Optimisation under Uncertainty
Session 1/4

I Workshop de Otimização sob Incerteza - UFSCar

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Outline of this lecture

The farmer's problem [Birge and Louveaux, 2011]

A farmer has available 500 acres of land available for raising wheat, corn, and sugar beets.

- She needs at least 200 tons of wheat and 240 tons of corn for cattle feed;
- Cattle feed amounts can be raised on the farm or bought from a wholesale market;
- Sugar beet is raised for profit only. However, production above a 6000 ton quota has a lower sales price.

	Wheat	**Corn**	**Sugar beets**
Yield (ton/acre)	2.5	3	20
Planting cost ($/acre)	150	230	260
Selling price ($/ton)	170	150	36 (under 6000 ton)
			10 (above 6000 ton)
Purchase price ($/ton)	238	210	-

Table: Framer's problem data

The farmer's problem [Birge and Louveaux, 2011]

Let $I = \{1 : \text{wheat}, 2 : \text{corn}, 3 : \text{sugar beets}\}$. Then

- x_i - acres devoted to i;
- y_i - tons of i purchased, $i \in I \setminus \{3\}$;
- w_i - tons of i sold, $i \in I \cup \{4\}$, $\{4 : \text{sugar beets (over quota)}\}$.

The farmer's problem is:

$$
\begin{aligned}
\text{min. } & 150x_1 + 230x_2 + 260x_3 + 238y_1 + 210y_2 \\
& - 170w_1 - 150w_2 - 36w_3 - 10w_4 \\
\text{s.t.: } & x_1 + x_2 + x_3 \leq 500 \\
& 2.5x_1 + y_1 - w_1 \geq 200 \\
& 3x_2 + y_2 - w_2 \geq 240 \\
& w_3 + w_4 \leq 20x_3 \\
& w_3 \leq 6000 \\
& x_i \geq 0, i \in I; y_i \geq 0, i \in I \setminus \{3\}; w_i \geq 0, i \in I \cup \{4\}.
\end{aligned}
$$

The farmer's problem [Birge and Louveaux, 2011]

The optimal solution is given by:

Optimal strategy:

1. plant sugar beets to reach quota,
2. satisfy minimum requirements
3. if in excess, sell the excess as wheat; if in shortage, purchase corn.

	Wheat	**Corn**	**Sugar beets**
Surface	120	80	300
Yield	300	240	6000
Sales	100	-	6000
Purchase	-	-	-
Overall profit:			$118,600

Table: Optimal solution: average yields

- The farmer is well aware that climate factors influence crop yields, which can fluctuate $\pm$ 20%.
- How can we take these scenarios into account for making land allocation decisions?

The farmer's problem [Birge and Louveaux, 2011]

Considering the scenarios individually, we obtain:

	Wheat	**Corn**	**Sugar beets**
Surface	183.33	66.67	250
Yield	550	240	6000
Sales	350	-	6000
Purchase	-	-	-
Overall profit:			\$167,667

Table: Optimal solution: 20% higher yields

	Wheat	**Corn**	**Sugar beets**
Surface	100	25	375
Yield	200	60	6000
Sales	-	-	6000
Purchase	-	180	-
Overall profit:			\$59,950

Table: Optimal solution: 20% lower yields

The farmer's problem [Birge and Louveaux, 2011]

As one may notice, the land allocation for sugar beets is the critical factor:

- Planting too much $\Rightarrow$ losses for selling above quota
- Planting too little $\Rightarrow$ opportunity losses

We can hedge against this uncertainty by taking a long-term perspective:

- We assume that each year one of these scenarios happens.
- We know they are equally likely to happen, but exactly which will happen cannot be known.
- Thus, maximise long-run profit $\Rightarrow$ maximise expected profit.

The farmer's problem [Birge and Louveaux, 2011]

Let $S = \{1 : \text{-20\%}, 2 : \text{avg.}, 3 : +20\%\}$ represent the yield scenarios.

