

Homework Assignment 3

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Problem 2.20. Apply the Fourier cosine transform to find the solution $u(x, y)$ of the problem

$$\begin{aligned}u_{xx} + u_{yy} &= 0, & 0 < x < \infty, & \quad 0 < y < \infty \\u(x, 0) &= H(a - x), & x < a \\u_x(0, y) &= 0, & 0 < x, y < \infty.\end{aligned}$$

Solution. Consider the function $u(x, y)$. The Fourier cosine transform of u with respect to x is defined as

$$\mathcal{F}_c \{u(x, y)\} = U_c(k, y) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x, y) \cos(kx) dx.$$

From this definition we see using the Leibniz integral rule that

$$\begin{aligned}\mathcal{F}_c \left\{ \frac{\partial^n u(x, y)}{\partial y^n} \right\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^n u(x, y)}{\partial y^n} \cos(kx) dx \\&= \frac{d^n}{dy^n} \left[\sqrt{\frac{2}{\pi}} \int_0^\infty u(x, y) \cos(kx) dx \right] \\&= \frac{d^n}{dy^n} [\mathcal{F}_c \{u(x, y)\}].\end{aligned}$$

The transforms of the partials of u with respect to x are not as easy to characterize. Nevertheless, we see from the properties of the Fourier cosine transform that

$$\mathcal{F}_c \left\{ \frac{\partial u(x, y)}{\partial x} \right\} = k \mathcal{F}_s \{u(x, y)\} - \sqrt{\frac{2}{\pi}} u(0, y)$$

and

$$\mathcal{F}_c \left\{ \frac{\partial^2 u(x, y)}{\partial x^2} \right\} = -k^2 \mathcal{F}_c \{u(x, y)\} - \sqrt{\frac{2}{\pi}} u_x(0, y)$$

Let $U_c(x, y) = \mathcal{F}_c \{u(x, y)\}$. Then, applying the Fourier cosine transform to the first differential equation shows that

$$\mathcal{F}_c \{u_{xx} + u_{yy}\} = -k^2 U_c(k, y) - \sqrt{\frac{2}{\pi}} u_x(0, y) + \frac{d^2}{dy^2} [U_c(k, y)] = 0 = \mathcal{F}_c \{0\}.$$

From the third equation we see that $u_x(0, y) = 0$ for all $0 < x, y < \infty$ which implies that the above equation reduces to

$$\frac{d^2}{dy^2} [U_c(k, y)] - k^2 U_c(k, y) = 0.$$

This is a second-order linear homogeneous differential equation, the solution to which is readily seen to be

$$U_c(k, y) = c_1 e^{-ky} + c_2 e^{ky}.$$

However, since $U_c(k, y) \rightarrow 0$ as $k \rightarrow \infty$, we must have that $c_2 = 0$. Thus, the solution to the previous differential equation is given by

$$U_c(k, y) = c_1 e^{-ky}. \quad (1)$$

We now apply the Fourier cosine transform to the second differential equation yielding

$$\mathcal{F}_c \{u(x, 0)\} = U_c(k, 0) = \mathcal{F}_c \{H(a - x)\}.$$

Using the form (1) of the solution to the transformed differential equation and a table of Fourier cosine transforms we see that

$$U_c(k, 0) = c_1 = \mathcal{F}_c \{H(a - x)\} = \sqrt{\frac{2}{\pi}} \left(\frac{\sin ak}{k} \right).$$

Thus, the solution to the transformed differential equation with the boundary conditions listed above is given by

$$U_c(k, y) = \mathcal{F}_c \{H(a - x)\} e^{-ky} = \sqrt{\frac{2}{\pi}} \left(\frac{\sin ak}{k} \right) e^{-ky}.$$

Therefore, taking the inverse Fourier cosine transform to both sides shows that the solution to the original differential equation is given by

$$\begin{aligned} u(x, y) &= \mathcal{F}_c^{-1} \{U_c(k, y)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{\sin ak}{k} \right) e^{-ky} \cos kx dk \\ &= \frac{2}{\pi} \int_0^\infty \left(\frac{\sin ak}{k} \right) e^{-ky} \cos kx dk. \end{aligned}$$

