Homework Assignment 11

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Problem 5.1. The sunspot numbers $\{X_t, t = 1, \dots, 100\}$, filed as `SUNSPOTS.TSM`, have sample autocovariances $\hat{\gamma}(0) = 1382.2$, $\hat{\gamma}(1) = 1114.4$, $\hat{\gamma}(2) = 591.73$, and $\hat{\gamma}(3) = 96.216$. Use these values to find the Yule-Walker estimates of ϕ_1, ϕ_2, and σ^2 in the model

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + Z_t, \quad \{Z_t\} \sim \mathrm{WN}(0, \sigma^2),$$

for the mean-corrected series $Y_t = X_t - 46.93$, $t = 1, \dots, 100$. Assuming the data really are a realization of an AR(2) process, find 95% confidence intervals for ϕ_1 and ϕ_2.

Solution. We wish to find $\hat{\phi}_1, \hat{\phi}_2$, and $\hat{\sigma}^2$ given $\hat{\gamma}(0)$, $\hat{\gamma}(1)$, and $\hat{\gamma}(2)$. By the Yule-Walker equations for sample autocovariances,

$$\begin{bmatrix} \hat{\gamma}(0) & \hat{\gamma}(1) \\ \hat{\gamma}(1) & \hat{\gamma}(0) \end{bmatrix} \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{bmatrix} = \begin{bmatrix} \hat{\gamma}(1) \\ \hat{\gamma}(2) \end{bmatrix}.$$

Solving this system yields $\hat{\phi}_1 = 1.31755$ and $\hat{\phi}_2 = -0.634168$. Using the Yule-Walker equation $\hat{\sigma}^2 = \hat{\gamma}(0) - \hat{\phi}_1 \hat{\gamma}(1) - \hat{\phi}_2 \hat{\gamma}(2)$, we see that $\hat{\sigma}^2 = 289.2$.

Since our sample size $n = 100$ is large, a 95% confidence interval for the parameter ϕ_j is given by

$$\hat{\phi}_j \pm \Phi_{1-\frac{\alpha}{2}} n^{-1/2} \hat{\nu}_{jj}^{1/2}$$

where $\Phi_{1-\frac{\alpha}{2}} = 1.96$ and $\hat{\nu}_{jj}$ is the j-th element on the diagonal of $\hat{\sigma}^2 \Gamma_2^{-1}$ for $j = 1, 2$. Using this formula, we see that $\nu_{jj} = 0.5979$ for $j = 1, 2$ and 95% confidence intervals for the model parameters are given by

$$\begin{aligned} (1.1660, 1.4961) &\quad \text{for } \phi_1 \\ (-0.7858, -0.4827) &\quad \text{for } \phi_2. \end{aligned}$$

□

Problem 5.2. From the information given in the previous problem, use the Durbin-Levinson algorithm to compute the sample partial autocorrelations $\hat{\phi}_{11}$, $\hat{\phi}_{22}$, and $\hat{\phi}_{33}$ of the sunspot series. Is the value of $\hat{\phi}_{33}$ compatible with the hypothesis that the data are generated by an AR(2) process? (Use significance level $\alpha = 0.05$.)

Solution. Note that the sample partial autocorrelation function is given by $\hat{\alpha}(0) = 1$ and $\hat{\alpha}(n) = \hat{\phi}_{nn}$. We can use the Durbin-Levinson algorithm to compute $\hat{\phi}_{nn}$. Following the recursive equations presented by the algorithm, we have that for $n = 1$,

$$\begin{aligned}\hat{\phi}_{11} &= \hat{\gamma}(1)/\hat{\gamma}(0) = 0.8063\\ \hat{\nu}_1 &= \hat{\nu}_0(1 - \hat{\rho}(1)^2) = 483.7140.\end{aligned}$$

Similarly, for $n = 2$, we have that

$$\begin{aligned}\hat{\phi}_{22} &= \nu_1^{-1}(\hat{\gamma}(2) - \hat{\phi}_{11}\hat{\gamma}(1)) = -0.6342\\ \hat{\phi}_{21} &= \hat{\phi}_{11} - \hat{\phi}_{22}\hat{\phi}_{11} = 1.3175\\ \hat{\nu}_2 &= \hat{\nu}_1(1 - \hat{\phi}_{22}^2) = 289.1791.\end{aligned}$$

