Exam 1

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April 8, 2017

Problem 1. Find the Fourier Transforms of the following functions:

a. $f(x) = x^2 e^{-a|x|}$, $a > 0$,

b. $f(x) = \left(1 - \frac{|x|}{2}\right) H\left(1 - \frac{|x|}{2}\right)$.

Solution. Recall that if $f(x) \in L^1(\mathbb{R})$, then the Fourier Transform of f is defined to be

$$\mathscr{F}\{f(x)\} = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx. \tag{1}$$

a. If $f(x) = x^2 e^{-a|x|}$, $a > 0$, we see from (1) that, by definition, the Fourier Transform of f is given by

$$\begin{aligned}\mathscr{F}\{f(x)\} = F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-a|x|} e^{-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} x^2 e^{-(-a+ik)x} dx + \int_{0}^{\infty} x^2 e^{-(a+ik)x} dx\right].\end{aligned}$$

Now, we see by integration by parts that

$$\begin{aligned}\int_c^d x^2 e^{-(b+ik)x} dx &= -\frac{x^2}{b+ik} e^{-(b+ik)x}\Big|_c^d + \frac{2}{b+ik} \int_c^d x e^{-(b+ik)x} dx \\ &= -\frac{x^2}{b+ik} e^{-(b+ik)x}\Big|_c^d + \frac{2}{b+ik}\left[-\frac{x}{b+ik} e^{-(b+ik)x}\Big|_c^d + \frac{1}{b+ik}\int_c^d e^{-(b+ik)x} dx\right] \\ &= -\frac{x^2}{b+ik} e^{-(b+ik)x}\Big|_c^d + \frac{2}{b+ik}\left[-\frac{x}{b+ik} e^{-(b+ik)x}\Big|_c^d - \frac{1}{(b+ik)^2} e^{-(b+ik)x}\Big|_c^d\right].\end{aligned} \tag{2}$$

Thus,

$$\int_{-\infty}^{0} x^2 e^{-(-a+ik)x} dx = -\frac{2}{(-a+ik)^3}$$

and

$$\int_0^\infty x^2 e^{-(a+ik)x} dx = \frac{2}{(a+ik)^3}.$$

Therefore,

$$\begin{aligned}
\mathscr{F}\{f(x)\} = F(k) &= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^0 x^2 e^{-(-a+ik)x} dx + \int_0^\infty x^2 e^{-(a+ik)x} dx \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[-\frac{2}{(-a+ik)^3} + \frac{2}{(a+ik)^3} \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{4a(a^2-3k)}{(a^2+k^2)^3} \right] \\
&= \sqrt{\frac{2}{\pi}} \left[\frac{2a(a^2-3k)}{(a^2+k^2)^3} \right].
\end{aligned}$$

b. Recall that the Heaviside function H is defined as

$$H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Thus,

$$H\left(1 - \frac{|x|}{2}\right) = \begin{cases} 1 & \text{if } |x| < 2 \\ 0 & \text{if } |x| > 2. \end{cases}$$

If $f(x) = \left(1 - \frac{|x|}{2}\right) H\left(1 - \frac{|x|}{2}\right)$, we see from (1) that, by definition, the Fourier Transform of f is given by

$$\begin{aligned}
\mathscr{F}\{f(x)\} = F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \left(1 - \frac{|x|}{2}\right) H\left(1 - \frac{|x|}{2}\right) e^{-ikx} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-2}^2 \left(1 - \frac{|x|}{2}\right) e^{-ikx} dx \\
&= \frac{1}{\sqrt{2\pi}} \left[\int_{-2}^0 \left(1 + \frac{x}{2}\right) e^{-ikx} dx + \int_0^2 \left(1 - \frac{x}{2}\right) e^{-ikx} dx \right].
\end{aligned}$$

Now, we see by integration by parts that

$$\begin{aligned}
\int_c^d \left(1 \pm \frac{x}{2}\right) e^{-ikx} dx &= \frac{i}{k} e^{-ikx} \Big|_c^d \pm \frac{1}{2} \int_c^d x e^{-ikx} dx \\
&= \frac{i}{k} e^{-ikx} \Big|_c^d \pm \frac{1}{2} \left[\frac{ix}{k} e^{-ikx} \Big|_c^d - \frac{i}{k} \int_c^d e^{-ikx} dx \right] \\
&= \frac{i}{k} e^{-ikx} \Big|_c^d \pm \frac{1}{2} \left[\frac{ix}{k} e^{-ikx} \Big|_c^d + \frac{e^{-ikx}}{k^2} \Big|_c^d \right].
\end{aligned}$$

Thus,

$$\int_{-2}^{0}\left(1+\frac{x}{2}\right)e^{-ikx}dx = \frac{1-e^{2ik}+2ik}{2k^2}$$

and

$$\int_{0}^{2}\left(1-\frac{x}{2}\right)e^{-ikx}dx = \frac{1-e^{-2ik}-2ik}{2k^2}$$

Therefore, using the definition of the complex exponential and various trigonometric identities, we have that

$$\begin{aligned}
\mathscr{F}\left\{f(x)\right\} = F(k) &= \frac{1}{\sqrt{2\pi}}\left[\int_{-2}^{0}\left(1+\frac{x}{2}\right)e^{-ikx}dx + \int_{0}^{2}\left(1-\frac{x}{2}\right)e^{-ikx}dx\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{1-e^{2ik}+2ik}{2k^2} + \frac{1-e^{-2ik}-2ik}{2k^2}\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{k^2} - \frac{e^{-2ik}+e^{2ik}}{2k^2}\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{1-\cos 2k}{k^2}\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{1-(\cos^2 k - \sin^2 k)}{k^2}\right] \\
&= \frac{2\sin^2 k}{\sqrt{2\pi}k^2}.
\end{aligned}$$

$\square$

Problem 2. Find the Laplace Transforms of the following functions:

a. $f(t) = \displaystyle\int_0^t \frac{\sin ax}{x} dx$,

b. $f(t) = tH(t-a)$.

