# Exam 2

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**Problem 1.** A function $f : \mathbb{C} \to \mathbb{C}$ is defined by $f(z) = z^8$. Find the fixed points of $f$. Use your calculations to find the real linear and quadratic factors of the polynomial $p(z) = z^7 - 1$.

*Solution.* The fixed points of $f$ are the solutions to the equation

$$f(z) - z = z^8 - z = z(z^7 - 1) = 0.$$

Thus, the fixed points of $f$ are $z = 0$ and the 7-th roots of unity, i.e. the points $z = e^{2\pi ki/7}$ for $k = 0, 1, \ldots, 6$.

Note that for $z, \alpha \in \mathbb{C}$, we have that

$$(z - \alpha)(z - \overline{\alpha}) = z^2 - \overline{\alpha}z - \alpha z + \alpha\overline{\alpha} = z^2 - 2\mathrm{Re}(\alpha)z + |\alpha|^2$$

is a polynomial with real coefficients.

Using the 7-th roots of unity, we can obtain the following factorization of $p(z)$:

$$p(z) = \prod_{k=0}^{6} \left(z - e^{2\pi ki/7}\right).$$

Let $\alpha_k = e^{2\pi ki/7}$. From the previous note, the real quadratic factors of $p(z)$ are obtained by multiplying each factor $(z - \alpha_k)$ with $(z - \overline{\alpha_k})$, if $\alpha_k$ and $\overline{\alpha_k}$ are both roots of $p(z)$. For $k = 1, \ldots, 6$, we have that $\alpha_k$ is a root of $p(z)$ and

$$\overline{\alpha_k} = e^{-2\pi ki/7} = e^{2\pi(7-k)i/7} = \alpha_{7-k},$$

which is also a root of $p(z)$. Therefore, the real linear and quadratic factors of $p(z)$ are given by

$$\begin{aligned} p(z) &= (z - \alpha_0)(z - \alpha_1)(z - \alpha_6)(z - \alpha_2)(z - \alpha_5)(z - \alpha_3)(z - \alpha_4) \\ &= (z - 1)(z - \alpha_1)(z - \overline{\alpha_1})(z - \alpha_2)(z - \overline{\alpha_2})(z - \alpha_3)(z - \overline{\alpha_3}) \\ &= (z - 1)\left(z^2 - 2\mathrm{Re}(\alpha_1)z + 1\right)\left(z^2 - 2\mathrm{Re}(\alpha_2)z + 1\right)\left(z^2 - 2\mathrm{Re}(\alpha_3)z + 1\right), \end{aligned}$$

where $\mathrm{Re}(\alpha_k) = \cos(2\pi k/7)$.

□

**Problem 2.** Let $K_c$ be the filled-in Julia set of $f_c(z) = z^2 + c$.

a. Find the fixed points and the period 2 points of $f_{-6}$.

b. Show that $2\sqrt{2} \in K_{-6}$ and find another point in $K_{-6}$, distinct from those found so far.

c. Do any of the points you have found lie in the Julia set of $f_{-6}$?

d. Is $-6 \in \mathcal{M}$ where $\mathcal{M}$ is the Mandelbrot set?

*Solution.* a. The fixed points of $f_{-6}$ are the solutions to

$$f_{-6}(z) - z = z^2 - z - 6 = 0.$$

Thus, the fixed points of $f_{-6}$ are $z_0 = 3$ and $z_1 = -2$. The period 2 points are the solutions to

$$f_{-6}^2(z) - z = (z^2 - 6)^2 - z - 6 = 0$$

that are also not fixed points of $f_{-6}$. Factoring $f_{-6}^2(z) - z$, we see that

$$f_{-6}^2(z) - z = (z - 3)(z + 2)(z^2 + z - 5).$$

Thus, the period 2 points of $f_{-6}$ are the solutions to $z^2 + z - 5 = 0$, i.e. the period 2 points of $f_{-6}$ are

$$z_2 = \frac{-1 - \sqrt{21}}{2}, \quad z_3 = \frac{-1 + \sqrt{21}}{2}.$$

b. Recall that for a polynomial $p(z)$ with $\deg(p) > 1$, the filled-in Julia set of $p(z)$ is the set of all points that do not converge to $\infty$ under iteration of $p$.

Note that $2\sqrt{2}$ is an eventual fixed point of $f_{-6}$. We see that $f_{-6}^2(2\sqrt{2}) = -2$ so that $f_{-6}^k(2\sqrt{2}) = -2$ for $k > 2$. This implies that $2\sqrt{2}$ does not converge to $\infty$ under iteration of $f_{-6}$ so that $2\sqrt{2}$ is in the filled-in Julia set of $f_{-6}$, i.e. $2\sqrt{2} \in K_{-6}$.

For reasons similar to those listed above, we see that $-3$ is an eventual fixed point of $f_{-6}$, i.e. $f_{-6}(-3) = 3$, so that $-3 \in K_{-6}$.

c. For a polynomial $p(z)$ with $\deg(p) > 1$, the Julia set of $p(z)$ is the boundary of the basin of attraction of $\infty$.

Since all of the points listed do not converge to $\infty$ under iteration of $f_{-6}$, we see that none of the listed points belong to the Julia set of $f_{-6}$.

d. The definition of the Mandelbrot set is the set of all $c \in \mathbb{C}$ such that the orbit of 0 is bounded under iteration by $f_c$. It was shown previously that $c \in \mathcal{M}$ if and only if $|f_c^n(0)| \leq 2$ for all $n > 0$. For $f_{-6}$, we see that $f_{-6}(0) = -6$ where $|f_{-6}(0)| > 2$. Therefore, we must have that $-6 \notin \mathcal{M}$.

