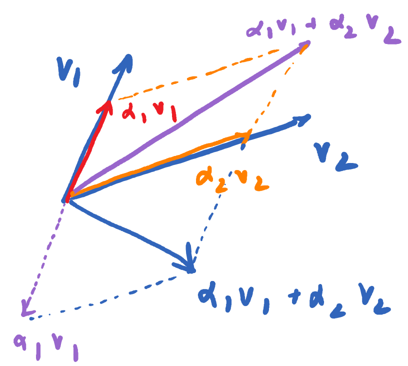
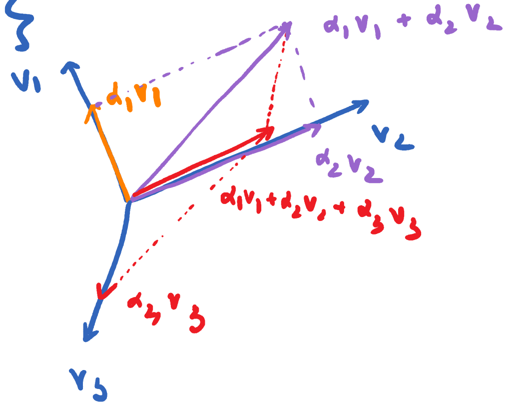
Def: Let $v_1, v_2, \dots, v_n$ be vectors of a vector space V.

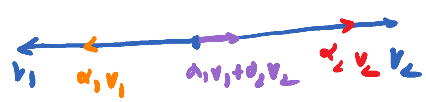
A sum $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$ is called a linear combination.

The set of all linear combinations of $v_1, v_2, \dots, v_n$ is called the span of $v_1, v_2, \dots, v_n$, denoted by

$$\operatorname{Span}\{v_1, v_2, \dots, v_n\}$$







Remark: $\operatorname{Span}\{v_1, v_2, \dots, v_n\}$ is a subspace of V.

ex: Consider $v_1 = (1,0,0)^T$, $v_2 = (0,1,0)^T$, and $v_3 = (0,0,1)^T$

$\operatorname{Span}(v_1, v_2, v_3) = \mathbb{R}^3$

Sol:
$$\begin{aligned}
\operatorname{span}\{v_1, v_2, v_3\} &= \{\underbrace{\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3} \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\} \\
&= \left\{\alpha_1 \begin{pmatrix}1\\0\\0\end{pmatrix} + \alpha_2 \begin{pmatrix}0\\1\\0\end{pmatrix} + \alpha_3 \begin{pmatrix}0\\0\\1\end{pmatrix} \,\middle|\, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\right\} \\
&= \left\{\begin{pmatrix}\alpha_1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\\alpha_2\\0\end{pmatrix} + \begin{pmatrix}0\\0\\\alpha_3\end{pmatrix} \,\middle|\, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\right\} \\
&= \left\{\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix} \,\middle|\, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\right\}
\end{aligned}$$

$$= \left\{ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \middle| \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \right\}$$

$$= \mathbb{R}^3 .$$

3.4 linear independence.

Def: Let $v_1, v_2, \dots, v_n$ be vectors in a vector space V.
$v_1, v_2, \dots, v_n$ are said to be linearly independent if
the equation : $c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$
implies $c_1 = c_2 = \dots = c_n = 0$ ($(0, 0, \dots, 0)$ is the only solution of the above system).

* **Remark**: If one of $c_1, c_2, \dots, c_n$ is different from zero, e.g., $c_1 \neq 0$, we have :

$$c_1 v_1 = -c_2 v_2 - c_3 v_3 - \dots - c_n v_n$$

$$v_1 = \underbrace{-\frac{c_2}{c_1} v_2 - \frac{c_3}{c_1} v_3 - \dots - \frac{c_n}{c_1} v_n}_{\text{a linear comb. of } v_2, \dots, v_n}$$

If $c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$ gives a solution $(c_1, c_2, \dots, c_n)$ not equal to zero $(0, \dots, 0)$, then $v_1, v_2, \dots, v_n$ are linearly dependent.

ex: $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \in \mathbb{R}^3$.

Check if v_1, v_2, v_3 are linearly independent,

Sol: Set the equation

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 \quad \text{(vector zero)}$$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 \quad \text{(vector zero)}$$

$$\begin{pmatrix} c_1 \\ c_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} c_2 \\ 2c_2 \\ c_2 \end{pmatrix} + \begin{pmatrix} c_3 \\ -c_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_1 + 2c_2 - c_3 &= 0 \\ c_1 + c_2 + 0 &= 0 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow[R_3 - R_1]{R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right)$$

R_3: $-c_3 = 0 \longrightarrow c_3 = 0$

R_2: $c_2 - 2c_3 = 0 \longrightarrow c_2 = 0$

R_1: $c_1 + c_2 + c_3 = 0 \longrightarrow c_1 = 0$

So v_1, v_2, v_3 are L.I.

ex: $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$.

