

Optimisation under Uncertainty

Session 1/4

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Outline of this lecture

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The farmer's problem [Birge and Louveau, 2011]

A farmer has available 500 acres of land available for raising wheat, corn, and sugar beets.

- ▶ She needs at least 200 tons of wheat and 240 tons of corn for cattle feed;
- ▶ Cattle feed amounts can be raised on the farm or bought from a wholesale market;
- ▶ Sugar beet is raised for profit only. However, production above a 6000 ton quota has a lower sales price.

	Wheat	Corn	Sugar beets
Yield (ton/acre)	2.5	3	20
Planting cost (\$/acre)	150	230	260
Selling price (\$/ton)	170	150	36 (under 6000 ton) 10 (above 6000 ton)
Purchase price (\$/ton)	238	210	-

Table: Farmer's problem data

The farmer's problem [Birge and Louveau, 2011]

Let $I = \{1 : \text{wheat}, 2 : \text{corn}, 3 : \text{sugar beets}\}$. Then

- ▶ x_i - acres devoted to i ;
- ▶ y_i - tons of i purchased, $i \in I \setminus \{3\}$;
- ▶ w_i - tons of i sold, $i \in I \cup \{4\}$, $\{4 : \text{sugar beets (over quota)}\}$.

The farmer's problem is:

$$\begin{aligned} \min. \quad & 150x_1 + 230x_2 + 260x_3 + 238y_1 + 210y_2 \\ & - 170w_1 - 150w_2 - 36w_3 - 10w_4 \end{aligned}$$

$$\text{s.t.: } x_1 + x_2 + x_3 \leq 500$$

$$2.5x_1 + y_1 - w_1 \geq 200$$

$$3x_2 + y_2 - w_2 \geq 240$$

$$w_3 + w_4 \leq 20x_3$$

$$w_3 \leq 6000$$

$$x_i \geq 0, i \in I; y_i \geq 0, i \in I \setminus \{3\}; w_i \geq 0, i \in I \cup \{4\}.$$

The farmer's problem [Birge and Louveau, 2011]

The optimal solution is given by:

Optimal strategy:

1. plant sugar beets to reach quota,
2. satisfy minimum requirements
3. if in excess, sell the excess as wheat; if in shortage, purchase corn.

	Wheat	Corn	Sugar beets
Surface	120	80	300
Yield	300	240	6000
Sales	100	-	6000
Purchase	-	-	-
Overall profit:			\$118,600

Table: Optimal solution: average yields

- ▶ The farmer is well aware that climate factors influence crop yields, which can fluctuate $\pm 20\%$.
- ▶ How can we take these scenarios into account for making land allocation decisions?

The farmer's problem [Birge and Louveaux, 2011]

Considering the scenarios individually, we obtain:

	Wheat	Corn	Sugar beets
Surface	183.33	66.67	250
Yield	550	240	6000
Sales	350	-	6000
Purchase	-	-	-
Overall profit:	\$167,667		

Table: Optimal solution: 20% higher yields

	Wheat	Corn	Sugar beets
Surface	100	25	375
Yield	200	60	6000
Sales	-	-	6000
Purchase	-	180	-
Overall profit:	\$59,950		

Table: Optimal solution: 20% lower yields

The farmer's problem [Birge and Louveaux, 2011]

As one may notice, the land allocation for sugar beets is the **critical** factor:

- ▶ Planting too much \Rightarrow losses for selling above quota
- ▶ Planting too little \Rightarrow opportunity losses

We can **hedge** against this uncertainty by taking a **long-term perspective**:

- ▶ We assume that each year one of these scenarios happens.
- ▶ We know they are **equally likely** to happen, but exactly which will happen cannot be known.
- ▶ Thus, maximise **long-run** profit \Rightarrow maximise **expected profit**.

The farmer's problem [Birge and Louveau, 2011]

Let $S = \{1 : -20\%, 2 : \text{avg.}, 3 : +20\%\}$ represent the **yield scenarios**.