Thus, for $n = 3$,

$$\hat{\phi}_{33} = \nu_2^{-1}(\hat{\gamma}(3) - \hat{\phi}_{21}\hat{\gamma}(2) - \hat{\phi}_{22}\hat{\gamma}(1)) = 0.0806.$$

Note that a process is an AR(2) process if $\alpha(n) = 0$ for $n > 2$. As we have a sample size of $n = 100$, we have for a significance level $\alpha = 0.05$ the identically-zero bounds $0 \pm 1.96/\sqrt{100}$. Using these bounds we see that $\hat{\alpha}(3) = \hat{\phi}_{33} = 0.0806$ which does in fact fall within our identically-zero bounds. So, the data suggests $\hat{\alpha}(3)$ is identically 0 which supports the hypothesis that the data is a realization of an AR(2) process. □

Problem 5.3. Consider the AR(2) process $\{X_t\}$ satisfying

$$X_t - \phi X_{t-1} - \phi^2 X_{t-2} = Z_t, \quad \{Z_t\} \sim \mathrm{WN}(0, \sigma^2).$$

a. For what values of ϕ is this a causal process?

b. The following sample moments were computed after observing $X_1, \ldots, X_{200}$:

$$\hat{\gamma}(0) = 6.06, \quad \hat{\rho}(1) = 0.687.$$

Find estimates of ϕ and σ^2 by solving the Yule-Walker equations. (If you find more than one solution, choose the one that is causal.)

Solution. The characteristic polynomial of this AR(2) process is given by $\phi(z) = 1 - \phi z - \phi^2 z^2$. The AR(2) process $\{X_t\}$ is causal if the roots of $\phi(z)$ occur outside the unit circle. Note that the roots of $\phi(z)$ are given by $z_1 = \frac{-1+\sqrt{5}}{2}$ and $z_2 = \frac{-1-\sqrt{5}}{2}$. Note that $|z_i| > 1$ for $i = 1, 2$ if $|\phi| < \frac{-1+\sqrt{5}}{2}$. Therefore, the process is causal if $|\phi| < \frac{-1+\sqrt{5}}{2} = 0.618034$.

Given $\hat{\gamma}(0) = 6.06$ and $\hat{\rho}(1) = \frac{\hat{\gamma}(1)}{\hat{\gamma}(0)} = 0.687$, we see that $\hat{\gamma}(1) = 4.16322$. By the Yule-Walker equations,

$$\begin{bmatrix}\hat{\gamma}(0) & \hat{\gamma}(1)\\ \hat{\gamma}(1) & \hat{\gamma}(0)\end{bmatrix}\begin{bmatrix}\phi\\ \phi^2\end{bmatrix} = \begin{bmatrix}\hat{\gamma}(1)\\ \hat{\gamma}(2)\end{bmatrix}.$$

The solution to this system of equations suggests that $\phi = 1.30106 - 0.214696\hat{\gamma}(2)$ and $\phi^2 = -0.893828 + 0.3125\hat{\gamma}(2)$. Therefore,

$$1.30106 - 0.214696\hat{\gamma}(2) = \sqrt{\phi^2} = \pm\sqrt{-0.893828 + 0.3125\hat{\gamma}(2)}.$$

This suggests that $\hat{\gamma}(2) = 15.2107$ or $\hat{\gamma}(2) = 3.68918$. For $\hat{\gamma}(2) = 15.2107$, we have that $\phi = -1.96462$. As this value of ϕ violates the causality of the process, we reject this value so $\hat{\gamma}(2) = 3.68918$ and $\phi = 0.509008$. These values of ϕ and $\hat{\gamma}(2)$ can be used to show that

$$\sigma^2 = \hat{\gamma}(0) - \phi\hat{\gamma}(1) - \phi^2\hat{\gamma}(2) = 2.98506.$$

