Exam 3

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Problem 1. Solve the non-homogeneous diffusion problem by the Hankel transform

$$
\begin{aligned}
&u_t = a\left(u_{rr} + \frac{1}{r}u_r\right) + Q(r,t), \qquad 0 < r < \infty, \quad 0 < t \\
&u(r,0) = f(r), \qquad 0 < r < \infty.
\end{aligned}
$$

Solution. Application of the 0-th order Hankel transform will transform the above Partial Differential Equation into an Ordinary Differential Equation. The following property of the 0-th order Hankel transform will aid in the application; if $\mathscr{H}_0\left\{u(r,t)\right\} = \tilde{u}_0(\kappa,t)$, then

$$
\mathscr{H}_0\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[u(r,t)\right] + \frac{\partial^2}{\partial r^2}\left[u(r,t)\right]\right\} = -\kappa^2\tilde{u}_0(\kappa,t). \tag{1}
$$

Now, with the above property, we see that applying the 0-th order Hankel transform to the diffusion problem yields

$$
\begin{aligned}
&\frac{d}{dt}\left[\tilde{u}_0(\kappa,t)\right] + a\kappa^2\tilde{u}_0(\kappa,t) = \tilde{Q}_0(\kappa,t), \qquad 0 < \kappa < \infty, \quad 0 < t \\
&\tilde{u}_0(\kappa,0) = \tilde{f}_0(\kappa), \qquad 0 < \kappa < \infty.
\end{aligned}
$$

This is a first order linear Ordinary Differential Equation, the solution to which is

$$
\tilde{u}_0(\kappa,t) = c_1(\kappa)e^{-a\kappa^2 t} + e^{-a\kappa^2 t}\int_0^t e^{a\kappa^2 x}\tilde{Q}_0(\kappa,x)dx.
$$

Thus, from this solution and the transformed boundary condition, we see that $c_1(\kappa) = \tilde{f}_0(\kappa)$ and the solution to the transformed boundary value problem is

$$
\tilde{u}_0(\kappa,t) = \tilde{f}_0(\kappa)e^{-a\kappa^2 t} + e^{-a\kappa^2 t}\int_0^t e^{a\kappa^2 x}\tilde{Q}_0(\kappa,x)dx.
$$

Therefore, the solution to the initial diffusion problem is

$$
\begin{aligned}
u(r,t) = \mathscr{H}_0^{-1}\left\{\tilde{u}_0(\kappa,t)\right\} &= \mathscr{H}_0^{-1}\left\{\tilde{f}_0(\kappa)e^{-a\kappa^2 t} + e^{-a\kappa^2 t}\int_0^t e^{a\kappa^2 x}\tilde{Q}_0(\kappa,x)dx\right\} \\
&= \int_0^\infty \kappa J_0(\kappa r)\left[\tilde{f}_0(\kappa)e^{-a\kappa^2 t} + e^{-a\kappa^2 t}\int_0^t e^{a\kappa^2 x}\tilde{Q}_0(\kappa,x)dx\right]d\kappa,
\end{aligned}
$$

where $J_0(\kappa r)$ is the Bessel function of order 0.

□

Problem 2.

Solution. $\square$

Problem 3. Solve the following integral equation by the Mellin transform

$$f(x) = \sin ax + \int_0^\infty \frac{f(xt)}{1+t^2}dt.$$

Solution. Let $g(x) = \dfrac{1}{1+x^2}$ and $h(x) = \sin ax$. Recall that $(f \circ g)(x)$ is defined to be

$$(f \circ g)(x) = \int_0^\infty f(xt)g(t)dt.$$

Thus, with this knowledge, the integral equation becomes

$$\begin{aligned} f(x) &= h(x) + \int_0^\infty f(xt)g(t)dt \\ &= h(x) + (f \circ g)(x). \end{aligned}$$

Let $\mathscr{M}\{f(x)\} = \tilde{f}(p)$, $\mathscr{M}\{g(x)\} = \tilde{g}(p)$, and $\mathscr{M}\{h(x)\} = \tilde{h}(p)$. Then from the Convolution Type theorem regarding the Mellin transform, we see that application of the Mellin transform to the integral equation yields

$$\begin{aligned} \tilde{f}(p) &= \mathscr{M}\{h(x)\} + \mathscr{M}\{(f \circ g)(x)\} \\ &= \tilde{h}(p) + \tilde{f}(p)\tilde{g}(1-p). \end{aligned}$$

