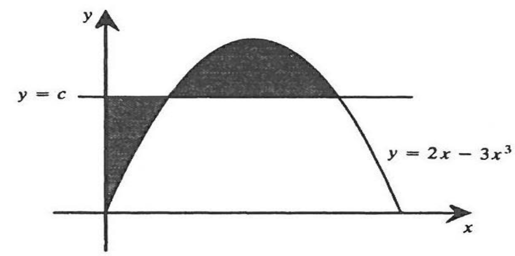
# MTH 3001 Problem Set 3: A Few Lead-Off Problems

Usually the first problems (A-1 and B-1) in the morning and afternoon sections are do-able. Here are a few such problem.All of the problems below are one-star (*) or two-star (**) problems. [Adapted by Professor Jerrold Grossman from material prepared by Professor Barry Turett, Oakland University. September 20, 2004.]

1. (1993, A-1) The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the figure. Find $c$ such that the areas of the two shaded regions are equal.




2. (1993, B-1) Find the smallest positive integer $n$ such that for every integer $m$ with $0 < m < 1993$, there exists an integer $k$ for which

$$\frac{m}{1993} < \frac{k}{n} < \frac{m+1}{1994}.$$

3. (1998, A-1) A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one of the faces of the cube is contained in the base of the cone. What is the side-length of the cube?

4. (1998, B-1) Find the minimum value of

$$\frac{(x+1/x)^6 - (x^6+1/x^6) - 2}{(x+1/x)^3 + (x^3+1/x^3)}$$

for $x > 0$.

5. (2000, A-1) Let $A$ be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, \ldots$ are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?

6. (2000, B-1) Let $a_j$, $b_j$, and $c_j$ be integers for $1 \leq j \leq N$. Assume that for each $j$, at least one of $a_j$, $b_j$, and $c_j$ is odd. Show that there exist integers $r$, $s$, and $t$ such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of $j$, $1 \leq j \leq N$.

Hints:

1. Write the condition on the area as a definite integral.

2. Try finding such an $n$ when replacing 1993 with a smaller number (such as $3, 4, 5$), then generalize.

3. Use similar triangles.

4. Notice that the last two terms of the numerator form a complete square, and then the numerator is a difference of squares we can factor.

5. Note that the sum of the squares is less than the square of the sum to find an upper bound on the sum of the squares. Use geometric series to get all possible values.

6. Note that only parity matters, so we can choose $r$, $s$, $t$ to be either 0 or 1 (but not all 1). Try all and see how many odd numbers you get.