

# Homework Assignment 2

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**Problem 1.4.1.** Find the fixed points and determine their stability for the function

$$f(x) = \frac{6}{x} - 1.$$

*Solution.* The fixed points of the function  $f(x)$  are the roots of the function

$$\begin{aligned} g(x) &= f(x) - x \\ &= \frac{6}{x} - 1 - x \\ &= -\frac{(x+3)(x+2)}{6}. \end{aligned}$$

We readily see that the roots of  $g(x)$ , which are the fixed points of  $f(x)$ , are given by  $x = -3$  and  $x = 2$ .

According to Theorem 1.4.4, since  $f(x)$  is a  $C^1$  function, we may use the derivative of  $f(x)$  to classify its fixed points. If  $c$  is a fixed point of  $f$  and  $|f'(c)| < 1$ , then  $c$  is an asymptotically stable fixed point, while  $|f'(c)| > 1$  indicates that  $c$  is a repelling (unstable) fixed point.

Note that  $f'(x) = -6/x^2$ . For the fixed point  $x = -3$ , we see that

$$|f'(-3)| = \left| -\frac{6}{(-3)^2} \right| = \frac{2}{3} < 1$$

from which we classify the point  $x = -3$  as an asymptotically stable fixed point. On the other hand, for the fixed point  $x = 2$ , we see that

$$|f'(2)| = \left| -\frac{6}{(2)^2} \right| = \frac{3}{2} > 1$$

from which we classify the point  $x = 2$  as a repelling (unstable) fixed point.  $\square$

**Problem 1.4.2.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ . If  $f'(x)$  exists with  $f'(x) \neq 1$  for all  $x \in \mathbb{R}$ , prove that  $f$  has at most one fixed point. (Hint: Use the Mean Value Theorem).

*Solution.* Suppose to the contrary that for all  $x \in \mathbb{R}$  we have that  $f'(x)$  exists with  $f'(x) \neq 1$ , but  $f$  has at least two distinct fixed points,  $c_1$  and  $c_2$ , say. The Mean Value Theorem states that if a function  $g$  is continuous on an interval  $[a, b]$  and differentiable on the interval  $(a, b)$ , then there exists a point  $c \in (a, b)$  such that

$$g'(c) = \frac{g(b) - g(a)}{b - a}.$$

By our supposition, we have that the function  $f$  is continuous and differentiable on any interval and, in particular, it is continuous on  $[c_1, c_2]$  and differentiable on  $(c_1, c_2)$ . By the Mean Value Theorem, there exists a point  $c_3 \in (c_1, c_2)$  such that

$$f'(c_3) = \frac{f(c_2) - f(c_1)}{c_2 - c_1}. \quad (1)$$

However, since  $c_1$  and  $c_2$  are fixed points of  $f$ , we know that  $f(c_2) - f(c_1) = c_2 - c_1$  and we gather from (1) that

$$f'(c_3) = \frac{f(c_2) - f(c_1)}{c_2 - c_1} = \frac{c_2 - c_1}{c_2 - c_1} = 1.$$

Note that this is in contradiction to our supposition that  $f'(x) \neq 1$  for any  $x \in \mathbb{R}$ . Therefore, we must conclude that for a function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , if for all  $x \in \mathbb{R}$  we have that  $f'(x)$  exists with  $f'(x) \neq 1$ , then  $f$  has at most one fixed point.  $\square$

**Problem 1.4.4.** Let  $S_\mu(x) = \mu \sin(x)$ ,  $0 \leq x \leq 2\pi$ ,  $0 < \mu \leq \pi$  and  $C_\mu(x) = \mu \cos(x)$ ,  $-\pi \leq x \leq \pi$  and  $-\pi \leq \mu \leq \pi$ ,  $\mu \neq 0$ .

- i. Show that  $S_\mu$  has a super-attracting fixed point at  $x = \pi/2$ , when  $\mu = \pi/2$ .
- ii. Find the corresponding values for  $C_\mu$  having a super-attracting fixed point.

*Solution.* Recall that if  $c$  is a fixed point of a differentiable function  $f$ , then  $c$  is a super-attracting fixed point if  $f'(c) = 0$ .

