Λ $A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{pmatrix}$

$N(A)$: $\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 3 & 4 & 0 \end{array}\right) \xrightarrow{R_3 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right)$

$\xrightarrow{R_3 - 2R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \to x_1 + x_3 = 0 \\ \to x_2 + x_3 = 0 \end{array}$

$\xrightarrow{\text{Set } x_3 = t} x = \begin{pmatrix} -t \\ -t \\ t \end{pmatrix} \longrightarrow N(A) = \text{span}\left\{\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}\right\}$

$R(A^T)$ = Range of (A^T) $= CS(A^T) \cong RS(A)$

$R(A)$ = Range of A $= CS(A)$

$\to R(A^T) = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$

6.1

1a) $A = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$

Solve: $\boxed{\det(A - \lambda I) = 0}$

$\det\left(\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = 0$

$\det \underbrace{\begin{pmatrix} 3-\lambda & 2 \\ 4 & 1-\lambda \end{pmatrix}}_{A - \lambda I} = 0$

$(3-\lambda)(1-\lambda) - 2 \cdot 4 = 0$

$3 - 3\lambda - \lambda + \lambda^2 - 8 = 0$

$\lambda^2 - 4\lambda - 5 = 0$

$(\lambda + 1)(\lambda - 5) = 0$

$\lambda = -1$ or $\lambda = 5$

$-1, 5$ are our eigenvalues.

- $\lambda = -1 \to$ Solve $(A - \lambda I)x = 0$

$\left(\begin{array}{cc|c} 4 & 2 & 0 \\ 4 & 2 & 0 \end{array}\right) \longrightarrow \ldots\ldots$

$x = \begin{pmatrix} -\frac{1}{2}t \\ t \end{pmatrix} \longrightarrow \text{span}\left\{\begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}\right\}$ = eigenvectors at eigenvalue $\lambda = -1$

- $\lambda = 5 \to$ Solve $(A - \lambda I)x = 0$

$\left(\begin{array}{cc|c} -2 & 2 & 0 \\ 4 & -4 & 0 \end{array}\right)$

$\to x = \begin{pmatrix} t \\ t \end{pmatrix} \longrightarrow \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$.

2/Review $A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -4 & 1 & 1 \\ -3 & 3 & 2 \\ 1 & -2 & -2 \end{pmatrix}$

(a) Angle between A and B:

$\cos\theta = \dfrac{\langle A, B \rangle}{\|A\| \cdot \|B\|}$ $\qquad x^T x \qquad x^T y$

$\|x\| = \sqrt{x^T x}$

$\langle A, B \rangle = -4 + 2 + 2 - 3 + 0 + 4 + 3 - 2 + 2$
$= 4$

$\|A\| = \sqrt{\langle A, A \rangle} = \sqrt{\underbrace{1^2 + 2^2 + 2^2}_{9} + \underbrace{1^2 + 0^2 + 2^2}_{5} + \underbrace{3^2 + 1^2}_{10} + \underbrace{(-1)^2}_{1}}$

$= \sqrt{25}$

$= 5$

$\|B\| =$

11/5.4 $\langle p, q \rangle = \sum_{i=1}^{5} p(x_i)\, q(x_i)$, $x_i = \dfrac{i-3}{2}$

$x_1 = -1,\ x_2 = -\frac{1}{2},\ x_3 = 0,\ x_4 = \frac{1}{2},\ x_5 = 1$

$x \perp x^2$

$\langle \underbrace{x}_{p}, \underbrace{x^2}_{q} \rangle = -1 \cdot (-1)^2 + -\frac{1}{2} \cdot \frac{1}{4} + 0 \cdot 0 + \frac{1}{2} \cdot \frac{1}{4} + 1 \cdot 1^2$

$= 0$

6/ $L(p) = p(x) - x\,p(1) - p(0)$

(a) Verify L is a linear trans.

$L(\alpha p + \beta q) = \alpha L(p) + \beta L(q)$

LHS $= (\alpha p + \beta q)(x) - x(\alpha p + \beta q)(1) - (\alpha p + \beta q)(0)$

$= \alpha p(x) + \beta q(x) - \alpha x\, p(1) - \beta x\, q(1) - \alpha p(0) - \beta q(0)$

RHS $= \alpha(p(x) - x\,p(1) - p(0)) + \beta(q(x) - x\,q(1) - q(0))$

$= \alpha p(x) - \alpha x\,p(1) - \alpha p(0) + \beta q(x) - \beta x\,q(1) - \beta q(0)$

b) Find ker L:

$\boxed{\text{Solve } L(p) = 0}$ $\qquad p(x) - x\,p(1) - p(0) = 0$

Set $p(x) = ax^2 + bx + c$

$L(p) = ax^2 + bx + c - x(a + b + c) - c$

$= ax^2 + bx + c - ax - bx - cx - c$

$= ax^2 - (a + c)x = 0$ (for any x)

$\hookrightarrow a = 0 \qquad a + c = 0 \to c = 0$

$p(x) = bx$

$\to \ker L = \text{span}\{x\} \to \dim(\ker L) = 1$

c) range L

$L(p) = ax^2 - ax - cx$

$= \underbrace{a(x^2 - x) - cx}$

a linear comb. $x^2 - x$, x

range $(L) = \text{span}\{x^2 - x, x\}$.

$\to \dim(\text{range}(L)) = 2$.

d) Matrix representation.

$E = [x^2, x, 1)$

$[p]_E = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \longrightarrow \begin{pmatrix} a \\ -a-c \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$(L(p))_E = \begin{pmatrix} a \\ -a-c \\ 0 \end{pmatrix}$

9/b $N(A) \overset{\text{(intersection)}}{\cap} R(A^T) = \{0\}$

pick any $x \in N(A) \cap R(A^T)$, i.e., $x \in N(A)$ and $x \in R(A^T)$

Fundamental Subspace Theorem

$N(A) \perp R(A^T)$

$\to x^T x = 0$

$\|x\| = 0 \to \boxed{x = 0}$