Homework Assignment 2

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Problem 1. Convert the following linear programming problem to *standard form*:

$$\begin{array}{rll} \text{maximize} & 2x_1 + x_2 & \\ \text{subject to} & 0 \le x_1 & \le 2 \\ & x_1 + x_2 & \le 3 \\ & x_1 + 2x_2 & \le 5 \\ & x_2 \ge 0 & \end{array}$$

Solution. In order to convert this linear programming problem into standard form, we must transform the objective from *maximize* to *minimize* and the constraints must be transformed from linear inequalities into linear equations.

Our first step will be to rewrite the objective function as a minimization problem and write each constraint as a linear inequality as so:

$$\begin{array}{rll} \text{minimize} & -2x_1 - x_2 & \\ \text{subject to} & x_1 & \le 2 \\ & x_1 + x_2 & \le 3 \\ & x_1 + 2x_2 & \le 5 \\ & x_1 \ge 0, x_2 \ge 0 & \end{array}$$

We can then introduce three slack variables x_3, x_4, x_5 to turn the linear inequalities into linear equations:

$$\begin{array}{rllllll} \text{minimize} & -2x_1 & -x_2 & & & & \\ \text{subject to} & x_1 & +x_2 & +x_3 & & & = 2 \\ & x_1 & +x_2 & & +x_4 & & = 3 \\ & x_1 & +2x_2 & & & +x_5 & = 5 \\ & x_1 \ge 0, & x_2 \ge 0, & x_3 \ge 0, & x_4 \ge 0, & x_5 \ge 0 & \end{array}$$

As the above linear programming problem is written as

$$\begin{array}{rl} \text{minimize} & \boldsymbol{c}^\intercal \boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \ge 0 \end{array}$$

where

$$\boldsymbol{c}^{\mathsf{T}} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}^{\mathsf{T}}, \quad A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

with $\boldsymbol{x} \geq 0$ and $\boldsymbol{b} \geq 0$ the linear programming problem is in standard form and we are done. $\square$

Problem 2. Solve the system $A\boldsymbol{x} = \boldsymbol{b}$ where

$$A = \begin{bmatrix} 2 & -1 & 2 & -1 & 3 \\ 1 & 2 & 3 & 1 & 0 \\ 1 & 0 & -2 & 0 & -5 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 14 \\ 5 \\ -10 \end{bmatrix}.$$

If possible, generate a non-basic feasible solution of the system from which you derive next a basic feasible one.

Solution. In order to solve the system $Ax = b$, we must perform row operations on the augmented matrix to reduce it to reduced row form. We perform these operations below:

$$\left[\begin{array}{ccccc|c} 2 & -1 & 2 & -1 & 3 & 14 \\ 1 & 2 & 3 & 1 & 0 & 5 \\ 1 & 0 & -2 & 0 & -5 & -10 \end{array}\right] \underset{\substack{[2]-(1/2)[1] \\ [3]-(1/2)[1] \\ \longrightarrow}}{\xrightarrow{(1/2)[1]}} \left[\begin{array}{ccccc|c} 1 & -1/2 & 1 & -1/2 & 3/2 & 7 \\ 0 & 5/2 & 2 & 3/2 & -3/2 & -2 \\ 0 & 1/2 & -3 & 1/2 & -13/2 & -17 \end{array}\right] \xrightarrow{(2/5)[2]}$$

$$\left[\begin{array}{ccccc|c} 1 & -1/2 & 1 & -1/2 & 3/2 & 7 \\ 0 & 1 & 4/5 & 3/5 & -3/5 & -4/5 \\ 0 & 1/2 & -3 & 1/2 & -13/2 & -17 \end{array}\right] \underset{[3]-(1/2)[2]}{\xrightarrow{[1]+(1/2)[2]}} \left[\begin{array}{ccccc|c} 1 & 0 & 7/5 & -1/5 & 6/5 & 33/5 \\ 0 & 1 & 4/5 & 3/5 & -3/5 & -4/5 \\ 0 & 0 & -17/5 & 1/5 & -31/5 & -83/5 \end{array}\right]$$

$$\xrightarrow{(-5/17)[3]} \left[\begin{array}{ccccc|c} 1 & 0 & 7/5 & -1/5 & 6/5 & 33/5 \\ 0 & 1 & 4/5 & 3/5 & -3/5 & -4/5 \\ 0 & 0 & 1 & -1/17 & 31/17 & 83/17 \end{array}\right] \underset{[2]-(4/5)[3]}{\xrightarrow{[1]-(7/5)[3]}} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -2/17 & -23/17 & -4/17 \\ 0 & 1 & 0 & 11/17 & -35/17 & -80/17 \\ 0 & 0 & 1 & -1/17 & 31/17 & 83/17 \end{array}\right].$$

