Exam 3

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Problem 1. Solve the non-homogeneous diffusion problem by the Hankel transform

$$u_t = a\left(u_{rr} + \frac{1}{r}u_r\right) + Q(r,t), \qquad 0 < r < \infty, \quad 0 < t$$
$$u(r,0) = f(r), \qquad 0 < r < \infty.$$

Solution. Application of the 0-th order Hankel transform will transform the above Partial Differential Equation into an Ordinary Differential Equation. The following property of the 0-th order Hankel transform will aid in the application; if $\mathscr{H}_0\left\{u(r,t)\right\} = \tilde{u}_0(\kappa,t)$, then

$$\mathscr{H}_0\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[u(r,t)\right] + \frac{\partial^2}{\partial r^2}\left[u(r,t)\right]\right\} = -\kappa^2\tilde{u}_0(\kappa,t). \tag{1}$$

Now, with the above property, we see that applying the 0-th order Hankel transform to the diffusion problem yields

$$\frac{d}{dt}\left[\tilde{u}_0(\kappa,t)\right] + a\kappa^2\tilde{u}_0(\kappa,t) = \tilde{Q}_0(\kappa,t), \qquad 0 < \kappa < \infty, \quad 0 < t$$
$$\tilde{u}_0(\kappa,0) = \tilde{f}_0(\kappa), \qquad 0 < \kappa < \infty.$$

This is a first order linear Ordinary Differential Equation, the solution to which is

$$\tilde{u}_0(\kappa,t) = c_1(\kappa)e^{-a\kappa^2 t} + e^{-a\kappa^2 t}\int_0^t e^{a\kappa^2 x}\tilde{Q}_0(\kappa,x)dx.$$

Thus, from this solution and the transformed boundary condition, we see that $c_1(\kappa) = \tilde{f}_0(\kappa)$ and the solution to the transformed boundary value problem is

$$\tilde{u}_0(\kappa,t) = \tilde{f}_0(\kappa)e^{-a\kappa^2 t} + e^{-a\kappa^2 t}\int_0^t e^{a\kappa^2 x}\tilde{Q}_0(\kappa,x)dx.$$

Therefore, the solution to the initial diffusion problem is

$$\begin{aligned} u(r,t) = \mathscr{H}_0^{-1}\left\{\tilde{u}_0(\kappa,t)\right\} &= \mathscr{H}_0^{-1}\left\{\tilde{f}_0(\kappa)e^{-a\kappa^2 t} + e^{-a\kappa^2 t}\int_0^t e^{a\kappa^2 x}\tilde{Q}_0(\kappa,x)dx\right\} \\ &= \int_0^\infty \kappa J_0(\kappa r)\left[\tilde{f}_0(\kappa)e^{-a\kappa^2 t} + e^{-a\kappa^2 t}\int_0^t e^{a\kappa^2 x}\tilde{Q}_0(\kappa,x)dx\right]d\kappa, \end{aligned}$$

where $J_0(\kappa r)$ is the Bessel function of order 0.

$\square$

Problem 2.

Solution. □

Problem 3. Solve the following integral equation by the Mellin transform
$$f(x) = \sin ax + \int_0^\infty \frac{f(xt)}{1+t^2} dt.$$

Solution. Let $g(x) = \dfrac{1}{1+x^2}$ and $h(x) = \sin ax$. Recall that $(f \circ g)(x)$ is defined to be
$$(f \circ g)(x) = \int_0^\infty f(xt) g(t) dt.$$
Thus, with this knowledge, the integral equation becomes
$$\begin{aligned} f(x) &= h(x) + \int_0^\infty f(xt) g(t) dt \\ &= h(x) + (f \circ g)(x). \end{aligned}$$

