Homework Assignment 4

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Problem 1. Find the first three terms in the asymptotic expansions of $x \to 0^+$ of the following integrals:

$$\int_x^1 \cos(xt)dt, \qquad \int_0^{1/x} e^{-t^2} dt.$$

Solution. If the function $f(t, x)$ possesses the asymptotic expansion

$$f(t,x) \sim \sum_{n=0}^{\infty} f_n(t)(x - x_0)^{\alpha n} \qquad \text{as } x \to x_0$$

for some $\alpha > 0$, uniformly for $a \leq t \leq b$, then the asymptotic expansion of the integral

$$I(x) = \int_a^b f(t,x)dt$$

as $x \to x_0$ is given by

$$I(x) \sim \sum_{n=0}^{\infty} (x - x_0)^{\alpha n} \int_a^b f_n(t)dt \qquad \text{as } x \to x_0.$$

We begin with finding the first three terms of the asymptotic expansion of the integral

$$I_1(x) = \int_x^1 \cos(xt)dt \qquad \text{as } x \to 0^+.$$

Note that $f(t, x) = \cos(xt)$ has the following asymptotic expansion as $x \to 0^+$:

$$f(t,x) = \cos(xt) \sim 1 - \frac{t^2x^2}{2} + \frac{t^4x^4}{24}.$$

This expansion converges uniformly for all $x \leq t \leq 1$ as $x \to 0^+$. Therefore, we have that the first three terms of the asymptotic expansion of $I_1(x)$ as $x \to 0^+$ are given by

$$I_1(x) \sim \int_x^1 dt - \frac{x^2}{2}\int_x^1 t^2dt + \frac{x^4}{24}\int_x^1 t^4dt = (1 - x) - \frac{x^2}{2}\left[\frac{1 - x^3}{3}\right] + \frac{x^4}{24}\left[\frac{1 - x^5}{5}\right].$$

Similar to what was shown above, we have that if

$$f(t,x) \sim f_0(t) \qquad \text{as } x \to x_0$$

uniformly for $a \leq t \leq b$, then the asymptotic expansion of the integral is given by

$$I(x) = \int_a^b f(t,x)dt \sim \int_a^b f_0(t)dt \qquad \text{as } x \to x_0.$$

Let us continue by finding the first three terms of the asymptotic expansion of the integral

$$I_2(x) = \int_0^{1/x} e^{-t^2} dt \qquad \text{as } x \to 0^+.$$

Note that $f(t,x) = e^{-t^2}$ has the following asymptotic expansion as $x \to 0^+$:

$$f(t,x) = e^{-t^2} \sim 1 - t^2 + \frac{t^4}{2}.$$

This expansion converges uniformly for all finite points, so it converges uniformly for $0 \leq t \leq 1/x$ as $x \to 0^+$. Therefore, we may integrate the expansion term by term and we have that the first three terms of the asymptotic expansion of $I_2(x)$ as $x \to 0^+$ are given by

$$I_2(x) \sim \int_0^{1/x} dt - \int_0^{1/x} t^2 dt + \frac{1}{2}\int_0^{1/x} t^4 dt = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{10x^5}.$$

$\square$

Problem 2. Find the full asymptotic behavior as $x \to 0^+$ of the following integral:

$$\int_0^1 \frac{e^{-t}}{1+x^2t^3}dt$$

Solution. Note that the function $f(t,x) = e^{-t}/(1+x^2t^3)$ has the asymptotic expansion

$$f(t,x) = \frac{e^{-t}}{1+x^2t^3} \sim e^{-t}\sum_{n=0}^{\infty}\left[(-1)^n t^{3n}\right]x^{2n} \qquad \text{as } x \to 0^+.$$

Note that this asymptotic expansion converges uniformly for $0 \le x \le t < 1-\epsilon$ for all $\epsilon > 0$. To see this, we note that for $0 < m < n$, we have that

$$\left|\sum_{k=m+1}^{n}(-1)^k(x^2t^3)^k\right| < \sum_{k=m+1}^{n}(1-\epsilon)^{5k}.$$

Since $(1-\epsilon)^5 < 1$, we have that its geometric series converges and we can make it as small as we wish. Thus, by the Cauchy criterion we have uniform convergence for $0 \le x \le t < 1-\epsilon$ for all $\epsilon > 0$.