Using the above row-reduced augmented matrix, we see that the solution to the system $A\boldsymbol{x} = \boldsymbol{b}$ is given by

$$\boldsymbol{x} = \begin{bmatrix} -4/17 \\ -80/17 \\ 83/17 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2/17 \\ -11/17 \\ 1/17 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 23/17 \\ 35/17 \\ -31/17 \\ 0 \\ 1 \end{bmatrix} \tag{1}$$

where $s \in \mathbb{R}$ and $t \in \mathbb{R}$.

Suppose that the matrix A is written such that $A = [\boldsymbol{a}_i]$ for $1 \le i \le 5$ where $\boldsymbol{a}_i$ corresponds to the i-th column of the original matrix A. Recall that a solution $\boldsymbol{x}_0 \ge \boldsymbol{0}$ of the system $A\boldsymbol{x} = \boldsymbol{b}$ is a basic feasible solution if the columns of the matrix A associated to the nonzero components of $\boldsymbol{x}_0$ are linearly independent. Otherwise the solution is a non-basic feasible solution.

Using solution (1), we see that for $s = 0$ and $t = 82/31$, we get the corresponding feasible solution $\boldsymbol{x}_0 = [1762/527, 390/527, 1/17, 0, 82/31]^\intercal$ to the system $A\boldsymbol{x} = \boldsymbol{b}$. We know that this is a non-basic feasible solution since the vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, and $\boldsymbol{a}_5$ must be linearly dependent as the rank$(A) = 3$, i.e. the maximum number of linearly independent columns of A is 3.

The Fundamental Theorem of LP prescribes how to move from this non-basic feasible solution $\boldsymbol{x}_0$ to a basic feasible solution $\boldsymbol{x}_1$. As $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, and $\boldsymbol{a}_5$ are linearly dependent, there exists constants y_1, y_2, y_3, y_5 not all zero such that

$$y_1\boldsymbol{a_1} + y_2\boldsymbol{a_2} + y_3\boldsymbol{a_3} + y_5\boldsymbol{a_5} = \boldsymbol{0},$$

namely $y_1 = 1$, $y_2 = 35/23$, $y_3 = -31/23$, and $y_5 = 17/23$. Thus, the vector $\epsilon\boldsymbol{y} = \epsilon[y_1, y_2, y_3, 0, y_5]^\intercal$ satisfies $A[\epsilon\boldsymbol{y}] = \mathbf{0}$. As such, the vector $\boldsymbol{x}_0 - \epsilon\boldsymbol{y}$ satisfies $A[\boldsymbol{x}_0 - \epsilon\boldsymbol{y}] = \boldsymbol{b}$, i.e. the vector $\boldsymbol{x}_0 - \epsilon\boldsymbol{y}$ is a solution of the original system $A\boldsymbol{x} = \boldsymbol{b}$. Choose

$$\epsilon = \min\{x_i/y_i | i = 1, 2, 3, 5\ y_i > 0\} = -23/527.$$

Then the vector $\boldsymbol{x}_1 = \boldsymbol{x}_0 - \epsilon\boldsymbol{y}$ will have 3 positive components and the rest of the components will be 0 showing that the vector $\boldsymbol{x}_1$ is a basic feasible solution. Therefore,

$$\boldsymbol{x}_1 = \boldsymbol{x}_0 - \epsilon\boldsymbol{y} = \begin{bmatrix} 105/31 \\ 25/31 \\ 0 \\ 0 \\ 83/31 \end{bmatrix}$$

is the desired basic feasible solution. $\square$

Problem 3. Does every linear programming problem in standard form have a nonempty feasible set? If "yes", provide a proof. If "no", provide a counter-example.

Does every linear programming problem in standard form (assuming a nonempty feasible set) have an optimal solution? If "yes", provide a proof. If "no", provide a counter-example.

