

Homework Assignment 4

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March 19, 2017

Problem 3.1. Find the Laplace transforms of the following functions:

b. $f(t) = (1 - 2t)e^{-2t}$

c. $f(t) = t \cos at$

d. $f(t) = t^{3/2}$

g. $f(t) = (t - 3)^2 H(t - 3)$

Solution. Recall that the Laplace transform of the function $f(t)$ defined for $t > 0$ is given by

$$\mathcal{L}\{f(t)\} = \bar{f}(s) = \int_0^\infty f(t)e^{-st}dt. \quad (1)$$

b. Let $g(t) = 1 - 2t$. Then $f(t) = (1 - 2t)e^{-2t} = g(t)e^{-2t}$. From the definition of the Laplace transform, we have that

$$\begin{aligned} \mathcal{L}\{g(t)\} &= \bar{g}(s) = \int_0^\infty (1 - 2t)e^{-st}dt \\ &= \int_0^\infty t^0 e^{-st}dt - 2 \int_0^\infty t^1 e^{-st}dt \\ &= \mathcal{L}\{t^0\} - 2\mathcal{L}\{t^1\}. \end{aligned}$$

From a previous theorem, we know for $n \in \mathbb{N}$ that

$$\mathcal{L}\{t^n\} = \int_0^\infty t^n e^{-st}dt = \frac{n!}{s^{n+1}}.$$

Thus,

$$\bar{g}(s) = \mathcal{L}\{t^0\} - 2\mathcal{L}\{t^1\} = \frac{1}{s} - \frac{2}{s^2} = \frac{s - 2}{s^2}.$$

From Heaviside's First Shifting Theorem, we know that for $\bar{g}(s) = \mathcal{L}\{g(t)\}$ that

$$\mathcal{L}\{g(t)e^{-at}\} = \bar{g}(s + a).$$

Therefore, the Laplace transform of $f(t) = (1 - 2t)e^{-2t} = g(t)e^{-2t}$ is

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)e^{-2t}\} = \bar{g}(s + 2) = \frac{s}{(s + 2)^2}.$$

- c. From the definition of the complex exponential, we have that $f(t) = t \cos at = \frac{t}{2} (e^{-iat} + e^{iat})$. From the definition of the Laplace transform, we have that

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \bar{f}(s) = \int_0^\infty \frac{t}{2} (e^{-iat} + e^{iat}) e^{-st} dt \\ &= \frac{1}{2} \left[\int_0^\infty t e^{-(s+ia)t} dt + \int_0^\infty t e^{-(s-ia)t} dt \right].\end{aligned}$$

We readily see by integrating by parts using $u = t$ and $dv = e^{-(s \pm ia)t} dt$ that

$$\begin{aligned}\int_0^\infty t e^{-(s \pm ia)t} dt &= -\frac{t}{s \pm ia} e^{-(s \pm ia)t} \Big|_0^\infty + \frac{1}{s \pm ia} \int_0^\infty e^{-(s \pm ia)t} dt \\ &= -\frac{1}{(s \pm ia)^2} e^{-(s \pm ia)t} \Big|_0^\infty \\ &= \frac{1}{(s \pm ia)^2}.\end{aligned}$$

Therefore, the Laplace transform of $f(t)$ is given by

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \bar{f}(s) = \frac{1}{2} \left[\int_0^\infty t e^{-(s+ia)t} dt + \int_0^\infty t e^{-(s-ia)t} dt \right] \\ &= \frac{1}{2} \left[\frac{1}{(s+ia)^2} + \frac{1}{(s-ia)^2} \right] \\ &= \frac{s^2 - a^2}{(s+ia)^2 (s-ia)^2} \\ &= \frac{s^2 - a^2}{(s^2 + a^2)^2}.\end{aligned}$$

- d. By definition, the Laplace transform of $f(t)$ is given by

$$\mathcal{L}\{f(t)\} = \bar{f}(s) = \int_0^\infty t^{3/2} e^{-st} dt.$$

Let $u = st$, then $du/s = dt$ and

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \bar{f}(s) = \frac{1}{s} \int_0^\infty \left(\frac{u}{s}\right)^{3/2} e^{-u} du \\ &= \frac{1}{s^{5/2}} \int_0^\infty u^{3/2} e^{-u} du.\end{aligned}$$

Recall that the definition of the Gamma function is given by

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du.$$

Therefore, the Laplace transform of $f(t) = t^{3/2}$ is

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \bar{f}(s) = \frac{1}{s^{5/2}} \int_0^\infty u^{5/2-1} e^{-u} dt \\ &= \frac{\Gamma\left(\frac{5}{2}\right)}{s^{5/2}}.\end{aligned}$$

- g. Let $g(t) = t^2$ and suppose that $\mathcal{L}\{g(t)\} = \bar{g}(s)$. Then Heaviside's Second Shifting Theorem shows that

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t-3)H(t-3)\} = e^{-3s}\bar{g}(s).$$

As shown previously, we know for $n \in \mathbb{N}$ that

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}.$$

Therefore, the Laplace transform of $f(t)$ is

$$\mathcal{L}\{f(t)\} = \bar{f}(s) = e^{-3s}\bar{g}(s) = \frac{2e^{-3s}}{s^3}.$$

□

Problem 3.3. The following is a result relating the Laplace transform of a function's derivative to the Laplace transform of that function:

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0). \quad (2)$$

Use the result to find

a. $\mathcal{L}\{\cos at\},$

b. $\mathcal{L}\{\sin at\}.$

Solution. a. Let $f(t) = \cos at$. Then $f'(t) = -a \sin at$ and from (2) we have

$$-a\mathcal{L}\{\sin at\} = s\mathcal{L}\{\cos at\} - 1. \quad (3)$$

Now let $g(t) = \sin at$. Then $g'(t) = a \cos at$ and applying (2) to $g(t)$ yields

$$a\mathcal{L}\{\cos at\} = s\mathcal{L}\{\sin at\}.$$

Therefore, from (3) we have that

$$-a\left(\frac{a}{s}\mathcal{L}\{\cos at\}\right) = s\mathcal{L}\{\cos at\} - 1$$

which implies that

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}.$$

b. Let $f(t) = \sin at$. Then $f'(t) = a \cos at$ and from (2) we have

$$a\mathcal{L}\{\cos at\} = s\mathcal{L}\{\sin at\}. \quad (4)$$

Now let $g(t) = \cos at$. Then $g'(t) = -a \sin at$ and applying (2) to $g(t)$ yields

$$-a\mathcal{L}\{\sin at\} = s\mathcal{L}\{\cos at\} - 1$$

which implies that

$$\mathcal{L}\{\cos at\} = \frac{1}{s} - \frac{a}{s}\mathcal{L}\{\sin at\}.$$

Therefore, from (4) we have that

$$a\left(\frac{1}{s} - \frac{a}{s}\mathcal{L}\{\sin at\}\right) = s\mathcal{L}\{\sin at\}$$

which implies that

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}.$$

□

Problem 3.6. Show that

$$\mathcal{L} \left\{ \int_0^t \frac{f(u)}{u} du \right\} = \frac{1}{s} \int_s^\infty \bar{f}(x) dx.$$

Solution. From the definition of the Laplace transform we see that

$$\mathcal{L} \left\{ \int_0^t \frac{f(u)}{u} du \right\} = \int_0^\infty e^{-st} \left[\int_0^t \frac{f(u)}{u} du \right] dt.$$

Interchanging the order of integration from u to t where $0 \leq t < \infty$, we see that $u \leq t < \infty$ as $0 \leq u < \infty$ and

$$\begin{aligned} \mathcal{L} \left\{ \int_0^t \frac{f(u)}{u} du \right\} &= \int_0^\infty e^{-st} \left[\int_0^t \frac{f(u)}{u} du \right] dt \\ &= \int_0^\infty \frac{f(u)}{u} \left[\int_u^\infty e^{-st} dt \right] du \\ &= \frac{1}{s} \int_0^\infty \frac{f(u)}{u} e^{-su} du. \end{aligned}$$

We note that $\frac{d}{ds} \left[\frac{e^{-su}}{u} \right] = -e^{-su}$ so that in particular we have that

$$- \int_s^\infty e^{-su} ds = - \frac{e^{-su}}{u} \Big|_s^\infty = - \frac{e^{-su}}{u}$$

or that

$$\int_s^\infty e^{-su} ds = \frac{e^{-su}}{u}.$$

Thus,

$$\begin{aligned} \mathcal{L} \left\{ \int_0^t \frac{f(u)}{u} du \right\} &= \frac{1}{s} \int_0^\infty \frac{f(u)}{u} e^{-su} du \\ &= \frac{1}{s} \int_0^\infty f(u) \left[\int_s^\infty e^{-su} ds \right] du. \end{aligned}$$

Interchanging the order of integration yet again from s to u where $s \leq u < \infty$ as $0 \leq u < \infty$, we see that the integration limits remain unchanged and therefore that

$$\begin{aligned} \mathcal{L} \left\{ \int_0^t \frac{f(u)}{u} du \right\} &= \frac{1}{s} \int_0^\infty f(u) \left[\int_s^\infty e^{-su} ds \right] du \\ &= \frac{1}{s} \int_s^\infty \left[\int_0^\infty f(u) e^{-su} du \right] ds \\ &= \frac{1}{s} \int_s^\infty \bar{f}(s) ds \\ &= \frac{1}{s} \int_s^\infty \bar{f}(x) dx, \end{aligned}$$

and we are done. □

Problem 3.7. Obtain the inverse Laplace transforms of the following functions:

b. $\bar{f}(s) = \frac{1}{s^2(s^2 + c^2)}.$

Solution. b. Let $\bar{f}(s) = \bar{g}(s)\bar{h}(s)$ where $\bar{g}(s) = \frac{1}{s^2}$ and $\bar{h}(s) = \frac{1}{s^2+c^2}.$

Using previous results, we know that

$$g(t) = \mathcal{L}^{-1}\{\bar{g}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

and

$$h(t) = \mathcal{L}^{-1}\{\bar{h}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{c}\left(\frac{c}{s^2+c^2}\right)\right\} = \frac{\sin ct}{c}.$$

Now, by the Convolution Theorem for the Laplace transform, we have that

$$f(t) = \mathcal{L}^{-1}\{\bar{f}(s)\} = \mathcal{L}^{-1}\{\bar{g}(s)\bar{h}(s)\} = (g * h)(t)$$

where

$$(g * h)(s) = \int_0^t g(t - \tau)h(\tau)d\tau.$$

Therefore,

$$\begin{aligned} f(t) &= (g * h)(t) = \frac{1}{c} \int_0^t (t - \tau) \sin c\tau d\tau \\ &= \frac{t}{c} \int_0^t \sin c\tau d\tau - \frac{1}{c} \int_0^t \tau \sin c\tau d\tau \\ &= \frac{t}{c} \left[\frac{1}{c} - \frac{\cos ct}{c} \right] - \frac{1}{c} \left[-\frac{t \cos ct}{c} + \frac{\sin ct}{c^2} \right] \\ &= \frac{t}{c^2} - \frac{\sin ct}{c^3}. \end{aligned}$$

□

Problem 3.8.*Solution.*

Problem 3.10.*Solution.*

Problem 3.12.*Solution.*

Problem 3.15.*Solution.*

Problem 3.18.*Solution.*

□