

Exam 1

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Problem 1. Find the Fourier Transforms of the following functions:

- a. $f(x) = x^2 e^{-a|x|}$, $a > 0$,
- b. $f(x) = \left(1 - \frac{|x|}{2}\right) H\left(1 - \frac{|x|}{2}\right)$.

Solution. Recall that if $f(x) \in L^1(\mathbb{R})$, then the Fourier Transform of f is defined to be

$$\mathcal{F}\{f(x)\} = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx. \quad (1)$$

- a. If $f(x) = x^2 e^{-a|x|}$, $a > 0$, we see from (1) that, by definition, the Fourier Transform of f is given by

$$\begin{aligned} \mathcal{F}\{f(x)\} = F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-a|x|} e^{-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^0 x^2 e^{-(-a+ik)x} dx + \int_0^{\infty} x^2 e^{-(a+ik)x} dx \right]. \end{aligned}$$

Now, we see by integration by parts that

$$\begin{aligned} \int_c^d x^2 e^{-(b+ik)x} dx &= -\frac{x^2}{b+ik} e^{-(b+ik)x} \Big|_c^d + \frac{2}{b+ik} \int_c^d x e^{-(b+ik)x} dx \\ &= -\frac{x^2}{b+ik} e^{-(b+ik)x} \Big|_c^d + \frac{2}{b+ik} \left[-\frac{x}{b+ik} e^{-(b+ik)x} \Big|_c^d + \frac{1}{b+ik} \int_c^d e^{-(b+ik)x} dx \right] \\ &= -\frac{x^2}{b+ik} e^{-(b+ik)x} \Big|_c^d + \frac{2}{b+ik} \left[-\frac{x}{b+ik} e^{-(b+ik)x} \Big|_c^d - \frac{1}{(b+ik)^2} e^{-(b+ik)x} \Big|_c^d \right]. \end{aligned} \quad (2)$$

Thus,

$$\int_{-\infty}^0 x^2 e^{-(-a+ik)x} dx = -\frac{2}{(-a+ik)^3}$$

and

$$\int_0^\infty x^2 e^{-(a+ik)x} dx = \frac{2}{(a+ik)^3}.$$

Therefore,

$$\begin{aligned} \mathcal{F}\{f(x)\} = F(k) &= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^0 x^2 e^{-(-a+ik)x} dx + \int_0^\infty x^2 e^{-(a+ik)x} dx \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[-\frac{2}{(-a+ik)^3} + \frac{2}{(a+ik)^3} \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{4a(a^2-3k)}{(a^2+k^2)^3} \right] \\ &= \sqrt{\frac{2}{\pi}} \left[\frac{2a(a^2-3k)}{(a^2+k^2)^3} \right]. \end{aligned}$$

b. Recall that the Heaviside function H is defined as

$$H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Thus,

$$H\left(1 - \frac{|x|}{2}\right) = \begin{cases} 1 & \text{if } |x| < 2 \\ 0 & \text{if } |x| > 2. \end{cases}$$

If $f(x) = \left(1 - \frac{|x|}{2}\right) H\left(1 - \frac{|x|}{2}\right)$, we see from (1) that, by definition, the Fourier Transform of f is given by

$$\begin{aligned} \mathcal{F}\{f(x)\} = F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \left(1 - \frac{|x|}{2}\right) H\left(1 - \frac{|x|}{2}\right) e^{-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-2}^2 \left(1 - \frac{|x|}{2}\right) e^{-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \left[\int_{-2}^0 \left(1 + \frac{x}{2}\right) e^{-ikx} dx + \int_0^2 \left(1 - \frac{x}{2}\right) e^{-ikx} dx \right]. \end{aligned}$$

Now, we see by integration by parts that

$$\begin{aligned} \int_c^d \left(1 \pm \frac{x}{2}\right) e^{-ikx} dx &= \frac{i}{k} e^{-ikx} \Big|_c^d \pm \frac{1}{2} \int_c^d x e^{-ikx} dx \\ &= \frac{i}{k} e^{-ikx} \Big|_c^d \pm \frac{1}{2} \left[\frac{ix}{k} e^{-ikx} \Big|_c^d - \frac{i}{k} \int_c^d e^{-ikx} dx \right] \\ &= \frac{i}{k} e^{-ikx} \Big|_c^d \pm \frac{1}{2} \left[\frac{ix}{k} e^{-ikx} \Big|_c^d + \frac{e^{-ikx}}{k^2} \Big|_c^d \right]. \end{aligned}$$

Thus,

$$\int_{-2}^0 \left(1 + \frac{x}{2}\right) e^{-ikx} dx = \frac{1 - e^{2ik} + 2ik}{2k^2}$$

and

$$\int_0^2 \left(1 - \frac{x}{2}\right) e^{-ikx} dx = \frac{1 - e^{-2ik} - 2ik}{2k^2}$$

