MTH 2775 Some practice problems for Exam I

1. Answer whether the following statements are true or false. Explain shortly your answers if they are true or give a counterexample if they are false.

   (a) Given two 2×2 matrices A and B. We always have $AB = BA$.

   (b) Let A be a nonsingular matrix. Then A^{-1} is also nonsingular.

   (c) Given a 3×4 matrix A, the elementary matrix corresponding to the row operation of interchanging row 1 and row 3 of A is $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

   (d) Given two $n \times n$ matrices A and B. If $AB = 0$, then either $A = 0_n$ or $B = 0_n$.

   (e) The sum of two matrices exist only if they have the same size.

   (f) Let A be a $n \times n$ matrix. If $A^3 = I_n$, then $\det(A) = 1$.

   (g) Equation $Ax = 0$ may have no solution.

   (h) Equation $Ax = b$ could have exactly two solution.

   (k) A is nonsingular, equation $Ax = b$ has a unique solution.

   (i) The set of (u_1, u_2) with $u_2 = |u_1|$ is a subspace of $\mathbb{R}^2$.

   (j) Let V be a vector space. For any $x, y \in V$, $x - y$ is also an element of V.

2. Solve the following equation system:

$$\begin{aligned} x_1 \quad\quad - x_3 + x_4 &= 5 \\ 2x_1 - 2x_2 + 3x_3 - x_4 &= 1 \\ x_1 - x_2 + x_3 + 3x_4 &= 2 \\ 4x_1 - 3x_2 + 3x_3 + 3x_4 &= 8 \end{aligned}$$

3. Consider the linear system

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 2 \\ 0 & 2 & 2 & -2 \\ 0 & 0 & a^2 - a & b - a \end{array}\right]$$

   Find all values of a and b such that the system has:

   (a) a unique solution.

   (b) no solution.

   (c) infinitely many solutions.

   (d) three solutions.

4. Given $A = \begin{bmatrix} 1 & 0 & 1 & 4 \\ 2 & 0 & 0 & 1 \\ 1 & 2 & 1 & 3 \\ 3 & 0 & 0 & 1 \end{bmatrix}$

   (a) Find A^{-1}.

   (b) Find $\det A$.

   (c) Find $\det A^{-1}$.

5. Given $A = \begin{bmatrix} 1 & 4 & 3 \\ 1 & 4 & 4 \\ 2 & 7 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & -1 & 1 & 0 \end{bmatrix}$
   (a) Find AB and BA.
   (b) Find $2A + B$ and $3B - A$.
   (c) Find A^{-1}
   (d) Solve $AX = B$ for X.

6. Determine whether or not the following sets are subspaces. If yes, show it. If no, provide a counterexample.
   (a) $X = \{x \in \mathbb{R}^4 |\ x_1 x_2 = x_3 + x_4\}$.
   (b) $Y = \{x \in \mathbb{R}^3 |\ x_1 + 3x_2 = x_3\}$.
   (c) $Z = \{x \in \mathbb{R}^3 |\ |2x_1 + x_3| = |x_2|\}$, where $|\ |$ is the absolute value.
   (d) $W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 |\ x_1 + x_2 = 3x_3, x_1 - x_2 + x_3 = 0\}$

7. Suppose that a nonsingular matrix $A \in \mathbb{R}^{n \times n}$ satisfies that $A = A^{-1}$. Find all possible values of $\det A$.

8. (a) State the definition of a nonsingular matrix A.
   (b) State four conditions that each of which is equivalent to A being nonsingular.
   (c) Suppose that $A^2 = I$. Verify that:
   (c1) A is nonsingular and $A^{-1} = A$.
   (c2) $A - I$ is singular.
   (c3) $A - 2I$ is nonsingular and $(A - 2I)^{-1} = -\frac{1}{3}(A + 2I)$.

9. Let A be an $m \times n$ matrix.
   (a) Explain why $A^T A$ and AA^T are possible.
   (b) Show that both $A^T A$ and AA^T are symmetric matrices.

10. Determine whether the set of functions f in $\mathcal{C}[-1, 1]$ such that $f(-1) = 0$ and $f(1) = 0$ is a subspace of $\mathcal{C}[-1, 1]$. Show all your work.

11. Given two $n \times n$ matrices A and B.
    (a) Suppose that $AB = 0_n$. Show that both A and B are singular.
    (b) Suppose that $A + B$=0. Show that $\det A = (-1)^n \det B$.
    (c) Show that $\det(A^T A) \geq 0$.

Answer keys.

1. (a) F (b) T (c) T (d) F (e) T (f) T (g) F (h) F (k) T (i) F (j) T.

2. $(-2+6t, -3+16t, -3+7t, t)$.

3. (a) $a \neq 0$ and $a \neq 1$ (b) $a = 0$ and $b \neq 0$, $a = 1$ and $b \neq 1$ (c) $a = 0$ and $b = 0$, $a = 1$ and $b = 1$ (d) No such a and b.

4. (a) $\begin{bmatrix} 0 & -1 & 0 & -1 \\ -1/2 & 3/2 & 1/2 & -1 \\ 1 & -11 & 0 & 7 \\ 0 & 3 & 0 & -2 \end{bmatrix}$ (b) -2 (c) $-1/2$.

5. (a) $AB = \begin{bmatrix} 8 & -3 & 4 & 9 \\ 9 & -4 & 5 & 9 \\ 15 & -6 & 8 & 16 \end{bmatrix}$, BA DNE (b) Both DNE (c) $\begin{bmatrix} -4 & -3 & 4 \\ 2 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$

(d) $A^{-1}B = \begin{bmatrix} -3 & 4 & 0 & -10 \\ 1 & 1 & 1 & 2 \\ 0 & 0 & -1 & 1 \end{bmatrix}$

6. (a) No (b) Yes (c) No (d) Yes

7. Hint: Use $\det(A^{-1}) = 1/\det(A)$

11. Hint: Use $\det(AB) = \det(A)\det(B)$.