The reformulated farmer's problem is:

$$
\begin{aligned}
\text{min. } & 150x_1 + 230x_2 + 260x_3 + \\
& \frac{1}{3}(238y_{11} + 210y_{21} - 170w_{11} - 150w_{21} - 36w_{31} - 10w_{41}) \\
& \frac{1}{3}(238y_{12} + 210y_{22} - 170w_{12} - 150w_{22} - 36w_{32} - 10w_{42}) \\
& \frac{1}{3}(238y_{13} + 210y_{23} - 170w_{13} - 150w_{23} - 36w_{33} - 10w_{43}) \\
\text{s.t.: } & x_1 + x_2 + x_3 \leq 500 \\
& 2x_1 + y_{11} - w_{11} \geq 200,\ 2.5x_1 + y_{12} - w_{12} \geq 200,\ 3x_1 + y_{13} - w_{13} \geq 200 \\
& 2.4x_2 + y_{21} - w_{21} \geq 240,\ 3x_2 + y_{22} - w_{22} \geq 240,\ 3.6x_2 + y_{23} - w_{23} \geq 240 \\
& w_{31} + w_{41} \leq 16x_3,\ w_{32} + w_{42} \leq 20x_3,\ w_{33} + w_{43} \leq 24x_3 \\
& w_{31} \leq 6000, w_{32} \leq 6000, w_{33} \leq 6000 \\
& x_i \geq 0, i \in I; y_{is} \geq 0, i \in I \setminus \{3\}, s \in S; w_{is} \geq 0, i \in I \cup \{4\}, s \in S.
\end{aligned}
$$

The farmer's problem [Birge and Louveaux, 2011]

The optimal solution becomes:

		Wheat	Corn	Sugar beets
	Surface	170	80	250
$s = 1$	**Yield**	340	192	4000
	Sales	140	-	4000
	Purchase	-	48	-
$s = 2$	**Yield**	422	240	5000
	Sales	225	-	5000
	Purchase	-	-	-
$s = 3$	**Yield**	510	288	6000
	Sales	310	48	6000
	Purchase	-	-	-
Overall profit:				\$108,390

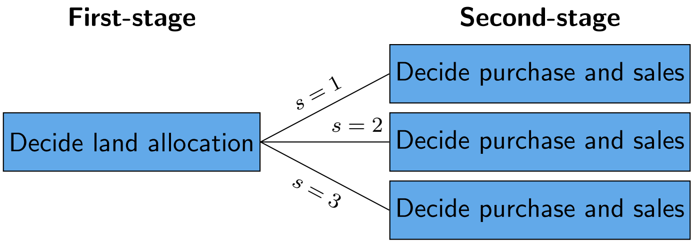
Table: Optimal solution: all scenario yields

The farmer's problem [Birge and Louveaux, 2011]

By doing so, the farmer takes into account all scenarios simultaneously.

1. The farmer exploits the timing between making decisions and observing uncertainties;
2. This "hedging" comes with a "price" that can be estimated against perfect information performance.

Effectively, this is achieved by incorporating decision stages into the model:



Two-stage stochastic programming

More formally, let:

- x - definite (first-stage) decisions;
- y - corrective or recourse (second-stage) decisions;
- ξ - random variable;
- $[q(\xi), T(\xi), W(\xi), h(\xi)]$ - random vector (data);

The formulation of a two-stage stochastic programming (2SSP) model is

$$\begin{aligned} \text{min. } & c^\top x + \mathcal{Q}(x) && \text{(1a)} \\ \text{s.t.: } & Ax = b && \text{(1b)} \\ & x \geq 0, && \text{(1c)} \end{aligned}$$

where $\mathcal{Q}(x) = \mathbb{E}_\xi \left[Q(x, \xi) \right]$ and

$$Q(x, \xi) = \left\{ \text{min. } q(\xi)^\top y : W(\xi)y = h(\xi) - T(\xi)x, y \geq 0 \right\}. \quad (2)$$

Two-stage stochastic programming

In essence, we are assuming the following decision process

$$x \to \xi \to y(\xi, x)$$

in which y is chosen to minimise $\mathbb{E}_\xi\left[Q(x,\xi)\right]$, assuming a known probability distribution for ξ with support Ξ.

