# Homework Assignment 1

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**Problem 2.1.** Find the Fourier transforms of each of the following functions:

c. $f(x) = \delta^{(n)}(x)$,

f. $f(x) = x \exp\left(-\dfrac{ax^2}{2}\right), a > 0$,

g. $f(x) = x^2 \exp\left(-\dfrac{x^2}{2}\right)$.

*Solution.* Recall that, by definition, we have that for a function $f(x) \in L^1(\mathbb{R})$, its Fourier transform is given by

$$\mathscr{F}\left\{f(x)\right\} = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx \tag{1}$$

where $k \in \mathbb{R}$.

c. The Dirac delta function $\delta(x)$ is defined such that for any good function $g(x)$ we have that

$$\int_{-\infty}^{\infty} \delta(x) g(x) dx = g(0).$$

A good function is defined as a function in $C^\infty(\mathbb{R})$ that decays sufficiently rapidly. Since it is clear that $\delta(x) \to 0$ as $|x| \to \infty$, we have by a previous theorem that

$$\mathscr{F}\left\{\delta'(x)\right\} = ik\mathscr{F}\left\{\delta(x)\right\}. \tag{2}$$

By (1) and the definition of the Dirac delta function, we see that

$$\mathscr{F}\left\{\delta(x)\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \delta(x) dx = \frac{1}{\sqrt{2\pi}}.$$

Thus, using (2), we can easily see by induction for $n > 1$ that

$$\mathscr{F}\left\{\delta^{(n)}(x)\right\} = ik\mathscr{F}\left\{\delta^{(n-1)}(x)\right\} = \cdots = \frac{(ik)^n}{\sqrt{2\pi}}.$$

f. From (1), we see that

$$\begin{aligned}
\mathscr{F}\left\{f(x)\right\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x\exp(-ikx)\exp\left(-\frac{ax^2}{2}\right)dx \\
&= \frac{\exp\left(\frac{(ik)^2}{2a}\right)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x\exp\left(-\frac{ax^2}{2} - ikx - \frac{(ik)^2}{2a}\right)dx \\
&= \frac{\exp\left(-\frac{k^2}{2a}\right)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x\exp\left(-\frac{a}{2}\left(x + \frac{ik}{a}\right)^2\right)dx.
\end{aligned}$$

Making the substitution $u = x + ik/a$, where $du = dx$, we have that

$$\begin{aligned}
\mathscr{F}\left\{f(x)\right\} &= \frac{\exp\left(-\frac{k^2}{2a}\right)}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(u - \frac{ik}{a}\right)\exp\left(-\frac{au^2}{2}\right)du \\
&= \frac{\exp\left(-\frac{k^2}{2a}\right)}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} u\exp\left(-\frac{au^2}{2}\right)du - \frac{ik}{a}\int_{-\infty}^{\infty}\exp\left(-\frac{au^2}{2}\right)du\right]. \qquad (3)
\end{aligned}$$

Since the function $g(x) = u\exp\left(-\frac{au^2}{2}\right)$ is odd, we know that

$$\int_{-\infty}^{\infty} u\exp\left(-\frac{au^2}{2}\right)du = 0.$$

Using the formula for the general Gaussian integral we have that

$$\int_{-\infty}^{\infty}\exp\left(-\frac{au^2}{2}\right) = \frac{\sqrt{2\pi}}{\sqrt{a}}$$

when $a > 0$.

Combining, we see from (3) that the Fourier transform of $f(x) = x\exp\left(-\frac{ax^2}{2}\right)$ for $a > 0$ is

$$\begin{aligned}
\mathscr{F}\left\{f(x)\right\} &= \frac{\exp\left(-\frac{k^2}{2a}\right)}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} u\exp\left(-\frac{au^2}{2}\right)du - \frac{ik}{a}\int_{-\infty}^{\infty}\exp\left(-\frac{au^2}{2}\right)du\right] \\
&= \frac{\exp\left(-\frac{k^2}{2a}\right)}{\sqrt{2\pi}}\left(-\frac{ik}{a}\right)\left(\frac{\sqrt{2\pi}}{\sqrt{a}}\right) \\
&= -\frac{ik\exp\left(-\frac{k^2}{2a}\right)}{a\sqrt{a}}.
\end{aligned}$$

g. From (1), we see that

$$\begin{aligned}
\mathscr{F}\left\{f(x)\right\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \exp(-ikx) \exp\left(-\frac{x^2}{2}\right) dx \\
&= \frac{\exp\left(\frac{(ik)^2}{2}\right)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2}{2} - ikx - \frac{(ik)^2}{2}\right) dx \\
&= \frac{\exp\left(-\frac{k^2}{2}\right)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{1}{2}\left(x+ik\right)^2\right) dx.
\end{aligned}$$

