

Homework Assignment 1

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Problem 1.1.2. Use Example 1.1.3 for affine maps to find the solutions to the difference equations:

i. $x_{n+1} - \frac{x_n}{3} = 2, x_0 = 2.$

ii. $x_{n+1} + 3x_n = 4, x_0 = -1.$

Solution. Consider the affine map $f : \mathbb{R} \rightarrow \mathbb{R}$ with $f(x) = ax + b$. Define the sequence $x_{n+1} = f(x_n) = ax_n + b$ where $x_0 \in \mathbb{R}$ is given. As was shown in the reading, the closed form solution to the above recurrence relation is given by

$$x_n = \left(x_0 - \frac{b}{1-a} \right) a^n + \frac{b}{1-a}. \quad (1)$$

Thus, the solutions to the provided difference equations can be solved by rewriting the equation in the form of an affine map, identifying a, b , and x_0 , and using the closed solution (1).

- i. For the difference equation $x_{n+1} - \frac{x_n}{3} = 2, x_0 = 2$, we readily see by rewriting the equation that $a = 1/3$ and $b = 2$ with $x_0 = 2$ given. Therefore, using (1), the solution to the difference equation is

$$\begin{aligned} x_n &= \left(x_0 - \frac{b}{1-a} \right) a^n + \frac{b}{1-a} \\ &= \left(2 - \frac{2}{1-1/3} \right) \left(\frac{1}{3} \right)^n + \frac{2}{1-1/3} \\ &= 3 - 3^{-n} \end{aligned}$$

- ii. For the difference equation $x_{n+1} + 3x_n = 4, x_0 = -1$, we readily see by rewriting the equation that $a = -3$ and $b = 4$ with $x_0 = -1$ given. Therefore, using (1), the solution to the difference equation is

$$\begin{aligned} x_n &= \left(x_0 - \frac{b}{1-a} \right) a^n + \frac{b}{1-a} \\ &= \left(-1 - \frac{4}{1-(-3)} \right) (-3)^n + \frac{4}{1-(-3)} \\ &= 1 - 2(-3)^n. \end{aligned}$$

□

Problem 1.1.3. A *logistic difference equation* is one of the form $x_{n+1} = \mu x_n(1 - x_n)$ for some fixed $\mu \in \mathbb{R}$. Find exact (closed form) solutions to the following logistic difference equations:

- i. $x_{n+1} = 2x_n(1 - x_n)$. Hint: Use the substitution $x_n = (1 - y_n)/2$ to transform the equation into a simpler equation that is easily solved.
- ii. $x_{n+1} = 4x_n(1 - x_n)$. Hint: Set $x_n = \sin^2(\theta_n)$ and simplify to get an equation that is easily solved.

Solution. i. Let $x_n = (1 - y_n)/2$ for $n \in \mathbb{N}$ with x_0 given. Substituting this expression into the original difference equation yields the new difference equation

$$\begin{aligned} \frac{1 - y_{n+1}}{2} &= 2 \left(\frac{1 - y_n}{2} \right) \left[1 - \left(\frac{1 - y_n}{2} \right) \right] \\ &= (1 - y_n) \left(\frac{1 + y_n}{2} \right) \\ &= \frac{1 - y_n^2}{2}. \end{aligned}$$

This new difference equation reduces to $y_{n+1} = y_n^2$ for $n \in \mathbb{N}$, the solution of which is readily seen to be $y_{n+1} = y_0^{2^{n+1}}$. Making the substitution $y_n = 1 - 2x_n$ shows that, for $n \in \mathbb{N}$, the solution to the original difference equation is given by

$$x_{n+1} = \frac{1 - (1 - 2x_0)^{2^{n+1}}}{2}.$$

- ii. Let $x_n = \sin^2(\theta_n)$ for $n \in \mathbb{N}$ with x_0 given. We may assume without loss of generality that $\theta_n \in [0, \pi)$ for if the angle θ_n isn't in the stated range, we can find an integer k such that $\theta_n + k\pi \in [0, \pi)$ and $\sin^2(\theta_n) = \sin^2(\theta_n + k\pi)$. We then declare the sum $\theta_n + k\pi$ to be the new angle θ_n . Substituting the above expression for x_n into the original difference equation yields the new difference equation

$$\begin{aligned} \sin^2(\theta_{n+1}) &= 4 \sin^2(\theta_n) (1 - \sin^2(\theta_n)) \\ &= (2 \sin(\theta_n) \cos(\theta_n))^2 \\ &= \sin^2(2\theta_n). \end{aligned}$$

