MTH 3001 Problem Set 9: Rolle, Taylor, and Mean Values

Rolle's theorem, the mean value theorem, and Taylor's theorem from calculus courses have been useful on past Putnam competitions. Here are some problems to try. [Adapted by Professor Jerrold Grossman from material prepared by Professor Barry Turett, Oakland University. November 8, 2004.]

1. (1980, B-1)** For which real numbers c is $(e^x + e^{-x})/2 \le e^{cx^2}$ for all real x?

2. (1988, A-3)** Determine, with proof, the set of real numbers x for which

$$\sum_{n=1}^{\infty} \left(\frac{\csc\left(\frac{1}{n}\right)}{n} - 1 \right)^x$$

converges.

3. (1990, A-2)* Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$, where n and m are whole numbers?

4. (1992, A-4)*** Let f be an infinitely differentiable real-valued function defined on the real numbers. If

$$f\left(\frac{1}{n}\right) = \frac{n^2}{n^2+1}, \qquad n = 1, 2, 3, \ldots,$$

compute the values of the derivatives $f^{(k)}(0),\ k = 1, 2, 3, \ldots$.

5. (1995, A-2)*** For what pairs (a, b) of positive real numbers does the improper integral

$$\int_b^{\infty} \left(\sqrt{\sqrt{x+a} - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x-b}} \right) dx$$

converge?

6. (2002, B-3)**** Show that for all integers $n > 1$,

$$\frac{1}{2ne} < \frac{1}{e} - \left(1 - \frac{1}{n}\right)^n < \frac{1}{ne}.$$

Hints:

1. Use Taylor series.

2. Use Taylor series to estimate $\frac{1}{n}\csc\left(\frac{1}{n}\right) - 1$.

3. Show that $\sqrt[3]{n+1} - \sqrt[3]{n}$ can be arbitrarily close to 0, so the numbers given are dense among positive reals.

4. Let $h(x) = f(x) - 1/(x^2+1)$. Use Rolle's theorem and continuity repeatedly to prove $h^{(n)}(0) = 0$ for all n.

5. Estimate the integrand using Taylor series. In particular, note that $\sqrt{1+t} = 1 + (t/2) + O(t^2)$ where $O(t^2)$ is a stand-in for a function $f(t)$ for which there exists a constant C such that $|f(t)| \leq Ct^2$ for all sufficiently large t. Alternately, multiply and divide with the conjugate several times to estimate the integrand.

6. Use the Taylor series for $\ln(1-x)$ for $0 < x < 1$.