MTH 2775 **Sample Exam II**

1. Answer whether the following statements are true or false. Explain shortly your answers if they are true or give a counterexample if they are false.

   (a) $(1,2,3)^T$, $(2,-1,0)^T$, $(-2,3,-1)^T$, $(-1,2,3)^T$ are linearly independent.

   (b) $E = \{(1,1)^T, (1,2)^T\}$ forms a basic of $\mathbb{R}^2$. Then $[x]_E = (1,1)$ if and only if $x = (2,3)^T$.

   (c) $\operatorname{rank} \begin{pmatrix} 1 & 2 & -1 \\ -1 & -2 & 1 \\ 2 & 4 & -2 \end{pmatrix} = 3.$

   (d) $L(x_1, x_2) = (x_1 x_2, x_2)$ is a linear operator from $\mathbb{R}^2$ to $\mathbb{R}^2$.

   (e) Let $\|\cdot\|$ is a norm in $\mathbb{R}^n$. Then $\|x\|\|y\| \geq |x^T y|$.

2. (10 pts) Whether the following vectors $x+2, x+1, x^2-1$ are linearly independent in P_3. What is the span of $\{x+2, x+1, x^2-1\}$?

3. Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & -3 & -2 \\ 3 & 3 & 0 & 2 \end{pmatrix}$.

(a) Find a basis of $N(A)$, row space of A, column space of A.

(b) Find nullity(A) and rank A.

4. Given $v = (1, -1, 1, 1)^T$ and $w = (4, 2, 2, 1)^T$.

   (a) Determine the angle between v and w.

   (b) Find the orthogonal complement of $V = \operatorname{span}\{v, w\}$.

5. Determine whether the following are linear transformation in $\mathcal{C}^1[-1, 1]$, the set of all differentiable functions in [-1,1]:

   (a) $L(f(x)) = x^2 + f(x)$ for $f \in \mathcal{C}^1[-1, 1]$.

   (b) $L(f(x)) = x^2 f(x) + f'(x)$ for $f \in \mathcal{C}^1[-1, 1]$.

6. (a) Define $L : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ by $L(A) = A + A^T$.

   (a) Show that L is a linear operator.

   (b) Find $\ker L$ and its dimension.

   (c) Find the matrix representation of L.

7. Let $\|\cdot\|$ be the Euclidean norm in $\mathbb{R}^n$. For any $x, y \in \mathbb{R}^n$,

(a) Show that $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2x^Ty$

(b) Show that $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

8. (5 pts for each) Given an 5×4 matrix A with $\text{rank}(A) = 4$.

(a) How many solutions are there for equation $Ax = 0$? Explain your answer.

(b) How many solutions are there for equation $A^Ty = 0$? Explain your answer.

Answer keys.

1. (a) F (b) T (c) F (d) F (e) T.

2. Yes.

3. (a) A basis of N(A) is $\{(3,-3,1,0)^T, (\frac{8}{3}, -\frac{10}{3}, 0, 1)^T\}$. A basis for row space is $\{(1,2,3,4)^T, (0,-3,-9,-10)^T$
   A basis for column space is $\{(1,2,3)^T, (0,1,1)^T\}$.
   (b) $\text{nullity}(A) = 2$ and $\text{rank}(A) = 2$.

4. (a) 60^o (b) $\text{span}\{(2/3, 1/3, 1, 0)^T, (1/2.1/2, 0, 1)^T\}$.

5. (a) No (b) Yes

6. (b) $\ker L = \left\{ \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \right\}$ is the set of all anti-symmetric matrices in $\mathbb{R}^2$ and its dimension is 1.
   (c) $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

7. Hint: $\|x+y\|^2 = (x+y)^T(x+y)$.

8. Use Rank-Nullity theorem (a) 1 solution (b) Infinitely many solutions.