Homework Assignment 4

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Problem 3.1. Find the Laplace transforms of the following functions:

b. $f(t) = (1 - 2t)e^{-2t}$

c. $f(t) = t\cos at$

d. $f(t) = t^{3/2}$

g. $f(t) = (t-3)^2 H(t-3)$

Solution. Recall that the Laplace transform of the function $f(t)$ defined for $t > 0$ is given by

$$\mathscr{L}\{f(t)\} = \bar{f}(s) = \int_0^\infty f(t)e^{-st}dt. \tag{1}$$

b. Let $g(t) = 1 - 2t$. Then $f(t) = (1-2t)e^{-2t} = g(t)e^{-2t}$. From the definition of the Laplace transform, we have that

$$\begin{aligned}\mathscr{L}\{g(t)\} = \bar{g}(s) &= \int_0^\infty (1-2t)e^{-st}dt \\ &= \int_0^\infty t^0 e^{-st}dt - 2\int_0^\infty t^1 e^{-st}dt \\ &= \mathscr{L}\left\{t^0\right\} - 2\mathscr{L}\left\{t^1\right\}.\end{aligned}$$

From a previous theorem, we know for $n \in \mathbb{N}$ that

$$\mathscr{L}\{t^n\} = \int_0^\infty t^n e^{-st}dt = \frac{n!}{s^{n+1}}.$$

Thus,

$$\bar{g}(s) = \mathscr{L}\left\{t^0\right\} - 2\mathscr{L}\left\{t^1\right\} = \frac{1}{s} - \frac{2}{s^2} = \frac{s-2}{s^2}.$$

From Heaviside's First Shifting Theorem, we know that for $\bar{g}(s) = \mathscr{L}\{g(t)\}$ that

$$\mathscr{L}\left\{g(t)e^{-at}\right\} = \bar{g}(s+a).$$

Therefore, the Laplace transform of $f(t) = (1-2t)e^{-2t} = g(t)e^{-2t}$ is

$$\mathscr{L}\{f(t)\} = \mathscr{L}\left\{g(t)e^{-2t}\right\} = \bar{g}(s+2) = \frac{s}{(s+2)^2}.$$

c. From the definition of the complex exponential, we have that $f(t) = t\cos at = \frac{t}{2}\left(e^{-iat} + e^{iat}\right)$. From the definition of the Laplace transform, we have that

$$\begin{aligned}\mathscr{L}\left\{f(t)\right\} = \bar{f}(s) &= \int_0^\infty \frac{t}{2}\left(e^{-iat} + e^{iat}\right)e^{-st}dt \\ &= \frac{1}{2}\left[\int_0^\infty te^{-(s+ia)t}dt + \int_0^\infty te^{-(s-ia)t}dt\right].\end{aligned}$$

We readily see by integrating by parts using $u = t$ and $dv = e^{-(s\pm ia)t}dt$ that

$$\begin{aligned}\int_0^\infty te^{-(s\pm ia)t}dt &= -\frac{t}{s\pm ia}e^{-(s\pm ia)t}\bigg|_0^\infty + \frac{1}{s\pm ia}\int_0^\infty e^{-(s\pm ia)t}dt \\ &= -\frac{1}{(s\pm ia)^2}e^{-(s\pm ia)t}\bigg|_0^\infty \\ &= \frac{1}{(s\pm ia)^2}.\end{aligned}$$

Therefore, the Laplace transform of $f(t)$ is given by

$$\begin{aligned}\mathscr{L}\left\{f(t)\right\} = \bar{f}(s) &= \frac{1}{2}\left[\int_0^\infty te^{-(s+ia)t}dt + \int_0^\infty te^{-(s-ia)t}dt\right] \\ &= \frac{1}{2}\left[\frac{1}{(s+ia)^2} + \frac{1}{(s-ia)^2}\right] \\ &= \frac{s^2 - a^2}{(s+ia)^2(s-ia)^2} \\ &= \frac{s^2 - a^2}{(s^2+a^2)^2}.\end{aligned}$$

d. By definition, the Laplace transform of $f(t)$ is given by

$$\mathscr{L}\left\{f(t)\right\} = \bar{f}(s) = \int_0^\infty t^{3/2}e^{-st}dt.$$

Let $u = st$, then $du/s = dt$ and

$$\begin{aligned}\mathscr{L}\left\{f(t)\right\} = \bar{f}(s) &= \frac{1}{s}\int_0^\infty \left(\frac{u}{s}\right)^{3/2} e^{-u}dt \\ &= \frac{1}{s^{5/2}}\int_0^\infty u^{3/2}e^{-u}dt.\end{aligned}$$