The reformulated farmer's problem is:

$$\text{min. } 150x_1 + 230x_2 + 260x_3 +$$

$$\frac{1}{3}(238y_{11} + 210y_{21} - 170w_{11} - 150w_{21} - 36w_{31} - 10w_{41})$$

$$\frac{1}{3}(238y_{12} + 210y_{22} - 170w_{12} - 150w_{22} - 36w_{32} - 10w_{42})$$

$$\frac{1}{3}(238y_{13} + 210y_{23} - 170w_{13} - 150w_{23} - 36w_{33} - 10w_{43})$$

$$\text{s.t.: } x_1 + x_2 + x_3 \leq 500$$

$$2x_1 + y_{11} - w_{11} \geq 200, \quad 2.5x_1 + y_{12} - w_{12} \geq 200, \quad 3x_1 + y_{13} - w_{13} \geq 200$$

$$2.4x_2 + y_{21} - w_{21} \geq 240, \quad 3x_2 + y_{22} - w_{22} \geq 240, \quad 3.6x_2 + y_{23} - w_{23} \geq 240$$

$$w_{31} + w_{41} \leq 16x_3, \quad w_{32} + w_{42} \leq 20x_3, \quad w_{33} + w_{43} \leq 24x_3$$

$$w_{31} \leq 6000, w_{32} \leq 6000, w_{33} \leq 6000$$

$$x_i \geq 0, i \in I; y_{is} \geq 0, i \in I \setminus \{3\}, s \in S; w_{is} \geq 0, i \in I \cup \{4\}, s \in S.$$

The farmer's problem [Birge and Louveaux, 2011]

The optimal solution becomes:

		Wheat	Corn	Sugar beets
$s = 1$	Surface	170	80	250
	Yield	340	192	4000
	Sales	140	-	4000
	Purchase	-	48	-
$s = 2$	Yield	422	240	5000
	Sales	225	-	5000
	Purchase	-	-	-
$s = 3$	Yield	510	288	6000
	Sales	310	48	6000
	Purchase	-	-	-
Overall profit:				\$108,390

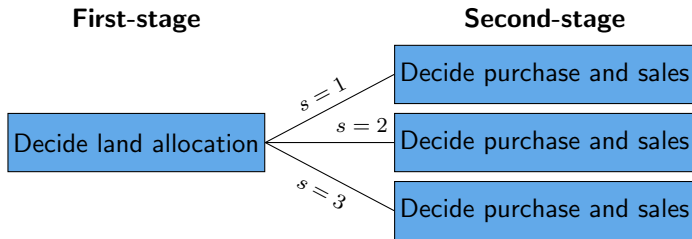
Table: Optimal solution: all scenario yields

The farmer's problem [Birge and Louveau, 2011]

By doing so, the farmer takes into account **all scenarios simultaneously**.

1. The farmer exploits the **timing** between making decisions and observing uncertainties;
2. This “hedging” comes with a “price” that can be **estimated** against perfect information performance.

Effectively, this is achieved by incorporating **decision stages** into the model:



Two-stage stochastic programming

More formally, let:

- ▶ x - definite (first-stage) decisions;
- ▶ y - corrective or recourse (second-stage) decisions;
- ▶ ξ - random variable;
- ▶ $[q(\xi), T(\xi), W(\xi), h(\xi)]$ - random vector (data);

The formulation of a two-stage stochastic programming (2SSP) model is

$$\min. c^\top x + \mathcal{Q}(x) \quad (1a)$$

$$\text{s.t.: } Ax = b \quad (1b)$$

$$x \geq 0, \quad (1c)$$

where $\mathcal{Q}(x) = \mathbb{E}_\xi [Q(x, \xi)]$ and

$$Q(x, \xi) = \left\{ \min. q(\xi)^\top y : W(\xi)y = h(\xi) - T(\xi)x, y \geq 0 \right\}. \quad (2)$$

Two-stage stochastic programming

In essence, we are assuming the following **decision process**

$$x \rightarrow \xi \rightarrow y(\xi, x)$$

in which y is chosen to minimise $\mathbb{E}_\xi [Q(x, \xi)]$, assuming a **known probability distribution** for ξ with support Ξ .

