Homework Assignment 7

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Problem 9.1. For the following problems for 9.1, suppose a function $f : [a, b] \to \mathbb{R}$ is only known at distinct sites $x = [x_1, x_2, \ldots, x_n]$ where $x_i \in [a, b]$, for $i = 1, 2, \ldots n$. Let $p_n(f, t)$ be the Lagrange interpolating polynomial at these sites.

Problem 9.1.1. Show that the basic quadrature $J(f) := \int_a^b p_n(f, t)\,\mathrm{d}t$ satisfies $J(f) = \sum_{j=1}^n w_j f(x_j)$ where the weights w_j depend on the Lagrange basis.

Solution. Note the Lagrange interpolating polynomial of f through the nodes $x_1, x_2, \ldots, x_n$ is given by

$$p_n(f, t) = \sum_{j=1}^{n} f(x_j) \prod_{\substack{i=1 \\ i \neq j}} \frac{t - x_i}{x_j - x_i}.$$

If $J(f) := \int_a^b p_n(f, t)$, then, using this definition of the Lagrange interpolating polynomial, it is clear that

$$\begin{aligned} J(f) = \int_a^b p_n(f, t)\,\mathrm{d}t &= \int_a^b \left[\sum_{j=1}^{n} f(x_j) \prod_{\substack{i=1 \\ i \neq j}} \frac{t - x_i}{x_j - x_i} \right] \mathrm{d}t \\ &= \sum_{j=1}^{n} \left[\int_a^b \prod_{\substack{i=1 \\ i \neq j}} \frac{t - x_i}{x_j - x_i}\,\mathrm{d}t \right] f(x_j) = \sum_{j=1}^{n} w_j f(x_j). \end{aligned}$$

Thus, $J(f)$ is of the form $\sum_{j=1}^n w_j f(x_j)$ where w_j depends on the Lagrange basis $l_j(t) = \prod_{\substack{i=1 \\ i \neq j}} \frac{t - x_i}{x_j - x_i}$. □