = 2$ or -2



4. Given $v = (1, -1, 1, 1)^T$ and $w = (4, 2, 2, 1)^T$.

(a) Determine the angle between v and w.

$$\cos\theta = \frac{v^T w}{\|v\| \cdot \|w\|} = \frac{1\cdot 4 - 2 + 2 + 1}{\sqrt{1+1+1+1} \cdot \sqrt{16+4+4+1}} = \frac{5}{\sqrt{4}\,\sqrt{25}} = \frac{5}{2\cdot 5} = \frac{1}{2}$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$

(b) Find the orthogonal complement of $V = \text{span}\{v, w\}$.

$$V^\perp = \{x \mid x \perp v \text{ for any } v \in V\}$$

$$= \{x \mid x \perp v \text{ and } x \perp w\}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad \begin{array}{l} x \perp v \longrightarrow x^T v = 0 \longrightarrow x_1 - x_2 + x_3 + x_4 = 0 \\ x \perp w \longrightarrow x^T w = 0 \longrightarrow 4x_1 + 2x_2 + 2x_3 + x_4 = 0 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 4 & 2 & 2 & 1 & 0 \end{array}\right) \xrightarrow{R_2 - 4R_1} \left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 0 & 6 & -2 & -3 & 0 \end{array}\right)$$

$$\xrightarrow{R_2/6} \left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{2} & 0 \end{array}\right) \xrightarrow{R_1 + R_2} \left(\begin{array}{cccc|c} 1 & 0 & \frac{2}{3} & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{3} & -\frac{1}{2} & 0 \end{array}\right)$$

$$\begin{cases} x_3 = t \\ x_4 = s \end{cases} \longrightarrow \begin{array}{l} x_2 = \frac{1}{3}t + \frac{1}{2}s \\ x_1 = -\frac{2}{3}t - \frac{1}{2}s \end{array}$$

$$\longrightarrow x = \begin{pmatrix} -\frac{2}{3}t - \frac{1}{2}s \\ \frac{1}{3}t + \frac{1}{2}s \\ t \\ s \end{pmatrix} \longrightarrow V^\perp$$

$$V^\perp = \text{span}\left\{ \begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{pmatrix} \right\} \to \dim V^\perp = 2$$

3. Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & -3 & -2 \\ 3 & 3 & 0 & 2 \end{pmatrix}$.

(a) Find a basis of $N(A)$, row space of A, column space of A.

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 2 & 1 & -3 & -2 & 0 \\ 3 & 3 & 0 & 2 & 0 \end{array}\right) \xrightarrow{\;\cdots\;} \left(\begin{array}{cccc|c} 1 & 0 & -3 & -\frac{8}{3} & 0 \\ 0 & 1 & 3 & \frac{10}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

column 1, 2

$$\begin{cases} x_3 = t \\ x_4 = s \end{cases} \quad \cdots$$

$$RS(A) = \operatorname{span}\left\{ \left(1, 0, -3, -\tfrac{8}{3}\right), \left(0, 1, 3, \tfrac{10}{3}\right) \right\}$$

$$CS(A) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \right.$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

7. Let $\|\cdot\|$ be the Euclidean norm in $\mathbb{R}^n$. For any $x, y \in \mathbb{R}^n$,

$$\|x\| = \sqrt{x^T x}$$
$$\|x\|^2 = x^T x$$

(a) Show that $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2x^T y$

$$\begin{aligned} LHS &= (x+y)^T (x+y) \\ &= (x^T + y^T)(x+y) \\ &= x^T x + x^T y + y^T x + y^T y \\ &= \|x\|^2 + 2x^T y + \|y\|^2 \end{aligned}$$

RHS

(b) Show that $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

$$\|x\|^2 + \|y\|^2 + 2x^T y + \|x\|^2 + \|y\|^2 - 2x^T y$$

$$2\|x\|^2 + 2\|y\|^2$$

RHS.