Recall that the definition of the Gamma function is given by

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}du.$$

Therefore, the Laplace transform of $f(t) = t^{3/2}$ is

$$\begin{aligned}\mathscr{L}\{f(t)\} = \bar{f}(s) &= \frac{1}{s^{5/2}} \int_0^\infty u^{5/2-1} e^{-u} dt \\ &= \frac{\Gamma\left(\frac{5}{2}\right)}{s^{5/2}}.\end{aligned}$$

g. Let $g(t) = t^2$ and suppose that $\mathscr{L}\{g(t)\} = \bar{g}(s)$. Then Heaviside's Second Shifting Theorem shows that

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{g(t-3)H(t-3)\} = e^{-3s}\bar{g}(s).$$

As shown previously, we know for $n \in \mathbb{N}$ that

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}.$$

Therefore, the Laplace transform of $f(t)$ is

$$\mathscr{L}\{f(t)\} = \bar{f}(s) = e^{-3s}\bar{g}(s) = \frac{2e^{-3s}}{s^3}.$$

$\square$

Problem 3.3. The following is a result relating the Laplace transform of a function's derivative to the Laplace transform of that function:

$$\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - f(0). \tag{2}$$

Use the result to find

a. $\mathscr{L}\{\cos at\}$,

b. $\mathscr{L}\{\sin at\}$.

Solution. a. Let $f(t) = \cos at$. Then $f'(t) = -a\sin at$ and from (2) we have

$$-a\mathscr{L}\{\sin at\} = s\mathscr{L}\{\cos at\} - 1. \tag{3}$$

Now let $g(t) = \sin at$. Then $g'(t) = a\cos at$ and applying (2) to $g(t)$ yields

$$a\mathscr{L}\{\cos at\} = s\mathscr{L}\{\sin at\}.$$

Therefore, from (3) we have that

$$-a\left(\frac{a}{s}\mathscr{L}\{\cos at\}\right) = s\mathscr{L}\{\cos at\} - 1$$

which implies that

$$\mathscr{L}\{\cos at\} = \frac{s}{s^2 + a^2}.$$

b. Let $f(t) = \sin at$. Then $f'(t) = a\cos at$ and from (2) we have

$$a\mathscr{L}\{\cos at\} = s\mathscr{L}\{\sin at\}. \tag{4}$$

Now let $g(t) = \cos at$. Then $g'(t) = -a\sin at$ and applying (2) to $g(t)$ yields

$$-a\mathscr{L}\{\sin at\} = s\mathscr{L}\{\cos at\} - 1$$

which implies that

$$\mathscr{L}\{\cos at\} = \frac{1}{s} - \frac{a}{s}\mathscr{L}\{\sin at\}.$$

Therefore, from (4) we have that

$$a\left(\frac{1}{s} - \frac{a}{s}\mathscr{L}\{\sin at\}\right) = s\mathscr{L}\{\sin at\}$$

which implies that

$$\mathscr{L}\{\sin at\} = \frac{a}{s^2 + a^2}.$$

□

Problem 3.6. Show that
$$\mathscr{L}\left\{\int_0^t \frac{f(u)}{u}du\right\} = \frac{1}{s}\int_s^\infty \bar{f}(x)dx.$$

Solution. From the definition of the Laplace transform we see that
$$\mathscr{L}\left\{\int_0^t \frac{f(u)}{u}du\right\} = \int_0^\infty e^{-st}\left[\int_0^t \frac{f(u)}{u}du\right]dt.$$
Interchanging the order of integration from u to t where $0 \leq t < \infty$, we see that $u \leq t < \infty$ as $0 \leq u < \infty$ and
$$\begin{aligned}\mathscr{L}\left\{\int_0^t \frac{f(u)}{u}du\right\} &= \int_0^\infty e^{-st}\left[\int_0^t \frac{f(u)}{u}du\right]dt \\ &= \int_0^\infty \frac{f(u)}{u}\left[\int_u^\infty e^{-st}dt\right]du \\ &= \frac{1}{s}\int_0^\infty \frac{f(u)}{u}e^{-su}du.\end{aligned}$$