- i. Suppose that  $\mu = \pi/2$ . Since  $S_\mu(\pi/2) = (\pi/2) \sin(\pi/2) = \pi/2$ , we readily see that if  $\mu = \pi/2$ , then  $x = \pi/2$  is a fixed point of  $S_\mu(x)$ . Note that  $S'_\mu(x) = \mu \cos(x)$ . From this we gather that if  $\mu = \pi/2$ , then for the fixed point  $x = \pi/2$ , we have that  $S'_\mu(x) = (\pi/2) \cos(\pi/2) = 0$ . Therefore, the fixed point  $x = \pi/2$  is a super-attracting fixed point.
- ii. We now investigate the super-attracting fixed points of  $C_\mu(x)$ . The definition of  $C_\mu(x)$  shows that  $C'_\mu(x) = -\mu \sin(x)$  from which we can gather that  $C'_\mu(x) = 0$  for  $x \in [-\pi, \pi]$  if  $x = k\pi$  for  $k \in \{-1, 0, 1\}$ . Note that these are the possible super-attracting fixed points of  $C_\mu(x)$ , we must still determine which of these possible super-attracting fixed points are indeed fixed points, i.e. we must determine which points satisfy  $C_\mu(x) = x$ . If  $x = k\pi$  for  $k \in \{-1, 0, 1\}$ , then

$$C_\mu(k\pi) = \mu \cos(k\pi) = (-1)^k \mu.$$

Thus,  $C_\mu(k\pi) = (-1)^k \mu = k\pi$ , if  $\mu = (-1)^k k\pi$ . Therefore, if  $x, \mu \in [-\pi, \pi]$  with  $\mu \neq 0$ , then the points  $x_1 = -\pi$  and  $x_2 = \pi$ , with corresponding  $\mu$ -values  $\mu_1 = \pi$  and  $\mu_2 = -\pi$ , are super-attracting fixed points. Note that  $x = 0$  is not a super-attracting fixed point since it is not a fixed point, that is  $C_\mu(0) = 0$  only if  $\mu = 0$ , which violates our initial conditions.

□

**Problem 1.4.7.** Let  $N_f$  be the Newton function of the map  $f(x) = x^2 + 1$ . Clearly there are no fixed points of the Newton function as there are no zeros of  $f$ . Show that there are points  $c$  where  $N_f^2(c) = c$  (called *period 2-points* of  $N_f$ ).

*Solution.* The Newton function for a function  $f$  is defined by

$$N_f(x) = x - \frac{f(x)}{f'(x)}.$$

Thus, for the function  $f(x) = x^2 + 1$ , we have that

$$\begin{aligned} N_f(x) &= x - \frac{f(x)}{f'(x)} \\ &= x - \frac{x^2 + 1}{2x} \\ &= \frac{x^2 - 1}{2x}. \end{aligned}$$

Using this definition of  $N_f(x)$ , we can readily see that

$$\begin{aligned} N_f^2(x) &= N_f(N_f(x)) = \frac{\left(\frac{x^2-1}{2x}\right)^2 - 1}{2\left(\frac{x^2-1}{2x}\right)} \\ &= \frac{x^4 - 2x^2 - 4x^2 + 1}{4x^2} \cdot \frac{x}{x^2 - 1} \\ &= \frac{x^4 - 6x^2 + 1}{4x(x^2 - 1)}. \end{aligned}$$

The points  $x$  such that  $N_f^2(x) = x$  are the solutions to the equation

$$N_f^2(x) - x = \frac{x^4 - 6x^2 + 1}{4x(x^2 - 1)} - x = \frac{-3x^4 - 2x^2 + 1}{4x(x^2 - 1)} = 0.$$

We readily see that  $x = \pm 3^{-1/2}$  are the real solutions to the above equation. Therefore,  $x = \pm 3^{-1/2}$  satisfy  $N_f^2(x) = x$  and are period 2-points.  $\square$

- Problem 1.4.8.** i. Suppose that  $f(c) = f'(c) = 0$  and  $f''(c) \neq 0$ . If  $f''(x)$  is continuous at  $x = c$ , show that the Newton function  $N_f(x)$  has a removable discontinuity at  $x = c$ . (Hint: Apply L'Hopital's rule to  $N_f$  at  $x = c$ .)
- ii. If in addition,  $f'''(x)$  is continuous at  $x = c$  with  $f'''(c) \neq 0$ , show that  $N'_f(c) = 1/2$ , so that  $x = c$  is not a super-attracting fixed point in this case.
- iii. Check the above for the function  $f(x) = x^3 - x^2$  with  $c = 0$ .

*Solution.* i. Suppose that  $\lim_{x \rightarrow c} N_f(x) = L$ . The function  $N_f(x)$  will have a removable discontinuity at  $x = c$  if  $N_f(c)$  is undefined or  $N_f(c) \neq L$  and the following function defined as

$$F(x) := \begin{cases} N_f(x) & \text{if } x \neq c \\ L & \text{if } x = c \end{cases} \quad (2)$$

is continuous everywhere. Throughout, we assume without loss of generality that  $x = c$  is the only point such that  $f'(x) = 0$ .