Per the discussion in Problem 1, using this uniformly convergent asymptotic expansion, we have that as $x \to 0^+$

$$\int_0^1 \frac{e^{-t}}{1+x^2t^3}dt \sim \sum_{n=0}^{\infty}(-1)^n x^{2n}\int_0^1 e^{-t}t^{3n}dt = \sum_{n=0}^{\infty}(-1)^n x^{2n}\left[\Gamma(3n+1) - \Gamma(3n+1,1)\right]$$

where $\Gamma(a,k) = \int_k^\infty t^{a-1}e^{-t}dt$. □

Problem 3. Find the full asymptotic expansion of $\int_0^x \mathrm{Bi}(t)dt$ as $x \to +\infty$.

Solution. Note that for $x \to +\infty$, the integral above can be written as

$$\int_0^x \mathrm{Bi}(t)dt = \int_0^1 \mathrm{Bi}(t)dt + \int_1^x \mathrm{Bi}(t)dt \tag{1}$$

Thus, the asymptotic expansion of the integral depends only on the second integral on the right. The Airy function $\mathrm{Bi}(t)$ satisfies the differential equation $y'' = ty$. Using this differential equation and integrating the integral on the right by parts we see that

$$\begin{aligned}\int_1^x \mathrm{Bi}(t)dt &= \int_1^x \frac{1}{t}\mathrm{Bi}''(t)dt \\ &= \frac{1}{x}\mathrm{Bi}'(x) - \mathrm{Bi}'(1) + \int_1^x \frac{1}{t^2}\mathrm{Bi}'(t)dt.\end{aligned}$$

Note that it is clear that as $x \to +\infty$ the following relations hold

$$\begin{aligned}\mathrm{Bi}'(1) &\ll \frac{1}{x}\mathrm{Bi}'(x) \\ \int_0^1 \mathrm{Bi}(t)dt &\ll \int_0^x \mathrm{Bi}(t)dt.\end{aligned}$$

Thus, from equation (1) and the above relations, we have that as $x \to +\infty$

$$\int_0^x \mathrm{Bi}(t)dt \sim \frac{1}{x}\mathrm{Bi}'(x) + \int_1^x \frac{1}{t^2}\mathrm{Bi}'(t)dt. \tag{2}$$

However, upon further investigation we see that as $x \to +\infty$

$$\int_1^x \frac{1}{t^2}\mathrm{Bi}'(t)dt \ll \frac{1}{x}\mathrm{Bi}'(x). \tag{3}$$

To see that this is true, we integrate the integral on the left by parts which yields

$$f(x) = \int_1^x \frac{1}{t^2}\mathrm{Bi}'(t)dt = x^{-2}\mathrm{Bi}(x) - \mathrm{Bi}(1) + 2\int_1^x t^{-3}\mathrm{Bi}(t)dt.$$

In comparing the function $f(x)$ with the function $g(x) = x^{-1}\mathrm{Bi}'(x)$ as $x \to +\infty$, we see that

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \frac{+\infty}{+\infty}$$

an indeterminate form. Thus, applying L'Hôpital's rule, we see that derivatives of $f(x)$ and $g(x)$ are

$$\begin{aligned}f'(x) &= -2x^{-3}\mathrm{Bi}(x) + x^{-2}\mathrm{Bi}'(x) + 2\left[x^{-3}\mathrm{Bi}(x) - \mathrm{Bi}(1)\right] \\ &= x^{-2}\mathrm{Bi}'(x) - 2\mathrm{Bi}(1) \\ g'(x) &= -x^{-2}\mathrm{Bi}'(x) + x^{-1}\mathrm{Bi}''(x)\end{aligned}$$

and that

$$\lim_{x\to+\infty}\frac{f'(x)}{g'(x)} = \frac{x^{-2}\mathrm{Bi}'(x) - 2\mathrm{Bi}(1)}{-x^{-2}\mathrm{Bi}'(x) + x^{-1}\mathrm{Bi}''(x)} = \frac{1}{1+\frac{x^{-1}\mathrm{Bi}''(x)}{x^{-2}\mathrm{Bi}'(x)}} = 0$$

Therefore, we must have that relation (3) is true and that relation (2) reduces to

$$\int_0^x \mathrm{Bi}(t)dt \sim \frac{1}{x}\mathrm{Bi}'(x) \qquad (x\to+\infty).$$

