Homework Assignment 3

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Problem 1.4. Let $\{Z_t\}$ be a sequence of independent normal random variables, each with mean 0 and variance σ^2, and let a, b, c be constants. Which, if any, of the following processes are stationary? For each stationary process specify the mean and autocovariance function.

a. $X_t = a + bZ_t + cZ_{t-2}$

b. $X_t = Z_1 \cos(ct) + Z_2 \sin(ct)$

c. $X_t = Z_t \cos(ct) + Z_{t-1} \sin(ct)$

d. $X_t = a + bZ_0$

e. $X_t = Z_0 \cos(ct)$

f. $X_t = Z_t Z_{t-1}$

Solution. In the following we assume that $Z_t \sim N(0, \sigma^2)$.

a. The mean function of this process is given by

$$\begin{aligned}\mu_X(t) &= \mathrm{E}(X_t) \\ &= \mathrm{E}(a + bZ_t + cZ_{t-2}) \\ &= a + b\mathrm{E}(Z_t) + c\mathrm{E}(Z_{t-2}) = a\end{aligned}$$

where $\mathrm{E}(Z_u) = 0$ since $Z_u \sim N(0, \sigma^2)$.

Using the linearity of the covariance, the covariance function of this process is given by

$$\begin{aligned}\gamma_X(t+h, t) &= \mathrm{Cov}(X_{t+h}, X_t) \\ &= \mathrm{Cov}(a + bZ_{t+h} + cZ_{t+h-2}, a + bZ_t + cZ_{t-2}) \\ &= \mathrm{Cov}(bZ_{t+h}, a + bZ_t + cZ_{t-2}) + \mathrm{Cov}(cZ_{t+h-2}, a + bZ_t + cZ_{t-2}) \\ &= b^2\mathrm{Cov}(Z_{t+h}, Z_t) + bc\mathrm{Cov}(Z_{t+h}, Z_{t-2}) + bc\mathrm{Cov}(Z_{t+h-2}, Z_t) + c^2\mathrm{Cov}(Z_{t+h-2}, Z_{t-2}).\end{aligned}$$

The independence of the random variables Z_u shows us that the autocovariance function is a function of h and that

$$\gamma_X(h) = \begin{cases} (b^2 + c^2)\sigma^2 & \text{if } h = 0 \\ (bc)\sigma^2 & \text{if } h = \pm 2 \\ 0 & \text{otherwise} \end{cases}.$$

Since the mean function does not depend on t and the covariance function does not depend on t for each h, this process is stationary.

b. The mean function of this process is given by

$$\begin{aligned}
\mu_X(t) &= \mathrm{E}(X_t) \\
&= \mathrm{E}(Z_1 \cos(ct) + Z_2 \sin(ct)) \\
&= \mathrm{E}(Z_1 \cos(ct)) + \mathrm{E}(Z_2 \sin(ct)) \\
&= \cos(ct)\mathrm{E}(Z_1) + \sin(ct)\mathrm{E}(Z_2) = 0
\end{aligned}$$

where $\mathrm{E}(Z_u) = 0$ since $Z_u \sim N(0, \sigma^2)$.

Using the linearity of the covariance, the covariance function of this process is given by

$$\begin{aligned}
\gamma_X(t+h, t) &= \mathrm{Cov}(X_{t+h}, X_t) \\
&= \mathrm{Cov}(Z_1 \cos(c(t+h)) + Z_2 \sin(c(t+h)), Z_1 \cos(ct) + Z_2 \sin(ct)) \\
&= \cos(c(t+h)) \cos(ct)\mathrm{Cov}(Z_1, Z_1) + \cos(c(t+h)) \sin(ct)\mathrm{Cov}(Z_1, Z_2) \\
&\quad + \sin(c(t+h)) \cos(ct)\mathrm{Cov}(Z_2, Z_1) + \sin(c(t+h)) \sin(ct)\mathrm{Cov}(Z_2, Z_2) \\
&= \cos(c(t+h)) \cos(ct)\sigma^2 + \sin(c(t+h)) \sin(ct)\sigma^2 \\
&= \sigma^2(\cos^2(ct) \cos(ch) - \sin(ct) \sin(ch) \cos(ct) \\
&\quad + \sin^2(ct) \cos(ch) + \cos(ct) \sin(ch) \sin(ct)) \\
&= \sigma^2 \cos(ch)
\end{aligned}$$

due to the independence of the random variables. Since the mean function does not depend on t and the covariance function does not depend on t for each h, this process is stationary.

c. The mean function of this process is given by

$$\begin{aligned}
\mu_X(t) &= \mathrm{E}(X_t) \\
&= \mathrm{E}(Z_t \cos(ct) + Z_{t-1} \sin(ct)) \\
&= \mathrm{E}(Z_t \cos(ct)) + \mathrm{E}(Z_{t-1} \sin(ct)) \\
&= \cos(ct)\mathrm{E}(Z_t) + \sin(ct)\mathrm{E}(Z_{t-1}) = 0
\end{aligned}$$

where $\mathrm{E}(Z_u) = 0$ since $Z_u \sim N(0, \sigma^2)$.