Solution. Recall that if $f(t) \in L^1(\mathbb{R})$, then the Laplace Transform of f is defined to be

$$\mathscr{L}\{f(t)\} = \bar{f}(s) = \int_0^\infty f(t)e^{-st}dt, \quad \text{Re } s > 0. \tag{3}$$

Note that it can be shown that the Laplace transform satisfies the important property

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\left[\mathscr{L}\{f(t)\}\right]. \tag{4}$$

a. Let $f(t) = \displaystyle\int_0^t \frac{\sin ax}{x} dx$. Then $f(0) = 0$ and $f'(t) = \dfrac{\sin at}{t}$ so that $tf'(t) = \sin at$. This implies that

$$\mathscr{L}\{tf'(t)\} = \mathscr{L}\{\sin at\} = \frac{a}{s^2 + a^2}.$$

The Laplace transform satisfies the following property that relates a function's derivative to its Laplace Transform:

$$\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - f(0).$$

This combined with (4) shows that

$$\mathscr{L}\{tf'(t)\} = -\frac{d}{ds}\left[\mathscr{L}\{f'(t)\}\right] = -\frac{d}{ds}\left[s\mathscr{L}\{f(t)\} - f(0)\right] = \frac{a}{s^2 + a^2},$$

or that

$$\frac{d}{ds}\left[s\mathscr{L}\{f(t)\}\right] = -\frac{a}{s^2 + a^2}.$$

Integrating both sides of the above equation yields that

$$s\mathscr{L}\{f(t)\} = -\int \frac{a}{s^2 + a^2} ds = -\tan^{-1}(s/a) + C.$$

In order to determine the constant of integration, we use the Initial Value Theorem which states that

$$\lim_{s\to\infty} s\mathscr{L}\{f(t)\} = \lim_{t\to 0} f(t) = f(0).$$

Since $f(0) = 0$, this implies that

$$\lim_{s\to\infty} s\mathscr{L}\{f(t)\} = \lim_{s\to\infty}\left[-\tan^{-1}(s/a) + C\right] = 0$$

or that $C = \frac{\pi}{2}$. Therefore, we have that

$$s\mathscr{L}\left\{f(t)\right\} = -\tan^{-1}(s/a) + \frac{\pi}{2} = \tan^{-1}(a/s)$$

so that

$$\mathscr{L}\left\{f(t)\right\} = \frac{\tan^{-1}(a/s)}{s}.$$

b. Let $f(t) = tH(t-a)$ and assume that $a > 0$. By property (4) we see that

$$\mathscr{L}\left\{tH(t-a)\right\} = -\frac{d}{ds}\left[\mathscr{L}\left\{H(t-a)\right\}\right].$$

From our table of Laplace Transforms, we have that

$$\mathscr{L}\left\{H(t-a)\right\} = \frac{e^{-as}}{s},$$

assuming that $a > 0$. Therefore,

$$\begin{aligned}\mathscr{L}\left\{f(t)\right\} = -\frac{d}{ds}\left[\mathscr{L}\left\{H(t-a)\right\}\right] &= -\frac{d}{ds}\left[s^{-1}e^{-as}\right] \\ &= -\left[-s^{-2}e^{-as} - as^{-1}e^{-as}\right] \\ &= \frac{(1+as)e^{-as}}{s^2}.\end{aligned}$$

□

Problem 3. Solve the following integral equations:

a. $\displaystyle\int_0^\infty f(x)\sin kx dx = \begin{cases} 1-k & k<1 \\ 0 & k>1 \end{cases},$

b. $\displaystyle\int_{-\infty}^\infty \frac{f(t)}{(x-t)^2+4}dt = \frac{1}{x^2+9}.$

Solution. a. Recall that the definition of the Fourier Sine Transform is given by

$$\mathscr{F}_s\left\{f(x)\right\} = F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin kx dx.$$

Thus, we see that

$$\int_0^\infty f(x)\sin kx dx = \sqrt{\frac{\pi}{2}}\mathscr{F}_s\left\{f(x)\right\} = \sqrt{\frac{\pi}{2}}F_s(k).$$

Let $G_s(k) = \begin{cases} 1-k & k<1 \\ 0 & k>1 \end{cases}$. Then the above integral equation becomes

$$F_s(k) = \sqrt{\frac{2}{\pi}}G_s(k).$$

Thus, applying the inverse Fourier Sine Transform, we have that

$$f(x) = \mathscr{F}_s^{-1}\left\{F_s(k)\right\} = \sqrt{\frac{2}{\pi}}\mathscr{F}_s^{-1}\left\{G_s(k)\right\}$$

where the inverse Fourier Sine Transform is defined as

$$g(x) = \mathscr{F}_s^{-1}\left\{G_s(k)\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty G_s(k)\sin kx dk. \tag{5}$$

