Homework Assignment 5

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Problem 1. Use the method of stationary phase to find the leading behavior of the following integral as $x \to +\infty$:

$$I(x) = \int_0^1 e^{ixt^2} \cosh t^2 dt.$$

Solution. We begin by noting that the integral $I(x)$ is a generalized Fourier integral which can be written as

$$I(x) = \int_0^1 f(t) e^{ix\psi(t)} dt$$

where $f(t) = \cosh t^2$ and $\psi(t) = t^2$. The leading asymptotic behavior of such integrals as $x \to +\infty$ may be found, in general, using integration by parts. However, this method may fail at *stationary points*, i.e. any point on the interval of definition such that $\psi'(t) = 0$. For the integral $I(x)$ we note that $t = 0$ is a stationary point. Thus, we proceed by writing $I(x)$ as follows:

$$I(x) = I_1(x) + I_2(x) = \int_0^\varepsilon f(t) e^{ix\psi(t)} dt + \int_\varepsilon^1 f(t) e^{ix\psi(t)} dt$$

for some $\varepsilon > 0$. Since $I_2(x)$ does not have any stationary points and the function $f(t) = \cosh t^2 \in L^1$ over the interval $[0, 1]$, i.e. we have that $\int_0^1 |f(t)|\, dt < +\infty$, integration by parts works on $I_2(x)$ and by the Riemann-Lebesgue lemma, $I_2(x) \to 0$ as $x \to +\infty$. Thus, as $x \to +\infty$,

$$I(x) \sim I_1(x) = \int_0^\varepsilon f(t) e^{ix\psi(t)} dt = \int_0^\varepsilon \cosh t^2 e^{ixt^2} dt.$$

We continue by replacing $f(t)$ with $f(0) = \cosh 0 = 1$ and ε with ∞, since these are the parts that contribute the most to the integral, introducing error terms that vanish as $x \to +\infty$ so that

$$I(x) \sim \int_0^\infty e^{ixt^2} dt$$

Making the substitution

$$t = e^{i\pi/4} \left[\frac{u}{x}\right]^{1/2}$$

yields that

$$\int_0^\infty e^{ixt^2} dt = e^{i\pi/4} \left[\frac{1}{x}\right]^{1/2} \frac{\Gamma(1/2)}{2} = \frac{e^{i\pi/4}}{2}\sqrt{\frac{\pi}{x}}.$$

Therefore, as $x \to +\infty$,

$$I(x) \sim \int_0^\infty e^{ixt^2} dt = \frac{e^{i\pi/4}}{2}\sqrt{\frac{\pi}{x}}.$$

$\square$

Problem 2. Use second-order perturbation theory to find approximations to the roots of the following equation:

$$x^3 + \varepsilon x^2 - x = 0.$$

Solution. If we assume that the roots of the above equation are functions of ε, then the roots x_i for $i = 0, 1, 2$ of the equation are of the form

$$x_i(\varepsilon) = \sum_{k=0}^{\infty} a_{i_k} \varepsilon^k.$$

Second-order perturbation theory prescribes that the roots are of the form

$$x_i(\varepsilon) = a_{i_0} + a_{i_1}\varepsilon + a_{i_2}\varepsilon^2 + O(\varepsilon^3)$$

where we disregard terms of order ε^3 or greater. Substituting $\varepsilon = 0$ into the equation yields the new equation $x^-3 - x = 0$, the roots of which are $-1, 0$, and 1 which we will say correspond to the coefficients $a_{0_0} = -1, a_{1_0} = 0$, and $a_{2_0} = 1$.

