Friday, April 18, 2025 4:36 PM

10a) A is diagonalizable.

$$A = X D X^{-1} \quad , D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & & \lambda_n \end{pmatrix}$$

$$A^k = X D^k X^{-1}$$

$$D^k = \underbrace{\begin{pmatrix} \lambda_1^k & & 0 \\ & \lambda_2^k & \\ & & \ddots \\ 0 & & & \lambda_n^k \end{pmatrix}}_{\text{diagonal matrix}}$$

A^k is diagonalizable.

1a) $x \perp y$, x, y are nonzero vectors

$$c_1 x + c_2 y = 0$$

$$c_1 \langle x, x \rangle + c_2 \underbrace{\langle y, x \rangle} = \underbrace{\langle 0, x \rangle}$$

$$c_1 \underbrace{\|x\|^2} + 0 = 0$$

$$\to c_1 = 0 \quad , \quad c_2 y = 0 \to c_2 = 0$$

$\to$ L.I.

3) $C[-\pi, \pi]$

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$$

1, cos x, sin x are orthonormal

$1+\cos x$, $\sin x$ are orthogonal.

$$\langle 1+\cos x, \sin x\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+\cos x)\sin x\, dx$$

$\boxed{u=f(x)}$

$du=f'(x)dx$

$$= \frac{-1}{\pi}\int_0^0 u\, du$$

$$= -\frac{1}{\pi}\frac{u^2}{2}\Big|_0^0$$

$$= 0$$

$u = 1+\cos x$

$\to du = -\sin x\, dx$

$x=-\pi \to u = 1+\cos(-\pi) = 0$

$x=\pi \to u = 1+\cos\pi = 0$

9b) $N(A) \cap R(A^T) = \{0\}$

Choose any $x \in N(A) \cap R(A^T)$, i.e.,

$x \in N(A)$ and $x \in R(A^T)$

As $N(A) \perp R(A^T)$, $x \perp x$

$$\langle x, x\rangle = 0$$

$$x = 0$$

1g) $L(f) = \int_{-\pi}^{\pi} f(x)\, dx$

$L(1) = \int^{\pi} 1\, dx$ $\qquad \int \alpha\, dx$

$\hookrightarrow \underline{L(1)} = \int_{-\pi}^{\pi} 1 \, dx$ $\int \alpha \, dx$

$= \alpha x$

$\int x^2$ $= x \Big|_{-\pi}^{\pi}$

False

$\int x$ $= \pi - (-\pi)$

$\int \sin x$ $= 2\pi$

$\rightarrow 1$ is NOT an element of Ker L.

10c) $A^2 = O$ and A is nonzero.

Show that A is NOT diagonalizable.

Suppose by contradiction that A is diagonalizable

$A = X D X^{-1}$

$A^2 = \boxed{X D^2 X^{-1} = O}$

$\rightarrow D = O$