MTH 2775

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26: Inner product space : $\|u\| = \sqrt{\langle u, u\rangle}$ or $\|u\|^2 = \langle u, u\rangle$

Show that

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$$

$$\underline{\underline{LHS}}: \langle u+v, u+v\rangle + \langle u-v, u-v\rangle$$

$$= \underbrace{\langle u,u\rangle}_{\|u\|^2} + \langle u,v\rangle + \langle v,u\rangle + \underbrace{\langle v,v\rangle}_{\|v\|^2} + \underbrace{\langle u,u\rangle}_{\|u\|^2} - \langle u,v\rangle - \langle v,u\rangle + \underbrace{\langle v,v\rangle}_{\|v\|^2}$$

$$= \|u\|^2 + \|v\|^2 + \|u\|^2 + \|v\|^2$$

$$= 2\|u\|^2 + 2\|v\|^2 = RHS.$$

Chapter 6 : Eigenvalues

6.1 Eigenvalues and eigenvectors.

$A_{m\times n} \cdot B_{n\times p}$

n entries

n entries

by using $2n$ operations.

$\to$ we need $m \cdot p \cdot 2n$ operations

$A^2 = A_{n\times n} \cdot A_{n\times n} \to 2n^3$ operations.

How to compute A^m

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^{1000}$$

not possible in principle by using our current knowledge on L.A.

Main idea to compute A^m

If A is a diagonal matrix:

$$A_{n\times n} = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ & & & & 0 \\ 0 & \cdots & & 0 & \lambda_n \end{pmatrix} \longrightarrow A^m = \begin{pmatrix} \lambda_1^m & & & 0 \\ & \lambda_2^m & & \\ & & \ddots & \\ 0 & & & \lambda_n^m \end{pmatrix}$$

If a matrix A can be represented by:

$$\boxed{A = X_{n\times n} D_{n\times n} X^{-1}}$$

, D is a diagonal matrix and X is an invertible matrix.

$$A^2 = A \cdot A$$

$$= \underbrace{X \cdot D \cdot X^{-1}} \; X \; D \; X^{-1}$$

$$= X \cdot D \cdot I_n \cdot D \; X^{-1}$$

$$X \; D \cdot D \; X^{-1}$$

$$= X \; D^2 \; X^{-1}.$$

$$A^m = X D^m X^{-1}$$

* If $A = X D X^{-1}$, we have

$$A X = X D X^{-1} X$$

$$\boxed{A X = X D}$$

$$X = \boxed{X} \qquad D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

$$\begin{pmatrix} | & | & & | \\ A \cdot \text{column 1 of } X & A \cdot \text{column 2 of } X & \cdots & A \cdot \text{column } n \text{ of } X \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ \lambda_1 \text{ Column 1 of } X & \lambda_2 \text{ Column 2 of } X & \cdots & \lambda_n \text{ Column } n \text{ of } X \\ | & | & & | \end{pmatrix}$$

To find X and $\lambda_1, \lambda_2, \ldots, \lambda_n$ we have to solve:

$$\boxed{A x = \lambda x}$$

<u>Def</u>: Let A be n $n \times n$ matrix. Consider the equation

$$A x = \lambda x, \iff (A - \lambda I_{n \times n}) x = 0$$

λ is called the eigenvalue of A if the above equation has a nonzero solution $x \neq 0$. In this case, the nonzero solution x is called an eigenvector of A at the eigenvalue λ.

$$\boxed{(A - \lambda I) x = 0,}$$

which is a homogeneous equation.

This has nonzero solution iff $A - \lambda I$ is Singular / or noninvertible.

iff $\boxed{\det (A - \lambda I) = 0}$

To find all the eigenvalues and eigenvectors of

ex: Find all the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Sol: We need to set up

$$\det(A - \lambda I) = 0$$

$$\det\left(\underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}} \right) = 0$$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det \begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(1-\lambda) - 1 \cdot 1 = 0$$

$$1 - \lambda - \lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - 2\lambda = 0$$

$$\lambda(\lambda - 2) = 0$$

$$\longrightarrow \underbrace{\lambda = 0 \quad \text{and} \quad \lambda = 2}$$

eigenvalues: 0, 2.

* To find eigenvectors:

- $\lambda = 0 \longrightarrow A x = 0$ solve for x

- $\lambda = 2 \xrightarrow{\text{solve}} \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right)$