Homework Assignment 6

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October 6, 2015

Problem 3.1. Determine which of the following ARMA processes are causal and which of them are invertible. (In each case $\{Z_t\}$ denotes white noise.)

a. $X_t + 0.2X_{t-1} - 0.48X_{t-2} = Z_t$

b. $X_t + 1.9X_{t-1} + 0.88X_{t-2} = Z_t + 0.2Z_{t-1} + 0.7Z_{t-2}$

c. $X_t + 0.6X_{t-1} = Z_t + 1.2Z_{t-1}$

d. $X_t + 1.8X_{t-1} + 0.81X_{t-2} = Z_t$

e. $X_t + 1.6X_{t-1} = Z_t - 0.4Z_{t-1} + 0.04Z_{t-2}$

Solution. A general ARMA(p, q) process $X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$ can be rewritten as $\phi(B)X_t = \theta(B)Z_t$, where B is the backshift operator and $\phi(x) = 1 - \phi_1 x - \cdots - \phi_p x^p$ and $\theta(x) = 1 + \theta_1 x + \cdots + \theta_q x^q$. A stationary solution of the above process exists if and only if $\phi(x) = 0$ for $|x| \neq 1$. Similarly, if a stationary solution to the process exists, then the process is causal if and only the solutions to $\phi(x) = 0$ have norm greater than 1 and the process is invertible if the solutions to $\theta(x) = 0$ have norm greater than 1.

a. For this process, $\phi(x) = 1 - (-0.2)x - (0.48)x^2$ and $\theta(x) = 1$. The solutions to $\phi(x) = 0$ are $x_1 = 1.\overline{66}$ and $x_2 = -1.25$ both of which have norm greater than 1. Therefore a stationary solution exists and the process is causal. Similarly, $\theta(x) \neq 0$ for any x so it is invertible as well.

b. For this process, $\phi(x) = 1 - (-1.9)x - (-0.88)x^2$ and $\theta(x) = 1 + 0.2x + 0.7x^2$. The solutions to $\phi(x) = 0$ are $x_1 = -0.\overline{90}$ and $x_2 = -1.25$. Therefore a stationary solution exists however since $|x_1| \leq 1$ the process is not causal. Similarly, the solutions to $\theta(x) = 0$ are $x_1 = -0.1429 + 1.1867i$ and $x_2 = -0.1429 - 1.1867i$ both of which have norm greater than 1. Therefore the process is invertible.

c. For this process, $\phi(x) = 1 - (-0.6)x$ and $\theta(x) = 1 + 1.2x$. The solution to $\phi(x) = 0$ is $x_1 = -1.\overline{66}$ which has norm greater than 1. Therefore a stationary solution exists and the process is causal. Similarly, the solution to $\theta(x) = 0$ is $x_1 = -0.8\overline{33}$ which has norm less than 1. Therefore the process is not invertible.

d. For this process, $\phi(x) = 1 - (-1.8)x - (-0.81)x^2$ and $\theta(x) = 1$. The solution to $\phi(x) = 0$ is $x_1 = -1.\overline{11}$ with multiplicity 2 which has norm greater than 1. Therefore a stationary solution exists and the process is causal. Similarly, $\theta(x) \neq 0$ for any x so it is invertible as well.

e. For this process, $\phi(x) = 1 - (-1.6)x$ and $\theta(x) = 1 + (-0.4)x + (0.04)x^2$. The solution to $\phi(x) = 0$ is $x_1 = -0.625$ which has norm less than 1. Therefore a stationary solution exists however the process is not causal. Similarly, the solution to $\theta(x) = 0$ is $x = 5$ with multiplicity 2 which has norm greater than 1. Therefore the process is invertible.

$\square$

Problem 3.3. For those processes in Problem 3.1 that are causal, compute the first six coefficients $\psi_0, \psi_1, \ldots, \psi_5$ in the causal representation $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ of $\{X_t\}$.

Solution. $\square$