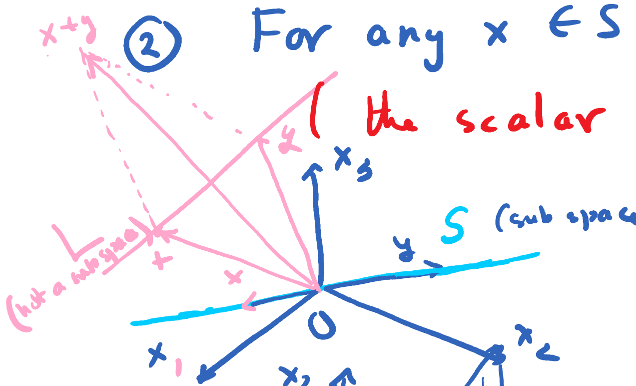
3.2 Subspaces

Thursday, February 6, 2025 5:36 PM

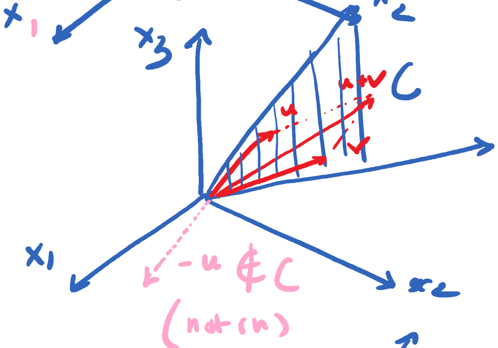
Def: Let V be a vector space with well-defined addition and scalar multiplication. A subset S of V is called a subspace if the following two conditions hold:

① For any $x, y \in S$, we have $x + y \in S$.
(the addition is closed on S)

② For any $x \in S$ and $\alpha \in \mathbb{R}$, we have $\alpha x \in S$.
(the scalar multiplication is closed on S).

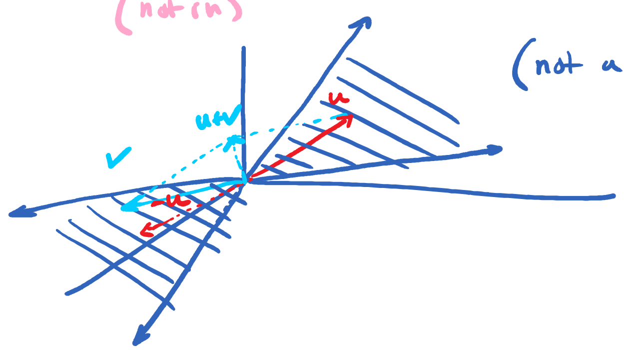


S (subspace, a line passing the origin point)

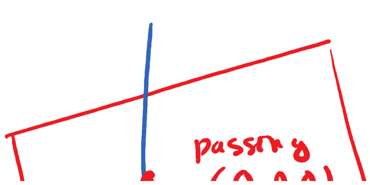


(scalar multiplication is not closed in C)

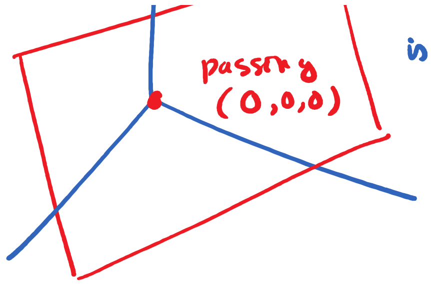
C is not a subspace in $\mathbb{R}^3$.



(not a subspace)



is a subspace



is a subspace

Remark: A subspace must contain vector zero b/c $0 \cdot x = \theta$ (vector zero).

$\mathbb{R}^3$ and $\{0\}$ are also two other trivial subspaces of $\mathbb{R}^3$.

ex: $S = \{ x \in \mathbb{R}^4 \mid x_1 x_2 - x_3 - x_4 = 1 \}$

Is it a subspace of $\mathbb{R}^4$?

As $0 \notin S$, S is not a subspace.

ex: $S = \{ x \in \mathbb{R}^4 \mid x_1 x_2 - x_3 - x_4 = 0 \}$

Is it a subspace of $\mathbb{R}^4$?

$u = (1, -1, 0, -1) \in S$ b/c $1 \cdot (-1) - 0 - (-1) = 0$

$\hookrightarrow -u = (-1, 1, 0, 1)$: $(-1) \cdot (1) - 0 - 1 = -2 \neq 0$.

i.e., $-u \notin S$ So the scalar multiplication is not closed in S, i.e., S is not a subspace.

ex: $S = \{ x \in S \mid \underbrace{x_1 + x_2 - x_3 - x_4 = 0}_{\text{linear equation}} \}$

Is it a subspace of $\mathbb{R}^4$?

Theorem: Let A be an an $m \times n$ matrix. The solution set of the homogeneous equation $Ax = 0$ is a subspace of $\mathbb{R}^n$, i.e., $S = \{x \in \mathbb{R}^n \mid Ax = 0\}$ is a subspace of $\mathbb{R}^n$.

