2.3 General Form of a LP Problem

Minimize (Maximize) $f(x) = \sum_{k=1}^{n} c_k x_k = c^T x$ subject to $Ax \leq (\geq, =) b$, $x \geq (\leq) 0$.

Here $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ is variable, $x^T = (x_1, x_2, \ldots, x_n)$, $A = (a_{ij})_{i=\overline{1,m}, j=\overline{1,n}} \in \mathbb{R}^{m \times n}$ is a constant matrix, $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m$, $c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n$ are constant vectors, $x \geq 0$ means

and $Ax =$

2.4 Standard Form of a LP Problem

(LP) Minimize $f(x) = c^T x$ subject to $Ax = b$, $x \geq 0$.

Here A is a matrix, b, c are vectors, and $b \geq 0$ (all constant).

Theorem 1 *Every general form can be reduced to a standard form.*

Proof. If our problem is a maximization problem then:

If $x_j \leq 0$ for a certain $j \in \overline{1,n}$ then

If $\sum_{i=1}^{n} a_{ij} x_j \leq b_i$ for a certain $i \in \overline{1,m}$ then

If $\sum_{i=1}^{n} a_{ij} x_j \geq b_i$ for a certain $i \in \overline{1,m}$ then

If $\sum_{i=1}^{n} a_{ij} x_j = b_i$ for a certain $i \in \overline{1,m}$ but $b_i < 0$ then

■

More general form reduction operations:

If $x_j \geq K_j$ for a certain $j \in \overline{1,n}$, where K is a constant vector but $K_j \neq 0$ then

If x_j is a free variable, for a certain $j \in \overline{1,n}$, that is, x_j has no constraint then

What do you do if $x_j \leq K_j$ for a certain $j \in \overline{1,n}$, where K is a constant vector but $K_j \neq 0$?

What do you do if $x_j \leq -1$?

What do you do if $x_j \geq -2$?

Example. Reduce the shipping problem studied in Math 111 to its standard form.

Example. Maximize $-5x_1 - 3x_2 + 7x_3$
subject to

$$\begin{gathered} 2x_1 + 4x_2 + 6x_3 = 7 \\ 3x_1 - 5x_2 + 3x_3 \le 5 \\ -4x_1 - 9x_2 + 4x_3 \le -4 \\ x_1 \ge -2 \\ 0 \le x_2 \le 4 \end{gathered}$$

2.5 Basic Solutions

Consider a linear program in standard form:

and suppose that $\operatorname{rank} A = m \leq n$. Note that this means that the linear system $Ax = b$ is solvable for all $b \in \mathbb{R}^m$ and that its solution set has the form

$$x = x_0 + \operatorname{Span}\{v_1, v_2, \ldots, v_{n-m}\}$$

for some vectors $v_1, v_2, \ldots, v_{n-m} \in \mathbb{R}^n$ (a basis for the null-space $\operatorname{Ker} A =$) and for some fixed arbitrary solution $x_0 \in \mathbb{R}^n$ of $Ax = b$.

A *basic solution* for (LP) is a point $x \in \mathbb{R}^n$ such that:

i) $Ax = b$,

ii) the columns of matrix A that correspond to nonzero components of x are linearly independent

(in which case these columns are called *basic columns* while the nonzero components of x are called *basic variables*).

If, in addition, a basic solution x satisfies the non-negativity constraint $x \geq 0$, then x (is a feasible point and) is called a *basic feasible solution.*

Note that a basic solution can have no more than m nonzero components, because A cannot have $m + 1$ linearly independent columns. Why?

Remark. When a basic solution has exactly m nonzero components, the respective columns of A form a basis for $\mathbb{R}^m$; and as previously seen, we call those columns of A basic columns and the respective variables basic variables. When a basic solution has fewer than m nonzero components, the respective columns of A no longer form a basis for $\mathbb{R}^m$, but we can get a basis by adding to them some columns corresponding to zero components of the solution x. Once we have constructed such a basis for $\mathbb{R}^m$, we call the columns of A that form the basis basic columns and the respective variables basic variables. When the basic solution has m nonzero components, they determine uniquely the basic variables, whereas when the basic solution has fewer than m nonzero components there are multiple ways to choose the basic variables. In the latter case, we say that the basic solution is *degenerate.*

A basic feasible solution that optimizes the objective function $f(x) = c^T x$ is called *optimal.* Equivalently, an optimal feasible solution that is basic is said to be an *optimal basic feasible solution.*

Example 15.13, p. 325 and the same example with $b = (6, 2)^T$

Theorem (Fundamental Theorem of LP). Consider a linear program in standard form.

1. If there exists a feasible solution, then there exists a basic feasible solution.
2. If the linear program has an optimal feasible solution, then it has an optimal basic feasible solution.

Method to find basic feasible solutions. explained on Example 15.14, p. 329

2.6 Geometric View of Linear Programs

A point x in a convex set Θ is called an *extreme point* of Θ if

$$\exists x_1, x_2 \in \Theta,\ \exists \alpha \in (0,1):\ x = \alpha x_1 + (1-\alpha)x_2 \in \Theta \Rightarrow x_1 = x_2$$

Equivalent formulation:

Geometric interpretation: x is not an point of a (non-degenerate) segment with end-points (contained) in Θ

What did we call an extreme point for $\Theta \subset \mathbb{R}^2$ (in the plane)?

Theorem. Let $\Omega :=$
be the feasible region of a linear program in standard form.

Then x is an extreme point of Ω iff x is a basic feasible solution of the linear program.

Proof. Read it on p. 331. ■

Lemma. (Krein-Milman) If a linear (continuous) functional $f(x) = c^T x$ attains its minimum (maximum) over a convex set Θ then it attains its minimum (maximum) at an extreme point of Θ.

What does this lemma in conjunction with the previous theorem provide?

Interpretation:

When solving linear programming problems we need only examine the extreme points of the constraint set

or, equivalently, the

Example 15.15, p. 333 Maximize $3x_1 + 5x_2$ $\longrightarrow$
subject to

$$\begin{array}{rcl} x_1 + 5x_2 & \leq & 40 \\ 2x_1 + x_2 & \leq & 20 \\ x_1 + x_2 & \leq & 12 \\ x_1, x_2 & \geq & 0 \end{array} \quad \longrightarrow \quad \left\{ \begin{array}{ccccccc} x_1 + 5x_2 & +x_3 & & & & = & 40 \\ 2x_1 + x_2 + & & x_4 & & & = & 20 \\ x_1 + x_2 + & & & & x_5 & = & 12 \\ & & x_1, x_2, x_3, & & x_4, x_5 & \geq & 0 \end{array} \right.$$

Some other useful definitions related to extreme points and basic feasible solutions:

• Two bases for an $m \times n$ linear program are adjacent if they share $m - 1$ variables;

• Two basic feasible solutions are adjacent if their underlying bases are adjacent;

• Two extreme points are adjacent if their respective basic feasible solutions are adjacent. This is equivalent to saying that the line segment connecting them is an edge of the boundary of the feasible region.