Homework Assignment 7

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April 18, 2016

Problem 1. State all of the KKT conditions for (N-max). More precisely state all of the following results for (N-max): KKT-FONC, KKT-FOSC, KKT-SONC, KKT-SOSC.

Solution. For the following theorems, we assume (N- max) has the following form

$$\begin{array}{llll} (N\text{- max}) & \text{maximize} & f(\boldsymbol{x}) & \\ & \text{subject to} & \boldsymbol{h}(\boldsymbol{x}) & = \mathbf{0} \\ & & \boldsymbol{g}(\boldsymbol{x}) & \leq \mathbf{0} \end{array}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^m$, and $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^p$ with $m \leq n$. Additionally, define the following Lagrangian function to be $\boldsymbol{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) := -f(\boldsymbol{x}) + \boldsymbol{\lambda}^\mathsf{T}\boldsymbol{h}(\boldsymbol{x}) + \boldsymbol{\mu}^\mathsf{T}\boldsymbol{g}(\boldsymbol{x})$.

Theorem 1 (KKT-FONC for (N- max))**.** Let f, $\boldsymbol{g}$, $\boldsymbol{h} \in C^1$ and let $\boldsymbol{x}^*$ be a regular point and local maximizer for the problem (N- max). Then, there exist $\boldsymbol{\lambda}^* \in \mathbb{R}^m$ and $\boldsymbol{\mu}^* \in \mathbb{R}^p$ such that:

i. $\boldsymbol{\mu}^* \geq \mathbf{0}$.

ii. $D_{\boldsymbol{x}}\boldsymbol{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) = -Df(\boldsymbol{x}^*) + \boldsymbol{\lambda}^{*\mathsf{T}}D\boldsymbol{h}(\boldsymbol{x}^*) + \boldsymbol{\mu}^{*\mathsf{T}}D\boldsymbol{g}(\boldsymbol{x}^*) = \mathbf{0}^\mathsf{T}$.

iii. $\mu^{*\mathsf{T}}\boldsymbol{g}(\boldsymbol{x}^*) = 0$.

Note that there are no explicit first-order conditions that are sufficient in general to show optimality.

Theorem 2 (KKT-SONC for (N- max))**.** Let f, $\boldsymbol{g}$, $\boldsymbol{h} \in C^2$ and let $\boldsymbol{x}^*$ be a regular point and local maximizer for the problem (N- max). Then, there exist $\boldsymbol{\lambda}^* \in \mathbb{R}^m$ and $\boldsymbol{\mu}^* \in \mathbb{R}^p$ such that:

i. $\boldsymbol{\mu}^* \geq \mathbf{0}$, $D_{\boldsymbol{x}}\boldsymbol{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) = \mathbf{0}^\mathsf{T}$, $\mu^{*\mathsf{T}}\boldsymbol{g}(\boldsymbol{x}^*) = 0$.

ii. For all $\boldsymbol{y} \in T(\boldsymbol{x}^*) = \{\boldsymbol{y} \mid D\boldsymbol{h}(\boldsymbol{x}^*)\boldsymbol{y} = \mathbf{0}, Dg_j(\boldsymbol{x}^*)\boldsymbol{y} = 0, j \in J(\boldsymbol{x}^*)\}$, we have that $\boldsymbol{y}^\mathsf{T}D^2_{\boldsymbol{x}}\boldsymbol{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)\boldsymbol{y} \leq 0$.

