Homework Assignment 2

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September 19, 2016

Problem 1.4.1. Find the fixed points and determine their stability for the function

$$f(x) = \frac{6}{x} - 1.$$

Solution. The fixed points of the function $f(x)$ are the roots of the function

$$\begin{aligned} g(x) &= f(x) - x \\ &= \frac{6}{x} - 1 - x \\ &= -\frac{(x+3)(x+2)}{6}. \end{aligned}$$

We readily see that the roots of $g(x)$, which are the fixed points of $f(x)$, are given by $x = -3$ and $x = 2$.

According to Theorem 1.4.4, since $f(x)$ is a C^1 function, we may use the derivative of $f(x)$ to classify its fixed points. If c is a fixed point of f and $|f'(c)| < 1$, then c is an asymptotically stable fixed point, while $|f'(c)| > 1$ indicates that c is a repelling (unstable) fixed point.

Note that $f'(x) = -6/x^2$. For the fixed point $x = -3$, we see that

$$|f'(-3)| = \left| -\frac{6}{(-3)^2} \right| = \frac{2}{3} < 1$$

from which we classify the point $x = -3$ as an asymptotically stable fixed point. On the other hand, for the fixed point $x = 2$, we see that

$$|f'(2)| = \left| -\frac{6}{(2)^2} \right| = \frac{3}{2} > 1$$

from which we classify the point $x = -3$ as a repelling (unstable) fixed point. □

Problem 1.4.2. Let $f : \mathbb{R} \to \mathbb{R}$. If $f'(x)$ exists with $f'(x) \neq 1$ for all $x \in \mathbb{R}$, prove that f has at most one fixed point. (Hint: Use the Mean Value Theorem).

Solution. Suppose to the contrary that for all $x \in \mathbb{R}$ we have that $f'(x)$ exists with $f'(x) \neq 1$, but f has at least two distinct fixed points, c_1 and c_2, say. The Mean Value Theorem states that if a function g is continuous on an interval $[a, b]$ and differentiable on the interval (a, b), then there exists a point $c \in (a, b)$ such that

$$g'(c) = \frac{g(b) - g(a)}{b - a}.$$

By our supposition, we have that the function f is continuous and differentiable on any interval and, in particular, it is continuous on $[c_1, c_2]$ and differentiable on (c_1, c_2). By the Mean Value Theorem, there exists a point $c_3 \in (c_1, c_2)$ such that

$$f'(c_3) = \frac{f(c_2) - f(c_1)}{c_2 - c_1}. \tag{1}$$

However, since c_1 and c_2 are fixed points of f, we know that $f(c_2) - f(c_1) = c_2 - c_1$ and we gather from (1) that

$$f'(c_3) = \frac{f(c_2) - f(c_1)}{c_2 - c_1} = \frac{c_2 - c_1}{c_2 - c_1} = 1.$$

Note that this is in contradiction to our supposition that $f'(x) \neq 1$ for any $x \in \mathbb{R}$. Therefore, we must conclude that for a function $f : \mathbb{R} \to \mathbb{R}$, if for all $x \in \mathbb{R}$ we have that $f'(x)$ exists with $f'(x) \neq 1$, then f has at most one fixed point. $\square$

Problem 1.4.4. Let $S_\mu(x) = \mu \sin(x)$, $0 \le x \le 2\pi$, $0 < \mu \le \pi$ and $C_\mu(x) = \mu \cos(x)$, $-\pi \le x \le \pi$ and $-\pi \le \mu \le \pi$, $\mu \ne 0$.

i. Show that S_μ has a super-attracting fixed point at $x = \pi/2$, when $\mu = \pi/2$.

ii. Find the corresponding values for C_μ having a super-attracting fixed point.

