19/1.4 Let A be an $n \times n$ matrix. Show that if $A^2 = 0$ then $I - A$ is invertible and $\underbrace{(I-A)^{-1} = I + A}$.

Review: B is the inverse of A iff
$AB = I$

the inverse of $I - A$ is $I + A$

We just need to check:

$$\underbrace{(I - A)(I + A)} \stackrel{?}{=} I$$

$$LHS = \underbrace{I \cdot I} + \underbrace{I \cdot A} - A \cdot I - \underbrace{A \cdot A}$$

$$= I + \underbrace{A - A}_{0} - 0$$

b/c $A^2 = 0$ in the initial assumption.

$$\boxed{= I}$$

$= RHS$ (true)

So $(I - A)^{-1} = I + A$ and $I - A$ is invertible.

Chapter 2: Determinants

2.1. The determinant of a matrix.

0 is not invertible, any nonzero number is invertible.

ex: $\underbrace{A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}}_{\text{invertible}}$ and $\underbrace{B = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}}_{\text{non invertible}}$.

ex: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. When is A invertible?

$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right) \xrightarrow{R_1/a} \left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} c & d & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} c & d & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 - cR_1} \left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \underbrace{d - \frac{bc}{a}}_{\frac{ad-bc}{a}} & -\frac{c}{a} & 1 \end{array}\right) \xrightarrow[ad-bc \neq 0]{R_2 / \frac{ad-bc}{a}} \left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right)$$

$$\xrightarrow{R_1 - \frac{b}{a}R_2} \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{a} + \frac{bc}{a(ad-bc)} & -\frac{b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right)$$

A is invertible iff $ad - bc \neq 0$

Def: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an 2×2 matrix. The determinant of A is defined by

$$\det A := ad - bc$$

(or $|A|$)

ex: $A = \begin{pmatrix} 1 & 4 \\ 3 & 5 \end{pmatrix} \to \det A = 5 - 12 = -7$

$\to$ A is invertible b/c $\det A \neq 0$

ex: $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \to \det A = 4 - 4 = 0$

$\to$ A is noninvertible b/c $\det A = 0$.

Def: The determinant of an 3×3 matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is computed by:

$$\det A = a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

determinants

ex: $A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 2 \\ 0 & -1 & 2 \end{pmatrix}$. Find $\det A$.

Sol: $\det A = +1 \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} - 0 \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix}$

$= 1(2-(-2)) - 0 - 1(-1-0)$

$= 1 \cdot 4 - 0 + 1$

$= 5.$

A is invertible b/c $\det A \neq 0$.

Def: Let A be an $n \times n$ matrix. Let M_{ij} be the $(n-1)\times(n-1)$ matrix obtained from A by deleting the row i and the column j. The determinant of M_{ij} is called the minor of a_{ij}. The cofactor A_{ij} is defined by

$$A_{ij} = (-1)^{i+j} \det(M_{ij})$$

The the determinant of A is

$$\det A = a_{11}\det(M_{11}) - a_{12}\det(M_{12}) + a_{13}\det(M_{13})$$

$$\cdots\cdots$$

$$= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} + \dots + a_{1n}A_{1n}$$

ex: $A = \begin{pmatrix} 1 & 1 & 0 & -2 \\ 1 & 1 & 1 & 0 \\ 0 & -1 & 2 & 3 \\ 0 & 1 & 0 & -1 \end{pmatrix}$. Find $\det A$.

Sol: $\det A = 1 \cdot \begin{vmatrix} 1 & 1 & 0 \\ -1 & 2 & 3 \\ \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \\ \end{vmatrix} + 0(\quad) - (-2)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ \end{vmatrix}$

Sol.

$$\det A = 1\cdot\begin{vmatrix} -1 & 2 & 3 \\ 1 & 0 & -1 \end{vmatrix} - 1\begin{vmatrix} 0 & 2 & 3 \\ 0 & 0 & -1 \end{vmatrix} + 0(\quad) - (-2)\begin{vmatrix} 0 & -1 & 2 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= 1\left[1\begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} - 1\begin{vmatrix} -1 & 3 \\ 1 & -1 \end{vmatrix} + 0\right] - 1\cdot\left[1\begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} - 1\cdot\begin{vmatrix} 0 & 3 \\ 0 & -1 \end{vmatrix} + 0\right]$$

$$+2\left[1\begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} + 1\begin{vmatrix} 0 & -1 \\ 0 & 1 \end{vmatrix}\right]$$

$$= 1(-2 + 2 + 0) - 1(-2 - 0 + 0)$$

$$+ 2(-2 - 0 + 0)$$

$$= 0 + 2 - 4$$

$$= -2.$$

* **Remark**: We can use any row or column to proceed / calculate the determinant:

$$\det A = (-1)^{i+1} A_{i1} + (-1)^{i+2} A_{i2} + \dots + (-1)^{i+n} A_{in}$$

We also need the matrix sign:

$$\begin{pmatrix} + & - & + & - & + & - & \cdots \\ - & + & - & + & - & + & \cdots \\ + & - & + & - & + & - & \cdots \\ - & & & & & & \\ + & & & & & & \\ - & & & & & & \\ \vdots & & & & & & \end{pmatrix}$$

ex: $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 3 \\ 0 & 2 & 0 & -2 \end{pmatrix}$. Find det A

<u>Use row 2 to proceed</u>:

Use row 2 to proceed :

$$\det A = -0 \cdot (\quad) + 1 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 0 & 0 & -2 \end{vmatrix} - 0 + 0$$

(two zeros)

$$= 1 \left[\underbrace{+0 - 0}_{} + (-2) \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \right]$$

$$= 1 \left(0 + -2 \, (0) \right)$$

$$= 0$$

$\boxed{\text{A is not invertible}}$