Therefore, the solution to the integral equation is

$$\begin{aligned} f(x) = \sqrt{\frac{2}{\pi}}\mathscr{F}_s^{-1}\left\{G_s(k)\right\} &= \frac{2}{\pi}\int_0^\infty G_s(k)\sin kx dk \\ &= \frac{2}{\pi}\int_0^1 (1-k)\sin kx dk \\ &= \frac{2}{\pi}\left[\frac{1-\cos x}{x} - \frac{\sin x - x\cos x}{x^2}\right] \\ &= \frac{2}{\pi}\left[\frac{x-\sin x}{x^2}\right]. \end{aligned}$$

b. Recall that the convolution of two functions f and g is defined such that

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi.$$

Let $g(x) = \dfrac{1}{x^2+2^2}$. Then we see that

$$\int_{-\infty}^{\infty} \frac{f(t)}{(x-t)^2+4}dt = \int_{-\infty}^{\infty} f(t)g(x-t)dt = \sqrt{2\pi}(g * f)(x) = \sqrt{2\pi}(f * g)(x).$$

Now, let $h(x) = \dfrac{1}{x^2+3^2}$. Then in light of the above remarks, the integral equation becomes

$$\int_{-\infty}^{\infty} \frac{f(t)}{(x-t)^2+4}dt = \sqrt{2\pi}(f * g)(x) = h(x) = \frac{1}{x^2+9}.$$

Applying the Fourier transform to the integral equation, we see by the Convolution Theorem that

$$\mathscr{F}\left\{\sqrt{2\pi}(f * g)(x)\right\} = \sqrt{2\pi}F(k)G(k) = H(k) = \mathscr{F}\left\{h(x)\right\},$$

where $\mathscr{F}\{f(x)\} = F(k), \mathscr{F}\{g(x)\} = G(k)$, and $\mathscr{F}\{h(x)\} = H(k)$, respectively. From our table of Fourier transforms, we see that for $a > 0$ we have that

$$\mathscr{F}\left\{\frac{1}{x^2+a^2}\right\} = \sqrt{\frac{\pi}{2}}\frac{e^{-a|k|}}{a}. \tag{6}$$

Thus, from (6) we have that

$$F(k) = \frac{1}{\sqrt{2\pi}}\frac{H(k)}{G(k)} = \frac{2}{3\sqrt{2\pi}}\frac{e^{-3|k|}}{e^{-2|k|}} = \frac{2e^{-|k|}}{3\sqrt{2\pi}}.$$

Applying the inverse Fourier Transform to this equation yields that

$$f(x) = \mathscr{F}^{-1}\left\{F(k)\right\} = \frac{2}{3\sqrt{2\pi}}\mathscr{F}^{-1}\left\{e^{-|k|}\right\}.$$

But from (6), we know that

$$\mathscr{F}^{-1}\left\{e^{-|k|}\right\} = \sqrt{\frac{2}{\pi}}\frac{1}{x^2+1}.$$

Therefore, the solution to the integral equation is given by

$$\begin{aligned} f(x) = \frac{2}{3\sqrt{2\pi}}\mathscr{F}^{-1}\left\{e^{-|k|}\right\} &= \frac{2}{3\sqrt{2\pi}}\left[\sqrt{\frac{2}{\pi}}\frac{1}{x^2+1}\right] \\ &= \frac{2}{3\pi}\left[\frac{1}{x^2+1}\right]. \end{aligned}$$

$\square$

Problem 4. Show that

$$\int_0^\infty F_s(k)G_c(k)\sin kxdk = \frac{1}{2}\int_0^\infty g(\xi)\left[f(\xi+x)-f(\xi-x)\right]d\xi$$

where

$$F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin kxdx$$

and

$$G_c(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\cos kxdx.$$

Solution. Using the definition of $G_c(k)$, we see that

$$\int_0^\infty F_s(k)G_c(k)\sin kxdk = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(k)\sin kx\left[\int_0^\infty g(\xi)\cos k\xi d\xi\right]dk$$

Interchanging the order of integration from ξ to k shows that

$$\begin{aligned}\int_0^\infty F_s(k)G_c(k)\sin kxdk &= \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(k)\sin kx\left[\int_0^\infty g(\xi)\cos k\xi d\xi\right]dk\\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty g(\xi)\left[\int_0^\infty F_s(k)\cos k\xi\sin kxdk\right]d\xi\end{aligned}$$

Using the following trigonometric identity

$$\cos k\xi\sin kx = \frac{\sin k(\xi+x)-\sin k(\xi-x)}{2},$$

we then see that

$$\begin{aligned}\int_0^\infty F_s(k)G_c(k)\sin kxdk &= \sqrt{\frac{2}{\pi}}\int_0^\infty g(\xi)\left[\int_0^\infty F_s(k)\cos k\xi\sin kxdk\right]d\xi\\ &= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty g(\xi)\left[\int_0^\infty F_s(k)\sin k(\xi+x)dk-\int_0^\infty F_s(k)\sin k(\xi-x)dk\right]d\xi\\ &= \frac{1}{2}\int_0^\infty g(\xi)\left[f(\xi+x)-f(\xi-x)\right]d\xi,\end{aligned}$$

where the last line follows using (5), the definition of the inverse Fourier Sine Transform.