Note that if  $x_0 \neq c$ , then the differentiability of  $f'$  implies that both  $f$  and  $f'$  are continuous so that

$$\lim_{x \rightarrow x_0} N_f(x) = \lim_{x \rightarrow x_0} x - \frac{f(x)}{f'(x)} = x_0 - \frac{f(x_0)}{f'(x_0)} = N_f(x_0),$$

i.e. the function  $N_f(x)$  is continuous.

On the other hand, at the point  $x = c$ , we see from the continuity of  $f$  and  $f'$  that

$$\begin{aligned} \lim_{x \rightarrow c} N_f(x) &= \lim_{x \rightarrow c} x - \frac{f(x)}{f'(x)} \\ &= \lim_{x \rightarrow c} \frac{xf'(x) - f(x)}{f'(x)} \\ &= \frac{cf'(c) - f(c)}{f'(c)}. \end{aligned}$$

From our supposition that  $f(c) = f'(c) = 0$ , we see that  $cf'(c) - f(c) = 0$  and  $f'(c) = 0$  implying that  $\lim_{x \rightarrow c} N_f(x) = 0/0$ , an indeterminate form. By L'Hospital's rule in conjunction with our supposition that  $f''$  is continuous and  $f''(c) \neq 0$ , since  $\lim_{x \rightarrow c} xf'(x) - f(x) = \lim_{x \rightarrow c} f'(x) = 0$ , we know that

$$\begin{aligned} \lim_{x \rightarrow c} N_f(x) &= \lim_{x \rightarrow c} \frac{xf'(x) - f(x)}{f'(x)} \\ &= \lim_{x \rightarrow c} \frac{[xf'(x) - f(x)]'}{[f'(x)]'} \\ &= \lim_{x \rightarrow c} \frac{xf''(x)}{f''(x)} = c. \end{aligned}$$

Therefore, the function

$$F(x) := \begin{cases} N_f(x) & \text{if } x \neq c \\ c & \text{if } x = c \end{cases}$$

is continuous everywhere and  $x = c$  is a removable discontinuity for the function  $N_f(x)$ .

- ii. Suppose that  $f'''(x)$  is continuous at  $x = c$  with  $f'''(c) \neq 0$ . From the definition of  $N_f(x)$ , we see that

$$N'_f(x) = \left[ \frac{xf'(x) - f(x)}{f(x)} \right]' = \frac{f(x)f''(x)}{f'(x)^2}.$$

Thus, we have from our initial assumptions that

$$\lim_{x \rightarrow c} N'_f(x) = \lim_{x \rightarrow c} \frac{f(x)f''(x)}{f'(x)^2} = \frac{0}{0},$$

an indeterminate form. The assumptions to use L'Hospital's rule are met and we see from the continuity of  $f$  and its derivatives combined with our initial suppositions that

$$\begin{aligned} \lim_{x \rightarrow c} N'_f(x) &= \lim_{x \rightarrow c} \frac{[f(x)f''(x)]'}{[f'(x)^2]'} \\ &= \lim_{x \rightarrow c} \frac{f'(x)f''(x) + f(x)f'''(x)}{2f'(x)f''(x)} = \frac{0}{0}, \end{aligned}$$

another indeterminate form. Applying L'Hospital's rule one last time we see that

$$\begin{aligned} \lim_{x \rightarrow c} N'_f(x) &= \lim_{x \rightarrow c} \frac{[f'(x)f''(x) + f(x)f'''(x)]'}{[2f'(x)f''(x)]'} \\ &= \lim_{x \rightarrow c} \frac{f''(x)^2 + 2f'(x)f'''(x) + f(x)f^{(4)}(x)}{2f''(x)^2 + 2f'(x)f'''(x)} = \frac{1}{2}. \end{aligned}$$

Therefore, since  $|N'_f(c)| \neq 0$ , the point  $x = c$  is not a super-attracting fixed point.

- iii. Let  $f(x) = x^3 - x^2$ . Note that  $f'(x) = 3x^2 - 2x$ ,  $f''(x) = 6x - 2$ , and  $f'''(x) = 6$ , all of which are continuous. Further, we know that

$$N_f(x) = x - \frac{f(x)}{f'(x)} = \frac{2x^2 - x}{3x - 2}.$$

It is clear that the point  $c = 0$  is a fixed point of the function  $f$ . Note that  $f(c) = f'(c) = 0$  with  $f''(c) \neq 0$  and  $f'''(c) \neq 0$ . However, the point  $x = c$  is not a removable discontinuity since  $\lim_{x \rightarrow c} N_f(x) = 0 = N_f(c)$ , i.e. the Newton function is actually continuous at  $x = c$  so there is no discontinuity to remove. Since  $|N'_f(c)| = 1/2$ , the result that  $c = 0$  is not a super-attracting fixed point still holds. □