

Homework Assignment 2

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Problem 1. Use the method of variation of parameters to find the general solution of

$$y'' + 2y' + 2y = \sin x.$$

Solution. Suppose that $Ly = y'' + 2y' + 2y$. The general solution to $Ly = \sin x$ is given by $y = y_0 + y_h$ where y_0 is a particular solution of $Ly = \sin x$ and y_h is the solution to the homogeneous equation $Ly = 0$.

The characteristic equation of the equation $Ly = 0$ is $m(x) = x^2 + 2x + 2$, the roots of which are $m_1 = -1 - i$ and $m_2 = -1 + i$. As the roots of the characteristic equation are complex, the solution to $Ly = 0$ is given by

$$y_h = c_1 e^{-x} \sin x + c_2 e^{-x} \cos x. \quad (1)$$

The method of variation of parameters can be used to find a particular solution y_0 . We wish to find functions $u_1(x), u_2(x)$ such that

$$y_0 = u_1(x)y_1(x) + u_2(x)y_2(x) \quad (2)$$

satisfies $Ly_0 = \sin x$ where $y_1(x)$ and $y_2(x)$ are solutions to the homogeneous equation $Ly = 0$. If the functions $u_1(x)$ and $u_2(x)$ are solutions to the system

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = \sin x \end{cases} \quad (3)$$

then (2) will satisfy the original differential equation $Ly = \sin x$ equation. The solution to the system (3) is

$$u_1(x) = - \int \frac{y_2(x) \sin x}{W[\{y_1, y_2\}]} dx \quad u_2(x) = \int \frac{y_1(x) \sin x}{W[\{y_1, y_2\}]} dx \quad (4)$$

where $W[\{y_1, y_2\}]$ is the Wronskian of the functions y_1 and y_2 .

Using (1), we know that $y_1(x) = e^{-x} \sin x$ and $y_2(x) = e^{-x} \cos x$ so the particular solution has the form $y_0 = u_1(x)e^{-x} \sin x + u_2(x)e^{-x} \cos x$. Further, the Wronskian of y_1 and y_2 is

$$W[\{y_1, y_2\}] = \begin{vmatrix} e^{-x} \sin x & e^{-x} \cos x \\ e^{-x} \cos x - e^{-x} \sin x & -e^{-x} \cos x - e^{-x} \sin x \end{vmatrix} = -e^{-2x}.$$

Thus, using (4), we know that

$$\begin{aligned} u_1(x) &= - \int \frac{y_2(x) \sin x}{W[\{y_1, y_2\}]} dx \\ &= \int \frac{e^{-x} \cos x \sin x}{e^{-2x}} dx \\ &= \frac{e^x}{10} (-2 \cos 2x + \sin 2x) + C \end{aligned}$$

and

$$\begin{aligned} u_2(x) &= \int \frac{y_1(x) \sin x}{W[\{y_1, y_2\}]} dx \\ &= - \int \frac{e^{-x} \sin^2 x}{e^{-2x}} dx \\ &= \frac{e^x}{10} (-5 + \cos 2x + 2 \sin 2x) + C. \end{aligned}$$

Therefore, a particular solution to $Ly = \sin x$ is

$$y_0(x) = \frac{1}{10} (-2 \cos 2x + \sin 2x) \sin x + \frac{1}{10} (-5 + \cos 2x + 2 \sin 2x) \cos x$$

and the general solution to $Ly = \sin x$ is

$$\begin{aligned} y(x) &= y_0(x) + y_h(x) \\ &= \frac{1}{10} (-2 \cos 2x + \sin 2x) \sin x + \frac{1}{10} (-5 + \cos 2x + 2 \sin 2x) \cos x \\ &\quad + c_1 e^{-x} \sin x + c_2 e^{-x} \cos x \end{aligned} \tag{5}$$

□

Problem 2. Find the Green function of the IVP

$$y'' + 2y' + 2y = f(x), \quad y(0) = y'(0) = 0.$$

Solution. Let $Ly = f(x)$ denote the differential equation $y'' + 2y' + 2y = f(x)$ together with the initial conditions $y(0) = y'(0) = 0$. The Green function $G(x, a)$ of the IVP $Ly = f(x)$ is defined by the equations

$$\frac{\partial^2 G(x, a)}{\partial x^2} + \frac{2\partial G(x, a)}{\partial x} + 2G(x, a) = \delta(x - a), \quad G(0, a) = 0, \quad \frac{\partial G}{\partial x}(0, a) = 0$$

where $\delta(x - a)$ is the Dirac Delta function such that $\int_{-\infty}^{\infty} \delta(x - a)f(x)dx = f(a)$. Note that $G(x, a)$ is continuous at $x = a$ and $\partial G/\partial x$ has a jump discontinuity of magnitude 1 at $x = a$.

