Exam 2

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Problem 1. A function $f : \mathbb{C} \to \mathbb{C}$ is defined by $f(z) = z^8$. Find the fixed points of f. Use your calculations to find the real linear and quadratic factors of the polynomial $p(z) = z^7 - 1$.

Solution. The fixed points of f are the solutions to the equation

$$f(z) - z = z^8 - z = z(z^7 - 1) = 0.$$

Thus, the fixed points of f are $z = 0$ and the 7-th roots of unity, i.e. the points $z = e^{2\pi k i/7}$ for $k = 0, 1, \ldots, 6$.

Note that for $z, \alpha \in \mathbb{C}$, we have that

$$(z - \alpha)(z - \overline{\alpha}) = z^2 - \overline{\alpha} z - \alpha z + \alpha\overline{\alpha} = z^2 - 2\mathrm{Re}(\alpha) z + |\alpha|^2$$

is a polynomial with real coefficients.

Using the 7-th roots of unity, we can obtain the following factorization of $p(z)$:

$$p(z) = \prod_{k=0}^{6} \left(z - e^{2\pi k i/7}\right).$$

Let $\alpha_k = e^{2\pi k i/7}$. From the previous note, the real quadratic factors of $p(z)$ are obtained by multiplying each factor $(z - \alpha_k)$ with $(z - \overline{\alpha_k})$, if α_k and $\overline{\alpha_k}$ are both roots of $p(z)$. For $k = 1, \ldots, 6$, we have that α_k is a root of $p(z)$ and

$$\overline{\alpha_k} = e^{-2\pi k i/7} = e^{2\pi(7-k)i/7} = \alpha_{7-k},$$

which is also a root of $p(z)$. Therefore, the real linear and quadratic factors of $p(z)$ are given by

$$\begin{aligned} p(z) &= (z - \alpha_0)(z - \alpha_1)(z - \alpha_6)(z - \alpha_2)(z - \alpha_5)(z - \alpha_3)(z - \alpha_4) \\ &= (z - 1)(z - \alpha_1)(z - \overline{\alpha_1})(z - \alpha_2)(z - \overline{\alpha_2})(z - \alpha_3)(z - \overline{\alpha_3}) \\ &= (z - 1)\left(z^2 - 2\mathrm{Re}(\alpha_1) z + 1\right)\left(z^2 - 2\mathrm{Re}(\alpha_2) z + 1\right)\left(z^2 - 2\mathrm{Re}(\alpha_3) z + 1\right), \end{aligned}$$

where $\mathrm{Re}(\alpha_k) = \cos(2\pi k/7)$.

$\square$

Problem 2. Let K_c be the filled-in Julia set of $f_c(z) = z^2 + c$.

a) Find the fixed points and the period 2 points of f_{-6}.

b) Show that $2\sqrt{2} \in K_{-6}$ and find another point in K_{-6}, distinct from those found so far.

c) Do any of the points you have found lie in the Julia set of f_{-6}?

d) Is $-6 \in \mathcal{M}$ where $\mathcal{M}$ is the Mandelbrot set?

Solution. a) The fixed points of f_{-6} are the solutions to

$$f_{-6}(z) - z = z^2 - z - 6 = 0.$$

Thus, the fixed points of f_{-6} are $z_0 = 3$ and $z_1 = -2$. The period 2 points are the solutions to

$$f_{-6}^2(z) - z = (z^2 - 6)^2 - z - 6 = 0$$

that are also not fixed points of f_{-6}. Factoring $f_{-6}^2(z) - z$, we see that

$$f_{-6}^2(z) - z = (z - 3)(z + 2)(z^2 + z - 5).$$

Thus, the period 2 points of f_{-6} are the solutions to $z^2 + z - 5 = 0$, i.e. the period 2 points of f_{-6} are

$$z_2 = \frac{-1 - \sqrt{21}}{2}, \quad z_3 = \frac{-1 + \sqrt{21}}{2}.$$

b) Recall that for a polynomial $p(z)$ with $\deg(p) > 1$, the filled-in Julia set of $p(z)$ is the set of all points that do not converge to ∞ under iteration of p.

