Property: $\det(A^T) = \det(A)$ for any $n\times n$ matrix A. Consequently, A is invertible iff A^T is invertible.

Property: If A is a triangular matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} \cdots & a_{2n} \\ 0 & 0 & a_{33} \cdots & a_{3n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}_{n\times n},$$ then

$$\det A = a_{11} \cdot a_{22} \cdots a_{nn}$$

Property: If A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det A}.$$

Theorem: Let A, B be 2 $n\times n$ matrices. Then

$$\det(AB) = (\det A)\cdot(\det B)$$

Theorem: A is invertible iff $\det A \neq 0$.

2.2 Properties of the determinants

Row operation I: Interchange 2 rows of a matrix

ex: $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \xleftrightarrow{R_1 \leftrightarrow R_2} B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

$\det A = 1\cdot 1 - 1\cdot 2$; $\det B = 1\cdot 2 - 1\cdot 1$

$= -1$ $\quad = 1$

$$\det A = 1\cdot 1 - 1\cdot 2 = -1 \quad ; \quad \det B = 1\cdot 2 - 1\cdot 1 = 1$$

After interchanging 2 rows of a matrix A, we have a new matrix B:

$$\det B = -\det A$$

* Row operation II: Multiply a row of A by a constant $\alpha \neq 0$

ex: $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \xrightarrow{R_2 \to R_2 \cdot 2} B = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$

$$\det A = -1 \quad ; \quad \det B = 1\cdot 2 - 2\cdot 2 = -2$$

After multiplying a row of A by a constant $\alpha \neq 0$, we have the new matrix B:

$$\det B = \alpha \det A .$$

$$\det(\alpha A) \overset{?}{=} \underbrace{\alpha\cdot\alpha\cdots\alpha}_{n \text{ times}} \det A \qquad (A \text{ is an } n\times n \text{ matrix})$$

$$= \alpha^n \det A$$

* Row operation III:

ex: $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} B = \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}$

$$\det A = 1\cdot 1 - 2\cdot 2 = -3 \quad ; \quad \det B = 1\cdot(-3) - 0\cdot 2 = -3$$

After adding a multiple of one row to another row of A, we have a new matrix B:

$$\det B = \det A .$$

ex: $A = \begin{pmatrix} 1 & 3 \\ \ldots \end{pmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} B = \begin{pmatrix} 1 & 3 \\ \ldots \end{pmatrix}$

$\det B = \det A$.

ex: $A = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} \xrightarrow[\text{Row operation III}]{R_2 \to R_2 - 3R_1} B = \begin{pmatrix} 1 & 3 \\ 0 & -7 \end{pmatrix}$

$\downarrow R_2 \to 2R_2 - R_1$ ← not a row operation III

$C = \begin{pmatrix} 1 & 3 \\ 5 & 1 \end{pmatrix}$

$\det A = 1 \cdot 2 - 3 \cdot 3 = -7$ | $\det B = 1 \cdot (-7) - 0 \cdot 3 = -7$

$\det C = 1 \cdot 1 - 5 \cdot 3 = -14$

ex: $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 2 & 1 & 1 & 2 \\ 3 & 1 & 1 & 4 \end{pmatrix}$. Find $\det A$.

By using the technique of Gauss elimination methods / row operations (row operation III)

Sol: $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 2 & 1 & 1 & 2 \\ 3 & 1 & 2 & 4 \end{pmatrix} \xrightarrow[R_3 - 2R_1,\ R_4 - 3R_1]{R_2 - R_1} \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -2 & -1 & 1 \end{pmatrix}}_{\det\ =\ x}$

Set $x = \det A$

$\xrightarrow[R_4 + 2R_2]{R_3 + R_2}$ (Row operation III) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix}$ $\det = x$ $\xrightarrow[\text{(Row operation I)}]{R_3 \leftrightarrow R_4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ ← triangular matrix (row-echelon matrix)

$\det = -x = 1 \cdot 1 \cdot 1 \cdot 1 = 1.$

So we have $-x = 1$, i.e., $x = -1$.

Then $\det(A) = -1$.

Chapter 3: Vector Spaces.

Chapter 3: Vector Spaces.

3.1. Definition and examples.

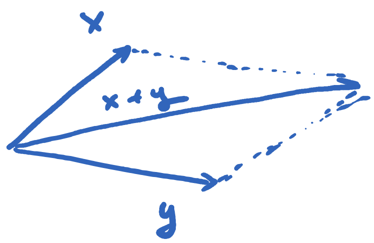
$\mathbb{R}^3$ is called the three-dimensional vector space.