Note that the asymptotic expansion of $\mathrm{Bi}(x)$ as $x\to+\infty$ is given by

$$\mathrm{Bi}(x) \sim \pi^{-1/2}x^{-1/4}\exp\left(\frac{2x^{3/2}}{3}\right)\sum_{n=0}^{\infty}c_n x^{-3n/2}$$

where

$$c_n = \frac{1}{2\pi}\left(\frac{3}{4}\right)^n\frac{\Gamma(n+5/6)\Gamma(n+1/6)}{n!}.$$

Thus, we see that as $x\to+\infty$,

$$\begin{aligned}\mathrm{Bi}'(x) &\sim \pi^{-1/2}\exp\left(\frac{2x^{3/2}}{3}\right)\left[\left(x^{3/2}-\frac{1}{4}\right)x^{-5/4}\sum_{n=0}^{\infty}c_n x^{-3n/2} + x^{-1/4}\sum_{n=0}^{\infty}\frac{-3nc_n}{2}x^{-3n/2-1}\right]\\ &= \pi^{-1/2}\exp\left(\frac{2x^{3/2}}{3}\right)\left(x^{3/2}+\frac{3}{4}\right)\sum_{n=0}^{\infty}\left(1-\frac{3n}{2}\right)c_n x^{-3n/2-5/4}.\end{aligned}$$

Therefore, we can readily see that the full asymptotic behavior as $x\to+\infty$ of the integral of the problem is given by

$$\int_0^x \mathrm{Bi}(t)dt \sim \frac{\mathrm{Bi}'(x)}{x} \sim \pi^{-1/2}\exp\left(\frac{2x^{3/2}}{3}\right)\left(x^{3/2}+\frac{3}{4}\right)\sum_{n=0}^{\infty}\left(1-\frac{3n}{2}\right)c_n x^{-3n/2-9/4}.$$

$\square$

Problem 4. Find the first five terms in the asymptotic expansion as $x \to +\infty$ of the integral

$$\int_0^{\pi/4} e^{-xt^2} \sqrt{\tan t} dt$$

a. by using a suitable change of variables and then applying Watson's lemma.

b. by applying Laplace's method directly to the given integral.

Solution. a. Watson's lemma provides a formula for an asymptotic expansion as $x \to +\infty$ for integrals of the form

$$I(x) = \int_0^b f(s)e^{-xs} ds \qquad b > 0 \tag{4}$$

where the function $f(s)$ is continuous on the interval $0 \leq s \leq b$ and has the asymptotic expansion

$$f(s) \sim s^\alpha \sum_{n=0}^{\infty} a_n s^{\beta n} \qquad (s \to 0^+)$$

with $\alpha > -1$ and $\beta > 0$. Given these assumptions, Watson's lemma states that

$$I(x) \sim \sum_{n=0}^{\infty} \frac{a_n \Gamma(\alpha + \beta n + 1)}{x^{\alpha+\beta n+1}} \qquad (x \to +\infty). \tag{5}$$

For the integral

$$I(x) = \int_0^{\pi/4} e^{-xt^2} \sqrt{\tan t} dt,$$

we proceed by making the change of variables $s = t^2$. The integral then becomes

$$I(x) = \int_0^{\sqrt{\pi}/2} 2^{-1} s^{-1/2} \sqrt{\tan s^{1/2}} e^{-xs} ds.$$

Identifying the function $f(s) = 2^{-1}s^{-1/2}\sqrt{\tan s^{1/2}}$, we see that the above integral is of the form (4) with $f(s)$ being continuous on $0 \leq s \leq \sqrt{\pi}/2$. Further, the function $f(s)$ has the following asymptotic expansion

$$f(s) \sim \frac{1}{2}s^{-1/4} + \frac{1}{12}s^{3/4} + \frac{19}{720}s^{7/4} + \frac{55}{6048}s^{11/4} + \frac{11813}{3628800}s^{15/4} \qquad (s \to 0^+).$$

