1.4 Matrix Algebra

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$AB \neq BA$ in general

$A + B = B + A$

<u>Theorem</u>: Let A, B, C be matrices such that the following operations are well-defined. We have

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $(AB)C = A(BC)$
4. $A(B + C) = AB + AC$
5. $AB \neq BA$ (in general)

<u>Zero matrix</u>: $O_{m \times n} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & & & \\ 0 & \cdots & & 0 \end{pmatrix}_{m \times n}$

$$A_{m \times n} + O_{m \times n} = A$$

$$A_{m \times n} \cdot O_{n \times p} = O_{m \times p}$$

<u>Identity matrix</u>: is an $n \times n$ (square) matrix defined by

$$I_{n \times n} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & & & & \\ 0 & 0 & \cdots & & 1 \end{pmatrix}$$

$$A_{m \times n} \cdot I_{n \times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix}$$

$$= A_{m\times n}$$

$$I_{m\times m} A_{m\times n} = A_{m\times n}$$

Given an $n \times n$ matrix:

$$A^2 := A \cdot A$$

Question: A is 3×4 matrix. Can we define A^2 this way?

$$A^2 = A_{3\times 4} \cdot A_{3\times 4} \quad \text{DNE}$$

$A_{m\times n} \cdot A_{m\times n}$ is well-defined iff $m = n$

* Given an $n\times n$ matrix A, the power of A

$$A^k = \underbrace{A \cdot A \cdot \ldots A}_{k \text{ times}}, \quad \text{where } k = 1, 2, 3, \ldots,$$

$$\boxed{B} \xrightarrow{\text{send}} \boxed{AB} \xrightarrow[B]{\text{get back}} \underbrace{A^{-1}A}\, B = B$$

Information

the inverse matrix of A.

$$2 \xrightarrow[\text{the inverse}]{\text{What is}} \frac{1}{2} : \quad \frac{1}{2} \cdot 2 = 2 \cdot \frac{1}{2} = 1.$$

Def: Let A be an $n\times n$ matrix. $B_{n\times n}$ is called the inverse of A if

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$$AB = BA = I_{n\times n}$$

ex: Given $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$. Find the inverse of A.

Sol: Let B be an 2×2 matrix, i.e., $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

As $AB = I_{2\times 2}$, $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2}$

$$\begin{pmatrix} a+2c & b+2d \\ a+c & b+d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So $a + 2c = 1$ and $b + 2d = 0$

$a + c = 0$ $\qquad$ $b + d = 1$

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ \underbrace{1 \quad 1}_{A} & & 0 \end{array}\right) \xrightarrow{R_2 - R_1} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -1 & -1 \end{array}\right)$$

$$\xrightarrow{R_2 \times (-1)} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \end{array}\right) \xrightarrow{R_1 - 2R_2} \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right)$$

$\longrightarrow a = -1$ and $c = 1$

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ \underbrace{1 \quad 1}_{A} & & 1 \end{array}\right) \xrightarrow{R_2 - R_1} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 \times (-1)} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & -1 \end{array}\right) \xrightarrow{R_1 - 2R_2} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right)$$

$b = 2$, $d = -1$.

So $B = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$

* To find an inverse of an $n\times n$ matrix A, we just need to find an $n\times n$ matrix B s.t.:

$$AB = I_{n\times n}$$

ex: Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$. Find the inverse of A.

Sol: Set $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Check $AB = I$

$$\begin{pmatrix} 1 & 1 \\ & \end{pmatrix}\begin{pmatrix} a & b \\ & \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ & \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a+c & b+d \\ 2a+2c & 2b+2d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$a+c = 1$$
$$2a+2c = 0$$

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 0 \end{array}\right) \xrightarrow{R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & -2 \end{array}\right)$$

no solution, DNE

$$b+d = 0$$
$$2b+2d = 1$$

The inverse of A DNE.

* Remark: Given an $n \times n$ matrix A. It is possible that the inverse of A DNE. In this case, we say A is singular or non-invertible.
Otherwise, if the inverse of A exists, A is called to be invertible or nonsingular. In this case, the inverse of A is denoted by A^{-1}.

$$A^{-1} A = I \quad \text{and} \quad A \cdot A^{-1} = I$$

whenever A is invertible.

ex: Back to $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. To find the inverse of A, we set the augmented matrix

$$\left(\underbrace{\begin{array}{cc} 1 & 2 \\ 1 & 1 \end{array}}_{A} \middle| \underbrace{\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}}_{I}\right) \xrightarrow{R_2 - R_1} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right)$$

$$R_2 \times (-1) \quad \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \end{array}\right) \xrightarrow{R_1 - 2R_2} \left(\begin{array}{cc|cc} 1 & 0 & -1 & 2 \end{array}\right)$$

$$\xrightarrow{R_2 \times (-1)} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{array}\right) \xrightarrow{R_1 - 2R_2} \left(\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right)$$

$$\left(\underbrace{\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}}_{I} \;\middle|\; \underbrace{\begin{array}{cc} -1 & 2 \\ 1 & -1 \end{array}}_{A^{-1}}\right)$$

General method of finding the inverse matrix $A_{n \times n}$

1. Set up the augmented matrix $(A \mid I)$

2. Use Gauss-Jordan method to turn it into the reduced row-echelon matrix $(\underbrace{E}_{\text{reduced row-echelon}} \mid B)$

3. If $E = I_{n \times n}$, then $B = A^{-1}$ (the inverse of A).

   If $E \neq I_{n \times n}$, A^{-1} DNE or A is not invertible.

ex: $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. Find A^{-1}.

Sol: ① Set up $(A \mid I)$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

② Use Gauss-Jordan

$$\xrightarrow{R_2 - R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 + R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right)$$

no solution

A^{-1} DNE.

ex: $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 3 & 4 \end{pmatrix}$. Find A^{-1}.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow[R_3 - 2R_1]{R_2 - R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 - 2R_1]{R_2 - R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right) \xrightarrow[R_1 - 2R_3]{R_2 + R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & 2 & -2 \\ 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 - R_2} \left(\underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}}_{I} \middle| \underbrace{\begin{array}{ccc} 5 & 2 & -3 \\ -2 & 0 & 1 \\ -1 & -1 & 1 \end{array}}_{A^{-1}} \right).$$

Theorem: Let A and B be 2 invertible $n \times n$ matrices. Then AB is also invertible and

$$(AB)^{-1} = B^{-1} A^{-1}.$$

Proof:
$$\begin{aligned} (AB)(B^{-1}A^{-1}) &= A \underbrace{(B B^{-1})} A^{-1} \\ &= \underbrace{A \quad I} \quad A^{-1} \\ &= \underbrace{A \cdot A^{-1}} \\ &= I. \end{aligned}$$

So $(AB)^{-1} = B^{-1} A^{-1}$.

$$B \longrightarrow A B \xrightarrow{\text{recovery}} A^{-1}(A \quad B)$$
$$= \underbrace{A^{-1} \; A} \quad B$$
$$= I \cdot B$$

Quiz 2: covers 1.3 & 1.4.

= B