1.3 Matrix Arithmetic.

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<u>Matrix notation</u>: An $m \times n$ matrix A of m rows and n columns is written by:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m \times n}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$ is a column vector.

$\vec{x} = (x_1, x_2, \ldots, x_n)$ is a row vector.

<u>Addition</u>: Given 2 $m \times n$ matrices A & B (same size)

$A + B$ is also an $m \times n$ matrix and

$$A \pm B = (a_{ij} \pm b_{ij})_{ij} \qquad 1 \le i \le m, \quad 1 \le j \le n$$

<u>ex</u>: $A = \begin{pmatrix} 2 & 1 & 3 \\ 2 & 1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -4 \end{pmatrix}_{2\times 3}$, $C = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}_{2 \times 2}$

$$A + B = \begin{pmatrix} 3 & 3 & 6 \\ 1 & -1 & -5 \end{pmatrix}$$

$B + C =$ Doesn't not exist (DNE)

$$A - B = \begin{pmatrix} 1 & -1 & 0 \\ 3 & 3 & 3 \end{pmatrix}$$

$A - C$ DNE

* <u>Scalar multiplication</u> : Given an $m \times n$ matrix A and a number α, scalar multiplication αA is also an $m \times n$ matrix :

$$\alpha A = (\alpha a_{ij})_{i,j}$$

<u>ex</u>: $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & 0 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 1 & 2 \\ 1 & 3 & 1 \\ -1 & -2 & -1 \end{pmatrix}$

Find $2A - B$ and $2A + 3B$.

<u>Sol</u> :

$$2A - B = \begin{pmatrix} 2 & 2 & 2 \\ 4 & 6 & 8 \\ -2 & 0 & 6 \end{pmatrix} - \begin{pmatrix} -1 & 1 & 2 \\ 1 & 3 & 1 \\ -1 & -2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 & 0 \\ 3 & 3 & 7 \\ -1 & 2 & 7 \end{pmatrix}$$

$$2A + 3B = \begin{pmatrix} 2 & 2 & 2 \\ 4 & 6 & 8 \\ -2 & 0 & 6 \end{pmatrix} + \begin{pmatrix} -3 & 3 & 6 \\ 3 & 9 & 3 \\ -3 & -6 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 5 & 8 \\ 7 & 15 & 11 \\ -5 & -6 & 3 \end{pmatrix}$$

* Matrix Multiplication :

Linear system:

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1$$
$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2$$
$$\cdots\cdots$$
$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = b_m$$

Matrix notation for this system:

$$\underbrace{A\,x}_{\text{a matrix } \times \text{ a column vector}} = b$$

$$A_{m\times n} \cdot x_{n\times 1} = b_{m\times 1}$$

$$A\,x = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{12}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{pmatrix}$$

$\boxed{\text{When } m = 1}$: $a_{1\times n} = (a_1, a_2, \dots, a_n)$

$$a_{1\times n} \cdot x_{n\times 1} = a_1 x_1 + a_2 x_2 + \dots + a_n x_n$$

$$(a_1, a_2, \dots, a_n) \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = a_1 x_1 + a_2 x_2 + \dots + a_n x_n$$

Back to the case $A\,x = \begin{pmatrix} \text{row 1 of } A \cdot x \\ \text{row 2 of } A \cdot x \\ \vdots \\ \text{row } m \text{ of } A \cdot x \end{pmatrix}$

ex: $A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ -1 & 0 & -1 \end{pmatrix}$ $\quad x = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Compute $A\,x$ and $A\,y$.

Sol: $A\,x = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\cdot 1 + 1\cdot(-1) + 3(1) \\ 1\cdot(1) + 2(-1) + 3\cdot 1 \\ -1\cdot 1 + 0\cdot(-1) + (-1)\cdot 1 \end{pmatrix}$

$$= \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$$

$A\,y = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ DNE.

$$Ay = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \end{pmatrix} \quad \text{DNE.}$$

* Given an $m \times n$ matrix A and an $n \times p$ matrix B,

AB is an $m \times p$ matrix defined by:

$$AB = (\text{row } i \text{ of } A \times \text{column } k \text{ of } B).$$

ex: $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ -1 & 3 \\ 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \end{pmatrix}$

Find AB, BC, AC and CA.

ex: $AB = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 3 \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 2 & 0 \\ 1 & 2+0-2 \\ -1 & 0+3+3 \end{pmatrix} =$$

$$= \begin{pmatrix} 2 & 0 \\ 1 & 0 \\ -1 & 6 \end{pmatrix}$$

$BC = \begin{pmatrix} 1 & 2 \\ -1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 1+0 & -1+2 & 2+6 \\ -1+0 & 1+3 & -2+9 \\ 0+0 & 0+1 & 0+3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 8 \\ -1 & 4 & 7 \\ 0 & 1 & 3 \end{pmatrix}$$

$AC = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \end{pmatrix}$ DNE.

$CA = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 1 \quad -1+0 & -1+0+2 & 1+2+6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - 1 + 0 & -1 + 0 + 2 & 1 + 2 + 6 \\ 0 + 1 + 0 & 0 + 0 + 3 & 0 - 2 + 9 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 9 \\ 1 & 3 & 7 \end{pmatrix}$$

$$\boxed{\begin{matrix}\text{Information} \\ A\end{matrix}} \xrightarrow[B \,=\, \text{the key.}]{\text{sending}} \boxed{A\,B} \xrightarrow[\text{by "dividing" } B]{\text{recovering } A}$$

<u>The transpose of matrix</u>:

Given $m \times n$ matrix A, the transpose of A is denoted by A^T, which is an $n \times m$ matrix

$$A^T = (b_{ji}) \quad \text{with} \quad \boxed{b_{ji} = a_{ij}}$$

<u>ex</u>: $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \end{pmatrix} \rightarrow A^T = \begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 1 \end{pmatrix}$

<u>ex</u>: $A = \begin{pmatrix} 2 & 1 \\ 2 & -1 \\ 0 & 1 \end{pmatrix} \rightarrow A^T = \begin{pmatrix} 2 & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix}$

<u>Def</u>: A matrix A is called symmetric if $A^T = A$.

<u>ex</u>: $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 1 & 4 \end{pmatrix}$ is a symmetric matrix.

It is symmetric over the main diagonal.

ex: Verify that if $A_{m\times n}$ is a symmetric matrix, it must be a square matrix, i.e., $m = n$.

Sol: $$A_{m\times n} = (A^T)_{n\times m}$$ b/c A is symmetric

Then they have the same size, i.e,

$m = n$ and $n = m$

$m = n$

So A is a square matrix.