Making the substitution $u = x + ik$, where $du = dx$, we have that

$$\begin{aligned}
\mathscr{F}\left\{f(x)\right\} &= \frac{\exp\left(-\frac{k^2}{2}\right)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(u-ik\right)^2 \exp\left(-\frac{u^2}{2}\right) du \\
&= \frac{\exp\left(-\frac{k^2}{2}\right)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(u^2 - 2iku - k^2\right) \exp\left(-\frac{u^2}{2}\right) du. \qquad (4)
\end{aligned}$$

After distributing the exponential term to the polynomial in (4) and splitting the integral using the operator's linearity, the first integral may be computed by parts by setting $w = u$ and $dv = -u\exp\left(-\frac{u^2}{2}\right) du$ as so:

$$\int_{-\infty}^{\infty} u^2 \exp\left(-\frac{u^2}{2}\right) du = -\left[u\exp\left(-\frac{u^2}{2}\right)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \exp\left(-\frac{u^2}{2}\right) du\right] = \sqrt{2\pi}$$

where the last equality follows from the formula for the general Gaussian integral.

The other two resulting integrals in (4) are calculated very similarly to their corresponding integrals in 2.1.f, i.e.

$$\begin{aligned}
2ik \int_{-\infty}^{\infty} u\exp\left(-\frac{u^2}{2}\right) du &= 0, \\
k^2 \int_{-\infty}^{\infty} \exp\left(-\frac{u^2}{2}\right) du &= k^2\sqrt{2\pi}.
\end{aligned}$$

Therefore, we have that the Fourier transform of $f(x) = x^2\exp\left(-\frac{x^2}{2}\right)$ is given by

$$\begin{aligned}
\mathscr{F}\left\{f(x)\right\} &= \frac{\exp\left(-\frac{k^2}{2}\right)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(u^2 - 2iku - k^2\right) \exp\left(-\frac{u^2}{2}\right) du \\
&= \frac{\exp\left(-\frac{k^2}{2}\right)}{\sqrt{2\pi}} \left[\sqrt{2\pi} - k^2\sqrt{2\pi}\right] \\
&= \left(1-k^2\right)\exp\left(-\frac{k^2}{2}\right).
\end{aligned}$$

$\square$

**Problem 2.2.** Show that

a. $\mathscr{F}\left\{\delta(x-ct)+\delta(x+ct)\right\} = \sqrt{\frac{2}{\pi}}\cos(kct)$,

b. $\mathscr{F}\left\{H\left(ct-|x|\right)\right\} = \mathscr{F}\left\{\chi_{[-ct,ct]}(x)\right\} = \sqrt{\frac{2}{\pi}}\frac{\sin(kct)}{k}$.

*Solution.* a. By the shifting property of the Fourier transform, we know that

$$\mathscr{F}\left\{\delta(x\pm ct)\right\} = e^{\pm ikct}\mathscr{F}\left\{\delta(x)\right\}.$$

As shown previously, we also know from the definition of the Fourier transform and the definition of the Dirac delta function that $\mathscr{F}\left\{\delta(x)\right\} = 1/\sqrt{2\pi}$. Combining, we see using the linearity of the Fourier transform that

$$\begin{aligned}\mathscr{F}\left\{\delta(x-ct)+\delta(x+ct)\right\} &= \mathscr{F}\left\{\delta(x-ct)\right\}+\mathscr{F}\left\{\delta(x+ct)\right\}\\ &= \left(e^{-ikct}+e^{ikct}\right)\mathscr{F}\left\{\delta(x)\right\}\\ &= \frac{2}{\sqrt{2\pi}}\left(\frac{e^{-ikct}+e^{ikct}}{2}\right)\\ &= \sqrt{\frac{2}{\pi}}\cos(kct)\end{aligned}$$

where the last equality follows using the definition of the complex exponential.

b. Recall from the definitions of the Heaviside function $H$ and the characteristic function $\chi$ that

$$H(ct-|x|) = \chi_{[-ct,ct]}(x) = \begin{cases} 1 & |x| < ct \\ 0 & |x| > ct \end{cases}.$$

Since the Fourier transform is a well-defined operator, this implies that

$$\mathscr{F}\left\{H\left(ct-|x|\right)\right\} = \mathscr{F}\left\{\chi_{[-ct,ct]}(x)\right\}.$$

