A Numerical Solution to a Second Order Ordinary Differential Equation

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1. Introduction

The authors were tasked by the client with finding the solution to the following family of differential equations

$$\begin{cases} -u''(x) + cu(x) = f(x) \\ 0 \leq x \leq 1 \\ u(0) = \epsilon \\ u(1) = \delta. \end{cases}$$

Additionally, the client has also requested to be provided with a means of plotting the solution once obtained.

Throughout this report, the above family of differential equations together with the interval of definition and initial conditions will be represented by $Lu = f$.

Assumptions were placed on this family so that $c \in \mathbb{R}$ with $c > 0$ and $f \in C^k([0,1])$ for sufficiently large k so that f is relatively well-behaved on the defined interval.

In this report we will detail the analytical solution to this family of differential equations showing the the above problem is well-posed and explain why this solution is not amenable to practical use. We therefore provide a numerical scheme to approximate the solution to the family of differential equations and examine the convergence, consistency and stability of the numerical scheme. Using the solution provided by the numerical scheme, we then explore the different options for plotting the solution.

2. Analytical Solution

The family of differential equations $Lu = f$ represents a second order linear differential equation and therefore well-known techniques can be used to find the solution $u(x)$.

The solution $u(x)$ is given by $u(x) = u_h(x)+u_p(x)$ where $u_h(x)$ is the solution to the homogeneous equation $-u''(x) + cu(x) = 0$ and $u_p(x)$ is a particular solution of $-u''(x) + cu(x) = f(x)$.

To find the homogeneous solution, note that the characteristic equation of this family of differential equations is given by $-m^2 + c = 0$, the roots of which are $m_1 = \sqrt{c} = \omega$ and $m_2 = -\sqrt{c} = -\omega$. Note that since $c > 0$, these roots are real and distinct suggesting that the homogeneous solution is given by

$$u_h(x) = c_1 e^{\omega x} + c_2 e^{-\omega x}. \tag{1}$$

To find the particular solution, we assume the particular solution is of the form $u_p(x) = \kappa(x)e^{\omega x}$ for some unknown function $\kappa(x)$. Thus,

$$u_p''(x) = \kappa''(x)e^{\omega x} + 2\omega\kappa'(x)e^{\omega x} + \omega^2\kappa(x)e^{\omega x}$$

and substituting the above into the original differential equation $Lu = f$ with $u_p(x) = \kappa(x)e^{\omega x}$ we have

$$\kappa''(x) + 2\omega\kappa'(x) = -f(x)e^{-\omega x}. \tag{2}$$

Making the substitution $\lambda(x) = \kappa'(x)$ into (2) we can reduce the above second order linear differential equation into the first order linear differential equation

$$\lambda'(x) + 2\omega\lambda(x) = -f(x)e^{-\omega x}. \tag{3}$$

The homogeneous solution to this first order differential equation is given by $\lambda_h(x) = c_3 e^{-2\omega x}$ suggesting the particular solution to the first order differential equation is of the form $\lambda_p(x) = \mu(x)e^{-2\omega x}$.

Repeating the same process as above, we see that

$$\lambda_p'(x) = \mu'(x)e^{-2\omega x} - 2\omega\mu(x)e^{-2\omega x}$$

and substituting into (3) with $\lambda_p(x) = \mu(x)e^{-2\omega x}$ we find that the first order linear differential equation becomes the separable first order differential equation

$$\mu'(x) = -f(x)e^{\omega x}.$$

We readily see the solution to the above differential equation is given by

$$\mu(x) = -\int_0^x f(r)e^{\omega r}dr.$$

As $\kappa'(x) = \lambda_p(x) = \mu(x)e^{-2\omega x}$, we deduce that

$$\kappa(x) = -\int_0^x e^{-2\omega s}\left[\int_0^s f(r)e^{\omega r}dr\right]ds$$

and

$$u_p(x) = \kappa(x)e^{\omega x} = -e^{\omega x}\int_0^x e^{-2\omega s}\left[\int_0^s f(r)e^{\omega r}dr\right]ds. \tag{4}$$

Combining the homogeneous solution (1) and the particular solution (4) we have that the general solution to $Lu = f$ is given by

$$\begin{aligned} u(x) &= u_h(x) + u_p(x) \\ &= c_1 e^{\omega x} + c_2 e^{-\omega x} - e^{\omega x}\int_0^x e^{-2\omega s}\left[\int_0^s f(r)e^{\omega r}dr\right]ds. \end{aligned} \tag{5}$$

Using the boundary values provided in $Lu = f$, the general solution is specified by the system of linear equations

$$\begin{aligned} u(0) &= c_1 + c_2 = \epsilon \\ u(1) &= c_1 e^{\omega} + c_2 e^{-\omega} - e^{\omega}\int_0^1 e^{-2\omega s}\left[\int_0^s f(r)e^{\omega r}dr\right]ds = \delta. \end{aligned}$$

The solution to this system in terms of the unknowns c_1 and c_2 is given by

$$c_1 = \frac{\epsilon e^{-\omega} - \delta - e^{\omega} \int_0^1 e^{-2\omega s} \left[\int_0^s f(r) e^{\omega r} dr \right] ds}{e^{-\omega} - e^{\omega}}$$
$$c_2 = \frac{-\epsilon e^{\omega} + \delta + e^{\omega} \int_0^1 e^{-2\omega s} \left[\int_0^s f(r) e^{\omega r} dr \right] ds}{e^{-\omega} - e^{\omega}}.$$

Using these constants in the general solution (5) gives us the unique analytical solution to the family of differential equation $Lu = f$. Furthermore, we deduce that the problem is in fact well-posed.