Proof: * **Addition**: Pick any $x, y \in S$:

$$\begin{array}{rl} & Ax = 0 \\ + & Ay = 0 \\ \hline & A(x+y) = 0 \end{array} \rightarrow x+y \in S.$$

* Scalar Multiplication: Pick any $x \in S$ and $\alpha \in \mathbb{R}$,

$$\alpha(Ax = 0)$$

$$A(\alpha x) = 0 \rightarrow \alpha x \in S$$

So S is a subspace of $\mathbb{R}^n$.

ex: $S = \{x \in \mathbb{R}^4 \mid x_1 + x_2 - x_3 - x_4 = 0\}$ is a subspace b/c $x_1 + x_2 - x_3 - x_4 = 0$ is a hom. eqn.

ex: $S = \left\{ x \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} x_1 + x_2 - 2x_3 - x_4 = 0 \\ x_1 - x_2 + x_3 - 3x_4 = 0 \end{array} \right\}$

$$\underbrace{\begin{array}{l} x_1 + x_2 - 2x_3 - x_4 = 0 \\ x_1 - x_2 + x_3 - 3x_4 = 0 \end{array}}_{Ax = 0}$$ is a hom. eqn.

$$A = \begin{pmatrix} 1 & 1 & -2 & -1 \\ 1 & -1 & 1 & -3 \end{pmatrix}$$

So S is a subspace.

ex: $S = \{ x \in \mathbb{R}^4 \mid x_1 + x_2 - x_3 - 2x_4 = 0$

$\underbrace{x_1^2 + x_2^2 = x_3^2 + x_4^2}_{x_1^2 + x_2^2 - x_3^2 - x_4^2 = 0 \text{ (non linear)}} \}$.

The feeling is S is not a subspace.

The we have to provide a counter example.

Pick $u = (1, 0, 1, 0) \in S$ b/c $1 + 0 - 1 - 2\cdot 0 = 0$, $1^2 + 0^2 = 1^2 + 0^2$

$v = (0, 1, 1, 0) \in S$ b/c $0 + 1 - 1 - 2\cdot 0 = 0$, $0 + 1^2 = 1^2 + 0^2$

$u + v = (1, 1, 2, 0) \notin S$ b/c $1^2 + 1^2 \neq 2^2 + 0^2$

Addition is NOT closed in S. S is not a subspace.

ex: Let $C^1(0,1)$ be a vector space of differentiable function on $(0,1)$. Define $S = \{ y \in C^1(0,1) \mid y' - 2xy = 0 \}$

Show that S is a subspace of $C^1(0,1)$. ($y = y(x)$)

Sol: * Addition: Pick any $y_1(x)$, $y_2(x)$ in S:

$$\begin{aligned} & y_1' - 2x\,y_1 = 0 \\ + \; & y_2' - 2x\,y_2 = 0 \\ \hline & (\underbrace{y_1 + y_2}_{y})' - 2x(\underbrace{y_1 + y_2}_{y}) = 0 \end{aligned}$$

So $y_1 + y_2 \in S$.

* Scalar multiplication: Pick any $y(x) \in S$ and $\alpha \in \mathbb{R}$,

$$\alpha(y' - 2x\,y = 0)$$

$(\alpha y)' - 2x(\alpha y) = 0$

So $\alpha y \in S$

S is a subspace of $C'(0,1)$.

ex: $\mathbb{R}^{n\times n}$ is the space of all $n \times n$ matrices.

$S = \{$ singular matrices in $\mathbb{R}^{n\times n}\}$

$T = \{$ invertible matrices in $\mathbb{R}^{n\times n}\}$

Whether S, T are subspaces of $\mathbb{R}^{n\times n}$?

Sol: T is not a subspace of $\mathbb{R}^{n\times n}$ b/c the matrix zero is not in T.

Counter example:

$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \in S$ b/c $\det A = 0$

$+$ $B = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \in S$ b/c $\det B = 0$

$A+B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \notin S$ b/c $\det(A+B) = 1\cdot 2 - 1\cdot 1 = 1$

$\rightarrow$ addition is not closed. S is not a subspace

- $A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ \vdots & & \ddots 1 & \vdots \\ 0 & \cdots & & 0 \end{pmatrix}_{n\times n} \in S$ b/c $\det A = 1\cdot 1 \cdots 1 \cdot 0 = 0$

$\left(0 \ 0 \ \cdots \ 0 \right.$

$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & 0 & \vdots \\ 0 & \cdots & & 1 \end{pmatrix}_{n \times n} \in S \quad b/c \quad \det B = 0 \cdot 0 \cdots 0 \cdot 1 = 0$$

$$A + B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & 1 \end{pmatrix} \notin S \qquad \det(A+B) = 1$$

S is not a subspace b/c the addition is not closed.

ex: Let $P_3 = \{ax^2 + bx + c\}$. Show that

$$S = \{ ax^2 + bx + c \mid a + b + c = 0 \}$$

is a subspace of P_3.

Proof:

* Addition: Pick any $p_1(x) = a_1x^2 + b_1x + c_1 \in S$, $a_1 + b_1 + c_1 = 0$

$p_2(x) = a_2x^2 + b_2x + c_2 \in S$, $a_2 + b_2 + c_2 = 0$

$$p_1(x) + p_2(x) = (a_1+a_2)x^2 + (b_1+b_2)x + c_1 + c_2$$

This is an element of S b/c

$$a_1 + a_2 + b_1 + b_2 + c_1 + c_2 = (a_1 + b_1 + c_1) + (a_2 + b_2 + c_2) = 0 + 0 = 0$$

Addition is closed.

* Scalar multiplication: Pick any $p(x) = ax^2 + bx + c \in S$, $a + b + c = 0$ and any $\alpha \in \mathbb{R}$:

$$\alpha \left(p(x) = ax^2 + bx + c \right)$$

$$(\alpha p)(x) = \alpha a x^2 + \alpha b x + \alpha c \in S \quad b/c$$

$$\alpha a + \alpha b + \alpha c = \alpha(a + b + c)$$

$= \alpha \cdot 0$

Scalar multiplication is closed. $= 0$

S is a subspace.

Exam I covers: 1.1, 1.2, 1.3, 1.4, 2.1, 2.2, 3.1, 3.2