Now, from the definition of the Fourier transform in (1), we see that

$$\begin{aligned}\mathscr{F}\left\{H\left(ct-|x|\right)\right\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}H\left(ct-|x|\right)dx\\ &= \frac{1}{\sqrt{2\pi}}\int_{-ct}^{ct} e^{-ikx}dx\\ &= \frac{2}{k\sqrt{2\pi}}\left(\frac{e^{ikct}-e^{-ikct}}{2i}\right)\\ &= \sqrt{\frac{2}{\pi}}\frac{\sin(kct)}{k}\end{aligned}$$

where again the last equality follows from the definition of the complex exponential.

□

**Problem 2.3.** Show that

a. $i\dfrac{d}{dk}F(k) = \mathscr{F}\left\{xf(x)\right\}$

b. $i^n\dfrac{d^n}{dk^n}F(k) = \mathscr{F}\left\{x^n f(x)\right\}$

*Solution.* a. Recall from the definition of the Fourier transform in (1) that

$$F(k) = \mathscr{F}\left\{f(x)\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x)dx.$$

Using the Leibniz integral rule, we have that

$$\begin{aligned}\frac{d}{dk}F(k) = \frac{d}{dk}\mathscr{F}\left\{f(x)\right\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial}{\partial k}\left[e^{-ikx}f(x)\right]dx \\ &= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} ixe^{-ikx}f(x)dx \\ &= -i\mathscr{F}\left\{xf(x)\right\}.\end{aligned}$$

Of course, this implies that

$$i\frac{d}{dk}F(k) = \mathscr{F}\left\{xf(x)\right\}.$$

b. Suppose that for $n > 1$ we have that

$$i^n\frac{d^n}{dk^n}F(k) = \mathscr{F}\left\{x^n f(x)\right\}. \tag{5}$$

Then by the Leibniz integral rule, we have that

$$\begin{aligned}i^n\frac{d^{n+1}}{dk^{n+1}}F(k) = \frac{d}{dk}\left[i^n\frac{d^n}{dk^n}F(k)\right] &= \frac{d}{dk}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}x^n e^{-ikx}f(x)dx\right] \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\partial}{\partial k}\left[x^n e^{-ikx}f(x)\right]dx \\ &= -i\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}x^{n+1}e^{-ikx}f(x)dx \\ &= -i\mathscr{F}\left\{x^{n+1}f(x)\right\}.\end{aligned}$$

This implies that

$$i^{n+1}\frac{d^{n+1}}{dk^{n+1}}F(k) = \mathscr{F}\left\{x^{n+1}f(x)\right\}$$

and (5) holds by induction.

□

**Problem 2.5.** Prove the following:

c. If $f(x)$ has a finite discontinuity at a point $x = a$, then

$$\mathscr{F}\left\{f'(x)\right\} = (ik)F(k) - \frac{1}{\sqrt{2\pi}}\exp(-ika)[f]_a,$$

where $[f]_a = f(a+0) - f(a-0)$.

Generalize this result for $\mathscr{F}\left\{f^{(n)}(x)\right\}$.

*Solution.* c. Suppose that $f$ has a finite discontinuity at the point $x = a$. From the definition of the Fourier transform, we have that

$$\begin{aligned}\mathscr{F}\left\{f'(x)\right\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}f'(x)dx \\ &= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{a} e^{-ikx}f'(x)dx + \int_{a}^{\infty} e^{-ikx}f'(x)dx\right],\end{aligned} \tag{6}$$

where we have separated the integral at the finite discontinuity. Since $f(x)$ and $e^{-ikx}$ are both continuous with continuous first derivatives on the intervals present in the above two integrals, we may apply integration by parts to compute these integrals. Set $u = e^{-ikx}$ and $dv = f'(x)dx$. Then $du = -ike^{-ikx}dx$ and $v = f(x)$ so that

$$\begin{aligned}\int_{-\infty}^{a} e^{-ikx}f'(x)dx &= e^{-ikx}f(x)\Big|_{-\infty}^{a^-} + ik\int_{-\infty}^{a} e^{-ikx}f(x)dx \\ &= e^{-ika}\lim_{x\to a^-} f(x) + ik\int_{-\infty}^{a} e^{-ikx}f(x)dx.\end{aligned}$$

