Thursday, March 13, 2025 5:31 PM

4.1 Linear Transformation:

$$L : V \longrightarrow W$$

vector spaces

$$L(\alpha v_1 + \beta v_2) = \alpha L(v_1) + \beta L(v_2)$$

$$\forall\, v_1, v_2 \in V \text{ and } \forall\, \alpha, \beta \in \mathbb{R}$$

Def: If $L : V \to V$ is a linear trans, we say L to be a linear operator.

ex: $L : P_3 \longrightarrow P_3$

$$p \longmapsto p - 2p'$$

$$ax^2+bx+c \longmapsto \underbrace{ax^2+bx+c - 2(2ax+b)}_{ax^2+bx+c-4ax-2b \,\in\, P_3}$$

Check if L is a linear trans.: Pick any $p, q \in P_3$ and $\alpha, \beta \in \mathbb{R}$, we need to check

$$L(\alpha p + \beta q) = \alpha L(p) + \beta L(q).$$

LHS: $\alpha p + \beta q - 2(\alpha p + \beta q)'$

$\alpha p + \beta q - 2\alpha p' - 2\beta q'$

same

RHS: $\alpha L(p) + \beta L(q) = \alpha(p - 2p') + \beta(q - 2q')$

$= \alpha p - 2\alpha p' + \beta q - 2\beta q'$

$$= \alpha p - 2\alpha p' + \beta q - 2\beta q'$$

$$LHS = RHS$$

This tells us that L is linear operator.

<u>Observation</u>: $\boxed{L(f) = f'}$, the derivative is actually a linear transformation. Indeed,

$\forall f, g \in C'$ (differentiable functions)

$\forall \alpha, \beta \in \mathbb{R}$

$$L(\alpha f + \beta g) = (\alpha f + \beta g)'$$
$$= \alpha f' + \beta g'$$
$$= \alpha L(f) + \beta L(g)$$

<u>ex</u>: $L : C[a,b] \longrightarrow \mathbb{R}$

the set of all continuous function on $[a,b]$

$$: f \longmapsto \int_a^b f(x)\,dx.$$

L is a linear transformation.

<u>Sol</u>: Pick any $f, g \in C[a,b]$ and any $\alpha, \beta \in \mathbb{R}$

We need to check:

$$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g).$$

$\underline{\text{LHS}}$: $L(\alpha f + \beta g) = \int_a^b \alpha f(x) + \beta g(x)\, dx$

RHS: $\alpha L(f) + \beta L(g) = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$

$= \int_a^b \alpha f(x) + \beta g(x)\,dy$

same

LHS = RHS

$\longrightarrow$ L is a linear transformation.

$\underline{\text{Theorem}}$: Let A be an $m \times n$ matrix. Define

$$L: \mathbb{R}^n \longrightarrow \mathbb{R}^m, \quad x \longmapsto Ax \qquad \text{, i.e., } L(x) = Ax$$

is a linear trans.

$\underline{\text{Proof}}$: Pick any $x, y \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$,

we need to check $L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)$.

$\underline{\text{LHS}}$: $L(\alpha x + \beta y) = A(\alpha x + \beta y)$

$= \alpha A x + \beta A y$

$\underline{\text{RHS}}$: $\alpha L(x) + \beta L(y) = \alpha A x + \beta A y$

same

LHS = RHS

$L(x) = Ax$ is a linear trans.

ex : $L : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$

$$A \longmapsto \det(A).$$

Whether A is linear transformation ?

$$\det(\alpha A) = \alpha^n \det(A)$$

No b/c Pick $A = I$, $B = I$

$$\begin{aligned}
L(A+B) &= L(I+I) \\
&= L(2I) \\
&= \det\begin{pmatrix} 2 & 0 & \cdots & \\ 0 & 2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 2 \end{pmatrix} \\
&= 2^n.
\end{aligned}$$

$$\begin{aligned}
L(A)+L(B) &= \underbrace{\det I} + \det I \\
&= 1 + 1 \\
&= 2
\end{aligned}$$

$$\rightarrow L(A+B) \neq L(A) + L(B)$$

$$2^n \neq 2 \qquad (n > 1)$$

L is NOT a linear trans.

ex : $L : P_4 \rightarrow P_{\cancel{2}4}$

$$p \longmapsto p - 2xp''$$

is a linear transformation.

$$L : P_4 \longmapsto P_4$$
$$p \longmapsto 2p + x^2 p'' - x p'.$$

is a linear trans.

$$L : P_4 \longmapsto P_8$$
$$p \longmapsto p^2 - x p'$$

not linear in p.

The image and kernel:

<u>Def</u>: Let $L : V \to W$ be a linear transformation.

The kernel of L is defined by:

$$\text{Ker } L = \{ v \mid L(v) = 0_W \}$$

(null space of L)

vector zero in W.

