2. (5 for each) Let $A := \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & -1 \end{bmatrix}$ 3×3, $B := \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ 2×3. Find each of the following items. If an item does not exist, say "DNE".

(a) AB and BA.

AB DNE

$$BA = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

(b) $\det(A)$ and $\det(B)$.

$$\det A = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & -1 \end{vmatrix} \overset{R_1 \leftrightarrow R_2}{=} - \begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= -0 \cdot + 1 \cdot \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} - 0$$

$$= 1(-1-0)$$

$$= -1$$

$$\det B = \det\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}_{2\times 3}$$ DNE b/c B is not a square matrix

(c) How many solutions to the linear system $Ax = 0$ are there? How many solutions to the linear system $By = 0$ are there? Explain in details your answer for full credit.

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 \end{array}\right) \xrightarrow{R_3 - 2R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right) \xrightarrow{R_3 + R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right)$$

$-x_3 = 0 \to x_3 = 0$ $\to$ a unique solution $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\left.\begin{array}{l} x_2 = 0 \\ x_1 + x_2 = 0 \end{array}\right\} \to x_1 = 0$

$$\left(\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 0 & -1 & 0 & 0 \end{array}\right)$$

x_3 = free variable → infinitely many solutions

$$\boxed{Ax = 0}$$

a unique solution, that is $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

b/c $0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is a solution.

6) $A^2 = 4I$.

(a) Verify A is invertible and $\boxed{A^{-1} = \frac{1}{4} A}$

<u>Review</u>: B is the inverse of A if

$$AB = I.$$

We need to check that:

$$A \cdot \frac{1}{4} A = I$$

$$\underline{LHS} = \frac{1}{4} A^2$$

$$= \frac{1}{4} \cdot 4I$$

$$= I$$

$$= RHS$$

(verified)

So $A^{-1} = \frac{1}{4} A$ and A is invertible.

* Another way to show A to be invertible is using

the determinants

$$A^2 = 4I$$

$$\det(A^2) = \det(4I) = \det\begin{pmatrix} 4 & & 0 \\ & \ddots & \\ 0 & & 4 \end{pmatrix}$$

$$\det(A \cdot A) = 4^n$$

$$\det A \cdot \det A = 4^n$$

$$(\det A)^2 = 4^n$$

$$\det A = \pm 2^n \neq 0$$

A is invertible

(b) $A - I$ is nonsingular and $(A-I)^{-1} = \frac{1}{3}(A+I)$

We need to check :

$$(A-I)\frac{1}{3}(A+I) = I$$

$(x-y)(x+y)$
$= x^2 - y^2$
(difference of squares)

$$\begin{aligned}
\text{LHS} &= \frac{1}{3}(A-I)(A+I) \\
&= \frac{1}{3}(A^2 + A \cdot I - I \cdot A - I \cdot I) \\
&= \frac{1}{3}(4I + A - A - I) \\
&= \frac{1}{3}(3I) \\
&= I \\
&= \text{RHS} \quad (\text{verified})
\end{aligned}$$

So $(A-I)^{-1} = \frac{1}{3}(A+I)$ and $A-I$ is invertible.

$$(A-I)^{-1} = \frac{1}{3}(A+I)$$

$$\underbrace{(A-I)(A-I)^{-1}}_{I} = \underbrace{(A-I)\frac{1}{3}(A+I)}_{I}$$

$$I = I$$

7. (7.5 pts for each) Determine whether the following sets are subspaces:

(a) $X = \{f \in C^1(0,1) | f(x) - 2f'(x) = 0\}$, where $C^1(0,1)$ is the set of all continuously differentiable functions in $(0,1)$.

Addition: Pick any f and g in X:

satisfy

$$f(x) - 2f'(x) = 0$$
$$g(x) - 2g'(x) = 0$$
$$\overline{(f+g)(x) - 2(f+g)'(x) = 0}$$

(sum rule)

So $f+g \in X$. Addition is closed.

Scalar Multiplication: Pick any $f \in X$ and any scalar $\alpha \in \mathbb{R}$,

$$\alpha\left(f(x) - 2f'(x) = 0\right)$$
$$(\alpha f)(x) - 2(\alpha f)'(x) = 0$$

$\rightarrow \alpha f \in X$. So scalar multiplication is closed.

X is a subspace.

(b) $Y = \{ A \in \mathbb{R}^{3\times 3} \mid \det A = 0 \}$

$\det(A+B) \neq \det(A) + \det B$

det is not a linear function.

Find a counter-example :

Pick some $A \in Y$: $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in Y$ ($\det A = 0$)

$B \in Y$: $B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \in Y$ ($\det B = 0$)

but $A + B \notin Y$

$$A + B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\det(A+B) = 0 - 1 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= 0 - 1 \cdot (-1) + 1 \cdot 1$$

$$= 2 \neq 0 .$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in Y, \quad \det A = 0$$

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \in Y, \quad \det B = 0 \cdot 2 \cdot 3 = 0$$

(triangular matrices)

$$A + B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \notin Y, \quad \det(A+B) = 1 \cdot 2 \cdot 3 = 6 \neq 0$$

3 (c) (5 pts) Using (b) above, solve the system $Ax = b$ with $b = (1, 2, 3, 4)^T$.

$b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$

$ax = b$

$x = \frac{b}{a}$

From (b), we are able to find A^{-1}.

$$A^{-1}(Ax = b)$$

$$\underbrace{A^{-1}A}_{I}\, x = A^{-1}b$$

$$I \cdot x = A^{-1}b$$

$$x = A^{-1}b$$

. . . . . .

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4. Consider the linear system: $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 1 & 4 & 2 & 4 \\ 0 & 6 & a^2 & a \end{array}\right]$

(a) (5 pts) Find all values of a such that the above system a unique solution.

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 1 & 4 & 2 & 4 \\ 0 & 6 & a^2 & a \end{array}\right) \xrightarrow{R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 3 & 2 & 1 \\ 0 & 6 & a^2 & a \end{array}\right) \xrightarrow{R_3 - 2R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & a^2-4 & a-2 \end{array}\right)$$

* Case 1: $a^2 - 4 \neq 0$, $a \neq \pm 2$

From R_3: $(a^2-4)x_3 = a-2 \rightarrow x_3 = \dfrac{a-2}{a^2-4}$

no free variable $\rightarrow$ a unique solution.

(b) (5 pts) Find all values of a such that the above system has infinitely many solutions. **Find all these solutions in this case.**

* Case 2: $a^2 - 4 = 0$ $\begin{cases} a = 2 \\ a = -2 \end{cases}$

$0x_3 = a - 2 \rightarrow \boxed{a = 2}$

- $a = 2$, x_3 = free variable $\rightarrow$ infinitely many solutions
- $a = -2$, $0 = -4$ (no solution).

8. (10 points) Let A be an $m \times n$ matrix. Explain why $A^T A$ and AA^T are possible.

$A_{m \times n} \cdot (A^T)_{n \times m}$ is an $m \times m$ matrix

$(A^T)_{n \times m} A_{m \times n}$ is an $n \times n$ matrix