

# Homework Assignment 4

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September 23, 2015

**Problem 2.3.** Find the ACVF of the time series  $X_t = Z_t + aZ_{t-1} + bZ_{t-2}$  where  $Z_t \sim WN(0, \sigma^2)$  when:

a.  $a = 0.3$ ,  $b = -0.4$ , and  $\sigma^2 = 1$ .

b.  $a = -1.2$ ,  $b = -1.6$ , and  $\sigma^2 = 0.25$ .

*Solution.* The ACVF of the time series  $\{X_t\}$ ,  $\gamma_X(h)$ , is by definition:

$$\begin{aligned}\gamma_X(h) &= \text{Cov}(X_{t+h}, X_t) \\ &= \text{Cov}(Z_{t+h} + aZ_{t+h-1} + bZ_{t+h-2}, Z_t + aZ_{t-1} + bZ_{t-2}) \\ &= \text{Cov}(Z_{t+h}, Z_t) + a\text{Cov}(Z_{t+h}, Z_{t-1}) + b\text{Cov}(Z_{t+h}, Z_{t-2}) \\ &\quad + a\text{Cov}(Z_{t+h-1}, Z_t) + a^2\text{Cov}(Z_{t+h-1}, Z_{t-1}) + ab\text{Cov}(Z_{t+h-1}, Z_{t-2}) \\ &\quad + b\text{Cov}(Z_{t+h-2}, Z_t) + ab\text{Cov}(Z_{t+h-2}, Z_{t-1}) + b^2\text{Cov}(Z_{t+h-2}, Z_{t-2}).\end{aligned}\tag{1}$$

Using (1), we can see that since  $Z_t \sim WN(0, \sigma^2)$ ,

$$\gamma_X(h) = \begin{cases} (1 + a^2 + b^2)\sigma^2 & \text{if } h = 0 \\ a(1 + b)\sigma^2 & \text{if } h = \pm 1 \\ b\sigma^2 & \text{if } h = \pm 2 \\ 0 & \text{otherwise} \end{cases}.$$

Therefore, when

a.  $a = 0.3$ ,  $b = -0.4$ , and  $\sigma^2 = 1$ , the ACVF of  $\{X_t\}$  is:

$$\begin{cases} 1.25 & \text{if } h = 0 \\ 0.18 & \text{if } h = \pm 1 \\ -0.4 & \text{if } h = \pm 2 \\ 0 & \text{otherwise} \end{cases}$$

b.  $a = -1.2$ ,  $b = -1.6$ , and  $\sigma^2 = 0.25$ , the ACVF of  $\{X_t\}$  is:

$$\begin{cases} 1.25 & \text{if } h = 0 \\ 0.18 & \text{if } h = \pm 1 \\ -0.4 & \text{if } h = \pm 2 \\ 0 & \text{otherwise} \end{cases}$$

□

**Problem 2.5.** Suppose that  $\{X_t, t = 0, \pm 1, \dots\}$  is stationary and that  $|\theta| < 1$ . Show that for each fixed  $n$  the sequence

$$S_m = \sum_{j=1}^m \theta^j X_{n-j}$$

is convergent absolutely and in mean square as  $m \rightarrow \infty$ .

*Solution.* Let  $a_j = \theta^j X_{n-j}$ . Since each  $X_i$  is a random variable, each  $X_i$  maps to a real, non-infinite value so let  $X = \max\{|X_i|\}$ . Then to see that  $S_m$  is convergent absolutely as  $m \rightarrow \infty$ , notice that

$$\begin{aligned} \sum_{j=1}^m |a_j| &= \sum_{j=1}^m |\theta^j X_{n-j}| \\ &= \sum_{j=1}^m |\theta|^j |X_{n-j}| \\ &\leq \sum_{j=1}^m X |\theta|^j = \sum_{j=1}^m b_j = T_m \end{aligned}$$

Since  $|\theta| < 1$ , we know that as  $m \rightarrow \infty$ , the partial sum  $\sum_{j=1}^m X |\theta|^j \rightarrow 0$  and it must hold that  $T_m \rightarrow 0$ . Thus, we know that as  $m \rightarrow \infty$ ,  $\sum_{j=1}^m |a_j|$  converges to some  $L$  since  $|a_j| \leq b_j$  and  $T_m$  is convergent. Therefore,  $S_m$  is convergent absolutely.

To see that  $S_m$  is convergent in the mean square, it suffices to show that  $E(S_m - S_l)^2 \rightarrow 0$  as  $m, l \rightarrow \infty$ .

