5.2 (cont):

Consider $\underbrace{Ax = 0}$, A is an $m \times n$ matrix

$$\underbrace{\begin{pmatrix} \sim\sim\sim\sim \\ \\ \\ \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \\ \\ \\ \end{pmatrix}}_{x} = 0$$

rows of A $\perp$ x

$N(A) = \{x \mid Ax = 0\}$ - null space of A.

So the $\underbrace{\text{row space of } A}$ is orthogonal to the null space of A.

$$CS(A^T) \perp N(A)$$

Theorem (Fundamental Theorem of Linear Algebra)

$$(CS(A^T))^\perp = N(A)$$

$$CS(A^T) = N(A)^\perp$$

$$(CS(A))^\perp = N(A^T)$$

$$\text{Range}(A) = CS(A) = N(A^T)^\perp$$

(Fundamental Subspace Theorem)

$$\text{Range}(A) = \{Ax \mid x \in \mathbb{R}^n\}$$

$$Ax = \underbrace{x_1(\text{Column}_1 \text{ of } A) + x_2(\text{Column}_2) + \cdots + x_n(\text{Column}_n)}_{\text{linear comb. of columns of } A}$$

$\hookrightarrow$ Range of A = Column space of A.

Theorem: Let S be a subspace of $\mathbb{R}^n$. Then

$$\dim S + \dim S^{\perp} = n$$

$*$ $S = N(A) \rightarrow \dim(S) = \text{Nullity}(A)$

$S^{\perp} = N(A)^{\perp} = CS(A^T) \rightarrow \dim(S^{\perp}) = \dim(CS(A^T))$

$= \text{rank}(A)$

$\dim S + \dim S^{\perp} = \text{Nullity}(A) + \text{rank}(A)$

$= n.$

Theorem: $(S^{\perp})^{\perp} = S$.

$\hookrightarrow$ $S + S^{\perp} = \mathbb{R}^n$

For any $x \in \mathbb{R}^n$, we can decompose it in the sum: $x = u + v$, where $u \in S$ and $v \in S^{\perp}$.

ex: let $A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \end{pmatrix}$. Find $N(A)^{\perp}$ and $N(A^T)^{\perp}$.

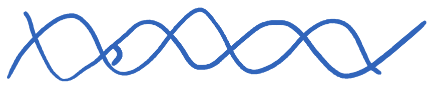
Sol: $N(A)^{\perp} = CS(A^T)$ (very much the $RS(A)$)

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & -2 & -2 \\ 0 & 1 & -1 & -2 \end{pmatrix} \xrightarrow[-2]{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$CS(A^T) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

$\rightarrow CS(A^T) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$

$$N(A^T)^{\perp} = CS(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\}$$



A signal : $2\cos x - 3\sin 2x$

5.4 Inner product Spaces

In $\mathbb{R}^n$: $x^T y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$

is the scalar product / inner product / dot product of 2 vectors $x, y \in \mathbb{R}^n$.

Def : Let V be a vector space. An inner product on V is an operation that assigns each pair (x, y) in V a real number $\langle x, y \rangle$ satisfying 3 conditions:

① $\langle x, x \rangle \geq 0 \quad \forall x \in V$ and

$\langle x, x \rangle = 0$ iff $x = \vec{0}$ (vector zero).

② $\langle x, y \rangle = \langle y, x \rangle \quad \forall x, y \in V$.

③ For any fixed $x \in V$, the function

$\langle x, \cdot \rangle : V \longrightarrow \mathbb{R}$ is a linear

transformation, i.e.,

$$\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$$

for any $y, z \in V$, $\alpha, \beta \in \mathbb{R}$

ex: **In $\mathbb{R}'$:**

$\langle x,y\rangle := 2xy$ for any $x, y \in \mathbb{R}$.

① $\langle x,x\rangle = 2x\cdot x = 2x^2 \geq 0 \quad \forall x \in \mathbb{R}$

$\langle x,x\rangle = 0$ iff $2x^2 = 0$ iff $x = 0$

② $\langle x,y\rangle = 2xy$
$\langle y,x\rangle = 2yx$
the same.

③ $\langle x,y\rangle = 2xy$ is a linear function $\boxed{\text{w.r.t. } y}$

All three conditions are satisfied. So $\langle x,y\rangle$ is an inner product.

ex: $\langle x,y\rangle := x + y$.

$\langle -1,-1\rangle = -2 < 0$

$\longrightarrow$ $\langle x,y\rangle$ is not an inner product

ex: $\langle x,y\rangle := (x+y)^2$

$\langle 1,-1\rangle = (1+-1)^2 = 0$, but $1,-1 \neq 0$

$\langle x,y\rangle$ fails the first condition.

$\langle x, y\rangle$ is NOT an inner product.

<u>ex</u> : $\underline{\mathbb{R}^n}$: $\boxed{\text{let } \alpha_1, \alpha_2, \dots, \alpha_n > 0}$ and define

$$\langle x, y\rangle := \alpha_1 x_1 y_1 + \alpha_2 x_2 y_2 + \dots + \alpha_n x_n y_n$$

<u>is an inner product</u>

<u>ex</u> : in $\mathbb{R}^2$

$$\langle x, y\rangle := 2x_1y_1$$

Choose $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \langle x, x\rangle = 2\cdot 0 \cdot 0 = 0$

but $x \neq 0$.

So $\langle x, y\rangle$ is NOT an inner product.

<u>ex</u> : <u>In $\mathbb{R}^{m\times n}$</u>,

$$\langle A, B\rangle := \underbrace{\sum_{\substack{1\le i\le m \\ 1\le j\le n}} a_{ij} b_{ij}}_{\text{a number.}}$$

(regular/standard inner product)

<u>ex</u> : For $C[a,b]$ = set of continuous functions on the interval $[a,b]$. For any $f, g \in C[a,b]$, define the inner product:

$$\langle f, g\rangle := \int_a^b f(x)\, g(x)\, dx$$

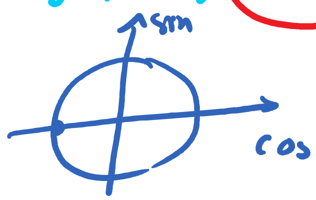
Consider $C[-\pi, \pi]^{a}$, compute

$$\langle \sin x, \cos x \rangle = \int_{-\pi}^{\pi} \sin x \underbrace{\cos x \, dx}.$$

Substitution

$$\int f(u(x)) \, u'(x) \, dx$$

$$= \int f(u) \, du$$



$u = \sin x \rightarrow du = \cos x \, dx$

$x = -\pi \rightarrow u = 0$

$x = \pi \rightarrow u = 0$

$$= \int_0^0 u \, du$$

$$= \frac{u^2}{2} \Big|_0^0 = 0$$