Thus, we can pose problem (1) as the semi-infinite problem

$$\begin{aligned}
\text{min. } & c^\top x + \mathbb{E}_\xi\left[Q(x,\xi)\right] && \text{(3a)}\\
\text{s.t.: } & Ax = b, x \geq 0 && \text{(3b)}\\
& T(\xi)x + W(\xi)y(\xi) = h(\xi),\ \forall \xi \in \Xi && \text{(3c)}\\
& y(\xi) \geq 0,\ \forall \xi \in \Xi. && \text{(3d)}
\end{aligned}$$

Two-stage stochastic programming

There are two complicating factors in (3):

1. $\mathbb{E}_\xi\left[Q(x,\xi)\right]$;
2. $\forall \xi \in \Xi$.

Those are treated by means of discretisation, that is:

- In general, we assume Ξ to be a discrete and finite set;
- Each realisations $\xi_s \in \Xi$, for $s \in S \equiv \{1, \dots, |\Xi|\}$ is a scenario;
- Thus, $[q(\xi), T(\xi), W(\xi), h(\xi)]$ becomes a discrete and finite set of parameters:

$$\begin{aligned}
&[q(\xi_1), T(\xi_1), W(\xi_1), h(\xi_1);\\
&\ q(\xi_2), T(\xi_2), W(\xi_2), h(\xi_2);\\
&\ \dots;\\
&\ q(\xi_{|\Xi|}), T(\xi_{|\Xi|}), W(\xi_{|\Xi|}), h(\xi_{|\Xi|})]\\
\Rightarrow\ &[q_s, T_s, W_s, h_s] = \xi_s, s \in S.
\end{aligned}$$

Two-stage stochastic programming

Considering a finite and discrete set of scenarios, we can restate (3) as its deterministic equivalent

$$
\begin{aligned}
\text{min. } & c^\top x + \sum_{s \in S} P_s q_s^\top y_s && \text{(4a)} \\
\text{s.t.: } & Ax = b, x \geq 0 && \text{(4b)} \\
& T_s x + W_s y_s = h_s, \ \forall s \in S && \text{(4c)} \\
& y_s \geq 0, \ \forall s \in S. && \text{(4d)}
\end{aligned}
$$

Remark: notice how discretisation solves the tractability issues:

1. P_s is the probability associated with scenario s ($P_s = P(\xi = \xi_s)$). Thus $\mathbb{E}_\xi \left[Q(x, \xi) \right] = \sum_{s \in S} P_s q_s^\top y_s$.
2. (4) has a finite number of variables and constraints.

Two-stage stochastic programming

In the farmer's problem, there we assumed that $s \in \{1, 2, 3\}$:

- $q_s = q_{s'}$, $W_s = W_{s'}$, and $h_s = h_{s'}$, $\forall s, s' \in S \mid s \neq s'$;
- $T_s = [t_1(s), t_2(s)]$.

And thus, we had that:

$$
\begin{aligned}
Q_s(x) = \text{min.} \quad & 238y_1(s) - 170w_1(s) + 210y_2(s) - 150w_2(s) \\
& \quad - 36w_3(s) - 10w_4(s) \\
\text{s.t.:} \quad & t_1(s)x_1 + y_1(s) - w_1(s) \geq 200 \\
& t_2(s)x_2 + y_2(s) - w_2(s) \geq 240 \\
& w_3(s) + w_4(s) \leq t_3(s)x_3 \\
& w_3(s) \leq 6000 \\
& y_1(s), w_1(s), y_2(s), w_2(s), w_3(s), w_4(s) \geq 0.
\end{aligned}
$$

The expected value of perfect information (EVPI)

The performance of the solution of a 2SSP can be compared against a so-called wait-and-see (WS) solution.

Let our 2SSP be compactly represented as

$$z = \min_{x}. \ \{\mathbb{E}\left[F(x,\xi)\right]\},$$

where $F(x,\xi) = \left\{c^\top x + \mathbb{E}\left[Q(x,\xi)\right] : x \in X\right\}$, $Q(x,\xi)$ is defined as (2), and $X = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$.