<u>ex</u>: Let $L : P_3 \to P_3$
$$p \longmapsto 2p - x p'$$

Find $\text{Ker}(L)$.

<u>Sol</u>: Take $p \in \text{Ker } L$, i.e., $\underbrace{L(p)}_{2p - xp' = 0} = 0$

$= ax^2 + bx + c$

$$2ax^2 + 2bx + 2c - x(2ax + b) = 0$$
$$2ax^2 + 2bx + 2c - 2ax^2 - bx = 0$$

$$bx + 2c = 0 \quad \text{(for any } x\text{)}$$

$$b = 0 \text{ and } c = 0$$

$$\boxed{p(x) = ax^2}$$

$$\text{Ker } L = \{ ax^2 \mid a \in \mathbb{R} \}$$

$$= \text{span}\{x^2\} \longrightarrow \dim(\text{Ker } L) = 1$$

ex: Let $L : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$

$$A \longmapsto A + A^T.$$

Find Ker L and its dimension.

Sol: Solve $L(A) = 0$

$$A + A^T = 0$$

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} + \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = 0_{3\times 3}$$

$$\begin{pmatrix} 2a_1 & a_2 + b_1 & a_3 + c_1 \\ b_1 + a_2 & 2b_2 & b_3 + c_2 \\ c_1 + a_3 & c_2 + b_3 & 2c_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\left.\begin{array}{l} \boxed{a_1 = b_2 = c_3 = 0} \\ a_2 + b_1 = 0 \\ a_3 + c_1 = 0 \\ b_3 + c_2 = 0 \end{array}\right\} \longrightarrow A = \begin{pmatrix} 0 & a_2 & a_3 \\ -a_2 & 0 & b_3 \\ -a_3 & -b_3 & 0 \end{pmatrix}$$

$$\text{So} \quad \ker L = \left\{ \begin{pmatrix} 0 & a_2 & a_3 \\ -a_2 & 0 & b_3 \\ -a_3 & -b_3 & 0 \end{pmatrix} \middle| \, a_2, a_3, b_3 \in \mathbb{R} \right\}.$$

$$= \operatorname{span}\left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}$$

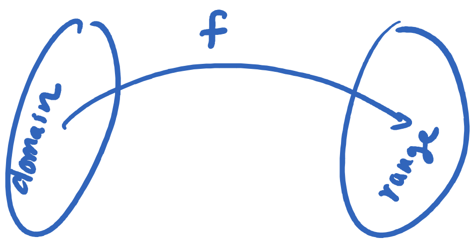
$\longrightarrow \dim \ker L = 3$.

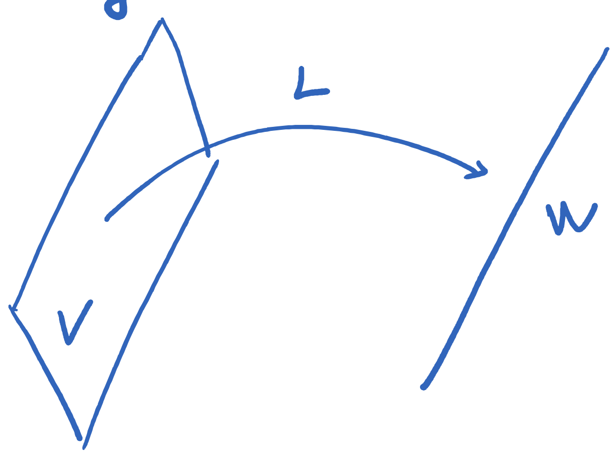
<u>Theorem</u>: $\ker L$ is a subspace of V when $L: V \to W$ is a linear transformation.

<u>Def</u>: Let $L: V \to W$ be a linear transformation and let S be a subspace of V. The image of S over L is denoted by $L(S)$ and defined by

$$L(S) = \{ L(v) \mid v \in S \}.$$

The range of L is defined by $L(V)$.





ex: $L: \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$
$A \longmapsto A + A^T$

Find the range of L and its dimension.

Sol: $\text{Range}(L) = \{ L(A) \mid A \in \mathbb{R}^{2\times 2} \}$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow L(A) = A + A^T$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$= \begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix}$$

$$= 2a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (b+c) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 2d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{Range}(L) = \text{span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$\longrightarrow \dim(\text{Range}(L)) = 3$.

ex: $L : P_3 \longrightarrow P_3$

$$p \longmapsto 2p - xp'.$$

Find Range (L) and its dimension.

Pick $p = ax^2 + bx + c$, $L(p) = 2p - xp'$

$$= 2ax^2 + 2bx + 2c - x(2ax + b)$$

$$= 2ax^2 + 2bx + 2c - 2ax^2 - bx$$

$$= bx + 2c.$$

$\text{Range } L = \{ bx + 2c \mid b, c \in \mathbb{R} \}$

$$= \operatorname{span}\{x, 2\} \longrightarrow \dim(\text{Range } L) = 2.$$