Decision Programming

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I Workshop de Otimização sob Incerteza - UFSCar

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Outline of this talk

Outline of this talk

Introduction

Decision Programming

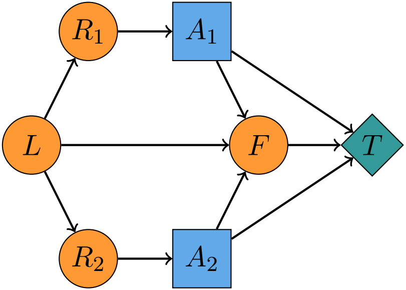
Computational experiments

Conclusions

Recent developments

Modelling decision problems

Influence diagrams are widely used to model decision problems under uncertainty.



- Circles denote chance events
- Squares denote decision events
- Diamonds denote value/utility calculation
- Arc represent influence (dependence).

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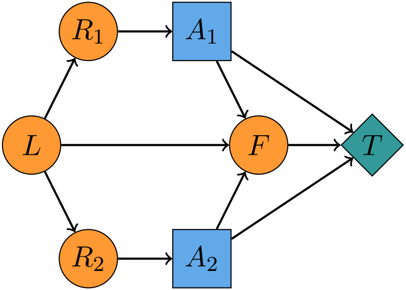

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A simple yet powerful tool that allows for representing a vast range of decision problems with endogenous uncertainties.

Influence diagrams

Despite its simplicity, obtaining solution strategies from influence diagrams is not trivial. Methods include:

- Form a decision tree and solve it (backward induction);
- Apply arc reversal/ node elimination methods;
- Apply Single Policy Update (SPU) (Lauritzen and Nilsson, 2001) or variant.

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Influence diagrams represent Markov decision processes and are, likewise, generally hard to solve.

- Solving an influence diagram is NP-Hard (Mauá et al., 2013)
- Even obtaining approximate solutions is NP-Hard (Mauá et al., 2014)

Influence diagrams

Moreover, several limitations arise from relying on influence diagrams as a modelling framework:

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Our contribution: a framework to address the above while being computationally reliable.

Outline of this talk

Decision Programming

In specific, we propose a framework that can:

1. exploit the expressiveness of influence diagrams
2. exploit linearity (i.e., solve Mixed-Integer Linear Programs - MIPs) as opposed to recursion.

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1. exploit the expressiveness of influence diagrams
2. exploit linearity (i.e., solve Mixed-Integer Linear Programs - MIPs) as opposed to recursion.

In a nutshell, Decision Programming combines:

- the structuring for decision problem under uncertainty from Decision Analysis with
- the structure of MIP formulation of deterministic equivalents for multistage Stochastic Programming problems.

Decision Programming

Information sets and paths

We represent an influence diagram as an acyclic graph $G(N, A)$.

- N consists of chance nodes $c \in C$, decision nodes $d \in D$, and value nodes $v \in V$. Let $n = |C| + |D|$.
- Arcs $A = \{(i, j) : i, j \in N\}$ represent dependencies between nodes.

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With these in mind, we define two key concepts:

- **Information sets:** $I(j)$ consists of nodes from which there is an arc to j.
- **Information states:** $s_{I(j)} \in S_{I(j)} = \prod_{i \in I(j)} S_i$ is a combination of states s_i for nodes in the information set of $i \in I(j)$.

Decision Programming

Information sets and paths

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$$\mathbb{P}(X_d = s_d \mid X_i = s_i, i \in I(d), Z_d) = 1 \iff Z_d(s_{I(d)}) = s_d$$

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Remark: A (global) decision strategy $Z = \prod_{d \in D} Z_d$ is the combination of all local decision strategies.

Information sets and paths

Another key concept: the notion of a path.

- Since G is acyclic, $i < j$ if $(i, j) \in A$ w.l.o.g.;
- A **path** of length k is a sequence $(s_1, s_2, \dots, s_k)$ such that $s_i \in S_i, i = 1, \dots, k$;
- Paths of length $n = |C| + |D|$ are denoted by

$$s = (s_1, \dots, s_n) \in S = \prod_{i \in C \cup D} S_i.$$

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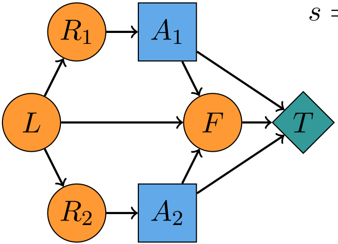

$$s = (s_1, \dots, s_n) \in S = \prod_{i \in C \cup D} S_i.$$

Example: assume that $L = R_1 = R_2 = ... = F = \{+, -\}$.

