

# Homework Assignment 5

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**Problem 5.2.** Suppose that you arrive at a single-teller bank to find five other customers in the bank, one being served and the other four waiting in line. You join the end of the line. If the service times are exponential with rate  $\mu$ , what is the expected amount of time you will spend in the bank?

*Solution.* Let  $T_i$  denote the time that the  $i$ -th person spends at the teller in order to complete his or her transaction. Then the total amount of time I will spend in the bank is the amount of time the other five customers spend at the teller plus the time that I, the sixth customer, will spend at the teller. If  $S$  denotes the total amount of time that I spend at the bank, then

$$S = \sum_{i=1}^6 T_i.$$

Note that even though the first customer is currently being served, the service time is exponentially distributed with rate  $\mu$ , i.e. the waiting time is memory-less so that the expected service time of the first customer is still  $1/\mu$ . Since the other  $T_i$  are exponential random variables with mean  $1/\mu$ , we have that

$$E[S] = \sum_{i=1}^6 E[T_i] = \frac{6}{\mu}.$$

□

**Problem 5.8.** If  $X$  and  $Y$  are independent exponential random variables with respective rates  $\lambda$  and  $\mu$ , what is the conditional distribution of  $X$  given that  $X < Y$ ?

*Solution.* Let  $E$  be the event such that  $X < Y$ . Recall from Bayes' Theorem that the conditional density function of  $X$  given  $E$  is given by

$$f_{X|E}(x | x < y) = \frac{f_X(x)P(E | X = x)}{\int_0^\infty f_X(s)P(E | X = s)ds}.$$

Note that since  $X$  and  $Y$  are independent we have that  $P(X < Y | X = x) = P(x < Y) = e^{-\mu x}$ . Thus,

$$\begin{aligned} f_{X|E}(x|E) &= \frac{\lambda e^{-\lambda x} e^{-\mu x}}{\int_0^\infty \lambda e^{-\lambda s} e^{-\mu s} ds} \\ &= \frac{\lambda e^{-\lambda x} e^{-\mu x}}{\frac{\lambda}{\lambda + \mu}} \\ &= (\lambda + \mu) e^{-(\lambda + \mu)x}, \end{aligned}$$

if  $x \geq 0$ , and we see that  $f_{X|E}(x | x < y)$  has the density function that of an exponential random variable with rate  $\lambda + \mu$ .

Therefore, the conditional distribution of  $X$  given  $E$  is

$$F_{X|E}(x | x < y) = \begin{cases} 1 - e^{-(\lambda + \mu)x} & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

□

**Problem 5.15.** One hundred items are simultaneously put on a life test. Suppose the lifetimes of the individual items are independent exponential random variables with mean 200 hours. The test will end when there have been a total of 5 failures. If  $T$  is the time at which the test ends, find  $E[T]$  and  $\text{Var}[T]$ .

*Solution.* Let  $T_i$  be the time between the  $(i - 1)$ -th and the  $i$ -th failure. Note that from a previous proposition, we have that if  $X_1, \dots, X_n$  are independent exponential random variables all with rate  $\lambda = 1/200$ , then  $\min_i X_i$  is exponential with rate  $n\lambda$ . Thus, the shortest lifetime among the 100 initial items is exponential with rate  $100\lambda$ , i.e. the time that it takes for the first failure to occur,  $T_1$ , is exponential with rate  $100\lambda$ .

In general, after the  $(i - 1)$ -th failure, there will be  $101 - i$  items left to test. Thus, the shortest lifetime among the  $101 - i$  items is exponential with rate  $(101 - i)\lambda$ , i.e. the time that it takes for the  $i$ -th failure to occur,  $T_i$ , is exponential with rate  $(101 - i)\lambda$ .

Since each  $T_i$  is exponential with rate  $(101 - i)\lambda$ , we have that

$$E[T_i] = \frac{1}{(101 - i)\lambda} = \frac{200}{(101 - i)}, \quad \text{Var}[T_i] = \frac{1}{(101 - i)^2\lambda^2} = \frac{200^2}{(101 - i)^2}.$$

If the test ends after 5 failures and if  $T$  is the total time that the test takes, then  $T = T_1 + \dots + T_5$  and

$$E[T] = \sum_{i=1}^5 E[T_i] = \sum_{i=1}^5 \frac{200}{(101 - i)} = 10.2062.$$

Since each  $T_i$  is independent, we have that

$$\text{Var}[T] = \sum_{i=1}^5 \text{Var}[T_i] = \sum_{i=1}^5 \frac{200^2}{(101 - i)^2} = 20.8377.$$

□

**Problem 5.43.** Customers arrive at a two-server service station according to a Poisson process with rate  $\lambda$ . Whenever a new customer arrives, any customer that is in the system immediately departs. A new arrival enters first with server 1 and then with server 2. If the service times are independent exponentials with respective rates  $\mu_1$  and  $\mu_2$ , what proportion of entering customers complete their service with server 2?

*Solution.* Let  $T_1$  denote the amount of time spent with server 1 and  $T_2$  denote the amount of time spent with server 2. Note that  $T_1$  and  $T_2$  are independent and exponentially distributed with rates  $\mu_1$  and  $\mu_2$ , respectively.

Let  $T$  be the amount of time before the next customer arrives. Since  $T$  is the arrival time of some event of a Poisson process with rate  $\lambda$ , we have that  $T$  is exponentially distributed with rate  $\lambda$ .

Let  $p$  be the proportion of customers that complete their service with server 2. Note that  $p$  will be the proportion of time between inter-arrivals that is greater than  $T_1 + T_2$ . This will be the probability that  $T > T_1 + T_2$ , i.e.

$$p = P(T > T_1 + T_2)$$

Since  $T$  is exponentially distributed and  $T_1$  and  $T_2$  are independent, we have that

$$\begin{aligned} p &= P(T > T_1 + T_2) \\ &= P(T > T_1)P(T > T_2) \\ &= \int_0^\infty P(T > T_1 \mid T = t)f_T(t)dt \int_0^\infty P(T > T_2 \mid T = t)f_T(t)dt \\ &= \int_0^\infty \lambda(1 - e^{-\mu_1 t})e^{-\lambda t}dt \int_0^\infty \lambda(1 - e^{-\mu_2 t})e^{-\lambda t}dt \\ &= \frac{\mu_1}{\mu_1 + \lambda} \frac{\mu_2}{\mu_2 + \lambda}. \end{aligned} \tag{1}$$

Therefore, (1) is the proportion of customers that complete their services with server 2.  $\square$

**Problem 5.44.***Solution.*

**Problem 5.50.***Solution.*