

Exam 3

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Problem 1. Solve the non-homogeneous diffusion problem by the Hankel transform

$$\begin{aligned} u_t &= a \left(u_{rr} + \frac{1}{r} u_r \right) + Q(r, t), \quad 0 < r < \infty, \quad 0 < t \\ u(r, 0) &= f(r), \quad 0 < r < \infty. \end{aligned}$$

Solution. Application of the 0-th order Hankel transform will transform the above Partial Differential Equation into an Ordinary Differential Equation. The following property of the 0-th order Hankel transform will aid in the application; if $\mathcal{H}_0 \{u(r, t)\} = \tilde{u}_0(\kappa, t)$, then

$$\mathcal{H}_0 \left\{ \frac{1}{r} \frac{\partial}{\partial r} [u(r, t)] + \frac{\partial^2}{\partial r^2} [u(r, t)] \right\} = -\kappa^2 \tilde{u}_0(\kappa, t). \quad (1)$$

Now, with the above property, we see that applying the 0-th order Hankel transform to the diffusion problem yields

$$\begin{aligned} \frac{d}{dt} [\tilde{u}_0(\kappa, t)] + a\kappa^2 \tilde{u}_0(\kappa, t) &= \tilde{Q}_0(\kappa, t), \quad 0 < \kappa < \infty, \quad 0 < t \\ \tilde{u}_0(\kappa, 0) &= \tilde{f}_0(\kappa), \quad 0 < \kappa < \infty. \end{aligned}$$

This is a first order linear Ordinary Differential Equation, the solution to which is

$$\tilde{u}_0(\kappa, t) = c_1(\kappa) e^{-a\kappa^2 t} + e^{-a\kappa^2 t} \int_0^t e^{a\kappa^2 x} \tilde{Q}_0(\kappa, x) dx.$$

Thus, from this solution and the transformed boundary condition, we see that $c_1(\kappa) = \tilde{f}_0(\kappa)$ and the solution to the transformed boundary value problem is

$$\tilde{u}_0(\kappa, t) = \tilde{f}_0(\kappa) e^{-a\kappa^2 t} + e^{-a\kappa^2 t} \int_0^t e^{a\kappa^2 x} \tilde{Q}_0(\kappa, x) dx.$$

Therefore, the solution to the initial diffusion problem is

$$\begin{aligned} u(r, t) &= \mathcal{H}_0^{-1} \{ \tilde{u}_0(\kappa, t) \} = \mathcal{H}_0^{-1} \left\{ \tilde{f}_0(\kappa) e^{-a\kappa^2 t} + e^{-a\kappa^2 t} \int_0^t e^{a\kappa^2 x} \tilde{Q}_0(\kappa, x) dx \right\} \\ &= \int_0^\infty \kappa J_0(\kappa r) \left[\tilde{f}_0(\kappa) e^{-a\kappa^2 t} + e^{-a\kappa^2 t} \int_0^t e^{a\kappa^2 x} \tilde{Q}_0(\kappa, x) dx \right] d\kappa, \end{aligned}$$

where $J_0(\kappa r)$ is the Bessel function of order 0.

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Problem 2.*Solution.*

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Problem 3. Solve the following integral equation by the Mellin transform

$$f(x) = \sin ax + \int_0^\infty \frac{f(xt)}{1+t^2} dt.$$

Solution. Let $g(x) = \frac{1}{1+x^2}$ and $h(x) = \sin ax$. Recall that $(f \circ g)(x)$ is defined to be

$$(f \circ g)(x) = \int_0^\infty f(xt)g(t)dt.$$

Thus, with this knowledge, the integral equation becomes

$$\begin{aligned} f(x) &= h(x) + \int_0^\infty f(xt)g(t)dt \\ &= h(x) + (f \circ g)(x). \end{aligned}$$

Let $\mathcal{M}\{f(x)\} = \tilde{f}(p)$, $\mathcal{M}\{g(x)\} = \tilde{g}(p)$, and $\mathcal{M}\{h(x)\} = \tilde{h}(p)$. Then from the Convolution Type theorem regarding the Mellin transform, we see that application of the Mellin transform to the integral equation yields

$$\begin{aligned} \tilde{f}(p) &= \mathcal{M}\{h(x)\} + \mathcal{M}\{(f \circ g)(x)\} \\ &= \tilde{h}(p) + \tilde{f}(p)\tilde{g}(1-p). \end{aligned}$$