Without loss of generality, assume that  $m > l > 0$ . Notice that  $S_m - S_l = \sum_{j=1}^m a_j - \sum_{j=1}^l a_j = \sum_{j=l+1}^m a_j$ . Thus,

$$E(S_m - S_l) = E\left(\sum_{j=l+1}^m a_j\right) = \sum_{j=l+1}^m E(a_j).$$

It is clear that  $E(a_j) = E(\theta^j X_{n-j}) = \theta^j E(X_{n-j})$ . Since  $\{X_t\}$  is a stationary time series, its expectation does not depend on  $t$ , so say  $E(X_{n-j}) = \mu_X$ . Then

$$\begin{aligned} E(S_m - S_l) &= \sum_{j=l+1}^m \theta^j E(X_{n-j}) \\ &= \mu_X \sum_{j=l+1}^m \theta^j \\ &= \frac{\mu_X \theta^{l+1} (1 - \theta^{m-l-1})}{1 - \theta} \end{aligned}$$

Since  $|\theta| < 1$ , it is clear then that  $E(S_m - S_l)^2 \rightarrow 0$  as  $m, l \rightarrow \infty$  showing that  $S_m$  is convergent in mean square for any  $n$ . □

**Problem 2.11.** Suppose that in a sample of size 100 from an AR(1) process with mean  $\mu$ ,  $\phi = 0.6$ , and  $\sigma^2 = 2$  we obtain  $\bar{x}_{100} = 0.271$ . Construct an approximate 95% confidence interval for  $\mu$ . Are the data compatible with the hypothesis that  $\mu = 0$ .

*Solution.* Note that since AR(1) is a linear model,  $\bar{X}_n$  is approximately normal with mean  $\mu$  for large  $n$  and an approximate 95% confidence interval for  $\mu$  is

$$\left( \bar{X}_n - \frac{1.96\nu^{1/2}}{\sqrt{n}}, \bar{X}_n + \frac{1.96\nu^{1/2}}{\sqrt{n}} \right)$$

where  $\nu = \sum_{|h|<\infty} \gamma_X(h)$ .

Since  $\{X_t\}$  is an AR(1) process, we know that  $\gamma_X(h) = \gamma_X(0)\phi^{|h|}$  where  $\gamma_X(0) = \sigma^2/(1 - \phi^2)$ . Thus

$$\begin{aligned} \nu &= \sum_{|h|<\infty} \gamma_X(h) = \sum_{|h|<\infty} \frac{\sigma^2 \phi^{|h|}}{1 - \phi^2} \\ &= \frac{\sigma^2}{1 - \phi^2} \left( 1 + 2 \sum_{h=1}^{\infty} \phi^h \right) \\ &= \frac{\sigma^2}{1 - \phi^2} \left( 1 + \frac{2\phi}{1 - \phi} \right) \\ &= \frac{\sigma^2(1 + \phi)}{(1 - \phi)(1 - \phi^2)} = \frac{\sigma^2}{(1 - \phi)^2} \end{aligned}$$

If  $\phi = 0.6$  and  $\sigma^2 = 2$ , then  $\nu = 2/(1 - 0.6)^2 = 12.5$ . Since  $n = 100$ ,  $\bar{x}_n = \bar{x}_{100} = 0.271$ , and an approximate 95% confidence interval for  $\mu$  is

$$\left( 0.271 - \frac{1.96(12.5)^{1/2}}{\sqrt{100}}, 0.271 + \frac{1.96(12.5)^{1/2}}{\sqrt{100}} \right)$$

or  $(-0.42197, 0.96397)$ . Given this confidence interval, it is plausible that  $\mu = 0$ . □

**Problem 2.12.** Suppose that in a sample of size 100 from an MA(1) process with mean  $\mu$ ,  $\theta = -0.6$ , and  $\sigma^2 = 1$  we obtain  $\bar{x}_{100} = 0.157$ . Construct an approximate 95% confidence interval for  $\mu$ . Are the data compatible with the hypothesis that  $\mu = 0$ .

*Solution.* Note that since MA(1) is a linear model,  $\bar{X}_n$  is approximately normal with mean  $\mu$  for large  $n$  and an approximate 95% confidence interval for  $\mu$  is

$$\left( \bar{X}_n - \frac{1.96\nu^{1/2}}{\sqrt{n}}, \bar{X}_n + \frac{1.96\nu^{1/2}}{\sqrt{n}} \right)$$

where  $\nu = \sum_{|h|<\infty} \gamma_X(h)$ .

