MTH 2775 Exam II (6 problems, 100 points) Name: ___________________

1. (3 pts for each) Answer True or False for the following statements. Give short explanations for your answers.

(a) $L : \mathbb{R}^2 \to \mathbb{R}^2$ with $L(x) = (x_1 + x_2, x_1 - x_2)^T\}$ for $x = (x_1, x_2)^T$ is a linear transformation.

$L(x) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ So it is a linear transformation.

True.

(b) Vectors $\{(1,0,0)^T, (0,-2,0)^T, (1,2,-1)^T\}$ are linear independent.

$\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 1 & 2 & -1 \end{pmatrix} = 1 \cdot -2 \cdot (-1) \neq 0$. So they are L.I.

True.

(c) The angle between vectors $(1,2,-3)^T$ and $(2,-1,1)^T$ is 90^o.

False. $(1,2,-3)\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 2 + (-2) - 3 = -3 \neq 0$

So they are not perpendicutar.

(d) Rank of a $\begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 2 & 4 \end{pmatrix}$ is 2.

$\begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 2 & 4 \end{pmatrix} \xrightarrow[R_3 - 2R_1]{R_2 - 2R_1} \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 0 & 0 \end{pmatrix} \longrightarrow$ has rank 2 (True)

(e) Let A be a 4×3 matrix. If nullity $(A) = 1$, then rank $(A) = 3$.

By Rank-Nullity Theorem:

$\text{rank}(A) = 3 - \text{nullity}(A)$
$= 3 - 1$
$= 2 \neq 3$ (False)

2. (7.5 pts for each) (a) Whether $\{(1, 1, -2)^T, (2, 2, -1)^T, (3, -1, 2)^T\}$ forms a basis for $\mathbb{R}^3$. Show all your work.

We just need to check if they are L.I by solving

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 2 & -1 & 0 \\ -2 & -1 & 2 & 0 \end{array}\right) \xrightarrow[R_3+2R_1]{R_2-R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 3 & 8 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 3 & 8 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right) \rightarrow c_1 = c_2 = c_3 = 0$$

They are L.I. Thus they form a basis of $\mathbb{R}^3$.

(b) Whether $\{2x^2 - x - 1, x^2 - 1, x - 1\}$ forms a basis of P_3. Show all your work.

Check linear independence by setting up

$$c_1(2x^2 - x - 1) + c_2(x^2 - 1) + c_3(x - 1) = 0$$

$$2c_1x^2 - c_1x - c_1 + c_2x^2 - c_2 + c_3x - c_3 = 0$$

$$(2c_1 + c_2)x^2 + (-c_1 + c_3)x - (c_1 + c_2 + c_3) = 0.$$

$$\rightarrow \begin{array}{l} 2c_1 + c_2 = 0 \\ -c_1 + c_3 = 0 \\ c_1 + c_2 + c_3 = 0 \end{array} \longrightarrow \left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{c_1 \leftrightarrow c_2} \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \xrightarrow[c_3+c_1]{c_2+2c_1} \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{R_3-R_2} \left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad c_3 = \text{free variable}$$

Linear Dependence. So they do not form a basis of P_3

3. (5 pts for each) Given $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \\ 3 & 4 & 7 & 10 \end{pmatrix}$.

(a) Find a basis for the null space $N(A)$.

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 1 & 1 & 2 & 3 & 0 \\ 3 & 4 & 7 & 10 & 0 \end{array}\right) \xrightarrow[R_3-3R_1]{R_2-R_1} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & -1 & -1 & -1 & 0 \\ 0 & -2 & -2 & -2 & 0 \end{array}\right) \xrightarrow{R_3-2R_2} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1+2R_2} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & -1 & -1 & -1 & 0 \end{array}\right) \qquad x_3 = t,\ x_4 = s \Rightarrow x_2 = -t-s, \quad x_1 = -t-2s$$

$$N(A) = \left\{ \begin{pmatrix} -t-2s \\ -t-s \\ t \\ s \end{pmatrix} \right\} = \operatorname{span}\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(b) Find a basis for the row space.

A basis of row space is

$\{(1,0,1,2),(0,-1,-1,-1)\}$

(c) Find a basis for the column space.

A basis of column space is:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \right\}$$

(d) Find nullity(A) and rank(A).

nullity $(A) = 2$ and rank $(A) = 2$

4. (a) (5 pts) State the Rank-Nullity Theorem.

