Homework Assignment 11

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Problem 14.2.2. Let $f_c : \mathbb{C} \to \mathbb{C}$, $f_c(z) = z^2 + c$ for $c \in \mathbb{C}$. Recall that a period n point z_0 is super attracting if $(f^n)'(z_0) = 0$.

i. If z_0 and z_1 are the fixed points of f_c, show that $f'(z_0) + f'(z_1) = 2$. Deduce that there can be at most one attracting fixed point. Give an example to show that f_c may not have any attracting fixed points.

ii. Show that if f_c has a super-attracting fixed point z_0, then $z_0 = 0$ and $c = 0$.

iii. Find the value of c such that f_c has a super-attracting 2-cycle and give the associated 2-cycle.

iv. Why is it that $z = 0$ is a point in the orbit of a cycle if and only if the cycle is super-attracting?

v. If f_c has a super-attracting 3-cycle, show that c satisfies the equation

$$c^3 + 2c + c + 1 = 0$$

Solution. i. If z_0 and z_1 are fixed points of f_c, then they are the roots of the equation $z^2 - z + c = 0$. Note that the solutions to this equation are of the form

$$z_0 = \frac{1 + \sqrt{1 - 4c}}{2}, \quad z_1 = \frac{1 - \sqrt{1 - 4c}}{2}. \tag{1}$$

Since $f_c'(z) = 2z$, we see from (1) that

$$f_c'(z_0) + f_c'(z_1) = 2(z_0 + z_1) = 2\left(\frac{1 + \sqrt{1 - 4c}}{2} + \frac{1 - \sqrt{1 - 4c}}{2}\right) = 2.$$

Suppose that $z_0 = r_0 e^{i\theta_0}$ is an attracting fixed point and let $z_1 = r_1 e^{i\theta_1}$. Since z_0 is attracting, we have that $|f_c'(z_0)| = 2|z_0| \leq 1$ which implies that $r_0 \leq 1/2$. Note that by the relation $f_c'(z_0) + f_c'(z_1) = 2$ we have that $z_1 = 1 - z_0 = 1 - r_0 e^{i\theta_0}$. Thus,

$$|z_1| = \left|1 - r_0 e^{i\theta_0}\right| \geq \left||1| - \left|r_0 e^{i\theta_0}\right|\right| = |1 - r_0|.$$

If $r_0 = 1/2$, then $z_1 = z_0$ and there is at most one fixed point, otherwise if $r_0 < 1/2$, then $|z_1| \geq |1 - r_0| > 1/2$ which implies that $|f_c'(z_1)| > 1$ or that z_1 is repelling.

If $c = 5/4$, then we see that $|f_c'(z_0)| = |f_c'(z_1)| = \sqrt{5}/2$ where $\sqrt{5}/2 > 1$. Thus, f_c may not have any attracting fixed points.

ii. Suppose that z_0 is a super attracting fixed point of f_c. Then $|f_c'(z_0)| = 2|z_0| = 0$. Since $|z_0| = 0$ if and only $z_0 = 0$, we readily see that $z_0 = 0$. Note that z_0 is of the form presented in (1). Thus, $(1 \pm \sqrt{1-4c})/2 = 0$ which implies that $1 - 4c = 1$ or that $c = 0$.

iii. The 2-cycles of f_c are solutions of the equation $f_c^2(z) - z = 0$ that are also not solutions of $f_c(z) - z = z^2 - z + c = 0$. Factoring $f_c^2(z) - z$ we see that

$$f_c^2(z) - z = (z^2 - z + c)(z^2 + z + c + 1) = 0$$

if and only if z is a fixed point or if

$$z_2 = \frac{-1 - \sqrt{-3-4c}}{2}, \quad z_3 = \frac{-1 + \sqrt{-3-4c}}{2}.$$

Thus, $\{z_2, z_3\}$ forms a 2-cycle of f_c. This 2-cycle will be super-attracting if and only if

$$\left|(f_c^2)'(z_2)\right| = |f_c'(z_2)f_c'(z_3)| = |\left(-1 - \sqrt{-3-4c}\right)\left(-1 + \sqrt{-3-4c}\right)| = 4|1+c| = 0$$

Thus, the 2-cycle is super-attracting if and only if $c = -1$. Therefore, the super attracting 2-cycle is $\{0, -1\}$.

iv. For an n-cycle $\{z_0, \ldots, z_{n-1}\}$ of f_c we see that

$$|(f_c^n)'(z_0)| = |f_c'(z_0) \cdots f_c'(z_{n-1})| = 2^n \left|z_0 \cdots z_{n-1}\right|. \tag{2}$$

Thus, from (2), we have that $\{z_0, \ldots, z_{n-1}\}$ is a super-attracting n-cycle of f_c if and only if $|(f_c^n)'(z_0)| = 0$ if and only if $z_i = 0$ for some $i = 0, \ldots, n-1$.

v. Suppose that $\{z_0, z_1, z_2\}$ is a super-attracting 3-cycle of f_c. Thus, we must have that

$$|f_c'(z_0)f_c'(z_1)f_c'(z_2)| = 2^3 \left|z_0 z_1 z_2\right| = 0$$

Without loss of generality, we may assume that $z_0 = 0$. Using the fact that $f_c(z_0) = z_1 \neq z_0$ and $f_c^2(z_0) = z_2 \neq z_0$, we see that

$$\begin{aligned} z_1 &= f_c(z_0) = z_0^2 + c = c \\ z_2 &= f_c^2(z_0) = (z_0^2 + c)^2 + c = c^2 + c \end{aligned}$$

In order for this to be a 3-cycle, we require that $f^3(z_0) = z_0 = 0$, i.e. we require that

$$\begin{aligned} f_c^3(z_0) = f_c(f_c^2(z_0)) &= (c^2 + c)^2 + c \\ &= c^4 + 2c^3 + c^2 + c \\ &= c(c^3 + 2c^2 + c + 1) = 0. \end{aligned}$$

However, we must have that $c \neq 0$ or $z_0 = 0$ would not generate a 3-cycle. Therefore, $\{z_0, z_1, z_2\}$ is a super-attracting 3-cycle if and only if $c^3 + 2c^2 + c + 1 = 0$.