Therefore, using the definition of the complex exponential and various trigonometric identities, we have that

$$\begin{aligned} \mathcal{F}\{f(x)\} = F(k) &= \frac{1}{\sqrt{2\pi}} \left[\int_{-2}^0 \left(1 + \frac{x}{2}\right) e^{-ikx} dx + \int_0^2 \left(1 - \frac{x}{2}\right) e^{-ikx} dx \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{1 - e^{2ik} + 2ik}{2k^2} + \frac{1 - e^{-2ik} - 2ik}{2k^2} \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{k^2} - \frac{e^{-2ik} + e^{2ik}}{2k^2} \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{1 - \cos 2k}{k^2} \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{1 - (\cos^2 k - \sin^2 k)}{k^2} \right] \\ &= \frac{2 \sin^2 k}{\sqrt{2\pi} k^2}. \end{aligned}$$

□

Problem 2. Find the Laplace Transforms of the following functions:

a. $f(t) = \int_0^t \frac{\sin ax}{x} dx,$

b. $f(t) = tH(t - a).$

Solution. Recall that if $f(t) \in L^1(\mathbb{R})$, then the Laplace Transform of f is defined to be

$$\mathcal{L}\{f(t)\} = \bar{f}(s) = \int_0^\infty f(t)e^{-st}dt, \quad \operatorname{Re} s > 0. \quad (3)$$

Note that it can be shown that the Laplace transform satisfies the important property

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} [\mathcal{L}\{f(t)\}]. \quad (4)$$

a. Let $f(t) = \int_0^t \frac{\sin ax}{x} dx$. Then $f(0) = 0$ and $f'(t) = \frac{\sin at}{t}$ so that $tf'(t) = \sin at$. This implies that

$$\mathcal{L}\{tf'(t)\} = \mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}.$$

The Laplace transform satisfies the following property that relates a function's derivative to its Laplace Transform:

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

This combined with (4) shows that

$$\mathcal{L}\{tf'(t)\} = -\frac{d}{ds} [\mathcal{L}\{f'(t)\}] = -\frac{d}{ds} [s\mathcal{L}\{f(t)\} - f(0)] = \frac{a}{s^2 + a^2},$$

or that

$$\frac{d}{ds} [s\mathcal{L}\{f(t)\}] = -\frac{a}{s^2 + a^2}.$$

Integrating both sides of the above equation yields that

$$s\mathcal{L}\{f(t)\} = -\int \frac{a}{s^2 + a^2} ds = -\tan^{-1}(s/a) + C.$$

In order to determine the constant of integration, we use the Initial Value Theorem which states that

$$\lim_{s \rightarrow \infty} s\mathcal{L}\{f(t)\} = \lim_{t \rightarrow 0} f(t) = f(0).$$

Since $f(0) = 0$, this implies that

$$\lim_{s \rightarrow \infty} s\mathcal{L}\{f(t)\} = \lim_{s \rightarrow \infty} [-\tan^{-1}(s/a) + C] = 0$$

or that $C = \frac{\pi}{2}$. Therefore, we have that

$$s\mathcal{L}\{f(t)\} = -\tan^{-1}(s/a) + \frac{\pi}{2} = \tan^{-1}(a/s)$$

so that

$$\mathcal{L}\{f(t)\} = \frac{\tan^{-1}(a/s)}{s}.$$

b. Let $f(t) = tH(t-a)$ and assume that $a > 0$. By property (4) we see that

$$\mathcal{L}\{tH(t-a)\} = -\frac{d}{ds} [\mathcal{L}\{H(t-a)\}].$$

From our table of Laplace Transforms, we have that

$$\mathcal{L}\{H(t-a)\} = \frac{e^{-as}}{s},$$

assuming that $a > 0$. Therefore,

$$\begin{aligned} \mathcal{L}\{f(t)\} &= -\frac{d}{ds} [\mathcal{L}\{H(t-a)\}] = -\frac{d}{ds} [s^{-1}e^{-as}] \\ &= -[-s^{-2}e^{-as} - as^{-1}e^{-as}] \\ &= \frac{(1+as)e^{-as}}{s^2}. \end{aligned}$$

□

Problem 3. Solve the following integral equations:

a. $\int_0^\infty f(x) \sin kx dx = \begin{cases} 1-k & k < 1 \\ 0 & k > 1 \end{cases},$

b. $\int_{-\infty}^\infty \frac{f(t)}{(x-t)^2 + 4} dt = \frac{1}{x^2 + 9}.$

Solution. a. Recall that the definition of the Fourier Sine Transform is given by

$$\mathcal{F}_s \{f(x)\} = F_s(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin kx dx.$$

Thus, we see that

$$\int_0^\infty f(x) \sin kx dx = \sqrt{\frac{\pi}{2}} \mathcal{F}_s \{f(x)\} = \sqrt{\frac{\pi}{2}} F_s(k).$$

Let $G_s(k) = \begin{cases} 1-k & k < 1 \\ 0 & k > 1 \end{cases}$. Then the above integral equation becomes

$$F_s(k) = \sqrt{\frac{2}{\pi}} G_s(k).$$

Thus, applying the inverse Fourier Sine Transform, we have that

$$f(x) = \mathcal{F}_s^{-1} \{F_s(k)\} = \sqrt{\frac{2}{\pi}} \mathcal{F}_s^{-1} \{G_s(k)\}$$

where the inverse Fourier Sine Transform is defined as

$$g(x) = \mathcal{F}_s^{-1} \{G_s(k)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty G_s(k) \sin kx dk. \quad (5)$$