We note that $\dfrac{d}{ds}\left[\dfrac{e^{-su}}{u}\right] = -e^{-su}$ so that in particular we have that
$$-\int_s^\infty e^{-su}ds = -\frac{e^{-su}}{u}\bigg|_s^\infty = -\frac{e^{-su}}{u}$$
or that
$$\int_s^\infty e^{-su}ds = \frac{e^{-su}}{u}.$$
Thus,
$$\begin{aligned}\mathscr{L}\left\{\int_0^t \frac{f(u)}{u}du\right\} &= \frac{1}{s}\int_0^\infty \frac{f(u)}{u}e^{-su}du \\ &= \frac{1}{s}\int_0^\infty f(u)\left[\int_s^\infty e^{-su}ds\right]du.\end{aligned}$$

Interchanging the order of integration yet again from s to u where $s \leq s < \infty$ as $0 \leq u < \infty$, we see that the integration limits remain unchanged and therefore that
$$\begin{aligned}\mathscr{L}\left\{\int_0^t \frac{f(u)}{u}du\right\} &= \frac{1}{s}\int_0^\infty f(u)\left[\int_s^\infty e^{-su}ds\right]du \\ &= \frac{1}{s}\int_s^\infty\left[\int_0^\infty f(u)e^{-su}du\right]ds \\ &= \frac{1}{s}\int_s^\infty \bar{f}(s)ds \\ &= \frac{1}{s}\int_s^\infty \bar{f}(x)dx,\end{aligned}$$
and we are done.

□

Problem 3.7. Obtain the inverse Laplace transforms of the following functions:

b. $\bar{f}(s) = \dfrac{1}{s^2(s^2+c^2)}$.

Solution. b. Let $\bar{f}(s) = \bar{g}(s)\bar{h}(s)$ where $\bar{g}(s) = \dfrac{1}{s^2}$ and $\bar{h}(s) = \frac{1}{s^2+c^2}$.

Using previous results, we know that

$$g(t) = \mathscr{L}^{-1}\{\bar{g}(s)\} = \mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

and

$$h(t) = \mathscr{L}^{-1}\left\{\bar{h}(s)\right\} = \mathscr{L}^{-1}\left\{\frac{1}{c}\left(\frac{c}{s^2+c^2}\right)\right\} = \frac{\sin ct}{c}.$$

Now, by the Convolution Theorem for the Laplace transform, we have that

$$f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} = \mathscr{L}^{-1}\left\{\bar{g}(s)\bar{h}(s)\right\} = (g*h)(t)$$

where

$$(g*h)(s) = \int_0^t g(t-\tau)h(\tau)d\tau.$$

Therefore,

$$\begin{aligned} f(t) = (g*h)(t) &= \frac{1}{c}\int_0^t (t-\tau)\sin c\tau d\tau \\ &= \frac{t}{c}\int_0^t \sin c\tau d\tau - \frac{1}{c}\int_0^t \tau \sin c\tau d\tau \\ &= \frac{t}{c}\left[\frac{1}{c} - \frac{\cos ct}{c}\right] - \frac{1}{c}\left[-\frac{t\cos ct}{c} + \frac{\sin ct}{c^2}\right] \\ &= \frac{t}{c^2} - \frac{\sin ct}{c^3}. \end{aligned}$$

□

Problem 3.8. Use the Convolution Theorem to fin the inverse Laplace transforms of the following functions:

a. $\bar{f}(s) = \dfrac{s^2}{(s^2+a^2)^2}$.

Solution. Let $\bar{f}(s) = \bar{g}(s)^2$ where $\bar{g}(s) = \dfrac{s}{s^2+a^2}$. Then we know that

$$g(t) = \mathscr{L}^{-1}\left\{\bar{g}(s)\right\} = \cos at.$$

From the Convolution Theorem, we then have that

$$f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} = \mathscr{L}^{-1}\left\{\bar{g}(s)\bar{g}(s)\right\} = (g * g)(t).$$

Therefore,

$$\begin{aligned} f(t) = (g*g)(t) &= \int_0^t \cos a(t-\tau)\cos a\tau d\tau \\ &= \cos at \int_0^t \cos^2 a\tau d\tau + \sin at \int_0^t \sin a\tau \cos a\tau d\tau \\ &= \cos at\left[\frac{2at + \sin 2at}{4a}\right] + \sin at\left[\frac{\sin^2 at}{2a}\right] \\ &= \frac{at\cos at + \sin at}{2a}. \end{aligned}$$

□

Problem 3.10.

Solution. □

Problem 3.12.

Solution. □

Problem 3.15.

Solution. □

Problem 3.18.

Solution. □