17b) $L(x) = (x_1, x_2, 0)^T$, $L: \mathbb{R}^3 \to \mathbb{R}^3$

<u>Kernel</u> : Solve $L(x) = 0$

$$\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow \begin{matrix} x_1 = 0 \\ x_2 = 0 \end{matrix}$$

$$\text{Ker } L = \{ x \mid L(x) = 0 \}$$

$$= \left\{ \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix} \mid x_3 \in \mathbb{R} \right\}$$

$$= \text{span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

<u>Range</u> $\text{Ran } L = \{ L(x) \mid x \in \mathbb{R}^3 \}$

$$= \left\{ \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}} \mid x_1, x_2 \in \mathbb{R} \right\}$$

$$= \left\{ \underbrace{x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} \mid x_1, x_2 \in \mathbb{R} \right\}$$

$$= \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

<u>4.2</u> Matrix representations of linear Transformation.

Suppose $L: V \to W$ is a linear transformation.

and $E = \{ \overbrace{v_1, v_2, \dots, v_n}^{n \text{ vectors}} \}$ forms a basis of V and

$F = \{ \underbrace{w_1, w_2, \dots, w_m}_{m \text{ vectors}} \}$ forms a basis of W.

For any $v \in V$, we can write $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ uniquely. We write $[v]_E = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$ - the coordinate of v with respect to E.

$$L(v) = L(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n)$$

$$\overset{\text{linearity}}{=} \alpha_1 L(v_1) + \alpha_2 L(v_2) + \cdots + \alpha_n L(v_n) \in W$$

$$[L(v)]_F = \alpha_1 [L(v_1)]_F + \alpha_2 [L(v_2)]_F + \cdots + \alpha_n [L(v_n)]_F$$

(m components)

$$= \underbrace{\begin{bmatrix} | & | & & | \\ [L(v_1)]_F & [L(v_2)]_F & \cdots & [L(v_n)]_F \\ | & | & & | \end{bmatrix}}_{A_{m \times n}} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

$$[L(v)]_F = A_{m \times n} [v]_E$$

Theorem: Let $L: V \to W$ be a linear transformation between 2 vectorspace V and W. Suppose that $E = \{v_1, v_2, \ldots, v_n\}$ is a basis of V and $F = \{w_1, w_2, \ldots, w_m\}$ is a basis of W. Then we have:

$$[L(v)]_F = A [v]_E,$$

where $A_{m \times n} = \begin{bmatrix} | & | & & | \\ [L(v_1)]_F & [L(v_2)]_F & \cdots & [L(v_n)]_F \end{bmatrix}$

where $A_{m\times n} = \left[[L(v_1)]_F \; [L(v_2)]_F \cdots [L(v_n)]_F \right]$

n columns .

ex: Let $L: P_3 \to P_3$

$$p(x) \mapsto p(x) - x\,p'(x)$$

Suppose $E = \{1, x, x^2\}$ is a basis of P_3. Find a matrix representation of L.

Sol: $p(x) = ax^2 + bx + c$

$$L(p) = p(x) - x\,p'(x)$$

$$= ax^2 + bx + c - x(2ax + b)$$

$$= ax^2 + bx + c - 2ax^2 - bx$$

$$= -ax^2 + c \qquad , \quad E = \{1, x, x^2\}$$

$$[L(p)]_E = \begin{pmatrix} c \\ 0 \\ -a \end{pmatrix} \quad \text{b/c} \quad -ax^2 + c = c\cdot 1 + 0\cdot x + (-a)\cdot x^2$$

$$[P]_E = [ax^2 + bx + c]_E$$

$$= \begin{pmatrix} c \\ b \\ a \end{pmatrix}$$

So the matrix representation

$$\begin{pmatrix} c \\ 0 \\ -a \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{\text{the matrix representation of } L.} \begin{pmatrix} c \\ b \\ a \end{pmatrix}$$

ex: Let $L: \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$

$$A \mapsto A + A^T$$

Find the matrix representation of this transformation with the $\underbrace{\text{standard basis of } \mathbb{R}^{2\times 2}}$.

$$E = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Sol: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$L(A) = A + A^T$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$= \begin{pmatrix} 2a & b+c \\ c+b & 2d \end{pmatrix} = 2a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (b+c)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + (c+b)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 2d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[L(A)]_E = \begin{pmatrix} 2a \\ b+c \\ c+b \\ 2d \end{pmatrix}$$

$$[A]_E = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$\begin{pmatrix} 2a \\ b+c \\ c+b \\ 2d \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}}_{\text{matrix representation}} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

Chapter 5: Orthogonality.

Chapter 5: Orthogonality.

5.1 The scalar product in $\mathbb{R}^n$

Given $x, y \in \mathbb{R}^n$, the scalar product (dot product) of x any y is defined by:

$$x^T y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

The "norm" (magnitude, length) of a vector x is defined by:

$$\|x\| = \sqrt{x^T x}. \quad \text{(Euclidean norm)}$$
$$= \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}. \quad (\ell_2\text{-norm})$$

The angle θ between 2 vectors x, y is defined by

$$\cos\theta = \frac{x^T y}{\|x\| \cdot \|y\|} \qquad 0 \leq \theta \leq \pi.$$

Cauchy-Schwarz inequality:

$$-1 \leq \frac{x^T y}{\|x\| \cdot \|y\|} \leq 1 \quad \text{or} \quad x^T y \leq \|x\| \cdot \|y\|$$

ex: let $x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $y = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$. Find the angle between x, y.

Sol: $\cos\theta = \dfrac{x^T y}{\|x\| \cdot \|y\|}$

$$x^T y = 1 \cdot 2 + 2 \cdot (-1) + 3 \cdot 1$$
$$= 2 - 2 + 3$$
$$= 3$$

$$\|x\| = \sqrt{1^2 + 2^2 + 3^2}$$
$$= \sqrt{14}$$

$$\|y\| = \sqrt{2^2 + (-1)^2 + (1)^2}$$
$$= \sqrt{6}$$

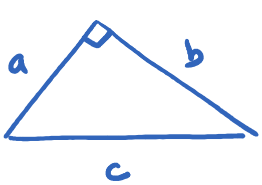
$$\cos\theta = \frac{x^T y}{\|x\| \cdot \|y\|}$$
$$= \frac{3}{\sqrt{14} \cdot \sqrt{6}} = \frac{3}{\sqrt{84}}$$

$$\theta = \arccos\left(\frac{3}{\sqrt{84}}\right) \quad \text{or} \quad \cos^{-1}\left(\frac{3}{\sqrt{84}}\right)$$

<u>Corollary</u>: x is perpendicular to y iff $x^T y = 0$
(orthogonal)

<u>Proof</u>: $x^T y = 0$ iff $\cos\theta = 0$

iff $\theta = \frac{\pi}{2}$ or 90°



Phythagorean theorem: $a^2 + b^2 = c^2$.