$\mathbb{R}^n$ is ______ the n-dimensional vector space.

$$\mathbb{R}^n = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \;\middle|\; x_1, x_2, \ldots, x_n \text{ are real numbers} \right\}$$

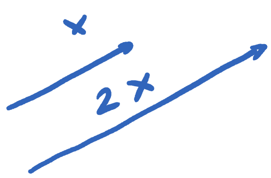
* **Addition**: For any $x, y \in \mathbb{R}^n$:

$$x + y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix} \in \mathbb{R}^n$$



* **Scalar Multiplication**: For any $x \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$,

$$\alpha x = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{pmatrix} \in \mathbb{R}^n.$$



* $\mathbb{R}^{m \times n}$ is the space of all $m \times n$ matrices.

$$\mathbb{R}^{m \times n} = \{ A \mid A \text{ is an } m \times n \text{ matrix} \}$$

- **Addition**: For any $A, B \in \mathbb{R}^{m \times n}$

q

$$A + B := (a_{ij} + b_{ij})_{1 \le i \le m} \in \mathbb{R}^{m \times n}$$

$$A + B := (a_{ij} + b_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}} \in \mathbb{R}^{m \times n}$$

($A + B$ is still an $m \times n$ matrix)

- Scalar multiplication: For any $A \in \mathbb{R}^{m \times n}$ and $\alpha \in \mathbb{R}$,

$$\alpha A := (\alpha a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

* P_n the space of polynomial of degree less than n.

$$P_3 = \{ a x^2 + b x + c \mid a, b, c \in \mathbb{R} \}$$

- Addition: For any $p, q \in P_n$

$$+\begin{cases} p(x) = a_0 x^{n-1} + a_1 x^{n-2} + \cdots + a_{n-1} \\ q(x) = b_0 x^{n-1} + b_1 x^{n-2} + \cdots + b_{n-1} \end{cases}$$

$$(p+q)(x) = (a_0 + b_0) x^{n-1} + (a_1 + b_1) x^{n-2} + \cdots + (a_{n-1} + b_{n-1})$$

So $p + q \in P_n$

- Scalar multiplication: For any $p \in P_n$ and $\alpha \in \mathbb{R}$,

$$p(x) = a_0 x^{n-1} + a_1 x^{n-2} + \cdots + a_{n-1}$$

$$(\alpha p)(x) = (\alpha a_0) x^{n-1} + (\alpha a_1) x^{n-2} + \cdots + (\alpha a_{n-1})$$

$$\alpha p \in P_n$$

* Vector spaces:

Def: Let V be a set on which the operations of addition and scalar multiplications are well-defined, i.e.,

- addition: $\forall$ (for any) $x, y \in V$: $x + y \in V$
- scalar multiplication: $\forall x \in V, \alpha \in \mathbb{R}$: $\alpha x \in V$.

Moreover, the addition and scalar multiplication above must satisfy the following conditions:

C1 : $x + y = y + x$ for any $x, y \in V$

C2 : $(x+y)+z = x + (y+z)$ ——— $x, y, z \in V$

C3 : There exists an element $\theta \in V$ (vector zero) s.t. $x + \theta = x$ for any $x \in V$.

(C4) : For any $x \in V$, there exists an element $(-x)$ s.t
$$x + (-x) = \theta \text{ (vector zero)}.$$

(C5) : $\alpha(x+y) = \alpha x + \alpha y$, for any $\alpha \in \mathbb{R}$, $x, y \in V$.

(C6) : $(\alpha+\beta) x = \alpha x + \beta x$, for any $\alpha, \beta \in \mathbb{R}$, $x \in V$

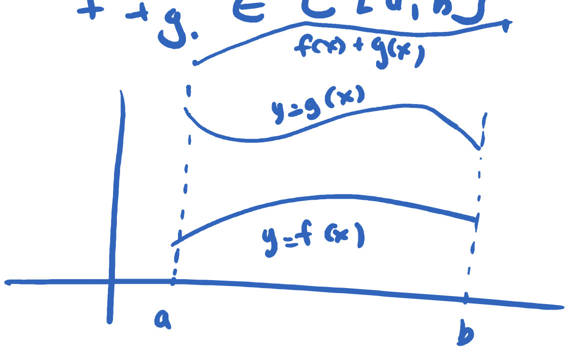
(C7) : $(\alpha\beta) x = \alpha(\beta x)$

(C8) : $\underbrace{1}_{\text{number } 1} x = x$ for any $x \in V$.

ex: $C[a,b]$ = the set of continuous function in $[a,b]$

Addition : For any $f, g \in C[a,b]$
$$(f+g)(x) := f(x) + g(x).$$

So $f+g \in C[a,b]$



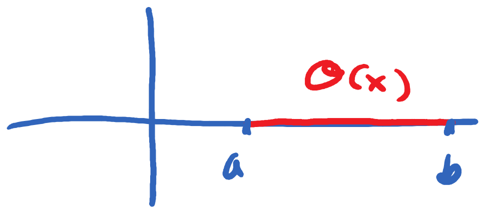
Scalar multiplication : for any $f \in C[a,b]$ and $\alpha \in \mathbb{R}$

Scalar multiplication: for any $f \in C[a,b]$ and $\alpha \in \mathbb{R}$

$$(\alpha f)(x) = \alpha f(x)$$

And $\alpha f \in C[a,b]$

Vector zero of $C[a,b]$: $\Theta(x) \equiv 0$ for any $x \in [a,b]$



$$(-f)(x) = -f(x) .$$

$C[a,b]$ is a vector space,