A WS solution can be obtained from a perfect-foresight version

$$z^{\mathsf{WS}} = \mathbb{E}_\xi\left[\min_{x}. \ \{F(x,\xi)\}\right] = \mathbb{E}_\xi\left[F(x(\xi),\xi)\right],$$

where $x(\xi) = \operatorname{argmin}_x \{F(x,\xi)\}$.

Then, the expected value of perfect information (EVPI) is:

$$EVPI = z - z^{\mathsf{WS}}.$$

Value of stochastic solution (VSS)

We can also compare the solution of a 2SSP against a reference (first-stage) solution.

For that, let $\overline{\xi}$ be a reference scenario (realisation). Then

$$x(\overline{\xi}) = \underset{x}{\operatorname{argmin}}\, F(x, \overline{\xi})$$

represents the optimal solution associated with that scenario.

We can then calculate the performance of $x(\overline{\xi})$ against $\xi \in \Xi$:

$$z^{\mathsf{EV}} = \mathbb{E}_{\xi}\left[F(x(\overline{\xi}), \xi)\right].$$

If $\overline{\xi} = \mathbb{E}\left[\xi\right]$, we have the value of the stochastic solution (VSS):

$$VSS = z^{\mathsf{EV}} - z.$$

EVPI and VSS - general remarks

Some relevant observations:

- With minimisation as a reference, we have that

$$z^{\mathsf{WS}} \leq z \leq z^{\mathsf{EV}} \Rightarrow EVPI \geq 0, VSS \geq 0.$$

- Stronger statements are possible if assumptions on the 2SSP problem structure or the uncertainty terms are made.
- VSS: the higher the better;
- EVPI: the lower the better.

For the farmer's example:

- $EVPI = -108,390 - \left(\frac{1}{3} \times -167,777 + \frac{1}{3} \times -118,000 + \frac{1}{3} \times -59,950\right) = \$7016$
- $z^{\mathsf{EV}} = -107,240$[1]; $VSS = -107,240 - (-108,390) = \$1150$

[1] Calculated by fixing the solution $x = (120, 80, 300)$ for each scenario and taking the average of the objective function values

Types of recourse problems

It is common to classify 2SSP according to their recourse problem

$$Q(x,\xi) = \left\{ \text{min. } q(\xi)^\top y : W(\xi)y = h(\xi) - T(\xi)x, \; y \geq 0 \right\}.$$

Most common structures:

1. **Fixed recourse:** means that $W(\xi) = W$, $\forall \xi \in \Xi$.
2. **Simple recourse:** In that case, $W = I$, reducing the recourse feasibility condition to $y = h(\xi) - T(\xi)x$.

   This implies that the recourse becomes

   $$Q(x,\xi) = h(\xi) - T(\xi)x.$$

Types of recourse problems

3. **Complete recourse:** relates to the feasibility of the recourse problem. If the 2SSP has complete recourse, then

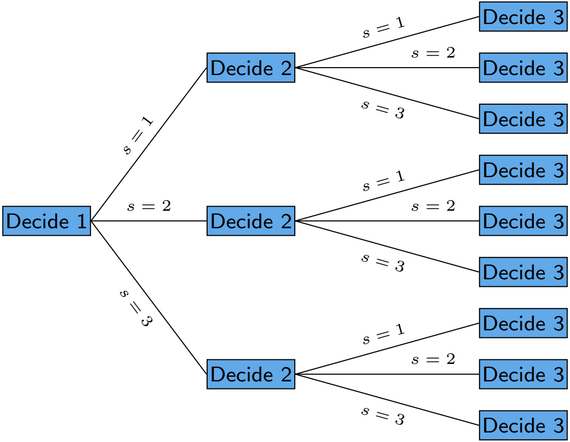
$$Q(x,\xi) < \infty,\ \forall \xi \in \Xi \iff \\ \{y : W(\xi)y = h(\xi) - T(\xi)x, y \geq 0\} \neq \emptyset,\ \forall \xi \in \Xi$$

1. **Relatively complete recourse:** in this case, the feasibility of the recourse problem is conditioned on $x \in X$

$$Q(x,\xi) < \infty,\ \forall \xi \in \Xi, x \in X \iff \\ \{y : W(\xi)y = h(\xi) - T(\xi)x, y \geq 0\} \neq \emptyset,\ \forall \xi \in \Xi, x \in X.$$

Beyond two decision stages

Many problems have in fact multiple decision points, in which decisions are made sequentially.



