Exam 1

Matthew Tiger

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Problem 1. Find the Fourier Transforms of the following functions:

   a. $f(x) = x^2 e^{-a|x|}$, $a > 0$,

   b. $f(x) = \left(1 - \dfrac{|x|}{2}\right) H\left(1 - \dfrac{|x|}{2}\right)$.

Solution. Recall that if $f(x) \in L^1(\mathbb{R})$, then the Fourier Transform of f is defined to be

$$\mathscr{F}\{f(x)\} = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx. \tag{1}$$

   a. If $f(x) = x^2 e^{-a|x|}$, $a > 0$, we see from (1) that, by definition, the Fourier Transform of f is given by

$$\mathscr{F}\{f(x)\} = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-a|x|} e^{-ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} x^2 e^{-(-a+ik)x} dx + \int_{0}^{\infty} x^2 e^{-(a+ik)x} dx \right].$$

Now, we see by integration by parts that

$$\int_{c}^{d} x^2 e^{-(b+ik)x} dx = -\frac{x^2}{b+ik} e^{-(b+ik)x} \Big|_{c}^{d} + \frac{2}{b+ik} \int_{c}^{d} x e^{-(b+ik)x} dx$$

$$= -\frac{x^2}{b+ik} e^{-(b+ik)x} \Big|_{c}^{d} + \frac{2}{b+ik} \left[-\frac{x}{b+ik} e^{-(b+ik)x} \Big|_{c}^{d} + \frac{1}{b+ik} \int_{c}^{d} e^{-(b+ik)x} dx \right]$$

$$= -\frac{x^2}{b+ik} e^{-(b+ik)x} \Big|_{c}^{d} + \frac{2}{b+ik} \left[-\frac{x}{b+ik} e^{-(b+ik)x} \Big|_{c}^{d} - \frac{1}{(b+ik)^2} e^{-(b+ik)x} \Big|_{c}^{d} \right].$$

Thus,

$$\int_{-\infty}^{0} x^2 e^{-(-a+ik)x} dx = -\frac{2}{(-a+ik)^3}$$

and

$$\int_0^\infty x^2 e^{-(a+ik)x} dx = \frac{2}{(a+ik)^3}.$$

Therefore,

$$\begin{aligned}
\mathscr{F}\{f(x)\} = F(k) &= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^0 x^2 e^{-(-a+ik)x} dx + \int_0^\infty x^2 e^{-(a+ik)x} dx \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[-\frac{2}{(-a+ik)^3} + \frac{2}{(a+ik)^3} \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{4a(a^2-3k)}{(a^2+k^2)^3} \right] \\
&= \sqrt{\frac{2}{\pi}} \left[\frac{2a(a^2-3k)}{(a^2+k^2)^3} \right].
\end{aligned}$$

b. Recall that the Heaviside function H is defined as

$$H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Thus,

$$H\left(1 - \frac{|x|}{2}\right) = \begin{cases} 1 & \text{if } |x| < 2 \\ 0 & \text{if } |x| > 2. \end{cases}$$

If $f(x) = \left(1 - \frac{|x|}{2}\right) H\left(1 - \frac{|x|}{2}\right)$, we see from (1) that, by definition, the Fourier Transform of f is given by

$$\begin{aligned}
\mathscr{F}\{f(x)\} = F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \left(1 - \frac{|x|}{2}\right) H\left(1 - \frac{|x|}{2}\right) e^{-ikx} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-2}^2 \left(1 - \frac{|x|}{2}\right) e^{-ikx} dx \\
&= \frac{1}{\sqrt{2\pi}} \left[\int_{-2}^0 \left(1 + \frac{x}{2}\right) e^{-ikx} dx + \int_0^2 \left(1 - \frac{x}{2}\right) e^{-ikx} dx \right].
\end{aligned}$$

Now, we see by integration by parts that

$$\begin{aligned}
\int_c^d \left(1 \pm \frac{x}{2}\right) e^{-ikx} dx &= \frac{i}{k} e^{-ikx} \Big|_c^d \pm \frac{1}{2} \int_c^d x e^{-ikx} dx \\
&= \frac{i}{k} e^{-ikx} \Big|_c^d \pm \frac{1}{2} \left[\frac{ix}{k} e^{-ikx} \Big|_c^d - \frac{i}{k} \int_c^d e^{-ikx} dx \right] \\
&= \frac{i}{k} e^{-ikx} \Big|_c^d \pm \frac{1}{2} \left[\frac{ix}{k} e^{-ikx} \Big|_c^d + \frac{e^{-ikx}}{k^2} \Big|_c^d \right].
\end{aligned}$$

