MATH 635 Final Assessment

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Problem 1. Provide a rigorous proof of the case $x_0 = a$ in the Fundamental Lemma of the Calculus of Variations:

Theorem 1 (Fundamental Lemma of the Calculus of Variations)**.** *Suppose $M(x)$ is a continuous function defined on the interval $a \leq x \leq b$. Suppose further that for every continuous function $\zeta(x)$,*

$$\int_a^b M(x)\zeta(x)dx = 0.$$

Then

$$M(x) = 0 \text{ for all } x \in [a, b].$$

Solution. Suppose to the contrary that $M(x) \neq 0$ at the point $x_0 = a$. In that case either $M(a) > 0$ or $M(a) < 0$. Let us first assume that $M(a) > 0$. Due to the continuity of $M(x)$ there is some neighborhood of a where the function is positive, i.e. there is some $\delta > 0$ such that if $|x - a| < \delta$ then

$$|M(x) - M(a)| < \frac{M(a)}{2} \quad \text{for } x \in [a, b].$$

Thus, $0 < M(a)/2 < M(x)$ for $x \in [a, a + \delta)$. Choose the function $\zeta(x)$ to be the linear spline interpolating the points $(a, 3M(a)/2)$ and $(a + \delta, 0)$ with support on $[a, a + \delta)$, i.e.

$$\zeta(x) := \begin{cases} \frac{-3M(a)}{2\delta}(x - (a + \delta)) & \text{if } a \leq x < a + \delta \\ 0 & \text{if } a + \delta \leq x \leq b. \end{cases}$$

Clearly $\zeta(x)$ is continuous and positive on the interval $[a, a + \delta)$. Thus,

$$\int_a^b M(x)\zeta(x)dx = \int_a^{a+\delta} M(x)\zeta(x)dx > \frac{M(a)}{2} \int_a^{a+\delta} \zeta(x)dx > 0.$$

However, by our supposition

$$\int_a^b M(x)\zeta(x)dx = 0,$$

a contradiction. Therefore, if $M(a) > 0$, the function $M(x) \equiv 0$ on the interval $[a, b]$.

If $M(a) < 0$, then we can repeat the argument above replacing $M(x)$ with $-M(x)$. To demonstrate, let us investigate the case when $M(a) < 0$. Due to the continuity of $M(x)$ there is some neighborhood of a where $-M(x)$ is positive, i.e. there is some $\delta > 0$ such that if $|x - a| < \delta$ then

$$|-M(x) + M(a)| < \frac{-M(a)}{2} \quad \text{for } x \in [a, b].$$

Thus, $0 < -M(a)/2 < -M(x)$ for $x \in [a, a + \delta)$. Choose the function $\zeta(x)$ to be the linear spline interpolating the points $(a, -3M(a)/2)$ and $(a + \delta, 0)$ with support on $[a, a + \delta)$, i.e.

$$\zeta(x) := \begin{cases} \frac{3M(a)}{2\delta}(x - (a + \delta)) & \text{if } a \leq x < a + \delta \\ 0 & \text{if } a + \delta \leq x \leq b. \end{cases}$$

Clearly $\zeta(x)$ is continuous and positive on the interval $[a, a+\delta)$. Thus,

$$\int_a^b -M(x)\zeta(x)dx = \int_a^{a+\delta} -M(x)\zeta(x)dx > \frac{-M(a)}{2}\int_a^{a+\delta} \zeta(x)dx > 0.$$

However, by our supposition

$$\int_a^b M(x)\zeta(x)dx = 0,$$

a contradiction. Therefore, if $M(a) < 0$, the function $M(x) \equiv 0$ on the interval $[a, b]$ and we have proven both cases. $\square$

Problem 2. Consider the differential equation

$$y'' - y = -x, \quad 0 < x < 1 \quad y(0) = y(1) = 0 \tag{1}$$

as in Example 15.12 on page 502. Use the basis $\{\phi_j(x)\} = \{x^j(1-x)^j\}$, as in section 15.5.1, to compute approximations to the exact solution using the finite-element method.