Similarly, we also see that

$$\begin{aligned}\int_{a}^{\infty} e^{-ikx}f'(x)dx &= e^{-ikx}f(x)\Big|_{a^+}^{\infty} + ik\int_{-\infty}^{a} e^{-ikx}f(x)dx \\ &= -e^{-ika}\lim_{x\to a^+} f(x) + ik\int_{a}^{\infty} e^{-ikx}f(x)dx.\end{aligned}$$

Combining, we see therefore from (6) that

$$\begin{aligned}\mathscr{F}\left\{f'(x)\right\} &= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{a} e^{-ikx}f'(x)dx + \int_{a}^{\infty} e^{-ikx}f'(x)dx\right] \\ &= \frac{1}{\sqrt{2\pi}}\left[e^{-ika}\lim_{x\to a^-} f(x) - e^{-ika}\lim_{x\to a^+} f(x) + ik\int_{-\infty}^{\infty} e^{-ikx}f(x)dx\right] \\ &= ikF(k) - \frac{1}{\sqrt{2\pi}}e^{-ika}\left(\lim_{x\to a^+} f(x) - \lim_{x\to a^-} f(x)\right) \\ &= ikF(k) - \frac{1}{\sqrt{2\pi}}e^{-ika}[f]_a\end{aligned}$$

where we have used $\lim_{x\to a^+} f(x) - \lim_{x\to a^-} f(x) = f(a+0) - f(a-0) = [f]_a$.

We now wish to show that, in general,

$$\mathscr{F}\left\{f^{(n)}(x)\right\} = (ik)^n F(k) - \frac{1}{\sqrt{2\pi}} e^{-ika} \sum_{j=1}^{n} (ik)^{n-j} [f^{(j-1)}]_a. \tag{7}$$

We have shown previously that this result holds for $n = 1$, so suppose this result holds for $n$. Then we see that

$$\begin{aligned}
\mathscr{F}\left\{f^{(n+1)}\right\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f^{(n+1)}(x)dx \\
&= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{a} e^{-ikx} f^{(n+1)}(x)dx + \int_{-\infty}^{a} e^{-ikx} f^{(n+1)}(x)dx\right].
\end{aligned}$$

Proceeding as before, we integrate by parts using the substitution $u = e^{-ikx}$ and $dv = f^{(n+1)}(x)dx$ yielding

$$\begin{aligned}
\int_{-\infty}^{a} e^{-ikx} f^{(n+1)}(x)dx &= e^{-ikx} f^{(n)}(x)\Big|_{-\infty}^{a^-} + ik \int_{-\infty}^{a} e^{-ikx} f^{(n)}(x)dx \\
&= e^{-ika} \lim_{x\to a^-} f^{(n)}(x) + ik \int_{-\infty}^{a} e^{-ikx} f^{(n)}(x)dx
\end{aligned}$$

and similarly

$$\begin{aligned}
\int_{a}^{\infty} e^{-ikx} f^{(n+1)}(x)dx &= e^{-ikx} f^{(n)}(x)\Big|_{a^+}^{\infty} + ik \int_{a}^{\infty} e^{-ikx} f^{(n)}(x)dx \\
&= -e^{-ika} \lim_{x\to a^+} f^{(n)}(x) + ik \int_{a}^{\infty} e^{-ikx} f^{(n)}(x)dx.
\end{aligned}$$

Therefore, combining we have that

$$\begin{aligned}
\mathscr{F}\left\{f^{(n+1)}\right\} &= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{a} e^{-ikx} f^{(n+1)}(x)dx + \int_{-\infty}^{a} e^{-ikx} f^{(n+1)}(x)dx\right] \\
&= \frac{ik}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f^{(n)}(x)dx - \frac{1}{\sqrt{2\pi}} e^{-ika} \left(\lim_{x\to a^+} f^{(n)}(x) - \lim_{x\to a^-} f^{(n)}(x)\right) \\
&= ik\mathscr{F}\left\{f^{(n)}(x)\right\} - \frac{1}{\sqrt{2\pi}} e^{-ika} [f^{(n)}]_a.
\end{aligned}$$

From our assumption, we then see that

$$\begin{aligned}
\mathscr{F}\left\{f^{(n+1)}\right\} &= ik\mathscr{F}\left\{f^{(n)}(x)\right\} - \frac{1}{\sqrt{2\pi}}e^{-ika}[f^{(n)}]_a \\
&= ik\left[(ik)^n F(k) - \frac{1}{\sqrt{2\pi}}e^{-ika}\sum_{j=1}^{n}(ik)^{n-j}[f^{(j-1)}]_a\right] - \frac{1}{\sqrt{2\pi}}e^{-ika}[f^{(n)}]_a \\
&= (ik)^{n+1}F(k) - \frac{1}{\sqrt{2\pi}}e^{-ika}[f^{(n)}]_a - \frac{1}{\sqrt{2\pi}}e^{-ika}\sum_{j=1}^{n}(ik)^{n-j+1}[f^{(j-1)}]_a \\
&= (ik)^{n+1}F(k) - \frac{1}{\sqrt{2\pi}}e^{-ika}\sum_{j=1}^{n+1}(ik)^{n+1-j}[f^{(j-1)}]_a.
\end{aligned}$$

Therefore, result (7) holds for $n+1$ and the result is true by induction.

