1) Solve :

$$x_1 - 3x_2 - 10x_3 + 5x_4 = 0$$
$$x_1 + 4x_2 + 11x_3 - 2x_4 = 0$$
$$x_1 + 3x_2 + 8x_3 - x_4 = 0$$

let S be the set of all solutions. Find a basis of S and its dimension.

$$\left(\begin{array}{cccc|c} 1 & -3 & -10 & 5 & 0 \\ 1 & 4 & 11 & -2 & 0 \\ 1 & 3 & 8 & -1 & 0 \end{array}\right) \xrightarrow[R_3 - R_1]{R_2 - R_1} \left(\begin{array}{cccc|c} 1 & -3 & -10 & 5 & 0 \\ 0 & 7 & 21 & -7 & 0 \\ 0 & 6 & 18 & -6 & 0 \end{array}\right)$$

$$\xrightarrow{R_2/7} \left(\begin{array}{cccc|c} 1 & -3 & -10 & 5 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 6 & 18 & -6 & 0 \end{array}\right) \xrightarrow{R_3 - 6R_2} \left(\begin{array}{cccc|c} 1 & -3 & -10 & 5 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 + 3R_2} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 3 & -1 & 0 \end{array}\right)$$

free variables: $x_3 = t$, $x_4 = s$

$$x_2 = -3t + s$$
$$x_1 = t - 2s$$

$$S = \left\{ \begin{pmatrix} t - 2s \\ -3t + s \\ t \\ s \end{pmatrix} \middle| \ t, s \in \mathbb{R} \right\}$$

$$t \begin{pmatrix} 1 \\ -3 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

↳ a basis is $\begin{pmatrix} 1 \\ -3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

dim = 2 .

2) Determine whether the following vectors are L.I.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

<u>Sol</u>: Set $c_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix} + \begin{pmatrix} c_2 & c_2 \\ c_2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} c_1 + c_2 & c_2 \\ c_2 + c_3 & c_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$c_1 + c_2 = 0$

$c_2 + c_3 = 0$ $\quad c_3 = 0$

$c_2 = 0$

$c_1 = 0$

$\longrightarrow$ A, B, C are L.I.

③ Find a basis of $S = \{\underbrace{ax^2 - 3bx + 2a - b}_{a(x^2+2) - b(3x+1)} \mid a, b \in \mathbb{R}\}$

$\longrightarrow S = \text{span}\{x^2 + 2,\ 3x+1\}$

Check L.I. of $x^2 + 2$, $3x + 1$:

Set $c_1(x^2+2) + c_2(3x+1) = 0$

$c_1 x^2 + 2c_1 + 3c_2 x + c_2 = 0$

$\longrightarrow$ $c_1 = 0$

$3c_2 = 0 \longrightarrow c_2 = 0$.

$3c_2 = 0 \longrightarrow c_2 = 0$ }.

$2c_1 + c_2 = 0$

$\rightarrow$ x^2+2, $3x+1$ are L.I.

So x^2+2, $3x+1$ form a basis of S and

$\dim S = 2$.

<u>3.6</u> (cont) Let A be an $m \times n$ matrix

$RS(A) =$ span { all rows of A }

$CS(A) =$ span { all columns of A }

If we consider system $Ax = 0$, we have

$$r = \operatorname{rank}(A) \left\{ \begin{array}{l} \dim(RS(A)) = \#\text{ leading variables.} \\ \quad \| \\ \dim(CS(A)) \end{array} \right.$$

ex: Let A be an 5×6 matrix and $\operatorname{rank}(A) = 2$.

$$A_{5\times 6} \xrightarrow{\text{row operations}} \begin{pmatrix} * & \cdots & & & & \\ 0 & * & \cdots & & & \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So any 3 rows of A are L.D.

any 3 columns of A are L.D.

$\dim(CS(A)) = 2$

$\cdot\ T$ Row operations $\begin{pmatrix} * & \cdots & \\ 0 & \cdots \end{pmatrix}$

$\dim(CS(A)) = \ldots$

$$A \to A^T \xrightarrow{\text{Row operations}} \begin{pmatrix} * & \cdots & & & \\ 0 & * & \cdots & & \\ 0 & 0 & \cdots & & \\ 0 & \cdots & & & 0 \\ 0 & \cdots & & & 0 \\ 0 & \cdots & & & 0 \end{pmatrix} = B$$

$$B^T = \begin{pmatrix} * & 0 & 0 & 0 & 0 & 0 \\ * & * & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & & & & \\ \vdots & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to A$$

$$\boxed{A_{m\times n}\, x_{n\times 1} = 0} \longrightarrow (A \mid 0) \xrightarrow{\text{row operations}}$$

rank = r - non zero rows

$$\left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \underbrace{\begin{pmatrix} * & \cdots & & & \\ 0 & * & \cdots & & \\ 0 & 0 & * & \cdots & \\ 0 & 0 & \cdots & & 0 \\ 0 & 0 & \cdots & & 0 \end{pmatrix}}_{m\times n} \left|\begin{matrix} 0 \\ 0 \\ \vdots \\ \\ 0 \end{matrix}\right)$$

$n - r$ free variables.