Beyond two decision stages

Multi-stage decision problems:

1. Consist of nested two-stage problems. This can be exploited in a dynamic programming fashion;
2. Likewise, presents exponential growth with scenarios per stage ($|\Xi|^{|H|-1}$, with $|\Xi|$-scenario stages $t \in [H]$[2]);
3. Trade-off: future flexibility versus computational cost.

Consider that we have H decision stages. Our decision process becomes:

$$x^1 \to \xi^2 \to x^2(\xi^2, x^1) \to \xi^3 \to x^3((\xi^2, \xi^3), (x^1, x^2)) \to \dots$$
$$\to \; \xi^H \to x^H((\xi^2, \dots, \xi^H), (x^1, \dots, x^{H-1}))$$

- ξ up to stage $t = 2, \dots, T$ represents a sequence of events
- Hereinafter, $x^t(\xi)$ is a shorthand for $x^t((\xi^2, \dots, \xi^t))$

[2] $n \in [N] = n \in \{1, \dots, N\}$.

Multi-stage decision problems

For $t = H$ we have:

$$\begin{aligned} Q^H(x^{H-1}, \xi^H) = \text{min. } & c^H(\xi)^\top x^H(\xi) \\ \text{s.t.: } & W^H(\xi)x^H(\xi) = h^H(\xi) - T^{H-1}(\xi)x^{H-1} \\ & x^H(\xi) \geq 0. \end{aligned}$$

For $t = 2, \dots, H-1$ we have:

$$\begin{aligned} Q^t(x^{t-1}, \xi^t) = \text{min. } & c^t(\xi)^\top x^t(\xi) + \mathcal{Q}^{t+1}(x^t) \\ \text{s.t.: } & W^t(\xi)x^t(\xi) = h^t(\xi) - T^{t-1}(\xi)x^{t-1} \\ & x^t(\xi) \geq 0. \end{aligned}$$

We want to solve

$$\begin{aligned} \text{min. } & c^{1\top} x^1 + \mathcal{Q}(x^1) \\ \text{s.t.: } & W^1 x^1 = h^1 \\ & x^1 \geq 0. \end{aligned}$$

Example: 3SSP

A 3-stage formulation is given as:

$$
\begin{aligned}
\text{min. } & c^{1\top} x^1 + \sum_{\xi_s^2 \in S^2} P(\xi_s^2) \Big[c^2(\xi_s^2)^\top x^2(\xi_s^2) + \\
& \sum_{\xi_s^3 \in S^3(\xi_s^2)} P(\xi_s^3|\xi_s^2) \left(c^3(\xi_s^3|\xi_s^2)^\top x^3(\xi_s^3|\xi_s^2) \right) \Big] \\
\text{s.t.: } & T^1 x^1 = h^1 \\
& T(\xi_s^2) x^1 + W(\xi_s^2) x^2(\xi_s^2) = h(\xi_s^2), \forall \xi_s^2 \\
& T(\xi_s^3|\xi_s^2) x^2(\xi_s^2) + W(\xi_s^3|\xi_s^2) x^3(\xi_s^3|\xi_s^2) = h(\xi_s^3|\xi_s^2), \forall \xi_s^2, \xi_s^3|\xi_s^2 \\
& x^1 \geq 0 \\
& x^2(\xi_s^2) \geq 0, \ \forall \xi_s^2 \\
& x^3(\xi_s^3|\xi_s^2) \geq 0, \ \forall \xi_s^2, \xi_s^3|\xi_s^2.
\end{aligned}
$$

References

Birge, J. R. and Louveaux, F. (2011).
Introduction to stochastic programming.
Springer Science & Business Media.