Therefore, identifying $\alpha = -1/4$ and $\beta = 1$, we see that by Watson's lemma the first five terms in the asymptotic expansion of $I(x)$ as $x \to +\infty$ is given by

$$I(x) \sim \frac{\Gamma\left(\frac{3}{4}\right)}{2}x^{-3/4} + \frac{\Gamma\left(\frac{7}{4}\right)}{12}x^{-7/4} + \frac{19\Gamma\left(\frac{11}{4}\right)}{720}x^{-11/4} + \frac{55\Gamma\left(\frac{15}{4}\right)}{6048}x^{-15/4} + \frac{11813\Gamma\left(\frac{19}{4}\right)}{3628800}x^{-19/4}.$$

b. Laplace's method states that, as $x \to +\infty$, for an integral of the form

$$I(x) = \int_a^b f(t)e^{x\phi(t)}dt$$

where $f(t)$ and $\phi(t)$ are real continuous functions, the integral $I(x)$ is asymptotic to the integral of $f(t)e^{x\phi(t)}$ over some small neighborhood of the point where $\phi(t)$ obtains its maximum over the interval $[a, b]$.

Identifying the function $f(t) = \sqrt{\tan t}$ and $\phi(t) = -t^2$, both real and continuous on the interval $[0, \pi/4]$, we see that $\phi(t)$ obtains its maximum at the point $t = 0$ on the same interval. However the function $f(t)$ vanishes at $t = 0$. Nevertheless Laplace's method may still be used since any contribution to the integral outside of the interval $[0, \epsilon]$ is subdominant for any $\epsilon > 0$. Thus, all of the assumptions of Laplace's method are satisfied and we have that for small $\epsilon > 0$,

$$I(x) = \int_0^{\pi/4} e^{-xt^2}\sqrt{\tan t}dt \sim \int_0^{\epsilon} e^{-xt^2}\sqrt{\tan t}dt \qquad (x \to +\infty).$$

Since $\epsilon > 0$ is small, we may replace the function $f(t)$ with the asymptotic expansion about $t = 0$

$$\sqrt{\tan t} \sim t^{1/2} + \frac{1}{6}t^{5/2} + \frac{19}{360}t^{9/2} + \frac{55}{3024}t^{13/2} + \frac{11813}{1814400}t^{17/2} \qquad (t \to 0^+)$$

so that, as $x \to +\infty$, the first five terms in the asymptotic expansion of the integral are

$$\begin{aligned} I(x) &\sim \int_0^{\epsilon} \left[t^{1/2} + \frac{1}{6}t^{5/2} + \frac{19}{360}t^{9/2} + \frac{55}{3024}t^{13/2} + \frac{11813}{1814400}t^{17/2}\right] e^{-xt^2}dt \\ &\sim \int_0^{\infty} \left[t^{1/2} + \frac{1}{6}t^{5/2} + \frac{19}{360}t^{9/2} + \frac{55}{3024}t^{13/2} + \frac{11813}{1814400}t^{17/2}\right] e^{-xt^2}dt \\ &= \frac{\Gamma\left(\frac{3}{4}\right)}{2}x^{-3/4} + \frac{\Gamma\left(\frac{7}{4}\right)}{12}x^{-7/4} + \frac{19\Gamma\left(\frac{11}{4}\right)}{720}x^{-11/4} + \frac{55\Gamma\left(\frac{15}{4}\right)}{6048}x^{-15/4} + \frac{11813\Gamma\left(\frac{19}{4}\right)}{3628800}x^{-19/4}. \end{aligned}$$

$\square$

Problem 5. Use Laplace's method of moving maxima to obtain the first two terms in the asymptotic expansion as $x \to +\infty$ of the integral

$$\int_0^\infty \exp\left[-t - \frac{x}{\sqrt{t}}\right] dt. \tag{6}$$

Solution. Identifying $f(t) = e^{-t}$ and $\phi(t) = -1/\sqrt{t}$, the integral (6) is of the form needed to apply Laplace's method. However, the maximum of $\phi(t)$ over the interval $[0, \infty)$ is in fact ∞ so Laplace's method is not directly applicable. As $t \to \infty$, the function $f(t)$ vanishes exponentially, suggesting we instead look for the maximum of $g(t) = \exp\left[-t - \frac{x}{\sqrt{t}}\right]$ over the non-negative real line.

