MTH 2775 **Exam II** (8 problems, 125 points) Name: ___________________

March 29, 2022

1. (3 pts for each) Answer True or False for the following statements. Give **short explanations** for your answers.

   (a) $S = \{x \in \mathbb{R}^2 |\ x_1 x_2 = 0\}$ is a subspace of $\mathbb{R}^2$.

   (b) Vectors $\{(1,0,0)^T, (2,-2,0)^T, (1,2,-1)^T\}$ form a basis for $\mathbb{R}^3$.

   (c) The angle between vectors $(1,2,-3)^T$ and $(2,-1,1)^T$ is 90^o.

   (d) Rank of a $\begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 2 & 4 \end{pmatrix}$ is 2.

   (e) Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be $L(x) = (x_1 - x_2, x_1 - 2x_2 + x_3)^T$. Then $(1,1,1)$ is an element of $\ker(L)$.

2. (a) (7.5 pts) Determine whether the set $S = \{f \in C[-1, 1] |\ f(-1) + f(1) = 2\}$ is a subspace of $C[-1, 1]$. **If yes, find its dimension.**

(b) Determine whether the set of all symmetric 2×2 matrices is a subspace of $\mathbb{R}^{2\times 2}$. **If yes, find its dimension.**

3. (15 pts) Find all possible choice of a that make the following matrix singular:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & a & 1 \\ 2 & 3 & 3 & a \\ 3 & 2 & 3 & 1 \end{pmatrix}$$

4. (a) (5 pts) State the Rank-Nullity Theorem.

(b) (10 pts) Given an 5×7 matrix A with $\text{rank}(A) = 5$ and let b be any vector in $\mathbb{R}^5$. Explain why the system $Ax = b$ must have infinitely many solutions.

5. (7.5 pts for each) (a) Whether $\{(1,1,-2)^T, (2,2,-1)^T, (3,-1,2)^T\}$ forms a basis for $\mathbb{R}^3$. Show all your work.

(b) Whether $\{2x^2 - x + 1, x^2 + 1, x + 1\}$ forms a basis of P_3. Show all your work.

6. (5 pts for each) Given $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \\ 3 & 4 & 7 & 10 \end{pmatrix}$.

(a) Find a basis for the null space $N(A)$.

(b) Find a basis for the row space.

(c) Find a basis for the column space.

(d) Find nullity(A) and rank(A).

7. Define $L : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ by $L(A) = A - A^T$ for any $A \in \mathbb{R}^{2\times 2}$.

   (a) (10 pts) Verify that L is a linear transformation.

   (b) (5 pts) Determine $\operatorname{Ker} L$ and its dimension.

   (c) (5 pts) Determine the range of L and its dimension.

8. (a) (5 pts) State the triangle inequality for the Euclidean norm $\| \cdot \|$ in $\mathbb{R}^n$.

(b) (5 pts) Use the triangle inequality to prove that

$$\|x - y\| \geq \max\{\|x\| - \|y\|, \|y\| - \|x\|\} \quad \text{for all} \quad x, y \in \mathbb{R}^n$$