Thus,

$$\int_{-2}^{0} \left(1 + \frac{x}{2}\right) e^{-ikx} dx = \frac{1 - e^{2ik} + 2ik}{2k^2}$$

and

$$\int_{0}^{2} \left(1 - \frac{x}{2}\right) e^{-ikx} dx = \frac{1 - e^{-2ik} - 2ik}{2k^2}$$

Therefore, using the definition of the complex exponential and various trigonometric identities, we have that

$$\begin{aligned}
\mathscr{F}\{f(x)\} = F(k) &= \frac{1}{\sqrt{2\pi}} \left[\int_{-2}^{0} \left(1 + \frac{x}{2}\right) e^{-ikx} dx + \int_{0}^{2} \left(1 - \frac{x}{2}\right) e^{-ikx} dx\right] \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{1 - e^{2ik} + 2ik}{2k^2} + \frac{1 - e^{-2ik} - 2ik}{2k^2}\right] \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{k^2} - \frac{e^{-2ik} + e^{2ik}}{2k^2}\right] \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{1 - \cos 2k}{k^2}\right] \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{1 - (\cos^2 k - \sin^2 k)}{k^2}\right] \\
&= \frac{2\sin^2 k}{\sqrt{2\pi} k^2}.
\end{aligned}$$

$\square$

Problem 2. Find the Laplace Transforms of the following functions:

a. $f(t) = \int_0^t \frac{\sin ax}{x} dx,$

b. $f(t) = tH(t - a).$

Solution.   □

Problem 3. Solve the following integral equations:

a. $\displaystyle\int_0^\infty f(x)\sin kx\,dx = \begin{cases} 1-k & k < 1 \\ 0 & k > 1 \end{cases}$,

b. $\displaystyle\int_{-\infty}^\infty \frac{f(t)}{(x-t)^2+4}\,dt = \frac{1}{x^2+9}$.

Solution.   ☐

Problem 4. Show that

$$\int_0^\infty F_s(k)G_c(k)\sin kx\,dx = \frac{1}{2}\int_0^\infty g(\xi)\left[f(\xi+x) - f(\xi-x)\right]d\xi$$

where

$$F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin kx\,dx$$

and

$$G_c(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\cos kx\,dx.$$

Solution.																	□

Problem 5. Apply the Fourier Transform to solve the following initial value problem for the heat equation:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad -\infty < x < \infty,$$
$$u(x,0) = \phi(x), \quad t > 0.$$

Solution.   □

Problem 6. Evaluate the following definite integrals:

a. $\displaystyle\int_0^\infty \frac{\sin ax \sin bx}{x^2}\,dx,$

b. $\displaystyle\int_0^\infty \frac{(a^2 - x^2)^2}{(x^2 + a^2)^4}\,dx, \quad a > 0.$

Solution. □

Problem 7. Use the Fourier Sine Transform to solve the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \infty$$

with the boundary data $0 < y < L$

$$u(x, L) = 0, \quad u(x, 0) = f(x),$$
$$u(0, y) = 0, \quad u(x, y) \to 0 \text{ as } x \to \infty \text{ uniformly in } y.$$

Solution.   □

Problem 8. Apply the Fourier Transform to solve the 3-dimensional wave problem

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \quad -\infty < x, y, z < \infty,$$

subject to the initial conditions

$$u(x, y, z, t)|_{t=0} = 0$$
$$\left. \frac{\partial u(x, y, z, t)}{\partial t} \right|_{t=0} = \delta(x, y, z).$$

Solution.   □

Problem 9. Show that if E is a solution of an m-th order partial differential equation

$$P(\partial)u = \sum_{k=0}^{m} a_k \partial^k u = \delta,$$

where δ is the Dirac delta function, then $E * f$ is the solution of the partial differential equation $P(\partial)u = f$, where $*$ is the convolution.

Solution.   □