The maximum of $g(t)$ occurs when $g'(t) = 0$ or when $\frac{x}{2t^{3/2}} - 1 = 0$, i.e. at the point $t = (x/2)^{2/3}$. This point is a movable maximum which suggests we make the change of variables $t = s(x/2)^{2/3}$ in the original integral. Doing so yields the integral

$$\begin{aligned} I(x) &= \left(\frac{x}{2}\right)^{2/3} \int_0^\infty \exp\left[-s\left(\frac{x}{2}\right)^{2/3} - \frac{x}{s^{1/2}\left(\frac{x}{2}\right)^{1/3}}\right] ds \\ &= \left(\frac{x}{2}\right)^{2/3} \int_0^\infty \exp\left[\left(-2^{-2/3}s - 2^{1/3}s^{-1/2}\right)x^{2/3}\right] ds \end{aligned}$$

which is in the form needed to apply Laplace's method. Identifying the functions $f(s) = 1$ and $\phi(s) = -2^{-2/3}s - 2^{1/3}s^{-1/2}$, we see that $\phi(s)$ is maximal when $s = 1$ so that it is only in a small neighborhood of this point that contributes to the integral. Thus, for small $\epsilon > 0$,

$$\begin{aligned} I(x) &\sim \left(\frac{x}{2}\right)^{2/3} \int_{1-\epsilon}^{1+\epsilon} \exp\left[\left(-2^{-2/3}s - 2^{1/3}s^{-1/2}\right)x^{2/3}\right] ds \qquad (x \to +\infty) \\ &\sim \left(\frac{x}{2}\right)^{2/3} \int_{1-\epsilon}^{1+\epsilon} \exp\left[\left(-\frac{3}{2^{2/3}} - \frac{3(s-1)^2}{2 \cdot 2^{5/3}} + \frac{15(s-1)^3}{6 \cdot 2^{8/3}} - \frac{105(s-1)^4}{24 \cdot 2^{11/3}}\right)x^{2/3}\right] ds \qquad (x \to +\infty). \end{aligned}$$

where we have replaced $\phi(s)$ with the approximation

$$\phi(s) \sim \phi(1) + \frac{\phi''(1)(s-1)^2}{2} + \frac{\phi^{(3)}(1)(s-1)^3}{6} + \frac{\phi^{(4)}(1)(s-1)^4}{24} \qquad (x \to +\infty).$$

$\square$

Problem 6. Let $f(x,t)$ be differentiable in x and continuous in (x,t) on $I \times J$, where I and J are intervals, and suppose that there exist functions $g(t)$ and $g_1(t)$ that are integrable on J such that for all $(x,t) \in I \times J$ we have that

$$|f(x,t)| \leq g(t) \quad \text{and} \quad |\partial_x f(x,t)| \leq g_1(t).$$

Then

$$\frac{d}{dx} \int_J f(x,t)dt = \int_J \partial_x f(x,t)dt.$$

a. Let $0 < a < b < \infty$. Use the above theorem to show that if $x \in (a,b)$, then

$$\frac{d^3}{dx^3} \int_0^\infty \exp\left[-t - \frac{x}{\sqrt{t}}\right] dt = -\int_0^\infty t^{-3/2} \exp\left[-t - \frac{x}{\sqrt{t}}\right] dt.$$

b. Use integration by parts to show that

$$\int_0^\infty \exp\left[-t - \frac{x}{\sqrt{t}}\right] dt = \frac{x}{2} \int_0^\infty t^{-3/2} \exp\left[-t - \frac{x}{\sqrt{t}}\right] dt.$$

c. Combine parts (a) and (b) to prove that integral (6) is a solution of the differential equation $xy''' + 2y = 0$ that also satisfies the initial condition $y(0) = 1$. Then use integration by parts to give an easy direct proof that the integral also satisfies the condition $y(+\infty) = 0$.

Solution. □

Problem 7. a. Find the leading behavior as $x \to +\infty$ of Laplace integrals of the form

$$\int_a^b (t-a)^\alpha g(t) e^{x\phi(t)} dt$$

where $\phi(t)$ has a maximum at $t = a$, $g(a) = 1$. Suppose further that $\alpha > -1$ and $\phi'(a) < 0$.

b. Repeat the analysis of part (a) when $\alpha > -1$ and $\phi'(a) = \phi''(a) = \cdots = \phi^{(p-1)}(a) = 0$ and $\phi^{(p)}(a) < 0$.

Solution. □