$\square$

**Problem 2.7.** Prove the following results for the convolution:

c. $\dfrac{d}{dx}\left[f(x) * g(x)\right] = f'(x) * g(x) = f(x) * g'(x)$,

d. $\displaystyle\int_{-\infty}^{\infty} (f * g)(x)dx = \int_{-\infty}^{\infty} f(u)du \int_{-\infty}^{\infty} g(v)dv$.

*Solution.* For two functions functions $f, g \in L^1(\mathbb{R})$, the convolution of $f$ and $g$, denoted by $(f * g)(x)$, is defined to be

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi.$$

c. Using the definition of the convolution of $f$ and $g$, we have that

$$\begin{aligned}\frac{d}{dx}\left[f(x) * g(x)\right] &= \frac{d}{dx}\int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi \\ &= \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left[f(x-\xi)g(\xi)\right]d\xi \\ &= \int_{-\infty}^{\infty} f'(x-\xi)g(\xi)d\xi \\ &= f'(x) * g(x).\end{aligned}$$

Since the convolution of two functions is commutative, we have using the above that

$$\begin{aligned}\frac{d}{dx}\left[f(x) * g(x)\right] &= \frac{d}{dx}\left[g(x) * f(x)\right] \\ &= g'(x) * f(x) \\ &= f(x) * g'(x).\end{aligned}$$

d. From the definition of the convolution of $f$ and $g$, we have that

$$\int_{-\infty}^{\infty} (f * g)(x)dx = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi\right]dx.$$

Since $f, g \in L^1(\mathbb{R})$, we may interchange the order of integration above. Doing so yields

$$\begin{aligned}\int_{-\infty}^{\infty} (f * g)(x)dx &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi\right]dx \\ &= \int_{-\infty}^{\infty} g(\xi)\left[\int_{-\infty}^{\infty} f(x-\xi)dx\right]d\xi.\end{aligned}$$

Making the substitution $u = x - \xi$ with $du = dx$, the above becomes

$$\begin{aligned}\int_{-\infty}^{\infty} (f * g)(x)dx &= \int_{-\infty}^{\infty} g(\xi)\left[\int_{-\infty}^{\infty} f(x-\xi)dx\right]d\xi \\ &= \int_{-\infty}^{\infty} g(\xi)\left[\int_{-\infty}^{\infty} f(u)du\right]d\xi \\ &= \int_{-\infty}^{\infty} g(\xi)d\xi \int_{-\infty}^{\infty} f(u)du.\end{aligned}$$

The variables $u$ and $\xi$ are arbitary; therefore

$$\int_{-\infty}^{\infty} (f * g)(x) dx = \int_{-\infty}^{\infty} f(u) du \int_{-\infty}^{\infty} g(v) dv$$

and we are done.

$\square$

**Problem 2.8.** Use the Fourier transform to solve the following ordinary differential equations for $-\infty < x < \infty$:

a. $y''(x) - y(x) + 2f(x) = 0$, where $f(x) = 0$ when $x < -a$ and when $x > a$ and its derivatives vanish at $x = \pm\infty$,

b. $2y''(x) + xy'(x) + y(x) = 0$.

*Solution.* Consider the following $n$-th order ordinary differential equation with constant coefficients:

$$Ly(x) = f(x),$$

where $L$ is the $n$-th order differential operation given by

$$L \equiv a_n \frac{d^n}{dx^n} + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1 \frac{d^n}{dx^n} + a_0.$$

