(cnt.)

ex: $P_3 = \{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\}$

Define $\langle p, q \rangle := p(-1)\,q(-1) + p(0)\cdot q(0) + p(1)\cdot q(1)$

for any $p, q \in P_3$.

① $\langle p, p \rangle = (p(-1))^2 + (p(0))^2 + (p(1))^2 \geq 0$

When $\langle p, p \rangle = 0$, i.e.,

$$\begin{cases} p(-1) = 0 \\ p(0) = 0 \\ p(1) = 0 \end{cases} \xleftrightarrow{p(x) = ax^2+bx+c} \begin{cases} a - b + c = 0 \\ c = 0 \\ a + b + c = 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \xrightarrow{R_3 - R_1} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right) \xleftrightarrow{R_3/2 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$a = b = c = 0$, i.e., $p(x) \equiv 0$.

② $\langle p, q \rangle = \langle q, p \rangle = q(-1)\,p(-1) + q(0)\,p(0) + q(1)\,p(1)$

$\forall\, p, q \in P_3$

③ Linearity:

$\langle p, \cdot \rangle$ is linear with fixed p.

Check

$\langle p, \underbrace{\alpha q_1 + \beta q_2} \rangle = \alpha \langle p, q_1 \rangle + \beta \langle p, q_2 \rangle$

LHS: $p(-1)\left(\alpha q_1(-1)+\beta q_2(-1)\right)+p(0)\left(\alpha q_1(0)+\beta q_2(0)\right)$
$+\,p(1)\left(\alpha q_1(1)+\beta q_2(1)\right)$

$= \alpha\, p(-1)q_1(-1)+\beta\, p(-1)q_2(-1)+\alpha\, p(0)q_1(0)+\beta\, p(0)q_2(0)$
$+\,\alpha\, p(1)q_1(1)+\beta\, p(1)q_2(1).$

RHS $= \underbrace{\alpha\langle p,q_1\rangle}+\beta\langle p,q_2\rangle$

$= \alpha\, p(-1)q_1(-1)+\alpha\, p(0)q_1(0)+\alpha\, p(1)q_1(1)+$
$\beta\, p(-1)q_2(-1)+\beta\, p(0)q_2(0)+\beta\, p(1)q_1(1).$

match

ex: For P_n

$$\langle p,q\rangle = p(\alpha_1)q(\alpha_1)+p(\alpha_2)q(\alpha_2)+\cdots+p(\alpha_n)q(\alpha_n),$$

where $\alpha_1,\alpha_2,\ldots,\alpha_n$ are known distinct numbers in $\mathbb{R}$.

ex: Consider P_3 with

$$\langle p,q\rangle = p(-1)q(-1)+p(0)q(0)+p(1)q(1).$$

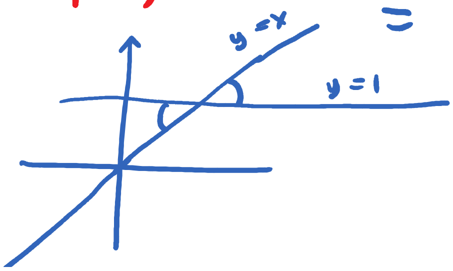
Find $\langle x,1\rangle$ and $\langle x^2,1\rangle$.

Sol: $\langle \underbrace{x}_{p(x)}, \underbrace{1}_{q(x)}\rangle = -1\cdot 1+0\cdot 1+1\cdot 1$

$= 0$

$\langle x^2,1\rangle = 1\cdot 1+0\cdot 1+1\cdot 1$

$= 2$



* **Norm**: Let v be a vector in the inner product space V. The norm of v is defined by

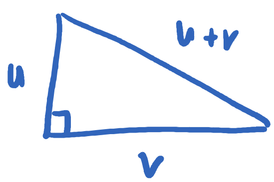
$$\|v\| = \sqrt{\langle v, v\rangle}.$$

* **Orthogonality**: In an inner product space V, u is called to be perpendicular to v iff

$$\langle u, v\rangle = 0.$$

Pythagorean's theorem: $u \perp v$ iff

$$\|u\|^2 + \|v\|^2 = \underbrace{\|u+v\|^2}.$$



Proof:
$$\begin{aligned}\|u+v\|^2 &= \langle u+v, u+v\rangle \\ &= \underbrace{\langle u,u\rangle} + \underbrace{\langle u,v\rangle} + \underbrace{\langle u,v\rangle} + \langle v,v\rangle \\ &= \|u\|^2 + 0 + 0 + \|v\|^2 \\ &= \|u\|^2 + \|v\|^2.\end{aligned}$$

ex: On $C[-\pi,\pi]$, consider:

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, g(x)\, dx$$

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi} f(x)\, g(x)\, dx$$

(a) Find $\|\sin x\|$, $\|\cos x\|$

(b) Find $\|\sin x + \cos x\|$.