Let $\mathscr{M}\{f(x)\} = \tilde{f}(p)$, $\mathscr{M}\{g(x)\} = \tilde{g}(p)$, and $\mathscr{M}\{h(x)\} = \tilde{h}(p)$. Then from the Convolution Type theorem regarding the Mellin transform, we see that application of the Mellin transform to the integral equation yields
$$\begin{aligned} \tilde{f}(p) &= \mathscr{M}\{h(x)\} + \mathscr{M}\{(f \circ g)(x)\} \\ &= \tilde{h}(p) + \tilde{f}(p)\tilde{g}(1-p). \end{aligned}$$
Solving the above algebraic equation shows that
$$\tilde{f}(p) = \frac{\tilde{h}(p)}{1 - \tilde{g}(1-p)}.$$
From our table of Mellin transforms we know that
$$\tilde{g}(p) = \frac{\pi}{2} \csc\left(\frac{\pi p}{2}\right)$$
and
$$\tilde{h}(p) = a^{-p}\Gamma(p) \sin\left(\frac{\pi p}{2}\right).$$

Therefore, we see that
$$\begin{aligned} \tilde{f}(p) &= \frac{a^{-p}\Gamma(p)\sin\left(\frac{\pi p}{2}\right)}{1 - \frac{\pi}{2}\csc\left(\frac{\pi(1-p)}{2}\right)} \\ &= \frac{2a^{-p}\Gamma(p)\sin\left(\frac{\pi p}{2}\right)}{2 - \pi \sec\left(\frac{\pi p}{2}\right)} \end{aligned}$$
and the solution to the integral equation is
$$\begin{aligned} f(x) = \mathscr{M}^{-1}\left\{\tilde{f}(p)\right\} &= \mathscr{M}^{-1}\left\{\frac{2a^{-p}\Gamma(p)\sin\left(\frac{\pi p}{2}\right)}{2 - \pi \sec\left(\frac{\pi p}{2}\right)}\right\} \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-p}\left[\frac{2a^{-p}\Gamma(p)\sin\left(\frac{\pi p}{2}\right)}{2 - \pi \sec\left(\frac{\pi p}{2}\right)}\right] dp. \end{aligned}$$
$\square$

Problem 4. Solve the following Partial Differential Equation by the Mellin transform

$$r^2\phi_{rr} + r\phi_r + \phi_{\theta\theta} = 0, \qquad 0 < r < \infty, \quad 0 < \theta < \pi$$
$$\phi(r,0) = \begin{cases} (1-r)^2 & 0 < r < 1 \\ 0 & 1 < r \end{cases}$$
$$\phi(r,\pi) = \begin{cases} 1 & 0 < r < 1 \\ 0 & 1 < r \end{cases},$$

Solution. Recall that if $\mathscr{M}\left\{\phi(r,\theta)\right\} = \tilde{\phi}(p,\theta)$, then the following property holds

$$\mathscr{M}\left\{r^2\frac{\partial^2}{\partial r^2}\left[\phi(r,\theta)\right] + r\frac{\partial}{\partial r}\left[\phi(r,\theta)\right]\right\} = p^2\tilde{\phi}(p,\theta).$$

Thus, applying the Mellin transform to the Partial Differential Equation and using our table of Mellin transforms, we see that

$$\frac{d^2}{d\theta^2}\left[\tilde{\phi}(p,\theta)\right] + p^2\tilde{\phi}(p,\theta) = 0, \qquad 0 < p < \infty, \quad 0 < \theta < \pi$$
$$\tilde{\phi}(p,0) = \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)}, \quad \tilde{\phi}(p,\pi) = \frac{1}{p}.$$

The solution to the resulting homogeneous linear Ordinary Differential Equation is

$$\tilde{\phi}(p,\theta) = c_1(p)\cos p\theta + c_2(p)\sin p\theta.$$

Using the above solution and the transformed boundary conditions, we see that

$$c_1(p) = \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)}$$
$$c_1(p)\cos p\pi + c_2(p)\sin p\pi = \frac{1}{p}.$$

Solving, we see that

$$\begin{aligned} c_2(p) &= \left(\frac{1}{p} - \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)}\cos p\pi\right)\csc p\pi \\ &= \frac{\csc p\pi}{p} - \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)}\cot p\pi. \end{aligned}$$

Thus, the solution to the transformed boundary value problem is

$$\tilde{\phi}(p,\theta) = \left[\frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)}\right]\cos p\theta + \left[\frac{\csc p\pi}{p} - \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)}\cot p\pi\right]\sin p\theta.$$

