1. I cant)

Thursday, January 9, 2025 5:45 PM

System of linear equations :

$$\underbrace{\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right\} m \text{ equations}}_{n \text{ variables.}}$$

- A is an $m \times n$ matrix :

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \left.\begin{array}{l} \to \text{row } 1 \\ \to \text{row } 2 \\ \vdots \\ \to \text{row } m \end{array}\right\} m \text{ rows}$$

$$\underbrace{\begin{array}{cccc} \downarrow & \downarrow & \cdots & \downarrow \\ \text{column } 1 & \text{column } 2 & & \text{column } n \end{array}}_{n \text{ columns.}}$$

- $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n \times 1}$ – the column vector (variable)

- $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}_{m \times 1}$ – the RHS column vector.

The above linear system can be written by:

$$\underset{\text{coefficient matrix}}{A}\,\underset{\text{variable vector}}{x} = \underset{\text{RHS vector.}}{b}$$

↳ The augmented matrix of the linear system is

$$(A \mid b)$$

ex: Solve

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \quad ① \\ 3x_1 - x_2 - 3x_3 &= -1 \quad ② \\ 2x_1 + 3x_2 + x_3 &= 4 \quad ③ \end{aligned}$$

→ Write the augmented matrix $(A \mid b)$

$$\left(\underbrace{\begin{matrix}1&2&1\\3&-1&-3\\2&3&1\end{matrix}}_{A}\left|\begin{matrix}3\\-1\\4\end{matrix}\right.\right) \xrightarrow[R_3\to R_3-2R_1]{\overset{(\text{row }1)}{R_2\to R_2-3R_1}} \left(\begin{array}{ccc|c}1&2&1&3\\0&-7&-6&-10\\0&-1&-1&-2\end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c}1&2&1&3\\0&-1&-1&-2\\0&-7&-6&-10\end{array}\right) \xrightarrow{R_3 \to R_3 - 7R_2} \left(\begin{array}{ccc|c}1&2&1&3\\0&-1&-1&-2\\0&0&1&4\end{array}\right)$$

Back substitution:

- R_3 means: $0x_1 + 0x_2 + 1\cdot x_3 = 4$

  $x_3 = 4.$

- R_2 means: $0 - x_2 - x_3 = -2$

  $-x_2 - 4 = -2 \rightarrow x_2 = -2.$

- R_1 means: $x_1 + 2x_2 + x_3 = 3$

  $x_1 - 4 + 4 = 3$

  $x_1 = 3$

So $x = \begin{pmatrix}3\\-2\\4\end{pmatrix}$ is the solution.

* We have 3 kinds of row operations:

① Interchange 2 rows

② Multiply one row by a nonzero number
(Divide)

③ Add a multiple of one row to another row.
(Substract)

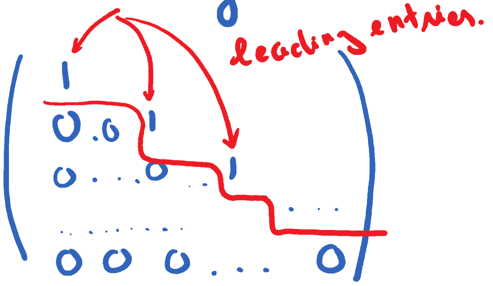
1.2 Row Echelon Form.

Def : An $m \times n$ matrix A is called a row-echelon matrix if

① The first nonzero entry in each nonzero row is 1.

② If row k does not consist entirely of zeros, the first nonzero element (from left to right) lies strictly to the right of the first nonzero element in the preceding row.

③ If there are rows of all zeros, they are below the rows having nonzero elements.



ex :

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} ; \quad B = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} ; \quad C = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 2 & 4 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

row-echelon. not row-echelon not row-echelon.

$$\downarrow R_2 \longleftrightarrow R_3$$

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\downarrow R_2 \cdot \frac{1}{2}$$

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & \frac{5}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ (row-echelon) $\begin{pmatrix} 0 & 1 & 2 & \frac{5}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (row-echelon)

* Given $m \times n$ matrix A and let E be its row-echelon reduction.

The first nonzero entry in E is called the leading entry.

If we consider $Ax = b$ or $(A|b)$, the variable corresponding to the leading entries are called the leading variables. Other variables are called the free variables.

ex:

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ 1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

Variables: x_1, x_2, x_3, x_4

Leading variables: x_1, x_3, x_4.

Free variable: x_2.

Back substitution:

From R_3: $x_4 = 1$

From R_2: $x_3 + 2x_4 = -3$

$x_3 + 2 = -3 \longrightarrow x_3 = -5$.

From R_1: $x_1 + 2x_2 + x_3 + \underbrace{x_4} = 1$

$x_1 + 2x_2 - 5 + 1 = 1$

$x_1 + 2x_2 = 5$

Set $x_2 = s$ (parameter)

$x_1 + 2x_2 = 5$

Set $x_2 = s$ (parameter)

$x_1 = 5 - 2s$

$$\rightarrow x = \begin{pmatrix} 5-2s \\ s \\ -5 \\ 1 \end{pmatrix}, \quad s = \text{parameter}.$$

* __Remark__: Free variables are just parameters (t, s, u, v ....) in the linear system

__ex__: Turn the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 4 \end{pmatrix}$$

to the row-echelon form by using __only__ row operations!.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 4 \end{pmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 2 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3/2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

row echelon.