Sol: $\|\sin x\| = \sqrt{\langle \sin x, \sin x \rangle}$

$$= \sqrt{\frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{\sin x \cdot \sin x}_{\sin^2 x}\, dx}.$$

Review

$$\int u\,dv = uv - \int v\,du$$

$$\int f(u)\,du$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$= \sqrt{\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1 - \cos 2x}{2}\, dx}$$

$$= \sqrt{\frac{1}{2\pi} \left(x - \frac{\sin 2x}{2} \right) \Big|_{-\pi}^{\pi}}$$

$$= \sqrt{\frac{1}{2\pi} \left[\underbrace{\pi - \frac{\sin 2\pi}{2} - \left(-\pi - \frac{\sin 2\pi}{2} \right)} \right.}$$

$$= \sqrt{\frac{1}{2\pi} \cdot 2\pi}$$

$$= 1.$$

$$\|\cos x\| = \sqrt{\langle \cos x, \cos x \rangle}$$

$$= \sqrt{\int^{\pi}}$$

$$= \sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi} \cos x \cdot \cos x \, dx}$$

$$= \sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1+\cos 2x}{2} \, dx}$$

$$= \sqrt{\frac{1}{2\pi}\left(x + \frac{\sin 2x}{2} \right)\Big|_{-\pi}^{\pi}}$$

$$= \sqrt{\frac{1}{2\pi}\left(\underline{\pi} + \cancel{\frac{\sin 2\pi}{2}} \underline{- (-\pi} + \cancel{\frac{\sin -2\pi}{2}} \right)}$$

$$= \sqrt{\frac{1}{2\pi} \cdot 2\pi}$$

$$= 1 .$$

b) $\| \sin x + \cos x \| = ?$

B/c $\sin x \perp \cos x$ (last lecture), by Pythagorean theorem

$$\| \sin x + \cos x \|^2 = \| \sin x \|^2 + \| \cos x \|^2 .$$

$$= 1 + 1$$

$$\rightarrow \| \sin x + \cos x \| = \sqrt{2} .$$

* In $\mathbb{R}^{m \times n}$, given a matrix A, the **Frobenius norm** of A

$$\|A\|_F = \sqrt{\sum_{i,j} a_{ij}^2} = \sqrt{\langle A, A\rangle}.$$

Theorem: (The Cauchy-Schwarz inequality).
If u and v are 2 vectors of V, then

$$|\langle u, v\rangle| \leq \|u\| \cdot \|v\|.$$

→ The angle between 2 vectors $u, v \in V$:

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\| \cdot \|v\|}, \quad 0 \leq \theta \leq \pi.$$

ex: Let $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

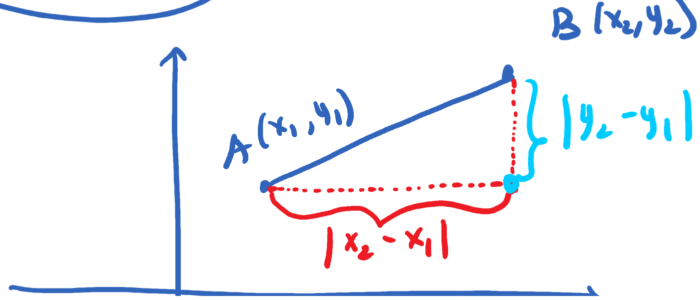
Find the angle between A and B.

Sol: $\langle A, B\rangle = \left\langle \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \right\rangle$

$$= 1 \cdot 1 + 1 \cdot (-1) + 1 \cdot 1 + (-1) \cdot (1)$$

$$= 0$$

$\theta = 90°$ or $\frac{\pi}{2}$.



$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Another "distance" from A to B:

to B:

$|x_2 - x_1| + |y_2 - y_1|$.

* **Norms**: A vector space V is said to be a normed linear space if for any $v \in V$, there is an associated number $\|v\|$, called the norm of v, satisfying three conditions:

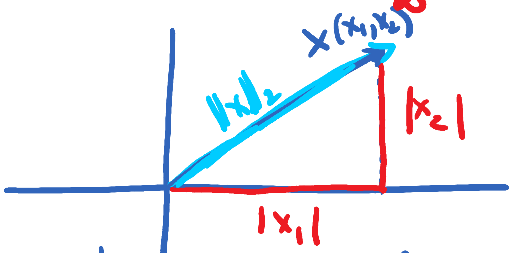
1. $\|v\| \geq 0$ and $\|v\| = 0$ iff $v = 0$
2. $\|\alpha v\| = |\alpha| \cdot \|v\| \quad \forall \alpha \in \mathbb{R}, \forall v \in V.$
3. $\|u + v\| \leq \|u\| + \|v\| \quad \forall u, v \in \mathbb{R}$ (Triangle Inequality)

ex: On $\mathbb{R}^2$:

Define $\|x\|_1 = |x_1| + |x_2|$ (ℓ_1-norm)

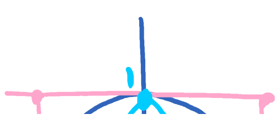
$\|x\|_2 = \sqrt{x_1^2 + x_2^2}$ (ℓ_2-norm)

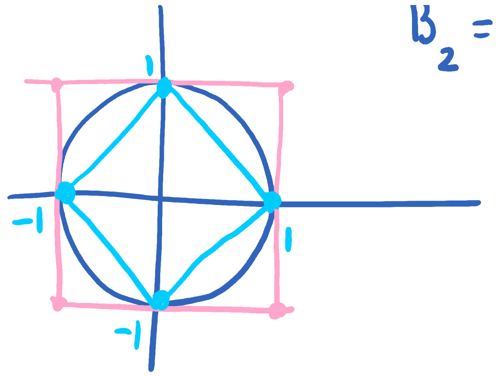
$\|x\|_\infty = \max\{|x_1|, |x_2|\}$ (ℓ_∞-norm)



A unit circle of those norms:

$B_2 = \{x \mid \|x\|_2 \leq 1\}$





$$B_2 = \{ x \mid \|x\|_2 \le 1 \}$$

$$B_1 = \{ x \mid \underbrace{\|x\|_1 \le 1}_{|x_1| + |x_2| \le 1} \}.$$

$$B_\infty = \{ x \mid \|x\|_\infty \le 1 \}$$

$$\max\{|x_1|, |x_2|\} \le 1 \}$$