A Numerical Solution to a Second Order Ordinary Differential Equation

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November 22, 2015

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1. Introduction

The authors were tasked by the client with finding the solution to the following family of differential equations

$$\begin{cases} -u''(x) + cu(x) = f(x) \\ 0 \leq x \leq 1 \\ u(0) = \epsilon \\ u(1) = \delta. \end{cases}$$

Additionally, the client has also requested to be provided with a means of plotting the solution once obtained.

Throughout this report, the above family of differential equations together with the interval of definition and initial conditions will be represented by $Lu = f$.

Assumptions were placed on this family so that $c \in \mathbb{R}$ with $c > 0$ and $f \in C^k([0,1])$ for sufficiently large k so that f is relatively well-behaved on the defined interval.

In this report we will detail the analytical solution to this family of differential equations showing the the above problem is well-posed and explain why this solution is not amenable to practical use. We therefore provide a numerical scheme to approximate the solution to the family of differential equations and examine the convergence, consistency and stability of the numerical scheme. Using the solution provided by the numerical scheme, we then explore the different options for plotting the solution.

2. Analytical Solution

The family of differential equations $Lu = f$ represents a second order linear differential equation and therefore well-known techniques can be used to find the solution $u(x)$.

The solution $u(x)$ is given by $u(x) = u_h(x) + u_p(x)$ where $u_h(x)$ is the solution to the homogeneous equation $-u''(x) + cu(x) = 0$ and $u_p(x)$ is a particular solution of $-u''(x) + cu(x) = f(x)$.

To find the homogeneous solution, note that the characteristic equation of this family of differential equations is given by $-m^2 + c = 0$ the roots of which are $m_1 = \sqrt{c} = \omega$ and $m_2 = -\sqrt{c} = -\omega$. Note that since $c > 0$, these roots are real and distinct suggesting that the homogeneous solution is given by

$$u_h(x) = c_1 e^{\omega x} + c_2 e^{-\omega x}. \tag{1}$$

To find the particular solution, we assume the particular solution is of the form $u_p(x) = \kappa(x)e^{\omega x}$ for some unknown function $\kappa(x)$. Thus,

$$u_p''(x) = \kappa''(x)e^{\omega x} + 2\omega\kappa'(x)e^{\omega x} + \omega^2\kappa(x)e^{\omega x},$$

and substituting the above into the original differential equation $Lu = f$ with $u_p(x) = \kappa(x)e^{\omega x}$ we have

$$\kappa''(x) + 2\omega\kappa'(x) = -f(x)e^{-\omega x}. \tag{2}$$

Making the substitution $\lambda(x) = \kappa'(x)$ into (2) we can reduce the above second order linear differential equation into the first order linear differential equation

$$\lambda'(x) + 2\omega\lambda(x) = -f(x)e^{-\omega x}. \tag{3}$$

The homogeneous solution to this first order differential equation is given by $\lambda_h(x) = c_3 e^{-2\omega x}$ suggesting the particular solution to the first order differential equation is of the form $\lambda_p(x) = \mu(x)e^{-2\omega x}$.

Repeating the same process as above, we see that

$$\lambda_p'(x) = \mu'(x)e^{-2\omega x} - 2\omega\mu(x)e^{-2\omega x}$$

and substituting into (3) with $\lambda_p(x) = \mu(x)e^{-2\omega x}$ we find that the first order linear differential equation becomes the separable first order differential equation

$$\mu'(x)e^{-2\omega x} = -f(x)e^{-\omega x}.$$

We readily see the solution to the above differential equation is given by

$$\mu(x) = -\int_0^x f(r)e^{\omega r}dr.$$

As $\kappa'(x) = \lambda_p(x) = \mu(x)e^{-2\omega x}$, we deduce that

$$\kappa(x) = -\int_0^x e^{-2\omega s}\left[\int_0^s f(r)e^{\omega r}dr\right]ds \tag{4}$$

and

$$u_p(x) = \kappa(x)e^{\omega x} = -e^{\omega x}\int_0^x e^{-2\omega s}\left[\int_0^s f(r)e^{\omega r}dr\right]ds. \tag{5}$$

Combining the homogeneous solution (1) and the particular solution (5) we have that the general solution to $Lu = f$ is given by

$$\begin{aligned} u(x) &= u_h(x) + u_p(x) \\ &= c_1 e^{\omega x} + c_2 e^{-\omega x} - e^{\omega x}\int_0^x e^{-2\omega s}\left[\int_0^s f(r)e^{\omega r}dr\right]ds. \end{aligned} \tag{6}$$

Using the boundary values provided in $Lu = f$, the general solution is specified by the system of linear equations

$$\begin{aligned} u(0) &= c_1 + c_2 = \epsilon \\ u(1) &= c_1 e^{\omega} + c_2 e^{-\omega} - e^{\omega}\int_0^1 e^{-2\omega s}\left[\int_0^s f(r)e^{\omega r}dr\right]ds = \delta. \end{aligned}$$

The solution to this system in terms of the unknowns c_1 and c_2 is given by

$$c_1 = \frac{\epsilon e^{-\omega} - \delta - e^{\omega} \int_0^1 e^{-2\omega s} \left[\int_0^s f(r) e^{\omega r} dr \right] ds}{e^{-\omega} - e^{\omega}}$$
$$c_2 = \frac{-\epsilon e^{-\omega} + \delta + e^{\omega} \int_0^1 e^{-2\omega s} \left[\int_0^s f(r) e^{\omega r} dr \right] ds}{e^{-\omega} - e^{\omega}}.$$

Using these constants in (6) gives us the unique analytical solution to the family of differential equation $Lu = f$. Furthermore, we deduce that the problem is in fact well-posed.

From this solution, we must make the following additional assumption on this problem:

- $f(x)$ must be integrable on the interval $[0, 1]$.

As the analytical solution depends on the symbolic integration of $f(x)$, we will be unable to use this solution for functions $f(x)$ in which the closed-form of the integral is not known.

3. Numerical Scheme

3.1. Description

3.2. Implementation

3.2.1. Discretized Solution

3.2.2. Plotting

4. Numerical Scheme Properties

4.1. Convergence

4.2. Consistency

4.3. Stability

5. Worked Example

A. Numerical Scheme Program