MTH 3001 Problem Set 2:

Evaluating Some Definite Integrals and Sums

Once in a while, you'll find a Putnam problem that asks you to evaluate a specific integral or a specific sum. This problem set is intended to give you some practice in evaluating some of these.

There aren't many series that you can really evaluate. If you are going to evaluate a series, usually it has to be a geometric series, a telescoping series, a Riemann sum, or a series that comes from a familiar Taylor series. Another possibility is to write out the first few partial sums, recognize and prove a formula for the nth partial sum via induction, and take a limit.

For example, suppose you wish to evaluate $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k^4+k^2+1}$. For $k \geq 1$,

$$k^4+k^2+1 = (k^4+2k^2+1)-k^2 = (k^2+k+1)(k^2-k+1)$$

and

$$\frac{2k}{k^4+k^2+1} = \frac{1}{k^2-k+1} - \frac{1}{k^2+k+1} = f(k-1)-f(k),$$

where $f(x) = \dfrac{1}{x^2+x+1}$. Thus the nth partial sum is

$$\sum_{k=1}^{n} \frac{k}{k^4+k^2+1} = \frac{1}{2}\sum_{k=1}^{n}(f(k-1)-f(k)) = \frac{1}{2}(f(0)-f(n)) = \frac{1}{2}\cdot\frac{n^2+n}{n^2+n+1}.$$

Letting $n \to \infty$ shows that $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k^4+k^2+1} = \frac{1}{2}$. Some of the integral problems are doable with a simple change of variable (and sometimes a bit of symmetry). For example consider

$$\int_0^{\pi/2} \frac{\sin^{2003} x}{\sin^{2003} x + \cos^{2003} x}\, dx.$$

Letting I denote the integral, letting $x = \frac{\pi}{2} - \theta$, and plugging in yields

$$\begin{aligned} I = \int_0^{\pi/2} \frac{\sin^{2003} x}{\sin^{2003} x + \cos^{2003} x}\, dx &= \int_{\pi/2}^{0} \frac{\cos^{2003}\theta}{\cos^{2003}\theta + \sin^{2003}\theta}(-d\theta) \\ &= \int_0^{\pi/2} \frac{\cos^{2003}\theta}{\cos^{2003}\theta + \sin^{2003}\theta}\, d\theta. \end{aligned}$$

Now adding the first and last expressions for I yields

$$2I = \int_0^{\pi/2} 1\; d\theta = \frac{\pi}{2},$$

so $I = \pi/4$. [Adapted by Professor Jerrold Grossman from material prepared by Professor Barry Turett, Oakland University. September 20, 2004.]

1.*** (1978, B-2) Express $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m^2n+mn^2+2mn}$ as a rational number.

2.** (1980, A-3) Evaluate $\int_0^{\pi/2}\frac{1}{1+(\tan x)^{\sqrt{2}}}\,dx$.

3.** (1982, A-3) Evaluate $\int_0^{\infty}\frac{\arctan(\pi x)-\arctan x}{x}\,dx$.

4.* (1984, A-2) Express $\sum_{k=1}^{\infty}\frac{6^k}{(3^{k+1}-2^{k+1})(3^k-2^k)}$ as a rational number.

5.*** (1985, A-5) Let $I_m=\int_0^{2\pi}\cos(x)\cos(2x)\cdots\cos(mx)\,dx$. For which integers m, $1\le m\le 10$, is $I_m\neq 0$?

6.** (1986, A-3) Evaluate $\sum_{n=0}^{\infty}\operatorname{arccot}(n^2+n+1)$ where $\operatorname{arccot} t$ for $t\ge 0$ denotes the number θ in the interval $0\le\theta\le\pi/2$ with $\cot\theta=t$.

7.** (1987, B-1) Evaluate $\int_2^4\frac{\sqrt{\ln(9-x)}\,dx}{\sqrt{\ln(9-x)}+\sqrt{\ln(x+3)}}$.

8.*** (1999, A-4) Sum the series

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2n}{3^m(n3^m+m3^n)}.$$

9.*** (2001, B-3) For any positive integer n, let $\langle n\rangle$ denote the closest integer to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty}\frac{2^{\langle n\rangle}+2^{-\langle n\rangle}}{2^n}.$$