Homework Assignment 2

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Problem 1. Use the method of variation of parameters to find the general solution of

$$y'' + 2y' + 2y = \sin x.$$

Solution. Suppose that $Ly = y'' + 2y' + 2y$. The general solution to $Ly = \sin x$ is given by $y = y_0 + y_h$ where y_0 is a particular solution of $Ly = \sin x$ and y_h is the solution to the homogeneous equation $Ly = 0$.

The characteristic equation of the equation $Ly = 0$ is $m(x) = x^2 + 2x + 2$, the roots of which are $m_1 = -1 - i$ and $m_2 = -1 + i$. As the roots of the characteristic equation are complex, the solution to $Ly = 0$ is given by

$$y_h = c_1 e^{-x} \sin x + c_2 e^{-x} \cos x. \tag{1}$$

The method of variation of parameters can be used to find a particular solution y_0. We wish to find functions $u_1(x), u_2(x)$ such that

$$y_0 = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{2}$$

satisfies $Ly_0 = \sin x$ where $y_1(x)$ and $y_2(x)$ are solutions to the homogeneous equation $Ly = 0$. If the functions $u_1(x)$ and $u_2(x)$ are solutions to the system

$$\left\{ \begin{array}{ccccc} u_1'y_1 & + & u_2'y_2 & = & 0 \\ u_1'y_1' & + & u_2'y_2' & = & \sin x \end{array} \right. \tag{3}$$

then (2) will satisfy the original differential equation $Ly = \sin x$ equation. The solution to the system (3) is

$$u_1(x) = -\int \frac{y_2(x)\sin x}{W[\{y_1, y_2\}]}dx \qquad u_2(x) = \int \frac{y_1(x)\sin x}{W[\{y_1, y_2\}]}dx \tag{4}$$

where $W[\{y_1, y_2\}]$ is the Wronskian of the functions y_1 and y_2.

Using (1), we know that $y_1(x) = e^{-x}\sin x$ and $y_2(x) = e^{-x}\cos x$ so the particular solution has the form $y_0 = u_1(x)e^{-x}\sin x + u_2(x)e^{-x}\cos x$. Further, the Wronskian of y_1 and y_2 is

$$W[\{y_1, y_2\}] = \left| \begin{array}{ll} e^{-x}\sin x & e^{-x}\cos x \\ e^{-x}\cos x - e^{-x}\sin x & -e^{-x}\cos x - e^{-x}\sin x \end{array} \right| = -e^{-2x}.$$

Thus, using (4), we know that

$$\begin{aligned} u_1(x) &= -\int \frac{y_2(x)\sin x}{W[\{y_1, y_2\}]}dx \\ &= \int \frac{e^{-x}\cos x \sin x}{e^{-2x}}dx \\ &= \frac{e^x}{10}\left(-2\cos 2x + \sin 2x\right) + C \end{aligned}$$

and

$$\begin{aligned} u_2(x) &= \int \frac{y_1(x)\sin x}{W[\{y_1, y_2\}]}dx \\ &= -\int \frac{e^{-x}\sin^2 x}{e^{-2x}}dx \\ &= \frac{e^x}{10}\left(-5 + \cos 2x + 2\sin 2x\right) + C. \end{aligned}$$

Therefore, a particular solution to $Ly = \sin x$ is

$$y_0(x) = \frac{1}{10}\left(-2\cos 2x + \sin 2x\right)\sin x + \frac{1}{10}\left(-5 + \cos 2x + 2\sin 2x\right)\cos x$$

and the general solution to $Ly = \sin x$ is

$$\begin{aligned} y(x) &= y_0(x) + y_h(x) \\ &= \frac{1}{10}\left(-2\cos 2x + \sin 2x\right)\sin x + \frac{1}{10}\left(-5 + \cos 2x + 2\sin 2x\right)\cos x \\ &\quad + c_1 e^{-x}\sin x + c_2 e^{-x}\cos x \end{aligned} \tag{5}$$