Therefore,

$$\int_0^\infty F_s(k)G_c(k)\sin kxdk = \frac{1}{2}\int_0^\infty g(\xi)\left[f(\xi+x)-f(\xi-x)\right]d\xi,$$

and we are done.

□

Problem 5. Apply the Fourier Transform to solve the following initial value problem for the heat equation:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad -\infty < x < \infty,$$
$$u(x,0) = \phi(x), \quad t > 0.$$

Solution. Consider the function $u(x,t)$. The Fourier transform of u with respect to x is defined as

$$\mathscr{F}\{u(x,t)\} = U(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} u(x,t) dx. \tag{7}$$

From this definition and the Leibniz integral rule, we can see by induction that

$$\begin{aligned}
\mathscr{F}\left\{\frac{\partial^n}{\partial t^n}[u(x,t)]\right\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^n}{\partial t^n}[u(x,t)]\, e^{-ikx} dx \\
&= \frac{d^n}{dt^n}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-ikx} dx\right] \\
&= \frac{d^n}{dt^n}\left[\mathscr{F}\{u(x,t)\}\right]. \qquad (8)
\end{aligned}$$

Similarly, we see from definition (7) and previous theorems regarding the Fourier transform that

$$\begin{aligned}
\mathscr{F}\left\{\frac{\partial^n}{\partial x^n}[u(x,t)]\right\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^n}{\partial x^n}[u(x,t)]\, e^{-ikx} dx \\
&= (ik)^n \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-ikx} dx\right] \\
&= (ik)^n \mathscr{F}\{u(x,y)\}. \qquad (9)
\end{aligned}$$

Now, applying the Fourier transform to the first equation yields that

$$\mathscr{F}\left\{\frac{\partial u}{\partial t}\right\} = \frac{d}{dt}[U(k,t)] = -(ak)^2 U(k,t) + F(k,t) = \mathscr{F}\left\{a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t)\right\}.$$

This results in a first-order non-homogeneous linear differential equation

$$\frac{d}{dt}[U(k,t)] + (ak)^2 U(k,t) = F(k,t).$$

Using well-established techniques, we see that the solution to the linear differential equation is

$$U(k,t) = c_1 e^{-(ak)^2 t} + e^{-(ak)^2 t} \int_0^t e^{(ak)^2 \xi} F(k,\xi) d\xi.$$

Applying the Fourier Transform to the initial value equation shows that

$$\mathscr{F}\{u(x,0)\} = U(k,0) = \Phi(k) = \mathscr{F}\{\phi(x)\}$$

Thus, from the transformed initial value equation, we see using the above solution that

$$U(k,0) = c_1 = \Phi(k)$$

Therefore, the solution to the transformed system of differential equations is

$$U(k,t) = \Phi(k)e^{-(ak)^2 t} + e^{-(ak)^2 t}\int_0^t e^{(ak)^2\xi}F(k,\xi)d\xi.$$

Applying the inverse Fourier transform to the solution of the transformed system of differential equations yields that the solution to the original system is

$$u(x,t) = \mathscr{F}^{-1}\{U(k,t)\} = \mathscr{F}^{-1}\left\{\Phi(k)e^{-(ak)^2 t}\right\} + \mathscr{F}^{-1}\left\{e^{-(ak)^2 t}\int_0^t e^{(ak)^2\xi}F(k,\xi)d\xi\right\}$$

Note from our table of Fourier transforms that for $b > 0$

$$\mathscr{F}\left\{e^{-bx^2}\right\} = \frac{1}{\sqrt{2b}}\exp\left(-\frac{k^2}{4b}\right)$$

Thus,

$$\mathscr{F}^{-1}\left\{e^{-a^2tk^2}\right\} = \mathscr{F}^{-1}\left\{\exp\left(-\frac{k^2}{4(1/4a^2t)}\right)\right\} = \sqrt{\frac{1}{2a^2t}}\exp\left(-\frac{x^2}{4a^2t}\right) = g(x,t)$$

Now, from the Convolution Theorem, we have that

$$\mathscr{F}^{-1}\left\{\Phi(k)e^{-(ak)^2 t}\right\} = (\phi * g)(x) = \int_{-\infty}^{\infty}\phi(x-\xi)g(\xi,t)d\xi.$$

Therefore, using this identity and the definition of the inverse Fourier Transform, the solution to the original system of differential equations is

$$\begin{aligned}u(x,t) &= \mathscr{F}^{-1}\left\{\Phi(k)e^{-(ak)^2 t}\right\} + \mathscr{F}^{-1}\left\{e^{-(ak)^2 t}\int_0^t e^{(ak)^2\xi}F(k,\xi)d\xi\right\}\\ &= \int_{-\infty}^{\infty}\phi(x-\xi)g(\xi,t)d\xi + \int_{-\infty}^{\infty}\left[\int_0^t e^{(ak)^2\xi}F(k,\xi)d\xi\right]e^{-(ak)^2 t}e^{ikx}dk.\end{aligned}$$

