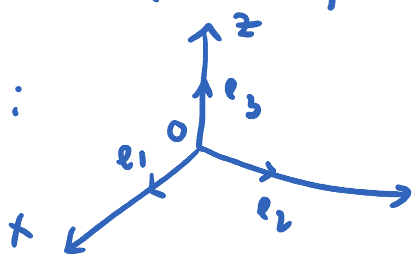
3.4 Bases and dimensions:

Def: Let V be a vector space. The vectors $v_1, v_2, \dots, v_n$ of V is called to be a basis of V if the following conditions are true:

(i) $v_1, v_2, \dots, v_n$ are L.I.

(ii) $\text{span}\{v_1, v_2, \dots, v_n\} = V$.

ex:



$\mathbb{R}^3$, $\underbrace{e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\substack{\text{unit vectors} \\ \text{(normal vectors)}}}$

e_1, e_2, e_3 are L.I b/c

$$\det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1 \neq 0.$$

$$\text{span}\{e_1, e_2, e_3\} = \{\underbrace{c_1 e_1 + c_2 e_2 + c_3 e_3}_{\underbrace{\begin{pmatrix} c_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ c_3 \end{pmatrix}}_{\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}}} \mid c_1, c_2, c_3 \in \mathbb{R}\}$$

$$= \mathbb{R}^3$$

So e_1, e_2, e_3 form a basis of $\mathbb{R}^3$.

* There are many bases of $\mathbb{R}^3$. Indeed, any L.I vectors v_1, v_2, v_3 form a basis of $\mathbb{R}^3$.

ex: In $\mathbb{R}^n$, $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

$e_1, e_2, \dots, e_n$ form a basis of $\mathbb{R}^n$.

Moreover, any n vectors in $\mathbb{R}^n$ that are L.I. form a basis of $\mathbb{R}^n$.

ex: $\mathbb{R}^{m \times n}$, $E_{ij} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1_{i,j} & & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$

$\{E_{ij}\}_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ form a basis of $\mathbb{R}^{m \times n}$.

$$\mathbb{R}^{2 \times 2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \right\}$$

$$= \left\{ a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ form a basis of $\mathbb{R}^{2 \times 2}$

ex: $P_n = \{$ all polynomial of degree $< n \}$

$$= \{ a_0 x^{n-1} + a_1 x^{n-2} + \cdots + a_{n-2} x + a_{n-1} \mid a_0, a_1, \ldots, a_{n-1} \in \mathbb{R} \}$$

$$= \operatorname{span}\{ \underbrace{x^{n-1}, x^{n-2}, \ldots, x, 1}_{\text{are L.I.}} \}$$

$\hookrightarrow$ $x^{n-1}, x^{n-2}, \ldots, x, 1$ form a (standard) basis of P_n.

ex: Find a basis of

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid \underbrace{x - y + 2z = 0}\}.$$

Sol: Solve $x - y + 2z = 0$

Set $y = t$ and $z = s$ (parameters)

$$x = y - 2z$$
$$= t - 2s$$

So
$$S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} t-2s \\ t \\ s \end{pmatrix}}_{\underbrace{\begin{pmatrix} t \\ t \\ 0 \end{pmatrix} + \begin{pmatrix} -2s \\ 0 \\ s \end{pmatrix}}_{t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}}} \;\middle|\; t, s \in \mathbb{R} \right\}$$

$$\longrightarrow S = \operatorname{span}\left\{ \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}} \right\}$$

$$\underbrace{\begin{pmatrix} \\ 0 \end{pmatrix}, \begin{pmatrix} \\ 1 \end{pmatrix}}_{}$$

To check L.I., we solve **L.I.**

$$t\begin{pmatrix}1\\1\\0\end{pmatrix} + s\begin{pmatrix}-2\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}t-2s\\t\\s\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \longrightarrow t=0,\ s=0$$

$\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\1\end{pmatrix}$ form a basis of S.

$\longrightarrow \dim S = 2$

ex: $S = \{ p \in P_3 \mid p(1) - 2p(0) = 0 \}$

Find a basis of S.

Sol: Pick: $p(x) = ax^2 + bx + c \in S$.

$$p(1) - 2p(0) = 0$$

$$a+b+c-2c = 0$$

$$a+b-c = 0$$

free variables: $b = t,\ c = s$ (parameters)

$$a = c - b$$
$$= s - t$$

So $p(x) = ax^2 + bx + c$

$$= (s-t)x^2 + tx + s$$

$$= sx^2 - tx^2 + tx + s$$

$$= s x^2 - t x^2 + t x + s$$

$$= s(x^2+1) + t(-x^2+x).$$

$$S = \text{span}\{x^2+1, -x^2+x\}$$

Check L.I of $x^2+1, -x^2+x$, we solve

$$s(x^2+1) + t(-x^2+x) = 0$$

$$(s-t)x^2 + tx + s = 0$$

all coefficients $s-t=0$ → $\dim S = 2$

$t = 0$ (unique solution)

$s = 0$

ex: $S = \{A \in \mathbb{R}^{2\times 2} \mid A = A^T\}$

symmetric matrices

Find a basis of S.

Sol: Pick any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$b = c$

$$\rightarrow A = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \overset{\text{separate}}{=} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$$

$$= a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S = \text{span}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

we need to check L.I of $\begin{pmatrix} 1 & 0 \\ \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \end{pmatrix}$

we need to check L.I. of $\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}$.

Set $a\begin{pmatrix}1&0\\0&0\end{pmatrix}+b\begin{pmatrix}0&1\\1&0\end{pmatrix}+d\begin{pmatrix}0&0\\0&1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$

$$\begin{pmatrix}a&b\\b&d\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$\rightarrow$ $a=0$, $b=0$, $d=0$ $\rightarrow$ L.I.

So $\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}$, and $\begin{pmatrix}0&0\\0&1\end{pmatrix}$ form a basis of S. $\rightarrow$ $\dim S = 3$

Def: If $v_1, v_2, \dots, v_n$ form a basis of a vector space V, then the dimension of V is

$\dim V = n$. (# of vectors in the basis)