MATH 635 Final Assessment

Matthew Tiger

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Problem 1. Provide a rigorous proof of the case $x_0 = a$ in the Fundamental Lemma of the Calculus of Variations:

Theorem 1 (Fundamental Lemma of the Calculus of Variations)**.** *Suppose $M(x)$ is a continuous function defined on the interval $a \leq x \leq b$. Suppose further that for every continuous function $\zeta(x)$,*

$$\int_a^b M(x)\zeta(x)dx = 0.$$

Then

$$M(x) = 0 \textit{ for all } x \in [a, b].$$

Solution. Suppose to the contrary that $M(x) \neq 0$ at the point $x_0 = a$. In that case either $M(a) > 0$ or $M(a) < 0$. Let us first assume that $M(a) > 0$. Due to the continuity of $M(x)$ there is some neighborhood of a where the function is positive, i.e. there is some $\delta > 0$ such that if $|x - a| < \delta$ then

$$|M(x) - M(a)| < \frac{M(a)}{2} \quad \text{for } x \in [a, b].$$

Thus, $0 < M(a)/2 < M(x)$ for $x \in [a, a + \delta)$. Choose the function $\zeta(x)$ to be the linear spline interpolating the points $(a, 3M(a)/2)$ and $(a + \delta, 0)$ with support on $[a, a + \delta)$, i.e.

$$\zeta(x) := \begin{cases} \frac{-3M(a)}{2\delta}(x - (a + \delta)) & \text{if } a \leq x < a + \delta \\ 0 & \text{if } a + \delta \leq x \leq b. \end{cases}$$

Clearly $\zeta(x)$ is continuous and positive on the interval $[a, a + \delta)$. Thus,

$$\int_a^b M(x)\zeta(x)dx = \int_a^{a+\delta} M(x)\zeta(x)dx > \frac{M(a)}{2}\int_a^{a+\delta} \zeta(x)dx > 0.$$

However, by our supposition

$$\int_a^b M(x)\zeta(x)dx = 0,$$

a contradiction. Therefore, if $M(a) > 0$, the function $M(x) \equiv 0$ on the interval $[a, b]$.

If $M(a) < 0$, then we can repeat the argument above replacing $M(x)$ with $-M(x)$. To demonstrate, let us investigate the case when $M(a) < 0$. Due to the continuity of $M(x)$ there is some neighborhood of a where $-M(x)$ is positive, i.e. there is some $\delta > 0$ such that if $|x - a| < \delta$ then

$$|-M(x) + M(a)| < \frac{-M(a)}{2} \quad \text{for } x \in [a, b].$$

Thus, $0 < -M(a)/2 < -M(x)$ for $x \in [a, a + \delta)$. Choose the function $\zeta(x)$ to be the linear spline interpolating the points $(a, -3M(a)/2)$ and $(a + \delta, 0)$ with support on $[a, a + \delta)$, i.e.

$$\zeta(x) := \begin{cases} \frac{3M(a)}{2\delta}(x - (a + \delta)) & \text{if } a \leq x < a + \delta \\ 0 & \text{if } a + \delta \leq x \leq b. \end{cases}$$

Clearly $\zeta(x)$ is continuous and positive on the interval $[a, a+\delta)$. Thus,

$$\int_a^b -M(x)\zeta(x)dx = \int_a^{a+\delta} -M(x)\zeta(x)dx > \frac{-M(a)}{2}\int_a^{a+\delta} \zeta(x)dx > 0.$$

However, by our supposition

$$\int_a^b M(x)\zeta(x)dx = 0,$$

a contradiction. Therefore, if $M(a) < 0$, the function $M(x) \equiv 0$ on the interval $[a, b]$ and we have proven both cases. $\square$

Problem 2. Consider the differential equation

$$y'' - y = -x, \quad 0 < x < 1 \quad y(0) = y(1) = 0 \tag{1}$$

as in Example 15.12 on page 502. Use the basis $\{\phi_j(x)\} = \{x^j(1-x)^j\}$, as in section 15.5.1, to compute approximations to the exact solution using the finite-element method.

Provide relative errors at the points 0.25, 0.50, and 0.75 of the approximations using the first $n = 2, 3, 4$ basis functions. Plot the corresponding approximations y_2, y_3, y_4, and the exact solution y. Then find the first value of j for which the relative error at all three points is less than 0.5%.

Solution. The differential equation presented in the problem is a second order linear differential equation. It is easily shown that the homogeneous solution is given by $y_h(x) = c_1e^{-x} + c_2e^x$ and that a particular solution is given by $y_p(x) = x$. Thus the general solution is $y(x) = c_1e^{-x} + c_2e^x + x$. Using the boundary conditions, we see that the exact solution is

$$y(x) = \frac{e^x e}{1-e^2} - \frac{e^{-x}e}{1-e^2} + x \tag{2}$$

We now wish to approximate the exact solution $y(x)$. Note that the exact solution to the differential equation a continuous function. This fact combined with the fact that $\{\phi_j(x)\}$ form a basis of the function space shows that the continuous function $y(x)$ can be approximated with a linear combination of the basis functions. Therefore, we wish to find an approximation $y_n(x)$ to the exact solution $y(x)$ where

$$y_n(x) = \sum_{j=1}^{n} a_j\phi_j(x). \tag{3}$$

Note that our basis functions $\{\phi_j(x)\}$ satisfy the boundary conditions, i.e. $\phi_j(0) = \phi_j(1) = 0$ so that $y_n(x)$ also satisfies the boundary conditions.