Check if v_1, v_2, v_3 are L.I.

Sol: Set the equation

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

$$\begin{pmatrix} c_1 \\ c_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} c_2 \\ -2c_2 \\ c_2 \end{pmatrix} + \begin{pmatrix} 2c_3 \\ -c_3 \\ 2c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} c_1 + c_2 + 2c_3 &= 0 \\ c_1 - 2c_2 - c_3 &= 0 \\ c_1 + c_2 + 2c_3 &= 0 \end{aligned}$$

$c_1 + c_2 + 2c_3 = 0$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & -2 & -1 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right) \xrightarrow[R_3 - R_1]{R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_3/-3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

c_3 = free variable

$c_3 = t \longrightarrow v_1, v_2, v_3$ are L. D.

R_2: $c_2 + c_3 = 0 \rightarrow c_2 = -t$

R_1: $c_1 + c_2 + 2c_3 = 0 \rightarrow c_1 = -c_2 - 2c_3$

$$= t - 2t$$

$$= -t$$

$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$

$\rightarrow -t v_1 - t v_2 + t v_3 = 0$

$\xrightarrow{\div -t} v_1 + v_2 - v_3 = 0$ or $v_3 = v_1 + v_2$

$v_1 = v_3 - v_2$

Theorem: Let $v_1, v_2, \dots, v_n$ be n vectors in $\mathbb{R}^n$. Then $v_1, v_2, \dots, v_n$ are L. I. iff the matrix

$$A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$$ is invertible, i.e.

$\det A \neq 0$.

The equation $\boxed{c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0}$ means

$A c = 0$, $c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix}$

$$\underset{n\times n}{A}\, c = 0, \quad c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

ex: $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \in \mathbb{R}^3$.

Check if v_1, v_2, v_3 are linearly independent.

$$\hookrightarrow A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\to \det A \underset{R_3 - R_1}{\overset{R_2 - R_1}{=}} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{vmatrix} = 1 \cdot 1 \cdot (-1) = -1 \neq 0$$

v_1, v_2, v_3 are L.I.

ex: $P_4 = \{ a x^3 + b x^2 + cx + d \mid a, b, c, d \in \mathbb{R} \}$

Check if $\underbrace{x^2 - 2x}_{v_1}$, $\underbrace{x^2}_{v_2}$, $\underbrace{x}_{v_3}$ are L.I.

Sol: Set $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 \xrightarrow[c_1, c_2, c_3]{\text{solve for}}$

$$c_1 (x^2 - 2x) + c_2 (x^2) + c_3 x = 0$$

- break all the parenthesis

$$c_1 x^2 - 2c_1 x + c_2 x^2 + c_3 x = 0$$

- combine like terms

$$(c_1 + c_2) x^2 + (c_3 - 2c_1) x = 0 \quad \text{(zero function)}$$

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_3 - 2c_1 &= 0 \end{aligned} \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \end{array} \right)$$

$$\xrightarrow{R_2 + 2R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array} \right)$$

$\xrightarrow{R_2+2R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right)$

c_3 = free variable

$\rightarrow$ v_1, v_2, v_3 are L. D.

x^2-2x, x^2, x are L. D.

$v_1 = v_2 - 2v_3$

ex : Test whether $p_1(x) = x^2 - 2x + 3$, $p_2(x) = 2x^2 + x + 8$, and $p_3(x) = x^2 + 8x + 7$ are L. I.

Sol : Set $c_1 p_1 + c_2 p_2 + c_3 p_3 = 0$

$$c_1(x^2 - 2x + 3) + c_2(2x^2 + x + 8) + c_3(x^2 + 8x + 7) = 0$$

$$c_1 x^2 - 2c_1 x + 3c_1 + 2c_2 x^2 + c_2 x + 8c_2 + c_3 x^2 + 8c_3 x + 7c_3 = 0$$

$$(c_1 + 2c_2 + c_3)x^2 + (-2c_1 + c_2 + 8c_3)x + (3c_1 + 8c_2 + 7c_3) = 0$$

$$\begin{aligned} c_1 + 2c_2 + c_3 &= 0 \\ -2c_1 + c_2 + 8c_3 &= 0 \\ 3c_1 + 8c_2 + 7c_3 &= 0 \end{aligned} \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -2 & 1 & 8 & 0 \\ 3 & 8 & 7 & 0 \end{array}\right)$$

$$\xrightarrow[R_3 - 3R_1]{R_2 + 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 5 & 10 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right) \xrightarrow{R_2/5} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 - 2R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

c_3 = free variable

$\rightarrow p_1, p_2, p_3$ are L. D.