Homework Assignment 3

Matthew Tiger

September 24, 2016

Problem 1.5.1. Find the fixed points of the following maps and use the appropriate theorems to determine whether they are asymptotically stable, semi-stable, or unstable:

i. $f(x) = \frac{x^3}{2} + \frac{x}{2}$,

ii. $f(x) = \arctan(x)$,

iii. $f(x) = x^3 + x^2 + x$,

iv. $f(x) = x^3 - x^2 + x$,

v. $f(x) = \begin{cases} 3x/4 & x \leq 1/2 \\ 3(1-x)/4 & x > 1/2 \end{cases}$.

Solution. Note that a point $x = c$ is a fixed point of f if c is a solution to the equation $g(x) = f(x) - x = 0$. If $x = c$ is a fixed point, then the behavior of the derivatives of f at the point $x = c$ will allow us to classify the stability of the fixed point.

i. The solutions to the equation

$$\begin{aligned} g(x) &= f(x) - x \\ &= \frac{x^3}{2} + \frac{x}{2} - x \\ &= \frac{x^3}{2} - \frac{x}{2} - x = 0 \end{aligned}$$

are given by $x = -1$, $x = 0$, and $x = 1$. Note that $f'(x) = 3x^2/2 + 1/2$.

For the fixed point $x = -1$, we see that $|f'(-1)| = 2 > 1$ so that $x = -1$ is a hyperbolic fixed point and by theorem 1.4.4, this fixed point is unstable.

For the fixed point $x = 0$, we see that $|f'(0)| = 1/2 < 1$ so that $x = 0$ is a hyperbolic fixed point and by theorem 1.4.4, this fixed point is stable.

For the fixed point $x = 1$, we see that $|f'(1)| = 2 > 1$ so that $x = 1$ is a hyperbolic fixed point and by theorem 1.4.4, this fixed point is unstable.

ii. Note that for any $x \in \mathbb{R}$, we have that $-\pi/2 < \arctan(x) < \pi/2$. Thus, if $|x| > \pi/2$, then $|\arctan(x)| < \pi/2 < |x|$ so that for any such x we have that $\arctan(x) \neq x$, i.e. $f(x) = \arctan(x)$ has no fixed points for $|x| > \pi/2$.

Since $f(x)$ is continuous on the interval $[-\pi/2, \pi/2]$, we know that $f(x)$ must have a fixed point on this interval. By the Mean Value Theorem, we know that if $x > 0$, then

$$0 < \frac{x}{x^2+1} < \arctan(x).$$

It can be shown that for $g(x) = \arctan(x) - x$, if $x > 0$, then $g'(x) < 0$. This implies that the function $g(x)$ is monotonically decreasing and that $g(x) < g(0) = 0$, i.e. $\arctan(x) < x$. Combining, we see that

$$0 < \arctan(x) < x.$$

From this inequality, we gather that if $x \in (0, \pi/2)$, we have that $\arctan(x) > 0$ and that

$$0 < f^n(x) < f^{n-1}(x) < \cdots < f(x) < x,$$

i.e. the iterates of f are monotonically decreasing and bounded below. Thus, the limit converges to the infimum, i.e. $\lim f^n(x) = 0$. Therefore, we must have $x = 0$ is a fixed point if $x \in (0, \pi/2)$.

Using a similar inequality, we can show that if $x \in (-\pi/2, 0)$, then the iterates of f form a monotonically increasing sequence that is bounded above. Thus, the limit in this case converges to the supremum, i.e. $\lim f^n(x) = 0$ and $x = 0$ is a fixed point if $x \in (-\pi/2, 0)$. Therefore, $x = 0$ is the only fixed point of $f(x) = \arctan(x)$.

Note that

$$f'(x) = 1/(x^2+1), \quad f''(x) = -2x/(1+x^2)^2, \quad f'''(x) = 8x^2/(1+x^2)^3 - 2/(1+x^2)^2.$$

Thus, for the fixed point $x = 0$, we see that $f'(0) = 1$, $f''(0) = 0$, and $f'''(0) = -2$. Therefore, according to theorem 1.5.3 (iii), this fixed point is non-hyperbolic and stable.

iii. The solutions to the equation

$$\begin{aligned} g(x) &= f(x) - x \\ &= x^3 + x^2 + x - x \\ &= x^2(x+1) = 0 \end{aligned}$$

are given by $x = -1$ and $x = 0$. Note that $f'(x) = 3x^2 + 2x + 1$, $f''(x) = 6x + 2$, and $f'''(x) = 6$.

For the fixed point $x = -1$, we see that $|f'(-1)| = 2 > 1$ so that $x = -1$ is a hyperbolic fixed point and by theorem 1.4.4, this fixed point is unstable.

