Homework Assignment 7

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Problem 9.1. For the following problems for 9.1, suppose a function $f : [a, b] \to \mathbb{R}$ is only known at distinct sites $x = [x_1, x_2, \ldots, x_n]$ where $x_i \in [a, b]$, for $i = 1, 2, \ldots n$. Let $p_n(f, t)$ be the Lagrange interpolating polynomial at these sites.

Problem 9.1.1. Show that the basic quadrature $J(f) := \int_a^b p_n(f, t)\,\mathrm{d}t$ satisfies $J(f) = \sum_{j=1}^n w_j f(x_j)$ where the weights w_j depend on the Lagrange basis.

Solution. Note the Lagrange interpolating polynomial of f through the nodes $x_1, x_2, \ldots, x_n$ is given by

$$p_n(f, t) = \sum_{j=1}^n f(x_j) \prod_{\substack{i=1 \\ i \neq j}} \frac{t - x_i}{x_j - x_i}.$$

If $J(f) := \int_a^b p_n(f, t)$, then, using this definition of the Lagrange interpolating polynomial, it is clear that

$$\begin{aligned} J(f) = \int_a^b p_n(f, t)\,\mathrm{d}t &= \int_a^b \left[\sum_{j=1}^n f(x_j) \prod_{\substack{i=1 \\ i \neq j}} \frac{t - x_i}{x_j - x_i} \right] \mathrm{d}t \\ &= \sum_{j=1}^n \left[\int_a^b \prod_{\substack{i=1 \\ i \neq j}} \frac{t - x_i}{x_j - x_i}\,\mathrm{d}t \right] f(x_j) = \sum_{j=1}^n w_j f(x_j). \end{aligned}$$

Thus, $J(f)$ is of the form $\sum_{j=1}^n w_j f(x_j)$ where w_j depends on the Lagrange basis $l_j(t) = \prod_{\substack{i=1 \\ i \neq j}} \frac{t - x_i}{x_j - x_i}$. $\square$

Problem 9.1.2. Show that $J(f)$ has degree of precision at least $n - 1$.

Solution. Let $q(t)$ be a polynomial of degree $n - 1$. Then,

$$q(t) = \sum_{j=1}^n q(x_i) \prod_{\substack{i=1 \\ i \neq j}} \frac{t - x_i}{x_j - x_i},$$

i.e. the Lagrange interpolating polynomial of q through the nodes $x_1, x_2, \ldots x_n$ is q itself. Hence, the exact integral of q, $I(q) = \int_a^b q(t)\,\mathrm{d}t$, satisfies

$$\begin{aligned} I(q) = \int_a^b q(t)\,\mathrm{d}t &= \int_a^b \sum_{j=1}^{n} q(x_j) \prod_{\substack{i=1\\ i\neq j}} \frac{t-x_i}{x_j-x_i}\,\mathrm{d}t \\ &= \sum_{j=1}^{n} \left[\int_a^b \prod_{\substack{i=1\\ i\neq j}} \frac{t-x_i}{x_j-x_i}\,\mathrm{d}t \right] q(x_j) = J(q). \end{aligned}$$

Since q is a polynomial of degree $n-1$ and $I(q) = J(q)$, we know that $J(f)$ has degree of precision at least $n-1$. $\square$

Problem 9.1.3. Show that if $f \in C^n[a,b]$, then the truncation error can be bounded in terms of the nodal polynomial as follows:

$$|R(f)| \leq \frac{1}{n!} \max_{t\in[a,b]} |f^{(n)}(t)| \int_a^b |\Pi_n(t)| \mathrm{d}t$$

Solution. Let $f \in C^n([a,b])$. Note the truncation error is given by $R(f) = I(f) - J(f)$. Since $f \in C^n([a,b])$ and the Lagrange interpolating polynomial p_n satisfies $p_n(f, x_i) = f(x_i)$ for $i = 1, 2, \ldots, n$, there is a point ξ_x in the smallest interval containing $[a,b]$ and every x_i such that

$$R(f) = I(f) - J(f) = \int_a^b f(t)\,\mathrm{d}t - \int_a^b p_n(f,t)\,\mathrm{d}t = \frac{1}{n!} \int_a^b f^{(n)}(\xi_x) \Pi_n(t)\,\mathrm{d}t$$

where $\Pi_n(t)$ is the nodal polynomial $\Pi_n(t) = \prod_{j=1}^{n} (t - x_j)$.

