MTH 3001 Problem Set 5: Polynomials

Another obvious source of problems is polynomials. Here are a few problems involving polynomials that have turned up on old exams. We've already seen a few examples on some of the earlier sheets.

One fact about polynomials that sometimes is useful is the division algorithm: if you divide a polynomial $f(x)$ by a polynomial $g(x)$ then $f(x) = q(x)g(x) + r(x)$ where the quotient $q(x)$ and the remainder $r(x)$ are polynomials and either r is the zero polynomial, or $0 \le \deg r(x) < \deg g(x)$. So, in particular, if the divisor $g(x) = x - a$, then the remainder is $f(a)$. [Adapted by Professor Jerrold Grossman from material prepared by Professor Barry Turett, Oakland University. October 11, 2004.]

1.* (1977, A-1) Consider all lines that meet the graph of

$$y = 2x^4 + 7x^3 + 3x - 5$$

in four distinct points, say (x_i, y_i), $i = 1, 2, 3, 4$. Show that

$$\frac{x_1 + x_2 + x_3 + x_4}{4}$$

is independent of the line and find its value.

2.*** (1990, B-5) Is there an infinite sequence $a_0, a_1, a_2, \ldots$ of nonzero real numbers such that for $n = 1, 2, 3, \ldots$ the polynomial $p_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ has exactly n distinct real roots?

3.** (1991, A-3) Find all real polynomials $p(x)$ of degree $n \ge 2$ for which there exist real numbers $r_1 < r_2 < \cdots < r_n$ such that

1. $p(r_i) = 0$ for $i = 1, 2, \ldots, n$ and
2. $p'\left(\frac{r_i + r_{i+1}}{2}\right) = 0$ for $i = 1, 2, \ldots, n-1$, where $p'(x)$ denotes the derivative of $p(x)$.

4.*** (1992, B-4) Let $p(x)$ be a nonzero polynomial of degree less than 1992 having no nonconstant factor in common with $x^3 - x$. Let

$$\frac{d^{1992}}{dx^{1992}}\left(\frac{p(x)}{x^3 - x}\right) = \frac{f(x)}{g(x)}$$

for polynomials $f(x)$ and $g(x)$. Find the smallest possible degree of $f(x)$.

5.*** (1994, B-2) Find all c such that the graph of the function $x^4 + 9x^3 + cx^2 + ax + b$ meets some line in four distinct points.

6.**** (2000, A-6) Let $f(x)$ be a polynomial with integer coefficients. Define a sequence $a_0, a_1, \ldots$ of integers such that $a_0 = 0$ and $a_{n+1} = f(a_n)$ for all $n \ge 0$. Prove that if there exists a positive integer m with $a_m = 0$ then either $a_1 = 0$ or $a_2 = 0$.

7.*** (2001, A-3) For each integer m, consider the polynomial

$$P_m(x) = x^4 - (2m + 4)x^2 + (m - 2)^2.$$

For what values of m is $P_m(x)$ the product of two non-constant polynomials with integer coefficients?

Hints:

1. Find the sum of the roots of some polynomial.

2. Define the a_n inductively with $|a_{n+1}|$ much less than $|a_n|$.

3. If $n > 2$, then $\frac{p'(x)}{p(x)} = \frac{1}{x-r_1} + \cdots + \frac{1}{x-r_n}$ is positive at $(r_{n-1} + r_n)/2$.

4. Use partial fractions.

5. By replacing x by $x - \frac{9}{4}$ and adding a linear polynomial, reduce to the analogous problem for $x^4 + ax^2$.

6. If p is a polynomial with integer coefficients, then $m - n$ divides $p(m) - p(n)$ for all integers m and n.

7. Note that $P_m(x) = (x^2 - (m-2))^2 - 8x^2$ and this has no integer roots.