□

Problem 5.4. Two hundred observations of a time series $X_1, \ldots, X_{200}$, gave the following sample statistics:

$$\begin{array}{ll} \text{sample mean:} & \bar{x}_{200} = 3.82; \\ \text{sample variance:} & \hat{\gamma}(0) = 1.15; \\ \text{sample ACF:} & \hat{\rho}(1) = 0.427; \\ & \hat{\rho}(2) = 0.475; \\ & \hat{\rho}(3) = 0.169. \end{array}$$

a. Based on these sample statistics, is it reasonable to suppose that $\{X_t - \mu\}$ is white noise?

b. Assuming $\{X_t - \mu\}$ can be modeled as an AR(2) process

$$X_t - \mu - \phi_1(X_{t-1} - \mu) - \phi_2(X_{t-2} - \mu) = Z_t,$$

where $\{Z_t\} \sim \mathrm{IID}(0, \sigma^2)$, find estimates of μ, ϕ_1, ϕ_2, σ^2.

c. Would you conclude that $\mu = 0$?

d. Construct 95% confidence intervals for ϕ_1 and ϕ_2.

e. Assuming that the data were generated from an AR(2) model, derive estimates for the PACF for all lags $h \geq 1$.

Solution. a. The process $\{X_t - \mu\}$ is white noise with mean 0 if $\rho(i) = 0$ for $i > 0$. As the process $\{X_t - \mu\}$ is zero-mean, a 95% confidence interval for zero with 200 samples is given by

$$\left(-\frac{1.96}{\sqrt{(200)}}, \frac{1.96}{\sqrt{(200)}} \right) = (-0.1386, 0.1386).$$

As the sample autocorrelations fall outside this interval, they are not identically 0 and we reject the hypothesis that the process $\{X_t - \mu\}$ is white noise.

b. By the Yule-Walker equations,

$$\begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_1 \end{bmatrix} = \begin{bmatrix} \hat{\rho}(0) & \hat{\rho}(1) \\ \hat{\rho}(1) & \hat{\rho}(0) \end{bmatrix}^{-1} \begin{bmatrix} \hat{\rho}(1) \\ \hat{\rho}(2) \end{bmatrix}$$

so we see that $\hat{\phi}_1 = 0.2742$ and $\hat{\phi}_2 = 0.3579$ and

$$\hat{\sigma}^2 = \hat{\gamma}(0) \left[1 - \begin{bmatrix} \hat{\rho}(1) \\ \hat{\rho}(2) \end{bmatrix}^{\intercal} = \begin{bmatrix} \hat{\rho}(0) & \hat{\rho}(1) \\ \hat{\rho}(1) & \hat{\rho}(0) \end{bmatrix}^{-1} \begin{bmatrix} \hat{\rho}(1) \\ \hat{\rho}(2) \end{bmatrix} \right] = 0.8199.$$

An estimate for μ is given by the sample mean, i.e. $\hat{\mu} = \bar{x}_{200} = 3.82$.

c. As our sample size $n = 200$ is large, the sample mean is normally distribute with mean μ and variance $\sum \gamma(i) \approx 3.61$. Thus, a 95% confidence interval for the mean is given by (3.5567, 4.0833). As $\mu = 0$ falls outside this interval, we reject the hypothesis that $\mu = 0$.

d. Since our sample size $n = 200$ is large, a 95% confidence interval for the parameter ϕ_j is given by

$$\hat{\phi}_j \pm \Phi_{1-\frac{\alpha}{2}} n^{-1/2} \hat{\nu}_{jj}^{1/2}$$

where $\Phi_{1-\frac{\alpha}{2}} = 1.96$ and $\hat{\nu}_{jj}$ is the j-th element on the diagonal of $\hat{\sigma}^2 \Gamma_2^{-1}$ for $j = 1, 2$. Using this formula, we see that $\nu_{jj} = 0.8719$ for $j = 1, 2$ and 95% confidence intervals for the model parameters are given by

$$\begin{aligned} &(0.1450, 0.4050) \quad \text{for } \phi_1 \\ &(0.2279, 0.4879) \quad \text{for } \phi_2. \end{aligned}$$

e. If the data were generated from an AR(2) process then, by the Durbin-Levinson algorithm, $\hat{\alpha}(1) = \hat{\phi}_{11} = \hat{\rho}(1) = 0.427$, $\hat{\alpha}(2) = \hat{\phi}_{22} = 0.3579$, and $\hat{\alpha}(h) = 0$ for $h \geq 3$.

$\square$