14) $L: V \to W \xrightarrow{\text{a basis}} F = \{w_1, w_2, \dots, w_m\}$

$V \downarrow$ a basis $E = \{v_1, v_2, \dots, v_n\}$

$[L(v)]_F = \underset{m \times n}{A} [v]_E$

$L: P_3 \to P_2$, $L(p) = p' + p(0)$

In P_3, $E = \{x^2, x, 1\}$

P_2, $F = \{2, 1-x\}$

Pick $p = ax^2 + bx + c \to [p]_E = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$L(p) = p' + p(0)$

$= 2ax + b + c$

$= \alpha \cdot 2 + \beta \cdot (1-x)$ a linear comb. of $2, 1-x$

$= 2\alpha + \beta - \beta x$

$\beta = -2a$

$2\alpha + \beta = b + c \to 2\alpha = b + c + 2a$

$\to \alpha = \dfrac{b+c+2a}{2}$.

$[L(p)]_F = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \frac{b+c+2a}{2} \\ -2a \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{2}{2} & \frac{1}{2} & \frac{1}{2} \\ -2 & 0 & 0 \end{pmatrix}}_{A} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

(a) $[L(x^2 + 2x - 3)]_F = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ -2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{2} \\ -2 \end{pmatrix}$.

$$= \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

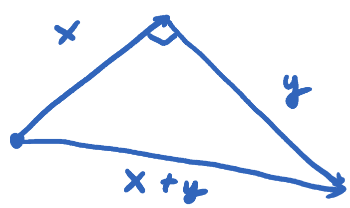
(d) $$[L(x^2+2x)]_F = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ -2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

* <u>Pythagorean theorem</u> : Suppose $x, y \in \mathbb{R}^n$ and $x \perp y$

Then we have

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2$$



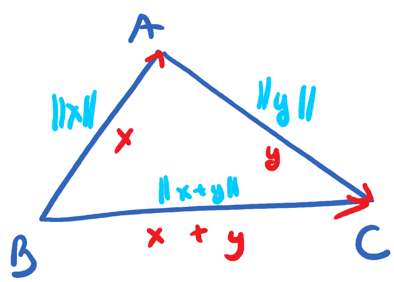
<u>Proof</u> : $\|x+y\|^2 = (x+y)^T(x+y)$

$$= \underbrace{x^Tx}_{} + \underbrace{x^Ty}_{} + \underbrace{y^Tx}_{} + y^Ty$$

$$= \|x\|^2 + 0 \quad 0 + \|y\|^2$$

$$\to \|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

* <u>Triangle Inequality</u> :



$$AB + AC \geq BC$$

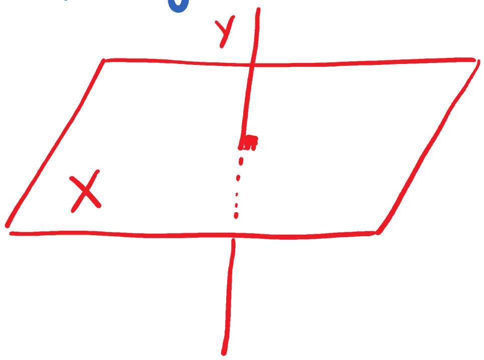
Given $x, y \in \mathbb{R}^n$, we have

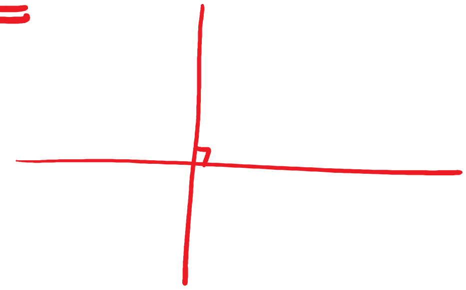
$$\|x\| + \|y\| \geq \|x \pm y\|$$

<u>5.2</u> Orthogonal Subspaces.