Thus, we can pose problem (1) as the **semi-infinite** problem

$$\min. \quad c^\top x + \mathbb{E}_\xi [Q(x, \xi)] \tag{3a}$$

$$\text{s.t.: } Ax = b, x \geq 0 \tag{3b}$$

$$T(\xi)x + W(\xi)y(\xi) = h(\xi), \quad \forall \xi \in \Xi \tag{3c}$$

$$y(\xi) \geq 0, \quad \forall \xi \in \Xi. \tag{3d}$$

Two-stage stochastic programming

There are two complicating factors in (3):

1. $\mathbb{E}_{\xi} [Q(x, \xi)];$
2. $\forall \xi \in \Xi.$

Those are treated by means of **discretisation**, that is:

- ▶ In general, we assume Ξ to be a **discrete** and **finite** set;
- ▶ Each realisations $\xi_s \in \Xi$, for $s \in S \equiv \{1, \dots, |\Xi|\}$ is a **scenario**;
- ▶ Thus, $[q(\xi), T(\xi), W(\xi), h(\xi)]$ becomes a discrete and finite set of parameters:

$$\begin{aligned} & [q(\xi_1), T(\xi_1), W(\xi_1), h(\xi_1); \\ & q(\xi_2), T(\xi_2), W(\xi_2), h(\xi_2); \\ & \dots; \\ & q(\xi_{|\Xi|}), T(\xi_{|\Xi|}), W(\xi_{|\Xi|}), h(\xi_{|\Xi|})] \\ \Rightarrow & [q_s, T_s, W_s, h_s] = \xi_s, s \in S. \end{aligned}$$

Two-stage stochastic programming

Considering a finite and discrete set of scenarios, we can restate (3) as its **deterministic equivalent**

$$\min. c^\top x + \sum_{s \in S} P_s q_s^\top y_s \quad (4a)$$

$$\text{s.t.: } Ax = b, x \geq 0 \quad (4b)$$

$$T_s x + W_s y_s = h_s, \forall s \in S \quad (4c)$$

$$y_s \geq 0, \forall s \in S. \quad (4d)$$

Remark: notice how discretisation solves the tractability issues:

1. P_s is the probability associated with scenario s ($P_s = P(\xi = \xi_s)$).
Thus $\mathbb{E}_\xi [Q(x, \xi)] = \sum_{s \in S} P_s q_s^\top y_s$.
2. (4) has a **finite** number of variables and constraints.

Two-stage stochastic programming

In the farmer's problem, there we assumed that $s \in \{1, 2, 3\}$:

- ▶ $q_s = q_{s'}$, $W_s = W_{s'}$, and $h_s = h_{s'}$, $\forall s, s' \in S \mid s \neq s'$;
- ▶ $T_s = [t_1(s), t_2(s)]$.

And thus, we had that:

$$\begin{aligned} Q_s(x) = \min. \quad & 238y_1(s) - 170w_1(s) + 210y_2(s) - 150w_2(s) \\ & - 36w_3(s) - 10w_4(s) \\ \text{s.t.:} \quad & t_1(s)x_1 + y_1(s) - w_1(s) \geq 200 \\ & t_2(s)x_2 + y_2(s) - w_2(s) \geq 240 \\ & w_3(s) + w_4(s) \leq t_3(s)x_3 \\ & w_3(s) \leq 6000 \\ & y_1(s), w_1(s), y_2(s), w_2(s), w_3(s), w_4(s) \geq 0. \end{aligned}$$

The expected value of perfect information (EVPI)

The performance of the solution of a 2SSP can be compared against a so-called **wait-and-see** (WS) solution.

Let our 2SSP be compactly represented as

$$z = \min_x \{ \mathbb{E} [F(x, \xi)] \},$$

where $F(x, \xi) = \{c^\top x + \mathbb{E} [Q(x, \xi)] : x \in X\}$, $Q(x, \xi)$ is defined as (2), and $X = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$.

