# Homework Assignment 2

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**Problem 2.10.** Solve the Cauchy problem for the Klein-Gordon equation

$$\begin{aligned} u_{tt} - c^2 u_{xx} + a^2 u &= 0, \qquad -\infty < x < \infty, \quad t > 0, \\ u(x,0) &= f(x) \qquad \text{for } -\infty < x < \infty, \\ \left[\frac{\partial u}{\partial t}\right]_{t=0} &= g(x) \qquad \text{for } -\infty < x < \infty. \end{aligned}$$

*Solution.* Consider the function $u(x,y)$. The Fourier transform of $u$ with respect to $x$ is defined as

$$\mathscr{F}\left\{u(x,y)\right\} = U(k,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} u(x,y) dx. \tag{1}$$

From this definition and the Leibniz integral rule, we can see by induction that

$$\begin{aligned} \mathscr{F}\left\{\frac{\partial^n}{\partial y^n}\left[u(x,y)\right]\right\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^n}{\partial y^n}\left[u(x,y)\right] e^{-ikx} dx \\ &= \frac{d^n}{dy^n}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,y) e^{-ikx} dx\right] \\ &= \frac{d^n}{dy^n}\left[\mathscr{F}\left\{u(x,y)\right\}\right]. \end{aligned} \tag{2}$$

Similarly, we see from definition (1) and previous theorems regarding the Fourier transform that

$$\begin{aligned} \mathscr{F}\left\{\frac{\partial^n}{\partial x^n}\left[u(x,y)\right]\right\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^n}{\partial x^n}\left[u(x,y)\right] e^{-ikx} dx \\ &= (ik)^n \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,y) e^{-ikx} dx\right] \\ &= (ik)^n \mathscr{F}\left\{u(x,y)\right\}. \end{aligned} \tag{3}$$

Now, applying the Fourier transform to the first equation, we see that

$$\begin{aligned} \mathscr{F}\left\{u_{tt} - c^2 u_{xx} + a^2 u\right\} &= \frac{d^2}{dt^2}\left[U(k,t)\right] - c^2 (ik)^2 U(k,t) + a^2 U(k,t) \\ &= \frac{d^2}{dt^2}\left[U(k,t)\right] + \left(c^2 k^2 + a^2\right) U(k,t). \end{aligned}$$

Thus, setting $\omega^2 = c^2k^2 + a^2$, we see that

$$\frac{d^2}{dt^2}\left[U(k,t)\right] + \omega^2 U(k,t) = 0.$$

This is a second-order linear homogeneous ordinary differential equation, the solution to which we readily see is given by

$$U(k,t) = c_1 e^{-i\omega t} + c_2 e^{i\omega t}. \tag{4}$$

Applying the Fourier transform to the last two equations yields

$$\mathscr{F}\left\{u(x,0)\right\} = U(k,0) = F(k) = \mathscr{F}\left\{f(x)\right\}$$

and

$$\mathscr{F}\left\{\left[\frac{\partial u}{\partial t}\right]_{t=0}\right\} = \frac{d}{dt}\left[U(k,t)\right]_{t=0} = G(k) = \mathscr{F}\left\{g(x)\right\}.$$

Using (4), we see that the first equation reduces to

$$c_1 + c_2 = F(k).$$

Taking the derivative of $U(k,t)$ with respect to $t$ yields

$$\frac{d}{dt}\left[U(k,t)\right] = -i\omega c_1 e^{-i\omega t} + i\omega c_2 e^{i\omega t}$$

and evaluating when $t = 0$ produces a second equation

$$i\omega(c_2 - c_1) = G(k).$$

This results in a system of two equations in two unknowns; the solution of which is given by

$$c_1 = \frac{\omega F(k) + iG(k)}{2\omega}, \qquad c_2 = \frac{\omega F(k) - iG(k)}{2\omega}.$$

Therefore, (4) becomes

$$U(k,t) = \left(\frac{\omega F(k) + iG(k)}{2\omega}\right)e^{-i\omega t} + \left(\frac{\omega F(k) - iG(k)}{2\omega}\right)e^{i\omega t}.$$