In order to find the values of the coefficients a_{i_k} for $k \geq 1$, we substitute the expression $x_i(\varepsilon) = a_{i_0} + a_{i_1}\varepsilon + a_{i_2}\varepsilon^2 + O(\varepsilon^3)$ into the original equation yielding

$$a_{i_0}^3 - a_{i_0} + (a_{i_0}^2 - a_{i_1} + 3a_{i_0}^2 a_{i_1})\varepsilon + (2a_{i_0}a_{i_1} + 3a_{i_0}a_{i_1}^2 - a_{i_2} + 3a_{i_0}^2 a_{i_2})\varepsilon^2 = O(\varepsilon^3).$$

Since ε is variable we must have that the coefficients of ε in the above equation are 0. This yields two equations for each root:

$$\begin{aligned} a_{i_0}^2 - a_{i_1} + 3a_{i_0}^2 a_{i_1} &= 0 \\ 2a_{i_0}a_{i_1} + 3a_{i_0}a_{i_1}^2 - a_{i_2} + 3a_{i_0}^2 a_{i_2} &= 0. \end{aligned}$$

For the root x_0, we have that $a_{0_0} = -1$ and the two equations become

$$\begin{aligned} (-1)^2 - a_{0_1} + 3(-1)^2 a_{0_1} &= 0 \\ -2a_{0_1} - 3a_{0_1}^2 - a_{0_2} + 3(-1)^2 a_{0_2} &= 0. \end{aligned}$$

The first equation yields that $a_{0_1} = -1/2$ and substituting into the second equation yields that $a_{0_2} = -1/8$. Thus, $x_0 = -1 + (-1/2)\varepsilon + (-1/8)\varepsilon^2 + O(\varepsilon^3)$.

For the root $x_1 = 0$, we see that $a_{1_0} = 0$ and consequently from the equations that $a_{1_1} = 0$ and $a_{1_2} = 0$. Thus, $x_1 = 0 + 0\varepsilon + 0\varepsilon^2 + O(\varepsilon^3)$.

Proceeding in the same way above we see for the root x_2, we have that $a_{2_0} = 1$ and the above two equations become

$$\begin{aligned} (1)^2 - a_{2_1} + 3(1)^2 a_{2_1} &= 0 \\ 2a_{2_1} + 3a_{2_1}^2 - a_{2_2} + 3(1)^2 a_{2_2} &= 0. \end{aligned}$$

The first equation yields that $a_{2_1} = -1/2$ and substituting into the second equation yields that $a_{2_2} = 1/8$. Therefore, $x_2 = 1 + (-1/2)\varepsilon + (1/8)\varepsilon^2 + O(\varepsilon^3)$ and we have found second-order approximations for all of the roots of the original equation.

□

Problem 3. Analyze in the limit $\varepsilon \to 0$ the roots of the polynomial

$$\varepsilon x^8 - \varepsilon^2 x^6 + x - 2 = 0.$$

Solution. $\square$

Problem 4. Solve perturbatively

$$\begin{cases} y'' = (\sin x)y \\ y(0) = 1 \\ y'(0) = 1 \end{cases}.$$

Is the resulting perturbation series uniformly valid for $0 \leq x \leq \infty$? Why?

Solution. $\square$

Problem 5. Find leading-order uniform asymptotic approximations to the solution of the following equation in the limit $\varepsilon \to 0^+$:

$$\begin{aligned} &\varepsilon y'' + (x^2 + 1)y' - x^3 y = 0 \\ &y(0) = 1, \ y(1) = 1. \end{aligned}$$

Solution. □

Problem 6. Obtain a uniform approximation accurate to order ε^2 as $\varepsilon \to 0^+$ for the problem

$$\begin{aligned} &\varepsilon y'' + (1+x)^2 y' + y = 0 \\ &y(0) = 1,\ y(1) = 1. \end{aligned}$$

Solution. □

Problem 7. For what real values of the constant α does the singular perturbation problem

$$\begin{aligned}&\varepsilon y''(x) + y'(x) - x^{\alpha} y(x) = 0\\ &y(0) = 1,\ y(1) = 1.\end{aligned}$$

have a solution with a boundary layer near $x = 0$ as $\varepsilon \to 0^+$?

Solution. $\square$