□

Problem 2. Find the Green function of the IVP

$$y'' + 2y' + 2y = f(x), \quad y(0) = y'(0) = 0.$$

Solution. Let $Ly = f(x)$ denote the differential equation $y'' + 2y' + 2y = f(x)$ together with the initial conditions $y(0) = y'(0) = 0$. The Green function $G(x, a)$ of the IVP $Ly = f(x)$ is defined by the equations

$$\frac{\partial^2 G(x,a)}{\partial x^2} + \frac{2\partial G(x,a)}{\partial x} + 2G(x,a) = \delta(x-a), \qquad G(0,a) = 0, \qquad \frac{\partial G}{\partial x}(0,a) = 0$$

where $\delta(x-a)$ is the Dirac Delta function such that $\int_{-\infty}^{\infty} \delta(x-a)f(x)dx = f(x)$. Note that $G(x,a)$ is continuous at $x = a$ and $\partial G/\partial x$ has a jump discontinuity of magnitude 1 at $x = a$.

If y_1 and y_2 are linearly independent solutions of the homogeneous equation $Ly = 0$, then

$$G(x,a) = \begin{cases} A_1 y_1 + A_2 y_2 & \text{if } x < a \\ B_1 y_1 + B_2 y_2 & \text{if } x > a \end{cases}$$

where A_1, A_2, B_1, and B_2 are undetermined functions. The continuity of $G(x,a)$ at $x = a$ gives the equation

$$A_1 y_1(a) + A_2 y_2(a) = B_1 y_1(a) + B_2 y_2(a).$$

Further, the fact that $\partial G/\partial x$ has a jump discontinuity of magnitude 1 at $x = a$ yields the second equation

$$(B_1 y_1'(a) + B_2 y_2'(a)) - (A_1 y_1'(a) + A_2 y_2'(a)) = 1.$$

Combining these equations, we see that A_1, A_2, B_1, and B_2 are given by

$$B_1 = A_1 - \frac{y_2(a)}{W[y_1(a), y_2(a)]}$$
$$B_2 = A_2 + \frac{y_1(a)}{W[y_1(a), y_2(a)]}$$

From (1), we know that the linearly independent solutions to the homogeneous equation $Ly = 0$ are $y_1(x) = e^{-x}\sin x$ and $y_2(x) = e^{-x}\cos x$. Also, the Wronskian of these solutions is $W[y_1(a), y_2(a)] = -e^{-2a}$. Thus,

$$B_1 = A_1 - \frac{y_2(a)}{W[y_1(a), y_2(a)]} = A_1 + e^a \cos a$$
$$B_2 = A_2 + \frac{y_1(a)}{W[y_1(a), y_2(a)]} = A_2 - e^a \sin a$$

Using the two initial conditions, we can uniquely determine A_1 and A_2 since $G(x,a) = A_1 y_1(a) + A_2 y_2(a)$ satisfies $LG = f(x)$. Since $y(0) = 0$ we see that $A_2 = 0$ and since $y'(0) = 0$ we see that $A_1 - A_2 = 0$ implying that $A_1 = A_2 = 0$. Therefore, the Green function for the IVP $Ly = f(x)$ is

$$G(x,a) = \begin{cases} 0 & \text{if } x < a \\ e^{a-x}(\sin x \cos a - \sin a \cos x) & \text{if } x > a \end{cases} \tag{6}$$

□

Problem 3. Use your answer to Problem 2 to solve the IVP

$$y'' + 2y' + 2y = \sin x, \quad y(0) = y'(0) = 0.$$

Solution. □

Problem 4. Show that if y_1, y_2, and y_3 are three linearly independent solutions of the linear ODE

$$y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = 0$$

and u_1, u_2, u_3 are solutions of the system

$$\left\{ \begin{array}{ccccccc} u_1'y_1 & + & u_2'y_2 & + & u_3'y_3 & = & 0, \\ u_1'y_1' & + & u_2'y_2' & + & u_3'y_3' & = & 0, \\ u_1'y_1'' & + & u_2'y_2'' & + & u_3'y_3'' & = & f(x), \end{array} \right. \tag{7}$$

