# MTH 3001 Problem Set 4: Some Geometry Problems

Not surprisingly, many Putnam Exams include geometry problems. Many of these deal with areas of common geometric figures while others deal with just classical plane geometry. Most of the time, what's needed to solve the problem is familiar to the solver. For example, it is useful to recall that, for two vectors $\overrightarrow{A}$ and $\overrightarrow{B}$, the area of the parallelogram determined by $\overrightarrow{A}$ and $\overrightarrow{B}$ is $||\overrightarrow{A} \times \overrightarrow{B}||$; or that the maximum area of a triangle that you can fit between a chord of a circle and the arc subtended by the chord is determined by the chord and the midpoint of the arc. (This is easy to see, so you should draw a picture and make sure it's clear to you why this holds.) Once in a while however (not really that often), a problem requires, or is greatly simplified by, knowing some geometric fact that may (or may not) have been familiar to you at one time. For example, most people have seen Heron's formula: the area $K$ of a triangle with sides $a$, $b$, and $c$ is given by $K = \sqrt{(s-a)(s-b)(s-c)}$ where $s = (a+b+c)/2$. They may not have seen the fact, due to the Hindu mathematician Brahmagupta, that the area $K$ of a cyclic quadrilateral (one inscribed in a circle) has area $K = \sqrt{s(s-a)(s-b)(s-c)(s-d)}$ where $s = (a+b+c+d)/2$ and that the area of a quadrilateral with sides $a$, $b$, $c$, $d$ is maximized by making it cyclic. A lot of people know Pick's theorem: If a simple polygon has vertices that are lattice points, then the formula for the area is $\frac{1}{2}B + I - 1$ where $B$ is the number of lattice points on the boundary and $I$ is the number of lattice points in the interior of the polygon. (A lattice point is a point $(x, y)$ with $x$, $y$ integers.) Once in a while, a geometric fact that most people probably never heard of is used. For example, did you know that if you have a triangle with sides of length $a$, $b$, and $c$ that is inscribed in a circle of radius $r$, then $abc = 4Kr$ where $K$ is the area of the triangle? It's not hard to prove using similar triangles but if you didn't know this fact (and I didn't), one of the problems below would seem hard to prove. Below is a selection of past Putnam problems, some of which use the above facts. [Adapted by Professor Jerrold Grossman from material prepared by Professor Barry Turett, Oakland University. October 6, 2004.]

1*. (1966, B-1) Let a convex polygon $P$ be contained in a square of side 1. Show that the sum of the squares of the sides of $P$ is less than or equal to 4.

2.* (1978, B-1) Find the area of a convex octagon that is inscribed in a circle and has four consecutive sides of length 3 units and the remaining four sides of length 2 units. Give the answer in the form $r + s\sqrt{t}$ wih $r$, $s$, and $t$ positive integers.

3.** (1990, A-3) Prove that any convex pentagon whose vertices (no three of which are collinear) have integer coordinates must have area greater than or equal to $5/2$.

4.** (1996, A-2) Let $C_1$ and $C_2$ be circles whose centers are 10 units apart and whose radii are 1 and 3. Find, with proof, the locus of all points $M$ for which there exists points $X$ on $C_1$ and $Y$ on $C_2$ such that $M$ is the midpoint of the line segment $\overline{XY}$.

5.**** (1998, A-6) Let $A$, $B$, and $C$ denote distinct points with integer coodinates in $\mathbf{R}^2$ (the real coordinate plane). Prove that if

$$(AB + BC)^2 < 8 \cdot [ABC] + 1,$$

then $A$, $B$, and $C$ are vertices of a square. Here $XY$ denotes the length of segment $\overline{XY}$ and $[ABC]$ is the area of triangle $ABC$.

6.** (1998, B-2) Given a point $(a, b)$ with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at $(a, b)$, one vertex on the $x$-axis, and one vertex on the line $y = x$. You may assume that a triangle of minimum perimeter exists.

7.** (2000, A-3) The octagon $P_1P_2P_3P_4P_5P_6P_7P_8$ is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon $P_1P_3P_5P_7$ is a square of area 5, and the polygon $P_2P_4P_6P_8$ is a rectangle of area 4, find the maximum possible area of the octagon.

8.*** (2000, A-5) Three distinct points with integer coordinates lie in the plane on a circle of radius $r > 0$. Show that two of these points are separated by a distance of at least $r^{1/3}$.

9.*** (2001, A-4) Triangle $ABC$ has area 1. Points $E$, $F$, and $G$ lie, respectively, on sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $\overline{AE}$ bisects $\overline{BF}$ at point $R$, $\overline{BF}$ bisects $\overline{CG}$ at point $S$, and $\overline{CG}$ bisects $\overline{AE}$ at point $T$. Find the area of the triangle $RST$.

Hints:

1. Pythagorean theorem

2. Find one-quarter of the area.

3. Use Pick's theorem and consider the parity of the coordinates.

4. Use vectors.

5. Show that there exists a point $C'$ such that $A, B, C'$ are vertices of a square and $|CC'| < 1$. To do this, work in a coordinate system in which $B = (0, 0)$ and $A = (s, 0)$ for some $s > 0$.

6. Use reflections.

7. Get a rectangle and then the square.

8. Relate the area and the sides of the triangle and the radius of of the circumcircle.

9. Use vectors, or use an affine transformation to assume that $A = (0, 1), B = (1, 0), C = (-1, 0)$.