(2) In $\mathbb{R}^2$, define
(5pts) $$\|x\|_\alpha = |x_1| + 2|x_2|$$
Show that it is a norm in $\mathbb{R}^2$.

(1) $\|x\|_\alpha \geq 0$ (trivial)

If $\|x\|_\alpha = 0$, we have $|x_1| + 2|x_2| = 0 \rightarrow x_1 = 0, x_2 = 0$, i.e., $x = 0$

(2) $$\|\lambda x\|_\alpha = |\lambda x_1| + 2|\lambda x_2|$$
$$= |\lambda|\,|x_1| + |\lambda|\,2|x_2| \qquad \text{for any } \lambda \in \mathbb{R}.$$
$$= |\lambda|\,\|x\|_\alpha$$

(3) $$\|x+y\|_\alpha = |x_1+y_1| + 2|x_2+y_2|$$
$$\leq |x_1| + |y_1| + 2|x_2| + 2|y_2|$$
$$= |x_1| + 2|x_2| + |y_1| + 2|y_2|$$
$$= \|x\|_\alpha + \|y\|_\alpha \qquad \text{(triangle inequality)}$$

(3) In $C[-\pi,\pi]$, we use the inner product
(5pts) $$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,g(x)\,dx$$

Verify that: $1+\cos x$ is perpendicular to $\sin x$.

$$\langle 1+\cos x, \sin x\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} (1+\cos x)\sin x \, dx$$

Define $u = 1+\cos x$

$du = -\sin x\, dx$

$x = -\pi \rightarrow u = 1+\cos(-\pi) = 0$

$x = \pi \rightarrow u = 1+\cos\pi = 0$

$$-\frac{1}{\pi}\int_0^0 u\,du$$
$$= \frac{-1}{\pi}\,\frac{u^2}{2}\Big|_0^0 = 0$$

Thus $1+\cos x \perp \sin x$.

Quiz 5 (20 points) Name

Show all your work for full credits.

① In $\mathbb{R}^{3\times 2}$, we use the inner product
(10 pts)
$$\langle A, B\rangle = \sum_{\substack{1\le i\le 3\\ 1\le j\le 2}} a_{ij} b_{ij} .$$

ⓐ Determine the value of $\langle A, B\rangle$, $\|A\|$, $\|B\|$

with
$$A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & -1 \\ 2 & 2 \\ 1 & -1 \end{pmatrix}$$

$$\langle A, B\rangle = 1\cdot 1 + 1\cdot(-1) + 2\cdot 2 + (-1)\cdot 2 + 0\cdot 1 + 1(-1)$$
$$= 1$$
$$\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{1^2 + 2^2 + 0^2 + 1^2 + (-1)^2 + 1^2} = \sqrt{8}$$
$$\|B\| = \sqrt{\langle B, B\rangle} = \sqrt{1^2 + 2^2 + 1^2 + (-1)^2 + 2^2 + (-1)^2} = \sqrt{12}$$

ⓑ Find the angle between A, B.
$$\cos\theta = \frac{\langle A, B\rangle}{\|A\|\,\|B\|} = \frac{1}{\sqrt{8\cdot 12}}$$
$$\theta = \arccos\left(\frac{1}{\sqrt{96}}\right).$$