Solving the above algebraic equation shows that

$$\tilde{f}(p) = \frac{\tilde{h}(p)}{1 - \tilde{g}(1-p)}.$$

From our table of Mellin transforms we know that

$$\tilde{g}(p) = \frac{\pi}{2}\csc\left(\frac{\pi p}{2}\right)$$

and

$$\tilde{h}(p) = a^{-p}\Gamma(p)\sin\left(\frac{\pi p}{2}\right).$$

Therefore, we see that

$$\begin{aligned} \tilde{f}(p) &= \frac{a^{-p}\Gamma(p)\sin\left(\frac{\pi p}{2}\right)}{1 - \frac{\pi}{2}\csc\left(\frac{\pi(1-p)}{2}\right)} \\ &= \frac{2a^{-p}\Gamma(p)\sin\left(\frac{\pi p}{2}\right)}{2 - \pi\sec\left(\frac{\pi p}{2}\right)} \end{aligned}$$

and the solution to the integral equation is

$$\begin{aligned} f(x) = \mathscr{M}^{-1}\left\{\tilde{f}(p)\right\} &= \mathscr{M}^{-1}\left\{\frac{2a^{-p}\Gamma(p)\sin\left(\frac{\pi p}{2}\right)}{2 - \pi\sec\left(\frac{\pi p}{2}\right)}\right\} \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-p}\left[\frac{2a^{-p}\Gamma(p)\sin\left(\frac{\pi p}{2}\right)}{2 - \pi\sec\left(\frac{\pi p}{2}\right)}\right]dp. \end{aligned}$$

□

Problem 4. Solve the following Partial Differential Equation by the Mellin transform

$$
\begin{aligned}
&r^2\phi_{rr} + r\phi_r + \phi_{\theta\theta} = 0, \qquad 0 < r < \infty, \quad 0 < \theta < \pi \\
&\phi(r,0) = \begin{cases} (1-r)^2 & 0 < r < 1 \\ 0 & 1 < r \end{cases} \\
&\phi(r,\pi) = \begin{cases} 1 & 0 < r < 1 \\ 0 & 1 < r \end{cases},
\end{aligned}
$$

Solution. Recall that if $\mathscr{M}\{\phi(r,\theta)\} = \tilde{\phi}(p,\theta)$, then the following property holds

$$\mathscr{M}\left\{r^2\frac{\partial^2}{\partial r^2}[\phi(r,\theta)] + r\frac{\partial}{\partial r}[\phi(r,\theta)]\right\} = p^2\tilde{\phi}(p,\theta).$$

Thus, applying the Mellin transform to the Partial Differential Equation and using our table of Mellin transforms, we see that

$$
\begin{aligned}
&\frac{d^2}{d\theta^2}\left[\tilde{\phi}(p,\theta)\right] + p^2\tilde{\phi}(p,\theta) = 0, \qquad 0 < p < \infty, \quad 0 < \theta < \pi \\
&\tilde{\phi}(p,0) = \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)}, \quad \tilde{\phi}(p,\pi) = \frac{1}{p}.
\end{aligned}
$$

The solution to the resulting homogeneous linear Ordinary Differential Equation is

$$\tilde{\phi}(p,\theta) = c_1(p)\cos p\theta + c_2(p)\sin p\theta.$$

Using the above solution and the transformed boundary conditions, we see that

$$
\begin{aligned}
c_1(p) &= \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)} \\
c_1(p)\cos p\pi + c_2(p)\sin p\pi &= \frac{1}{p}.
\end{aligned}
$$

Solving, we see that

$$
\begin{aligned}
c_2(p) &= \left(\frac{1}{p} - \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)}\cos p\pi\right)\csc p\pi \\
&= \frac{\csc p\pi}{p} - \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)}\cot p\pi.
\end{aligned}
$$

Thus, the solution to the transformed boundary value problem is

$$\tilde{\phi}(p,\theta) = \left[\frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)}\right]\cos p\theta + \left[\frac{\csc p\pi}{p} - \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)}\cot p\pi\right]\sin p\theta.$$

