Homework Assignment 2

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February 21, 2016

Problem 1. Use the method of variation of parameters to find the general solution of

$$y'' + 2y' + 2y = \sin x.$$

Solution. Suppose that $Ly = y'' + 2y' + 2y$. The general solution to $Ly = \sin x$ is given by $y = y_0 + y_h$ where y_0 is a particular solution of $Ly = \sin x$ and y_h is the solution to the homogeneous equation $Ly = 0$.

The characteristic equation of the equation $Ly = 0$ is $m(x) = x^2 + 2x + 2$, the roots of which are $m_1 = -1 - i$ and $m_2 = -1 + i$. As the roots of the characteristic equation are complex, the solution to $Ly = 0$ is given by

$$y_h = c_1 e^{-x} \sin x + c_2 e^{-x} \cos x. \tag{1}$$

The method of variation of parameters can be used to find a particular solution y_0. We wish to find functions $u_1(x), u_2(x)$ such that

$$y_0 = u_1(x) y_1(x) + u_2(x) y_2(x) \tag{2}$$

satisfies $Ly_0 = \sin x$ where $y_1(x)$ and $y_2(x)$ are solutions to the homogeneous equation $Ly = 0$. If the functions $u_1(x)$ and $u_2(x)$ are solutions to the system

$$\left\{ \begin{array}{ccccc} u_1' y_1 & + & u_2' y_2 & = & 0 \\ u_1' y_1' & + & u_2' y_2' & = & \sin x \end{array} \right. \tag{3}$$

then (2) will satisfy the original differential equation $Ly = \sin x$ equation. The solution to the system (3) is

$$u_1(x) = -\int \frac{y_2(x) \sin x}{W[\{y_1, y_2\}]} dx \qquad u_2(x) = \int \frac{y_1(x) \sin x}{W[\{y_1, y_2\}]} dx \tag{4}$$

where $W[\{y_1, y_2\}]$ is the Wronskian of the functions y_1 and y_2.

Using (1), we know that $y_1(x) = e^{-x} \sin x$ and $y_2(x) = e^{-x} \cos x$ so the particular solution has the form $y_0 = u_1(x) e^{-x} \sin x + u_2(x) e^{-x} \cos x$. Further, the Wronskian of y_1 and y_2 is

$$W[\{y_1, y_2\}] = \begin{vmatrix} e^{-x} \sin x & e^{-x} \cos x \\ e^{-x} \cos x - e^{-x} \sin x & -e^{-x} \cos x - e^{-x} \sin x \end{vmatrix} = -e^{-2x}.$$

Thus, using (4), we know that

$$\begin{aligned} u_1(x) &= -\int \frac{y_2(x)\sin x}{W[\{y_1, y_2\}]}dx \\ &= \int \frac{e^{-x}\cos x \sin x}{e^{-2x}}dx \\ &= \frac{e^x}{10}\left(-2\cos 2x + \sin 2x\right) + C \end{aligned}$$

and

$$\begin{aligned} u_2(x) &= \int \frac{y_1(x)\sin x}{W[\{y_1, y_2\}]}dx \\ &= -\int \frac{e^{-x}\sin^2 x}{e^{-2x}}dx \\ &= \frac{e^x}{10}\left(-5 + \cos 2x + 2\sin 2x\right) + C. \end{aligned}$$

Therefore, a particular solution to $Ly = \sin x$ is

$$y_0(x) = \frac{1}{10}\left(-2\cos 2x + \sin 2x\right)\sin x + \frac{1}{10}\left(-5 + \cos 2x + 2\sin 2x\right)\cos x$$

and the general solution to $Ly = \sin x$ is

$$\begin{aligned} y(x) &= y_0(x) + y_h(x) \\ &= \frac{1}{10}\left(-2\cos 2x + \sin 2x\right)\sin x + \frac{1}{10}\left(-5 + \cos 2x + 2\sin 2x\right)\cos x \\ &\quad + c_1 e^{-x}\sin x + c_2 e^{-x}\cos x \end{aligned} \tag{5}$$

$\square$

Problem 2. Find the Green function of the IVP

$$y'' + 2y' + 2y = f(x), \quad y(0) = y'(0) = 0.$$

Solution. $\square$

Problem 3. Use your answer to Problem 2 to solve the IVP

$$y'' + 2y' + 2y = \sin x, \quad y(0) = y'(0) = 0.$$

Solution. $\square$

Problem 4. Show that if y_1, y_2, and y_3 are three linearly independent solutions of the linear ODE

$$y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = 0$$

and u_1, u_2, u_3 are solutions of the system

$$\begin{cases} u_1'y_1 & + & u_2'y_2 & + & u_3'y_3 & = & 0, \\ u_1'y_1' & + & u_2'y_2' & + & u_3'y_3' & = & 0, \\ u_1'y_1'' & + & u_2'y_2'' & + & u_3'y_3'' & = & f(x), \end{cases}$$

then the function $u = u_1y_1 + u_2y_2 + u_3y_3$ is a solution of

$$y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = f(x)$$

Solution. $\square$

Problem 5. Find the eigenvalues and the respective eigenfunctions for the BVP

$$x^2y'' + xy' + \lambda y = 0, \quad y'(1) = 0, \quad y'(b) = 0$$

where $b > 1$.

Solution. $\square$