Optimisation under Uncertainty
Session 3/4

I Workshop de Otimização sob Incerteza - UFSCar

Fabricio Oliveira

Systems Analysis Laboratory
Department of Mathematics and Systems Analysis

Aalto University
School of Science

August 17, 2023

Outline of this lecture

Introduction

Robust optimisation

Adjustable robust optimisation

Outline of this lecture

Introduction

Robust optimisation

Adjustable robust optimisation

What is robust optimisation

An alternative paradigm for taking uncertainty into account:

- Permeated by the notion of worst-case;
- Control of the degree of conservatism;
- Parallels with chance constraints and risk measures.

What is robust optimisation

An alternative paradigm for taking uncertainty into account:

- Permeated by the notion of worst-case;
- Control of the degree of conservatism;
- Parallels with chance constraints and risk measures.

In robust optimisation, feasibility is the key concern

- Can be extended to objective function performance requirements;
- May or may not be scenario-based;
- Static v. adaptable: the presence of recourse decisions;
- Exception: distributionally robust optimisation.

Robust optimisation approaches

The key notion in robust optimisation is tie to that of an uncertainty set

- The "region" U within the uncertainty support Ξ within which parameter realisation does not turn the solution infeasible;
- Tractability is closely tied to the geometry of such uncertainty sets.

Robust optimisation approaches

The key notion in robust optimisation is tie to that of an uncertainty set

- The "region" U within the uncertainty support Ξ within which parameter realisation does not turn the solution infeasible;
- Tractability is closely tied to the geometry of such uncertainty sets.

$$
\begin{aligned}
\min_x.\ & c^\top x \\
\text{s.t.:}\ & Ax \leq b \\
& x \in X
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{aligned}
\min_x.\ & c^\top x \\
\text{s.t.:}\ & A(\eta)x \leq b,\ \forall \eta \in U \subseteq \Xi \\
& x \in X
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{aligned}
\min_x.\ & c^\top x \\
\text{s.t.:}\ & \max_{\eta \in U \subseteq \Xi} A(\eta)x \leq b \\
& x \in X.
\end{aligned}
$$

Outline of this lecture

Introduction

Robust optimisation

Adjustable robust optimisation

Robust counterparts

Let $\tilde{a}_{ij} \in J_i$ be the uncertain elements in the matrix $A_{m \times n}$

- Random variables $\tilde{a}_{ij}$ with "central value" a_{ij} and "maximum deviation" $\hat{a}_{ij}$;
- symmetric and with bounded support $\tilde{a}_{ij} \in [a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}]$.

[1] Assuming, w.l.g., $a_{ij} \geq 0$, $\forall i \in [m], \forall j \in [n]$

Robust counterparts

Let $\tilde{a}_{ij} \in J_i$ be the uncertain elements in the matrix $A_{m \times n}$

- Random variables $\tilde{a}_{ij}$ with "central value" a_{ij} and "maximum deviation" $\hat{a}_{ij}$;
- symmetric and with bounded support $\tilde{a}_{ij} \in [a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}]$.

Let $\eta_{ij} = \frac{(\tilde{a}_{ij} - a_{ij})}{\hat{a}_{ij}}$. Thus $\eta_{ij} \in [-1, 1]$ and follows the same distribution as $\tilde{a}_{ij}$, but centred in zero and scaled.

Our robust counterpart[1] is the following bilevel problem:

$$\begin{aligned} \min_x . \ & c^\top x \\ \text{s.t.: } & a_{ij} x_j + \max_{\eta_i \in U_i} \left\{ \sum_{j \in J_i} \eta_{ij} \hat{a}_{ij} x_j \right\} \leq b_i, \forall i \in [m] \\ & x_j \geq 0, \ \forall j \in [n]. \end{aligned} \tag{RC}$$

[1] Assuming, w.l.g., $a_{ij} \geq 0$, $\forall i \in [m], \forall j \in [n]$

Uncertainty set geometries

Box uncertainty set [Soyster, 1973]

- Maximum protection level;
- All parameters take their worst-possible value;
- Simple, but highly conservative.

The uncertainty set is

$$U_i = \{\eta_i : ||\eta_i||_1 \leq |J_i|\} \equiv \{\eta_{ij} : |\eta_{ij}| \leq 1, \forall j \in J_i\}$$

Uncertainty set geometries

Box uncertainty set [Soyster, 1973]

- Maximum protection level;
- All parameters take their worst-possible value;
- Simple, but highly conservative.

