MTH 2775 Exam I (7 problems, 100 points) Name: ________________

1. (3 points for each) Answer whether the following statements are true or false. Explain shortly your answers.

(a) Given two 3×2 matrices A and B, we always have $A + B = B + A$.

True, As they have the same size

(b) Let A be an $m \times n$ matrix. If $A \cdot A$ is well-defined, A must be a square matrix.

$A_{m \times n} \cdot A_{m \times n}$ is well-define when $m = n$, ie, A is square matrix.

(c) The matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

is not invertible.

$\det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0$

True, A is not invertible.

(d) The homogeneous system $Ax = 0$ always has at least a solution.

True, it alway has a solution zero, as $A0 = 0$

(e) The set $S = \{(x_1, x_2) | \, |x_1| \geq |x_2|\}$ is a subspace of $\mathbb{R}^2$.

False. Pick $(1,1) \in S$ as $|1| \geq |1|$

$+ \; (-1,1) \in S$ as $|-1| \geq |1|$

but $(0,2) \notin S$ as $|0| < |2|$

Addition is not closed. S is not a subspace

2. Let $A := \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$, $B := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -2 \\ 1 & -1 & 0 \end{bmatrix}$. Find each of the following items. If an item does not exist, say "DNE".

(a) (6 pts) AB and BA.

$$AB = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -2 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -2 \\ 1 & 0 & -2 \\ 0 & 1 & -2 \end{pmatrix}$$

$$BA = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -2 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 2 \\ 1 & 0 & 0 \end{pmatrix}$$

(b) (6 pts) A^2 and B^2.

$$A^2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B^2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -2 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -2 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ -2 & 3 & 0 \\ -1 & 1 & 2 \end{pmatrix}$$

(c) (8 pts) Is $(A - B)(A + B) = A^2 - B^2$? Explain your answer.

$(A-B)(A+B) = A^2 + AB - BA - B^2$

As $AB \neq BA$ on a,

$(A-B)(A+B) \neq A^2 - B^2$

3. Given $A = \begin{array}{c} \begin{array}{cccc} + & - & + & - \end{array} \\ \begin{bmatrix} 1 & 0 & 2 & 0 \\ 2 & 1 & 4 & 0 \\ 0 & 3 & 2 & 1 \\ 1 & 0 & 3 & 0 \end{bmatrix} \end{array}$

(a) (10 pts) Find $\det(A)$.

$$\det A = -0 + 0 - 1 \cdot \begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 4 \\ 1 & 0 & 3 \end{vmatrix} + 0$$

$$= -1 \cdot 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} =$$

$$= -1 \cdot (1)$$

$$= -1$$

(b) (10 pts) Find A^{-1}.

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_4 - R_1]{R_2 - 2R_1} \left(\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 - 3R_2} \left(\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 6 & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 \leftrightarrow R_4} \left(\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 6 & -3 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{R_4 - 2R_3} \left(\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 8 & -3 & 1 & -2 \end{array}\right) \xrightarrow{R_1 - 2R_3} \left(\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 3 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 8 & -3 & 1 & -2 \end{array}\right)$$

$$\underbrace{\begin{array}{cccc} 3 & 0 & 0 & -2 \\ -2 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 8 & -3 & 1 & -2 \end{array}}_{A^{-1}}$$

4. (15 pts) Find all solutions to the following linear system:

$$\begin{aligned} x_1 - 3x_2 - 10x_3 + 5x_4 &= 6 \\ x_1 + 4x_2 + 11x_3 - 2x_4 &= -1 \\ x_1 + 3x_2 + 8x_3 - x_4 &= 0 \\ 2x_1 + 7x_2 + 19x_3 - 3x_4 &= -1 \end{aligned}$$

$$\left(\begin{array}{cccc|c} 1 & -3 & -10 & 5 & 6 \\ 1 & 4 & 11 & -2 & -1 \\ 1 & 3 & 8 & -1 & 0 \\ 2 & 7 & 19 & -3 & -1 \end{array}\right) \xrightarrow[\substack{R_3 - R_1 \\ R_4 - 2R_1}]{R_2 - R_1} \left(\begin{array}{cccc|c} 1 & -3 & -10 & 5 & 6 \\ 0 & 7 & 21 & -7 & -7 \\ 0 & -6 & 18 & -6 & -6 \\ 0 & 13 & 39 & -13 & -13 \end{array}\right)$$

$$\xrightarrow{R_2/7} \left(\begin{array}{cccc|c} 1 & -3 & -10 & 5 & 6 \\ 0 & 1 & 3 & -1 & -1 \\ 0 & -6 & -18 & -6 & -6 \\ 0 & 13 & 39 & -13 & 13 \end{array}\right) \xrightarrow[R_4 - 13R_2]{R_3 + 6R_2} \left(\begin{array}{cccc|c} 1 & -3 & -10 & 5 & 6 \\ 0 & 1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 + 3R_2} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 3 \\ 0 & 1 & 3 & -1 & -1 \end{array}\right)$$

Set $x_3 = t$, $x_4 = s$ $\rightarrow x_2 = -3t + s$, $x_1 = t - s$

$$\rightarrow x = \begin{pmatrix} t - s \\ -3t + s \\ t \\ s \end{pmatrix}$$

5. (5 pts for each) Determine whether the following sets are subspace of $\mathbb{R}^4$:

(a) $S = \{(x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4 | \; x_1 + 2x_2 = x_3 x_4\}$.

Pick $x = (1, 1, 1, 3) \in S$ as $1 + 2 \cdot 1 = 1 \cdot 3$

We have $2x = (2, 2, 2, 6) \notin S$ as $\underbrace{2 + 2 \cdot 2}_{6} \neq 2 \cdot 6$

Scalar multiplication fails.

So S is not a subspace.

(b) $V = \{(x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4 | \; \underbrace{x_1 + 2x_2 = 3x_3 - x_4}\}$.

$x_1 + 2x_2 - 3x_3 + x_4 = 0$ is a homo. eqn. Thus V is a subspace.

6. (10 pts) Let $\mathbb{R}^{2\times 2}$ be the vector space of all 2×2 matrices. Show that the set $S = \{A \in \mathbb{R}^{2\times 2} | \; A = -A^T\}$ is a subspace of $\mathbb{R}^{2\times 2}$.

- Addition: Pick any $A, B \in S$:

$$A = -A^T$$
$$B = -B^T$$
$$A + B = -A^T + B^T$$
$$= -(A+B)^T$$

Addition is closed

- Scalar Multiplication: Pick any $A \in S$ and $\alpha \in \mathbb{R}$:

$$A = -A^T$$
$$\rightarrow \alpha A = -\alpha A^T$$
$$= -(\alpha A)^T$$

$\rightarrow$ Scalar multiplication is closed.

7. (5 pts for each) Let A be an $n \times n$ matrix such that $A^2 = 2A$.

(a) Find all possible values of $\det(A)$.

$$\det(A^2) = \det(2A)$$

$$(\det A)^2 = 2^n \det A$$

$$\rightarrow \det A = 0 \quad \text{or} \quad \det A = 2^n$$

(b) Verify that $A - \mathbb{I}$ is invertible and $(A - \mathbb{I})^{-1} = A - \mathbb{I}$.

$$\begin{aligned}(A - I)(A - I) &= A^2 - A - A + I \\ &= \underbrace{A^2 - 2A}_{} + I \\ &= 0 + I \\ &= I\end{aligned}$$

Thus $(A - I)^{-1} = A - I$.