MTH 3001 Problem Set 8:
Functional Equations

Sometimes linear recurrence relations with constant coefficients appear on Putnam Competitions. If the sequence $x_0, x_1, x_2, \ldots$ satisfies the linear recursion

$$x_{n+k} + b_{k-1}x_{n+k-1} + \cdots + b_1 x_{n+1} + b_0 x_n = 0,$$

then the solutions x_n are sums of terms of the form $(a_i n^i + \cdots + a_1 n + a_0)b^n$, where b is a root with multiplicity $i+1$ of the characteristic equation

$$x^k + b_{k-1}x^{k-1} + \cdots + b_1 x + b_0 = 0.$$

. For example, if $f : \mathbf{N} \cup \{0\} \to \mathbf{R}$ satisfies $f(n) = 5f(n-1) - 6f(n-2)$, $f(0) = 3$, and $f(1) = 8$, try to find solutions of the form $c_1 b_1^n + c_2 b_2^n$, where b_1 and b_2 are roots of the equation $x^2 - 5x + 6 = 0$. As another example, if $f(n) = 4f(n-1) - 4f(n-2)$, $f(0) = 2$, and $f(1) = 6$, try to find solutions of the form $(c_1 n + c_2)b^n$, where b is a root of multiplicity 2 of $x^2 - 4x + 4 = 0$. More about linear recurrence relations can be found in books on discrete mathematics or combinatorics. A few of the problems below use these ideas. [Adapted by Professor Jerrold Grossman from material prepared by Professor Barry Turett, Oakland University. November 1, 2004.]

1. (1984, B-1)* Let n be a positive integer, and define

$$f(n) = 1! + 2! + \cdots + n!.$$

Find polynomials $P(x)$ and $Q(x)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n),$$

for all $n \geq 1$.

2. (1986, B-5)**** Let $f(x,y,z) = x^2 + y^2 + z^2 + xyz$. Let $p(x,y,z)$, $q(x,y,z)$, and $r(x,y,z)$ be polynomials with real coefficients satisfying

$$f(p(x,y,z), q(x,y,z), r(x,y,z)) = f(x,y,z).$$

Prove or disprove the assertion that the sequence p, q, r consists of some permutation of $\pm x, \pm y, \pm z$, where the number of minus signs is 0 or 2.

3. (1988, A-5)*** Prove that there exists a *unique* function f from the set $\mathbf{R}^+$ of positive real numbers to $\mathbf{R}^+$ such that

$$f(f(x)) = 6x - f(x) \qquad \text{and} \qquad f(x) > 0 \;\text{ for all } x > 0.$$

4. (1992, A-1)** Prove that $f(n) = 1 - n$ is the only integer-valued function defined on the integers that satisfies the following conditions:

(i) $f(f(n)) = n$ for all integers n; (ii) $f(f(n+2)+2) = n$ for all integers n; (iii) $f(0) = 1$.

5. (1999, A-1)* Find polynomials $f(x)$, $g(x)$, and $h(x)$, if they exist, such that, for all x,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x + 2 & \text{if } -1 \le x \le 0 \\ -2x + 2 & \text{if } x > 0 \end{cases}$$

6. (2001, B-5)***** Let a and b be real numbers in the interval $(0, \frac{1}{2})$ and let g be a continuous real-valued function such that $g(g(x)) = ag(x) + bx$ for all real x. Prove that $g(x) = cx$ for some constant c.

Hints:

1. Try to get the last term in $f(n+2)$ from the last term of $f(n+1)$, then get the rest from $f(n)$ and $f(n+1)$.

2. It's false. Can you find a slightly different solution than $p(x, y, z) = x$, $q(x, y, z) = y$, and $r(x, y, z) = z$?

3. For any x, the sequence $x, f(x), f(f(x)), f(f(f(x))), \ldots$ is linearly recursive.

4. Apply f to (ii) and then use (i) on the left-hand side.

5. Guess the form of f, g, and h, trying to find linear polynomials.

6. This problem seems really hard: Only two people in the top 200 got significant points on it. Therefore this hint is somewhat of a sketch. Try to fill in the details. Start by showing that g is one-to-one. Since one-to-one continuous functions are strictly monotonic, g is either strictly increasing or strictly decreasing. Assuming $\lim_{x\to\pm\infty} g(x)$ exists, get a contradiction to show g is onto $\mathbf{R}$. Take x_0 arbitrary and define $x_{n+1} = g(x_n)$ for $n > 0$ and $x_{n-1} = g^{-1}(x_n)$ for $n < 0$. Use this to get $x_n = c_1 r_1^n + c_2 r_2^n$. The nature of r_1 and r_2 will be useful. If g is strictly increasing, show $g(x) = r_1 x$. If g is strictly decreasing, show $g(x) = r_2 x$.