The uncertainty set is

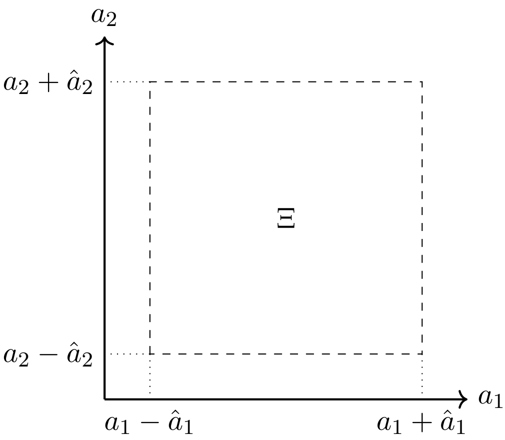
$$U_i = \{\eta_i : ||\eta_i||_1 \leq |J_i|\} \equiv \{\eta_{ij} : |\eta_{ij}| \leq 1, \forall j \in J_i\}$$

The lower-level problem becomes

$$\max_{\eta_i \in U_i} \left\{ \sum_{j \in J_i} \eta_{ij} \hat{a}_{ij} x_j : |\eta_{ij}| \leq 1, \forall j \in J_i \right\} = \sum_{j \in J_i} \hat{a}_{ij} x_j.$$

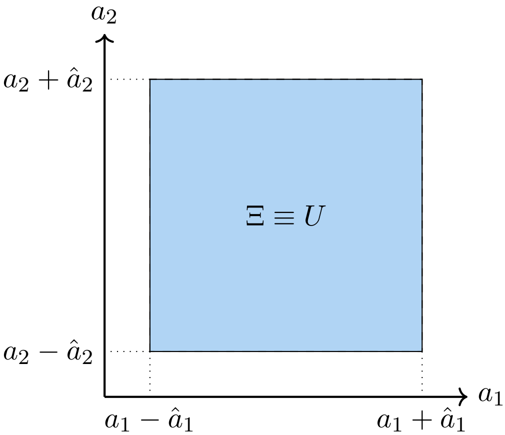
Uncertainty set geometries

Box uncertainty set [Soyster, 1973]



Uncertainty set geometries

Box uncertainty set [Soyster, 1973]



Uncertainty set geometries

Ellipsoidal uncertainty set [Ben-Tal and Nemirovski, 1999]

- Softens extreme-case protection;
- Parametrically controlled;
- Leads to smooth sets;
- (MI)SOCPs which are more computationally demanding.

The uncertainty set is

$$U_i = \{\eta_i : ||\eta_i||_2 \leq \Gamma\} \equiv \left\{\eta_{ij} : \sum_{j \in J_i} \eta_{ij}^2 \leq \Gamma^2, \forall j \in J_i\right\}$$

Uncertainty set geometries

Ellipsoidal uncertainty set [Ben-Tal and Nemirovski, 1999]

- Softens extreme-case protection;
- Parametrically controlled;
- Leads to smooth sets;
- (MI)SOCPs which are more computationally demanding.

The uncertainty set is

$$U_i = \{\eta_i : ||\eta_i||_2 \leq \Gamma\} \equiv \left\{ \eta_{ij} : \sum_{j \in J_i} \eta_{ij}^2 \leq \Gamma^2, \forall j \in J_i \right\}$$

The lower-level problem becomes

$$\max_{\eta_i \in U_i} \left\{ \sum_{j \in J_i} \eta_{ij} \hat{a}_{ij} x_j : \sum_{j \in J_i} \eta_{ij}^2 \leq \Gamma^2, \forall j \in J_i \right\}.$$

Uncertainty set geometries

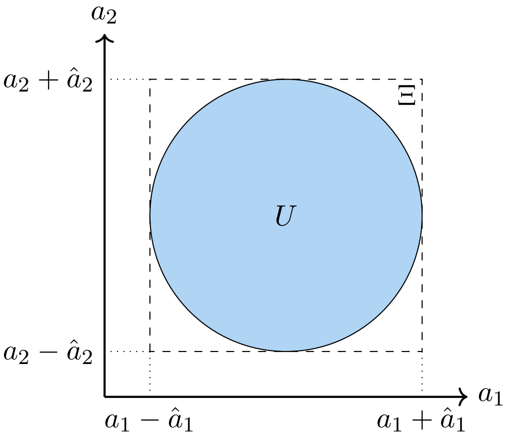
Ellipsoidal uncertainty set [Ben-Tal and Nemirovski, 1999]

