1.2 (cant)

Tuesday, January 14, 2025 5:39 PM

* Gaussian elimination method for solving linear equations

Let's consider $Ax = b$

Step 0: Write the augmented form for the above system

$$(A \mid b)$$

Step 1: Use row operations to transform the above matrix into the row-echelon form:

$$(A \mid b) \xrightarrow[\text{operations}]{\text{row}} (\underbrace{E}_{\text{row-echelon matrix}} \mid c)$$

Step 2: Use back-substitution to find $x_n, x_{n-1}, \dots, x_1$ from $(E \mid c)$

ex: Solve

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 &= 4 \\ 3x_1 + 8x_2 + 7x_3 + 2x_4 &= 20 \\ 2x_1 + 7x_2 + 9x_3 + x_4 &= 23. \end{aligned}$$

Sol:

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 4 \\ 3 & 8 & 7 & 2 & 20 \\ 2 & 7 & 9 & 1 & 23 \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 3R_1} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 4 \\ 0 & 2 & 4 & -1 & 8 \\ 0 & 3 & 7 & -1 & 15 \end{array}\right)$$

we want this leading entry to be 1

unit column

(Use R_1 to change R_2 & R_3)

$$\xrightarrow{R_2 \to R_2 - R_3} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 4 \\ 0 & -1 & -3 & 0 & -7 \\ 0 & 3 & 7 & -1 & 15 \end{array}\right) \xrightarrow{R_3 \to R_3 + 3R_2} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 4 \\ 0 & -1 & -3 & 0 & -7 \\ 0 & 0 & -2 & -1 & -6 \end{array}\right)$$

turn it to zero

turn this leading entry to 1

$$\xrightarrow[R_2 \to R_2 \cdot (-1)]{R_3 \to R_3 / (-2)} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \\ \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 4 \\ 0 & 1 & 3 & 0 & 7 \\ 0 & 0 & 1 & .5 & 3 \end{array}\right) \end{array}$$

row-echelon form

row-echelon form

* Back Substitution:

- Determine leading variables & free variables.

  leading variables: x_1, x_2, x_3.

  free variables: x_4

- Set free variables as parameter:

  set $x_4 = t$

- Find other leading variables by back subs:

From R_3: $x_3 + 0.5x_4 = 3$

$x_3 + 0.5t = 3$

$x_3 = 3 - 0.5t$.

From R_2: $x_2 + 3x_3 = 7$

$x_2 + 9 - 1.5t = 7$

$x_2 = -2 + 1.5t$

From R_1: $x_1 + 2x_2 + x_3 + x_4 = 4$

$x_1 - 4 + 3t + 3 - 0.5t + t = 4$

$x_1 + 3.5t - 1 = 4$

$x_1 = 5 - 3.5t$.

So $x = \begin{pmatrix} 5-3.5t \\ -2+1.5t \\ 3-0.5t \\ t \end{pmatrix}$ (infinitely many solutions)

(ex) Solve

$$\begin{aligned} x_1 + x_2 + x_3 - x_4 &= 2 \\ 2x_1 - x_2 + x_3 + x_4 &= 3 \\ 3x_1 - x_2 + 2x_3 - x_4 &= 3 \\ 5x_1 - 2x_2 + 3x_3 &= 6 \end{aligned}$$

Solution: $\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 2 \end{array}\right) \xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 2 \end{array}\right)$

Solution:

clear them

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 2 \\ 2 & -1 & 1 & 1 & 3 \\ 3 & -1 & 2 & -1 & 3 \\ 5 & -2 & 3 & 0 & 6 \end{array}\right) \xrightarrow[R_4 \to R_4 - 5R_1]{R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 2 \\ 0 & -3 & -1 & 3 & -1 \\ 0 & -4 & -1 & 2 & -3 \\ 0 & -7 & -2 & 5 & -4 \end{array}\right)$$

$$\xrightarrow{R_2 \to R_2 - R_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 2 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & -4 & -1 & 2 & -3 \\ 0 & -7 & -2 & 5 & -4 \end{array}\right) \xrightarrow[R_4 \to R_4 + 7R_2]{R_3 \to R_3 + 4R_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 2 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & -1 & 6 & 5 \\ 0 & 0 & -2 & 12 & 10 \end{array}\right)$$

need to be cleared

clear this

(Keep tracking leading entries and clear entries below it)

$$\xrightarrow{R_4 \to R_4 - 2R_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 2 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & -1 & 6 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

* **Back subs**:

leading variables: x_1, x_2, x_3

free ———: x_4

Set $x_4 = t$.

From R_3: $-x_3 + 6x_4 = 5$

$-x_3 + 6t = 5$

$x_3 = 6t - 5$.

—— R_2: $x_2 + x_4 = 2$

$x_2 + t = 2$

$x_2 = 2 - t$.

—— R_1: $x_1 + x_2 + x_3 - x_4 = 2$

$x_1 + 2 - t + 6t - 5 - t = 2$

$$x_1 + 4t - 3 = 2$$

$$x_1 = 5 - 4t.$$

So $x = \begin{pmatrix} 5-4t \\ 2-t \\ 6t-5 \\ t \end{pmatrix}$ (infinitely many solutions)

* Jordan Step :

clear them

clear it

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 2 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & -1 & 6 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 \to R_1 + R_3} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 5 & 7 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & -1 & 6 & 5 \end{array}\right)$$

unit column

$$\xrightarrow[R_3 \to R_3(-1)]{R_1 \to R_1 - R_2} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 5 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -6 & -5 \end{array}\right)$$

R_3: $x_3 - 6x_4 = -5$

$x_3 = 6t - 5$ (no substitution)

R_2 : $x_2 + t = 2 \to x_2 = 2 - t$

R_1 : $x_1 + 4t = 5 \to x_1 = 5 - 4t$

* Reduced row echelon matrix :

. The reduced row-echelon matrix is a row-echelon matrix, at which the first nonzero entry (leading entry) of a row is the only nonzero entry in its column.

ex : $\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$; $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \to R_2 - R_3} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

reduced row-echelon ; row-echelon but not reduced row-echelon ; (reduced)

Gauss-Jordan method for transforming a matrix A to

Gauss-Jordan method for transforming a matrix A to a reduced row-echelon form

- Step I: Gaussian step: we turn the matrix A to the row-echelon form E by using row operations
  (clearing entries below the leading entries)
- Step II: Jordan step: we transform E to the reduced row-echelon format by using row operations
  (clearing entries above the leading entries)

$\boxed{\text{Quiz 1 covers 1.1 \& 1.2}}$