# MTH 3001 Problem Set 10: Some Linear Algebra

Here are some facts that may be useful in the problems below:

1. If two rows (or two columns) of an $n \times n$ matrix are identical, the determinant of the matrix is zero.
2. If two rows (or two columns) of an $n \times n$ matrix are interchanged, the value of the determinant of the resulting matrix is the negative of the value of the determinant of the original matrix.
3. An $n \times n$ matrix $\mathbf{A}$ is invertible if and only if $\det \mathbf{A} \neq 0$. When $\mathbf{A}$ is invertible, the unique inverse of $\mathbf{A}$ is $\mathbf{A}^{-1} = (\det \mathbf{A})^{-1} \mathrm{adj}\, \mathbf{A}$.
4. (Cayley-Hamilton Theorem) If $p$ is the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$, then $p(\mathbf{A}) = \mathbf{0}$.
5. Similar matrices have the same trace. So, the trace of a linear transformation on $\mathbf{R}^n$ is independent of the basis.

Now for some problems. [Adapted by Professor Jerrold Grossman from material prepared by Professor Barry Turett, Oakland University. November 15, 2004.]

1. (1985, B-6)**** Let $G$ be a finite set of real $n \times n$ matrices $\{\mathbf{M}_i\}$, $1 \leq i \leq r$, which form a group under matrix multiplication. Suppose $\sum_{i=1}^{r} \mathrm{tr}(\mathbf{M}_i) = 0$, where $\mathrm{tr}(\mathbf{A})$ denotes the trace of the matrix $\mathbf{A}$. Prove that $\sum_{i=1}^{r} \mathbf{M}_i$ is the $n \times n$ zero matrix.

2. (1988, A-6)*** If a linear transformation $\mathbf{A}$ on an $n$-dimensional vector space has $n+1$ eigenvectors such that every set of $n$ of them are linearly independent, does it follow that $\mathbf{A}$ is a scalar multiple of the identity? Prove your answer.

3. (1991, A-2)** Let $\mathbf{A}$ and $\mathbf{B}$ be different $n \times n$ matrices with real entries. If $\mathbf{A}^3 = \mathbf{B}^3$ and $\mathbf{A}^2\mathbf{B} = \mathbf{B}^2\mathbf{A}$, can $\mathbf{A}^2 + \mathbf{B}^2$ be invertible?

4. (1992, B-5)*** Let $D_n$ denote the value of the $(n-1) \times (n-1)$ determinant

$$\begin{pmatrix} 3 & 1 & 1 & 1 & \dots & 1 \\ 1 & 4 & 1 & 1 & \dots & 1 \\ 1 & 1 & 5 & 1 & \dots & 1 \\ 1 & 1 & 1 & 6 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \dots & n+1 \end{pmatrix}$$

Is the set $\{D_n/n!\}_{n \geq 2}$ bounded?

5. (1995, B-3)*** To each positive integer with $n^2$ decimal digits, we associate the determinant of the matrix obtained by writing the digits in order across the rows. For example, for $n = 2$, to the integer 8617 we associate

$$\det \begin{pmatrix} 8 & 6 \\ 1 & 7 \end{pmatrix} = 50.$$

Find, as a function of $n$, the sum of all determinants associated with $n^2$-digit integers. (Leading digits are assumed to be nonzero; for example, for $n = 2$, there are 9000 determinants.)

6. (1996, B-4)*** For any square matrix $\mathbf{A}$, we can define $\sin \mathbf{A}$ by the usual power series:

$$\sin \mathbf{A} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \mathbf{A}^{2n+1}.$$

Prove or disprove: there exists a $2 \times 2$ matrix $\mathbf{A}$ with real entries such that

$$\sin \mathbf{A} = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix}.$$

Hints:

1. What is $(\sum_{i=1}^{r} \mathbf{M}_i)^2$?

2. The trace of $\mathbf{A}$ is independent of the choice of basis. Alternatively, write $x_{n+1}$ as a linear combination of the other elements and apply $\mathbf{A}$.

3. Show that $\mathbf{A}^2 + \mathbf{B}^2$ times something nonzero is zero. There is only one nonzero matrix in the problem statement.

4. Use row operations to make most entries zero, then use column operations to make the matrix upper or lower triangular.

5. Use properties of the determinants stated earlier.

6. Recall that $\sin \mathbf{A}$ and $\cos \mathbf{A}$ are defined by power series; that definition can be used to prove that certain trigonometric identities still hold for matrices. Alternatively, use a bit of linear algebra to conjugate $\mathbf{A}$ into a simple form before computing $\sin \mathbf{A}$ and $\cos \mathbf{A}$.