Again, this uncertainty set has a closed-form solution:

$$
\begin{aligned}
&\max_{\eta_i \in U_i} \left\{ \sum_{j \in J_i} \eta_{ij} \hat{a}_{ij} x_j : \sum_{j \in J_i} \eta_{ij}^2 \leq \Gamma^2, \forall j \in J_i \right\} \\
=&\max_{\eta_i \in U_i} \left\{ \sqrt{\left(\sum_{j \in J_i} \eta_{ij} \hat{a}_{ij} x_j \right)^2} : \sum_{j \in J_i} \eta_{ij}^2 \leq \Gamma^2, \forall j \in J_i \right\} \\
=&\max_{\eta_i \in U_i} \left\{ \sqrt{\left(\sum_{j \in J_i} \eta_{ij} \right)^2 \left(\sum_{j \in J_i} \hat{a}_{ij} x_j \right)^2} : \sum_{j \in J_i} \eta_{ij}^2 \leq \Gamma^2, \forall j \in J_i \right\} \\
=&\Gamma \sqrt{\sum_{j \in J_i} \hat{a}_{ij}^2 x_j^2}
\end{aligned}
$$

Uncertainty set geometries

Ellipsoidal uncertainty set [Ben-Tal and Nemirovski, 1999]



Uncertainty set geometries

Polyhedral uncertainty set [Bertsimas and Sim, 2004]

- Allows for controlling conservatism;
- Retains problem complexity;
- Budget of uncertainty lacks interpretability.

The uncertainty set is

$$U_i = \{\eta_i : ||\eta_i||_1 \leq \Gamma_i\} \equiv \left\{\eta_{ij} : \sum_{j \in J_i} \eta_{ij} \leq \Gamma_i, \forall j \in J_i\right\}.$$

Uncertainty set geometries

Polyhedral uncertainty set [Bertsimas and Sim, 2004]

- Allows for controlling conservatism;
- Retains problem complexity;
- Budget of uncertainty lacks interpretability.

The uncertainty set is

$$U_i = \{\eta_i : ||\eta_i||_1 \leq \Gamma_i\} \equiv \left\{\eta_{ij} : \sum_{j \in J_i} \eta_{ij} \leq \Gamma_i, \forall j \in J_i\right\}.$$

The lower-level problem becomes

$$\max_{\eta_i \in U_i} \left\{\sum_{j \in J_i} \eta_{ij} \hat{a}_{ij} x_j : \sum_{j \in J_i} \eta_{ij} \leq \Gamma_i,\ 0 \leq \eta_{ij} \leq 1,\ \forall j \in J_i\right\}.$$

Uncertainty set geometries

Polyhedral uncertainty set [Bertsimas and Sim, 2004]

In this case, the lower-level problem does not admit a closed form. However, it is a linear program.

- Strong duality (primal-dual equivalence) is available;
- True for any convex[2] lower-level problem.

[2] Satisfying some constraint qualification.

Uncertainty set geometries

Polyhedral uncertainty set [Bertsimas and Sim, 2004]

In this case, the lower-level problem does not admit a closed form. However, it is a linear program.

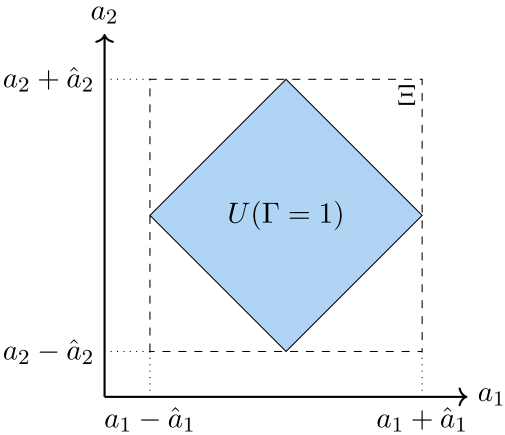
- Strong duality (primal-dual equivalence) is available;
- True for any convex[2] lower-level problem.