$\square$

Problem 6. Evaluate the following definite integrals:

a. $\displaystyle\int_0^\infty \frac{\sin ax \sin bx}{x^2} dx,$

b. $\displaystyle\int_0^\infty \frac{(a^2 - x^2)^2}{(x^2 + a^2)^4} dx, \quad a > 0.$

Solution. Suppose that $F_c(k) = \mathscr{F}_c\{f(x)\}$ and $G_c(k) = \mathscr{F}_c\{g(x)\}$. Then Parseval's relation derived from the Convolution Theorem for the Fourier Cosine Transform states that

$$\int_0^\infty F_c(k) G_c(k) dk = \int_0^\infty f(x) g(x) dx. \tag{10}$$

a. Let $F_c(k) = \dfrac{\sin ak}{k}$ and $G_c(k) = \dfrac{\sin bk}{k}$. Then Parseval's theorem shows that

$$\int_0^\infty \frac{\sin ak \sin bk}{k^2} dk = \int_0^\infty F_c(k) G_c(k) dk = \int_0^\infty f(x) g(x) dx$$

where $f(x) = \mathscr{F}_c^{-1}\{F_c(k)\}$ and $f(x) = \mathscr{F}_c^{-1}\{G_c(k)\}$. From our table of Fourier Cosine Transforms, we see that for $p \in \mathbb{R}$,

$$\mathscr{F}_c\{H(p - x)\} = \sqrt{\frac{2}{\pi}} \frac{\sin pk}{k}.$$

This implies that

$$\mathscr{F}_c^{-1}\left\{\frac{\sin pk}{k}\right\} = \sqrt{\frac{\pi}{2}} H(p - x).$$

Thus, we have that

$$\int_0^\infty \frac{\sin ak \sin bk}{k^2} dk = \int_0^\infty F_c(k) G_c(k) dk = \frac{\pi}{2} \int_0^\infty H(a - x) H(b - x) dx.$$

Now, we note from the definition of the Heaviside function that

$$H(a - x) H(b - x) = \begin{cases} 1 & \text{if } x < \min(a, b) \\ 0 & \text{if } x > \min(a, b) \end{cases}.$$

Therefore, we have that

$$\int_0^\infty \frac{\sin ak \sin bk}{k^2} dk = \frac{\pi}{2} \int_0^\infty H(a - x) H(b - x) dx = \frac{\pi}{2} \int_0^{\min(a,b)} dx = \frac{\pi}{2} \min(a, b).$$

b. Let $F_c(k) = \dfrac{a^2 - k^2}{(k^2 + a^2)^2}$. Then Parseval's theorem shows that

$$\int_0^\infty \frac{(a^2 - k^2)^2}{(k^2 + a^2)^4} dk = \int_0^\infty F_c(k) F_c(k) dk = \int_0^\infty f(x) f(x) dx$$

where $f(x) = \mathscr{F}_c^{-1}\left\{F_c(k)\right\}$. From our table of Fourier Cosine Transforms, we see that for $a > 0$,

$$\mathscr{F}_c\left\{xe^{-ax}\right\} = \sqrt{\frac{2}{\pi}}\frac{a^2-k^2}{(k^2+a^2)^2}.$$

This implies that

$$\mathscr{F}_c^{-1}\left\{\frac{a^2-k^2}{(k^2+a^2)^2}\right\} = \sqrt{\frac{\pi}{2}}.$$

Thus, we have that

$$\int_0^\infty \frac{(a^2-k^2)^2}{(k^2+a^2)^4}dk = \int_0^\infty F_c(k)F_c(k)dk = \frac{\pi}{2}\int_0^\infty x^2e^{-2ax}dx.$$

From (2), we see by setting $b = 2a$ and $k = 0$ that

$$\int_0^\infty x^2e^{-2ax}dx = \frac{1}{4a^3}.$$

Therefore, we have that

$$\int_0^\infty \frac{(a^2-k^2)^2}{(k^2+a^2)^4}dk = \frac{\pi}{2}\int_0^\infty x^2e^{-2ax}dx = \frac{\pi}{(2a)^3}.$$

$\square$

Problem 7. Use the Fourier Sine Transform to solve the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \infty$$

with the boundary data $0 < y < L$

$$u(x, L) = 0, \quad u(x, 0) = f(x),$$
$$u(0, y) = 0, \quad u(x, y) \to 0 \text{ as } x \to \infty \text{ uniformly in } y.$$

Solution. Consider the function $u(x, y)$. The Fourier Sine Transform of u with respect to x is defined as

$$\mathscr{F}_s \{u(x, y)\} = U_s(k, y) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x, y) \sin(kx) dx.$$

From this definition we see using the Leibniz integral rule that

$$\begin{aligned} \mathscr{F}_s \left\{ \frac{\partial^n u(x, y)}{\partial y^n} \right\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^n u(x, y)}{\partial y^n} \sin(kx) dx \\ &= \frac{d^n}{dy^n} \left[\sqrt{\frac{2}{\pi}} \int_0^\infty u(x, y) \sin(kx) dx \right] \\ &= \frac{d^n}{dy^n} \left[\mathscr{F}_s \{u(x, y)\} \right]. \end{aligned}$$

