Exam 3

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May 20, 2017

Problem 1. Solve the non-homogeneous diffusion problem by the Hankel transform

$$\begin{aligned} &u_t = a\left(u_{rr} + \frac{1}{r}u_r\right) + Q(r,t), \qquad 0 < r < \infty, \quad 0 < t \\ &u(r,0) = f(r), \qquad 0 < r < \infty. \end{aligned}$$

Solution. Application of the 0-th order Hankel transform will transform the above Partial Differential Equation into an Ordinary Differential Equation. The following property of the 0-th order Hankel transform will aid in the application; if $\mathscr{H}_0\left\{u(r,t)\right\} = \tilde{u}_0(\kappa,t)$, then

$$\mathscr{H}_0\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[u(r,t)\right] + \frac{\partial^2}{\partial r^2}\left[u(r,t)\right]\right\} = -\kappa^2\tilde{u}_0(\kappa,t). \tag{1}$$

Now, with the above property, we see that applying the 0-th order Hankel transform to the diffusion problem yields

$$\begin{aligned} &\frac{d}{dt}\left[\tilde{u}_0(\kappa,t)\right] + a\kappa^2\tilde{u}_0(\kappa,t) = \tilde{Q}_0(\kappa,t), \qquad 0 < \kappa < \infty, \quad 0 < t \\ &\tilde{u}_0(\kappa,0) = \tilde{f}_0(\kappa), \qquad 0 < \kappa < \infty. \end{aligned}$$

This is a first order linear Ordinary Differential Equation, the solution to which is

$$\tilde{u}_0(\kappa,t) = c_1(\kappa)e^{-a\kappa^2 t} + e^{-a\kappa^2 t}\int_0^t e^{a\kappa^2 x}\tilde{Q}_0(\kappa,x)dx.$$

Thus, from this solution and the transformed boundary condition, we see that $c_1(\kappa) = \tilde{f}_0(\kappa)$ and the solution to the transformed boundary value problem is

$$\tilde{u}_0(\kappa,t) = \tilde{f}_0(\kappa)e^{-a\kappa^2 t} + e^{-a\kappa^2 t}\int_0^t e^{a\kappa^2 x}\tilde{Q}_0(\kappa,x)dx.$$

Therefore, the solution to the initial diffusion problem is

$$\begin{aligned} u(r,t) = \mathscr{H}_0^{-1}\left\{\tilde{u}_0(\kappa,t)\right\} &= \mathscr{H}_0^{-1}\left\{\tilde{f}_0(\kappa)e^{-a\kappa^2 t} + e^{-a\kappa^2 t}\int_0^t e^{a\kappa^2 x}\tilde{Q}_0(\kappa,x)dx\right\} \\ &= \int_0^\infty \kappa J_0(\kappa r)\left[\tilde{f}_0(\kappa)e^{-a\kappa^2 t} + e^{-a\kappa^2 t}\int_0^t e^{a\kappa^2 x}\tilde{Q}_0(\kappa,x)dx\right]d\kappa, \end{aligned}$$

where $J_0(\kappa r)$ is the Bessel function of order 0.

□

Problem 2.

Solution. □

Problem 3. Solve the following integral equation by the Mellin transform

$$f(x) = \sin ax + \int_0^\infty \frac{f(xt)}{1+t^2} dt.$$

Solution. Let $g(x) = \dfrac{1}{1+x^2}$ and $h(x) = \sin ax$. Recall that $(f \circ g)(x)$ is defined to be

$$(f \circ g)(x) = \int_0^\infty f(xt)g(t)dt.$$

Thus, with this knowledge, the integral equation becomes

$$\begin{aligned} f(x) &= h(x) + \int_0^\infty f(xt)g(t)dt \\ &= h(x) + (f \circ g)(x). \end{aligned}$$

Let $\mathscr{M}\{f(x)\} = \tilde{f}(p)$, $\mathscr{M}\{g(x)\} = \tilde{g}(p)$, and $\mathscr{M}\{h(x)\} = \tilde{h}(p)$. Then from the Convolution Type theorem regarding the Mellin transform, we see that application of the Mellin transform to the integral equation yields

$$\begin{aligned} \tilde{f}(p) &= \mathscr{M}\{h(x)\} + \mathscr{M}\{(f \circ g)(x)\} \\ &= \tilde{h}(p) + \tilde{f}(p)\tilde{g}(1-p). \end{aligned}$$

Solving the above algebraic equation shows that

$$\tilde{f}(p) = \frac{\tilde{h}(p)}{1 - \tilde{g}(1-p)}.$$

From our table of Mellin transforms we know that

$$\tilde{g}(p) = \frac{\pi}{2} \csc\left(\frac{\pi p}{2}\right)$$

and

$$\tilde{h}(p) = a^{-p}\Gamma(p) \sin\left(\frac{\pi p}{2}\right).$$

Therefore, we see that

$$\begin{aligned} \tilde{f}(p) &= \frac{a^{-p}\Gamma(p)\sin\left(\frac{\pi p}{2}\right)}{1 - \frac{\pi}{2}\csc\left(\frac{\pi(1-p)}{2}\right)} \\ &= \frac{2a^{-p}\Gamma(p)\sin\left(\frac{\pi p}{2}\right)}{2 - \pi \sec\left(\frac{\pi p}{2}\right)} \end{aligned}$$

and the solution to the integral equation is

$$\begin{aligned} f(x) = \mathscr{M}^{-1}\left\{\tilde{f}(p)\right\} &= \mathscr{M}^{-1}\left\{\frac{2a^{-p}\Gamma(p)\sin\left(\frac{\pi p}{2}\right)}{2 - \pi\sec\left(\frac{\pi p}{2}\right)}\right\} \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-p}\left[\frac{2a^{-p}\Gamma(p)\sin\left(\frac{\pi p}{2}\right)}{2 - \pi\sec\left(\frac{\pi p}{2}\right)}\right] dp. \end{aligned}$$

$\square$

Problem 4.

Solution. □

Problem 5.

Solution. □

Problem 6.

Solution. □

Problem 7.

Solution. □

Problem 8.

Solution. □