For the fixed point $x = 0$, we see that $f'(0) = 1$ so that $x = 0$ is a non-hyperbolic fixed point. Since $f''(0) = 2 > 0$, we have by theorem 1.5.3 (i)(a) that this fixed point is one-sided stable to the left of $x = 0$.

iv. The solutions to the equation

$$\begin{aligned} g(x) &= f(x) - x \\ &= x^3 - x^2 + x - x \\ &= x^2(x-1) = 0 \end{aligned}$$

are given by $x = 1$ and $x = 0$. Note that $f'(x) = 3x^2 - 2x + 1$, $f''(x) = 6x - 2$, and $f'''(x) = 6$.

For the fixed point $x = 1$, we see that $|f'(1)| = 2 > 1$ so that $x = 1$ is a hyperbolic fixed point and by theorem 1.4.4, this fixed point is unstable.

For the fixed point $x = 0$, we see that $f'(0) = 1$ so that $x = 0$ is a non-hyperbolic fixed point. Since $f''(0) = -2 < 0$, we have by theorem 1.5.3 (i)(b) that this fixed point is one-sided stable to the right of $x = 0$.

v. If $x \leq 1/2$, then

$$f(x) - x = \frac{3x}{4} - x = -\frac{x}{4} = 0$$

if $x = 0$. Since $x = 0 \leq 1/2$, we have that $x = 0$ is a fixed point of $f(x)$.

If $x > 1/2$, then

$$f(x) - x = \frac{3(1-x)}{4} - x = \frac{3-7x}{4} = 0$$

if $x = 3/7$. Since $3/7 < 1/2$, we have that $x = 3/7$ is not a fixed point of $f(x)$.

If $x \leq 1/2$, then $f'(x) = 3/4$. Thus, for the fixed point $x = 0$, we see that $|f'(0)| < 1$ and $x = 0$ is a non-hyperbolic stable fixed point by theorem 1.4.4.

$\square$

Problem 1.5.2. Consider the family of quadratic maps $f_c(x) = x^2 + c$ where $x \in \mathbb{R}$.

i. Use the theorems of section 1.5 to determine the stability of the hyperbolic fixed points of the the family of maps for all possible values of c.

ii. Find any values of c such that f_c has a non-hyperbolic fixed point and determine the stability of these fixed points.

Solution. As was shown in problem 1.2.1, we know that $f_c : \mathbb{R} \to \mathbb{R}$ with $f_c(x) = x^2 + c$ has two fixed points given by

$$x_1 = \frac{1 - \sqrt{1 - 4c}}{2}, \qquad x_2 = \frac{1 + \sqrt{1 - 4c}}{2} \tag{1}$$

provided that $c \leq 1/4$.

i. Suppose that $c \leq 1/4$. Then the fixed points of f_c are provided by (1). Recall that a fixed point $x = a$ is a hyperbolic fixed point of a function g if $|g(a)| \neq 1$. In particular, $x = a$ will be asymptotically stable if $|g(a)| < 1$ and unstable if $|g(a)| > 1$.

   We begin by assuming the fixed point of the function f_c has the form x_1. Then x_1 will be stable if

   $$|f_c'(x_1)| = |1 - \sqrt{1 - 4c}| < 1. \tag{2}$$

   However, this is only true if $-3/4 < c < 1/4$. Thus, x_1 will be asymptotically stable if $-3/4 < c < 1/4$. Similarly, by reversing the inequality in (2), we can easily see that the fixed point x_1 will be unstable if $c < -3/4$.

   Now, assuming that the fixed point of f_c has the form x_2, then the fixed point x_2 will be stable if

   $$|f_c'(x_2)| = |1 + \sqrt{1 - 4c}| < 1.$$

   However, this has no real solutions if $c \leq 1/4$. On the other hand, we can see that

   $$|f_c'(x_2)| = |1 + \sqrt{1 - 4c}| > 1$$

   if $c < 1/4$. Therefore, every hyperbolic fixed point of f_c of the form x_2 is unstable.

ii. A fixed point $x = a$ is a non-hyperbolic fixed point of a function g if $|g(a)| = 1$.

   We first investigate fixed points of the form x_1. Assuming the fixed point of f_c is of the form x_1, then x_1 is non-hyperbolic if

   $$|f_c'(x_1)| = \left|1 - \sqrt{1 - 4c}\right| = 1$$

   from which we see that $1 - \sqrt{1 - 4c} = 1$ if $c = 1/4$ and that $1 - \sqrt{1 - 4c} = -1$ if $c = -3/4$. Thus, x_1 is a non-hyperbolic fixed point if $c = 1/4$ or $c = -3/4$.