Note that $2\sqrt{2}$ is an eventual fixed point of f_{-6}. We see that $f_{-6}^2(2\sqrt{2}) = -2$ so that $f_{-6}^k(2\sqrt{2}) = -2$ for $k > 2$. This implies that $2\sqrt{2}$ does not converge to ∞ under iteration of f_{-6} so that $2\sqrt{2}$ is in the filled-in Julia set of f_{-6}, i.e. $2\sqrt{2} \in K_{-6}$.

For reasons similar to those listed above, we see that -3 is an eventual fixed point of f_{-6}, i.e. $f_{-6}(-3) = 3$, so that $-3 \in K_{-6}$.

c) For a polynomial $p(z)$ with $\deg(p) > 1$, the Julia set of $p(z)$ is the boundary of the basin of attraction of ∞.

Since all of the points listed do not converge to ∞ under iteration of f_{-6}, we see that none of the listed points belong to the Julia set of f_{-6}.

d) The definition of the Mandelbrot set is the set of all $c \in \mathbb{C}$ such that the orbit of 0 is bounded under iteration by f_c. It was shown previously that $c \in \mathcal{M}$ if and only if $|f_c^n(0)| \leq 2$ for all $n > 0$. For f_{-6}, we see that $f_{-6}(0) = -6$ where $|f_{-6}(0)| > 2$. Therefore, we must have that $-6 \notin \mathcal{M}$.

$\square$

Problem 3. Let $f_c(z) = z^2 + c$. Find the values of c so that $z = i$ is a period 2 point. Find the fixed points in each case and determine their stability. Is $c \in \mathcal{M}$?

Solution. As was shown previously, the fixed points of $f_c(z) = z^2 + c$ are the solutions to $f_c(z) - z = 0$ which are the points

$$z_0 = \frac{1+\sqrt{1-4c}}{2}, \quad z_1 = \frac{1-\sqrt{1-4c}}{2}. \tag{1}$$

The period 2 points of f_c are the solutions to $f_c^2(z) - z = 0$ that are also not the fixed points (1). The period 2 points are thus given by

$$z_2 = \frac{-1-\sqrt{-3-4c}}{2}, \quad z_3 = \frac{-1+\sqrt{-3-4c}}{2}.$$

We wish to find the values of $c \in \mathbb{C}$ such that $z_2 = i$ or $z_3 = i$. Using Mathematica, we see that the only value of $c \in \mathbb{C}$ such that $z_2 = i$ or $z_3 = i$ is $c = -i$. If $c = -i$, we see that $f_c(i) = -1 - i$ and $f_c^2(i) = i$ so that $z = i$ is in fact a period 2 point.

From (1), the fixed points of f_c when $c = -i$ are given by

$$z_0 = \frac{1+\sqrt{1+4i}}{2}, \quad z_1 = \frac{1-\sqrt{1+4i}}{2}.$$

For a differentiable function f, the fixed point z of f is asymptotically stable if $|f'(z)| < 1$ and asymptotically unstable if $|f'(z)| > 1$. For $f_c(z) = z^2 + c$, we note that $f_c'(z) = 2z$. Consider $z_0 = \dfrac{1+\sqrt{1+4i}}{2}$. Note that

$$|f'(z_0)| = \left|1 + \sqrt{1+4i}\right| = \sqrt{1 + \sqrt{17} + \sqrt{2(1+\sqrt{17})}} > 1$$

so that z_0 is an unstable fixed point. Now consider $z_1 = \dfrac{1-\sqrt{1+4i}}{2}$. Then we have that

$$|f'(z_1)| = \left|1 - \sqrt{1+4i}\right| = \sqrt{1 + \sqrt{17} - \sqrt{2(1+\sqrt{17})}} > 1$$

so that z_1 is also an unstable fixed point.

Note that 0 is an eventual periodic point of f_{-i}, i.e. $f_{-i}(0) = -i$ and $f_{-i}^2(0) = -1 - i$ which is a period 2 point of f_{-i}. Thus, the orbit of 0 under iteration of f_{-i} will be bounded and we have that $-i \in \mathcal{M}$.