Therefore, the solution to the original boundary value problem is

$$
\begin{aligned}
\phi(r,\theta) &= \mathscr{M}^{-1}\left\{\left[\frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)}\right]\cos p\theta + \left[\frac{\csc p\pi}{p} - \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)}\cot p\pi\right]\sin p\theta\right\} \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} r^{-p}\left\{\left[\frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)}\right]\cos p\theta + \left[\frac{\csc p\pi}{p} - \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)}\cot p\pi\right]\sin p\theta\right\} dp.
\end{aligned}
$$

$\square$

Problem 5. Show that

$$\int_0^\infty e^{-ax}\left(\frac{\cos px - \cos qx}{x}\right)dx = \frac{1}{2}\log\frac{q^2+a^2}{p^2+a^2}$$

Solution. Recall that the Laplace transform of a function $f(t)$ is defined as

$$\bar{f}(s) = \mathscr{L}\left\{f(t)\right\}_s = \int_0^\infty f(t)e^{-st}dt$$

where we explicitly note the transformation variable s. Thus, we see that the integral above becomes

$$\begin{aligned}\int_0^\infty e^{-ax}\left(\frac{\cos px - \cos qx}{x}\right)dx &= \int_0^\infty e^{-ax}\left(\frac{\cos px}{x}\right)dx - \int_0^\infty e^{-ax}\left(\frac{\cos qx}{x}\right)dx \\ &= \mathscr{L}\left\{\frac{\cos px}{x}\right\}_a - \mathscr{L}\left\{\frac{\cos qx}{x}\right\}_a. \end{aligned} \tag{2}$$

From a previous theorem, if $\mathscr{L}\left\{f(t)\right\} = \bar{f}(s)$, then we have that

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(z)dz.$$

Thus, we see that (2) becomes

$$\begin{aligned}\int_0^\infty e^{-ax}\left(\frac{\cos px - \cos qx}{x}\right)dx &= \mathscr{L}\left\{\frac{\cos px}{x}\right\}_a - \mathscr{L}\left\{\frac{\cos qx}{x}\right\}_a \\ &= \int_a^\infty \mathscr{L}\left\{\cos px\right\}_z dz - \int_a^\infty \mathscr{L}\left\{\cos qx\right\}_z dz\end{aligned}$$

From the table of Laplace transforms, we have that

$$\mathscr{L}\left\{\cos bx\right\}_z = \frac{z}{b^2+z^2}.$$

Thus, we have that

$$\begin{aligned}\int_0^\infty e^{-ax}\left(\frac{\cos px - \cos qx}{x}\right)dx &= \int_a^\infty \mathscr{L}\left\{\cos px\right\}_z dz - \int_a^\infty \mathscr{L}\left\{\cos qx\right\}_z dz \\ &= \int_a^\infty \frac{z}{p^2+z^2}dz - \int_a^\infty \frac{z}{q^2+z^2}dz\end{aligned}$$

Using a u-substitution to solve the resulting integrals, we have that

$$\begin{aligned}\int_0^\infty e^{-ax}\left(\frac{\cos px - \cos qx}{x}\right)dx &= \int_a^\infty \frac{z}{p^2+z^2}dz - \int_a^\infty \frac{z}{q^2+z^2}dz \\ &= \frac{1}{2}\log p^2+z^2\Big|_a^\infty - \frac{1}{2}\log p^2+z^2\Big|_a^\infty.\end{aligned}$$

Using the properties of logarithms, we therefore see that

$$\begin{aligned}
\int_0^\infty e^{-ax}\left(\frac{\cos px - \cos qx}{x}\right) dx &= \frac{1}{2}\log p^2 + z^2\Big|_a^\infty - \frac{1}{2}\log p^2 + z^2\Big|_a^\infty \\
&= \frac{1}{2}\left[\log \frac{p^2+z^2}{q^2+z^2}\Bigg|_{z=\infty} - \log \frac{p^2+z^2}{q^2+z^2}\Bigg|_{z=a}\right] \\
&= -\frac{1}{2}\log \frac{p^2+a^2}{q^2+a^2} \\
&= \frac{1}{2}\log \frac{q^2+a^2}{p^2+a^2}.
\end{aligned}$$

$\square$

Problem 6.

Solution. $\square$

Problem 7.

Solution. □

Problem 8.

Solution. □