Solution. Recall that if c is a fixed point of a differentiable function f, then c is a super-attracting fixed point if $f'(c) = 0$.

i. Suppose that $\mu = \pi/2$. Since $S_\mu(\pi/2) = (\pi/2)\sin(\pi/2) = \pi/2$, we readily see that if $\mu = \pi/2$, then $x = \pi/2$ is a fixed point of $S_\mu(x)$. Note that $S'_\mu(x) = \mu\cos(x)$. From this we gather that if $\mu = \pi/2$, then for the fixed point $x = \pi/2$, we have that $S'_\mu(x) = (\pi/2)\cos(\pi/2) = 0$. Therefore, the fixed point $x = \pi/2$ is a super-attracting fixed point.

ii. We now investigate the super-attracting fixed points of $C_\mu(x)$. The definition of $C_\mu(x)$ shows that $C'_\mu(x) = -\mu\sin(x)$ from which we can gather that $C'_\mu(x) = 0$ for $x \in [-\pi, \pi]$ if $x = k\pi$ for $k \in \{-1, 0, 1\}$. Note that these are the possible super-attracting fixed points of $C_\mu(x)$, we must still determine which of these possible super-attracting fixed points are indeed fixed points, i.e. we must determine which points satisfy $C_\mu(x) = x$. If $x = k\pi$ for $k \in \{-1, 0, 1\}$, then

$$C_\mu(k\pi) = \mu\cos(k\pi) = (-1)^k\mu.$$

Thus, $C_\mu(k\pi) = (-1)^k\mu = k\pi$, if $\mu = (-1)^k k\pi$. Therefore, if $x, \mu \in [-\pi, \pi]$ with $\mu \ne 0$, then the points $x_1 = -\pi$ and $x_2 = \pi$, with corresponding μ-values $\mu_1 = \pi$ and $\mu_2 = -\pi$, are super-attracting fixed points. Note that $x = 0$ is not a super-attracting fixed point since it is not a fixed point, that is $C_\mu(0) = 0$ only if $\mu = 0$, which violates our initial conditions.

□

Problem 1.4.7. Let N_f be the Newton function of the map $f(x) = x^2 + 1$. Clearly there are no fixed points of the Newton function as there are no zeros of f. Show that there are points c where $N_f^2(c) = c$ (called *period 2-points* of N_f).

Solution. The Newton function for a function f is defined by

$$N_f(x) = x - \frac{f(x)}{f'(x)}.$$

Thus, for the function $f(x) = x^2 + 1$, we have that

$$\begin{aligned} N_f(x) &= x - \frac{f(x)}{f'(x)} \\ &= x - \frac{x^2+1}{2x} \\ &= \frac{x^2-1}{2x}. \end{aligned}$$

Using this definition of $N_f(x)$, we can readily see that

$$\begin{aligned} N_f^2(x) = N_f(N_f(x)) &= \frac{\left(\frac{x^2-1}{2x}\right)^2 - 1}{2\left(\frac{x^2-1}{2x}\right)} \\ &= \frac{x^4 - 2x^2 - 4x^2 + 1}{4x^2} \cdot \frac{x}{x^2-1} \\ &= \frac{x^4 - 6x^2 + 1}{4x(x^2-1)}. \end{aligned}$$

The points x such that $N_f^2(x) = x$ are the solutions to the equation

$$N_f^2(x) - x = \frac{x^4 - 6x^2 + 1}{4x(x^2-1)} - x = \frac{-3x^4 - 2x^2 + 1}{4x(x^2-1)} = 0.$$

We readily see that $x = \pm 3^{-1/2}$ are the real solutions to the above equation. Therefore, $x = \pm 3^{-1/2}$ satisfy $N_f^2(x) = x$ and are period 2-points. $\square$

Problem 1.4.8. i. Suppose that $f(c) = f'(c) = 0$ and $f''(c) \neq 0$. If $f''(x)$ is continuous at $x = c$, show that the Newton function $N_f(x)$ has a removable discontinuity at $x = c$. (Hint: Apply L'Hopital's rule to N_f at $x = c$.)

ii. If in addition, $f'''(x)$ is continuous at $x = c$ with $f'''(c) \neq 0$, show that $N_f'(c) = 1/2$, so that $x = c$ is not a super-attracting fixed point in this case.

iii. Check the above for the function $f(x) = x^3 - x^2$ with $c = 0$.

Solution. i. Suppose that $\lim_{x\to c} N_f(x) = L$. The function $N_f(x)$ will have a removable discontinuity at $x = c$ if $N_f(c)$ is undefined or $N_f(c) \neq L$ and the following function defined as

$$F(x) := \begin{cases} N_f(x) & \text{if } x \neq c \\ L & \text{if } x = c \end{cases} \tag{2}$$

is continuous everywhere. Throughout, we assume without loss of generality that $x = c$ is the only point such that $f'(x) = 0$.