$\square$

**Problem 3.** Let $f_c(z) = z^2 + c$. Find the values of $c$ so that $z = i$ is a period 2 point. Find the fixed points in each case and determine their stability. Is $c \in \mathcal{M}$?

*Solution.* As was shown previously, the fixed points of $f_c(z) = z^2 + c$ are the solutions to $f_c(z) - z = 0$ which are the points

$$z_0 = \frac{1 + \sqrt{1 - 4c}}{2}, \quad z_1 = \frac{1 - \sqrt{1 - 4c}}{2}. \tag{1}$$

The period 2 points of $f_c$ are the solutions to $f_c^2(z) - z = 0$ that are also not the fixed points (1). The period 2 points are thus given by

$$z_2 = \frac{-1 - \sqrt{-3 - 4c}}{2}, \quad z_3 = \frac{-1 + \sqrt{-3 - 4c}}{2}.$$

We wish to find the values of $c \in \mathbb{C}$ such that $z_2 = i$ or $z_3 = i$. Using Mathematica, we see that the only value of $c \in \mathbb{C}$ such that $z_2 = i$ or $z_3 = i$ is $c = -i$. If $c = -i$, we see that $f_c(i) = -1 - i$ and $f_c^2(i) = i$ so that $z = i$ is in fact a period 2 point.

From (1), the fixed points of $f_c$ when $c = -i$ are given by

$$z_0 = \frac{1 + \sqrt{1 + 4i}}{2}, \quad z_1 = \frac{1 - \sqrt{1 + 4i}}{2}.$$

For a differentiable function $f$, the fixed point $z$ of $f$ is asymptotically stable if $|f'(z)| < 1$ and asymptotically unstable if $|f'(z)| > 1$. For $f_c(z) = z^2 + c$, we note that $f_c'(z) = 2z$. Consider $z_0 = \dfrac{1 + \sqrt{1 + 4i}}{2}$. Note that

$$|f'(z_0)| = \left|1 + \sqrt{1 + 4i}\right| = \sqrt{1 + \sqrt{17} + \sqrt{2(1 + \sqrt{17})}} > 1$$

so that $z_0$ is an unstable fixed point. Now consider $z_1 = \dfrac{1 - \sqrt{1 + 4i}}{2}$. Then we have that

$$|f'(z_1)| = \left|1 - \sqrt{1 + 4i}\right| = \sqrt{1 + \sqrt{17} - \sqrt{2(1 + \sqrt{17})}} > 1$$

so that $z_1$ is also an unstable fixed point.

Note that 0 is an eventual periodic point of $f_{-i}$, i.e. $f_{-i}(0) = -i$ and $f_{-i}^2(0) = -1 - i$ which is a period 2 point of $f_{-i}$. Thus, the orbit of 0 under iteration of $f_{-i}$ will be bounded and we have that $-i \in \mathcal{M}$.

$\square$

**Problem 4.** Show that the function $H(z) = \dfrac{z - i}{z + i}$ gives a conjugacy between the Newton map $N_{f_1}$ of $f_1(z) = z^2 + 1$ and the function $f_0(z) = z^2$. Deduce the Julia set of $N_{f_1}$ and show that it is chaotic on its Julia set.

*Solution.* $\square$

**Problem 5.** Let $p(z)$ be a polynomial of degree $d > 1$ with Newton function

$$N_p(z) = z - \frac{p(z)}{p'(z)}.$$

a. If $p(\alpha) = 0$ and $p'(\alpha) \neq 0$, show that $\alpha$ is a fixed point of multiplicity two for $N_p$, i.e. there is a rational function $k(z) = m(z)/n(z)$ with $n(\alpha) \neq 0$ and $N_p(z) - \alpha = (z - \alpha)^2 k(z)$.

b. If $p(\alpha) = 0$, $p'(\alpha) \neq 0$, and $p''(\alpha) = 0$, show that $\alpha$ is a fixed point of multiplicity three for $N_p$.

*Solution.* □

**Problem 6.** a. Show that for $p_\alpha(z) = z(z-1)(z-\alpha)$, the Newton function $N_{p_\alpha}$ has a critical point where $z = (\alpha+1)/3$.

b. For what values of $\alpha$ does $p_\alpha$ satisfy $p(\alpha) = 0$, $p'(\alpha) \neq= 0$, and $p''(\alpha) = 0$?

*Solution.* □

**Problem 7.** Let $0 < \mu < \lambda < 1$ and let $h : [0,1] \to [0,1]$ be a homeomorphism with $h \circ L_\mu(x) = L_\lambda \circ h(x)$ for all $x \in [0,1]$.

a. Show that $h$ is orientation-preserving.

b. Show that $h(x) + h(1-x) = 1$ for all $x \in [0,1]$. Deduce that $h(1/2) = 1/2$.

c. Show that $h(\mu/4) = \lambda/4$ and $h(x) > x$ for $0 < x < 1/2$ and $h(x) < x$ for $1/2 < x < 1$.

*Solution.* $\square$

**Problem 8.** Prove that if $f_c(z) = z^2 + c$ has an attracting periodic point, then $c \in \mathcal{M}$, the Mandelbrot set.

*Solution.* $\square$