Then $s = (l, r_1, r_2, a_1, a_2, f) = (+,+,+,+,+,+)$ is a path.

Information sets and paths

We can now formally state (recursively) the probability of a path given a decision strategy Z

$$\mathbb{P}(s_{1:k} \mid Z) = \left(\prod_{i \in C : i \leq k} \mathbb{P}\big(X_i = s_i \mid X_{I(i)} = s_{I(i)}\big) \right) \left(\prod_{j \in D : j \leq k} \mathbb{I}\big(Z_j(s_{I(j)}) = s_j\big) \right),$$

where $\mathbb{I}(\,\cdot\,)$ is defined so that

$$\mathbb{I}(Z_j(s_{I(j)}) = s_j) = \begin{cases} 1, & \text{if } Z_j(s_{I(j)}) = s_j, \\ 0, & \text{otherwise.} \end{cases}$$

Towards an MIP formulation

Our objective is to encode this logic into decision variables.

- We represent decisions with variables $z(s_j \mid s_{I(j)}) \in \{0, 1\}$.

$$Z_j(s_{I(j)}) = s_j \iff z(s_j \mid s_{I(j)}) = 1, \forall\, j \in D,\ s_j \in S_j,\ s_{I(j)} \in S_{I(j)}. \tag{1}$$

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- Mutual exclusivity implies

$$\sum_{s_j \in S_j} z(s_j \mid s_{I(j)}) = 1, \forall j \in D, s_{I(j)} \in S_{I(j)} \tag{2}$$

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$$\sum_{s_j \in S_j} z(s_j \mid s_{I(j)}) = 1, \forall j \in D, s_{I(j)} \in S_{I(j)} \tag{2}$$

- And we define $\pi_k(s) \in [0, 1]$ to represent the path probability.

$$\pi_k(s) = \mathbb{P}\left(X_k = s_k \mid X_{I(k)} = s_{I(k)}\right) \pi_{k-1}(s), \tag{3}$$

For $k \in D$ being a decision node, we have that

$$\pi_k(s) = \begin{cases} \pi_{k-1}(s), & \text{if } z(s_k \mid s_{I(k)}) = 1 \\ 0, & \text{if } z(s_k \mid s_{I(k)}) = 0. \end{cases} \tag{4}$$

Information sets and paths

Theorem 1

Let $Z \in \mathbb{Z}$ be a decision strategy and choose a path $s \in S$. If $\pi_k(s)$, $k = 1, \dots, n$, and $z(s_j \mid s_{I(j)})$, $\forall j \in D$, satisfy the constraints (1) – (4), then

$$\pi_k(s) = \mathbb{P}(X_{1:k} = s_{1:k} \mid Z), \forall\, k = 1, \dots, n$$

In particular, $\pi(s) \overset{def}{=} \pi_n(s)$ is the probability of the path s for the strategy Z.

Towards an MIP formulation

Variables $\pi_k(s)$ can be defined by the inequalities

$$\max\{0, \pi_{k-1}(s) + z(s_k \mid s_{I(k)}) - 1\} \leq \pi_k(s) \leq \min\{\pi_{k-1}(s), z(s_k \mid s_{I(k)})\},$$

which are equivalent to the linear inequalities

$$\begin{aligned} \pi_k(s) &\leq \pi_{k-1}(s) && (5) \\ \pi_k(s) &\leq z(s_k \mid s_{I(k)}) && (6) \\ \pi_k(s) &\geq 0 && (7) \\ \pi_k(s) &\geq \pi_{k-1}(s) + z(s_k \mid s_{I(k)}) - 1. && (8) \end{aligned}$$

Towards a MIP formulation

We want to maximise expected utilities using $\mathcal{U} : S_{I(v)} \to \mathbb{R}$.

$$\text{max.}_{Z \in \mathbb{Z}} \sum_{s \in S} \pi_n(s) \mathcal{U}(s)$$

which only involve $\pi_n(s) = \pi(s)$.

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which only involve $\pi_n(s) = \pi(s)$. Notice that these can be pre-calculated for any given strategy $Z \in \mathbb{Z}$.

$$p(s) = \prod_{j \in C} \mathbb{P}(X_j = s_j \mid X_{I(j)} = s_{I(j)}).$$

And then

- if Z is compatible with $s \in S$ (i.e., if Z maps to path $s \in S$), then $\pi(s) = p(s)$
- otherwise, $\pi(s) = 0$.