Theorem 3 (KKT-SOSC for (N- max))**.** Let f, $\boldsymbol{g}$, $\boldsymbol{h} \in C^2$ and suppose there exists a feasible point $\boldsymbol{x}^*$ and vectors $\boldsymbol{\lambda}^* \in \mathbb{R}^m$ and $\boldsymbol{\mu}^* \in \mathbb{R}^p$ such that:

i. $\boldsymbol{\mu}^* \geq \mathbf{0}$, $D_x\boldsymbol{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) = \mathbf{0}^\mathsf{T}$, $\mu^{*\mathsf{T}}\boldsymbol{g}(\boldsymbol{x}^*) = 0$.

ii. For all

$$\boldsymbol{y} \in \widetilde{T}(\boldsymbol{x}^*, \boldsymbol{\mu}^*) = \{\boldsymbol{y} \mid D\boldsymbol{h}(\boldsymbol{x}^*)\boldsymbol{y} = \boldsymbol{0}, Dg_i(\boldsymbol{x}^*)\boldsymbol{y} = 0, \text{for } i \in \{i \mid g_i(\boldsymbol{x}^*) = 0, \mu_i^* > 0\}\},$$

with $\boldsymbol{y} \neq \boldsymbol{0}$, we have that $\boldsymbol{y}^\top D_{\boldsymbol{x}}^2 \boldsymbol{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)\boldsymbol{y} < 0$.

Then $\boldsymbol{x}^*$ is a strict local maximizer for the problem (N- max).

□

Problem 2. Find local minimizers for

$$\begin{array}{lrl} (N\text{- min}) & \text{minimize} & x_1^2 + 6x_1x_2 - 4x_1 - 2x_2 \\ & \text{subject to} & x_1^2 + 2x_2 \leq 1 \\ & & 2x_1 - 2x_2 \leq 1. \end{array}$$

Solution. We begin by rewriting the above problem as follows:

$$\begin{array}{lrl} (N\text{- min}) & \text{minimize} & f(\boldsymbol{x}) = x_1^2 + 6x_1x_2 - 4x_1 - 2x_2 \\ & \text{subject to} & g_1(\boldsymbol{x}) = x_1^2 + 2x_2 - 1 \leq 0 \\ & & g_2(\boldsymbol{x}) = 2x_1 - 2x_2 - 1 \leq 0. \end{array}$$

We proceed by using the KKT-FONC to determine the possible local minimizers for this problem. The Lagrangian associated to this problem is given by

$$\begin{aligned} \boldsymbol{L}(\boldsymbol{x}, \boldsymbol{\mu}) &= f(\boldsymbol{x}) + \boldsymbol{\mu}^\mathsf{T} \boldsymbol{g}(\boldsymbol{x}) \\ &= f(\boldsymbol{x}) + \mu_1 g_1(\boldsymbol{x}) + \mu_2 g_2(\boldsymbol{x}) \\ &= x_1^2 + 6x_1x_2 - 4x_1 - 2x_2 + \mu_1(x_1^2 + 2x_2 - 1) + \mu_2(2x_1 - 2x_2 - 1). \end{aligned}$$

This implies that

$$D_{\boldsymbol{x}} L(\boldsymbol{x}, \boldsymbol{\mu}) = \begin{bmatrix} 2x_1 + 6x_2 - 4 + 2\mu_1 x_1 + 2\mu_2 \\ 6x_1 - 2 + 2\mu_1 - 2\mu_2 \end{bmatrix}^\mathsf{T} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}^\mathsf{T}. \tag{1}$$

Thus, the KKT-FONC are then written as

i. $\mu_1, \mu_2 \geq 0$.

ii. $2x_1 + 6x_2 - 4 + 2\mu_1 x_1 + 2\mu_2 = 0$.

iii. $6x_1 - 2 + 2\mu_1 - 2\mu_2 = 0$.

iv. $\mu_1 g_1(\boldsymbol{x}) + \mu_2 g_2(\boldsymbol{x}) = \mu_1(x_1^2 + 2x_2 - 1) + \mu_2(2x_1 - 2x_2 - 1) = 0$.

v. $g_1(\boldsymbol{x}) = x_1^2 + 2x_2 - 1 \leq 0$.

vi. $g_2(\boldsymbol{x}) = 2x_1 - 2x_2 - 1 \leq 0$.