A WS solution can be obtained from a **perfect-foresight** version

$$z^{\text{WS}} = \mathbb{E}_\xi \left[\min_x \{F(x, \xi)\} \right] = \mathbb{E}_\xi [F(x(\xi), \xi)],$$

where $x(\xi) = \operatorname{argmin}_x \{F(x, \xi)\}$.

Then, the expected value of perfect information (EVPI) is:

$$EVPI = z - z^{\text{WS}}.$$

Value of stochastic solution (VSS)

We can also compare the solution of a 2SSP against a **reference (first-stage) solution**.

For that, let $\bar{\xi}$ be a reference scenario (realisation). Then

$$x(\bar{\xi}) = \underset{x}{\operatorname{argmin}} F(x, \bar{\xi})$$

represents the optimal solution associated with that scenario.

We can then calculate the performance of $x(\bar{\xi})$ against $\xi \in \Xi$:

$$z^{\text{EV}} = \mathbb{E}_{\xi} [F(x(\bar{\xi}), \xi)] .$$

If $\bar{\xi} = \mathbb{E} [\xi]$, we have the **value of the stochastic solution (VSS)**:

$$VSS = z^{\text{EV}} - z.$$

EVPI and VSS - general remarks

Some relevant observations:

- ▶ With minimisation as a reference, we have that

$$z^{\text{WS}} \leq z \leq z^{\text{EV}} \Rightarrow EVPI \geq 0, VSS \geq 0.$$

- ▶ Stronger statements are possible if assumptions on the 2SSP problem structure or the uncertainty terms are made.
- ▶ VSS: the **higher** the better;
- ▶ EVPI: the **lower** the better.

For the farmer's example:

- ▶ $EVPI = -108,390 - \left(\frac{1}{3} \times -167,777 + \frac{1}{3} \times -118,000 + \frac{1}{3} \times -59,950\right) = \7016
- ▶ $z^{\text{EV}} = -107,240^1$; $VSS = -107,240 - (-108,390) = \1150

¹Calculated by fixing the solution $x = (120, 80, 300)$ for each scenario and taking the average of the objective function values

Types of recourse problems

It is common to classify 2SSP according to their **recourse problem**

$$Q(x, \xi) = \left\{ \min. q(\xi)^\top y : W(\xi)y = h(\xi) - T(\xi)x, y \geq 0 \right\}.$$

Most common structures:

1. **Fixed recourse:** means that $W(\xi) = W, \forall \xi \in \Xi$.
2. **Simple recourse:** In that case, $W = I$, reducing the recourse feasibility condition to $y = h(\xi) - T(\xi)x$.

This implies that the recourse becomes

$$Q(x, \xi) = h(\xi) - T(\xi)x.$$

Types of recourse problems

3. **Complete recourse:** relates to the **feasibility** of the recourse problem. If the 2SSP has complete recourse, then

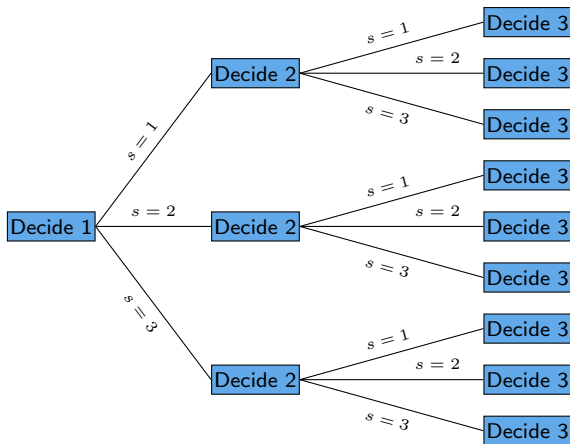
$$Q(x, \xi) < \infty, \forall \xi \in \Xi \iff \\ \{y : W(\xi)y = h(\xi) - T(\xi)x, y \geq 0\} \neq \emptyset, \forall \xi \in \Xi$$

4. **Relatively complete recourse:** in this case, the feasibility of the recourse problem is **conditioned** on $x \in X$

$$Q(x, \xi) < \infty, \forall \xi \in \Xi, x \in X \iff \\ \{y : W(\xi)y = h(\xi) - T(\xi)x, y \geq 0\} \neq \emptyset, \forall \xi \in \Xi, x \in X.$$

Beyond two decision stages

Many problems have in fact multiple decision points, in which decisions are made **sequentially**.