□

Problem 4. Show that the function $H(z) = \dfrac{z-i}{z+i}$ gives a conjugacy between the Newton map N_{f_1} of $f_1(z) = z^2+1$ and the function $f_0(z) = z^2$. Deduce the Julia set of N_{f_1} and show that it is chaotic on its Julia set.

Solution. Note that the Newton function N_{f_1} of $f_1(z) = z^2+1$ is given by

$$N_{f_1}(z) = z - \frac{f(z)}{f'(z)} = z - \frac{z^2+1}{2z} = \frac{z^2-1}{2z}.$$

Let $D = \{w \in \mathbb{C} \mid |w| > 1\}$ and consider $f_0(z) = z^2$. Note that, $B_{f_0}(\infty)$, the basin of attraction of infinity for f_0, is D. Define $H(z) = \dfrac{z-i}{z+i}$. Then $H : H^{-1}(D) \to D$ is a homeomorphism, where $H^{-1}(D) = \{z \in \mathbb{C} \mid w = H(z),\ |w| > 1\}$.

To see this, we will show that H is a continuous bijection with continuous inverse. Suppose first that $H(z_1) = H(z_2)$. Then we have that

$$H(z_1) = \frac{z_1-i}{z_1+i} = \frac{z_2-i}{z_2+i} = H(z_2).$$

This implies that

$$z_1z_2 + iz_1 - iz_2 + 1 = z_1z_2 - iz_1 + iz_2 + 1$$

or that $2i(z_1 - z_2) = 0$. Since the complex numbers form an integral domain, we must have that $z_1 - z_2 = 0$ or that $z_1 = z_2$. Thus, H is injective.

Let $w \in D$ and let $z = -\dfrac{i(w+1)}{w-1} \in H^{-1}(D)$. Then we see that

$$H(z) = H\left(-\frac{i(w+1)}{w-1}\right) = \frac{-\frac{i(w+1)}{w-1} - i}{-\frac{i(w+1)}{w-1} + i} = w$$

so that H is surjective.

Thus H is a bijection and we see that $H^{-1} : D \to H^{-1}(D)$ defined by

$$H^{-1}(w) = -\frac{i(w+1)}{w-1}$$

is the inverse of H. It is clear that H is continuous at all points except at $z = -i$. However, $z = -i \notin H^{-1}(D)$ and so H is continuous everywhere in its domain. Similarly, H^{-1} is continuous everywhere except at $w = 1$, but $w = 1 \notin D$. Therefore, H^{-1} is continuous everywhere in its domain and H is a homeomorphism.

Now, the function H will give a conjugacy between N_{f_1} and f_0 if $f_0 \circ H = H \circ N_{f_1}$. We can easily verify that

$$f_0 \circ H(z) = f_0\left(\frac{z-i}{z+i}\right) = \frac{(z-i)^2}{(z+i)^2}$$

and

$$\begin{aligned} H \circ N_{f_1}(z) = H\left(\frac{z^2-1}{2z}\right) &= \frac{\frac{z^2-1}{2z} - i}{\frac{z^2-1}{2z} + i} \\ &= \frac{\frac{(z-i)^2}{2z}}{\frac{(z+i)^2}{2z}} \\ &= \frac{(z-i)^2}{(z+i)^2}. \end{aligned}$$

Therefore, $f_0 \circ H = H \circ N_{f_1}$ and H gives a conjugacy between N_{f_1} and f_0.

Since D is the basin of attraction of infinity of f_0 and H is a conjugacy between N_{f_1} and f_0, we must have that $H^{-1}(D)$ is the basin of attraction of infinity for N_{f_1}. By definition, $K(N_{f_1})$, the filled-in Julia set of N_{f_1}, must be $K(N_{f_1}) = \mathbb{C} \setminus H^{-1}(D)$. The Julia set is then the boundary of this set.

$\square$

Problem 5. Let $p(z)$ be a polynomial of degree $d > 1$ with Newton function

$$N_p(z) = z - \frac{p(z)}{p'(z)}.$$

a) If $p(\alpha) = 0$ and $p'(\alpha) \neq 0$, show that α is a fixed point of multiplicity two for N_p, i.e. there is a rational function $k(z) = m(z)/n(z)$ with $n(\alpha) \neq 0$ and $N_p(z) - \alpha = (z-\alpha)^2 k(z)$.

b) If $p(\alpha) = 0$, $p'(\alpha) \neq 0$, and $p''(\alpha) = 0$, show that α is a fixed point of multiplicity three for N_p.