Knowing that for $x, y \in [0, \pi)$ we have that $\sin^2(x) = \sin^2(y)$ if and only if $x = y$, the new difference equation reduces to $\theta_{n+1} = 2\theta_n$ for $n \in \mathbb{N}$ where it is implicitly understood that θ_{n+1} will be mapped to the corresponding angle between 0 and π if $2\theta_n \geq \pi$. Using the closed form solution for difference equations in the form of linear maps, the solution to the reduced difference equation is given by $\theta_{n+1} = 2^{n+1}\theta_0$ for $n \in \mathbb{N}$. Making the substitution $\theta_n = \sin^{-1}(\sqrt{x_n})$ shows that, for $n \in \mathbb{N}$, the solution to the original difference equation is given by

$$x_{n+1} = \sin^2(2^{n+1} \sin^{-1}(\sqrt{x_0})).$$

□

Problem 1.1.4. You borrow $\$P$ at $r\%$ per annum and pay off $\$M$ at the end of each subsequent month. Write down a difference equation for the amount owing $A(n)$ at the end of each month (so $A(0) = P$). Solve the equation to find a closed form for $A(n)$. If $P = 100,000$, $M = 1,000$, and $r = 4$, after how long will the loan be paid off?

Solution. Let $A(n)$ be the amount owed on the loan at the end of month n . If the principal amount of the loan is $\$P$, then $A(0) = P$. If the annual interest rate is $r\%$, then the monthly interest rate is $r/12\%$. Assuming each month a payment of $\$M$ is made on the loan, a difference equation representing the amount owed on the loan at the end of month n is given by

$$\begin{aligned} A(n+1) &= A(n) + A(n) \left[\frac{r}{12(100)} \right] - M \\ &= \left[1 + \frac{r}{12(100)} \right] A(n) - M \end{aligned}$$

for $n \in \mathbb{N}$.

Using the closed form solution for difference equations in the form of affine maps, the solution to the difference equation is given by

$$\begin{aligned} A(n) &= \left(A(0) + \frac{M}{1 - \left(1 + \frac{r}{12(100)} \right)} \right) \left(1 + \frac{r}{12(100)} \right)^n - \frac{M}{1 - \left(1 + \frac{r}{12(100)} \right)} \\ &= \left(P - \frac{1200M}{r} \right) \left(1 + \frac{r}{1200} \right)^n + \frac{1200M}{r}. \end{aligned}$$

The loan will be paid off after $k \in \mathbb{R}$ months when $A(k) = 0$ from which we can gather that the loan will be paid off after $n = \lceil k \rceil$ full months. Solving

$$A(k) = \left(100000 - \frac{1200(1000)}{4} \right) \left(1 + \frac{4}{1200} \right)^n + \frac{1200(1000)}{4} = 0$$

shows that $k = 121.842$. Therefore, the loan will be paid off in full after 122 months. \square

Problem 1.1.7. Let $f(x) = x^2 + bx + c$. Give conditions on b and c for $f : [0, 1] \rightarrow [0, 1]$ to be a dynamical system. Hint: Recall that the maximum and minimum values of a continuous function defined on a closed interval $[a, b]$ occur either at the end points or at the critical points of the function.

Solution. The function $f(x) = x^2 + bx + c$ for $f : [0, 1] \rightarrow [0, 1]$ is a dynamical system if the image of the function is contained in its domain, i.e. if $f([0, 1]) \subseteq [0, 1]$. The minimum and maximum values of a continuous function occur either at the end points of the domain or at the critical points of the function. Thus, for the continuous function f , if we ensure that the evaluation of f at $x = 0$, $x = 1$, and the critical points of f are contained in $[0, 1]$ then the image of f will necessarily be contained in $[0, 1]$ and f will be a dynamical system.

At the end points of the domain we have that $f(0) = c$ and $f(1) = b + c + 1$. Thus, in order for f to be a dynamical system, we must have that $c \in [0, 1]$ and $b + c \in [-1, 0]$.