Therefore, the solution to the integral equation is

$$\begin{aligned} f(x) &= \sqrt{\frac{2}{\pi}} \mathcal{F}_s^{-1} \{G_s(k)\} = \frac{2}{\pi} \int_0^\infty G_s(k) \sin kx dk \\ &= \frac{2}{\pi} \int_0^1 (1-k) \sin kx dk \\ &= \frac{2}{\pi} \left[\frac{1 - \cos x}{x} - \frac{\sin x - x \cos x}{x^2} \right] \\ &= \frac{2}{\pi} \left[\frac{x - \sin x}{x^2} \right]. \end{aligned}$$

b. Recall that the convolution of two functions f and g is defined such that

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - \xi)g(\xi)d\xi.$$

Let $g(x) = \frac{1}{x^2 + 2^2}$. Then we see that

$$\int_{-\infty}^{\infty} \frac{f(t)}{(x - t)^2 + 4} dt = \int_{-\infty}^{\infty} f(t)g(x - t)dt = \sqrt{2\pi}(g * f)(x) = \sqrt{2\pi}(f * g)(x).$$

Now, let $h(x) = \frac{1}{x^2 + 3^2}$. Then in light of the above remarks, the integral equation becomes

$$\int_{-\infty}^{\infty} \frac{f(t)}{(x - t)^2 + 4} dt = \sqrt{2\pi}(f * g)(x) = h(x) = \frac{1}{x^2 + 9}.$$

Applying the Fourier transform to the integral equation, we see by the Convolution Theorem that

$$\mathcal{F} \left\{ \sqrt{2\pi}(f * g)(x) \right\} = \sqrt{2\pi}F(k)G(k) = H(k) = \mathcal{F} \{h(x)\},$$

where $\mathcal{F} \{f(x)\} = F(k)$, $\mathcal{F} \{g(x)\} = G(k)$, and $\mathcal{F} \{h(x)\} = H(k)$, respectively. From our table of Fourier transforms, we see that for $a > 0$ we have that

$$\mathcal{F} \left\{ \frac{1}{x^2 + a^2} \right\} = \sqrt{\frac{\pi}{2}} \frac{e^{-a|k|}}{a}. \quad (6)$$

Thus, from (6) we have that

$$F(k) = \frac{1}{\sqrt{2\pi}} \frac{H(k)}{G(k)} = \frac{2}{3\sqrt{2\pi}} \frac{e^{-3|k|}}{e^{-2|k|}} = \frac{2e^{-|k|}}{3\sqrt{2\pi}}.$$

Applying the inverse Fourier Transform to this equation yields that

$$f(x) = \mathcal{F}^{-1} \{F(k)\} = \frac{2}{3\sqrt{2\pi}} \mathcal{F}^{-1} \{e^{-|k|}\}.$$

But from (6), we know that

$$\mathcal{F}^{-1} \{e^{-|k|}\} = \sqrt{\frac{2}{\pi}} \frac{1}{x^2 + 1}.$$

Therefore, the solution to the integral equation is given by

$$\begin{aligned} f(x) &= \frac{2}{3\sqrt{2\pi}} \mathcal{F}^{-1} \{e^{-|k|}\} = \frac{2}{3\sqrt{2\pi}} \left[\sqrt{\frac{2}{\pi}} \frac{1}{x^2 + 1} \right] \\ &= \frac{2}{3\pi} \left[\frac{1}{x^2 + 1} \right]. \end{aligned}$$

□

Problem 4. Show that

$$\int_0^\infty F_s(k)G_c(k) \sin kx dk = \frac{1}{2} \int_0^\infty g(\xi) [f(\xi + x) - f(\xi - x)] d\xi$$

where

$$F_s(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin kx dx$$

and

$$G_c(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \cos kx dx.$$

Solution. Using the definition of $G_c(k)$, we see that

$$\int_0^\infty F_s(k)G_c(k) \sin kx dk = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(k) \sin kx \left[\int_0^\infty g(\xi) \cos k\xi d\xi \right] dk$$

Interchanging the order of integration from ξ to k shows that

$$\begin{aligned} \int_0^\infty F_s(k)G_c(k) \sin kx dk &= \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(k) \sin kx \left[\int_0^\infty g(\xi) \cos k\xi d\xi \right] dk \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty g(\xi) \left[\int_0^\infty F_s(k) \cos k\xi \sin kx dk \right] d\xi \end{aligned}$$

Using the following trigonometric identity

$$\cos k\xi \sin kx = \frac{\sin k(\xi + x) - \sin k(\xi - x)}{2},$$

we then see that

$$\begin{aligned} \int_0^\infty F_s(k)G_c(k) \sin kx dk &= \sqrt{\frac{2}{\pi}} \int_0^\infty g(\xi) \left[\int_0^\infty F_s(k) \cos k\xi \sin kx dk \right] d\xi \\ &= \frac{1}{2} \sqrt{\frac{2}{\pi}} \int_0^\infty g(\xi) \left[\int_0^\infty F_s(k) \sin k(\xi + x) dk - \int_0^\infty F_s(k) \sin k(\xi - x) dk \right] d\xi \\ &= \frac{1}{2} \int_0^\infty g(\xi) [f(\xi + x) - f(\xi - x)] d\xi, \end{aligned}$$

where the last line follows using (5), the definition of the inverse Fourier Sine Transform.