Therefore, the solution to the original boundary value problem is

$$\begin{aligned} \phi(r,\theta) &= \mathscr{M}^{-1}\left\{\left[\frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)}\right]\cos p\theta + \left[\frac{\csc p\pi}{p} - \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)}\cot p\pi\right]\sin p\theta\right\} \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} r^{-p}\left\{\left[\frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)}\right]\cos p\theta + \left[\frac{\csc p\pi}{p} - \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)}\cot p\pi\right]\sin p\theta\right\}dp. \end{aligned}$$

□

Problem 5. Show that

$$\int_0^\infty e^{-ax}\left(\frac{\cos px - \cos qx}{x}\right)dx = \frac{1}{2}\log\frac{q^2+a^2}{p^2+a^2}$$

Solution. Recall that the Laplace transform of a function $f(t)$ is defined as

$$\bar{f}(s) = \mathscr{L}\left\{f(t)\right\}_s = \int_0^\infty f(t)e^{-st}dt$$

where we explicitly note the transformation variable s. Thus, we see that the integral above becomes

$$\begin{aligned}\int_0^\infty e^{-ax}\left(\frac{\cos px - \cos qx}{x}\right)dx &= \int_0^\infty e^{-ax}\left(\frac{\cos px}{x}\right)dx - \int_0^\infty e^{-ax}\left(\frac{\cos qx}{x}\right)dx \\ &= \mathscr{L}\left\{\frac{\cos px}{x}\right\}_a - \mathscr{L}\left\{\frac{\cos qx}{x}\right\}_a. \qquad (2)\end{aligned}$$

From a previous theorem, if $\mathscr{L}\left\{f(t)\right\} = \bar{f}(s)$, then we have that

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(z)dz.$$

Thus, we see that (2) becomes

$$\begin{aligned}\int_0^\infty e^{-ax}\left(\frac{\cos px - \cos qx}{x}\right)dx &= \mathscr{L}\left\{\frac{\cos px}{x}\right\}_a - \mathscr{L}\left\{\frac{\cos qx}{x}\right\}_a \\ &= \int_a^\infty \mathscr{L}\left\{\cos px\right\}_z dz - \int_a^\infty \mathscr{L}\left\{\cos qx\right\}_z dz\end{aligned}$$

From the table of Laplace transforms, we have that

$$\mathscr{L}\left\{\cos bx\right\}_z = \frac{z}{b^2+z^2}.$$

Thus, we have that

$$\begin{aligned}\int_0^\infty e^{-ax}\left(\frac{\cos px - \cos qx}{x}\right)dx &= \int_a^\infty \mathscr{L}\left\{\cos px\right\}_z dz - \int_a^\infty \mathscr{L}\left\{\cos qx\right\}_z dz \\ &= \int_a^\infty \frac{z}{p^2+z^2}dz - \int_a^\infty \frac{z}{q^2+z^2}dz\end{aligned}$$

Using a u-substitution to solve the resulting integrals, we have that

$$\begin{aligned}\int_0^\infty e^{-ax}\left(\frac{\cos px - \cos qx}{x}\right)dx &= \int_a^\infty \frac{z}{p^2+z^2}dz - \int_a^\infty \frac{z}{q^2+z^2}dz \\ &= \frac{1}{2}\log p^2+z^2\bigg|_a^\infty - \frac{1}{2}\log p^2+z^2\bigg|_a^\infty.\end{aligned}$$

Using the properties of logarithms, we therefore see that

$$\begin{aligned}
\int_0^\infty e^{-ax}\left(\frac{\cos px - \cos qx}{x}\right)dx &= \frac{1}{2}\log p^2 + z^2\Big|_a^\infty - \frac{1}{2}\log p^2 + z^2\Big|_a^\infty \\
&= \frac{1}{2}\left[\log\frac{p^2+z^2}{q^2+z^2}\Big|_{z=\infty} - \log\frac{p^2+z^2}{q^2+z^2}\Big|_{z=a}\right] \\
&= -\frac{1}{2}\log\frac{p^2+a^2}{q^2+a^2} \\
&= \frac{1}{2}\log\frac{q^2+a^2}{p^2+a^2}.
\end{aligned}$$