<u>Def</u> : 2 subspaces X and Y of $\mathbb{R}^n$ are said to be orthogonal if :

$$x^T y = 0 \quad \text{for any} \quad x \in X \quad \text{and} \quad y \in Y$$





ex: Given $X = \{ x \mid \underbrace{x_1 + x_2 + 2x_3 = 0}\}$

and $Y = \text{span}\{(1, 1, 2)^T\}$.

Then $X \perp Y$.

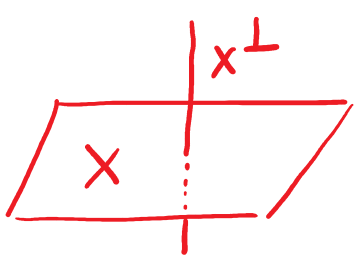
Sol: Pick any $x \in X$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

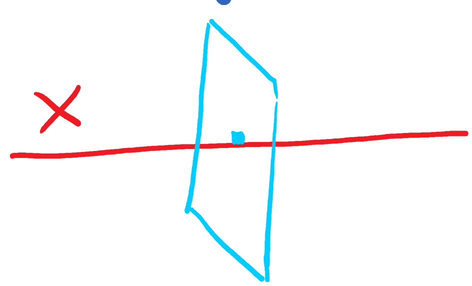
any $y \in Y$, $y = \begin{pmatrix} \alpha \\ \alpha \\ 2\alpha \end{pmatrix}$ α = a scalar

$$\begin{aligned} x^T y &= \alpha x_1 + \alpha x_2 + 2\alpha x_3 \\ &= \alpha(\underbrace{x_1 + x_2 + 2x_3}) \\ &= \alpha \cdot 0 \\ &= 0 \end{aligned}$$

Def: Let X be a subspace of $\mathbb{R}^n$. We define the orthogonal space of X by

$$X^\perp = \{ y \in \mathbb{R}^n \mid x^T y = 0 \text{ for any } x \in X \}$$





ex: Let $X = \left\{ x \;\middle|\; \begin{matrix} x_1 + x_2 - x_3 = 0 \\ x_1 - x_2 + 3x_3 = 0 \end{matrix} \right\}$.

Find $X^\perp$.

Sol: Find X: $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & -1 & 3 & 0 \end{array}\right) \xrightarrow{R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 4 & 0 \end{array}\right)$

$\xrightarrow{R_2 / -2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right) \xrightarrow{R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right)$

Free variable $x_3 = t \to x_2 = 2t, \quad x_1 = -t$

$$\to X = \left\{ \begin{pmatrix} -t \\ 2t \\ t \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

$x^T y = 0$

$$X^\perp = \left\{ y \in \mathbb{R}^3 \;\middle|\; -t \cdot y_1 + 2t\, y_2 + t\, y_3 = 0 \right\}$$

$$t\,(\underbrace{-y_1 + 2y_2 + y_3}) = 0$$

$$\to -y_1 + 2y_2 + y_3 = 0$$

$$\to X^\perp = \{ y \in \mathbb{R}^3 \mid -y_1 + 2y_2 + y_3 = 0 \}.$$

ex: $Y = \operatorname{span}\{ (1,2,1)^T, (1,2,2)^T \}$. Find $Y^\perp$.

$$Y^\perp = \{ x \mid \underbrace{x \perp (1,2,1)^T}_{x_1 + 2x_2 + x_3 = 0},\; \underbrace{x \perp (1,2,2)^T}_{x_1 + 2x_2 + 2x_3} \}$$

$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 2 & 2 & 0 \end{array}\right) \xrightarrow{R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$

$$\left.\begin{array}{l} \text{free variable } x_2 = t \\ x_3 = 0 \\ x_1 = -2t \end{array}\right\} \to Y^\perp = \left\{ \begin{pmatrix} -2t \\ t \\ 0 \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

$\{ \ldots \}^T$

$x_3 = 0$

$x_1 = -2t$

$= \text{span}\{(-2, 1, 0)^T\}.$