   In the case that $c = 1/4$, then $f_c'(x_1) = 1$ and $f_c''(x_1) = 2$. Thus, since $f_c''(x_2) > 0$, applying theorem 1.5.3 (i) (a), we see that this fixed point is one-sided stable to the left of x_1. On the other hand, if $c = -3/4$, then $f_c'(x_1) = -1$ with $f_c''(x_1) = 2$ and $f_c'''(x_1) = 0$. Since $f_c'(x_1) = -1$, the Schwarzian derivative of f_c is given by

$$Sf_c(x) = -f_c'''(x) - \frac{3\left(f_c''(x)\right)^2}{2} = -6.$$

Note that $Sf_c(x_1) < 0$, so applying theorem 1.5.7 (i) we find that the fixed point x_1 is asymptotically stable if $c = -3/4$.

We now investigate fixed points of the form x_2. Assuming the fixed point of f_c is of the form x_2, then

$$|f_c'(x_2)| = \left|1 + \sqrt{1-4c}\right| = 1$$

only if $c = 1/4$. Thus, x_2 is a non-hyperbolic fixed point if $c = 1/4$.

In this case, we see that $f_c'(x_2) = 1$ and $f_c''(x_2) = 2$. Thus, since $f_c''(x_2) > 0$, applying theorem 1.5.3 (i) (a), we see that this fixed point is one-sided stable to the left of x_2 if $c = 1/4$.

$\square$

Problem 1.5.3.

i. Show that $f(x) = -2x^3 + 2x^2 + x$ has two non-hyperbolic fixed points and determine their stability.

ii. If $x = 0$ and $x = 1$ are non-hyperbolic fixed points for $f : \mathbb{R} \to \mathbb{R}$ for $f(x) = ax^3 + bx^2 + cx + d$, find all possible values of a, b, c, and d.

iii. Write down the function $f(x)$ in each case of (ii) above and determine the stability of the fixed points.

Solution. $\square$

Problem 1.5.6. Find the Schwarzian derivative of both $f(x) = e^x$ and $g(x) = \sin(x)$ and show that they are always negative.

Solution. Recall that the Schwarzian derivative of a function $h(x)$ is given by

$$Sh(x) = \frac{h'''(x)}{h'(x)} - \frac{3}{2}\left[\frac{h''(x)}{h'(x)}\right]^2$$

and this derivative exists if $h'''(x)$ exists and $h'(x) \neq 0$.

Suppose that $f(x) = e^x$. Then we know that $f^{(n)}(x) = e^x = f(x)$ for any positive integer n. Therefore,

$$Sf(x) = \frac{e^x}{e^x} - \frac{3}{2}\left[\frac{e^x}{e^x}\right]^2 = 1 - \frac{3}{2} = -\frac{1}{2} < 0$$

and we are done.

Now suppose that $g(x) = \sin(x)$. The successive derivatives of g are given by

$$\begin{aligned} g'(x) &= \cos(x) \\ g''(x) &= -\sin(x) \\ g'''(x) &= -\cos(x). \end{aligned}$$

Computing the Schwarzian derivative of $g(x)$, we see that

$$\begin{aligned} Sg(x) &= -\frac{\cos(x)}{\cos(x)} - \frac{3}{2}\left[-\frac{\sin(x)}{\cos(x)}\right]^2 \\ &= -1 - \frac{3\tan^2(x)}{2}. \end{aligned}$$

Since $\tan^2(x) \geq 0$ for any $x \in \mathbb{R}$, we have that $1 + (3/2)\tan^2(x) \geq 1$ so that

$$Sg(x) = -1 - \frac{3\tan^2(x)}{2} \leq -1 < 0$$

and we are done. $\square$

Problem 1.5.9. Let $f(x)$ be a polynomial such that $f(c) = c$. (Recall that a polynomial $p(x)$ has $(x-c)^2$ as a factor if and only if both $p(c) = 0$ and $p'(c) = 0$.)

i. If $f'(c) = 1$, show that $(x-c)^2$ is a factor of $g(x) = f(x) - x$.

ii. If $|f'(c)| = 1$, show that $(x-c)^2$ is a factor of $h(x) = f^2(x) - x$.

iii. Show in the case that $f'(c) = -1$, we actually have that $(x-c)^3$ is a factor of $h(x) = f^2(x) - x$.

iv. Check that (iii) holds for the non-hyperbolic fixed point $x = 2/3$ of the logistic map $L_3(x) = 3x(1-x)$.

v. Check that (i), (ii), (iii) hold for the non-hyperbolic fixed points of the polynomial $f(x) = -2x^3 + 2x^2 + x$.

Solution. $\square$