4) c) A is 5×7 matrix with $\operatorname{rank}(A) = 5$. Explain why $Ax = b$ is consistent and has infinitely many solutions.

$$\underbrace{\operatorname{rank}(A)}_{5} + \underbrace{\operatorname{nullity}(A)}_{} = n$$
$$= 7$$

$\operatorname{nullity}(A) = 2$ — we have 2 free variables

$$(A \mid b) \xrightarrow{\text{row operations}} \left(\begin{array}{ccccc:c:c} * & & & & & & \\ & * & & & & & \\ & & * & & & & \\ & & & * & & & \\ & & & & * & & \end{array} \right)$$

non zeros

There is no zero rows

So $Ax = b$ is always "consistent", i.e., it always has solutions.

B/c we also have 2 free variables, $Ax = b$ must have infinitely many solutions.

ex: In $\mathbb{R}^n$, we define:

$$\|x\|_1 := |x_1| + |x_2| + \cdots + |x_n| \text{ as the } \ell_1\text{-norm.}$$

Verify that $\|x\|_1$ satisfies the three conditions of norm.

① $\|x\|_1 \geq 0$ (trivial)

Set $\|x\|_1 = 0 \rightarrow |x_1| + |x_2| + \cdots + |x_n| = 0$

$\rightarrow x_1 = 0, x_2 = 0, \ldots, x_n = 0$

$\rightarrow x = 0$.

② We need to check $\|\alpha x\|_1 = |\alpha| \cdot \|x\|_1$

LHS $= |\alpha x_1| + |\alpha x_2| + \cdots + |\alpha x_n|$

$$= |\alpha| \left(\underbrace{|x_1| + |x_2| + \cdots + |x_n|} \right)$$

$$= |\alpha| \ \|x\|_1$$

$$= \text{RHS}$$

③ We need to check $\|x+y\|_1 \leq \|x\|_1 + \|y\|_1$

$$\text{LHS} = \underbrace{|x_1+y_1|} + \underbrace{|x_2+y_2|} + \cdots + \underbrace{|x_n+y_n|}$$

Note that (Triangle inequality for numbers)

$|a+b| \leq |a| + |b|$ for any $a, b \in \mathbb{R}$

$\Leftarrow$ $a^2+b^2+2ab \leq a^2 + 2|a|\cdot|b| + b^2$

$\Leftarrow$ $2ab \leq \underbrace{2|a|\,|b|}_{2|ab|}$ (True)

So $\text{LHS} \leq |x_1| + |y_1| + |x_2| + |y_2| + \cdots + |x_n| + |y_n|$

$$= \|x\|_1 + \|y\|_1$$

$$= \text{RHS}$$

This verifies the triangle inequality.

ex: Similarly the ℓ_∞ norm of x:

$$\|x\|_\infty := \max\{|x_1|, |x_2|, \ldots, |x_n|\}$$

is also a norm on $\mathbb{R}^n$.

ex: $x \in \mathbb{R}^3$ and $x = (1, -1, 2)$

$$\|x\|_1 = |1| + |-1| + |2|$$

$$= 4$$

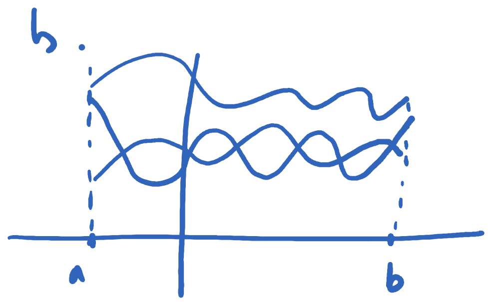
$$\|x\|_2 = \sqrt{1^2 + (-1)^2 + 2^2}$$

$$\|x\|_2 = \sqrt{1^2+(-1)^2+2^2} = \sqrt{6}$$

$$\|x\|_\infty = \max\{|1|, |-1|, |2|\} = 2$$

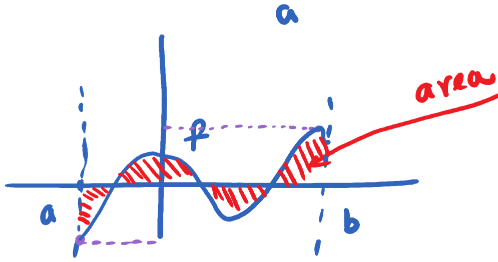
$$\left.\right\} \|x\|_1 > \|x\|_2 > \|x\|_\infty$$

<u>ex</u>: For $C[a,b]$ the set of all continuous functions from a to b.