then the function $u = u_1y_1 + u_2y_2 + u_3y_3$ is a solution of

$$Ly = y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = f(x)$$

Solution. We wish to show that $y = \sum_{i=1}^3 u_iy_i$ is a solution of the equation $Ly = f(x)$ given that y_i are linearly independent solutions of the homogeneous equation $Ly = 0$ and u_i are solutions of the system (7). Using the form $y = \sum_{i=1}^3 u_iy_i$, we see that

$$\begin{aligned} y' &= \sum_{i=1}^3 u_iy_i' + u_i'y_i \\ y'' &= \sum_{i=1}^3 u_iy_i'' + 2u_i'y_i' + u_i''y_i \\ y''' &= \sum_{i=1}^3 u_iy_i''' + 3u_i'y_i'' + 3u_i''y_i' + u_i'''y_i. \end{aligned}$$

Thus, we find that for $y = \sum_{i=1}^3 u_iy_i$,

$$\begin{aligned} Ly = \sum_{i=1}^3 u_iy_i''' + 3u_i'y_i'' + 3u_i''y_i' + u_i'''y_i + p_2(x)\sum_{i=1}^3 u_iy_i'' + 2u_i'y_i' + u_i''y_i \\ + p_1(x)\sum_{i=1}^3 u_iy_i' + u_i'y_i + p_0(x)\sum_{i=1}^3 u_iy_i \\ = \sum_{i=1}^3 u_i[y_i''' + p_2(x)u_i'' + p_1(x)y_i' + p_0(x)y_i] \\ + \sum_{i=1}^3 3u_i'y_i'' + 3u_i''y_i' + u_i'''y_i + 2p_2(x)u_i'y_i' + p_2(x)u_i''y_i + p_1(x)u_i'y_i. \end{aligned}$$

Since y_i are solutions of the homogeneous equation $Ly = 0$, we see that the first sum is 0 and

$$Ly = \sum_{i=1}^3 3u_i'y_i'' + 3u_i''y_i' + u_i'''y_i + 2p_2(x)u_i'y_i' + p_2(x)u_i''y_i + p_1(x)u_i'y_i. \tag{8}$$

We also know that since u_1, u_2, and u_3 are solutions of the system (7) the following implications are true

$$\sum_{i=1}^{3} u_i' y_i = 0 \implies \sum_{i=1}^{3} u_i'' y_i + u_i' y_i' = 0$$
$$\sum_{i=1}^{3} u_i'' y_i + u_i' y_i' = 0 \implies \sum_{i=1}^{3} u_i''' y_i + 2u_i'' y_i' + u_i' y_i'' = 0$$
$$\sum_{i=1}^{3} u_i' y_i' = 0 \implies \sum_{i=1}^{3} u_i'' y_i' + u_i' y_i'' = 0$$

Rearranging the terms of (8) and using the above relations we see that

$$\begin{aligned} Ly &= \sum_{i=1}^{3} u_i' y_i'' + [u_i''' y_i + 2u_i'' y_i' + u_i' y_i''] + [u_i' y_i'' + u_i'' y_i'] \\ &\quad + \sum_{i=1}^{3} p_2(x)[u_i' y_i' + u_i'' y_i] + p_2(x)[u_i' y_i'] + p_1(x)[u_i' y_i] \\ &= \sum_{i=1}^{3} u_i' y_i'' \end{aligned}$$

where every term in brackets is 0 as a consequence of the above derived relations or the fact that u_1, u_2, and u_3 are solutions of the system (7). From the third equation of the system (7) we know that $\sum_{i=1}^{3} u_i' y_i'' = f(x)$. Therefore, for $y = \sum_{i=1}^{3} u_i y_i$ satisfying the assumptions of the problem,

$$Ly = \sum_{i=1}^{3} u_i' y_i'' = f(x)$$

showing that y is a solution of the equation $Ly = f(x)$. $\square$

Problem 5. Find the eigenvalues and the respective eigenfunctions for the BVP

$$x^2y'' + xy' + \lambda y = 0, \quad y'(1) = 0, \quad y'(b) = 0$$

where $b > 1$.

Solution. $\square$