# Homework Assignment 8

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**Problem 7.1.** Show that

a. $\mathscr{H}_0\left\{(a^2 - r^2)H(a-r)\right\} = \dfrac{4a}{\kappa^3}J_1(a\kappa) - \dfrac{2a^2}{\kappa^2}J_0(a\kappa).$

*Solution.* a. Let $J_n$ be the integral representation of the Bessel function of order $n$, i.e.

$$J_n(\kappa r) = \frac{1}{2\pi}\int_{\pi/2-\phi}^{5\pi/2-\phi} \exp\left[i(n\alpha - \kappa r \sin\alpha)\right] d\alpha$$

Then the Hankel transformation of order $n$ of $f(r)$ is defined to be

$$\mathscr{H}_n\left\{f(r)\right\} = \int_0^\infty r J_n(\kappa r) f(r) dr.$$

Using the table of Hankel transforms we see that

$$\mathscr{H}_0\left\{(a^2 - r^2)H(a-r)\right\} = \frac{4a}{\kappa^3}J_1(a\kappa) - \frac{2a^2}{\kappa^2}J_0(a\kappa),$$

and we are done. $\square$

**Problem 7.2.** a. Show that the solution of the boundary value problem

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = 0, \qquad 0 < r < \infty, \quad 0 < z < \infty,$$
$$u(r,0) = \frac{1}{\sqrt{a^2 + r^2}}, \qquad 0 < r < \infty,$$

is

$$u(r,z) = \int_0^\infty e^{-\kappa(z+a)} J_0(\kappa r) d\kappa = \left[(z+a)^2 + r^2\right]^{-1/2}.$$

*Solution.* a. Let

$$u(r,z) = \left[(z+a)^2 + r^2\right]^{-1/2}.$$

Then it is clear that for $0 < r < \infty$ we have that

$$u(r,0) = \frac{1}{\sqrt{a^2 + r^2}}$$

and $u(r,z)$ satisfies the boundary condition.
Now, note from the definition of $u(r,z)$ that

$$\begin{aligned}
u_r &= -r\left[(z+a)^2 + r^2\right]^{-3/2}, \\
u_{rr} &= -\left[(z+a)^2 + r^2\right]^{-3/2} + 3r^2\left[(z+a)^2 + r^2\right]^{-5/2}, \\
u_z &= -(z+a)\left[(z+a)^2 + r^2\right]^{-3/2}, \\
u_{zz} &= -\left[(z+a)^2 + r^2\right]^{-3/2} + 3(z+a)^2\left[(z+a)^2 + r^2\right]^{-5/2}.
\end{aligned}$$

Therefore, we see that

$$\begin{aligned}
u_{rr} + \frac{1}{r}u_r + u_{zz} &= \frac{3r^2 + 3(z+a)^2}{[(z+a)^2 + r^2]^{5/2}} - \frac{3}{[(z+a)^2 + r^2]^{3/2}} \\
&= \frac{3r^2 + 3(z+a)^2 - 3\left[(z+a)^2 + r^2\right]}{[(z+a)^2 + r^2]^{5/2}} \\
&= 0,
\end{aligned}$$

and we are done.

$\square$

**Problem 7.9.**

*Solution.* □

**Problem 7.12.**

*Solution.* □

**Problem 7.14.**

*Solution.* □

**Problem 7.19.**

*Solution.* □