MTH 3001 Problem Set 11: Games

Several Putnam Competitions have contained "games" for you to play. The competition in 2002 had three games. (One of them, (2002, A-4) was discussed earlier, so it is omitted below.) Here are some games from previous competitions. A good strategy to solve these type of problems is to look at small cases of the game, figure out the best strategy, then move to larger cases and hope to notice a pattern, which can be proven by induction. [Adapted by Professor Jerrold Grossman from material prepared by Professor Barry Turett, Oakland University. November 22, 2004.]

1. (1989, A-4)** If α is an irrational number, $0 < \alpha < 1$, is there a finite game with an honest coin such that the probability of one player's winning the game is α? (An honest coin is one for which the probability of heads and the probability of tails are both $\frac{1}{2}$. A game is finite if with probability 1 it must end in a finite number of moves.)

2. (1995, B-5)** A game starts with four heaps of beans, containing 3, 4, 5, and 6 beans. The two players move alternately. A move consists of taking **either** one bean from a heap, provided at least two beans are left behind in that heap, **or** a complete heap of two or three beans. The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy.

3. (1997, A-2)* Players $1, 2, 3, \ldots, n$ are seated around a table, and each has a single penny. Player 1 passes a penny to Player 2, who then passes two pennies to Player 3. Player 3 then passes one penny to Player 4, who passes two pennies to Player 5, and so on, players alternately passing one penny or two to the next player who still has some pennies. A player who runs out of pennies drops out of the game and leaves the table. Find an infinite set of numbers n for which some player ends up with all n pennies.

4. (2002, B-2)* Consider a polyhedron with at least five faces such that exactly three edges emerge from each of its vertices. Two players play the following game: Each player, in turn, signs his or her name on a previously unsigned face. The winner is the player who first succeeds in signing three faces that share a common vertex. Show that the player who signs first will always win by playing as well as possible.

5. (2002, B-4)** An integer n, unknown to you, has been randomly chosen in the interval $[1, 2002]$ with uniform probability. Your objective is to select n is an **odd** number of guesses. After each incorrect guess, you are informed whether n is higher or lower, and you **must** guess an integer on your next turn among the numbers that are still feasibly correct. Show that you have a strategy so that the chance of winning is greater than 2/3.

Hints:

1. Let coin flips determine digits past the decimal point in the binary expansion of a real number.

2. Use parity. Argue that if there is a heap with 3 beans then you can win.

3. Determine how the game progresses by induction.

4. Player A should sign the face with the most edges. Show that if that face has at least four edges, then first player can win on the third move. If every face is a triangle, use Euler's formula to get a contradiction.

5. Try to guarantee that on all (but the first) odd-numbered guesses you have two possible ways to win and, on an even-numbered guess, you have only one way to win. Alternatively, try to find good strategies to win in odd or even number of steps for n chosen from intervals $[1, 2]$, $[1, 3]$, $[1, 4]$, $\ldots$, and find the pattern for the probabilities.