Using the linearity of the covariance, the covariance function of this process is given by

$$\begin{aligned}
\gamma_X(t+h, t) &= \mathrm{Cov}(X_{t+h}, X_t) \\
&= \mathrm{Cov}(Z_{t+h} \cos(c(t+h)) + Z_{t+h-1} \sin(c(t+h)), Z_t \cos(ct) + Z_{t-1} \sin(ct)) \\
&= \cos(c(t+h)) \cos(ct)\mathrm{Cov}(Z_{t+h}, Z_t) + \cos(c(t+h)) \sin(ct)\mathrm{Cov}(Z_{t+h}, Z_{t-1}) \\
&\quad + \sin(c(t+h)) \cos(ct)\mathrm{Cov}(Z_{t+h-1}, Z_t) + \sin(c(t+h)) \sin(ct)\mathrm{Cov}(Z_{t+h-1}, Z_{t-1}).
\end{aligned}$$

The independence of the random variables shows that

$$\gamma_X(t+h,t) = \begin{cases} (\cos^2(ct) + sin^2(ct))\sigma^2 = \sigma^2 & \text{if } h=0 \\ \sin(c(t+1))\cos(ct)\sigma^2 & \text{if } h=1 \\ \cos(c(t-1))\sin(ct)\sigma^2 & \text{if } h=-1 \\ 0 & \text{otherwise} \end{cases}.$$

It is apparent that the covariance function depends on t so this process is not stationary.

d. The mean function of this process is given by

$$\begin{aligned} \mu_X(t) &= \mathrm{E}(X_t) \\ &= \mathrm{E}(a + bZ_0) = a + b\mathrm{E}(Z_0) = a \end{aligned}$$

where $\mathrm{E}(Z_0) = 0$ since $Z_0 \sim N(0, \sigma^2)$.

It is clear that the covariance function is given by

$$\begin{aligned} \gamma_X(t+h,t) &= \mathrm{Cov}(X_{t+h}, X_t) \\ &= \mathrm{Cov}(a + bZ_0, a + bZ_0) \\ &= \mathrm{Cov}(a, a + bZ_0) + \mathrm{Cov}(bZ_0, a + bZ_0) \\ &= \mathrm{Cov}(bZ_0, a) + \mathrm{Cov}(bZ_0, bZ_0) \\ &= b^2\mathrm{Cov}(Z_0, Z_0) = b^2\sigma^2. \end{aligned}$$

Therefore, the autocovariance function is given by $\gamma_X(h) = b^2\sigma^2$. As the covariance function does not depend on t for any h and the mean function does not depend on t, this process is stationary.

e. The mean function of this process is given by

$$\begin{aligned} \mu_X(t) &= \mathrm{E}(X_t) \\ &= \mathrm{E}(Z_0\cos(ct)) = \cos(ct)\mathrm{E}(Z_0) = 0 \end{aligned}$$

where $\mathrm{E}(Z_0) = 0$ since $Z_0 \sim N(0, \sigma^2)$.

The covariance function of this process is given by

$$\begin{aligned} \gamma_X(t+h,t) &= \mathrm{Cov}(X_{t+h}, X_t) \\ &= \mathrm{Cov}(Z_0\cos(c(t+h)), Z_0\cos(ct)) \\ &= \cos(c(t+h))\cos(ct)\mathrm{Cov}(Z_0, Z_0) = \cos(c(t+h))\cos(ct)\sigma^2. \end{aligned}$$

As the covariance function depends on t, this is not a stationary process.

f. The mean function of this process is given by

$$\begin{aligned} \mu_X(t) &= \mathrm{E}(X_t) \\ &= \mathrm{E}(Z_tZ_{t-1}) = \mathrm{E}(Z_t)\mathrm{E}(Z_{t-1}) = 0 \end{aligned}$$

where $\mathrm{E}(Z_u) = 0$ since $Z_u \sim N(0, \sigma^2)$ and $\mathrm{E}(Z_t Z_{t-1}) = \mathrm{E}(Z_t)\mathrm{E}(Z_{t-1})$ due to the independence of the random variables.

It is clear that the covariance function is given by

$$\begin{aligned}\gamma_X(t+h,t) &= \mathrm{Cov}(X_{t+h}, X_t) \\ &= \mathrm{E}(X_{t+h} - \mathrm{E}(X_{t+h}))\mathrm{E}(X_t - \mathrm{E}(X_t)) \\ &= \mathrm{E}(X_{t+h})\mathrm{E}(X_t) = \mu_X(t+h)\mu_X(t) = 0.\end{aligned}$$

Therefore the autocovariance function $\gamma_X(h) = 0$. As the mean function does not depend on t and the autocovariance function does not depend on t for any h, this process is stationary.