If y_1 and y_2 are linearly independent solutions of the homogeneous equation $Ly = 0$, then

$$G(x, a) = \begin{cases} A_1 y_1 + A_2 y_2 & \text{if } x < a \\ B_1 y_1 + B_2 y_2 & \text{if } x > a \end{cases}$$

where A_1, A_2, B_1 , and B_2 are undetermined functions. The continuity of $G(x, a)$ at $x = a$ gives the equation

$$A_1 y_1(a) + A_2 y_2(a) = B_1 y_1(a) + B_2 y_2(a).$$

Further, the fact that $\partial G/\partial x$ has a jump discontinuity of magnitude 1 at $x = a$ yields the second equation

$$(B_1 y_1'(a) + B_2 y_2'(a)) - (A_1 y_1'(a) + A_2 y_2'(a)) = 1.$$

Combining these equations, we see that A_1, A_2, B_1 , and B_2 are given by

$$B_1 = A_1 - \frac{y_2(a)}{W[y_1(a), y_2(a)]}$$

$$B_2 = A_2 + \frac{y_1(a)}{W[y_1(a), y_2(a)]}$$

From (1), we know that the linearly independent solutions to the homogeneous equation $Ly = 0$ are $y_1(x) = e^{-x} \sin x$ and $y_2(x) = e^{-x} \cos x$. Also, the Wronskian of these solutions is $W[y_1(a), y_2(a)] = -e^{-2a}$. Thus,

$$B_1 = A_1 - \frac{y_2(a)}{W[y_1(a), y_2(a)]} = A_1 + e^a \cos a$$

$$B_2 = A_2 + \frac{y_1(a)}{W[y_1(a), y_2(a)]} = A_2 - e^a \sin a$$

Using the two initial conditions, we can uniquely determine A_1 and A_2 since $G(x, a) = A_1 y_1(a) + A_2 y_2(a)$ satisfies $LG = f(x)$. Since $y(0) = 0$ we see that $A_2 = 0$ and since $y'(0) = 0$ we see that $A_1 - A_2 = 0$ implying that $A_1 = A_2 = 0$. Therefore, the Green function for the IVP $Ly = f(x)$ is

$$G(x, a) = \begin{cases} 0 & \text{if } x < a \\ e^{a-x} (\sin x \cos a - \sin a \cos x) & \text{if } x > a \end{cases} \quad (6)$$

□

Problem 3. Use your answer to Problem 2 to solve the IVP

$$y'' + 2y' + 2y = \sin x, \quad y(0) = y'(0) = 0.$$

Solution.

□

Problem 4. Show that if y_1 , y_2 , and y_3 are three linearly independent solutions of the linear ODE

$$y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = 0$$

and u_1 , u_2 , u_3 are solutions of the system

$$\begin{cases} u_1'y_1 + u_2'y_2 + u_3'y_3 = 0, \\ u_1'y_1' + u_2'y_2' + u_3'y_3' = 0, \\ u_1'y_1'' + u_2'y_2'' + u_3'y_3'' = f(x), \end{cases} \quad (7)$$

then the function $u = u_1y_1 + u_2y_2 + u_3y_3$ is a solution of

$$Ly = y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = f(x)$$

Solution. We wish to show that $y = \sum_{i=1}^3 u_i y_i$ is a solution of the equation $Ly = f(x)$ given that y_i are linearly independent solutions of the homogeneous equation $Ly = 0$ and u_i are solutions of the system (7). Using the form $y = \sum_{i=1}^3 u_i y_i$, we see that

$$\begin{aligned} y' &= \sum_{i=1}^3 u_i y_i' + u_i' y_i \\ y'' &= \sum_{i=1}^3 u_i y_i'' + 2u_i' y_i' + u_i'' y_i \\ y''' &= \sum_{i=1}^3 u_i y_i''' + 3u_i' y_i'' + 3u_i'' y_i' + u_i''' y_i. \end{aligned}$$

Thus, we find that for $y = \sum_{i=1}^3 u_i y_i$,

$$\begin{aligned} Ly &= \sum_{i=1}^3 u_i y_i''' + 3u_i' y_i'' + 3u_i'' y_i' + u_i''' y_i + p_2(x) \sum_{i=1}^3 u_i y_i'' + 2u_i' y_i' + u_i'' y_i \\ &\quad + p_1(x) \sum_{i=1}^3 u_i y_i' + u_i' y_i + p_0(x) \sum_{i=1}^3 u_i y_i \\ &= \sum_{i=1}^3 u_i [y_i''' + p_2(x)u_i'' + p_1(x)y_i' + p_0(x)y_i] \\ &\quad + \sum_{i=1}^3 3u_i' y_i'' + 3u_i'' y_i' + u_i''' y_i + 2p_2(x)u_i' y_i' + p_2(x)u_i'' y_i + p_1(x)u_i' y_i. \end{aligned}$$

