MTH 2775 Some practice problems for Final Exam

1. Answer whether the following statements are true or false. Explain shortly your answers if they are true or give a counterexample if they are false.

   (a) Let x, y be two nonzero orthogonal vectors in $\mathbb{R}^n$. Then x, y are linearly independent.

   (b) Let $C[-\pi, \pi]$ be the set of continuous functions on $[-\pi, \pi]$ with the inner product

   $$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx.$$

   Then $\langle 1, 1\rangle = 1$.

   (c) $\|(1, -3, -2, 4)^T\|_1 = 0$.

   (d) Eigenvalues of $\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$ are 2 and 3.

   (e) The nullity of a square matrix is always 0.

   (f) Matrix $A = \begin{pmatrix} 3 & 4 \\ 3 & 2 \end{pmatrix}$ is diagonalizable.

   (g) Let $L(f) = \int_{-\pi}^{\pi} f(x)dx$ be the linear transformation from $C[-\pi, \pi]$ to $\mathbb{R}$. Then 1 is an element of $\operatorname{Ker} L$.

   (h) $\operatorname{span}\{(1, 1, 2, 1)^T, (1, 2, 1, 0)^T, (2, 2, 3, 1)^T\}$ forms a vector space with dimension 3.

   (k) Given a 3×4 matrix A with $\operatorname{rank} A = 3$. Equation $Ax = 0$ has only one solution.

   (i) When the equation $Ax = b$ is consistent, b is an element of range A.

2. Given $A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -4 & 1 & 1 \\ -3 & 3 & 2 \\ 1 & -2 & -2 \end{pmatrix}$.

   (a) Find the angle between A and B.

   (b) Verify that $\|A\|_F + \|B\|_F \geq \|A + B\|_F$.

3. Let $C[-\pi, \pi]$ be the set of continuous functions on $[-\pi, \pi]$ with the inner product

   $$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx \quad \text{for all} \quad f, g \in C[-\pi, \pi].$$

   Verify that vectors $\{\frac{1}{\sqrt{2}}, \sin x\}$ are orthogonal and their norms are 1.

4. (a) Verify that $\|x\| := |x_1| + 2|x_2|$ for any $x = (x_1, x_2)^T$ is a norm in $\mathbb{R}^2$.

   (b) Verify that $\|p\| := |p(0)| + |p(1)| + |p(-1)|$ for any $p \in P_3$ be a norm in P_3.

5. (a) Find all the eigenvalues and eigenvectors of $\begin{pmatrix} 4 & -5 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$.

   (b) Is this matrix diagonalizable? If so, what is its diagonal matrix.

6. Let $L : \mathbb{P}_3 \to \mathbb{P}_3$ be defined by

$$L(p) = p(x) - xp(1) - p(0).$$

   (a) Verify that L is a linear transformation.

   (b) Find a basis for $\ker(L)$ and its dimension.

   (c) Find a basis for $\text{range}(L)$ and its dimension.

   (d) Find the matrix representation of this linear transformation with the standard basis $E = \{x^2, x, 1\}$.

7. (10 for each) Let P_3 be associated with the following operation

$$\langle p, q\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1) \quad \text{for all} \quad p, q \in P_3.$$

   (a) Let S be the set of polynomials of the form $ax^2 + bx + 3a + 2b$. Verify that S is a subspace of P_3 and find its dimension.

   (b) Find the angle between $1 + x$ and x^2.

8. Let A be the matrix $\begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 2 & -1 & 2 \\ 0 & 4 & 2 & 6 \\ 1 & 6 & 5 & 10 \end{pmatrix}$

   (a) Find a basis for the null space of A

   (b) Find a basis for $N(A)^\perp$.

   (c) Find rank of A^T and nullity of A^T.

   (d) Determine whether A^{1000} is nonsingular.

9. (a) State the fundamental subspace theorem for a matrix.

   (b) Given an $m \times n$ matrix A. Verify that $N(A) \cap R(A^T) = \{0\}$.

   (c) State the Rank-Nullity Theorem.

   (d) If A is a 4×5 with rank 4. How many solutions do we have for $Ax = 0$ and $A^T y = 0$?

10. (a) Let A be a diagonalizable matrix. Show that A^k are also diagonalizable matrices for any positive integers k.

    (b) Let A be an $n \times n$ nonsingular matrix with known eigenvalues $\lambda_1, \ldots, \lambda_n$. Find all the eigenvalues of A^{-1}.

    (c) Suppose that A is a nonzero $n \times n$ matrix satisfying $A^2 = O_n$. Show that A is not diagonalizable.

    (d) Suppose that $\lambda_1, \ldots \lambda_n$ be eigenvalues of an $n \times n$ matrix A. Show that $\lambda_1 \ldots \lambda_n = \det(A)$.

11. Let A be a diagonalizable matrix such that that $A^{2025} = A$. Find all real-valued eigenvalues of A.

12. Let V be an inner product space. Verify that

$$\|u + v\|^2 - \|u - v\|^2 = 4\langle u, v\rangle \quad \text{for all} \quad u, v \in V.$$

13. Let V be a normed space with norm $\|\cdot\|$.

 (a) State the triangle inequality.

 (b) Show that

$$\|u+v\| + \|u-v\| \geq 2\|v\| \quad \text{for any} \quad u, v \in V.$$

Answer keys.

1. (a) T (b) F (c) F (d) T (e) F (f) T (g) T (h) T (k) F (i) T.
2. (a) 83.4°
3. (b) $2\sin x$
4. (a) Use the inequality $|a| + |b| \geq |a+b|$ (b) Hint: When $p(0) = p(1) = p(-1)$, the quadratic polynomial p must be 0.
5. (a) $\lambda = 0$ and $x = (1,1,1)^T$, $\lambda = 1$ and $x = (3,2,1)^T$, $\lambda = 2$ and $x = (5,3,1)^T$ (b) Yes.
6. (b)span $\{x\}$ (c)span $\{x^2 - x, x\}$ (d) $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$.
7. (a) dimension 2 (b) $3/\sqrt{10}$.
8. (a) $\{(2,-0.5,1,0)^T, (1,-1.5,0,1)^T\}$ (b)$\{(1,2,3,4)^T, (0,2,1,3)^T\}$ (c)rank $(A) = 2$ and Nullity $(A^T) = 2$ (d) A^{10} is singular.
9. (d) Infinitely many solutions and a unique solution, respectively.
10. (b) $1/\lambda_1, \ldots, 1/\lambda_n$.
11. $0, 1, -1$.