Applying the Fourier transform to both sides of the ordinary differential equation gives

$$\left[a_n (ik)^n + a_{n-1}(ik)^{n-1} + \cdots + a_1(ik) + a_0\right] Y(k) = F(k) \tag{8}$$

where $Y(k) = \mathscr{F}\{y(x)\}$ and $F(k) = \mathscr{F}\{f(x)\}$. Set $P(z) = \sum_{k=0}^{n} a_k z^k$. Then (8) may be rewritten as $P(ik)Y(k) = F(k)$, or equivalently

$$Y(k) = \frac{F(k)}{P(ik)} = F(k)Q(k)$$

where $Q(k) = 1/P(ik)$. The Convolution theorem then tell us that $y(x)$, the solution to the ordinary differential equation, is given by

$$y(x) = \mathscr{F}^{-1}\{F(k)Q(k)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) q(x - \xi) d\xi,$$

where $q(x) = \mathscr{F}^{-1}\{Q(k)\}$ is known explicitly.

a. The ordinary differential equation $y''(x) - y(x) + 2f(x) = 0$ may be rewritten as

$$\left[a_2 \frac{d^2}{dx^2} + a_0\right] y(x) = g(x)$$

where $a_2 = 1$, $a_0 = -1$, and $g(x) = -2f(x)$. Applying the Fourier transform to this ordinary differential equation results in the following equation

$$\left[a_2 (ik)^2 + a_0\right] Y(k) = G(k),$$

where $Y(k) = \mathscr{F}\{y(x)\}$ and $G(k) = \mathscr{F}\{g(x)\}$. Thus, we have that

$$Y(k) = G(k)Q(k)$$

where

$$Q(k) = -\frac{1}{k^2+1}.$$

By definition, the inverse Fourier transform of a function $Q(k)$ is given by

$$\mathscr{F}^{-1}\left\{Q(k)\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} Q(k) dk.$$

Using Cauchy's Residue theorem allows us to evaluate this integral for $Q(k)$. The theorem states that the integral of $h(k) = e^{ikx}Q(k)$ around any closed contour is given in terms of the residues of $h$ at all singular points bounded by the contour.

So, suppose first that $x > 0$ and consider the contour $C$ that winds counter clockwise along the straight line segment $[-a, a]$, say $C_1$, through the counter $C_2$ defined to be the half circle such that $z = i$ is bounded by this contour. Then, $h$ is analytic everywhere except at $k = i$ and by Cauchy's Residue theorem we have that

$$\int_C h(k)dk = \int_{C_1} h(k)dk + \int_{C_2} h(k)dk = 2\pi i \lim_{k\to i}(k-i)h(k) = 2\pi i \left[\frac{e^{-x}}{2i}\right] = \pi e^{-x}.$$

Thus,

$$\int_{-a}^{a} h(k)dk = \pi e^{-x} - \int_{C_2} h(k)dk.$$

It can be shown that

$$\left|\int_{C_2} h(k)dk\right| \le \frac{a\pi}{a^2-1}.$$

As $a \to \infty$, we have that $a\pi/(a^2-1) \to 0$ so

$$\int_{-\infty}^{\infty} h(k)dk = \pi e^{-x}.$$

A similar argument follows when $x < 0$ by changing the contour to wind around $z = -i$ in a clockwise fashion to yield that

$$\int_{-\infty}^{\infty} h(k)dk = \pi e^{x}.$$

Therefore, we have that

$$\begin{aligned}\mathscr{F}^{-1}\left\{Q(k)\right\} &= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{ikx}}{k^2+1} dk \\ &= -\frac{\pi e^{-|x|}}{\sqrt{2\pi}}.\end{aligned}$$

Now, from the previous remarks, this implies that the solution to the ordinary differential equation is given by

$$\begin{aligned} y(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\xi) q(x-\xi) d\xi \\ &= \frac{1}{\sqrt{2\pi}} \frac{2\pi}{\sqrt{2\pi}} \int_{-a}^{a} f(\xi) e^{-|x-\xi|} d\xi \\ &= \int_{-a}^{a} f(\xi) e^{-|x-\xi|} d\xi. \end{aligned}$$

b. We begin by applying the Fourier transform to the ordinary differential equation $2y''(x) + xy'(x) + y(x) = 0$. Using the linearity and other properties of the Fourier transform, we see that the left hand side becomes

$$\begin{aligned} \mathscr{F}\{2y''(x) + xy'(x) + y(x)\} &= 2\mathscr{F}\{y''(x)\} + \mathscr{F}\{xy'(x)\} + \mathscr{F}\{y(x)\} \\ &= (1-2k^2)\mathscr{F}\{y(x)\} + \mathscr{F}\{xy'(x)\}. \end{aligned}$$