Since y_i are solutions of the homogeneous equation $Ly = 0$, we see that the first sum is 0 and

$$Ly = \sum_{i=1}^3 3u_i' y_i'' + 3u_i'' y_i' + u_i''' y_i + 2p_2(x)u_i' y_i' + p_2(x)u_i'' y_i + p_1(x)u_i' y_i. \quad (8)$$

We also know that since u_1 , u_2 , and u_3 are solutions of the system (7) the following implications are true

$$\begin{aligned} \sum_{i=1}^3 u'_i y_i = 0 &\implies \left[\sum_{i=1}^3 u'_i y_i \right]' = \sum_{i=1}^3 u''_i y_i + u'_i y'_i = 0 \\ \sum_{i=1}^3 u''_i y_i + u'_i y'_i = 0 &\implies \left[\sum_{i=1}^3 u''_i y_i + u'_i y'_i \right]' = \sum_{i=1}^3 u'''_i y_i + 2u''_i y'_i + u'_i y''_i = 0 \\ \sum_{i=1}^3 u'_i y'_i = 0 &\implies \left[\sum_{i=1}^3 u'_i y'_i \right]' = \sum_{i=1}^3 u''_i y'_i + u'_i y''_i = 0 \end{aligned}$$

Rearranging the terms of (8) and using the above relations we see that

$$\begin{aligned} Ly &= \sum_{i=1}^3 u'_i y''_i + \left[\sum_{i=1}^3 u'''_i y_i + 2u''_i y'_i + u'_i y''_i \right] + \left[\sum_{i=1}^3 u'_i y''_i + u''_i y'_i \right] \\ &\quad + p_2(x) \left[\sum_{i=1}^3 u'_i y'_i + u''_i y_i \right] + p_2(x) \left[\sum_{i=1}^3 u'_i y'_i \right] + p_1(x) \left[\sum_{i=1}^3 u'_i y_i \right] \\ &= \sum_{i=1}^3 u'_i y''_i \end{aligned}$$

where every term in brackets is 0 as a consequence of the above derived relations or the fact that u_1 , u_2 , and u_3 are solutions of the system (7). From the third equation of the system (7) we know that $\sum_{i=1}^3 u'_i y''_i = f(x)$. Therefore, for $y = \sum_{i=1}^3 u_i y_i$ satisfying the assumptions of the problem,

$$Ly = \sum_{i=1}^3 u'_i y''_i = f(x)$$

showing that y is a solution of the equation $Ly = f(x)$. □

Problem 5. Find the eigenvalues and the respective eigenfunctions for the BVP

$$x^2 y'' + xy' + \lambda y = 0, \quad y'(1) = 0, \quad y'(b) = 0$$

where $b > 1$.

Solution. The differential equation stated in this problem is an Euler differential equation. The equation can be transformed into a constant coefficient second order linear differential equation by making the substitution $x(t) = e^t$ and rewriting the differential equation in terms of the independent variable t .

To see this, we note that

$$\begin{aligned} \frac{d}{dt} [y(x(t))] &= \frac{dy(x(t))}{dx} \frac{dx(t)}{dt} \\ &= \frac{dy(x(t))}{dx} \frac{dx(t)}{dt} \\ &= y'(x(t))x(t) \end{aligned}$$

since $x'(t) = [e^t]' = e^t = x(t)$. Similarly, using the above relation,

$$\begin{aligned} \frac{d^2}{dt^2} [y(x(t))] &= \frac{d}{dt} \left[\frac{dy(x(t))}{dx} \right] \\ &= \frac{d}{dt} \left[\frac{dy(x(t))}{dx} \right] x(t) + \frac{dy(x(t))}{dx} \frac{d}{dt} [x(t)] \\ &= \left[\frac{dy(x(t))}{dx} \frac{dx(t)}{dt} \right] x(t) + \left[\frac{dy(x(t))}{dx} \right] x(t) \\ &= x(t)^2 \frac{d^2 y(x(t))}{dx^2} + x(t) \frac{dy(x(t))}{dx} \\ &= x(t)^2 y''(x(t)) + x(t) y'(x(t)). \end{aligned}$$

Thus, the original differential equation in the independent variable x can be written as the following differential equation in the independent variable t after making the substitution $x(t) = e^t$:

$$[x^2 y''(x) + xy'(x)] + \lambda y(x) = [y''(x(t))] + \lambda y(x(t)) = 0. \quad (9)$$

The characteristic equation of the homogeneous second order linear differential equation in the variable t is given by

$$m(z) = z^2 + \lambda.$$

The roots of $m(z)$ are $z_1 = \sqrt{-\lambda}$ and $z_2 = -\sqrt{-\lambda}$. The solution to (9) is thus dependent on the value of λ and as such there are three cases to consider, when $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$.

Case 1: $\lambda < 0$

If $\lambda < 0$, then $\sqrt{-\lambda}$ is a positive real number and the roots of $m(z)$ are real and distinct. Thus, the solution to (9) is

Case 2: $\lambda = 0$

Case 3: $\lambda > 0$

□