MTH 2775 **Quiz 4 (2 problems, 20 points)** **Name:**____________________

Directions: Show all work for full credit. Turn off all electronic devices. Using calculator to make matrix operations is prohibited.

1. (12 pts) Consider the following mapping $L : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ with $L(A) = A - A^T$ for any $A \in \mathbb{R}^{2\times 2}$.

   (a) (5pts) Verify that it is a linear transformation.

For any $A, B \in \mathbb{R}^{2\times 2}$ and α, β:

$$L(\alpha A + \beta B) = (\alpha A + \beta B) - (\alpha A + \beta B)^T$$
$$= \alpha A + \beta B - \alpha A^T - \beta B^T$$

$$\alpha L(A) + \beta L(B) = \alpha(A - A^T) + \beta(B - B^T)$$
$$= \alpha A - \alpha A^T + \beta B - \beta B^T$$

$$\to L(\alpha A + \beta B) = \alpha L(A) + \beta L(B)$$

   (b) (4 pts) Find the kernel of L and its dimension.

$\operatorname{Ker} L = \{A \mid L(A) = 0\}$

Set $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to L(A) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$$= \begin{pmatrix} 0 & b-c \\ c-b & 0 \end{pmatrix} \to b = c$$

$$\to \operatorname{Ker} L = \left\{ \begin{pmatrix} a & b \\ b & d \end{pmatrix} \right\}$$

$$= \left\{ a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$= \operatorname{span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

   (c) (3 pts) Find the range of L and its dimension.

$$\to L(A) = \left\{ \begin{pmatrix} 0 & b-c \\ c-b & 0 \end{pmatrix} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

2. Consider the following mapping $L : P_3 \to P_3, L(p) = p(1) + x^2 p(0)$.

(a) (4pts) Verify that it is a linear transformation.

For any $\alpha, \beta \in \mathbb{R}$ and $p, q \in P_3$:

$L(\alpha p + \beta q) = (\alpha p + \beta q)(1) + x^2(\alpha p + \beta q)(0)$

$= \alpha p(1) + \beta q(1) + \alpha x^2 (p(0)) + \beta x^2 q(0)$

$\alpha L(p) + \beta L(q) = \alpha(p(1) + x^2 p(0)) + \beta(q(1) + x^2 q(0))$

$= \alpha p(1) + \alpha x^2 p(0) + \beta q(1) + \beta x^2 q(0)$

(b) (4 pts) With the standard basis $E = \{x^2, x, 1\}$, find the matrix representation of L.

Given $p(x) = ax^2 + bx + c \rightarrow [p]_E = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

$p(1) = a + b + c, \quad p(0) = c$

$L(p) = a + b + c + x^2 c$

$\rightarrow [L(p)]_E = \begin{pmatrix} c \\ 0 \\ a+b+c \end{pmatrix}$

$\rightarrow \begin{pmatrix} c \\ 0 \\ a+b+c \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.