We have shown previously that $\mathscr{F}\{xg(x)\} = i\frac{d}{dk}\mathscr{F}\{g(x)\}$ and $\mathscr{F}\{g'(x)\} = ik\mathscr{F}\{g(x)\}$. Thus, we have that

$$\begin{aligned} \mathscr{F}\{xy'(x)\} &= i\frac{d}{dk}\left[\mathscr{F}\{y'(x)\}\right] \\ &= i^2\frac{d}{dk}\left[kY(k)\right] \\ &= -Y(k) - kY'(k). \end{aligned}$$

So, we see that

$$\begin{aligned} \mathscr{F}\{2y''(x) + xy'(x) + y(x)\} &= (1-2k^2)\mathscr{F}\{y(x)\} + \mathscr{F}\{xy'(x)\} \\ &= (1-2k^2)Y(k) - Y(k) - kY'(k) \\ &= -2k^2Y(k) - kY'(k). \end{aligned}$$

Since $\mathscr{F}\{0\} = 0$, the ordinary differential equation under the Fourier transform becomes

$$-2k^2Y(k) - kY'(k) = 0$$

or, equivalently,

$$Y'(k) = -2kY(k).$$

This is a separable differential equation, the solution to which we readily see is

$$Y(k) = c_1 e^{-k^2}.$$

Thus, we have that

$$y(x) = \mathscr{F}^{-1}\left\{c_1 e^{-k^2}\right\} = c_1 \mathscr{F}^{-1}\left\{e^{-k^2}\right\}$$

Using a table of Fourier transforms, we have that

$$\mathscr{F}^{-1}\left\{\frac{1}{\sqrt{2a}} \exp\left(-\frac{k^2}{4a}\right)\right\} = \exp(-ax^2).$$

Setting $a = 1/4$, we have that

$$\sqrt{2}\mathscr{F}^{-1}\left\{\exp\left(-k^2\right)\right\} = \exp\left(-\frac{x^2}{4}\right).$$

Therefore, the solution to the ordinary differential equation is given by

$$y(x) = c_1 \mathscr{F}^{-1}\left\{\exp\left(-k^2\right)\right\} = \frac{1}{c_1\sqrt{2}} e^{-\frac{x^2}{4}}.$$

$\square$

**Problem 2.9.** Solve the following integral equations for an unknown function $f(x)$:

a. $\displaystyle\int_{-\infty}^{\infty} \phi(x-t)f(t)dt = g(x)$,

b. $\displaystyle\int_{-\infty}^{\infty} \exp(-at^2)f(x-t)dt = \exp(-bt^2), a > b > 0$,

d. $\displaystyle\int_{-\infty}^{\infty} f(x-t)f(t)dt = \frac{b}{x^2+b^2}$.

*Solution.* Recall the definition of the convolution

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi.$$

Using the definition of the convolution and the Fourier transform, we derive the following Convolution Theorem

$$\mathscr{F}\{(f*g)(x)\} = \mathscr{F}\{f(x)\}\mathscr{F}\{g(x)\} = F(k)G(k).$$

It is this theorem that allows us to solve the above integral equations.

a. From the definition of the convolution, we see that

$$\int_{-\infty}^{\infty} \phi(x-t)f(t)dt = \sqrt{2\pi}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi(x-t)f(t)dt\right] = \sqrt{2\pi}(\phi * f)(x) = g(x).$$

Applying the Fourier transform to this equation and using the Convolution Theorem, we see that

$$\mathscr{F}\left\{\sqrt{2\pi}(\phi * f)(x)\right\} = \mathscr{F}\{g(x)\}$$

or, equivalently,

$$\sqrt{2\pi}\Phi(k)F(k) = G(k).$$

Thus, we see that

$$F(k) = \frac{1}{\sqrt{2\pi}}\frac{G(k)}{\Phi(k)}.$$

Applying the inverse Fourier transform to this equation we see that

$$\mathscr{F}^{-1}\{F(k)\} = \mathscr{F}^{-1}\left\{\frac{1}{\sqrt{2\pi}}\frac{G(k)}{\Phi(k)}\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{G(k)}{\Phi(k)}e^{ikx}dk.$$

Therefore, the solution to the integral equation is given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{G(k)}{\Phi(k)}e^{ikx}dk. \tag{9}$$

b. Set $\phi(x) = e^{-at^2}$ and $g(x) = e^{-bx^2}$. Then the integral equation becomes

$$\sqrt{2\pi}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-t)\phi(t)dt\right] = \sqrt{2\pi}(f*\phi)(x) = \sqrt{2\pi}(\phi*f)(x) = g(x).$$