Therefore,

$$\int_0^\infty F_s(k)G_c(k) \sin kx dk = \frac{1}{2} \int_0^\infty g(\xi) [f(\xi + x) - f(\xi - x)] d\xi,$$

and we are done. □

Problem 5. Apply the Fourier Transform to solve the following initial value problem for the heat equation:

$$\begin{aligned}\frac{\partial u}{\partial t} &= a^2 \frac{\partial^2 u}{\partial x^2} + f(x, t), \quad -\infty < x < \infty, \\ u(x, 0) &= \phi(x), \quad t > 0.\end{aligned}$$

Solution. Consider the function $u(x, t)$. The Fourier transform of u with respect to x is defined as

$$\mathcal{F}\{u(x, t)\} = U(k, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} u(x, t) dx. \quad (7)$$

From this definition and the Leibniz integral rule, we can see by induction that

$$\begin{aligned}\mathcal{F}\left\{\frac{\partial^n}{\partial t^n}[u(x, t)]\right\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^n}{\partial t^n}[u(x, t)] e^{-ikx} dx \\ &= \frac{d^n}{dt^n} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t) e^{-ikx} dx \right] \\ &= \frac{d^n}{dt^n} [\mathcal{F}\{u(x, t)\}].\end{aligned} \quad (8)$$

Similarly, we see from definition (7) and previous theorems regarding the Fourier transform that

$$\begin{aligned}\mathcal{F}\left\{\frac{\partial^n}{\partial x^n}[u(x, t)]\right\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^n}{\partial x^n}[u(x, t)] e^{-ikx} dx \\ &= (ik)^n \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t) e^{-ikx} dx \right] \\ &= (ik)^n \mathcal{F}\{u(x, y)\}.\end{aligned} \quad (9)$$

Now, applying the Fourier transform to the first equation yields that

$$\mathcal{F}\left\{\frac{\partial u}{\partial t}\right\} = \frac{d}{dt}[U(k, t)] = -(ak)^2 U(k, t) + F(k, t) = \mathcal{F}\left\{a^2 \frac{\partial^2 u}{\partial x^2} + f(x, t)\right\}.$$

This results in a first-order non-homogeneous linear differential equation

$$\frac{d}{dt}[U(k, t)] + (ak)^2 U(k, t) = F(k, t).$$

Using well-established techniques, we see that the solution to the linear differential equation is

$$U(k, t) = c_1 e^{-(ak)^2 t} + e^{-(ak)^2 t} \int_0^t e^{(ak)^2 \xi} F(k, \xi) d\xi.$$

Applying the Fourier Transform to the initial value equation shows that

$$\mathcal{F}\{u(x, 0)\} = U(k, 0) = \Phi(k) = \mathcal{F}\{\phi(x)\}$$

Thus, from the transformed initial value equation, we see using the above solution that

$$U(k, 0) = c_1 = \Phi(k)$$

Therefore, the solution to the transformed system of differential equations is

$$U(k, t) = \Phi(k)e^{-(ak)^2t} + e^{-(ak)^2t} \int_0^t e^{(ak)^2\xi} F(k, \xi) d\xi.$$

Applying the inverse Fourier transform to the solution of the transformed system of differential equations yields that the solution to the original system is

$$u(x, t) = \mathcal{F}^{-1} \{U(k, t)\} = \mathcal{F}^{-1} \left\{ \Phi(k)e^{-(ak)^2t} \right\} + \mathcal{F}^{-1} \left\{ e^{-(ak)^2t} \int_0^t e^{(ak)^2\xi} F(k, \xi) d\xi \right\}$$

Note from our table of Fourier transforms that for $b > 0$

$$\mathcal{F} \left\{ e^{-bx^2} \right\} = \frac{1}{\sqrt{2b}} \exp \left(-\frac{k^2}{4b} \right)$$

Thus,

$$\mathcal{F}^{-1} \left\{ e^{-a^2tk^2} \right\} = \mathcal{F}^{-1} \left\{ \exp \left(-\frac{k^2}{4(1/4a^2t)} \right) \right\} = \sqrt{\frac{1}{2a^2t}} \exp \left(-\frac{x^2}{4a^2t} \right) = g(x, t)$$

Now, from the Convolution Theorem, we have that

$$\mathcal{F}^{-1} \left\{ \Phi(k)e^{-(ak)^2t} \right\} = (\phi * g)(x) = \int_{-\infty}^{\infty} \phi(x - \xi)g(\xi, t)d\xi.$$