Towards an MIP formulation

Corollary 2

The expected utility is maximised by the strategy $Z \in \mathbb{Z}$ which solves the optimisation problem

$$\max_{Z \in \mathbb{Z}} \sum_{s \in S} \pi(s)\mathcal{U}(s)$$

subject to constraints (1) – (3) *and* (5) – (8) *on decision variables* $z(s_k|s_{I(k)}) \in \{0,1\}$, $\forall k \in D$, $s_k \in S_k, s_{I(k)} \in S_{I(k)}$ *and path probabilities* $\pi_k(s) \in [0,1], \forall s \in S$.

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The formulation recursively simplified to only consider $k = n$, since

- (5) – (8) imply that $\pi_j(s) = \pi_{j-1}(s)$ for each $j \in D$ if $z(s_j \mid s_{I(j)}) = 1$
- Analogously, if the strategy Z is not compatible with s, $\pi_n(s) \leq \pi_j(s) = 0$ if $z(s_j \mid s_{I(j)}) = 0$ for some $j \in D$.

Towards a MIP formulation

The complete formulation is given by

$$\text{max.}_{Z \in \mathbb{Z}} \quad \sum_{s \in S} \pi(s)\mathcal{U}(s)$$

s.t.:

$$\begin{aligned}
&\sum_{s_j \in S_j} z(s_j \mid s_{I(j)}) = 1, && \forall\, j \in D, s_{I(j)} \in S_{I(j)} \\
&0 \leq \pi(s) \leq p(s), && \forall\, s \in S \\
&\pi(s) \leq z(s_j \mid s_{I(j)}), && \forall\, j \in D, s \in S \\
&\pi(s) \geq p(s) + \sum_{j \in D} z(s_j \mid s_{I(j)}) - |D|, && \forall\, s \in S \\
&z(s_j \mid s_{I(j)}) \in \{0, 1\}, && \forall\, j \in D,\ s_j \in S_j,\ s_{I(j)} \in S_{I(j)}.
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MIP formulation: key features

Some points worth highlighting:

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   but the formulation obtained was weaker (in terms of LP relaxation).
3. The model has exploitable structure. For example, we use (as lazy constraints) probability cuts of the form

$$\sum_{s \in S} \pi(s) = 1.$$

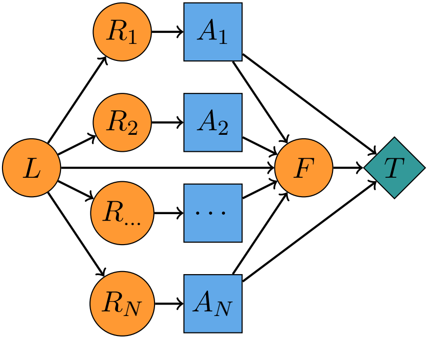
Outline of this talk

Examples

N-monitoring problem

N agents independently intervening without sharing information.

- Independent parallel measures;
- Decisions that can't be communicated;
- No no-forgetting: each action can be seen as taken by independent decision makers.



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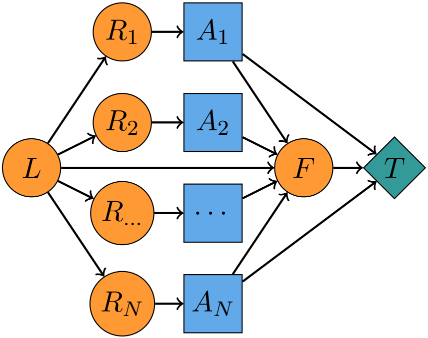

Remark: can be shown to not be soluble (Lauritzen and Nilsson, 2001), a sufficient condition for SPU to converge to optimal strategies.

Computational experiments[1]

N-monitoring problem

	Number of variables		No probability cuts		With probability cuts	
# Nodes	Binary	Real	A	SD	A	SD
2	8	64	0.01	0.01	0.01	0.00
3	12	256	0.12	0.08	0.02	0.01
4	16	1 024	0.79	0.53	0.07	0.02
5	20	4 096	5.94	2.80	0.35	0.19
6	24	16 384	77.35	46.31	2.44	1.63
7	28	65 536	676.35	468.09	20.58	17.48
8	32	262 144	8 474.00	7 377.28	268.93	330.89
9	36	1 048 576	-	-	1 727.19	2 880.20

Table: Solution times (s) for 10 randomly generated instances.