Solving the above algebraic equation shows that

$$\tilde{f}(p) = \frac{\tilde{h}(p)}{1 - \tilde{g}(1-p)}.$$

From our table of Mellin transforms we know that

$$\tilde{g}(p) = \frac{\pi}{2} \csc\left(\frac{\pi p}{2}\right)$$

and

$$\tilde{h}(p) = a^{-p}\Gamma(p) \sin\left(\frac{\pi p}{2}\right).$$

Therefore, we see that

$$\begin{aligned} \tilde{f}(p) &= \frac{a^{-p}\Gamma(p) \sin\left(\frac{\pi p}{2}\right)}{1 - \frac{\pi}{2} \csc\left(\frac{\pi(1-p)}{2}\right)} \\ &= \frac{2a^{-p}\Gamma(p) \sin\left(\frac{\pi p}{2}\right)}{2 - \pi \sec\left(\frac{\pi p}{2}\right)} \end{aligned}$$

and the solution to the integral equation is

$$\begin{aligned} f(x) &= \mathcal{M}^{-1}\{\tilde{f}(p)\} = \mathcal{M}^{-1}\left\{\frac{2a^{-p}\Gamma(p) \sin\left(\frac{\pi p}{2}\right)}{2 - \pi \sec\left(\frac{\pi p}{2}\right)}\right\} \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-p} \left[\frac{2a^{-p}\Gamma(p) \sin\left(\frac{\pi p}{2}\right)}{2 - \pi \sec\left(\frac{\pi p}{2}\right)} \right] dp. \end{aligned}$$

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Problem 4. Solve the following Partial Differential Equation by the Mellin transform

$$\begin{aligned} r^2 \phi_{rr} + r \phi_r + \phi_{\theta\theta} &= 0, \quad 0 < r < \infty, \quad 0 < \theta < \pi \\ \phi(r, 0) &= \begin{cases} (1-r)^2 & 0 < r < 1 \\ 0 & 1 < r \end{cases} \\ \phi(r, \pi) &= \begin{cases} 1 & 0 < r < 1 \\ 0 & 1 < r \end{cases}, \end{aligned}$$

Solution. Recall that if $\mathcal{M}\{\phi(r, \theta)\} = \tilde{\phi}(p, \theta)$, then the following property holds

$$\mathcal{M}\left\{r^2 \frac{\partial^2}{\partial r^2} [\phi(r, \theta)] + r \frac{\partial}{\partial r} [\phi(r, \theta)]\right\} = p^2 \tilde{\phi}(p, \theta).$$

Thus, applying the Mellin transform to the Partial Differential Equation and using our table of Mellin transforms, we see that

$$\begin{aligned} \frac{d^2}{d\theta^2} [\tilde{\phi}(p, \theta)] + p^2 \tilde{\phi}(p, \theta) &= 0, \quad 0 < p < \infty, \quad 0 < \theta < \pi \\ \tilde{\phi}(p, 0) &= \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)}, \quad \tilde{\phi}(p, \pi) = \frac{1}{p}. \end{aligned}$$

The solution to the resulting homogeneous linear Ordinary Differential Equation is

$$\tilde{\phi}(p, \theta) = c_1(p) \cos p\theta + c_2(p) \sin p\theta.$$

Using the above solution and the transformed boundary conditions, we see that

$$\begin{aligned} c_1(p) &= \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)} \\ c_1(p) \cos p\pi + c_2(p) \sin p\pi &= \frac{1}{p}. \end{aligned}$$

Solving, we see that

$$\begin{aligned} c_2(p) &= \left(\frac{1}{p} - \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)} \cos p\pi \right) \csc p\pi \\ &= \frac{\csc p\pi}{p} - \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)} \cot p\pi. \end{aligned}$$

Thus, the solution to the transformed boundary value problem is

$$\tilde{\phi}(p, \theta) = \left[\frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)} \right] \cos p\theta + \left[\frac{\csc p\pi}{p} - \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)} \cot p\pi \right] \sin p\theta.$$

Therefore, the solution to the original boundary value problem is

$$\begin{aligned} \phi(r, \theta) &= \mathcal{M}^{-1} \left\{ \left[\frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)} \right] \cos p\theta + \left[\frac{\csc p\pi}{p} - \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)} \cot p\pi \right] \sin p\theta \right\} \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} r^{-p} \left\{ \left[\frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)} \right] \cos p\theta + \left[\frac{\csc p\pi}{p} - \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)} \cot p\pi \right] \sin p\theta \right\} dp. \end{aligned}$$

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Problem 5.*Solution.*

Problem 6.*Solution.*

Problem 7.*Solution.*

Problem 8.*Solution.*

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