MTH 3001 Problem Set 6:
A Few "Second" Problems

Like problems labeled A-1 and B-1, the second problems in the morning and afternoon sessions (A-2 and B-2) are usually doable. Here are a few such problems. [Adapted by Professor Jerrold Grossman from material prepared by Professor Barry Turett, Oakland University. October 18, 2004.]

1. (1993, A-2)* Let $(x_n)_{n\geq 0}$ be a sequence of nonzero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for $n = 1, 2, 3, \ldots$. Prove that there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$.

2. (1993, B-2)** Consider the following game played with a deck of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of the two players, A and B. Beginning with A, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n + 1$. The last person to discard wins the game. Assuming optimal strategy by both A and B, what is the probability that A wins?

3. (1998, A-2)* Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A + B$ depends only on the arc length, and not on the position, of s.

4. (1997, B-2)*** Let f be a twice-differentiable real-valued function satisfying $f(x) + f''(x) = -xg(x)f'(x)$, where $g(x) \geq 0$ for all real x. Prove that $|f(x)|$ is bounded.

5. (1994, A-2)** Let A be the area of the region in the first quadrant bounded by the line $y = \frac{1}{2}x$, the x-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$. Find the positive number m such that A is equal to the area of the region in the first quadrant bounded by the line $y = mx$, the y-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$.

6. (2000, B-2)** Prove that the expression

$$\frac{\gcd(m, n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$.

Hints:

1. Let $a_n = (x_{n+1} + x_{n-1})/x_n$ and show $a_{n+1} = a_n$.
2. Can a player make each move so that the other cannot possibly win on the next move?
3. The result can be obtained simply by manipulating areas, without evaluating any integrals.
4. Multiply by $f'(x)$.
5. Transform the ellipse into a circle by a change of variables.
6. Recall that $\gcd(m, n)$ can be written as $am + bn$ for some integers a and b.