$\square$

Problem 6. Suppose that $I_n f(x)$ denotes the n-th repeated integral of $f(x)$ defined by

$$I_n f(x) = \int_x^\infty I_{n-1} f(t) dt$$

and that $\mathscr{M}\{f(x)\} = \tilde{f}(p)$. Show that

a. $\mathscr{M}\left\{\int_x^\infty f(t)dt\right\} = \frac{1}{p}\tilde{f}(p+1)$,

b. $\mathscr{M}\{I_n f(x)\} = \frac{\Gamma(p)}{\Gamma(p+n)}\tilde{f}(p+n)$

Solution. Recall that the Mellin transform of the function $f(x)$ is defined as

$$\mathscr{M}\{f(x)\} = \int_0^\infty x^{p-1} f(x) dx. \tag{3}$$

a. From the definition of the Mellin transform (3), we see that

$$\mathscr{M}\left\{\int_x^\infty f(t)dt\right\} = \int_0^\infty x^{p-1}\left[\int_x^\infty f(t)dt\right]dx.$$

Interchanging the order of integration from t to x, we see that

$$\begin{aligned}\mathscr{M}\left\{\int_x^\infty f(t)dt\right\} &= \int_0^\infty x^{p-1}\left[\int_x^\infty f(t)dt\right]dx \\ &= \int_0^\infty f(t)\left[\int_0^t x^{p-1}dx\right]dt \\ &= \frac{1}{p}\int_0^\infty t^p f(t)dt.\end{aligned}$$

If $\tilde{f}(p) = \mathscr{M}\{f(x)\}$, then from the definition of the Mellin transform (3), we see that the above integral becomes

$$\begin{aligned}\mathscr{M}\left\{\int_x^\infty f(t)dt\right\} &= \frac{1}{p}\int_0^\infty t^p f(t)dt \\ &= \frac{1}{p}\tilde{f}(p+1),\end{aligned}$$

and we are done.

b. We will now prove the relation

$$\mathscr{M}\{I_n f(x)\} = \frac{\Gamma(p)}{\Gamma(p+n)}\tilde{f}(p+n)$$

by induction. The results of the previous exercise show that

$$\mathscr{M}\{I_1 f(x)\} = \mathscr{M}\left\{\int_x^\infty f(t)dt\right\} = \frac{1}{p}\tilde{f}(p+1) = \frac{\Gamma(p)}{\Gamma(p+1)}\tilde{f}(p+1)$$

and the base step is established.

Now assume the relation holds for n, i.e. assume that

$$\mathscr{M}\left\{I_n f(x)\right\} = \frac{\Gamma(p)}{\Gamma(p+n)}\tilde{f}(p+n).$$

Now, we see from the definition of the Mellin transform (3) that

$$\begin{aligned}\mathscr{M}\left\{I_{n+1} f(x)\right\} &= \mathscr{M}\left\{\int_x^\infty I_n f(t)dt\right\} \\ &= \int_0^\infty x^{p-1}\left[\int_x^\infty I_n f(t)dt\right]dx.\end{aligned}$$

Let $g(t) = I_n f(t)$. Then, proceeding as we did in establishing the result of the base step, interchanging the order of integration from t to x yields

$$\begin{aligned}\mathscr{M}\left\{I_{n+1} f(x)\right\} &= \int_0^\infty x^{p-1}\left[\int_x^\infty g(t)dt\right]dx \\ &= \int_0^\infty g(t)\left[\int_0^t x^{p-1}dx\right]dt \\ &= \frac{1}{p}\int_0^\infty t^p g(t)dt \\ &= \frac{1}{p}\tilde{g}(p+1) \qquad (4)\end{aligned}$$

where $\tilde{g}(p) = \mathscr{M}\left\{g(t)\right\}$.