Solving the system (1) for x_1, x_2 yields that

$$\begin{aligned} x_1 &= \frac{\mu_2 - \mu_1 + 1}{3} \\ x_2 &= \frac{\mu_1^2 - \mu_1\mu_2 - 4\mu_2 + 5}{9} \end{aligned} \tag{2}$$

with $\mu_1, \mu_2 \geq 0$. Using these representations of x_1, x_2 we see that condition iv. yields three possible solutions in terms of μ_1, μ_2:

$$\begin{aligned} \text{Case 1} : \mu_2 &= \frac{13 + 12\mu_1 + 6\mu_1^2 - \sqrt{169 + 200\mu_1 + 388\mu_1^2}}{2(14 + 3\mu_1)} \\ \text{Case 2} : \mu_2 &= \frac{13 + 12\mu_1 + 6\mu_1^2 + \sqrt{169 + 200\mu_1 + 388\mu_1^2}}{2(14 + 3\mu_1)} \\ \text{Case 3} : \mu_1 &= -\frac{14}{3},\ \mu_2 = -\frac{3220}{789} \end{aligned}$$

We readily see that Case 3 cannot happen in light of condition i.

Assuming Case 1 is true and using the representations of x_1, x_2 in (2), we see that $g_1(\boldsymbol{x}) < 0$ for $\mu_1, \mu_2 \geq 0$ implying that this constraint is inactive and that $\mu_1 = 0$. This implies that $\mu_2 = 0$ which in turn implies that $x_1 = 1/3$, $x_2 = 5/9$. However, $g_1(x_1, x_2) = 2/9 \not\leq 0$ violating condition v. Thus, Case 1 cannot happen.

Assuming Case 2 is true and using the representations of x_1, x_2 in (2), we again see that $g_1(\boldsymbol{x}) < 0$ for $\mu_1, \mu_2 \geq 0$ implying that this constraint is inactive and that $\mu_1 = 0$. This implies that $\mu_2 = 13/14$ which in turn implies that $x_1 = 9/14$, $x_2 = 1/7$. These values of x_1, x_2 satisfy conditions v. and vi.

Therefore, the only vector $\boldsymbol{x}^*$ that satisfies conditions i. - vi., i.e. the only possible local minimizer for this problem is

$$\boldsymbol{x}^* = \begin{bmatrix} 9/14 \\ 1/7 \end{bmatrix}$$

with associated KKT multiplier

$$\boldsymbol{\mu}^* = \begin{bmatrix} 0 \\ 13/14 \end{bmatrix}.$$

To verify whether or not this vector is a strict local minimizer, we check the KKT-SOSC. For the vectors $\boldsymbol{x}^*$ and $\boldsymbol{\mu}^*$ defined above, we see that $\{i \mid g_i(\boldsymbol{x}^*) = 0, \mu_i^* > 0\} = \{2\}$ and that

$$\begin{aligned} \widetilde{T}(\boldsymbol{x}^*, \boldsymbol{\mu}^*) &= \{\boldsymbol{y} \in \mathbb{R}^2 \mid Dg_2(\boldsymbol{x}^*)\boldsymbol{y} = 0\} \\ &= \{\boldsymbol{y} \in \mathbb{R}^2 \mid [2, -2]\boldsymbol{y} = 0\} \\ &= \{\boldsymbol{y} = [y_1, y_2]^\mathsf{T} \in \mathbb{R}^2 \mid y_1 = y_2\}. \end{aligned}$$

Further, we have, for these vectors, that

$$D_{\boldsymbol{x}}^2 L(\boldsymbol{x}^*, \boldsymbol{\mu}^*) = \begin{bmatrix} 2 + 2\mu_1 & 6 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 6 & 0 \end{bmatrix}.$$