Solution. a) If $N_p(z)$ is the Newton function of the polynomial $p(z)$ of degree $n > 1$, then

$$N_p(z) - \alpha = z - \alpha - \frac{p(z)}{p'(z)} = \frac{(z-\alpha)p'(z) - p(z)}{p'(z)}. \tag{2}$$

Note that if $p(\alpha) = 0$, then

$$p(z) = (z-\alpha)\sum_{k=0}^{n-1} a_k z^k.$$

This implies that

$$\begin{aligned} p'(z) &= \sum_{k=0}^{n-1} a_k z^k + (z-\alpha)\sum_{k=1}^{n-1} k a_k z^{k-1} \\ &= \sum_{k=0}^{n-1} a_k z^k + (z-\alpha)m(z), \end{aligned}$$

say. Thus, from (2), we see that

$$\begin{aligned} N_p(z) - \alpha &= \frac{(z-\alpha)p'(z) - p(z)}{p'(z)} \\ &= \frac{p(z) + (z-\alpha)^2 m(z) - p(z)}{p'(z)} \\ &= \frac{(z-\alpha)^2 m(z)}{p'(z)}. \end{aligned}$$

Since $p'(\alpha) \neq 0$, set $n(z) = p'(z)$. Then we have from the above equation that

$$N_p(z) - \alpha = (z-\alpha)^2 \frac{m(z)}{n(z)},$$

or that α is a fixed point of multiplicity two for N_p.

b) Suppose that $p(\alpha) = 0$ and $p'(\alpha) \neq 0$. Then as shown previously,

$$\begin{aligned} p'(z) &= \sum_{k=0}^{n-1} a_k z^k + (z-\alpha) \sum_{k=1}^{n-1} k a_k z^{k-1} \\ &= \sum_{k=0}^{n-1} a_k z^k + (z-\alpha) m(z), \end{aligned}$$

and

$$N_p(z) - \alpha = (z-\alpha)^2 \frac{m(z)}{p'(z)}.$$

Thus, α will be a fixed point of multiplicity three for $N_p(z)$ if $(z-\alpha)$ is a factor of $m(z) = \sum_{k=1}^{n-1} k a_k z^{k-1}$.

Suppose that $p''(\alpha) = 0$. Then from previous calculations, we see that

$$\begin{aligned} p''(z) &= \left(\sum_{k=0}^{n-1} a_k z^k + (z-\alpha) \sum_{k=1}^{n-1} k a_k z^{k-1} \right)' \\ &= \sum_{k=1}^{n-1} k a_k z^{k-1} + \sum_{k=1}^{n-1} k a_k z^{k-1} + (z-\alpha) \sum_{k=2}^{n-1} k(k-1) a_k z^{k-2} \\ &= 2m(z) + (z-\alpha) m'(z) \end{aligned}$$

$\square$

Problem 6. a) Show that for $p_\alpha(z) = z(z-1)(z-\alpha)$, the Newton function N_{p_α} has a critical point where $z = (\alpha+1)/3$.

b) For what values of α does p_α satisfy $p(\alpha) = 0$, $p'(\alpha) \neq= 0$, and $p''(\alpha) = 0$?

Solution. □

Problem 7. Let $0 < \mu < \lambda < 1$ and let $h : [0, 1] \to [0, 1]$ be a homeomorphism with $h \circ L_\mu(x) = L_\lambda \circ h(x)$ for all $x \in [0, 1]$.

a) Show that h is orientation-preserving.

b) Show that $h(x) + h(1 - x) = 1$ for all $x \in [0, 1]$. Deduce that $h(1/2) = 1/2$.

c) Show that $h(\mu/4) = \lambda/4$ and $h(x) > x$ for $0 < x < 1/2$ and $h(x) < x$ for $1/2 < x < 1$.

Solution. $\square$

Problem 8. Prove that if $f_c(z) = z^2 + c$ has an attracting periodic point, then $c \in \mathcal{M}$, the Mandelbrot set.

Solution. $\square$