From this identity, it is clear that

$$\begin{aligned} |R(f)| &= \left| \frac{1}{n!} \int_a^b f^{(n)}(\xi_x)\, \Pi_n(t)\,\mathrm{d}t \right| \\ &\leq \frac{1}{n!} \left| f^{(n)}(\xi_x) \right| \int_a^b \Pi_n(t)\,\mathrm{d}t \\ &\leq \frac{1}{n!} \max_{t\in[a,b]} \left| f^{(n)}(t) \right| \int_a^b \Pi_n(t)\,\mathrm{d}t \end{aligned}$$

since $\left|f^{(n)}(\xi_x)\right| \leq \max_{t\in[a,b]} \left|f^{(n)}(t)\right|$ as $\xi_x \in [a,b]$ and we are done. $\square$

Problem 9.3.1. In the following, for a function $f : [a,b] \to \mathbb{R}$, f_i is shorthand for $f(x_i)$, with $x_i = a + (i-1)(b-a)/(n-1)$. For $n = 4$, consider **Simpson's 3/8 rule**

$$J_{S38}(f) = \frac{b-a}{8}(f_1 + 3f_2 + 3f_3 + f_4).$$

Choose the interval $[0,1]$. Find the exact degree of precision. The error is given by $R_{S38}(f) = c_{S38} f^{(4)}(\xi)$. Find c_{S38} using MATLAB and a polynomial for f.

Solution. Note that on the interval [0, 1],

$$J_{S38}(f) = \frac{b-a}{f_1 + 3f_2 + 3f_3 + f_4} = \frac{1}{8}(f(0) + 3f(1/3) + 3f(2/3) + f(1)).$$

To see that the exact degree of precision of this quadrature is 3, note that

$$I(x^3) = \int_0^1 x^3 \, \mathrm{d}x = \frac{1}{4} = \frac{1}{8}\left((0)^3 + 3(1/3)^3 + 3(2/3)^3 + (1)^3\right) = J_{S38}(x^3)$$

and similarly

$$I(x^2) = \int_0^1 x^2 \, \mathrm{d}x = \frac{1}{3} = \frac{1}{8}\left((0)^2 + 3(1/3)^2 + 3(2/3)^2 + (1)^2\right) = J_{S38}(x^2)$$
$$I(x^1) = \int_0^1 x^1 \, \mathrm{d}x = \frac{1}{2} = \frac{1}{8}\left((0)^1 + 3(1/3)^1 + 3(2/3)^1 + (1)^1\right) = J_{S38}(x^1)$$
$$I(x^0) = \int_0^1 x^0 \, \mathrm{d}x = 1 = \frac{1}{8}\left((0)^0 + 3(1/3)^0 + 3(2/3)^0 + (1)^0\right) = J_{S38}(x^0)$$

but

$$I(x^4) = \int_0^1 x^4 \, \mathrm{d}x = \frac{1}{5} \neq \frac{11}{54} = \frac{1}{8}\left((0)^4 + 3(1/3)^4 + 3(2/3)^4 + (1)^4\right) = J_{S38}(x^4).$$

Since $I(x^i) = J_{S38}(x^i)$ for all $0 \leq i \leq 3$ the quadrature rule is the same as the integral for all polynomial of degree 3 or less, but $I(x^4) \neq J_{S38}(x^4)$, the exact degree of precision must be 3.

To find c_{S38}, choose $f(x) = x^4$. Then, as shown above, $I(x^4) = 1/5$ and $J_{S38} = 11/54$, so $R_{S38}(f) = I_{S38}(f) - J_{S38}(f) = -0.0037037 = c_{S38} f^{(4)}(\xi)$. Since $f^{(4)}(\xi) = 24$ for our choice of f, it follows that $c_{S38} = -0.00015432$. □

Problem 9.4.1. Given two real numbers α_1 and α_2 and two values t_1, t_2 in [-1, 1]. For a function $f : [-1, 1] \to \mathbb{R}$, consider the basic quadrature $J_1(f) := \alpha_1 f(t_1) + \alpha_2 f(t_2)$ to approximate $\int_{-1}^{1} f(t) \, \mathrm{d}t$ How do we choose the four numbers α_1, α_2, t_1, and t_2 to obtain the maximum degree of precision?

Solution. For the quadrature $J_1(f)$ as defined in the problem, we have that

$$\int_{-1}^{1} t^0 \, \mathrm{d}t = 2 = \alpha_1 f(t_1) + \alpha_2 f(t_2) = \alpha_1 + \alpha_2$$
$$\int_{-1}^{1} t^1 \, \mathrm{d}t = 0 = \alpha_1 f(t_1) + \alpha_2 f(t_2) = \alpha_1 t_1 + \alpha_2 t_2$$
$$\int_{-1}^{1} t^2 \, \mathrm{d}t = \frac{2}{3} = \alpha_1 f(t_1) + \alpha_2 f(t_2) = \alpha_1 t_1^2 + \alpha_2 t_2^2$$
$$\int_{-1}^{1} t^3 \, \mathrm{d}t = 0 = \alpha_1 f(t_1) + \alpha_2 f(t_2) = \alpha_1 t_1^3 + \alpha_2 t_2^3.$$

This gives us a system of four equations in four unknowns. Solving the system yields the solution $\alpha_1 = \alpha_2 = 1$, and $t_1 = \frac{1}{\sqrt{3}}$ and $t_2 = -\frac{1}{\sqrt{3}}$.

Since the quadrature rule is such that $I(p) = J_1(p)$ for any polynomial of degree less than 4, by our own definition, this quadrature rule has maximal degree of precision 3 as the rule does not hold for polynomials of degree greater than or equal to 4. $\square$