MTH 3001 Problem Set 7:
Some Number Theory Problems

A few number theory problems have appeared on problem sets #3 and #6. Here are a few more. [Adapted by Professor Jerrold Grossman from material prepared by Professor Barry Turett, Oakland University. October 26, 2004.]

1. (1974, B-3)*** Prove that if α is a real number such that $\cos(\pi\alpha) = 1/3$, then α is irrational. (The angle $\pi\alpha$ is in radians.)

2. (1988, B-1)* A *composite* (positive integer) is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \ldots\}$. Show that every composite is expressible as $xy + xz + yz + 1$ with x, y, and z positive integers.

3. (1995, A-3)** The number $d_1 d_2 \ldots d_9$ has nine (not necessarily distinct) decimal digits. The number $e_1 e_2 \ldots e_9$ is such that each of the nine 9-digit numbers formed by replacing just one of the digits d_i in $d_1 d_2 \ldots d_9$ by the corresponding digit e_i $(1 \leq i \leq 9)$ is divisible by 7. The number $f_1 f_2 \ldots f_9$ is related to $e_1 e_2 \ldots e_9$ in the same way: that is, each of the nine numbers formed by replacing one of the e_i by the corresonding f_i is divisible by 7. Show that, for each i, $d_i - f_i$ is divisible by 7. [For example, if $d_1 d_2 \ldots d_9 = 199501996$, then e_6 may be 2 or 9 since 199502996 and 199509996 are multiples of 7.]

4. (1998, A-4)** Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example, $A_3 = A_2 A_1 = 10$, $A_4 = A_3 A_2 = 101$, $A_5 = A_4 A_3 = 10110$, and so forth. Determine all n such that 11 divides A_n.

5. (1998, B-5)*** Let N be the positive integer with 1998 decimal digits all of them 1; that is,

$$N = 1111\ldots11.$$

Find the thousandth digit after the decimal point of $\sqrt{N}$.

6. (1998, B-6)*** Prove that, for every three integers a, b, and c there exists a positive integer n such that $\sqrt{n^3 + an^2 + bn + c}$ is not an integer.

7. (2001, A-5)**** Prove that there are unique positive integers a and n such that

$$a^{n+1} - (a+1)^n = 2001.$$

Hints:

1. Let $\alpha = m/n$. Show that there are only finitely many possible values of $\cos(k\pi\alpha)$ for integer values of k. Use the double-angle formula to show that there are infinitely many.

2. Let $z = 1$ and factor.

3. Let $D = d_1 d_2 \dots d_9$ and $E = e_1 e_2 \dots e_9$. Write the given conditions on D and E and sum them.

4. Find the number of digits in A_n, and find a recursion formula for A_n modulo 11.

5. Note that $N = (10^{1998} - 1)/9$ and that the 1000th digit after the decimal point of $\sqrt{N}$ is $\lfloor 10^{1000}\sqrt{N} \rfloor$. Or use a Taylor expansion to approximate $\sqrt{N}$.

6. All perfect squares are congruent to 0 or 1 modulo 4.

7. Considering the equation modulo a implies that a divides 2002. Then consider the equation modulo 3, modulo $a + 1$, and finally modulo 8.