Let A be an $m \times n$ matrix. Then we have

$\text{rank}(A) + \text{nullity}(A) = n$

(b) (5 pts) Given an 5×7 matrix A with $\text{rank}(A) = 5$. How many solutions are there for $A^T y = 0$.

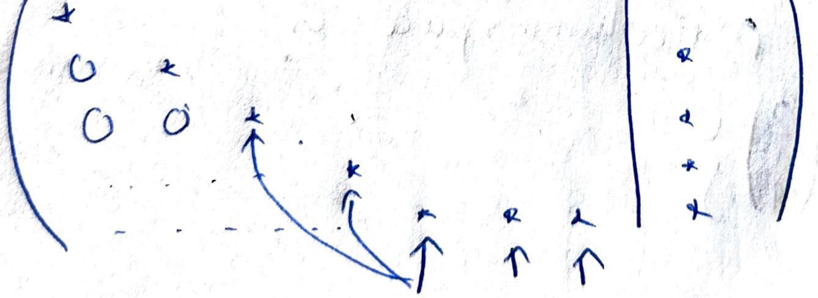
$\text{rank}(A^T) + \text{nullity}(A^T) = 5$

$\rightarrow$ $\text{nullity}(A^T) = 0$

$\rightarrow$ $A^T y = 0$ has a unique solution

(c) (5 pts) Given an 5×7 matrix A with $\text{rank}(A) = 5$ and let b be any vector in $\mathbb{R}^5$. Explain why the system $Ax = b$ is consistent and has infinitely many solutions.

As $\text{rank}(A) = 5$, after using Gaussian elimination method we have



There is no zero row, non zeros

So $Ax = b$ is consistent. Moreover

Nullity $(A) = 2$ means we have 2 free variables. Thus $Ax = b$ has infinitely many solutions.

5. Define $L : P_3 \to P_3$ by

$$L(p) = 2p - xp' \quad \text{for any} \quad p \in P_3.$$

(a) (10 pts) Verify that L is a linear transformation.

$L(\alpha p + \beta q) = 2(\alpha p + \beta q) - x(\alpha p + \beta q)'$

$= 2\alpha p + 2\beta q - x\alpha p' - x\beta q'$

$\alpha L(p) + \beta L(q) = \alpha(2p - xp') + \beta(2q - xq')$

$= 2\alpha p - \alpha x p' + 2\beta q - \beta x q'$

$\to L(\alpha p + \beta q) = \alpha L(p) + \beta L(q)$

(b) (5 pts) Determine Ker L and its dimension.

Solve $\cancel{ke}$ $(p) = 0$. Pick $p(x) = ax^2 + bx + c$

$L(p) = 2ax^2 + 2bx + 2c - x(2ax + b)$

$= bx + 2c = 0 \to b = 0$ and $c = 0$

$\to p = ax^2 \to \text{Ker } L = \text{span}\{x^2\}$

$\dim(\text{Ker } L) = 1$

(c) (5 pts) Determine the range of L and its dimension.

$\text{range}(L) = \{bx + 2c \mid b, c \in \mathbb{R}\}$

$= \text{span}\{x, 2\}$

$\to \dim(\text{range } L) = 2$

(c) (5 pts) By using the standard basis $E = \{x^2, x, 1\}$, find the matrix represetation of L.

$(L(p))_E = \begin{pmatrix} 0 \\ b \\ 2c \end{pmatrix}$ and $(p)_E = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. So

$L(p)_E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} (p)_E$

6. (10 pts) Let $V = \text{span}\{(1,-1,1,1)^T, (1,1,3,-1)^T\}$ be a supspace of $\mathbb{R}^4$. Find $V^{\perp}$ and its dimension

$V^{\perp} = \{ x \in \mathbb{R}^4 \mid (1,-1,1,1)^T \perp x \text{ and } (1,1,3,-1)^T \perp x \}$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 1 & 1 & 3 & 1 & 0 \end{array}\right) \xrightarrow{R_2 - R_1} \left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_2/2} \left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow{R_1 + R_2} \left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right)$$

$x_3 = t, \quad x_4 = s \rightarrow x_2 = -t, \quad x_1 = -2t - s$

$$x = \begin{pmatrix} -2t - s \\ -t \\ t \\ s \end{pmatrix} \rightarrow V^{\perp} = \text{span}\left\{ \begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$