Solution. Not every linear programming problem (LP) in standard form has a nonempty feasible set. Take for instance the following LP:

$$\begin{array}{rl} \text{minimize} & 2x_1 \\ \text{subject to} & x_1 \leq -1 \\ & x_1 \geq 0 \end{array}$$

Clearly this problem has an empty feasible set due to the contradictory constraints.

The LP in standard form is stated as:

$$\begin{array}{rl} \text{minimize} & 2x_1 \\ \text{subject to} & -x_1 - x_2 = 1 \\ & x_1 \geq 0, x_2 \geq 0 \end{array}$$

The equation $-x_1 - x_2 = 1$ for $x_1, x_2 \geq 0$ has no solutions and the feasible set of the standard form LP is empty. Therefore, not every LP in standard form has a nonempty feasible set.

Additionally, not every LP in standard form with a nonempty feasible set has an optimal solution. Take for instance the following LP:

$$\begin{array}{rl} \text{minimize} & -3x_1 \\ \text{subject to} & x_1 \geq 1 \\ & x_1 \geq 0 \end{array}$$

The problem can be written in standard form as so:

$$\begin{array}{rl} \text{minimize} & -3x_1 \\ \text{subject to} & x_1 - x_2 = 1 \\ & x_1 \geq 0, x_2 \geq 0 \end{array}$$

The equation $x_1 - x_2 = 1$ for $x_1, x_2 \geq 0$ has an infinite number of solutions so that x_1 can be chosen arbitrarily large. Consequently, $-x_1$ can be made arbitrarily small and no point in the feasible set of this LP in standard form is the smallest, i.e optimal. Therefore, not every LP in standard form with a nonempty feasible set has an optimal solution. $\square$

Problem 4. a. Solve the following linear program graphically:

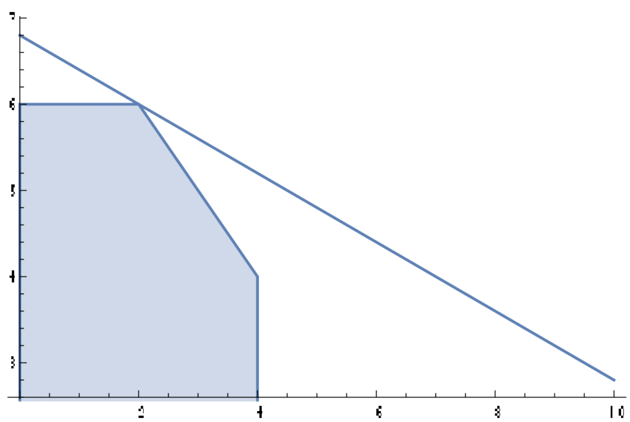
$$\begin{array}{ll} \text{maximize} & 2x_1 + 5x_2 \\ \text{subject to} & 0 \leq x_1 \leq 4 \\ & 0 \leq x_2 \leq 6 \\ & x_1 + x_2 \leq 8 \end{array}$$

b. Solve the linear program in (b) the same way Example 15.15 was solved in class. Compute only the vertices that lead to the optimal vertex found at (a).

Solution. a. In order to find the solution to this linear program, we must first plot the feasible region of this problem. Note, that the equation associated to the objective function forms a family of straight lines $O = \{2x_1 + 5x_2 = b \mid b \in \mathbb{R}\}$. We then wish to find a value b such that the line $2x_1 + 5x_2 = b$ is within the feasible region, i.e. satisfies the constraints and the value b is the largest such value that allows x_1 and x_2 to satisfy the constraints.

After plotting the feasible region with Mathematica and trying various values of b, we see that an objective function value $b = 34$ is associated to the optimal solution of this LP as the plot below shows:

```
Show[Plot[(-2 / 5) x1 + (34 / 5), {x1, 0, 10}],
 RegionPlot[{0 ≤ x1 ≤ 4 && 0 ≤ x2 ≤ 6 && x1 + x2 ≤ 8}, {x1, 0, 8}, {x2, 0, 8}]]
```



It is easy to see that any $b > 34$ will cause the objective function to leave the feasible region, providing a geometric proof that $(x_1, x_2) = (2, 6)$ is the optimal solution of this linear program leading to an optimal objective function value of 34.