□

Problem 14.3.1. Show that if $p(z)$ is a polynomial having degree at least 2, then $p(\infty) = \infty$. Use the definition of $p'(\infty)$ to show that $|p'(\infty)| < 1$ so that ∞ is an attracting point of p. What happens if $p(z) = a_1 z + a_0$ for some $a_1, a_0 \in \mathbb{C}$.

Solution. If $p(z)$ is a polynomial of degree at least 2, then p is conjugate to the polynomial q under the map $h(z) = 1/z$ where

$$q(z) = h^{-1} \circ p \circ h(z) = \frac{z^n}{a_0 z^n + a_1 z^{n-1} + \cdots + a_n}. \tag{3}$$

Since $q(0) = 0$, we see that 0 is a fixed point of q. Note that p and q are conjugate so that the map h preserves fixed points. Under h, we see that 0 maps to ∞ so that ∞ is a fixed point of p, i.e. $p(\infty) = \infty$.

From (3), we see that

$$|q'(z)| = \left| \frac{nz^{n-1} \sum_{k=0}^{n} a_k z^{n-k} - z^n \sum_{k=0}^{n} a_k (n-k) z^{n-k-1}}{\left(\sum_{k=0}^{n} a_k z^{n-k}\right)^2} \right|.$$

Thus, by definition, we have that $|p'(\infty)| = |q'(0)| = 0 < 1$ and ∞ is an attracting fixed point.

If $p(z) = a_1 z + a_0$ for some $a_1, a_0 \in \mathbb{C}$, then we see that p is conjugate to q under the map $h(z) = 1/z$ where

$$q(z) = \frac{z}{a_0 z + a_1}.$$

Since $q(0) = 0$, we know that $p(\infty) = \infty$ so that ∞ is a fixed point of p. However, we see that

$$q'(z) = \frac{(a_0 z + a_1) - a_0 z}{(a_0 z + a_1)^2} = \frac{a_1}{(a_0 z + a_1)^2}$$

which implies that $|p'(\infty)| = |q'(0)| = |a_1|$. Thus, ∞ will be attracting under p if and only if $|a_1| < 1$. $\square$

Problem 14.3.3. Let $p(z)$ be a polynomial of degree $n \geq 2$. Show that $z = \infty$ is a repelling fixed point for the Newton function $N_p(z)$, with $N_p'(\infty) = n/(n-1) > 1$. What happens if $p(z) = a_1 z + a_0$ for some $a_0, a_1 \in \mathbb{C}$?

Solution. Note that the Newton function of a polynomial $p(z)$ is given by

$$N_p(z) = z - \frac{p(z)}{p'(z)}.$$

Let $p(z) = \sum_{k=0}^{n} a_k z^k$. Note that N_p is conjugate to the polynomial q under $h(z) = 1/z$ where

$$\begin{aligned} q(z) = h^{-1} \circ N_p \circ h(z) &= \frac{1}{N_p(1/z)} \\ &= \frac{zp'(1/z)}{p'(1/z) - zp(1/z)} \\ &= \frac{z \sum_{k=1}^{n} k a_k z^{n-k}}{\sum_{k=1}^{n} k a_k z^{n-k} - z \sum_{k=0}^{n} a_k z^{n-k-1}} \\ &= \frac{\sum_{k=1}^{n} k a_k z^{n-k+1}}{\sum_{k=1}^{n} a_k (k-1) z^{n-k} - a_0 z^n} \end{aligned} \tag{4}$$

Note that $q(0) = 0$ so that 0 is a fixed point of q. Since q and N_p are conjugate, h maps fixed points of q to fixed points of N_p and since $h(0) = \infty$ we have $z = \infty$ is a fixed point of N_p, i.e. $N_p(\infty) = \infty$.

Using (4) we see that

$$\begin{aligned} q'(z) = &\frac{\left[\sum_{k=1}^{n} a_k (k-1) z^{n-k} - a_0 z^n\right] \left[\sum_{k=1}^{n} k(n-k+1) a_k z^{n-k}\right]}{\left[\sum_{k=1}^{n} a_k (k-1) z^{n-k} - a_0 z^n\right]^2} - \\ &\frac{\left[\sum_{k=1}^{n} k a_k z^{n-k+1}\right] \left[\sum_{k=1}^{n} a_k (k-1)(n-k) z^{n-k-1} - n a_0 z^{n-1}\right]}{\left[\sum_{k=1}^{n} a_k (k-1) z^{n-k} - a_0 z^n\right]^2}. \end{aligned}$$

In particular, we see that

$$q'(0) = \frac{n(n-1)a_n^2}{(n-1)^2 a_n^2} = \frac{n}{n-1}.$$

Thus, $|N_p'(\infty)| = |q'(0)| = n/(n-1) > 1$, since $n > 2$, and $z = \infty$ is a repelling fixed point.

If $p(z) = a_1 z + a_0$ for some $a_0, a_1 \in \mathbb{C}$, then

$$N_p(z) = z - \frac{a_1 z + a_0}{a_1} = -\frac{a_0}{a_1}.$$

Note that in this case $z = \infty$ is not a fixed point of N_p as the only fixed point of $N_p(z)$ is $-a_0/a_1$. $\square$