Combining we see that for $\boldsymbol{0} \neq \boldsymbol{y} \in \widetilde{T}(\boldsymbol{x}^*, \boldsymbol{\mu}^*)$, we have that

$$\boldsymbol{y}^\mathsf{T} D_{\boldsymbol{x}}^2 L(\boldsymbol{x}^*, \boldsymbol{\mu}^*)\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_1 \end{bmatrix}^\mathsf{T} \begin{bmatrix} 2 & 6 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_1 \end{bmatrix} = 14y_1^2 > 0$$

for $y_1 \neq 0$. Therefore, $\boldsymbol{x}^* = [9/14, 1/7]^\mathsf{T}$ is a strict local minimizer. □

Problem 3. Consider the problem of optimizing

$$\begin{array}{lll} (N) & \text{minimize (maximize)} & (x_1-2)^2+(x_2-1)^2 \\ & & x_2 - x_1^2 \quad\quad \geq 0 \\ & \text{subject to} & 2 - x_1 - x_2 \;\; \geq 0 \\ & & x_1 \quad\quad\quad\quad \geq 0. \end{array}$$

Let $\boldsymbol{x}^* = [0, 0]^\top$.

a. Does $\boldsymbol{x}^*$ satisfy the KKT-FONC for minimization or maximization? What are the KKT multipliers?

b. Does $\boldsymbol{x}^*$ satisfy the KKT-SOSC? Justify your answer.

Solution. We begin by rewriting the problem (N) as

$$\begin{array}{lll} (N_1) & \text{minimize (maximize)} & f(\boldsymbol{x}) = (x_1-2)^2+(x_2-1)^2 \\ & & g_1(\boldsymbol{x}) = -x_2 + x_1^2 \quad\quad \leq 0 \\ & \text{subject to} & g_2(\boldsymbol{x}) = -2 + x_1 + x_2 \;\; \leq 0 \\ & & g_3(\boldsymbol{x}) = -x_1 \quad\quad\quad\quad \leq 0. \end{array}$$

For both problems, the vector $\boldsymbol{x}^* = [0, 0]^\top$ is a regular point. To see this, we note that $\boldsymbol{x}^*$ is feasible and the constraints $g_1(\boldsymbol{x}^*) \leq 0$ and $g_3(\boldsymbol{x}^*) \leq 0$ are both active for this vector. Since $\nabla g_1(\boldsymbol{x}^*) = [0, -1]^\top$ and $\nabla g_3(\boldsymbol{x}^*) = [-1, 0]^\top$ are linearly independent, we have that $\boldsymbol{x}^*$ is a regular point as desired.

The Lagrangian function associated to problem (N_1-min) is given by

$$\begin{aligned} L_{\min}(\boldsymbol{x}, \boldsymbol{\mu}) &= f(\boldsymbol{x}) + \mu_1 g_1(\boldsymbol{x}) + \mu_2 g_2(\boldsymbol{x}) + \mu_3 g_3(\boldsymbol{x}) \\ &= (x_1-2)^2 + (x_2-1)^2 + \mu_1(-x_2 + x_1^2) + \mu_2(-2 + x_1 + x_2) + \mu_3(-x_1) \end{aligned}$$

while the Lagrangian associated to the problem (N_1-max) is given by

$$\begin{aligned} L_{\max}(\boldsymbol{x}, \boldsymbol{\mu}) &= -f(\boldsymbol{x}) + \mu_1 g_1(\boldsymbol{x}) + \mu_2 g_2(\boldsymbol{x}) + \mu_3 g_3(\boldsymbol{x}) \\ &= -(x_1-2)^2 - (x_2-1)^2 + \mu_1(-x_2 + x_1^2) + \mu_2(-2 + x_1 + x_2) + \mu_3(-x_1). \end{aligned}$$

a. Note that for problem (N_1-min), we have that

$$D_{\boldsymbol{x}} L_{\min}(\boldsymbol{x}, \boldsymbol{\mu}) = \begin{bmatrix} 2(x_1-2) + 2\mu_1 x_1 + \mu_2 - \mu_3 \\ 2(x_2-1) + \mu_2 - \mu_1 \end{bmatrix}^\top,$$

while for the problem (N_1-max), we have that

$$D_{\boldsymbol{x}} L_{\max}(\boldsymbol{x}, \boldsymbol{\mu}) = \begin{bmatrix} -2(x_1-2) + 2\mu_1 x_1 + \mu_2 - \mu_3 \\ -2(x_2-1) + \mu_2 - \mu_1 \end{bmatrix}^\top.$$

