# Homework Assignment 4

Matthew Tiger

March 19, 2017

**Problem 3.1.** Find the Laplace transforms of the following functions:

b. $f(t) = (1-2t)e^{-2t}$

c. $f(t) = t\cos at$

d. $f(t) = t^{3/2}$

g. $f(t) = (t-3)^2 H(t-3)$

*Solution.* Recall that the Laplace transform of the function $f(t)$ defined for $t > 0$ is given by

$$\mathscr{L}\{f(t)\} = \bar{f}(s) = \int_0^\infty f(t)e^{-st}dt. \tag{1}$$

b. Let $g(t) = 1-2t$. Then $f(t) = (1-2t)e^{-2t} = g(t)e^{-2t}$. From the definition of the Laplace transform, we have that

$$\begin{aligned}\mathscr{L}\{g(t)\} = \bar{g}(s) &= \int_0^\infty (1-2t)e^{-st}dt \\ &= \int_0^\infty t^0 e^{-st}dt - 2\int_0^\infty t^1 e^{-st}dt \\ &= \mathscr{L}\left\{t^0\right\} - 2\mathscr{L}\left\{t^1\right\}.\end{aligned}$$

From a previous theorem, we know for $n \in \mathbb{N}$ that

$$\mathscr{L}\{t^n\} = \int_0^\infty t^n e^{-st}dt = \frac{n!}{s^{n+1}}.$$

Thus,

$$\bar{g}(s) = \mathscr{L}\left\{t^0\right\} - 2\mathscr{L}\left\{t^1\right\} = \frac{1}{s} - \frac{2}{s^2} = \frac{s-2}{s^2}.$$

From Heaviside's First Shifting Theorem, we know that for $\bar{g}(s) = \mathscr{L}\{g(t)\}$ that

$$\mathscr{L}\left\{g(t)e^{-at}\right\} = \bar{g}(s+a).$$

Therefore, the Laplace transform of $f(t) = (1-2t)e^{-2t} = g(t)e^{-2t}$ is

$$\mathscr{L}\{f(t)\} = \mathscr{L}\left\{g(t)e^{-2t}\right\} = \bar{g}(s+2) = \frac{s}{(s+2)^2}.$$

c. From the definition of the complex exponential, we have that $f(t) = t\cos at = \frac{t}{2}\left(e^{-iat} + e^{iat}\right)$. From the definition of the Laplace transform, we have that

$$\begin{aligned}
\mathscr{L}\left\{f(t)\right\} = \bar{f}(s) &= \int_0^\infty \frac{t}{2}\left(e^{-iat} + e^{iat}\right)e^{-st}dt \\
&= \frac{1}{2}\left[\int_0^\infty te^{-(s+ia)t}dt + \int_0^\infty te^{-(s-ia)t}dt\right].
\end{aligned}$$

We readily see by integrating by parts using $u = t$ and $dv = e^{-(s\pm ia)t}dt$ that

$$\begin{aligned}
\int_0^\infty te^{-(s\pm ia)t}dt &= -\frac{t}{s\pm ia}e^{-(s\pm ia)t}\Big|_0^\infty + \frac{1}{s\pm ia}\int_0^\infty e^{-(s\pm ia)t}dt \\
&= -\frac{1}{(s\pm ia)^2}e^{-(s\pm ia)t}\Big|_0^\infty \\
&= \frac{1}{(s\pm ia)^2}.
\end{aligned}$$

Therefore, the Laplace transform of $f(t)$ is given by

$$\begin{aligned}
\mathscr{L}\left\{f(t)\right\} = \bar{f}(s) &= \frac{1}{2}\left[\int_0^\infty te^{-(s+ia)t}dt + \int_0^\infty te^{-(s-ia)t}dt\right] \\
&= \frac{1}{2}\left[\frac{1}{(s+ia)^2} + \frac{1}{(s-ia)^2}\right] \\
&= \frac{s^2 - a^2}{(s+ia)^2(s-ia)^2} \\
&= \frac{s^2 - a^2}{(s^2+a^2)^2}.
\end{aligned}$$

d. By definition, the Laplace transform of $f(t)$ is given by

$$\mathscr{L}\left\{f(t)\right\} = \bar{f}(s) = \int_0^\infty t^{3/2}e^{-st}dt.$$

Let $u = st$, then $du/s = dt$ and

$$\begin{aligned}
\mathscr{L}\left\{f(t)\right\} = \bar{f}(s) &= \frac{1}{s}\int_0^\infty \left(\frac{u}{s}\right)^{3/2} e^{-u}dt \\
&= \frac{1}{s^{5/2}}\int_0^\infty u^{3/2}e^{-u}dt.
\end{aligned}$$

Recall that the definition of the Gamma function is given by

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}du.$$

Therefore, the Laplace transform of $f(t) = t^{3/2}$ is

$$\begin{aligned}\mathscr{L}\{f(t)\} = \bar{f}(s) &= \frac{1}{s^{5/2}}\int_0^\infty u^{5/2-1}e^{-u}dt \\ &= \frac{\Gamma\left(\frac{5}{2}\right)}{s^{5/2}}.\end{aligned}$$

g. Let $g(t) = t^2$ and suppose that $\mathscr{L}\{g(t)\} = \bar{g}(s)$. Then Heaviside's Second Shifting Theorem shows that

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{g(t-3)H(t-3)\} = e^{-3s}\bar{g}(s).$$

As shown previously, we know for $n \in \mathbb{N}$ that

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}.$$

Therefore, the Laplace transform of $f(t)$ is

$$\mathscr{L}\{f(t)\} = \bar{f}(s) = e^{-3s}\bar{g}(s) = \frac{2e^{-3s}}{s^3}.$$

$\square$

**Problem 3.3.** The following is a result relating the Laplace transform of a function's derivative to the Laplace transform of that function:

$$\mathscr{L}\left\{f'(t)\right\} = s\mathscr{L}\left\{f(t)\right\} - f(0). \tag{2}$$

Use the result to find

a. $\mathscr{L}\left\{\cos at\right\}$,

b. $\mathscr{L}\left\{\sin at\right\}$.