$$
\begin{aligned}
\max_{\eta_i \in U_i} \ & \sum_{j \in J_i} \eta_{ij} \hat{a}_{ij} x_j \\
\text{s.t.: } & \sum_{j \in J_i} \eta_{ij} \leq \Gamma_i \ (\pi_i) \\
& 0 \leq \eta_{ij} \leq 1, \ (p_{ij}) \ \forall j \in J_i
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
\min_{\pi_i, p_i} \ & \Gamma_i \pi_i + \sum_{j \in J_i} p_{ij} \\
\text{s.t.: } & \pi_i + p_{ij} \geq \hat{a}_{ij} x_j, \forall j \in J_i \\
& p_{ij} \geq 0, \ \forall j \in J_i \\
& \pi_i \geq 0.
\end{aligned}
$$

[2] Satisfying some constraint qualification.

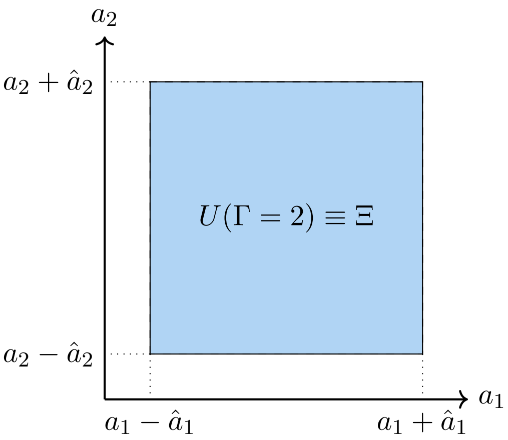
Uncertainty set geometries

Polyhedral uncertainty set [Bertsimas and Sim, 2004]



Uncertainty set geometries

Polyhedral uncertainty set [Bertsimas and Sim, 2004]



Robust counterparts

Let us consider a knapsack problem of the form

$$\begin{aligned}
\text{min. } & c^\top x \\
\text{s.t.: } & \sum_{j \in [n]} a_j x_j \leq b \\
& 0 \leq x_j \leq 0, \ \forall j \in [n].
\end{aligned}$$

Robust counterparts

Let us consider a knapsack problem of the form

$$\begin{aligned} \text{min. } & c^\top x \\ \text{s.t.: } & \sum_{j \in [n]} a_j x_j \leq b \\ & 0 \leq x_j \leq 0, \ \forall j \in [n]. \end{aligned}$$

The robust counterparts for the previous uncertainty sets are:

Box

$$\begin{aligned} \text{min. } & c^\top x \\ \text{s.t.: } & \sum_{j \in [n]} (a_j + \hat{a}_{ij}) x_j \leq b \\ & 0 \leq x_j \leq 1, \ \forall j \in [n]. \end{aligned}$$

Ellipsoid

$$\begin{aligned} \text{min. } & c^\top x \\ \text{s.t.: } & \sum_{j \in [n]} a_j x_j + \Gamma \sqrt{\sum_{j \in [n]} (\hat{a} x_j)^2} \leq b \\ & 0 \leq x_j \leq 1, \ \forall j \in [n]. \end{aligned}$$

Robust counterparts

Let us consider a knapsack problem of the form

$$\begin{aligned} \text{min. } & c^\top x \\ \text{s.t.: } & \sum_{j \in [n]} a_j x_j \leq b \\ & 0 \leq x_j \leq 1, \ \forall j \in [n]. \end{aligned}$$

The robust counterparts for the previous uncertainty sets are:

Polyhedral

$$\begin{aligned} \text{min. } & c^\top x \\ \text{s.t.: } & \sum_{j \in [n]} a_j x_j + \Gamma \pi + \sum_{j \in [n]} p_j \leq b \\ & \pi + p_j \geq \hat{a}_j x_j, \ \forall j \in [n] \\ & 0 \leq x_j \leq 1, p_j \geq 0, \ \forall j \in [n] \\ & \pi \geq 0. \end{aligned}$$

On constraint violation probabilities

Arguably, Bertsimas & Sim (2004) raised attention to robust optimisation with "the price of robustness".

- The price refers to the optimality traded off for feasibility guarantees;
- Quantifying these trade-offs can be done:
  1. Using theoretical bounds;
  2. Via simulating solution performance.
- In my own experience, theoretical bounds are often loose.

On constraint violation probabilities

Arguably, Bertsimas & Sim (2004) raised attention to robust optimisation with "the price of robustness".

