Homework Assignment 1

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Problem 2.1. Find the Fourier transforms of each of the following functions:

c. $f(x) = \delta^{(n)}(x)$,

f. $f(x) = x \exp\left(-\frac{ax^2}{2}\right)$, $a > 0$,

g. $f(x) = x^2 \exp\left(-\frac{1}{2}x^2\right)$.

Solution. Recall that, by definition, we have that for a function $f(x) \in L^1(\mathbb{R})$, its Fourier transform is given by

$$\mathscr{F}\{f(x)\} = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx \tag{1}$$

where $k \in \mathbb{R}$.

c. The Dirac delta function $\delta(x)$ is defined such that for any good function $g(x)$ we have that

$$\int_{-\infty}^{\infty} \delta(x) g(x) dx = g(0).$$

A good function is defined as a function in C^∞ that decays sufficiently rapidly. Since it is clear that $\delta(x) \to 0$ as $|x| \to \infty$, we have by a previous theorem that

$$\mathscr{F}\{\delta'(x)\} = ik\mathscr{F}\{\delta(x)\}. \tag{2}$$

By (1) and the definition of the Dirac delta function, we see that

$$\mathscr{F}\{\delta(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \delta(x) dx = \frac{1}{\sqrt{2\pi}}.$$

Thus, using (2), we can easily see by induction for $n > 1$ that

$$\mathscr{F}\left\{\delta^{(n)}(x)\right\} = ik\mathscr{F}\left\{\delta^{(n-1)}(x)\right\} = \cdots = \frac{(ik)^n}{\sqrt{2\pi}}.$$

f. Using (1), we see that

$$\begin{aligned}
\mathscr{F}\left\{f(x)\right\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x\exp(-ikx)\exp\left(-\frac{ax^2}{2}\right)dx \\
&= \frac{\exp\left(\frac{(ik)^2}{2a}\right)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x\exp\left(-\frac{ax^2}{2} - ikx - \frac{(ik)^2}{2a}\right)dx \\
&= \frac{\exp\left(-\frac{k^2}{2a}\right)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x\exp\left(-\frac{a}{2}\left(x+\frac{ik}{a}\right)^2\right)dx.
\end{aligned}$$

Making the substitution $u = x + ik/a$, where $du = dx$, we have that

$$\begin{aligned}
\mathscr{F}\left\{f(x)\right\} &= \frac{\exp\left(-\frac{k^2}{2a}\right)}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(u - \frac{ik}{a}\right)\exp\left(-\frac{au^2}{2}\right)du \\
&= \frac{\exp\left(-\frac{k^2}{2a}\right)}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} u\exp\left(-\frac{au^2}{2}\right)du - \frac{ik}{a}\int_{-\infty}^{\infty}\exp\left(-\frac{au^2}{2}\right)du\right]. \qquad (3)
\end{aligned}$$

Since the function $g(x) = u\exp\left(-\frac{au^2}{2}\right)$ is odd, we know that

$$\int_{-\infty}^{\infty} u\exp\left(-\frac{au^2}{2}\right)du = 0.$$

Using the formula for the general Gaussian integral we have that

$$\int_{-\infty}^{\infty}\exp\left(-\frac{au^2}{2}\right) = \frac{\sqrt{2\pi}}{\sqrt{a}}$$

when $a > 0$.

Combining, we see from (3) that the Fourier transform of $f(x) = x\exp\left(-\frac{ax^2}{2}\right)$ for $a > 0$ is

$$\begin{aligned}
\mathscr{F}\left\{f(x)\right\} &= \frac{\exp\left(-\frac{k^2}{2a}\right)}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} u\exp\left(-\frac{au^2}{2}\right)du - \frac{ik}{a}\int_{-\infty}^{\infty}\exp\left(-\frac{au^2}{2}\right)du\right] \\
&= \frac{\exp\left(-\frac{k^2}{2a}\right)}{\sqrt{2\pi}}\left(-\frac{ik}{a}\right)\left(\frac{\sqrt{2\pi}}{\sqrt{a}}\right) \\
&= -\frac{ik\exp\left(-\frac{k^2}{2a}\right)}{a\sqrt{a}}.
\end{aligned}$$

g.

$\square$

Problem 2.2. Show that

a. $\mathscr{F}\left\{\delta(x-ct)+\delta(x+ct)\right\} = \sqrt{\frac{2}{\pi}}\cos(kct)$,

b. $\mathscr{F}\left\{H\left(ct-|x|\right)\right\} = \mathscr{F}\left\{\chi_{[-ct,ct]}(x)\right\} = \sqrt{\frac{2}{\pi}}\frac{\sin(kct)}{k}$.

Solution. $\square$

Problem 2.3. Show that

a. $i\dfrac{d}{dk}F(k) = \mathscr{F}\left\{xf(x)\right\}$

b. $i^n\dfrac{d^n}{dk^n}F(k) = \mathscr{F}\left\{x^n f(x)\right\}$

Solution. $\square$

Problem 2.5. Prove the following:

c. If $f(x)$ has a finite discontinuity at a point $x = a$, then

$$\mathscr{F}\{f'(x)\} = (ik)F(k) - \frac{1}{\sqrt{2\pi}}\exp(-ika)[f]_a,$$

where $[f]_a = f(a+0) - f(a-0)$.

Generalize this result for $\mathscr{F}\left\{f^{(n)}(x)\right\}$.

Solution. □

Problem 2.7. Prove the following results for the convolution:

c. $\dfrac{d}{dx}\left[f(x) * g(x)\right] = f'(x) * g(x) = f(x) * g'(x)$,

d. $\displaystyle\int_{-\infty}^{\infty} (f * g)(x)dx = \int_{-\infty}^{\infty} f(u)du \int_{-\infty}^{\infty} g(v)dv$.

Solution. □

Problem 2.8. Use the Fourier transform to solve the following ordinary differential equations for $-\infty < x < \infty$:

a. $y''(x) - y(x) + 2f(x) = 0$, where $f(x) = 0$ when $x < -a$ and when $x > a$ and its derivatives vanish at $x = \pm\infty$,

b. $2y''(x) + xy'(x) + y(x) = 0$.

Solution. □

Problem 2.9. Solve the following integral equations for an unknown function $f(x)$:

a. $\displaystyle\int_{-\infty}^{\infty} \phi(x-t)f(t)dt = g(x),$

b. $\displaystyle\int_{-\infty}^{\infty} \exp(-at^2)f(x-t)dt = \exp(-at^2), a > b > 0,$

d. $\displaystyle\int_{-\infty}^{\infty} f(x-t)f(t)dt = \frac{b}{x^2+b^2}.$

Solution. □