Homework Assignment 1

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Problem 1.4.1. Find the fixed points and determine their stability for the function

$$f(x) = \frac{6}{x} - 1.$$

Solution. The fixed points of the function $f(x)$ are the roots of the function

$$\begin{aligned} g(x) &= f(x) - x \\ &= \frac{6}{x} - 1 - x \\ &= -\frac{(x+3)(x+2)}{6}. \end{aligned}$$

We readily see that the roots of $g(x)$, which are the fixed points of $f(x)$, are given by $x = -3$ and $x = 2$.

According to Theorem 1.4.4, since $f(x)$ is a C^1 function, we may use the derivative of $f(x)$ to classify its fixed points. If c is a fixed point of f and $|f'(c)| < 1$, then c is an asymptotically stable fixed point, while $|f'(c)| > 1$ indicates that c is a repelling (unstable) fixed point.

Note that $f'(x) = -6/x^2$. For the fixed point $x = -3$, we see that

$$|f'(-3)| = \left| -\frac{6}{(-3)^2} \right| = \frac{2}{3} < 1$$

from which we classify the point $x = -3$ as an asymptotically stable fixed point. On the other hand, for the fixed point $x = 2$, we see that

$$|f'(2)| = \left| -\frac{6}{(2)^2} \right| = \frac{3}{2} > 1$$

from which we classify the point $x = -3$ as a repelling (unstable) fixed point. $\square$

Problem 1.4.2. Let $f : \mathbb{R} \to \mathbb{R}$. If $f'(x)$ exists with $f'(x) \neq 1$ for all $x \in \mathbb{R}$, prove that f has at most one fixed point. (Hint: Use the Mean Value Theorem).

Solution. Suppose to the contrary that for all $x \in \mathbb{R}$ we have that $f'(x)$ exists with $f'(x) \neq 1$, but f has at least two distinct fixed points, c_1 and c_2, say. The Mean Value Theorem states that if a function g is continuous on an interval $[a, b]$ and differentiable on the interval (a, b), then there exists a point $c \in (a, b)$ such that

$$g'(c) = \frac{g(b) - g(a)}{b - a}.$$

By our supposition, we have that the function f is continuous and differentiable on any interval and, in particular, it is continuous on $[c_1, c_2]$ and differentiable on (c_1, c_2). By the Mean Value Theorem, there exists a point $c_3 \in (c_1, c_2)$ such that

$$f'(c_3) = \frac{f(c_2) - f(c_1)}{c_2 - c_1}. \tag{1}$$

However, since c_1 and c_2 are fixed points of f, we know that $f(c_2) - f(c_1) = c_2 - c_1$ and we gather from (1) that

$$f'(c_3) = \frac{f(c_2) - f(c_1)}{c_2 - c_1} = \frac{c_2 - c_1}{c_2 - c_1} = 1.$$

However, this is in contradiction to our supposition that $f'(x) \neq 1$ for any $x \in \mathbb{R}$. Therefore, we must conclude that for a function $f : \mathbb{R} \to \mathbb{R}$, if for all $x \in \mathbb{R}$ we have that $f'(x)$ exists with $f'(x) \neq 1$, then f has at most one fixed point. $\square$

Problem 1.4.4. Let $S_\mu(x) = \mu \sin(x)$, $0 \le x \le 2\pi$, $0 < \mu \le \pi$ and $C_\mu(x) = \mu \cos(x)$, $-\pi \le x \le \pi$ and $-\pi \le \mu \le \pi$, $\mu \ne 0$.

i. Show that S_μ has a super-attracting fixed point at $x = \pi/2$, when $\mu = \pi/2$.

ii. Find the corresponding values for C_μ having a super-attracting fixed point.

Solution. Recall that if c is a fixed point of a differentiable function f, then c is a super-attracting fixed point if $f'(c) = 0$.

i. Suppose that $\mu = \pi/2$. Since $S_\mu(\pi/2) = (\pi/2)\sin(\pi/2) = \pi/2$, we readily see that if $\mu = \pi/2$, then $x = \pi/2$ is a fixed point of $S_\mu(x)$. Note that $S'_\mu(x) = \mu\cos(x)$. From this we gather that if $\mu = \pi/2$, then for the fixed point $x = \pi/2$, we have that $S'_\mu(x) = (\pi/2)\cos(\pi/2) = 0$. Therefore, the fixed point $x = \pi/2$ is a super-attracting fixed point.

ii. We now investigate the super-attracting fixed points of $C_\mu(x)$. The definition of $C_\mu(x)$ shows that $C'_\mu(x) = -\mu\sin(x)$ from which we can gather that $C'_\mu(x) = 0$ for $x \in [-\pi, \pi]$ if $x = k\pi$ for $k \in \{-1, 0, 1\}$. Note that these are the possible super-attracting fixed points of $C_\mu(x)$, we must still determine which of these possible super-attracting fixed points are indeed fixed points, i.e. we must determine which points satisfy $C_\mu(x) = x$. If $x = k\pi$ for $k \in \{-1, 0, 1\}$, then

$$C_\mu(k\pi) = \mu\cos(k\pi) = (-1)^k\mu.$$

Thus, $C_\mu(k\pi) = (-1)^k\mu = k\pi$, if $\mu = (-1)^k k\pi$. Therefore, if $x, \mu \in [-\pi, \pi]$ with $\mu \ne 0$, then the points $x_1 = -\pi$ and $x_2 = \pi$, with corresponding μ-values $\mu_1 = \pi$ and $\mu_2 = -\pi$, are super-attracting fixed points. Note that $x = 0$ is not a super-attracting fixed point since it is not a fixed point, that is $C_\mu(0) = 0$ only if $\mu = 0$, which violates our initial conditions.

□

Problem 1.4.7. Let N_f be the Newton function of the map $f(x) = x^2 + 1$. Clearly there are no fixed points of the Newton function as there are no zeros of f. Show that there are points c where $N_f^2(c) = c$ (called *period 2-points* of N_f).

Solution. $\square$

Problem 1.4.8. i. Suppose that $f(c) = f'(c) = 0$ and $f''(c) \neq 0$. If $f''(x)$ is continuous at $x = c$, show that the Newton function $N_f(x)$ has a removable discontinuity at $x = c$. (Hint: Apply LHopitals rule to N_f at $x = c$.)

ii. If in addition, $f'''(x)$ is continuous at $x = c$ with $f'''(c) \neq 0$, show that $N'_f(c) = 1/2$, so that $x = c$ is not a super-attracting fixed point in this case.

iii. Check the above for the function $f(x) = x^3 x^2$ with $c = 0$.

Solution. □