- The price refers to the optimality traded off for feasibility guarantees;
- Quantifying these trade-offs can be done:
  1. Using theoretical bounds;
  2. Via simulating solution performance.
- In my own experience, theoretical bounds are often loose.

For example, [Bertsimas and Sim, 2004] show the probability of violation of constraint $i \in [m]$ to be

$$P^{\text{vio}} = P\left(a_{ij}x_j + \max_{\eta_i \in U_i}\left\{\sum_{j \in J_i} \eta_{ij}\hat{a}_{ij}x_j\right\} > b_i\right) \leq e^{\frac{-\Gamma^2}{2|J_i|}},$$

Outline of this lecture

Introduction

Robust optimisation

Adjustable robust optimisation

Multi-stage robust optimisation

We focus on 2-stage adjustable robust optimisation (ARO) problems:

$$
\begin{aligned}
\min\ & c^\top x + \max_{\xi \in U \subset \Xi} \min_y q^\top y(\xi) \\
\text{s.t.:}\ & Ax = b,\ x \geq 0 \\
& T(\xi)x + Wy(\xi) = h(\xi),\ \forall \xi \in U \subset \Xi \\
& y(\xi) \geq 0,\ \forall \xi \in \Xi.
\end{aligned} \tag{ARO}
$$

Multi-stage robust optimisation

We focus on 2-stage adjustable robust optimisation (ARO) problems:

$$\begin{aligned}
\min \ & c^\top x + \max_{\xi \in U \subset \Xi} \min_{y} q^\top y(\xi) \\
\text{s.t.: } & Ax = b, \ x \geq 0 \\
& T(\xi)x + Wy(\xi) = h(\xi), \ \forall \xi \in U \subset \Xi \\
& y(\xi) \geq 0, \ \forall \xi \in \Xi.
\end{aligned} \tag{ARO}$$

- Only RHS uncertainty: $T(\xi) = T$, $W(\xi) = W$, and $q(\xi) = q \ \forall \xi \in \Xi$;
- Assumption often necessary to eliminate quadratic dependence between ξ and decision variables;
- Not necessary if the uncertainty set is discrete and finite (scenarios)

A side note: min-max, minimum regret and related

If the uncertainty set is a finite and discrete set of scenarios, we have that

$$\begin{aligned} \min_{x} \ & c^\top x + \max_{s \in U} \ \min_{y} \ q_s^\top y_s \\ \text{s.t.:} \ & T_s x + W_s y_s \leq h_s, \ \forall s \in U \\ & x \in X \end{aligned}$$

is a tractable ARO [Mulvey et al., 1995].

A side note: min-max, minimum regret and related

If the uncertainty set is a finite and discrete set of scenarios, we have that

$$\begin{aligned}
\min_x .\ & c^\top x + \max_{s \in U} .\ \min_y .\ q_s^\top y_s \\
\text{s.t.:}\ & T_s x + W_s y_s \leq h_s,\ \forall s \in U \\
& x \in X
\end{aligned}$$

is a tractable ARO [Mulvey et al., 1995]. Variants include:

Min-max

$$\begin{aligned}
\min_x .\ & c^\top x + \theta \\
& \theta \geq q_s^\top y_s,\ \forall s \in U \\
\text{s.t.:}\ & T_s x + W_s y_s \leq h_s,\ \forall s \in U \\
& x \in X
\end{aligned}$$

Min-regret

$$\begin{aligned}
\min_x .\ & c^\top x + \theta \\
& \theta \geq q_s^\top y_s - q_s^\top y_s^\star,\ \forall s \in U \\
\text{s.t.:}\ & T_s x + W_s y_s \leq h_s,\ \forall s \in U \\
& x \in X
\end{aligned}$$

where $y_s^\star$ is optimal for $s \in U$.

Affinely adjustable robust optimisation [Ben-Tal et al., 2004]

One idea for modelling adjustability is using affine policies:

- Replace $y(\xi)$ with $\alpha + \beta\xi$;
- $h(\xi)$ is assumed affinely dependent on ξ, e.g.: $h(\xi) = h + \hat{h}\xi$.

Affinely adjustable robust optimisation [Ben-Tal et al., 2004]

One idea for modelling adjustability is using affine policies:

- Replace $y(\xi)$ with $\alpha + \beta\xi$;
- $h(\xi)$ is assumed affinely dependent on ξ, e.g.: $h(\xi) = h + \hat{h}\xi$.