The KKT-FONC for problem (N_1-min) then require that the following conditions hold

i a. $\mu_1, \mu_2, \mu_3 \geq 0$.

ii a. $2(x_1 - 2) + 2\mu_1 x_1 + \mu_2 - \mu_3 = 0$.

iii a. $2(x_2 - 1) + \mu_2 - \mu_1 = 0$.

iv a. $\mu_1 g_1(\boldsymbol{x}) + \mu_2 g_2(\boldsymbol{x}) + \mu_3 g_3(\boldsymbol{x}) = \mu_1(-x_2 + x_1^2) + \mu_2(-2 + x_1 + x_2) + \mu_3(-x_1) = 0$.

v a. $g_1(\boldsymbol{x}) = -x_2 + x_1^2 \leq 0$.

vi a. $g_2(\boldsymbol{x}) = -2 + x_1 + x_2 \leq 0$.

vii a. $g_2(\boldsymbol{x}) = -x_1 \leq 0$.

while the KKT-FONC for problem (N_1-max) require that the following similar conditions hold

i b. $\mu_1, \mu_2, \mu_3 \geq 0$.

ii b. $-2(x_1 - 2) + 2\mu_1 x_1 + \mu_2 - \mu_3 = 0$.

iii b. $-2(x_2 - 1) + \mu_2 - \mu_1 = 0$.

iv b. $\mu_1 g_1(\boldsymbol{x}) + \mu_2 g_2(\boldsymbol{x}) + \mu_3 g_3(\boldsymbol{x}) = \mu_1(-x_2 + x_1^2) + \mu_2(-2 + x_1 + x_2) + \mu_3(-x_1) = 0$.

v b. $g_1(\boldsymbol{x}) = -x_2 + x_1^2 \leq 0$.

vi b. $g_2(\boldsymbol{x}) = -2 + x_1 + x_2 \leq 0$.

vii b. $g_2(\boldsymbol{x}) = -x_1 \leq 0$.

Now suppose that $\boldsymbol{x}^* = [0, 0]^\top$. For both problems, since $\boldsymbol{x}^*$ is a regular point, conditions v a. - vii a. and v b. - vii b. are satisfied. Also, for both problems, since the constraint $g_2(\boldsymbol{x}^*)$ is inactive we have that by condition iv a. and iv b. that $\mu_2 = 0$.

For the problem (N_1-min), conditions ii a. and iii a. imply that $\mu_2 - \mu_3 = -\mu_3 = 4$ and $\mu_2 - \mu_1 = -\mu_1 = 2$ or that $\mu_1 = -2$, $\mu_2 = 0$, and $\mu_3 = -4$. However, this violates condition i a. so the point $\boldsymbol{x}^*$ does not satisfy the KKT-FONC for the problem (N_1-min).

For the problem (N_1-min), conditions ii a. and iii a. imply that $\mu_2 - \mu_3 = -\mu_3 = -4$ and $\mu_2 - \mu_1 = -\mu_1 = -2$ or that $\mu_1 = 2$, $\mu_2 = 0$, and $\mu_3 = 4$. Therefore, the vector $\boldsymbol{x}^* = [0, 0]^\top$ satisfies the KKT-FONC for the problem (N_1-max) with associated KKT multiplier $\boldsymbol{\mu}^* = [2, 0, 4]^\top$.