The transforms of the partials of u with respect to x are not as easy to characterize. Nevertheless, we see from the properties of the Fourier Sine Transform that

$$\mathscr{F}_s \left\{ \frac{\partial u(x, y)}{\partial x} \right\} = -k \mathscr{F}_c \{u(x, y)\}$$

and

$$\mathscr{F}_s \left\{ \frac{\partial^2 u(x, y)}{\partial x^2} \right\} = -k^2 \mathscr{F}_s \{u(x, y)\} + k \sqrt{\frac{2}{\pi}} u(0, y).$$

Let $U_s(k, y) = \mathscr{F}_s \{u(x, y)\}$. Then, applying the Fourier Sine Transform to the first differential equation shows that

$$\mathscr{F}_s \left\{ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right\} = \frac{d^2}{dy^2} [U_s(k, y)] - k^2 U_s(k, y) + k \sqrt{\frac{2}{\pi}} u(0, y) = 0 = \mathscr{F}_s \{0\}.$$

From the boundary equation $u(0, y) = 0$ we see that the above equation becomes

$$\frac{d^2}{dy^2} [U_s(k, y)] - k^2 U_s(k, y) = 0.$$

This is a linear, second-order homogeneous differential equation, the solution of which we readily see is

$$U_s(k,y) = c_1 e^{-ky} + c_2 e^{ky}. \tag{11}$$

Applying the Fourier Sine Transform to the boundary equations, we see that

$$U_s(k,L) = 0, \quad U_s(k,0) = F_s(k), \quad \text{for } 0 < k < \infty,\, 0 < y < L.$$

Using these equations and (11), the solution to the homogeneous equation, we see that

$$\begin{aligned} U_s(k,0) &= c_1 + c_2 = F_s(k) \\ U_s(k,L) &= c_1 e^{-kL} + c_2 e^{kL} = 0. \end{aligned}$$

Solving, we see that

$$\begin{aligned} c_1 &= -\frac{e^{2kL} F_s(k)}{1 - e^{2kL}} \\ c_2 &= \frac{F_s(k)}{1 - e^{2kL}}. \end{aligned}$$

Thus, the solution to the transformed system of differential equations is

$$\begin{aligned} U_s(k,y) &= -\frac{e^{2kL} F_s(k) e^{-ky}}{1 - e^{2kL}} + \frac{F_s(k) e^{ky}}{1 - e^{2kL}} \\ &= F_s(k) \left(\frac{e^{-kL}}{e^{-kL}}\right) \left(\frac{-e^{ky} + e^{2kL-ky}}{-1 + e^{2kL}}\right) \\ &= F_s(k) \frac{\sinh k(L-y)}{\sinh kL}. \end{aligned}$$

Applying the inverse Fourier Sine Transform gives that the solution to the original system of differential equations is

$$\begin{aligned} u(x,y) &= \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(k) \frac{\sinh k(L-y)}{\sinh kL} \sin kx dk \\ &= \frac{2}{\pi} \int_0^\infty \left[\int_0^\infty f(\xi) \sin k\xi d\xi\right] \frac{\sinh k(L-y)}{\sinh kL} \sin kx dk. \end{aligned}$$

It is easy to see from the definition of the hyperbolic sine function that $\frac{\sinh k(L-y)}{\sinh kL} \sim e^{-ky}$ as $kL \to \infty$. Thus, the above problem reduces to a simpler problem in the quarter plane instead of the semi-infinite strip. Therefore, the solution to the original differential equation is

$$\begin{aligned} u(x,y) &= \frac{2}{\pi} \int_0^\infty f(\xi) d\xi \int_0^\infty \sin k\xi \sin kx e^{-ky} dk \\ &= \frac{1}{\pi} \int_0^\infty f(\xi) d\xi \int_0^\infty \left[\cos k(x-\xi) - \cos k(x+\xi)\right] e^{-ky} dk \\ &= \frac{1}{\pi} \int_0^\infty f(\xi) \left[\frac{y}{(x-\xi)^2 + y^2} - \frac{y}{(x+\xi)^2 + y^2}\right] d\xi. \end{aligned}$$

□

Problem 8. Apply the Fourier Transform to solve the 3-dimensional wave problem

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right), \quad -\infty < x, y, z < \infty,$$

subject to the initial conditions

$$\begin{aligned} u(x,y,z,t)|_{t=0} &= 0 \\ \left.\frac{\partial u(x,y,z,t)}{\partial t}\right|_{t=0} &= \delta(x,y,z). \end{aligned}$$

Solution. Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and suppose that $u(\boldsymbol{x}, t)$ is given. The Fourier transform of $u(\boldsymbol{x}, t)$ with respect to $\boldsymbol{x}$ is defined to be

$$\mathscr{F}\{u(\boldsymbol{x},t)\} = U(\boldsymbol{k},t) = \frac{1}{(2\pi)^{n/2}}\int_{-\infty}^{\infty} u(\boldsymbol{x},t)e^{-i\boldsymbol{x}\cdot\boldsymbol{k}}d\boldsymbol{x} \tag{12}$$

where $\boldsymbol{k} \in \mathbb{R}^n$.