Therefore, using this identity and the definition of the inverse Fourier Transform, the solution to the original system of differential equations is

$$\begin{aligned} u(x, t) &= \mathcal{F}^{-1} \left\{ \Phi(k)e^{-(ak)^2t} \right\} + \mathcal{F}^{-1} \left\{ e^{-(ak)^2t} \int_0^t e^{(ak)^2\xi} F(k, \xi) d\xi \right\} \\ &= \int_{-\infty}^{\infty} \phi(x - \xi)g(\xi, t)d\xi + \int_{-\infty}^{\infty} \left[\int_0^t e^{(ak)^2\xi} F(k, \xi) d\xi \right] e^{-(ak)^2t} e^{ikx} dk. \end{aligned}$$

□

Problem 6. Evaluate the following definite integrals:

- a. $\int_0^\infty \frac{\sin ax \sin bx}{x^2} dx,$
 b. $\int_0^\infty \frac{(a^2 - x^2)^2}{(x^2 + a^2)^4} dx, \quad a > 0.$

Solution. Suppose that $F_c(k) = \mathcal{F}_c\{f(x)\}$ and $G_c(k) = \mathcal{F}_c\{g(x)\}$. Then Parseval's relation derived from the Convolution Theorem for the Fourier Cosine Transform states that

$$\int_0^\infty F_c(k)G_c(k)dk = \int_0^\infty f(x)g(x)dx. \quad (10)$$

- a. Let $F_c(k) = \frac{\sin ak}{k}$ and $G_c(k) = \frac{\sin bk}{k}$. Then Parseval's theorem shows that

$$\int_0^\infty \frac{\sin ak \sin bk}{k^2} dk = \int_0^\infty F_c(k)G_c(k)dk = \int_0^\infty f(x)g(x)dx$$

where $f(x) = \mathcal{F}_c^{-1}\{F_c(k)\}$ and $g(x) = \mathcal{F}_c^{-1}\{G_c(k)\}$. From our table of Fourier Cosine Transforms, we see that for $p \in \mathbb{R}$,

$$\mathcal{F}_c\{H(p-x)\} = \sqrt{\frac{2}{\pi}} \frac{\sin pk}{k}.$$

This implies that

$$\mathcal{F}_c^{-1}\left\{\frac{\sin pk}{k}\right\} = \sqrt{\frac{\pi}{2}} H(p-x).$$

Thus, we have that

$$\int_0^\infty \frac{\sin ak \sin bk}{k^2} dk = \int_0^\infty F_c(k)G_c(k)dk = \frac{\pi}{2} \int_0^\infty H(a-x)H(b-x)dx.$$

Now, we note from the definition of the Heaviside function that

$$H(a-x)H(b-x) = \begin{cases} 1 & \text{if } x < \min(a, b) \\ 0 & \text{if } x > \min(a, b) \end{cases}.$$

Therefore, we have that

$$\int_0^\infty \frac{\sin ak \sin bk}{k^2} dk = \frac{\pi}{2} \int_0^\infty H(a-x)H(b-x)dx = \frac{\pi}{2} \int_0^{\min(a,b)} dx = \frac{\pi}{2} \min(a, b).$$

- b. Let $F_c(k) = \frac{a^2 - k^2}{(k^2 + a^2)^2}$. Then Parseval's theorem shows that

$$\int_0^\infty \frac{(a^2 - k^2)^2}{(k^2 + a^2)^4} dk = \int_0^\infty F_c(k)F_c(k)dk = \int_0^\infty f(x)f(x)dx$$

where $f(x) = \mathcal{F}_c^{-1}\{F_c(k)\}$. From our table of Fourier Cosine Transforms, we see that for $a > 0$,

$$\mathcal{F}_c\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}} \frac{a^2 - k^2}{(k^2 + a^2)^2}.$$

This implies that

$$\mathcal{F}_c^{-1}\left\{\frac{a^2 - k^2}{(k^2 + a^2)^2}\right\} = \sqrt{\frac{\pi}{2}}.$$

Thus, we have that

$$\int_0^\infty \frac{(a^2 - k^2)^2}{(k^2 + a^2)^4} dk = \int_0^\infty F_c(k) F_c(k) dk = \frac{\pi}{2} \int_0^\infty x^2 e^{-2ax} dx.$$

From (2), we see by setting $b = 2a$ and $k = 0$ that

$$\int_0^\infty x^2 e^{-2ax} dx = \frac{1}{4a^3}.$$

Therefore, we have that

$$\int_0^\infty \frac{(a^2 - k^2)^2}{(k^2 + a^2)^4} dk = \frac{\pi}{2} \int_0^\infty x^2 e^{-2ax} dx = \frac{\pi}{(2a)^3}.$$

□

Problem 7. Use the Fourier Sine Transform to solve the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \infty$$

with the boundary data $0 < y < L$

$$\begin{aligned} u(x, L) &= 0, & u(x, 0) &= f(x), \\ u(0, y) &= 0, & u(x, y) &\rightarrow 0 \text{ as } x \rightarrow \infty \text{ uniformly in } y. \end{aligned}$$