From this solution, we must make the following additional assumption on this problem: $f(x)$ must be integrable on the interval $[0, 1]$.

As the analytical solution depends on the symbolic integration of $f(x)$, we will be unable to use this solution for functions $f(x)$ in which the closed-form of the integral is not known.

3. Numerical Scheme

As mentioned in the previous section, the analytical solution is not practical to use for most functions $f(x)$. Thus, we present a numerical solution to approximate the analytical solution for the problem $Lu = f$.

3.1. Description

Our solution is derived from the method of finite differences. We define a finite set of points on the interval $[0, 1]$ called the grid D_h where the parameter h is the size of the grid where a smaller h denotes a finer grid. For our purposes, we consider $h = 1/N$ for positive N and create the uniform grid

$$D_h = \{x_n | x_n = hn \text{ for } 0 \leq n \leq N\}.$$

Define on this grid the discretized solution to the problem $Lu = f$ as $[u]_h = \{u(x_n)\}$ and define the discretized function $f^{(h)} = \{f(x_n)\}$. We wish to create a scheme L_h that computes an approximate solution $u^{(h)} = \left\{ u_0^{(h)}, u_1^{(h)}, \ldots, u_N^{(h)} \right\}$ to the problem $Lu = f$, i.e. a scheme such that $L_h u^{(h)} = f^{(h)}$.

Finding an approximation to $u''(x)$ should suggest how to construct the scheme L_h. To find an approximation for $u''(x)$, we investigate the Taylor expansion of $u(x)$ centered at h and $-h$. These expansions are given by

$$u(x+h) = u(x) + hu'(x) + \frac{h^2 u''(x)}{2} + \frac{h^3 u^{(3)}(x)}{3!} + O(h^4)$$
$$u(x-h) = u(x) - hu'(x) + \frac{h^2 u''(x)}{2} - \frac{h^3 u^{(3)}(x)}{3!} + O(h^4).$$

Adding these two expressions and solving for $u''(x)$ shows that

$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} - O(h^2). \tag{6}$$

This suggests that we should define our numerical scheme by replacing $u''(x)$ in $Lu = f$ with the approximation

$$u''(x) \approx \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}.$$

Therefore, we define the numerical scheme as

$$L_h u^{(h)} = f^{(h)} := \begin{cases} \frac{-u_{n+1}+2u_n-u_{n-1}}{h^2} + cu_n = f_n & \text{for } n = 1, \dots, N-1 \\ u_0 = \epsilon \\ u_N = \delta \end{cases}.$$

For $n = 1, \dots, N-1$, the scheme presents us with the recurrence relation

$$-u_{n-1} + (2 + ch^2)u_n - u_{n+1} = h^2 f_n$$

with initial conditions $u_0 = \epsilon$ and $u_N = \delta$. This recurrence relation is represented by the following system of equations

$$\begin{aligned} (2 + ch^2)u_1 - u_2 &= h^2 f_1 + u_0 \\ -u_1 + (2 + ch^2)u_2 - u_3 &= h^2 f_2 \\ -u_2 + (2 + ch^2)u_3 - u_4 &= h^2 f_3 \\ &\vdots \\ -u_{N-2} + (2 + ch^2)u_{N-1} &= h^2 f_{N-1} + u_N. \end{aligned}$$

In matrix form, this system of equations becomes

$$\begin{bmatrix} 2+ch^2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2+ch^2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2+ch^2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 2+ch^2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N-1} \end{bmatrix} = \begin{bmatrix} h^2 f_1 + u_0 \\ h^2 f_2 \\ h^2 f_3 \\ \vdots \\ h^2 f_{N-1} + u_N \end{bmatrix}$$

The solution to this system of equations paired with the initial conditions allows us to explicitly find $u^{(h)}$, our scheme's solution.

In section 4 we examine the convergence, consistency, and stability of this scheme in order to determine its usefulness in approximating the analytical solution to the problem $Lu = f$.

3.2. Implementation

We will now discuss how we implemented the above numerical scheme so that we can use the scheme to numerically approximate the solution to the problem $Lu = f$.

3.2.1. Discretized Solution

3.2.2. Plotting

4. Numerical Scheme Properties

There are three main properties of the numerical scheme presented in section 3 that are important to the validity of the numerical solution, namely the convergence, consistency, and stability of the numerical scheme.

4.1. Convergence

4.2. Consistency

The consistency can be gained by looking at the taylor expansion and the discretized solution. It is clear that the truncation error goes to 0 as h goes to 0 implying consistency.

4.3. Stability

5. Worked Example

A. Numerical Scheme Program