The only critical point of the function f is found when $f'(x) = 0$ or when $x = -b/2$. Thus, we require that $f(-b/2) = -b^2/4 + c \in [0, 1]$. This reduces to requiring that $4c - 4 \leq b^2 \leq 4c$. Thus, when $b \in \mathbb{R}$, we must have that $b \in [-2\sqrt{c}, 2\sqrt{c}]$.

Combining all of these inequalities shows that in order for the image of f to be contained in the domain of f , we must have that $c \in [0, 1]$ and $b \in [-2\sqrt{c}, -c]$, i.e. the function $f(x) = x^2 + bx + c$ for $f : [0, 1] \rightarrow [0, 1]$ is a dynamical system if $0 \leq c \leq 1$ and $-2\sqrt{c} \leq b \leq -c$. \square

Problem 1.2.1. Give conditions on b and c for the map $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2 + bx + c$ to have a fixed point. Use these conditions to show that $f_c(x) = x^2 + c$ has a fixed point provided $c \leq 1/4$.

Solution. Let $g(x) = f(x) - x = x^2 + (b - 1)x + c$ for $g : \mathbb{R} \rightarrow \mathbb{R}$. From our definition of g , it is clear that the roots of the function g are the fixed points of the function f . Note that $g(x) = 0$ if

$$x = \frac{-b + 1 \pm \sqrt{(b - 1)^2 - 4c}}{2}. \quad (2)$$

However, in order for x to be a root of $g(x)$, we must have that $x \in \mathbb{R}$, i.e. we must have that $(b - 1)^2 - 4c \geq 0$. Thus, x is a fixed point of the function f if x is of the form (2) and for $b, c \in \mathbb{R}$ we have that $c \leq (b - 1)^2/4$.

Take the function $f_c(x) = x^2 + c$ for $f_c : \mathbb{R} \rightarrow \mathbb{R}$. Note that f_c has the same form as the function f if $b = 0$. Thus, according to the conditions described above, we see that f_c has a fixed point if $c \leq (0 - 1)^2/4 = 1/4$.

□

Problem 1.2.6. Consider the eventual fixed points of the logistic map $L_\mu : [0, 1] \rightarrow [0, 1]$, $L_\mu(x) = \mu x(1 - x)$ for $0 < \mu < 4$.

- i. Show that there are no eventual fixed points associated with the fixed point $x = 0$, other than $x = 1$.
- ii. Show that for $1 < \mu \leq 2$, the only eventual fixed point associated with the fixed point $x = 1 - 1/\mu$ is $x = 1/\mu$.
- iii. Show that there are additional eventual fixed points associated with $x = 1 - 1/\mu$ when $2 < \mu < 3$.
- iv. Investigate the eventual fixed points of the logistic map when $\mu = 5/2$.

Solution. i. It is clear that $x = 1$ is an eventual fixed point since $x = 0$ is a fixed point and $L_\mu(1) = 0$. This is the only eventual fixed point associated to $x = 0$ since no point in the interval $(0, 1)$ maps to either 0 or 1 under L_μ , i.e. for $y \in (0, 1)$, the equations $L_\mu(y) = 0$ and $L_\mu(y) = 1$ have no solutions. Therefore, since no $y \in D_{L_\mu} = [0, 1]$ besides $y = 0$ and $y = 1$ maps to 0 or 1, there are no other eventual fixed points associated to $x = 0$.

ii. Let $1 < \mu \leq 2$. It is clear that $x = 1/\mu$ is an eventual fixed point since $x = 1 - 1/\mu$ is a fixed point and $L_\mu(1/\mu) = 1 - 1/\mu$. We will now demonstrate that this is the only eventual fixed point associated to $x = 1 - 1/\mu$. Note that for $x \in [0, 1]$, the only solution to $L_\mu(x) = 1 - 1/\mu$ is $x = 1/\mu$. Therefore, in order for a point $x \in [0, 1]$ to be an eventual fixed point associated to $x = 1 - 1/\mu$, we must have that x either maps to $1/\mu$ or eventually maps to $1/\mu$, i.e. for $x \in [0, 1]$ we must have that $L_\mu(x) = 1/\mu$ has a solution. However, if $1 < \mu \leq 2$, then $L_\mu(x) = 1/\mu$ has no real solutions for $x \in [0, 1]$ and so there are no other eventual fixed points associated to $x = 1 - 1/\mu$.