[1] **Computational setting:** Intel Xeon E3-1230 @ 3.40 GHz with 32 GB RAM; coded in Julia 1.1.0 (JuMP 0.18.6); solved with Gurobi 8.1.0.

Examples

Pig farm problem

The original problem introducing LIMID as not soluble.

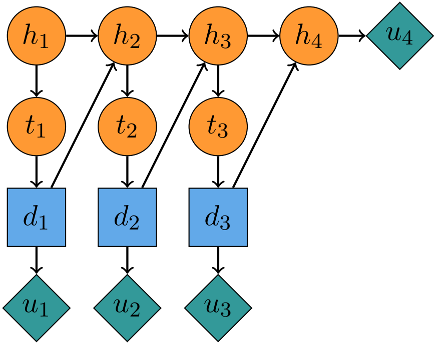

Figure: The pig farm problem with 4 periods (Lauritzen and Nilsson, 2001).

- Each month pigs are tested for a disease
- Decide whether to inject curative/preventive drug.
- Sick pigs worth less at the end.
- No record is kept for individual pigs.

Computational experiments[2]

Pig farm problem

Obtaining optimal solutions is fairly easy.

# Months	Optimal value (DKK)	Solution time (s)
3	764	0.01
4	727	0.04
5	703	0.62
6	686	19.52
7	674	617.21

Table: Results for the pig farm problem for different numbers of periods.

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We extend the example incorporating risk aversion (CVaR) and calculating all non-dominated strategies, originally not possible.

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Extra: Pig farm problem with risk considerations

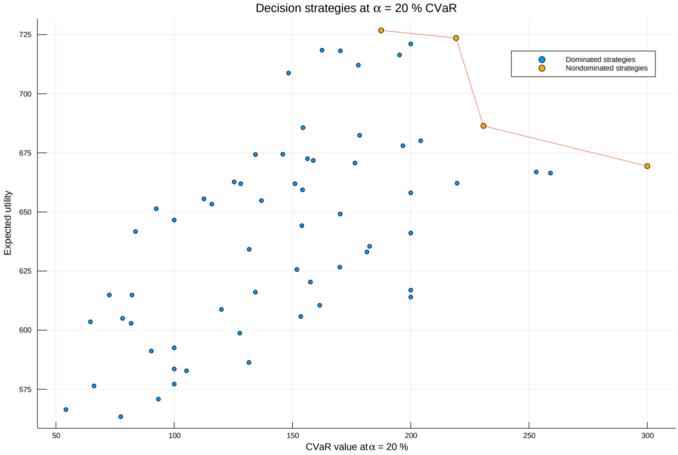

Figure: Expected utilities and conditional expectations in the lower $\alpha = 0.20$ tail for all 64 strategies of the 4-month pig problem.

Outline of this talk

Key points and takeaways

Decision Programming =

Decision Analysis + Mathematical Programming

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Decision Analysis + Mathematical Programming

- Decision Programming exploits linearity instead of recursion to solve decision diagrams.
- Pre-calculating the path probabilities $p(s)$ and utilities $\mathcal{U}$ can be done efficiently (in parallel).
- Mathematical programming as underpinning framework allows for flexibility in terms of imposing constraints.
- "Future" work: modelling endogenously uncertain problems and solution methods (preprocessing and heuristics).

To learn more:

Main reference: Salo et al. (2022), Decision programming for multi-stage optimization under uncertainty, EJOR, 299 (2), 550-565. DOI: 10.1016/j.ejor.2021.12.013

Julia package with many other examples:
github.com/gamma-opt/DecisionProgramming.jl

Some newer WiP:

Andelmin, Juho, et al. "DecisionProgramming.jl - A framework for modelling decision problems using mathematical programming." arXiv preprint arXiv:2307.13299 (2023).

Herrala, Olli, Tommi Ekholm, and Fabricio Oliveira. "A decomposition strategy for decision problems with endogenous uncertainty using mixed-integer programming." arXiv preprint arXiv:2304.02338 (2023).