Then ARO becomes:

$$\begin{aligned}
\min_{x,\alpha,\beta} \quad & c^\top x + \theta \\
\text{s.t.:} \quad & \theta \geq q^\top(\alpha + \beta\xi), \ \forall \xi \in U \\
& Ax = b, \ x \geq 0 \\
& Tx + W(\alpha + \beta\xi) \leq h(\xi), \ \forall \xi \in U \\
& y(\xi) \geq 0, \ \forall \xi \in \Xi.
\end{aligned}$$

Affinely adjustable robust optimisation [Ben-Tal et al., 2004]

One idea for modelling adjustability is using affine policies:

- Replace $y(\xi)$ with $\alpha + \beta\xi$;
- $h(\xi)$ is assumed affinely dependent on ξ, e.g.: $h(\xi) = h + \hat{h}\xi$.

Then ARO becomes:

$$
\begin{aligned}
\min_{x,\alpha,\beta} \quad & c^\top x + \theta \\
\text{s.t.:} \quad & \theta \geq q^\top(\alpha + \beta\xi), \ \forall \xi \in U \\
& Ax = b, \ x \geq 0 \\
& Tx + W(\alpha + \beta\xi) \leq h(\xi), \ \forall \xi \in U \\
& y(\xi) \geq 0, \ \forall \xi \in \Xi.
\end{aligned}
$$

Similar to the static case, computational tractability can be achieved:

- Requires that U is a box or ellipsoidal set;
- For a practical example, see [Ben-Tal et al., 2005]

Adjustable robust optimisation

An alternative approach: looking closer at the inner problem as a bilevel optimisation problem.

Let us restate our ARO in a simplified notation. For that, let

- $X = \{x \in \mathbb{R}^{n_1} : Ax = b, x \geq 0\}$;
- $Y = \{y \in \mathbb{R}^{n_2} : y \geq 0\}$;
- Uncertainty in RHS only, with $h(\xi) = h - \hat{h}\xi$, and $\xi \in \left[\, \underline{\xi}, \overline{\xi} \,\right]$

Adjustable robust optimisation

An alternative approach: looking closer at the inner problem as a bilevel optimisation problem.

Let us restate our ARO in a simplified notation. For that, let

- $X = \{x \in \mathbb{R}^{n_1} : Ax = b, x \geq 0\}$;
- $Y = \{y \in \mathbb{R}^{n_2} : y \geq 0\}$;
- Uncertainty in RHS only, with $h(\xi) = h - \hat{h}\xi$, and $\xi \in \left[\,\underline{\xi}, \overline{\xi}\,\right]$

Then we have that ARO is equivalent to

$$\min_{x \in X} c^\top x + \mathcal{Q}(x), \tag{ARO}$$

where

$$\mathcal{Q}(x) = \left\{ \underset{\xi \in U}{\text{max.}} \ \underset{y \in Y}{\text{min.}} \ q^\top y : Tx = (h - \hat{h}\xi) - Wy \right\}.$$

Adjustable robust optimisation & CCG

Let us assume that an oracle is available such that, for a given $\overline{x} \in X$ it evaluates $\mathcal{Q}(x)$ and returns associated $(\overline{\xi}, \overline{y})$, if they exist.

Adjustable robust optimisation & CCG

Let us assume that an oracle is available such that, for a given $\overline{x} \in X$ it evaluates $\mathcal{Q}(x)$ and returns associated $(\overline{\xi}, \overline{y})$, if they exist.

In addition, let us assume that the uncertainty set is finitely representable:

- Scenarios, but an intractable amount of them (e.g., samples, or data)
- Polyhedral set (finite extreme points and rays)

Adjustable robust optimisation & CCG

Let us assume that an oracle is available such that, for a given $\overline{x} \in X$ it evaluates $\mathcal{Q}(x)$ and returns associated $(\overline{\xi}, \overline{y})$, if they exist.