□

Problem 2.23. Use the Parseval formula to evaluate the following integrals with $a > 0$ and $b > 0$:

a. $\int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)^2},$

c. $\int_{-\infty}^{\infty} \frac{\sin^2 ax}{x^2} dx.$

Solution. Suppose that $f \in L^2(\mathbb{R})$ and that $F(k) = \mathcal{F}\{f(x)\}$. Then Parseval's relation states that

$$\int_{-\infty}^{\infty} f(x) \overline{f(x)} dx = \int_{-\infty}^{\infty} F(k) \overline{F(k)} dk.$$

a. Let $f(x) = \frac{1}{x^2 + a^2}$. Then from our table of Fourier transforms we see that

$$\mathcal{F}\{f(x)\} = F(k) = \sqrt{\frac{\pi}{2}} \left(\frac{e^{-a|k|}}{a} \right).$$

From Parseval's relation, we see that

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)^2} = \int_{-\infty}^{\infty} f(x) \overline{f(x)} dx = \int_{-\infty}^{\infty} F(k) \overline{F(k)} dk = \frac{\pi}{2a^2} \int_{-\infty}^{\infty} e^{-2a|k|} dk.$$

Therefore, we have that

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)^2} &= \frac{\pi}{2a^2} \int_{-\infty}^{\infty} e^{-2a|k|} dk \\ &= \frac{\pi}{a^2} \int_0^{\infty} e^{-2ak} dk \\ &= \frac{\pi}{a^2} \left[-\frac{e^{-2ak}}{2a} \right]_0^{\infty} \\ &= \frac{\pi}{2a^3}. \end{aligned}$$

c. Let $f(x) = \frac{\sin ax}{x}$. Then from our table of Fourier transforms we see that

$$\mathcal{F}\{f(x)\} = F(k) = \sqrt{\frac{\pi}{2}} H(a - |k|).$$

From Parseval's relation, we see that

$$\int_{-\infty}^{\infty} \frac{\sin^2 ax}{x^2} dx = \int_{-\infty}^{\infty} f(x) \overline{f(x)} dx = \int_{-\infty}^{\infty} F(k) \overline{F(k)} dk = \frac{\pi}{2} \int_{-\infty}^{\infty} H(a - |k|)^2 dk.$$

Therefore, we have using the definition of the Heaviside function that

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\sin^2 ax}{x} dx &= \frac{\pi}{2} \int_{-\infty}^{\infty} H(a - |k|)^2 dk \\ &= \frac{\pi}{2} \int_{-a}^a dk \\ &= a\pi.\end{aligned}$$

□

Problem 2.47. Apply the Fourier transform to solve the equation

$$u_{xxxx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad 0 \leq y$$

satisfying the conditions

$$u(x, 0) = f(x), \quad u_y(x, 0) = 0, \quad \text{for } -\infty < x < \infty$$

where $u(x, y)$ and its partial derivatives vanish as $|x| \rightarrow \infty$.

Solution.

□

Problem 2.48. The transverse vibration of a thin membrane of great extent satisfies the wave equation

$$c^2(u_{xx} + u_{yy}) = u_{tt}, \quad -\infty < x, y < \infty, \quad 0 < t,$$

with the initial and boundary conditions

$$\begin{aligned} u(x, y, t) &\rightarrow 0 \quad \text{as } |x| \rightarrow \infty, |y| \rightarrow \infty \quad \text{for all } t \geq 0, \\ u(x, y, 0) &= f(x, y), \quad u_t(x, y, 0) = 0 \quad \text{for all } x, y. \end{aligned}$$

Solution.

□

Problem 2.54. Solve the following equations

a. $u_{xxxx} - u_{yy} + 2u = f(x, y),$

b. $u_{xx} + 2u_{yy} + 3u_x - 4u = f(x, y),$

where $f(x, y)$ is a given function.

Solution.

□