Corollary 15.2 suggests that if

$$\int_0^1 (y_n'' - y_n + x)\phi_i(x)dx = 0 \quad \text{for } i = 1, \ldots, n$$

then $y_n'' - y_n + x = 0$, i.e $y_n(x)$ satisfies the differential equation (1). If $y_n(x)$ satisfies the differential equation and the boundary conditions, then we know that $y_n(x)$ approximates the exact solution $y(x)$.

Therefore, we choose the coefficients a_j such that they satisfy the system of equations

$$\sum_{j=1}^{n} a_j \int_0^1 \phi_j''(x)\phi_i(x) - \phi_j(x)\phi_i(x)dx = -\int_0^1 x\phi_i(x)dx \quad \text{for } i = 1, \ldots, n. \tag{4}$$

The above system unnecessarily uses the second derivative of the basis functions. We can rewrite the coefficients of the above system to use only the first derivative of the basis functions. To see this, note that we can rewrite the differential equation (1) in the form

$$(p(x)y')' + q(x)y' + r(x)y = f(x) \tag{5}$$

by choosing $p(x) = 1$, $q(x) = 0$, $r(x) = -1$, and $f(x) = -x$. With this form of the differential equation we would require the approximation (3) to satisfy the following equations

$$\int_0^1 ((p(x)y_n')' + r(x)y_n)\phi_i(x)dx = \int_0^1 f(x)\phi_i(x)dx \quad \text{for } i = 1, \ldots, n.$$

Making use of the fact that the basis functions are 0 on the boundary we see that

$$\begin{aligned}\int_0^1 (p(x)y_n')'\phi_i(x)dx &= \phi_i(x)p(x)y_n'|_0^1 - \int_0^1 p(x)y_n'\phi_i'(x)dx \\ &= -\int_0^1 p(x)y_n'\phi_i'(x)dx.\end{aligned}$$

With this and the definitions of the functions $p(x)$, $r(x)$, and $f(x)$, the system of equations (4) becomes

$$\sum_{j=1}^{n} a_j \int_0^1 -\phi_j'(x)\phi_i'(x) - \phi_j(x)\phi_i(x)dx = -\int_0^1 x\phi_i(x)dx \quad \text{for } i = 1, \ldots, n. \tag{6}$$

Finding the solution to the system of equations (6) identifies the coefficients a_j that define our approximation.

In this instance, we have chosen the basis $\{\phi_j(x)\}_{j=1}^{n}$ where $\phi_j(x) = x^j(1-x)^j$. Thus,

$$\begin{aligned}\phi_j'(x) &= \left(x^j\right)'(1-x)^j + x^j\left((1-x)^j\right)' \\ &= jx^{j-1}(1-x)^j - jx^j(1-x)^{j-1}\end{aligned}$$

for $j = 1, \ldots, n$.

Using the MATLAB function `approximation.m`, we construct the above system of equations and solve them arriving at approximations to the exact solution for $n = 2, 3, 4$. The tables comparing the exact solution to these approximations at the points $x = 0.25, 0.50, 0.75$ can be found below.

x	$y(x)$	$y_2(x)$	$\lvert y(x) - y_2(x)\rvert$	$\frac{100\lvert y(x)-y_2(x)\rvert}{\lvert y(x)\rvert}$
0.25	3.504760e-02	4.266210e-02	7.614504e-03	2.172618e 01
0.50	5.659056e-02	5.659010e-02	4.598883e-07	8.126590e-04
0.75	5.027579e-02	4.266210e-02	7.613681e-03	1.514383e 01

Table 1: Comparison of approximation y_2 to solution y. All computations are rounded to 6 significant digits.

We also provide the graphs of these comparisons in Figure 1.

The first value of n such that the relative error of the approximation at each of the points $x = 0.25, 0.50, 0.75$ is less than 0.5e-03 is quite large. In fact it is given by $n =$. This suggests

x	$y(x)$	$y_3(x)$	$\lvert y(x) - y_3(x)\rvert$	$\frac{100\lvert y(x)-y_3(x)\rvert}{\lvert y(x)\rvert}$
0.25	3.504760e-02	4.266169e-02	7.614092e-03	2.172500e 01
0.50	5.659056e-02	5.659056e-02	5.029993e-10	8.888397e-07
0.75	5.027579e-02	4.266169e-02	7.614093e-03	1.514465e 01

Table 2: Comparison of approximation y_3 to solution y. All computations are rounded to 6 significant digits.