Outline of this talk

More recent developments

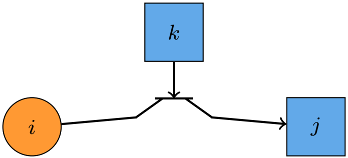
1. Modelling long-term endogenous climate uncertainty

- Consider decision-dependent (endogenous) uncertainties
- Take into account continuous decision spaces

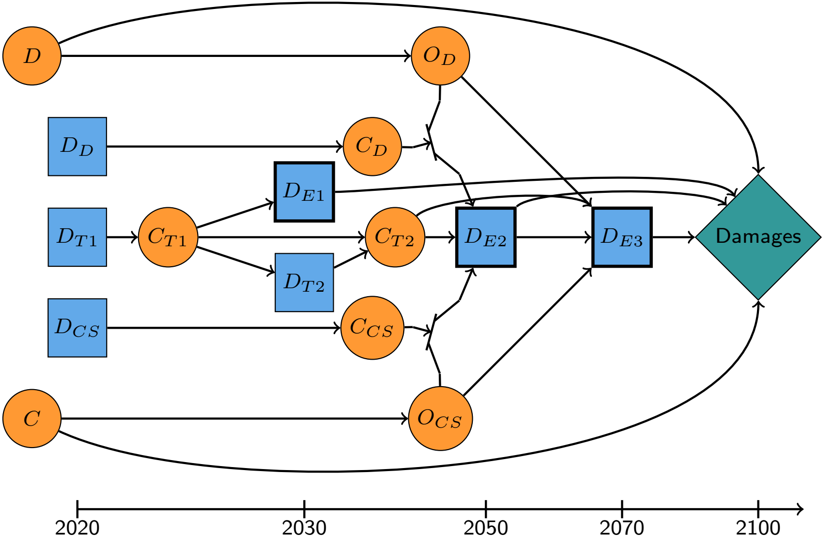
We develop the notions of extended value node

$$U_v(s_{I(v)}) := \max_y\{f_{s_{I(v)}}(y) \mid y \in Y_{s_{I(v)}}\}, \text{ for } v \in V,$$

and conditional arcs



Climate change mitigation



More recent developments

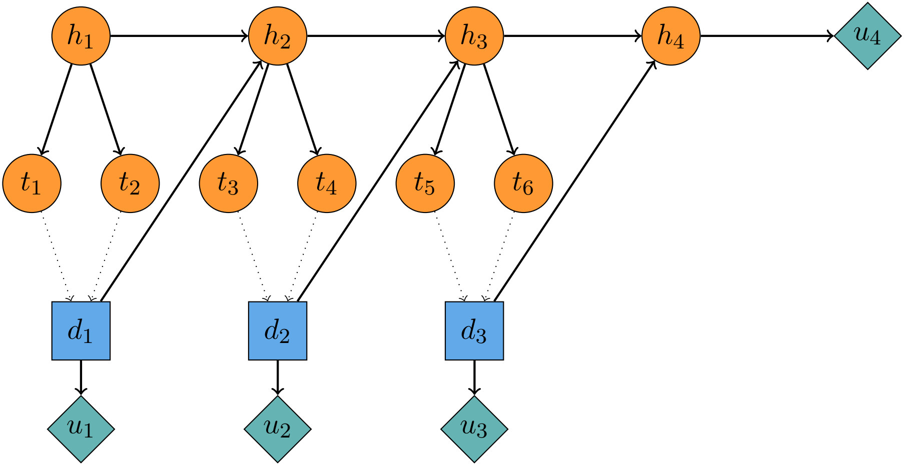
2. Optimal information structures

- We are interested in knowing what information to acquire and when
- Find optimal information structure and decision strategy

We propose three alternative formulations:

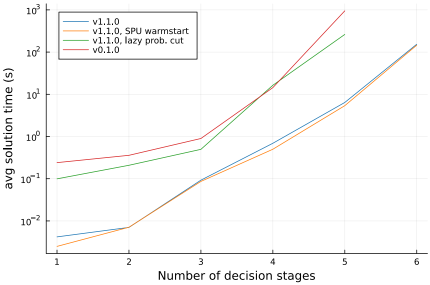
1. Constraints on path probabilities
2. Constraints on local decisions
3. Extended state space

The extended pig farm problem

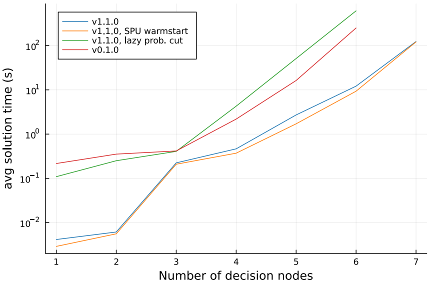


Better formulations

New formulations: stronger formulations in which we can replace indicator variables $\pi(s)$ with continuous variables.



(a) The pig farm problem



(b) The N-monitoring problem

Figure: The solution times of the two example problems with different number of decision nodes using different formulations. Notice the logarithmic y-axis.

Decision Programming

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