In addition, let us assume that the uncertainty set is finitely representable:

- Scenarios, but an intractable amount of them (e.g., samples, or data)
- Polyhedral set (finite extreme points and rays)

In that case, we can employ column-and-constraint generation (CCG) [Zeng and Zhao, 2013] to solve ARO:

Main problem M^k: $\overline{x}^{k+1}$

$$
\begin{aligned}
\min_{x,y,\theta} \ & c^\top x + \theta \\
& \theta \geq q^\top y_l, \ l \in [k] \\
& x \in X \\
& Tx = h - \hat{h}\overline{\xi}_l - W y_l, \ l \in [k] \\
& y_l \in Y, \ l \in [k].
\end{aligned}
$$

Oracle $\mathcal{Q}(\overline{x}^{k+1})$: $\overline{\xi}^{k+1}$

$$
\begin{aligned}
\max_{\xi \in U} \ \min_{y \in Y} \ & q^\top y \\
\text{s.t.: } & T\overline{x}^{k+1} = (h - \hat{h}\xi) - W y.
\end{aligned}
$$

Adjustable robust optimisation & CCG

In summary, the CCG method can be stated as

1. **Initialisation.** $LB = -\infty$, $UB = \infty$, $k = 0$.

Adjustable robust optimisation & CCG

In summary, the CCG method can be stated as

1. **Initialisation.** $LB = -\infty$, $UB = \infty$, $k = 0$.
2. **Solve the main problem M^k.** Let $\underline{z}^k = c^\top x^k + \theta^k$, where $\operatorname{argmin} M^k = (x^k, \theta^k, (y_l^k)_{l=1}^k)$. Make $LB = \underline{z}^k$.

Adjustable robust optimisation & CCG

In summary, the CCG method can be stated as

1. **Initialisation.** $LB = -\infty$, $UB = \infty$, $k = 0$.
2. **Solve the main problem M^k.** Let $\underline{z}^k = c^\top x^k + \theta^k$, where $\operatorname{argmin} M^k = (x^k, \theta^k, (y_l^k)_{l=1}^k)$. Make $LB = \underline{z}^k$.
3. **Solve** $\mathcal{Q}(x^k)$. Let $\operatorname{argmin} \mathcal{Q}(x^k) = (\overline{\xi}^{k+1}, \overline{y}^{k+1})$, if it exists. Let $\overline{z}^k = c^\top x^k + \mathcal{Q}(x^k)$. Make $UB = \min \left\{UB, \overline{z}^k\right\}$. **If** $UB = LB < \epsilon$, **return** x^k.

Adjustable robust optimisation & CCG

In summary, the CCG method can be stated as

1. **Initialisation.** $LB = -\infty$, $UB = \infty$, $k = 0$.
2. **Solve the main problem M^k.** Let $\underline{z}^k = c^\top x^k + \theta^k$, where $\operatorname{argmin} M^k = (x^k, \theta^k, (y_l^k)_{l=1}^k)$. Make $LB = \underline{z}^k$.
3. **Solve** $\mathcal{Q}(x^k)$. Let $\operatorname{argmin} \mathcal{Q}(x^k) = (\overline{\xi}^{k+1}, \overline{y}^{k+1})$, if it exists. Let $\overline{z}^k = c^\top x^k + \mathcal{Q}(x^k)$. Make $UB = \min\left\{UB, \overline{z}^k\right\}$. **If** $UB = LB < \epsilon$, **return** x^k.
4. **Add column and constraints to M^k.** If $\mathcal{Q}(x^k)$ is feasible, create columns y_{k+1} and, together with the constraints

$$\theta \geq q^\top y_{k+1} \tag{1}$$

$$Tx = h - \hat{h}\overline{\xi}_{k+1} - W y_{k+1}, \; y_{k+1} \in Y, \tag{2}$$

add them to M^k, forming M^{k+1}. Make $k = k + 1$ and return to Step 2. If $\mathcal{Q}(x^k)$ is not feasible, then only (2) is created.

Adjustable robust optimisation & CCG

Practical remarks

Essentially, CCG for ARO is a delayed-generation approach of the min-max formulation

- Can thus be useful when too many scenarios are available;
- Convergence relies on a finiteness argument on the uncertainty set.

Adjustable robust optimisation & CCG

Practical remarks

Essentially, CCG for ARO is a delayed-generation approach of the min-max formulation

- Can thus be useful when too many scenarios are available;
- Convergence relies on a finiteness argument on the uncertainty set.

CCG can be seen as a primal equivalent to Benders decomposition

- One can use the same column generation approach in the context of the L-shaped method [Van Slyke and Wets, 1969];
- This can help as a way to transmit "recourse information" to the main problem.