Solution. Consider the function $u(x, y)$. The Fourier Sine Transform of u with respect to x is defined as

$$\mathcal{F}_s \{u(x, y)\} = U_s(k, y) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x, y) \sin(kx) dx.$$

From this definition we see using the Leibniz integral rule that

$$\begin{aligned} \mathcal{F}_s \left\{ \frac{\partial^n u(x, y)}{\partial y^n} \right\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^n u(x, y)}{\partial y^n} \sin(kx) dx \\ &= \frac{d^n}{dy^n} \left[\sqrt{\frac{2}{\pi}} \int_0^\infty u(x, y) \sin(kx) dx \right] \\ &= \frac{d^n}{dy^n} [\mathcal{F}_s \{u(x, y)\}]. \end{aligned}$$

The transforms of the partials of u with respect to x are not as easy to characterize. Nevertheless, we see from the properties of the Fourier Sine Transform that

$$\mathcal{F}_s \left\{ \frac{\partial u(x, y)}{\partial x} \right\} = -k \mathcal{F}_c \{u(x, y)\}$$

and

$$\mathcal{F}_s \left\{ \frac{\partial^2 u(x, y)}{\partial x^2} \right\} = -k^2 \mathcal{F}_s \{u(x, y)\} + k \sqrt{\frac{2}{\pi}} u(0, y).$$

Let $U_s(k, y) = \mathcal{F}_s \{u(x, y)\}$. Then, applying the Fourier Sine Transform to the first differential equation shows that

$$\mathcal{F}_s \left\{ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right\} = \frac{d^2}{dy^2} [U_s(k, y)] - k^2 U_s(k, y) + k \sqrt{\frac{2}{\pi}} u(0, y) = 0 = \mathcal{F}_s \{0\}.$$

From the boundary equation $u(0, y) = 0$ we see that the above equation becomes

$$\frac{d^2}{dy^2} [U_s(k, y)] - k^2 U_s(k, y) = 0.$$

This is a linear, second-order homogeneous differential equation, the solution of which we readily see is

$$U_s(k, y) = c_1 e^{-ky} + c_2 e^{ky}. \quad (11)$$

Applying the Fourier Sine Transform to the boundary equations, we see that

$$U_s(k, L) = 0, \quad U_s(k, 0) = F_s(k), \quad \text{for } 0 < k < \infty, 0 < y < L.$$

Using these equations and (11), the solution to the homogeneous equation, we see that

$$\begin{aligned} U_s(k, 0) &= c_1 + c_2 = F_s(k) \\ U_s(k, L) &= c_1 e^{-kL} + c_2 e^{kL} = 0. \end{aligned}$$

Solving, we see that

$$\begin{aligned} c_1 &= -\frac{e^{2kL} F_s(k)}{1 - e^{2kL}} \\ c_2 &= \frac{F_s(k)}{1 - e^{2kL}}. \end{aligned}$$

Thus, the solution to the transformed system of differential equations is

$$\begin{aligned} U_s(k, y) &= -\frac{e^{2kL} F_s(k) e^{-ky}}{1 - e^{2kL}} + \frac{F_s(k) e^{ky}}{1 - e^{2kL}} \\ &= F_s(k) \left(\frac{e^{-kL}}{e^{-kL}} \right) \left(\frac{-e^{ky} + e^{2kL - ky}}{-1 + e^{2kL}} \right) \\ &= F_s(k) \frac{\sinh k(L - y)}{\sinh kL}. \end{aligned}$$

Applying the inverse Fourier Sine Transform gives that the solution to the original system of differential equations is

$$\begin{aligned} u(x, y) &= \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(k) \frac{\sinh k(L - y)}{\sinh kL} \sin kx dk \\ &= \frac{2}{\pi} \int_0^\infty \left[\int_0^\infty f(\xi) \sin k\xi d\xi \right] \frac{\sinh k(L - y)}{\sinh kL} \sin kx dk. \end{aligned}$$

It is easy to see from the definition of the hyperbolic sine function that $\frac{\sinh k(L - y)}{\sinh kL} \sim e^{-ky}$ as $kL \rightarrow \infty$. Thus, the above problem reduces to a simpler problem in the quarter plane instead of the semi-infinite strip. Therefore, the solution to the original differential equation is

$$\begin{aligned} u(x, y) &= \frac{2}{\pi} \int_0^\infty f(\xi) d\xi \int_0^\infty \sin k\xi \sin kx e^{-ky} dk \\ &= \frac{1}{\pi} \int_0^\infty f(\xi) d\xi \int_0^\infty [\cos k(x - \xi) - \cos k(x + \xi)] e^{-ky} dk \\ &= \frac{1}{\pi} \int_0^\infty f(\xi) \left[\frac{y}{(x - \xi)^2 + y^2} - \frac{y}{(x + \xi)^2 + y^2} \right] d\xi. \end{aligned}$$