Provide relative errors at the points 0.25, 0.50, and 0.75 of the approximations using the first $n = 2, 3, 4$ basis functions. Plot the corresponding approximations y_2, y_3, y_4, and the exact solution y. Then find the first value of j for which the relative error at all three points is less than 0.5%.

Solution. The exact solution to the differential equation (1), $y(x)$, is a continuous function. This fact combined with the fact that $\{\phi_j(x)\}$ form a basis of the function space shows that the continuous function $y(x)$ can be approximated with a linear combination of the basis functions. Therefore, we wish to find an approximation $y_n(x)$ to the exact solution $y(x)$ where

$$y_n(x) = \sum_{j=1}^{n} a_j \phi_j(x). \tag{2}$$

Note that the basis functions $\phi_j(x) = x^j(1-x)^j$ satisfy the boundary conditions $\phi_j(0) = \phi_j(1) = 0$ so that $y_n(x)$ also satisfies the boundary conditions.

Corollary 15.2 suggests that if

$$\int_0^1 (y_n'' - y_n + x)\phi_i(x)dx = 0 \quad \text{for } i = 1, \ldots, n$$

then $y_n'' - y_n + x = 0$, i.e $y_n(x)$ satisfies the differential equation (1). If $y_n(x)$ satisfies the differential equation and the boundary conditions, then we know that $y_n(x)$ approximates the exact solution $y(x)$.

Therefore, we choose the coefficients a_j such that they satisfy the system of equations

$$\sum_{j=1}^{n} a_j \int_0^1 \phi_j''(x)\phi_i(x) - \phi_j(x)\phi_i(x)dx = -\int_0^1 x\phi_i(x)dx \quad \text{for } i = 1, \ldots, n. \tag{3}$$

The above system unnecessarily uses the second derivative of the basis functions. We can rewrite the coefficients of the above system to use only the first derivative of the basis functions. To see this, note that we can rewrite the differential equation (1) in the form

$$(p(x)y')' + q(x)y' + r(x)y = f(x) \tag{4}$$

by choosing $p(x) = 1$, $q(x) = 0$, $r(x) = -1$, and $f(x) = -x$. With this form of the differential equation we would require the approximation (2) to satisfy the following equations

$$\int_0^1 ((p(x)y_n')' + r(x)y_n)\phi_i(x)dx = \int_0^1 f(x)\phi_i(x)dx \quad \text{for } i = 1, \ldots, n.$$

Making use of the fact that the basis functions are 0 on the boundary we see that

$$\begin{aligned}\int_0^1 (p(x)y_n')'\phi_i(x)dx &= \phi_i(x)p(x)y_n'|_0^1 - \int_0^1 p(x)y_n'\phi_i'(x)dx \\ &= -\int_0^1 p(x)y_n'\phi_i'(x)dx.\end{aligned}$$

With this and the definitions of the functions $p(x)$, $r(x)$, and $f(x)$, the system of equations (3) becomes

$$\sum_{j=1}^{n} a_j \int_0^1 -\phi_j'(x)\phi_i'(x) - \phi_j(x)\phi_i(x)dx = -\int_0^1 x\phi_i(x)dx \quad \text{for } i = 1, \dots, n. \tag{5}$$

Finding the solution to the system of equations (5) identifies the coefficients a_j that define our approximation.

In this instance, we have chosen the basis $\{\phi_j(x)\}_{j=1}^{n}$ where $\phi_j(x) = x^j(1-x)^j$. Thus,

$$\begin{aligned}\phi_j'(x) &= \left(x^j\right)'(1-x)^j + x^j\left((1-x)^j\right)' \\ &= jx^{j-1}(1-x)^j - jx^j(1-x)^{j-1}\end{aligned}$$

for $j = 1, \dots, n$. □