□

Problem 1.5. Let $\{X_t\}$ be the moving-average process of order 2 given by

$$X_t = Z_t + \theta Z_{t-2}$$

where $\{Z_t\}$ is $\mathrm{WN}(0,1)$.

a. Find the autocovariance and autocorrelation functions for this process when $\theta = 0.8$.

b. Compute the variance of the sample mean $(X_1 + X_2 + X_3 + X_4)/4$ when $\theta = 0.8$.

c. Repeat (b) when $\theta = -0.8$ and compare your answer with the result obtained in (b).

Solution. For the following, let $\{Z_t\}$ be $\mathrm{WN}(0,1)$.

a. The covariance function for this process for any θ is given by

$$\begin{aligned}\gamma_X(t+h,t) &= \mathrm{Cov}(X_{t+h}, X_t) \\ &= \mathrm{Cov}(Z_{t+h} + \theta Z_{t+h-2}, Z_t + \theta Z_{t-2}) \\ &= \mathrm{Cov}(Z_{t+h}, Z_t) + \theta \mathrm{Cov}(Z_{t+h}, Z_{t-2}) + \theta \mathrm{Cov}(Z_{t+h-2}, Z_t) + \theta^2 \mathrm{Cov}(Z_{t+h-2}, Z_{t-2})\end{aligned}$$

Since $\{Z_t\}$ is $\mathrm{WN}(0,1)$, the random variables are independent and the autocovariance function is given by

$$\gamma_X(h) = \begin{cases} 1+\theta^2 & \text{if } h = 0 \\ \theta & \text{if } h = \pm 2 \\ 0 & \text{otherwise} \end{cases}$$

Knowing the autocorrelation function, we know that the autocovariance function is

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = \begin{cases} 1 & \text{if } h = 0 \\ \frac{\theta}{1+\theta^2} & \text{if } h = \pm 2 \\ 0 & \text{otherwise} \end{cases}$$

Substituting $\theta = 0.8$ will reveal the desired autocovariance and autocorrelation functions.

b. Let the sample mean be defined as $\bar{x} = (X_1 + X_2 + \cdots + X_n)/n$. Then the variance of $\bar{x}$ is given by

$$\begin{aligned}
\text{Var}(\bar{x}) &= \text{Cov}(\bar{x}, \bar{x}) \\
&= \frac{1}{n^2}\text{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\text{Cov}(X_i, X_j) \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_X(i, j)
\end{aligned}$$

where

$$\gamma_X(i,j) = \begin{cases} 1+\theta^2 & \text{if } i = j \\ \theta & \text{if } i = j+2 \text{ or } i = j-2 \\ 0 & \text{otherwise} \end{cases}.$$

Using this covariance function we know that $\gamma_X(i,j) = 0$ if $i \neq j$ or i does not differ from j by 2, so that we can partition the sum as

$$\begin{aligned}
\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_X(i,j) &= \frac{1}{n^2}\left(\sum_{k=1}^{n}\gamma_X(k,k) + \sum_{k=1}^{n-2}\gamma_X(k,k+2) + \sum_{k=3}^{n}\gamma_X(k,k-2)\right) \\
&= \frac{1}{n^2}\left(n(1+\theta^2) + (n-2)\theta + (n-2)\theta\right) \\
&= \frac{n(1+\theta^2) + 2(n-2)\theta}{n^2}
\end{aligned}$$

Therefore, $\text{Var}(\bar{x}) = (n(1+\theta^2) + 2(n-2)\theta)/n^2$. As we wish to know the variance of the sample mean $\bar{x} = (X_1 + X_2 + X_3 + X_4)/4$, we can replace n with 4 and θ with 0.8 so that $\text{Var}(\bar{x}) = 0.61$.

c. Using the formula derived in the previous problem with $\theta = -0.8$ and $n = 4$, it is easy to see that $\text{Var}(\bar{x}) = 0.21$.

$\square$

Problem 1.6. Let $\{X_t\}$ be the AR(1) process defined in Example 1.4.5.

a. Compute the variance of the sample mean $(X_1 + X_2 + X_3 + X_4)/4$ when $\phi = 0.9$ and $\sigma^2 = 1$.

b. Repeat (a) when $\phi = -0.9$ and compare your answer with the result obtained in (a).