In order to investigate the Fourier transform of partials of $u(\boldsymbol{x}, t)$ with respect to a given component of $\boldsymbol{x}$, define the Fourier transform of $u(\boldsymbol{x}, t)$ with respect to x_j as the following

$$\mathscr{F}_{[x_j]}\{u(\boldsymbol{x},t)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(\boldsymbol{x},t)e^{-ix_jk_j}dx_j.$$

Further, we will also use the function $\pi_j : \mathbb{R}^n \to \mathbb{R}^{n-1}$ defined as

$$\pi_j(\boldsymbol{x}) := (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$$

to aid in our description of the Fourier transform of partials of $u(\boldsymbol{x}, t)$. Now from definition (12) and Leibniz's integral rule we see that

$$\begin{aligned} \mathscr{F}\left\{\frac{\partial^n u(\boldsymbol{x},t)}{\partial t^n}\right\} &= \frac{1}{(2\pi)^{n/2}}\int_{-\infty}^{\infty}\frac{\partial^n}{\partial t^n}\left[u(\boldsymbol{x},t)\right]e^{-i\boldsymbol{x}\cdot\boldsymbol{k}}d\boldsymbol{x} \\ &= \frac{d^n}{dt^n}\left[\frac{1}{(2\pi)^{n/2}}\int_{-\infty}^{\infty}u(\boldsymbol{x},t)e^{-i\boldsymbol{x}\cdot\boldsymbol{k}}d\boldsymbol{x}\right] \\ &= \frac{d^n}{dt^n}\left[\mathscr{F}\{u(\boldsymbol{x},t)\}\right]. \end{aligned}$$

Similarly, from definition (12) and previous results about the Fourier transform, we see that

$$\begin{aligned} \mathscr{F}\left\{\frac{\partial^n u(\boldsymbol{x},t)}{\partial x_j^n}\right\} &= \frac{1}{(2\pi)^{n/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{\partial^n}{\partial x_j^n}\left[u(\boldsymbol{x},t)\right]e^{-ix_1k_1}\cdots e^{-ix_nk_n}dx_1\ldots dx_n \\ &= \frac{1}{(2\pi)^{(n-1)/2}}\int_{-\infty}^{\infty}\mathscr{F}_{[x_j]}\left\{\frac{\partial^n}{\partial x_j^n}\left[u(\boldsymbol{x},t)\right]\right\}e^{-i\pi_j(\boldsymbol{x})\cdot\pi_j(\boldsymbol{k})}d\pi_j(\boldsymbol{x}) \\ &= \frac{(ik_j)^n}{(2\pi)^{(n-1)/2}}\int_{-\infty}^{\infty}\mathscr{F}_{[x_j]}\{u(\boldsymbol{x},t)\}e^{-i\pi_j(\boldsymbol{x})\cdot\pi_j(\boldsymbol{k})}d\pi_j(\boldsymbol{x}) \\ &= (ik_j)^n\mathscr{F}\{u(\boldsymbol{x},t)\}. \end{aligned}$$

Now, define $\boldsymbol{x} = (x_1, x_2, x_3) = (x, y, z) \in \mathbb{R}^3$. The the system of differential equations of the function $u(\boldsymbol{x}, t) = u(x_1, x_2, x_3, t)$ becomes

$$a^2 \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} \right) - \frac{\partial^2 u}{\partial t^2} = 0, \quad -\infty < x_1, x_2, x_3 < \infty,$$

subject to the initial conditions

$$u(\boldsymbol{x}, t)|_{t=0} = 0, \qquad \left. \frac{\partial u(\boldsymbol{x}, t)}{\partial t} \right|_{t=0} = \delta(\boldsymbol{x}).$$

Applying the Fourier transform with respect to $\boldsymbol{x}$ to the first equation yields

$$\mathscr{F} \left\{ a^2 \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} \right) - \frac{\partial^2 u}{\partial t^2} \right\} = -a^2 \left\| \boldsymbol{x} \right\|^2 U(\boldsymbol{k}, t) - \frac{d^2}{dt^2} \left[U(\boldsymbol{k}, t) \right] = 0 = \mathscr{F} \left\{ 0 \right\}$$

where $U(\boldsymbol{k}, t) = \mathscr{F} \left\{ u(\boldsymbol{x}, t) \right\}$. Similarly, we deduce that the transformed initial conditions become

$$\begin{aligned} &\mathscr{F} \left\{ u(\boldsymbol{x}, t)|_{t=0} \right\} = U(\boldsymbol{k}, t)|_{t=0} = 0 = \mathscr{F} \left\{ 0 \right\}, \\ &\mathscr{F} \left\{ \left. \frac{\partial u(\boldsymbol{x}, t)}{\partial t} \right|_{t=0} \right\} = \left. \frac{d}{dt} \left[U(\boldsymbol{k}, t) \right] \right|_{t=0} = \frac{1}{(2\pi)^{3/2}} = \mathscr{F} \left\{ \delta(\boldsymbol{x}) \right\}. \end{aligned}$$

We see that the first transformed equation is a second-order linear homogeneous ordinary differential equation, from which we readily see that the solution is

$$U(\boldsymbol{k}, t) = c_1 \cos(a \left\| \boldsymbol{k} \right\| t) + c_2 \sin(a \left\| \boldsymbol{k} \right\| t).$$