Taking the Inverse Fourier transform yields that the solution to original differential equation is given by

$$\begin{aligned}
u(x,t) &= \mathscr{F}^{-1}\left\{\left(\frac{\omega F(k) + iG(k)}{2\omega}\right)e^{-i\omega t} + \left(\frac{\omega F(k) - iG(k)}{2\omega}\right)e^{i\omega t}\right\} \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\left(\frac{\omega F(k) + iG(k)}{2\omega}\right)e^{-i\omega t} + \left(\frac{\omega F(k) - iG(k)}{2\omega}\right)e^{i\omega t}\right]e^{ikx}dk \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{\omega F(k) + iG(k)}{2\omega}\right)e^{i(kx-\omega t)} + \left(\frac{\omega F(k) - iG(k)}{2\omega}\right)e^{i(kx+\omega t)}dk.
\end{aligned}$$

$\square$

**Problem 2.12.** Solve the equation

$$
\begin{gathered}
u_{tt} + u_{xxxx} = 0, \qquad -\infty < x < \infty, \quad t > 0 \\
u(x,0) = f(x), \quad u_t(x,0) = 0 \qquad \text{for } -\infty < x < \infty.
\end{gathered}
$$

*Solution.* □

**Problem 2.14.** Obtain the Fourier cosine transforms of the following functions:

a. $xe^{-ax}$, $a > 0$.

*Solution.* Recall that the definition of the Fourier cosine transform of a function $f(x)$ is given by

$$\mathscr{F}_c\left\{f(x)\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos kx f(x)dx.$$

a. From the definition of the Fourier cosine transform we have that

$$\mathscr{F}_c\left\{xe^{-ax}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty xe^{-ax}\cos kx dx.$$

Using the definition of the complex exponential, we see that

$$\begin{aligned}\mathscr{F}_c\left\{xe^{-ax}\right\} &= \sqrt{\frac{2}{\pi}}\int_0^\infty xe^{-ax}\left[\frac{e^{-ikx}+e^{ikx}}{2}\right]dx \\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty x\left[e^{-(a+ik)x}+e^{-(a-ik)x}\right]dx.\end{aligned}$$

Now, for $w = a \pm ik$, we see using integration by parts with $u = x$ and $dv = e^{-wx}dx$ that

$$\int_0^\infty xe^{-wx}dx = -\frac{xe^{-wx}}{w}\bigg|_0^\infty + \frac{1}{w}\int_0^\infty e^{-wx}dx.$$

Note that

$$\lim_{x\to\infty}\left|e^{-wx}\right| = \lim_{x\to\infty}\left|e^{-(a\pm ik)x}\right| = \lim_{x\to\infty}\left|e^{-ax}\right|\left|e^{\mp ikx}\right| \leq \lim_{x\to\infty}\left|e^{-ax}\right| = 0.$$

This implies that $\lim_{x\to\infty} e^{-wx} = 0$. Thus,

$$\begin{aligned}\int_0^\infty xe^{-wx}dx &= -\frac{xe^{-wx}}{w}\bigg|_0^\infty + \frac{1}{w}\int_0^\infty e^{-wx}dx \\ &= -\frac{1}{w^2}\left[e^{-wx}\big|_0^\infty\right] \\ &= \frac{1}{w^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathscr{F}_c\left\{xe^{-ax}\right\} &= \frac{1}{\sqrt{2\pi}}\left[\int_0^\infty xe^{-(a+ik)x}dx + \int_0^\infty xe^{-(a-ik)x}dx\right] \\ &= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{(a+ik)^2}+\frac{1}{(a-ik)^2}\right] \\ &= \sqrt{\frac{2}{\pi}}\frac{a^2-k^2}{(a^2+k^2)^2}.\end{aligned}$$

□

**Problem 2.15.** Find the Fourier sine transform of the following functions:

a. $xe^{-ax}$, $a > 0$.

b. $\dfrac{1}{x}e^{-ax}$, $a > 0$.

*Solution.* $\square$

**Problem 2.20.** Apply the Fourier cosine transform to find the solution $u(x, y)$ of the problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \qquad 0 < x < \infty, \quad 0 < y < \infty \\ u(x, 0) &= H(a - x), \qquad x < a \\ u_x(0, y) &= 0, \qquad 0 < x, \quad y < \infty. \end{aligned}$$

*Solution.* $\square$

**Problem 2.22.** Solve the diffusion equation in the semi-infinite line

$$u_t = \kappa u_x x, \qquad 0 \leq x < \infty, \quad t > 0,$$

with the boundary and initial data

$$\begin{aligned} u(0,t) &= 0 && \text{for } t > 0, \\ u(x,t) &\to 0 && \text{as } x \to \infty \text{ for } t > 0, \\ u(x,0) &= f(x) && \text{for } 0 < x < \infty. \end{aligned}$$

*Solution.* $\square$