b. In order to solve this problem, we must first transform the problem into standard form through the addition of slack variables and changing the objective as follows:

$$\begin{array}{lllllll} \text{minimize} & -2x_1 & -5x_2 & & & & \\ \text{subject to} & x_1 & & +x_3 & & & = 4 \\ & & x_2 & & +x_4 & & = 6 \\ & x_1 & +x_2 & & & +x_5 & = 8 \\ & x_1, & x_2, & x_3, & x_4, & x_5 & \geq 0 \end{array}.$$

In matrix form, this linear program is represented as

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^\intercal \boldsymbol{x} = [-2, 5, 0, 0, 0][x_1, x_2, x_3, x_4, x_5]^\intercal \\ \text{subject to} \quad & A\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix} = \boldsymbol{b} \ . \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

Note that we can represent the matrix A in terms of its columns as follows:

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{a_1} \boldsymbol{a_2} \boldsymbol{a_3} \boldsymbol{a_4} \boldsymbol{a_5} \end{bmatrix} .$$

We begin by finding a basic feasible solution of the system. In order to find the optimal value for this linear program, we will move from this basic feasible solution to an adjacent basic feasible solution. We keep moving to adjacent basic feasible solutions until the objective function does not decrease any further. The solution at which this occurs will be optimal.

Note that $\boldsymbol{x} = [0, 0, 4, 6, 8]^\intercal$ is a basic feasible solution as $\boldsymbol{a_3}, \boldsymbol{a_4}, \boldsymbol{a_5}$ are linearly independent and $\boldsymbol{x} \geq 0$. The objective function's value for this basic feasible solution is $\boldsymbol{c}^\intercal \boldsymbol{x} = 0$.

To choose an adjacent basic feasible solution, we choose $\boldsymbol{a_2}$ as a basic column in the new basis. We can express $\boldsymbol{a_2}$ as a linear combination of the old basic columns $\boldsymbol{a_3}, \boldsymbol{a_4}, \boldsymbol{a_5}$:

$$\boldsymbol{a_2} = 0\boldsymbol{a_3} + 1\boldsymbol{a_4} + 1\boldsymbol{a_5}.$$

We now wish to find an $\epsilon > 0$ such that

$$0\boldsymbol{a_1} + \epsilon\boldsymbol{a_2} + (4 - 0\epsilon)\boldsymbol{a_3} + (6 - 1\epsilon)\boldsymbol{a_4} + (8 - 1\epsilon)\boldsymbol{a_5} = \boldsymbol{b}$$

and the coefficients of the above system are non-negative while also eliminating either $\boldsymbol{a_3}, \boldsymbol{a_4}, \boldsymbol{a_5}$. The choice of $\epsilon = 6$ satisfies the above requirements leading to the new basic feasible solution

$$\boldsymbol{x} = [0, 6, 4, 0, 2]^\intercal.$$

We now repeat the procedure choosing to make $\boldsymbol{a_1}$ the new basic column. We write $\boldsymbol{a_1}$ as a linear combination of the basic columns $\boldsymbol{a_2}, \boldsymbol{a_4}, \boldsymbol{a_5}$ as follows:

$$\boldsymbol{a_1} = 0\boldsymbol{a_2} + 1\boldsymbol{a_3} + 1\boldsymbol{a_5}.$$

We now need to find $\epsilon > 0$ such that

$$\epsilon\boldsymbol{a_1} + (6 - 0\epsilon)\boldsymbol{a_2} + (4 - 1\epsilon)\boldsymbol{a_3} + 0\boldsymbol{a_4} + (2 - 1\epsilon)\boldsymbol{a_5} = \boldsymbol{b}$$

that maintains the non-negativity of the solution and eliminates either $\boldsymbol{a_2}, \boldsymbol{a_3}$, or $\boldsymbol{a_5}$. The choice of $\epsilon = 2$ satisfies these requirements leading us to the new basic feasible solution

$$\boldsymbol{x} = [2, 6, 2, 0, 0]^\intercal \tag{2}$$

Choosing either $\boldsymbol{a_4}$ or $\boldsymbol{a_5}$ to be the new basic columns will reduce the value of x_1 or x_2 in (2) thereby leading to an objective function value greater than the current one. This would mean that the next solution could not be optimal. Therefore, (2) is the optimal solution to the linear program with associated objective function value -34, or 34 in terms of the objective function of the original linear program.

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