From 2.9.a and (9), we see that the solution to the integral equation is given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{G(k)}{\Phi(k)}e^{ikx}dk.$$

Using a table of Fourier transforms, we have that

$$\mathscr{F}\left\{e^{-cx^2}\right\} = \frac{1}{\sqrt{2c}}e^{-k^2/4c} \quad \text{for } c > 0.$$

Thus,

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\sqrt{2a}}{\sqrt{2b}}\frac{e^{-k^2/4b}}{e^{-k^2/4a}}e^{ikx}dk\right] \\ &= \frac{\sqrt{a}}{\sqrt{2\pi b}}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{k^2}{4}\frac{a-b}{ab}}e^{ikx}dk\right] \\ &= \frac{\sqrt{a}}{\sqrt{2\pi b}}\mathscr{F}^{-1}\left\{e^{-\frac{k^2}{4\frac{ab}{a-b}}}\right\}. \end{aligned}$$

From our same table of Fourier transforms, we have that

$$\mathscr{F}^{-1}\left\{\frac{1}{2c}e^{-k^2/4c}\right\} = e^{-cx^2}$$

or that

$$\mathscr{F}^{-1}\left\{e^{-k^2/4c}\right\} = \sqrt{2c}e^{-cx^2}.$$

Since $a > b > 0$, we have that $\frac{ab}{a-b} > 0$ and

$$\mathscr{F}^{-1}\left\{e^{-\frac{k^2}{4\frac{ab}{a-b}}}\right\} = \sqrt{\frac{2ab}{a-b}}e^{-\left(\frac{ab}{a-b}\right)x^2}.$$

Therefore, the solution to the integral equation is given by

$$\begin{aligned} f(x) &= \frac{\sqrt{a}}{\sqrt{2\pi b}}\mathscr{F}^{-1}\left\{e^{-\frac{k^2}{4\frac{ab}{a-b}}}\right\} \\ &= \frac{\sqrt{a}}{\sqrt{2\pi b}}\sqrt{\frac{2ab}{a-b}}e^{-\left(\frac{ab}{a-b}\right)x^2} \\ &= \frac{a}{\sqrt{\pi(a-b)}}e^{-\left(\frac{ab}{a-b}\right)x^2}. \end{aligned}$$

d. Let $\phi(x) = \dfrac{1}{x^2 + a^2}$ and $g(x) = \dfrac{1}{x^2 + b^2}$. Then the integral equation becomes

$$\int_{-\infty}^{\infty} f(x-t)\phi(t)dt = \sqrt{2\pi}(\phi * f)(x) = g(x).$$

From 2.9.a and (9), we see that the solution to the integral equation is given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{G(k)}{\Phi(k)} e^{ikx} dk.$$

Using a table of Fourier transforms, we have that

$$\mathscr{F}\left\{\frac{1}{x^2 + c^2}\right\} = \sqrt{\frac{\pi}{2}}\frac{e^{-c|k|}}{c}.$$

Thus,

$$\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{G(k)}{\Phi(k)} e^{ikx} dk \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{a}{b}\frac{e^{-b|k|}}{e^{-a|k|}} e^{ikx} dk \\
&= \frac{a}{b\sqrt{2\pi}}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(b-a)|k|} e^{ikx} dk\right] \\
&= \frac{a}{b\sqrt{2\pi}}\mathscr{F}^{-1}\left\{e^{-(b-a)|k|}\right\}.
\end{aligned}$$

From our same table of Fourier transforms, we have that

$$\mathscr{F}^{-1}\left\{\sqrt{\frac{\pi}{2}}\frac{e^{-c|k|}}{c}\right\} = \frac{1}{x^2 + c} \quad \text{for } c > 0$$

or that

$$\mathscr{F}^{-1}\left\{e^{-c|k|}\right\} = c\sqrt{\frac{2}{\pi}}\left(\frac{1}{x^2 + c}\right).$$

Since $b > a > 0$, we have that $b - a > 0$ and that

$$\mathscr{F}^{-1}\left\{e^{-(b-a)|k|}\right\} = \sqrt{\frac{2}{\pi}}\left(\frac{b-a}{x^2 + (b-a)^2}\right).$$

Therefore, the solution to the integral equation is given by

$$\begin{aligned}
y(x) &= \frac{a}{b\sqrt{2\pi}}\mathscr{F}^{-1}\left\{e^{-(b-a)|k|}\right\} \\
&= \frac{a(b-a)}{\pi b}\left(\frac{1}{x^2 + (b-a)^2}\right).
\end{aligned}$$

$\square$