Using this solution we see from the first transformed initial condition that

$$U(\boldsymbol{k}, t)|_{t=0} = \left. c_1 \cos(a \left\| \boldsymbol{k} \right\| t) + c_2 \sin(a \left\| \boldsymbol{k} \right\| t) \right|_{t=0} = c_1 = 0.$$

From the second transformed initial condition, we see using the above solution that

$$\begin{aligned} \left. \frac{d}{dt} \left[U(\boldsymbol{k}, t) \right] \right|_{t=0} &= \left. -a \left\| \boldsymbol{k} \right\| c_1 \sin(a \left\| \boldsymbol{k} \right\| t) + a \left\| \boldsymbol{k} \right\| c_2 \cos(a \left\| \boldsymbol{k} \right\| t) \right|_{t=0} \\ &= a \left\| \boldsymbol{k} \right\| c_2 \\ &= \frac{1}{(2\pi)^{3/2}}, \end{aligned}$$

or that $c_2 = \dfrac{1}{a \left\| \boldsymbol{k} \right\| (2\pi)^{3/2}}$. Therefore, the solution to the transformed system of differential equations is

$$U(\boldsymbol{k}, t) = \frac{\sin(a \left\| \boldsymbol{k} \right\| t)}{a \left\| \boldsymbol{k} \right\| (2\pi)^{3/2}}.$$

Applying the inverse Fourier Transform to the above solution gives the solution to the original system

$$
\begin{aligned}
u(\boldsymbol{x}, t) &= \mathscr{F}^{-1}\left\{\frac{\sin(a\left\|\boldsymbol{k}\right\| t)}{a\left\|\boldsymbol{k}\right\|(2\pi)^{3/2}}\right\} \\
&= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{\sin(a\left\|\boldsymbol{k}\right\| t)}{a\left\|\boldsymbol{k}\right\|}e^{i\boldsymbol{k}\cdot\boldsymbol{x}}d\boldsymbol{k} \\
&= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sin(a\left\|\boldsymbol{k}\right\| t)}{a\left\|\boldsymbol{k}\right\|}e^{ik_1x_1}e^{ik_2x_2}e^{ik_3x_3}dk_1dk_2dk_3.
\end{aligned}
$$

$\square$

Problem 9. Show that if E is a solution of an m-th order partial differential equation

$$P(\partial)u = \sum_{n=0}^{m} a_n \partial^n u = \sqrt{2\pi}\delta,$$

where δ is the Dirac delta function, then $E * f$ is the solution of the partial differential equation $P(\partial)u = f$, where $*$ is the convolution.

Solution. Suppose that E is a solution of the partial differential equation

$$P(\partial)u = \sum_{n=0}^{m} a_n \partial^n u = \sqrt{2\pi}\delta,$$

i.e. $P(\partial)E = \sqrt{2\pi}\delta$. Then applying the Fourier Transform shows that

$$\mathscr{F}\left\{P(\partial)E\right\} = \sum_{n=0}^{m} a_n \mathscr{F}\left\{\partial^n E\right\} = \sum_{n=0}^{m} a_n (ik)^n \mathscr{F}\left\{E\right\} = 1 = \mathscr{F}\left\{\sqrt{2\pi}\delta\right\}. \tag{13}$$

Now, we apply the Fourier Transform to the differential equation replacing u with $E * f$. Doing so yields

$$\begin{aligned}\mathscr{F}\left\{P(\partial)(E * f)\right\} &= \sum_{n=0}^{m} a_n \mathscr{F}\left\{\partial^n (E * f)\right\} \\ &= \sum_{n=0}^{m} a_n (ik)^n \mathscr{F}\left\{E * f\right\}.\end{aligned}$$

The Convolution Theorem states that $\mathscr{F}\left\{E * f\right\} = \mathscr{F}\left\{E\right\}\mathscr{F}\left\{f\right\}$ which implies that

$$\begin{aligned}\mathscr{F}\left\{P(\partial)(E * f)\right\} &= \sum_{n=0}^{m} a_n (ik)^n \mathscr{F}\left\{E * f\right\} \\ &= \sum_{n=0}^{m} a_n (ik)^n \mathscr{F}\left\{E\right\}\mathscr{F}\left\{f\right\} \\ &= \mathscr{F}\left\{f\right\}\sum_{n=0}^{m} a_n (ik)^n \mathscr{F}\left\{E\right\} \\ &= \mathscr{F}\left\{f\right\}\mathscr{F}\left\{P(\partial)E\right\}.\end{aligned}$$

Thus, from (13) we see that

$$\mathscr{F}\left\{P(\partial)(E * f)\right\} = \mathscr{F}\left\{f\right\}\mathscr{F}\left\{P(\partial)E\right\} = \mathscr{F}\left\{f\right\}.$$

Therefore, applying the inverse Fourier Transform we have that

$$P(\partial)(E * f) = \mathscr{F}^{-1}\left\{\mathscr{F}\left\{P(\partial)(E * f)\right\}\right\} = \mathscr{F}^{-1}\left\{\mathscr{F}\left\{f\right\}\right\} = f$$

or that $E * f$ is a solution of the partial differential equation $P(\partial)u = f$, and we are done. □