From our assumption, we have that

$$\tilde{g}(p) = \mathscr{M}\left\{g(t)\right\} = \mathscr{M}\left\{I_n f(x)\right\} = \frac{\Gamma(p)}{\Gamma(p+n)}\tilde{f}(p+n).$$

Thus, the integral in (4) becomes

$$\begin{aligned}\mathscr{M}\left\{I_{n+1} f(x)\right\} &= \frac{1}{p}\tilde{g}(p+1) \\ &= \frac{\Gamma(p+1)}{p\Gamma(p+n+1)}\tilde{f}(p+n+1) \\ &= \frac{\Gamma(p)}{\Gamma(p+n+1)}\tilde{f}(p+n+1).\end{aligned}$$

Therefore, the result holds for $n+1$ and the relation holds in general for all $n > 0$.

$\square$

Problem 7. Solve the following Initial Value Problem by the Z-transform

$$\begin{aligned} &f(n+2) - f(n+1) - 6f(n) = \sin\left(\frac{n\pi}{2}\right), \qquad n \geq 2 \\ &f(0) = 0, \quad f(1) = 1. \end{aligned}$$

Solution. Recall that if $Z\{f(n)\} = F(z)$ and $m \geq 0$, then the following property holds:

$$Z\{f(n+m)\} = z^m \left[F(z) - \sum_{r=0}^{m-1} f(r) z^{-r} \right].$$

Thus, applying the Z-transform to the Initial Value Problem, we have that

$$z^2 F(z) - z^2 f(0) - z f(1) - zF(z) + z f(0) - 6F(z) = Z\left\{\sin\left(\frac{n\pi}{2}\right)\right\}.$$

In light of the initial data, this reduces to

$$(z-3)(z+2)F(z) - z = Z\left\{\sin\left(\frac{n\pi}{2}\right)\right\}.$$

From our table of Z-transforms, we know that

$$Z\left\{\sin\left(\frac{n\pi}{2}\right)\right\} = \frac{z\sin\frac{\pi}{2}}{z^2 - 2z\cos\frac{\pi}{2} + 1} = \frac{z}{z^2+1}$$

Thus, the solution to the transformed equation is

$$\begin{aligned} F(z) &= \frac{Z\left\{\sin\left(\frac{n\pi}{2}\right)\right\}}{(z-3)(z+2)} + \frac{z}{(z-3)(z+2)} \\ &= \frac{z}{(z^2+1)(z-3)(z+2)} + \frac{z}{(z-3)(z+2)}. \end{aligned}$$

Applying the method of partial fraction decomposition to this transformed function shows that

$$\begin{aligned} F(z) &= \frac{z}{(z^2+1)(z-3)(z+2)} + \frac{z}{(z-3)(z+2)} \\ &= z\left[\frac{a_1 z + a_2}{z^2+1} + \frac{a_3}{z-3} + \frac{a_4}{z+2}\right] + z\left[\frac{b_1}{z-3} + \frac{b_2}{z+2}\right] \\ &= \frac{1}{50}\left[\frac{z(z-7)}{z^2+1}\right] + \frac{1}{50}\left[\frac{z}{z-3}\right] - \frac{1}{25}\left[\frac{z}{z+2}\right] + \frac{1}{5}\left[\frac{z}{z-3}\right] - \frac{1}{5}\left[\frac{z}{z+2}\right] \\ &= \frac{1}{50}\left[\frac{z(z-7)}{z^2+1}\right] + \frac{11}{50}\left[\frac{z}{z-3}\right] - \frac{6}{25}\left[\frac{z}{z+2}\right]. \end{aligned}$$

Therefore, using the fact that

$$Z\{a^n\} = \frac{z}{z-a}$$

and a computer algebra system we see that the solution to the original finite difference equation is

$$
\begin{aligned}
f(n) = Z^{-1}\left\{F(z)\right\} &= \frac{1}{50}Z^{-1}\left\{\frac{z(z-7)}{z^2+1}\right\} + \frac{11}{50}Z^{-1}\left\{\frac{z}{z-3}\right\} - \frac{6}{25}Z^{-1}\left\{\frac{z}{z+2}\right\} \\
&= \frac{1}{50}\left[i^n\left(\frac{1}{2}+\frac{7i}{2}\right) + (-i)^n\left(\frac{1}{2}-\frac{7i}{2}\right)\right] + \frac{11}{50}3^n - \frac{6}{25}(-2)^n.
\end{aligned}
$$

□

Problem 8. Find the sum of the following series

a. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{e^{-n}}{n+1}$,

b. $\displaystyle\sum_{n=0}^{\infty} n\sin nx$.