b. We now check to see if $\boldsymbol{x}^* = [0, 0]^\top$ satisfies the KKT-SOSC for the problem (N_1-max). Note that for $\boldsymbol{x}^* = [0, 0]^\top$, we have that

$$D_{\boldsymbol{x}}^2 L_{\max}(\boldsymbol{x}^*, \boldsymbol{\mu}^*) = \begin{bmatrix} -2 + 2\mu_1 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}.$$

We also see that for the vectors $\boldsymbol{x}^*$ and $\boldsymbol{\mu}^*$ defined above, $\{i \mid g_i(\boldsymbol{x}^*) = 0, \mu_i^* > 0\} = \{1, 3\}$, and that

$$\begin{aligned} \widetilde{T}(\boldsymbol{x}^*, \boldsymbol{\mu}^*) &= \{\boldsymbol{y} \in \mathbb{R}^2 \mid Dg_1(\boldsymbol{x}^*)\boldsymbol{y} = 0, Dg_3(\boldsymbol{x}^*)\boldsymbol{y} = 0\} \\ &= \{\boldsymbol{y} \in \mathbb{R}^2 \mid [0, -1]\boldsymbol{y} = 0, [-1, 0]\boldsymbol{y} = 0\} \\ &= \{\boldsymbol{0} \in \mathbb{R}^2\}. \end{aligned}$$

Therefore, we trivially have that the second condition in the KKT-SOSC is satisfied and $\boldsymbol{x}^* = [0, 0]^\top$ is a strict local maximizer.

□

Problem 4. Consider the problem with equality constraint

$$\begin{array}{rl}\text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}.\end{array}$$

We can convert the above into the equivalent optimization problem

$$\begin{array}{rl}\text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \frac{1}{2}\left\|\boldsymbol{h}(\boldsymbol{x})\right\|^2 \le 0.\end{array}$$

Write down the KKT condition for the equivalent problem and explain why the KKT theorem cannot be applied in this case.

Solution. Assume $f : \mathbb{R}^n \to \mathbb{R}$ and $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^m$ with $m \le n$. The Lagrangian associated to the equivalent problem is given by

$$\begin{aligned} L(\boldsymbol{x}, \boldsymbol{\mu}) &= f(\boldsymbol{x}) + \frac{1}{2}\boldsymbol{\mu}^\mathsf{T}\left\|\boldsymbol{h}(\boldsymbol{x})\right\|^2 \\ &= f(\boldsymbol{x}) + \frac{\mu_1}{2}h_1(\boldsymbol{x})^2 + \cdots + \frac{\mu_m}{2}h_m(\boldsymbol{x})^2. \end{aligned}$$

From this we readily see that

$$D_{\boldsymbol{x}}L(\boldsymbol{x}, \boldsymbol{\mu}) = \begin{bmatrix} \frac{\partial f(\boldsymbol{x})}{\partial x_1} + \mu_1 h_1(\boldsymbol{x})\frac{\partial h_1(\boldsymbol{x})}{\partial x_1} + \cdots + \mu_m h_m(\boldsymbol{x})\frac{\partial h_m(\boldsymbol{x})}{\partial x_1} \\ \frac{\partial f(\boldsymbol{x})}{\partial x_2} + \mu_1 h_1(\boldsymbol{x})\frac{\partial h_1(\boldsymbol{x})}{\partial x_2} + \cdots + \mu_m h_m(\boldsymbol{x})\frac{\partial h_m(\boldsymbol{x})}{\partial x_2} \\ \vdots \\ \frac{\partial f(\boldsymbol{x})}{\partial x_m} + \mu_1 h_1(\boldsymbol{x})\frac{\partial h_1(\boldsymbol{x})}{\partial x_m} + \cdots + \mu_m h_m(\boldsymbol{x})\frac{\partial h_m(\boldsymbol{x})}{\partial x_m} \end{bmatrix}^\mathsf{T} = \begin{bmatrix} \frac{\partial f(\boldsymbol{x})}{\partial x_1} + \sum_{i=1}^m \mu_i h_i(\boldsymbol{x})\frac{\partial h_i(\boldsymbol{x})}{\partial x_1} \\ \frac{\partial f(\boldsymbol{x})}{\partial x_2} + \sum_{i=1}^m \mu_i h_i(\boldsymbol{x})\frac{\partial h_i(\boldsymbol{x})}{\partial x_2} \\ \vdots \\ \frac{\partial f(\boldsymbol{x})}{\partial x_m} + \sum_{i=1}^m \mu_i h_i(\boldsymbol{x})\frac{\partial h_i(\boldsymbol{x})}{\partial x_m} \end{bmatrix}^\mathsf{T}.$$

