

Feb 8, 2022

1. (3 for each) Answer whether the following statements are true or false. Explain shortly your answers if they are true or give a counterexample if they are false.

(a) Given two 3×2 matrices A and B , we always have $A + B = B + A$.

(b) Let A be a nonsingular matrix. Equation $Ax = 0$ only has trivial solution.

(c) Given an 3×3 matrix A , the elementary matrix corresponding to the row operation of multiplying row 3 of A by 2 is $\begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

(d) Given two $n \times n$ matrices A and B with $AB = 4\mathbb{I}$. Then A is a nonsingular matrix.

(e) The determinant of the matrix $\begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ is 2.

2. Given $A = \begin{bmatrix} 1 & 4 & 3 \\ 1 & 4 & 4 \\ 2 & 7 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & -1 & 1 & 0 \end{bmatrix}$

(a) (5pts) Find AB and BA .

(b) (5 pts) Find $2A + B$ and $3B - A$.

(c) (10 pts) Find A^{-1} and B^{-1} .

(d) (5 pts) Solve $AX = B$ for X .

3. (10 pts) Find all solutions to the following linear system:

$$x_1 - 3x_2 - 10x_3 + 5x_4 = 6$$

$$x_1 + 4x_2 + 11x_3 - 2x_4 = -1$$

$$x_1 + 3x_2 + 8x_3 - x_4 = 0$$

4. (a) (4 pts) State the definition of singular matrix.

- (b) (6 pts) State three conditions that each of which is equivalent to A being singular.

5. Consider the linear system

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 2 \\ 2 & -4 & 2 & 2 \\ 1 & -3 & a^2 - a & b - a \end{array} \right]$$

Find all values of a and b such that the system has:

(a) (10 pts) a unique solution.

(b) (5 pts) no solution.

(c) (5 pts) infinitely many solutions.

(d) (5 pts) three solutions.

6. (10 pts) Given

$$A := \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & 4 & 4 & 0 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

Find $\det(A)$. Determine whether A is nonsingular or not.

7. (10 pts) Let A and B be two matrices satisfying $AB = BA$. Explain why A and B must be square matrices with the same size.

8. Let A be an $n \times n$ matrix satisfying that $A^2 = O_{n \times n}$
- (a) (5 pts) Verify that A is singular.

- (b) (5 pts) Verify that $A - 2\mathbb{I}$ is nonsingular with $(A - 2\mathbb{I})^{-1} = -\frac{1}{4}(A + 2\mathbb{I})$.