Solution. a. Let $f(n) = e^{-n}$, $g(n) = \dfrac{f(n)}{n+1}$, and $h(n) = (-1)^n g(n)$. Suppose that $H(z) = Z\{h(n)\}$. From a previous theorem, we know that

$$\sum_{n=0}^{\infty}(-1)^n\frac{e^{-n}}{n+1} = \sum_{n=0}^{\infty} h(n) = \lim_{z\to 1} H(z). \tag{5}$$

Thus, we merely need to find the Z-transform of $h(n)$ and evaluate the above limit to find the sum of the series.

From the table of Z-transforms, we know that

$$F(z) = Z\{f(n)\} = Z\left\{e^{-n}\right\} = \frac{z}{z-e^{-1}}.$$

A previous theorem states that if $F(z) = Z\{f(n)\}$, then

$$Z\left\{\frac{f(n)}{n+1}\right\} = z\int_z^{\infty}\frac{F(\xi)}{\xi^2}d\xi$$

Thus, the Z-transform of $g(n)$ is

$$\begin{aligned} G(z) = Z\{g(n)\} = Z\left\{\frac{f(n)}{n+1}\right\} &= z\int_z^{\infty}\frac{F(\xi)}{\xi^2}d\xi \\ &= z\int_z^{\infty}\frac{d\xi}{(\xi-e^{-1})\xi} \\ &= ze\left[1-\log\left(e-z^{-1}\right)\right]. \end{aligned}$$

Finally, there is another theorem that will aid in finding $H(z)$, namely the multiplication theorem of Z-transforms; if $F(z) = Z\{f(n)\}$, then

$$Z\{a^n f(n)\} = F\left(\frac{z}{a}\right).$$

Thus, we have that

$$\begin{aligned} H(z) = Z\{h(n)\} = Z\{(-1)^n g(n)\} &= G(-z) \\ &= -ze\left[1-\log\left(e+z^{-1}\right)\right]. \end{aligned}$$

Therefore, by (5), we have that

$$\sum_{n=0}^{\infty}(-1)^n\frac{e^{-n}}{n+1} = \lim_{z\to 1} H(z) = -e\left[1-\log(e+1)\right].$$

b. Let $f(n) = \sin nx$ and $g(n) = nf(n)$ and suppose that $G(z) = Z\{g(n)\}$. As shown previously, there is a theorem that states that

$$\sum_{n=0}^{\infty} n \sin nx = \sum_{n=0}^{\infty} g(n) = \lim_{z \to 1} G(z). \tag{6}$$

Thus, we need only find $G(z)$ and evaluate the above limit to find the sum of the series. The table of Z-transforms show that

$$F(z) = Z\{f(n)\} = \frac{z \sin x}{z^2 - 2z \cos x + 1}.$$

From the multiplication theorem of Z-transforms, we know that

$$G(z) = Z\{g(n)\} = Z\{nf(n)\} = -z \frac{d}{dz}\left[F(z)\right].$$

Thus, we have that

$$\begin{aligned} G(z) &= -z \frac{d}{dz}\left[F(z)\right] \\ &= -z \frac{d}{dz}\left[\frac{z \sin x}{z^2 - 2z \cos x + 1}\right] \\ &= -\frac{(z^2 - 1)\sin x}{(z^2 - 2z \cos x + 1)^2}. \end{aligned}$$

Therefore, by (7), we have the sum of the series is given by

$$\begin{aligned} \sum_{n=0}^{\infty} n \sin nx &= \lim_{z \to 1} G(z) \\ &= -\lim_{z \to 1} \frac{(z^2 - 1)\sin x}{(z^2 - 2z \cos x + 1)^2} \\ &= 0 \end{aligned}$$

□