Note that if $x_0 \neq c$, then the differentiability of f' implies that both f and f' are continuous so that

$$\lim_{x\to x_0} N_f(x) = \lim_{x\to x_0} x - \frac{f(x)}{f'(x)} = x_0 - \frac{f(x_0)}{f'(x_0)} = N_f(x_0),$$

i.e. the function $N_f(x)$ is continuous.

On the other hand, at the point $x = c$, we see from the continuity of f and f' that

$$\begin{aligned} \lim_{x\to c} N_f(x) &= \lim_{x\to c} x - \frac{f(x)}{f'(x)} \\ &= \lim_{x\to c} \frac{xf'(x) - f(x)}{f'(x)} \\ &= \frac{cf'(c) - f(c)}{f'(c)}. \end{aligned}$$

From our supposition that $f(c) = f'(c) = 0$, we see that $cf'(c) - f(c) = 0$ and $f'(c) = 0$ implying that $\lim_{x\to c} N_f(x) = 0/0$, an indeterminate form. By L'Hospital's rule in conjunction with our supposition that f'' is continuous and $f''(c) \neq 0$, since $\lim_{x\to c} xf'(x) - f(x) = \lim_{x\to c} f'(x) = 0$, we know that

$$\begin{aligned} \lim_{x\to c} N_f(x) &= \lim_{x\to c} \frac{xf'(x) - f(x)}{f'(x)} \\ &= \lim_{x\to c} \frac{[xf'(x) - f(x)]'}{[f'(x)]'} \\ &= \lim_{x\to c} \frac{xf''(x)}{f''(x)} = c. \end{aligned}$$

Therefore, the function

$$F(x) := \begin{cases} N_f(x) & \text{if } x \neq c \\ c & \text{if } x = c \end{cases}$$

is continuous everywhere and $x = c$ is a removable discontinuity for the function $N_f(x)$.

ii. Suppose that $f'''(x)$ is continuous at $x = c$ with $f'''(c) \neq 0$. From the definition of $N_f(x)$, we see that

$$N_f'(x) = \left[\frac{xf'(x) - f(x)}{f(x)}\right]' = \frac{f(x)f''(x)}{f'(x)^2}.$$

Thus, we have from our initial assumptions that

$$\lim_{x\to c} N_f'(x) = \lim_{x\to c} \frac{f(x)f''(x)}{f'(x)^2} = \frac{0}{0},$$

an indeterminate form. The assumptions to use L'Hospital's rule are met and we see from the continuity of f and its derivatives combined with our initial suppositions that

$$\begin{aligned} \lim_{x\to c} N_f'(x) &= \lim_{x\to c} \frac{[f(x)f''(x)]'}{[f'(x)^2]'} \\ &= \lim_{x\to c} \frac{f'(x)f''(x) + f(x)f'''(x)}{2f'(x)f''(x)} = \frac{0}{0}, \end{aligned}$$

another indeterminate form. Applying L'Hospital's rule one last time we see that

$$\begin{aligned} \lim_{x\to c} N_f'(x) &= \lim_{x\to c} \frac{[f'(x)f''(x) + f(x)f'''(x)]'}{[2f'(x)f''(x)]'} \\ &= \lim_{x\to c} \frac{f''(x)^2 + 2f'(x)f'''(x) + f(x)f^{(4)}(x)}{2f''(x)^2 + 2f'(x)f'''(x)} = \frac{1}{2}. \end{aligned}$$

Therefore, since $|N_f'(c)| \neq 0$, the point $x = c$ is not a super-attracting fixed point.

iii. Let $f(x) = x^3 - x^2$. Note that $f'(x) = 3x^2 - 2x$, $f''(x) = 6x - 2$, and $f'''(x) = 6$, all of which are continuous. Further, we know that

$$N_f(x) = x - \frac{f(x)}{f'(x)} = \frac{2x^2 - x}{3x - 2}.$$

It is clear that the point $c = 0$ is a fixed point of the function f. Note that $f(c) = f'(c) = 0$ with $f''(c) \neq 0$ and $f'''(c) \neq 0$. However, the point $x = c$ is not a removable discontinuity since $\lim_{x\to c} N_f(x) = 0 = N_f(c)$, i.e. the Newton function is actually continuous at $x = c$ so there is no discontinuity to remove. Since $|N_f'(c)| = 1/2$, the result that $c = 0$ is not a super-attracting fixed point still holds.

$\square$