Solution. a. Let the sample mean be defined as $\bar{x} = (X_1 + X_2 + \cdots + X_n)/n$. We know that autocovariance function is given by $\gamma_X(h) = (\sigma^2\phi^{|h|})/(1-\phi^2)$ using Example 1.4.5. Note that,

$$\begin{aligned}
\mathrm{Var}(\bar{x}) &= \mathrm{Cov}(\bar{x}, \bar{x}) \\
&= \frac{1}{n^2}\mathrm{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_X(i,j) \\
&= \frac{1}{n^2}\sum_{h=-n}^{n}(n-|h|)\gamma_X(h) \\
&= \frac{1}{n}\left[\gamma_X(0) + 2\sum_{h=1}^{n}\left(1-\frac{h}{n}\right)\gamma_X(h)\right].
\end{aligned}$$

Using the autocovariance function $\gamma_X(h)$, we can see that

$$\begin{aligned}
\mathrm{Var}(\bar{x}) &= \frac{1}{n}\left[\gamma_X(0) + 2\sum_{h=1}^{n}\left(1-\frac{h}{n}\right)\gamma_X(h)\right] \\
&= \frac{1}{n}\left[\frac{\sigma^2}{1-\phi^2} + \frac{2\sigma^2}{1-\phi^2}\sum_{h=1}^{n}\left(1-\frac{h}{n}\right)\phi^h\right] \\
&= \frac{\sigma^2}{n(1-\phi^2)}\left[1 + 2\left(\sum_{h=1}^{n}\phi^h - \frac{1}{n}\sum_{h=1}^{n}h\phi^h\right)\right] \\
&= \frac{\sigma^2}{n(1-\phi^2)}\left[1 + 2\left(\frac{1-\phi^n}{1-\phi} - \frac{(n\phi-n-1)\phi^{n+1}+\phi}{n(1-\phi)^2}\right)\right]
\end{aligned}$$

Now, when $\phi = 0.9$ and $\sigma^2 = 1$, we set $n = 4$ and $\mathrm{Var}(\bar{x}) = 5.3366$.

b. Using the formula above with $\phi = -0.9$ and $\sigma^2 = 1$, it is clear that for sample mean $\bar{x} = (X_1 + X_2 + X_3 + X_4)/4$, $\mathrm{Var}(\bar{x}) = -0.0148$.

□

Problem 1.7. If $\{X_t\}$ and $\{Y_t\}$ are uncorrelated stationary sequences, i.e., if X_r and Y_s are uncorrelated for every r and s, show that $\{X_t + Y_t\}$ is stationary with autocovariance function equal to the sum of the autocovariance functions of $\{X_t\}$ and $\{Y_t\}$.

Solution. Define $\{Z_t\} = X_t + Y_t$. We wish to prove that $\{Z_t\}$ is a stationary process. If $\mu_X(t)$ and $\mu_Y(t)$ are the mean functions of $\{X_t\}$ and $\{Y_t\}$, respectively, then the mean function of this new process is

$$\begin{aligned}
\mu_Z(t) &= \mathrm{E}(Z_t) = \mathrm{E}(X_t + Y_t) \\
&= \mathrm{E}(X_t) + \mathrm{E}(Y_t) = \mu_X(t) + \mu_Y(t).
\end{aligned}$$

Since $\{X_t\}$ and $\{Y_t\}$ are stationary processes, $\mu_X(t)$ and $\mu_Y(t)$ do not depend on t, thus their sum does not depend on t. Hence, the mean function of $\{Z_t\}$ does not depend on t.

If $\gamma_X(h)$ and $\gamma_Y(h)$ are the autocovariance functions of $\{X_t\}$ and $\{Y_t\}$, respectively, then the covariance function of $\{Z_t\}$ is

$$\begin{aligned}\gamma_Z(t+h,t) &= \operatorname{Cov}(Z_{t+h}, Z_t)\\ &= \operatorname{Cov}(X_{t+h} + Y_{t+h}, X_t + Y_t)\\ &= \operatorname{Cov}(X_{t+h}, X_t + Y_t) + \operatorname{Cov}(Y_{t+h}, X_t + Y_t)\\ &= \operatorname{Cov}(X_{t+h}, X_t) + \operatorname{Cov}(X_{t+h}, Y_t) + \operatorname{Cov}(Y_{t+h}, X_t) + \operatorname{Cov}(Y_{t+h}, Y_t)\\ &= \operatorname{Cov}(X_{t+h}, X_t) + \operatorname{Cov}(Y_{t+h}, Y_t) = \gamma_X(h) + \gamma_Y(h)\end{aligned}$$

due to the fact that X_r and Y_s are uncorrelated for any r or s. Since $\{X_t\}$ and $\{Y_t\}$ are stationary processes, $\gamma_X(h)$ and $\gamma_Y(h)$ do not depend on t for any h, thus their sum does not depend on t for any h. Hence, the covariance function of $\{Z_t\}$ does not depend on t for any h and the autocovariance function is $\gamma_Z(h) = \gamma_X(h) + \gamma_Y(h)$.

Since the mean function of $\{Z_t\}$ does not depend on t and the autocovariance function does not depend on t for any h, this process is stationary. $\square$