□

Problem 8. Apply the Fourier Transform to solve the 3-dimensional wave problem

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \quad -\infty < x, y, z < \infty,$$

subject to the initial conditions

$$\begin{aligned} u(x, y, z, t)|_{t=0} &= 0 \\ \frac{\partial u(x, y, z, t)}{\partial t} \Big|_{t=0} &= \delta(x, y, z). \end{aligned}$$

Solution. Let $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and suppose that $u(\mathbf{x}, t)$ is given. The Fourier transform of $u(\mathbf{x}, t)$ with respect to \mathbf{x} is defined to be

$$\mathcal{F}\{u(\mathbf{x}, t)\} = U(\mathbf{k}, t) = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} u(\mathbf{x}, t) e^{-i\mathbf{x} \cdot \mathbf{k}} d\mathbf{x} \quad (12)$$

where $\mathbf{k} \in \mathbb{R}^n$.

In order to investigate the Fourier transform of partials of $u(\mathbf{x}, t)$ with respect to a given component of \mathbf{x} , define the Fourier transform of $u(\mathbf{x}, t)$ with respect to x_j as the following

$$\mathcal{F}_{[x_j]}\{u(\mathbf{x}, t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(\mathbf{x}, t) e^{-ix_j k_j} dx_j.$$

Further, we will also use the function $\pi_j : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ defined as

$$\pi_j(\mathbf{x}) := (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$$

to aid in our description of the Fourier transform of partials of $u(\mathbf{x}, t)$. Now from definition (12) and Leibniz's integral rule we see that

$$\begin{aligned} \mathcal{F}\left\{\frac{\partial^n u(\mathbf{x}, t)}{\partial t^n}\right\} &= \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \frac{\partial^n}{\partial t^n} [u(\mathbf{x}, t)] e^{-i\mathbf{x} \cdot \mathbf{k}} d\mathbf{x} \\ &= \frac{d^n}{dt^n} \left[\frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} u(\mathbf{x}, t) e^{-i\mathbf{x} \cdot \mathbf{k}} d\mathbf{x} \right] \\ &= \frac{d^n}{dt^n} [\mathcal{F}\{u(\mathbf{x}, t)\}]. \end{aligned}$$

Similarly, from definition (12) and previous results about the Fourier transform, we see that

$$\begin{aligned} \mathcal{F}\left\{\frac{\partial^n u(\mathbf{x}, t)}{\partial x_j^n}\right\} &= \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{\partial^n}{\partial x_j^n} [u(\mathbf{x}, t)] e^{-ix_1 k_1} \dots e^{-ix_n k_n} dx_1 \dots dx_n \\ &= \frac{1}{(2\pi)^{(n-1)/2}} \int_{-\infty}^{\infty} \mathcal{F}_{[x_j]}\left\{\frac{\partial^n}{\partial x_j^n} [u(\mathbf{x}, t)]\right\} e^{-i\pi_j(\mathbf{x}) \cdot \pi_j(\mathbf{k})} d\pi_j(\mathbf{x}) \\ &= \frac{(ik_j)^n}{(2\pi)^{(n-1)/2}} \int_{-\infty}^{\infty} \mathcal{F}_{[x_j]}\{u(\mathbf{x}, t)\} e^{-i\pi_j(\mathbf{x}) \cdot \pi_j(\mathbf{k})} d\pi_j(\mathbf{x}) \\ &= (ik_j)^n \mathcal{F}\{u(\mathbf{x}, t)\}. \end{aligned}$$

Now, define $\mathbf{x} = (x_1, x_2, x_3) = (x, y, z) \in \mathbb{R}^3$. The the system of differential equations of the function $u(\mathbf{x}, t) = u(x_1, x_2, x_3, t)$ becomes

$$a^2 \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} \right) - \frac{\partial^2 u}{\partial t^2} = 0, \quad -\infty < x_1, x_2, x_3 < \infty,$$

subject to the initial conditions

$$u(\mathbf{x}, t)|_{t=0} = 0, \quad \left. \frac{\partial u(\mathbf{x}, t)}{\partial t} \right|_{t=0} = \delta(\mathbf{x}).$$

Applying the Fourier transform with respect to \mathbf{x} to the first equation yields

$$\mathcal{F} \left\{ a^2 \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} \right) - \frac{\partial^2 u}{\partial t^2} \right\} = -a^2 \|\mathbf{x}\|^2 U(\mathbf{k}, t) - \frac{d^2}{dt^2} [U(\mathbf{k}, t)] = 0 = \mathcal{F} \{0\}$$

where $U(\mathbf{k}, t) = \mathcal{F} \{u(\mathbf{x}, t)\}$. Similarly, we deduce that the transformed initial conditions become

$$\begin{aligned} \mathcal{F} \{u(\mathbf{x}, t)|_{t=0}\} &= U(\mathbf{k}, t)|_{t=0} = 0 = \mathcal{F} \{0\}, \\ \mathcal{F} \left\{ \left. \frac{\partial u(\mathbf{x}, t)}{\partial t} \right|_{t=0} \right\} &= \left. \frac{d}{dt} [U(\mathbf{k}, t)] \right|_{t=0} = \frac{1}{(2\pi)^{3/2}} = \mathcal{F} \{\delta(\mathbf{x})\}. \end{aligned}$$