iii. Now suppose that $2 < \mu < 3$. Recall that $x = 1/\mu$ is an eventual fixed point. Note that

$$y = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4}{\mu^2}} \in [0, 1]$$

satisfies $L_\mu(y) = 1/\mu$. Thus, $L_\mu^2(y) = 1 - 1/\mu$ where $1 - 1/\mu$ is a fixed point. Therefore, if $2 < \mu < 3$, then there are additional eventual fixed points associated to $x = 1 - 1/\mu$ besides $x = 1/\mu$.

iv. We proceed to describe all eventual fixed points of L_μ by first identifying all fixed points of the function. Suppose $\{x_{0_n}\}$ is the set of fixed points of L_μ where n is the number of fixed points. To find all eventual fixed points associated to the fixed point x_{0_k} , first find the pre-image of x_{0_k} minus the point x_{0_k} , i.e. find

$$L_\mu^{-1}(x_{0_k}) = \{x \neq x_{0_k} \mid L_\mu(x) = x_{0_k}\}.$$

Note that each point in $L_\mu^{-1}(x_{0_k})$ will be an eventual fixed point of x_{0_k} . Denote the set of eventual fixed points associated to x_{0_k} by x_{1_k} . If the set is empty, then there are no eventual fixed points associated to x_{0_k} . If the set is non-empty, continue the process by

finding for each point in the set of eventual fixed points x_{1_k} , the set of eventual fixed points associated to the eventual fixed point x_{1_k} , i.e. $x_{2_k} = L_\mu^{-1}(x_{1_k}) = L_\mu^{-1}(L_\mu^{-1}(x_{0_k}))$. Again, every point in x_{2_k} is an eventual fixed point of x_{0_k} . Continue this process indefinitely until the pre-image x_{m_k} is empty.

Now suppose that $\mu = 5/2$. The fixed points of L_μ are found by finding the roots of $g(x) = L_\mu(x) - x = (3/2)x - (5/2)x^2$ for $g : [0, 1] \rightarrow [0, 1]$. It is clear that the two roots of g are given by $x = 0$ and $x = 3/5$. Thus, denote the fixed points of L_μ by $\{x_{0_0}, x_{0_1}\} = \{0, 3/5\}$.

We will now find all eventual fixed points associated to the fixed point $x_{0_0} = 0$. As was shown previously, the only eventual fixed points associated to x_{0_0} is $x = 1$. Thus, $x_{1_0} = \{1\}$ and we are done.

Now we will find all eventual fixed points associated to the fixed point $x_{0_1} = 3/5$. Solving the equation $L_\mu(x) = x_{0_1}$ shows us that the pre-image of x_{0_1} minus the point x_{0_1} is given by $L_\mu^{-1}(x_{0_1}) = \{2/5\}$, i.e. $x_{1_1} = \{2/5\}$. Continuing, we solve the equation $L_\mu(x) = 2/5$ and find that $L_\mu^{-1}(x_{1_1}) = \{1/5, 4/5\}$. Thus $x_{2_1} = \{1/5, 4/5\}$. Since, $L_\mu(x) = 4/5$ has no real solutions, there are no eventual fixed points associated to the eventual fixed point $4/5$. However, solving $L_\mu(x) = 1/5$ we see that there are real solutions associated to this eventual fixed point so $x_{3_1} = \{1/10(5 - \sqrt{17}), 1/10(5 + \sqrt{17})\}$.

Under further investigation the above sequence continues in a similar pattern. The equation $L_\mu(x) = 1/10(5 + \sqrt{17})$ will have no real solutions but $L_\mu(x) = 1/10(5 - \sqrt{17})$ does. Thus, we see using these solutions that $x_{4_1} = \{1/10(5 - \sqrt{5 + 4\sqrt{17}}), 1/10(5 + \sqrt{5 + 4\sqrt{17}})\}$. This process repeats indefinitely so that two more eventual fixed points are found with the set of eventual fixed points x_{k_1} . The union of all such points as described above make up the eventual fixed points associated to $x = 3/5$.

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