The KKT condition for the equivalent problem can be stated as $\boldsymbol{x}^*$ is a local minimizer if

i. The functions $f, \boldsymbol{h} \in C^1$.

ii. The point $\boldsymbol{x}^*$ is a regular point.

iii. There exist $\boldsymbol{\mu}^* \in \mathbb{R}^m$ such that

  (a) $\boldsymbol{\mu}^* \ge \boldsymbol{0}$.

  (b) $D_{\boldsymbol{x}}L(\boldsymbol{x}^*, \boldsymbol{\mu}^*) = 0$.

  (c) $\boldsymbol{\mu}^{*\mathsf{T}}\frac{1}{2}\left\|\boldsymbol{h}(\boldsymbol{x}^*)\right\|^2 = \mu_1 h_1(\boldsymbol{x})^2 + \cdots + \mu_m h_m(\boldsymbol{x})^2 = 0$.

The KKT condition may not be applied here since no feasible point is also a regular point. To see why this is true, assume the point $\boldsymbol{x}$ is feasible. Then $(1/2)\left\|\boldsymbol{h}(\boldsymbol{x})\right\|^2 \le \boldsymbol{0}$ or

$$h_1(\boldsymbol{x})^2 + \cdots + h_m(\boldsymbol{x})^2 \le 0.$$

This implies that $h_i(\boldsymbol{x}) = 0$ for $1 \leq i \leq m$. Hence, the constraint is active for this problem. Note that

$$\nabla \frac{1}{2} \left\| \boldsymbol{h}(\boldsymbol{x}) \right\|^2 = \begin{bmatrix} h_1(\boldsymbol{x}) \frac{\partial h_1(\boldsymbol{x})}{\partial x_1} + \cdots + h_m(\boldsymbol{x}) \frac{\partial h_m(\boldsymbol{x})}{\partial x_1} \\ h_1(\boldsymbol{x}) \frac{\partial h_1(\boldsymbol{x})}{\partial x_2} + \cdots + h_m(\boldsymbol{x}) \frac{\partial h_m(\boldsymbol{x})}{\partial x_2} \\ \vdots \\ h_1(\boldsymbol{x}) \frac{\partial h_1(\boldsymbol{x})}{\partial x_m} + \cdots + h_m(\boldsymbol{x}) \frac{\partial h_m(\boldsymbol{x})}{\partial x_m} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^m h_i(\boldsymbol{x}) \frac{\partial h_i(\boldsymbol{x})}{\partial x_1} \\ \sum_{i=1}^m h_i(\boldsymbol{x}) \frac{\partial h_i(\boldsymbol{x})}{\partial x_2} \\ \vdots \\ \sum_{i=1}^m h_i(\boldsymbol{x}) \frac{\partial h_i(\boldsymbol{x})}{\partial x_m} \end{bmatrix}.$$

From this we clearly see that since $h_i(\boldsymbol{x}) = 0$ for $1 \leq i \leq m$, we have that $\nabla \frac{1}{2} \left\| \boldsymbol{h}(\boldsymbol{x}) \right\|^2 = \mathbf{0}$ or that the vector $\nabla \frac{1}{2} \left\| \boldsymbol{h}(\boldsymbol{x}) \right\|^2$ is linearly dependent. Therefore, no feasible point is regular and the KKT condition is not applicable. $\square$