We see that the first transformed equation is a second-order linear homogeneous ordinary differential equation, from which we readily see that the solution is

$$U(\mathbf{k}, t) = c_1 \cos(a \|\mathbf{k}\| t) + c_2 \sin(a \|\mathbf{k}\| t).$$

Using this solution we see from the first transformed initial condition that

$$U(\mathbf{k}, t)|_{t=0} = c_1 \cos(a \|\mathbf{k}\| t) + c_2 \sin(a \|\mathbf{k}\| t)|_{t=0} = c_1 = 0.$$

From the second transformed initial condition, we see using the above solution that

$$\begin{aligned} \left. \frac{d}{dt} [U(\mathbf{k}, t)] \right|_{t=0} &= -a \|\mathbf{k}\| c_1 \sin(a \|\mathbf{k}\| t) + a \|\mathbf{k}\| c_2 \cos(a \|\mathbf{k}\| t)|_{t=0} \\ &= a \|\mathbf{k}\| c_2 \\ &= \frac{1}{(2\pi)^{3/2}}, \end{aligned}$$

or that $c_2 = \frac{1}{a \|\mathbf{k}\| (2\pi)^{3/2}}$. Therefore, the solution to the transformed system of differential equations is

$$U(\mathbf{k}, t) = \frac{\sin(a \|\mathbf{k}\| t)}{a \|\mathbf{k}\| (2\pi)^{3/2}}.$$

Applying the inverse Fourier Transform to the above solution gives the solution to the original system

$$\begin{aligned}
 u(\mathbf{x}, t) &= \mathcal{F}^{-1} \left\{ \frac{\sin(a \|\mathbf{k}\| t)}{a \|\mathbf{k}\| (2\pi)^{3/2}} \right\} \\
 &= \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{\sin(a \|\mathbf{k}\| t)}{a \|\mathbf{k}\|} e^{i\mathbf{k} \cdot \mathbf{x}} d\mathbf{k} \\
 &= \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\sin(a \|\mathbf{k}\| t)}{a \|\mathbf{k}\|} e^{ik_1 x_1} e^{ik_2 x_2} e^{ik_3 x_3} dk_1 dk_2 dk_3.
 \end{aligned}$$

□

Problem 9. Show that if E is a solution of an m -th order partial differential equation

$$P(\partial)u = \sum_{n=0}^m a_n \partial^n u = \sqrt{2\pi} \delta,$$

where δ is the Dirac delta function, then $E * f$ is the solution of the partial differential equation $P(\partial)u = f$, where $*$ is the convolution.

Solution. Suppose that E is a solution of the partial differential equation

$$P(\partial)u = \sum_{n=0}^m a_n \partial^n u = \sqrt{2\pi} \delta,$$

i.e. $P(\partial)E = \sqrt{2\pi} \delta$. Then applying the Fourier Transform shows that

$$\mathcal{F}\{P(\partial)E\} = \sum_{n=0}^m a_n \mathcal{F}\{\partial^n E\} = \sum_{n=0}^m a_n (ik)^n \mathcal{F}\{E\} = 1 = \mathcal{F}\{\sqrt{2\pi} \delta\}. \quad (13)$$

Now, we apply the Fourier Transform to the differential equation replacing u with $E * f$. Doing so yields

$$\begin{aligned} \mathcal{F}\{P(\partial)(E * f)\} &= \sum_{n=0}^m a_n \mathcal{F}\{\partial^n (E * f)\} \\ &= \sum_{n=0}^m a_n (ik)^n \mathcal{F}\{E * f\}. \end{aligned}$$

The Convolution Theorem states that $\mathcal{F}\{E * f\} = \mathcal{F}\{E\} \mathcal{F}\{f\}$ which implies that

$$\begin{aligned} \mathcal{F}\{P(\partial)(E * f)\} &= \sum_{n=0}^m a_n (ik)^n \mathcal{F}\{E * f\} \\ &= \sum_{n=0}^m a_n (ik)^n \mathcal{F}\{E\} \mathcal{F}\{f\} \\ &= \mathcal{F}\{f\} \sum_{n=0}^m a_n (ik)^n \mathcal{F}\{E\} \\ &= \mathcal{F}\{f\} \mathcal{F}\{P(\partial)E\}. \end{aligned}$$

Thus, from (13) we see that

$$\mathcal{F}\{P(\partial)(E * f)\} = \mathcal{F}\{f\} \mathcal{F}\{P(\partial)E\} = \mathcal{F}\{f\}.$$

Therefore, applying the inverse Fourier Transform we have that

$$P(\partial)(E * f) = \mathcal{F}^{-1}\{\mathcal{F}\{P(\partial)(E * f)\}\} = \mathcal{F}^{-1}\{\mathcal{F}\{f\}\} = f$$

or that $E * f$ is a solution of the partial differential equation $P(\partial)u = f$, and we are done. \square