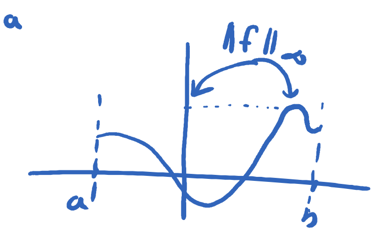
Given $f \in C[a,b]$, we define

$$\|f\|_1 := \int_a^b |f(x)|\,dx$$



$$\|f\|_2 = \sqrt{\langle f, f\rangle} = \sqrt{\int_a^b f^2(x)\,dx}$$

$$\|f\|_\infty = \max_{x\in[a,b]} |f(x)|$$



<u>ex</u>: $A \in \mathbb{R}^{m\times n}$

$$\|A\|_1 = \sum_{1\le i\le m} |a_{ij}| \qquad (\ell_1\text{-norm})$$

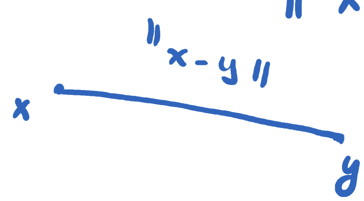
$$\|A\|_\infty = \max_{1 \le j \le n}\{|a_{ij}|\}$$

$$\|A\|_F = \sqrt{\sum_{\substack{1\le i\le m\\ 1\le j\le n}} a_{ij}^2}$$

They are norms of $\mathbb{R}^{m\times n}$.

__Def__: Let x, y be vectors in a normed space V. The distance between x and y is

$$\|x-y\|.$$



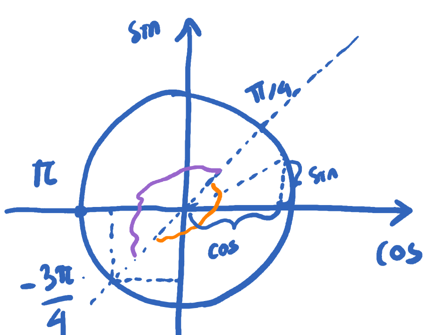
__ex__: In $C[-\pi,\pi]$, we use the norm

$$\|f\|_1 = \int_{-\pi}^{\pi} |f(x)|\,dx$$

$$\|f\|_\infty = \max_{-\pi\le x\le\pi} |f(x)|$$

Find $\|\sin x - \cos x\|_1$ and $\|\sin x - \cos x\|_\infty$

__Solution__: $\|\sin x - \cos x\| = \int_{-\pi}^{\pi} |\sin x - \cos x|\,dx.$



(unit circle)

$\cos x - \sin x \ge 0$ if $-\frac{3\pi}{4} \le x \le \frac{\pi}{4}$

$\cos x - \sin x \le 0$ if $\frac{\pi}{4} \le x \le \pi$

and $-\pi \le x \le -\frac{3\pi}{4}$.

$= \int^{\pi/4} \cos x - \sin x\,dx + \int^{\pi}$

4 1 and $-\pi \le x \le 4$
(unit circle)

$$= \int_{-\frac{3\pi}{4}} \cos x - \sin x \, dx + \int_{\pi/4}^{\pi} \sin x - \cos x \, dx$$

$$+ \int_{-\pi}^{-\frac{3\pi}{4}} \sin x - \cos x \, dx .$$

$$= \sin x + \cos x \Big|_{-\frac{3\pi}{4}}^{\pi/4} + -\cos x - \sin x \Big|_{\pi/4}^{\pi} + -\cos x - \sin x \Big|_{-\pi}^{-3\pi/4}$$

$$= + \frac{\sqrt{2}}{2} - \left(\frac{\sqrt{2}}{2} - -\frac{\sqrt{2}}{2} \right) + \sqrt{2}\left(1 - 0 - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)\right)$$

$$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - \left(-(-1) - 0\right)$$

$$= 2\sqrt{2} + 1 + \sqrt{2} + \sqrt{2} - 1$$

$$= 4\sqrt{2} .$$

ex: let $p \in P_3$.

Define $\|p\| := |p(-1)| + |p(0)| + |p(1)|$.

Find $\|\underbrace{1+x-x^2}_{p}\|$.

Sol: $\|1+x-x^2\| = |p(-1)| + |p(0)| + |p(1)|$

$$= |1+(-1)-(-1)^2| + |1+0-0^2| + |1+1-1^2|$$

$$= 1 + 1 + 1$$

$$= 3$$