Adjustable robust optimisation & CCG

On solving $\mathcal{Q}(x)$

Recall that $\mathcal{Q}(x)$ is of the form

$$
\begin{aligned}
\mathcal{Q}(x) = \max_{u}\, & q^\top y \\
\text{s.t.: }& u \in \mathcal{U} \\
& y \in \operatorname*{argmin}_{y}\, q^\top y \\
& \qquad \text{s.t.: } Tx = h - \hat{h}\xi - Wy \\
& \qquad\qquad y \in Y.
\end{aligned}
$$

Adjustable robust optimisation & CCG

On solving $\mathcal{Q}(x)$

Recall that $\mathcal{Q}(x)$ is of the form

$$\begin{aligned}
\mathcal{Q}(x) = \max_{u}\, & q^\top y \\
\text{s.t.: } & u \in \mathcal{U} \\
& y \in \operatorname*{argmin}_{y}\, q^\top y \\
& \qquad \text{s.t.: } Tx = h - \hat{h}\xi - Wy \\
& \qquad\qquad y \in Y.
\end{aligned}$$

This is a bilevel model and can be solved using dedicated methods.

- Most techniques rely on posing optimality conditions of the lower-level problem to yield an equivalent single-level (tractable) problem;
- Thus, lower-level convexity (plus CQ) is often a requirement.

Adjustable robust optimisation & CCG

On solving $\mathcal{Q}(x)$

Example: assume that $Y = \mathbb{R}^{n_2}_+$. We can use strong duality to reformulate the lower-level problem, obtaining

$$\begin{aligned}
\mathcal{Q}(x) = &\max_{\xi,\pi} (h - \hat{h}\xi - Tx)^\top \pi \\
\text{s.t.: } &\pi^\top W \leq q^\top \\
&\xi \in U.
\end{aligned}$$

Adjustable robust optimisation & CCG

On solving $\mathcal{Q}(x)$

Example: assume that $Y = \mathbb{R}^{n_2}_+$. We can use strong duality to reformulate the lower-level problem, obtaining

$$
\begin{aligned}
\mathcal{Q}(x) = & \max_{\xi,\pi} (h - \hat{h}\xi - Tx)^\top \pi \\
\text{s.t.: } & \pi^\top W \leq q^\top \\
& \xi \in U.
\end{aligned}
$$

$\mathcal{Q}(x)$ is solvable, if:

1. ξ is integer or has a discrete domain, since $\xi^\top \pi$ can be reformulated exactly (e.g., [Rintamäki et al., 2023]);
2. if $-(\hat{h}\xi)^\top \pi + (h - Tx)^\top \pi$ is a concave bilinear function in π and ξ;
3. if applying a global solver (e.g., Gurobi's spatial branch-and-bound method) is feasible from a computational standpoint.

References I

Ben-Tal, A., Golany, B., Nemirovski, A., and Vial, J.-P. (2005).
Retailer-supplier flexible commitments contracts: A robust optimization approach.
Manufacturing & Service Operations Management, 7(3):248–271.

Ben-Tal, A., Goryashko, A., Guslitzer, E., and Nemirovski, A. (2004).
Adjustable robust solutions of uncertain linear programs.
Mathematical programming, 99(2):351–376.

Ben-Tal, A. and Nemirovski, A. (1999).
Robust solutions of uncertain linear programs.
Operations research letters, 25(1):1–13.

Bertsimas, D. and Sim, M. (2004).
The price of robustness.
Operations research, 52(1):35–53.

References II

Mulvey, J. M., Vanderbei, R. J., and Zenios, S. A. (1995).
Robust optimization of large-scale systems.
Operations research, 43(2):264–281.

Rintamäki, T., Oliveira, F., Siddiqui, A. S., and Salo, A. (2023).
Achieving emission-reduction goals: Multi-period power-system expansion under short-term operational uncertainty.
IEEE Transactions on Power Systems.

Soyster, A. L. (1973).
Convex programming with set-inclusive constraints and applications to inexact linear programming.
Operations research, 21(5):1154–1157.

References III

Van Slyke, R. M. and Wets, R. (1969).
L-shaped linear programs with applications to optimal control and stochastic programming.
SIAM journal on applied mathematics, 17(4):638–663.

Zeng, B. and Zhao, L. (2013).
Solving two-stage robust optimization problems using a column-and-constraint generation method.
Operations Research Letters, 41(5):457–461.