*Solution.* a. Let $f(t) = \cos at$. Then $f'(t) = -a\sin at$ and from (2) we have

$$-a\mathscr{L}\left\{\sin at\right\} = s\mathscr{L}\left\{\cos at\right\} - 1. \tag{3}$$

Now let $g(t) = \sin at$. Then $g'(t) = a\cos at$ and applying (2) to $g(t)$ yields

$$a\mathscr{L}\left\{\cos at\right\} = s\mathscr{L}\left\{\sin at\right\}.$$

Therefore, from (3) we have that

$$-a\left(\frac{a}{s}\mathscr{L}\left\{\cos at\right\}\right) = s\mathscr{L}\left\{\cos at\right\} - 1$$

which implies that

$$\mathscr{L}\left\{\cos at\right\} = \frac{s}{s^2 + a^2}.$$

b. Let $f(t) = \sin at$. Then $f'(t) = a\cos at$ and from (2) we have

$$a\mathscr{L}\left\{\cos at\right\} = s\mathscr{L}\left\{\sin at\right\}. \tag{4}$$

Now let $g(t) = \cos at$. Then $g'(t) = -a\sin at$ and applying (2) to $g(t)$ yields

$$-a\mathscr{L}\left\{\sin at\right\} = s\mathscr{L}\left\{\cos at\right\} - 1$$

which implies that

$$\mathscr{L}\left\{\cos at\right\} = \frac{1}{s} - \frac{a}{s}\mathscr{L}\left\{\sin at\right\}.$$

Therefore, from (4) we have that

$$a\left(\frac{1}{s} - \frac{a}{s}\mathscr{L}\left\{\sin at\right\}\right) = s\mathscr{L}\left\{\sin at\right\}$$

which implies that

$$\mathscr{L}\left\{\sin at\right\} = \frac{a}{s^2 + a^2}.$$

$\square$

**Problem 3.6.** Show that
$$\mathscr{L}\left\{\int_0^t \frac{f(u)}{u}du\right\} = \frac{1}{s}\int_s^\infty \bar{f}(x)dx.$$
*Solution.* From the definition of the Laplace transform we see that
$$\mathscr{L}\left\{\int_0^t \frac{f(u)}{u}du\right\} = \int_0^\infty e^{-st}\left[\int_0^t \frac{f(u)}{u}du\right]dt.$$
Interchanging the order of integration from $u$ to $t$ where $0 \leq t < \infty$, we see that $u \leq t < \infty$ as $0 \leq u < \infty$ and
$$\begin{aligned}\mathscr{L}\left\{\int_0^t \frac{f(u)}{u}du\right\} &= \int_0^\infty e^{-st}\left[\int_0^t \frac{f(u)}{u}du\right]dt \\ &= \int_0^\infty \frac{f(u)}{u}\left[\int_u^\infty e^{-st}dt\right]du \\ &= \frac{1}{s}\int_0^\infty \frac{f(u)}{u}e^{-su}du.\end{aligned}$$
We note that $\frac{d}{ds}\left[\frac{e^{-su}}{u}\right] = -e^{-su}$ so that in particular we have that
$$-\int_s^\infty e^{-su}ds = -\left.\frac{e^{-su}}{u}\right|_s^\infty = -\frac{e^{-su}}{u}$$
or that
$$\int_s^\infty e^{-su}ds = \frac{e^{-su}}{u}.$$
Thus,
$$\begin{aligned}\mathscr{L}\left\{\int_0^t \frac{f(u)}{u}du\right\} &= \frac{1}{s}\int_0^\infty \frac{f(u)}{u}e^{-su}du \\ &= \frac{1}{s}\int_0^\infty f(u)\left[\int_s^\infty e^{-su}ds\right]du.\end{aligned}$$
Interchanging the order of integration yet again from $s$ to $u$ where $s \leq s < \infty$ as $0 \leq u < \infty$, we see that the integration limits remain unchanged and therefore that
$$\begin{aligned}\mathscr{L}\left\{\int_0^t \frac{f(u)}{u}du\right\} &= \frac{1}{s}\int_0^\infty f(u)\left[\int_s^\infty e^{-su}ds\right]du \\ &= \frac{1}{s}\int_s^\infty\left[\int_0^\infty f(u)e^{-su}du\right]ds \\ &= \frac{1}{s}\int_s^\infty \bar{f}(s)ds \\ &= \frac{1}{s}\int_s^\infty \bar{f}(x)dx,\end{aligned}$$
and we are done.

□

**Problem 3.7.**

*Solution.* □

**Problem 3.8.**

*Solution.* □

**Problem 3.10.**

*Solution.* □

**Problem 3.12.**

*Solution.* □

**Problem 3.15.**

*Solution.* □

**Problem 3.18.**

*Solution.* □