x	$y(x)$	$y_4(x)$	$\lvert y(x) - y_4(x)\rvert$	$\frac{100\lvert y(x)-y_4(x)\rvert}{\lvert y(x)\rvert}$
0.25	3.504760e-02	4.266169e-02	7.614093e-03	2.172500e 01
0.50	5.659056e-02	5.659056e-02	3.451059e-13	6.098294e-10
0.75	5.027579e-02	4.266169e-02	7.614092e-03	1.514465e 01

Table 3: Comparison of approximation y_4 to solution y. All computations are rounded to 6 significant digits.

that this basis does not give a practical approximation to the exact solution at all points of the interval of definition. However, as the relative error of the approximation is very small for $n = 2$ at the point $x = 0.50$, this suggests that the approximation would be useful for neighborhoods with small radius centered at 0.50.

All programming code used to create the approximations, tables, and graphs for this basis can be found here:

https://github.com/gammadistribution/gradschool/tree/master/MATH635/final/programs $\square$

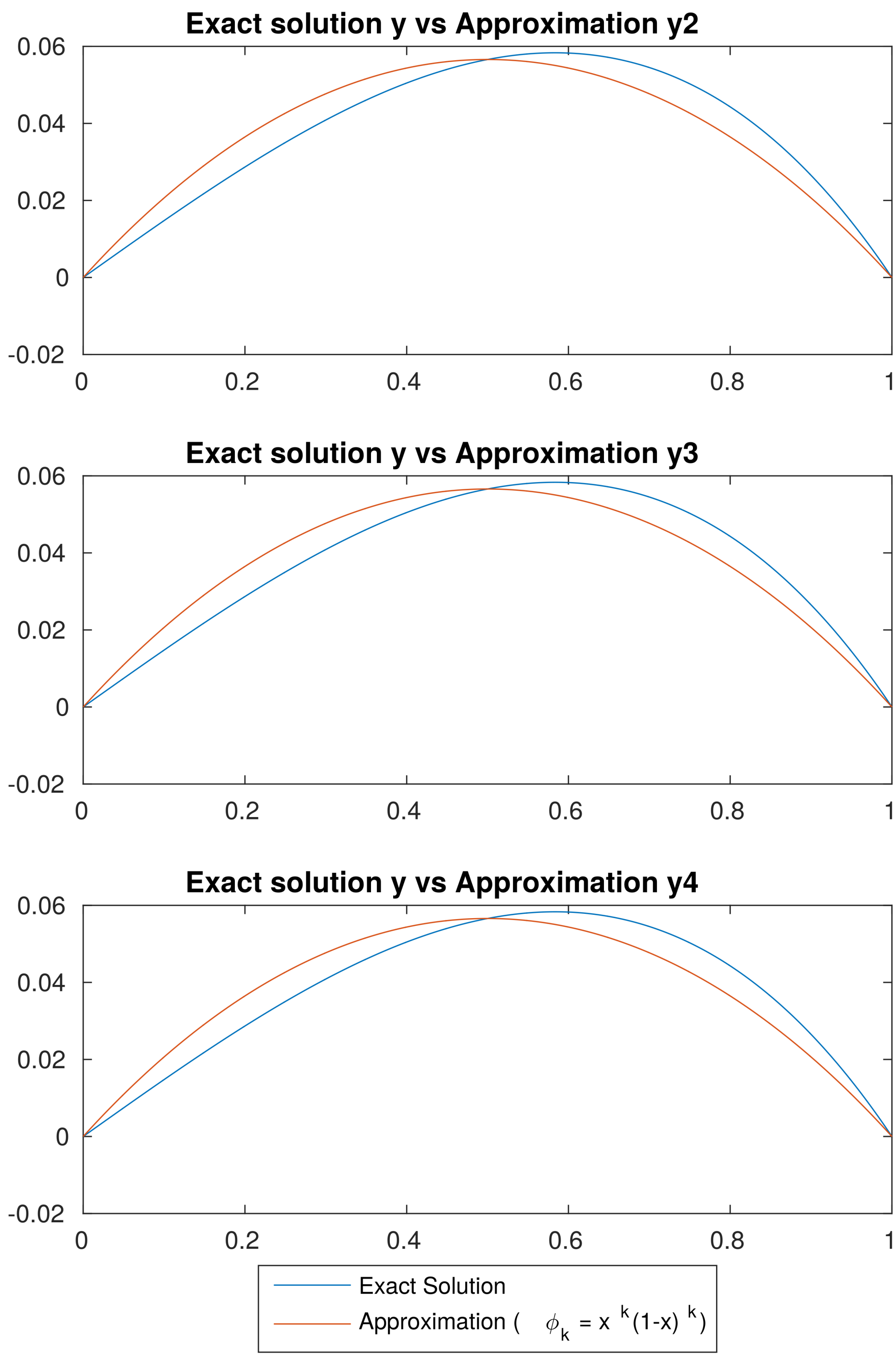

Figure 1: Plots of exact solution y and approximation y_n over the interval $[0, 1]$.