Beyond two decision stages

Multi-stage decision problems:

1. Consist of **nested two-stage problems**. This can be exploited in a **dynamic programming** fashion;
2. Likewise, presents **exponential growth** with scenarios per stage ($|\Xi|^{|\mathcal{H}|-1}$, with $|\Xi|$ -scenario stages $t \in [H]^2$);
3. Trade-off: future **flexibility** versus computational cost.

Consider that we have H decision stages. Our decision process becomes:

$$\begin{aligned} x^1 \rightarrow \xi^2 \rightarrow x^2(\xi^2, x^1) \rightarrow \xi^3 \rightarrow x^3((\xi^2, \xi^3), (x^1, x^2)) \rightarrow \dots \\ \rightarrow \xi^H \rightarrow x^H((\xi^2, \dots, \xi^H), (x^1, \dots, x^{H-1})) \end{aligned}$$

- ▶ ξ up to stage $t = 2, \dots, T$ represents a **sequence** of events
- ▶ Hereinafter, $x^t(\xi)$ is a shorthand for $x^t((\xi^2, \dots, \xi^t))$

$$^2n \in [N] = n \in \{1, \dots, N\}.$$

Multi-stage decision problems

For $t = H$ we have:

$$\begin{aligned} Q^H(x^{H-1}, \xi^H) = \min. & \ c^H(\xi)^\top x^H(\xi) \\ \text{s.t.:} & \ W^H(\xi)x^H(\xi) = h^H(\xi) - T^{H-1}(\xi)x^{H-1} \\ & \ x^H(\xi) \geq 0. \end{aligned}$$

For $t = 2, \dots, H - 1$ we have:

$$\begin{aligned} Q^t(x^{t-1}, \xi^t) = \min. & \ c^t(\xi)^\top x^t(\xi) + Q^{t+1}(x^t) \\ \text{s.t.:} & \ W^t(\xi)x^t(\xi) = h^t(\xi) - T^{t-1}(\xi)x^{t-1} \\ & \ x^t(\xi) \geq 0. \end{aligned}$$

We want to solve

$$\begin{aligned} \min. & \ c^1{}^\top x^1 + Q(x^1) \\ \text{s.t.:} & \ W^1 x^1 = h^1 \\ & \ x^1 \geq 0. \end{aligned}$$

Example: 3SSP

A 3-stage formulation is given as:

$$\min. c^1 \top x^1 + \sum_{\xi_s^2 \in S^2} P(\xi_s^2) \left[c^2(\xi_s^2) \top x^2(\xi_s^2) + \sum_{\xi_s^3 \in S^3(\xi_s^2)} P(\xi_s^3 | \xi_s^2) \left(c^3(\xi_s^3 | \xi_s^2) \top x^3(\xi_s^3 | \xi_s^2) \right) \right]$$

$$\text{s.t.: } T^1 x^1 = h^1$$

$$T(\xi_s^2) x^1 + W(\xi_s^2) x^2(\xi_s^2) = h(\xi_s^2), \forall \xi_s^2$$

$$T(\xi_s^3 | \xi_s^2) x^2(\xi_s^2) + W(\xi_s^3 | \xi_s^2) x^3(\xi_s^3 | \xi_s^2) = h(\xi_s^3 | \xi_s^2), \forall \xi_s^2, \xi_s^3 | \xi_s^2$$

$$x^1 \geq 0$$

$$x^2(\xi_s^2) \geq 0, \forall \xi_s^2$$

$$x^3(\xi_s^3 | \xi_s^2) \geq 0, \forall \xi_s^2, \xi_s^3 | \xi_s^2.$$

References



Birge, J. R. and Louveaux, F. (2011).
Introduction to stochastic programming.
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