Null space of A:

$$N(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$$

$\hookrightarrow \dim(N(A)) = \#$ of free variables

$= n - \operatorname{rank}(A)$

$$\dim \boxed{N(A) + \operatorname{rank}(A) = n}$$

Theorem: Let A be an $m\times n$ matrix. Then

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n,$$

where $\operatorname{nullity}(A) = \dim(N(A))$

where nullity(A) = dim(N(A))

(Rank - Nullity Theorem)

leading variables   # free variables.

ex: Let A be an 5×6 matrix with rank(A)=2.
Find nullity(A) and nullity(A^T).

Sol:
$$\text{nullity}(A) = 6 - \text{rank}(A) = 6 - 2 = 4.$$

$$\underset{6\times5}{\text{nullity}(A^T)} = 5 - \text{rank}(A^T) = 5 - 2 = 3$$

Remark: $\text{rank}(A) = \text{rank}(A^T)$ b/c
$\dim(CS(A)) = \dim(RS(A)) = \text{rank}(A)$.

Chapter 4: Linear Transformation.

$(x, y) \longmapsto 2x - 3y$ is a linear function.

$(x, y) \longmapsto \begin{pmatrix} 2x - 3y \\ x + y \end{pmatrix}$ is a linear mapping.

$A_{m\times n} \longmapsto A^T$

$L: V \to W$

Def: Let V, W be 2 vector spaces. A mapping L from V

Def: Let V, W be 2 vector spaces. A mapping L from V to W is called a linear transformation if

$$\boxed{L(\alpha v_1 + \beta v_2) = \alpha L(v_1) + \beta L(v_2)}$$ for any $v_1, v_2 \in V$ and $\alpha, \beta \in \mathbb{R}$.

or (i) (Addition) $L(v_1 + v_2) = L(v_1) + L(v_2)$ for any $v_1, v_2 \in V$

(ii) (Scalar Multiplication) $L(\alpha v) = \alpha L(v)$ for any $\alpha \in \mathbb{R}$ (scalar) and $v \in V$.

ex: $L: \mathbb{R}^{m \times n} \to \mathbb{R}^{n \times m}$ is a linear transformation.

$A \mapsto A^T$

Sol: Pick any $A, B \in \mathbb{R}^{m \times n}$ and any $\alpha, \beta \in \mathbb{R}$, (scalars)

we need to check:

$$L(\alpha A + \beta B) = \alpha L(A) + \beta L(B)$$

LHS: $L(\alpha A + \beta B) = (\alpha A + \beta B)^T$

$$= \alpha A^T + \beta B^T$$

RHS $\alpha L(A) + \beta L(B) = \alpha A^T + \beta B^T$

same

LHS = RHS $\longrightarrow$ L is a linear transformation.

ex: $L: \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation.

ex: $L: \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 - x_2 + 2x_3 \\ x_1 - 2x_2 - x_3 \end{pmatrix}$$

Sol: Pick any x and y in $\mathbb{R}^3$ and any $\alpha, \beta \in \mathbb{R}$.

We need to check:

$$L(\alpha x + \beta y) = \alpha L(x) + \beta L(y).$$

LHS $\quad L(\alpha x + \beta y) = L\begin{pmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \\ \alpha x_3 + \beta y_3 \end{pmatrix}$

$$= \begin{pmatrix} \alpha x_1 + \beta y_1 - (\alpha x_2 + \beta y_2) + 2(\alpha x_3 + \beta y_3) \\ \alpha x_1 + \beta y_1 + 2(\alpha x_2 + \beta y_2) - (\alpha x_3 + \beta y_3) \end{pmatrix}$$

$$= \begin{pmatrix} \alpha x_1 + \beta y_1 - \alpha x_2 - \beta y_2 + 2\alpha x_3 + 2\beta y_3 \\ \alpha x_1 + \beta y_1 + 2\alpha x_2 + 2\beta y_2 - \alpha x_3 - \beta y_3 \end{pmatrix}$$

RHS $= \alpha L(x) + \beta L(y) = \alpha \begin{pmatrix} x_1 - x_2 + 2x_3 \\ x_1 + 2x_2 - x_3 \end{pmatrix} + \beta \begin{pmatrix} y_1 - y_2 + 2y_3 \\ y_1 + 2y_2 - y_3 \end{pmatrix}$

Same

$$= \begin{pmatrix} \alpha x_1 - \alpha x_2 + 2\alpha x_3 + \beta y_1 - \beta y_2 + 2\beta y_3 \\ \alpha x_1 + 2\alpha x_2 - \alpha x_3 + \beta y_1 + 2\beta y_2 - \beta y_3 \end{pmatrix}$$

So LHS = RHS

$\rightarrow$ L is linear transformation.