Since  $\{X_t\}$  is an MA(1) process, we know that

$$\gamma_X(h) = \begin{cases} \sigma^2(1 + \theta^2) & \text{if } h = 0 \\ \sigma^2\theta & \text{if } h = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$\nu = \sum_{|h|<\infty} \gamma_X(h) = \sigma^2(1 + \theta^2) + 2\sigma^2\theta = \sigma^2(1 + \theta)^2$$

If  $\theta = -0.6$  and  $\sigma^2 = 1$ , then  $\nu = (1 - 0.6)^2 = 0.16$ . Since  $n = 100$ ,  $\bar{x}_n = \bar{x}_{100} = 0.157$ , and an approximate 95% confidence interval for  $\mu$  is

$$\left( 0.157 - \frac{1.96(0.16)^{1/2}}{\sqrt{100}}, 0.157 + \frac{1.96(0.16)^{1/2}}{\sqrt{100}} \right)$$

or  $(0.15198, 0.16202)$ . Given this confidence interval, it is not plausible that  $\mu = 0$ .  $\square$

**Problem 2.13.** Suppose that in a sample of size 100, we obtain  $\hat{\rho}(1) = 0.438$  and  $\hat{\rho}(2) = 0.145$ .

- Assuming that the data were generated from an AR(1) model, construct approximate 95 % confidence intervals for both  $\rho(1)$  and  $\rho(2)$ . Based on these two confidence intervals, are the data consistent with an AR(1) model with  $\phi = 0.8$ ?
- Assuming that the data were generated from an MA(1) model, construct approximate 95 % confidence intervals for both  $\rho(1)$  and  $\rho(2)$ . Based on these two confidence intervals, are the data consistent with an MA(1) model with  $\theta = 0.6$ ?

*Solution.* Note that since AR(1) and MA(1) are linear models, we know that for large  $n$ , the vector  $\hat{\boldsymbol{\rho}} = (\hat{\rho}(1), \dots, \hat{\rho}(k))^T$  is a multivariate normal random variable with mean  $\boldsymbol{\rho} = (\rho(1), \dots, \rho(k))^T$  and covariance matrix  $n^{-1}W$  where  $w_{ij}$  is defined by Bartlett's formula.

Hence, an approximate 95% confidence interval for  $\rho(i)$  is

$$(\hat{\rho}(i) - 1.96\text{Var}(\hat{\rho}(i))^{1/2}, \hat{\rho}(i) + 1.96\text{Var}(\hat{\rho}(i))^{1/2}).$$

Since  $\hat{\rho}(i) \sim N(\rho(i), n^{-1}w_{ii})$ , we know  $\text{Var}(\hat{\rho}(i)) = n^{-1}w_{ii}$  and the approximate 95% confidence interval for  $\rho(i)$  is

$$\left( \hat{\rho}(i) - \frac{1.96w_{ii}^{1/2}}{\sqrt{n}}, \hat{\rho}(i) + \frac{1.96w_{ii}^{1/2}}{\sqrt{n}} \right). \quad (2)$$

- If the data were generated from an AR(1) model, then

$$w_{ii} = (1 - \phi^{2i})(1 + \phi^2)(1 - \phi^2)^{-1} - 2i\phi^{2i}$$

If  $\phi = 0.8$ , then  $w_{11} = 0.36$  and  $w_{22} = 1.0512$ . Using (2), we know that the 95% confidence interval for  $\rho(1)$  is  $(0.3204, 0.556)$  and the 95% confidence interval for  $\rho(2)$  is  $(-0.0560, 0.3640)$ . If  $\phi = 0.8$ , then the actual values of  $\rho(1) = 0.8$  and  $\rho(2) = 0.64$  suggest that the data are not consistent with an AR(1) model with  $\phi = 0.8$ .

- If the data were generated from an MA(1) model, then  $w_{11} = 1 - 3\rho(1)^2 + 4\rho(1)^4$  and  $w_{22} = 1 + 2\rho(1)^2$  where  $\rho(1) = \theta/(1 + \theta^2)$ . If  $\theta = 0.6$ , then  $w_{11} = 0.5676$  and  $w_{22} = 1.3893$ . Using (2), we know that the 95% confidence interval for  $\rho(1)$  is  $(0.2903, 0.5857)$  and the 95% confidence interval for  $\rho(2)$  is  $(-0.0860, 0.3760)$ . If  $\theta = 0.6$ , then the actual values of  $\rho(1) = 0.4412$  and  $\rho(2) = 0$  suggest that the data are consistent with an MA(1) model with  $\theta = 0.6$ .  $\square$