Homework Assignment 2

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Problem 4.5. A Markov chain $\{X_n, n \geq 0\}$ with states 0,1,2, has the transition probability matrix

$$\boldsymbol{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

If $P\{X_0 = 0\} = P\{X_0 = 1\} = 1/4$, find $E[X_3]$.

Solution. $\square$

Problem 4.6. Let the transition probability matrix of a two-state Markov chain be given, as in Example 4.2, by

$$\boldsymbol{P} = \left\| \begin{matrix} p & 1-p \\ 1-p & p \end{matrix} \right\|.$$

Show by mathematical induction that

$$\boldsymbol{P}^{(n)} = \left\| \begin{matrix} \frac{1}{2} + \frac{1}{2}(2p-1)^n & \frac{1}{2} - \frac{1}{2}(2p-1)^n \\ \frac{1}{2} - \frac{1}{2}(2p-1)^n & \frac{1}{2} + \frac{1}{2}(2p-1)^n \end{matrix} \right\|.$$

Solution. □

Problem 4.8. Suppose that coin 1 has probability 0.7 of coming up heads and coin 2 has probability 0.6 of coming up heads. If the coin flipped today comes up heads, then we select coin 1 to flip tomorrow and if it comes up tails then we select coin 2 to flip tomorrow. If the coin initially flipped is equally likely to be coin 1 or coin 2, then what is the probability that the coin flipped on the third day after the initial flip is coin 1? Suppose that the coin flipped on Monday comes up heads. What is the probability that the coin flipped on Friday of the same week also comes up heads?

Solution. □

Problem 4.14. Specify the classes of the following Markov chains and determine whether they are transient or recurrent:

$$\boldsymbol{P_1} = \begin{Vmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{Vmatrix} \qquad \boldsymbol{P_2} = \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{Vmatrix}$$

$$\boldsymbol{P_3} = \begin{Vmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{Vmatrix} \quad \boldsymbol{P_4} = \begin{Vmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{Vmatrix}$$

Solution. □

Problem 4.16. Show that if state i is recurrent and state i does not communicate with state j, then $P_{ij} = 0$. This implies that once a process enters a recurrent class of states it can never leave that class. For this reason, a recurrent class is often referred to as a *closed* class.

Solution. □