MTH 3001, Problem Set 1: Mathematical Induction

The principles of mathematical induction have occurred frequently on past Putnam exams. Unlike the types of problems that you see in high school classes or in MTH 3002, more is usually required than just the verification of a formula. Many times you need to guess the correct formula and, for some problems, there are no formulas. In some of the problems below, you need to know some other mathematical facts as well. For a few of the problems, simple probabilistic facts such as Bayes' rule can be used; and, for another problem, the fact that if two numbers m and n are both the sum of two squares, then their product mn is also the sum of two squares. (This fact is easy to prove. Try it.) The number of stars is proportional to the difficulty level of the problem. [Adapted by Professor Jerrold Grossman from material prepared by Professor Barry Turett, Oakland University. September 14, 2004.]

1.*** (1956, B-5) Consider a set of $2n$ points in space, $n > 1$. Suppose they are joined by at least $n^2 + 1$ segments. Show that at least one triangle is formed. Show that for each n it is possible to have $2n$ points joined by n^2 segments without any triangles being formed.

2.** (1985, B-2) Define polynomials $f_n(x)$ for $n \geq 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \geq 1$, and

$$\frac{d}{dx}(f_{n+1}(x)) = (n+1)f_n(x+1)$$

for $n \geq 0$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes.

3.* (1990, A-1) Let $T_0 = 2$, $T_1 = 3$, $T_2 = 6$, and for $n \geq 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$

The first few terms are 2, 3, 6, 14, 40, 152, 784, 5168, 40576, 363392. Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where (A_n) and (B_n) are well-known sequences.

4.*** (1999, A-2) Let $p(x)$ be a polynomial that is nonnegative for all x. Prove that, for some k, there are polynomials $f_1(x), \ldots, f_k(x)$ such that

$$p(x) = \sum_{j=1}^{k} (f_j(x))^2.$$

5.** (2000, A-2) Prove that there exist infinitely many integers n such that n, $n+1$, and $n+2$ are each the sum of the squares of two integers. [Example: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, and $2 = 1^2 + 1^2$.]

6.** (2001, A-2) You have coins $C_1, C_2, \ldots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answers as a rational function of n.

7.** (2002, A-1) Let k be a fixed positive integer. The nth derivative of $\dfrac{1}{x^k - 1}$ has the form $\dfrac{P_n(x)}{(x^k-1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

8.**** (2002, A-5) Define a sequence by $a_0 = 1$, together with the rules $a_{2n+1} = a_n$ and $a_{2n+2} = a_n + a_{n+1}$ for each integer $n \geq 0$. Prove that every positive rational number appears in the set

$$\left\{ \frac{a_{n-1}}{a_n} : n \geq 1 \right\} = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \ldots \right\}.$$

9.** (2002, B-1) Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability that she hits exactly 50 of her first 100 tosses?