Exam 2

Matthew Tiger

December 11, 2016

Problem 1. A function $f : \mathbb{C} \to \mathbb{C}$ is defined by $f(z) = z^8$. Find the fixed points of f. Use your calculations to find the real linear and quadratic factors of the polynomial $p(z) = z^7 - 1$.

Solution. The fixed points of f are the solutions to the equation

$$f(z) - z = z^8 - z = z(z^7 - 1) = 0.$$

Thus, the fixed points of f are $z = 0$ and the 7-th roots of unity, i.e. the points $z = e^{2\pi ki/7}$ for $k = 0, 1, \ldots, 6$.

Note that for $z, \alpha \in \mathbb{C}$, we have that

$$(z - \alpha)(z - \overline{\alpha}) = z^2 - \overline{\alpha}z - \alpha z + \alpha\overline{\alpha} = z^2 - 2\mathrm{Re}(\alpha)z + |\alpha|^2$$

is a polynomial with real coefficients.

Using the 7-th roots of unity, we can obtain the following factorization of $p(z)$:

$$p(z) = \prod_{k=0}^{6} \left(z - e^{2\pi ki/7}\right).$$

Let $\alpha_k = e^{2\pi ki/7}$. From the previous note, the real quadratic factors of $p(z)$ are obtained by multiplying each factor $(z - \alpha_k)$ with $(z - \overline{\alpha_k})$, if α_k and $\overline{\alpha_k}$ are both roots of $p(z)$. For $k = 1, \ldots, 6$, we have that α_k is a root of $p(z)$ and

$$\overline{\alpha_k} = e^{-2\pi ki/7} = e^{2\pi(7-k)i/7} = \alpha_{7-k},$$

which is also a root of $p(z)$. Therefore, the real linear and quadratic factors of $p(z)$ are given by

$$\begin{aligned}
p(z) &= (z - \alpha_0)(z - \alpha_1)(z - \alpha_6)(z - \alpha_2)(z - \alpha_5)(z - \alpha_3)(z - \alpha_4) \\
&= (z - 1)(z - \alpha_1)(z - \overline{\alpha_1})(z - \alpha_2)(z - \overline{\alpha_2})(z - \alpha_3)(z - \overline{\alpha_3}) \\
&= (z - 1)\left(z^2 - 2\mathrm{Re}(\alpha_1)z + 1\right)\left(z^2 - 2\mathrm{Re}(\alpha_2)z + 1\right)\left(z^2 - 2\mathrm{Re}(\alpha_3)z + 1\right),
\end{aligned}$$

where $\mathrm{Re}(\alpha_k) = \cos(2\pi k/7)$.

□

Problem 2. Let K_c be the filled-in Julia set of $f_c(z) = z^2 + c$.

a. Find the fixed points and the period 2 points of f_{-6}.

b. Show that $2\sqrt{2} \in K_{-6}$ and find another point in K_{-6}, distinct from those found so far.

c. Do any of the points you have found lie in the Julia set of f_{-6}?

d. Is $-6 \in \mathcal{M}$ where $\mathcal{M}$ is the Mandelbrot set?

Solution. a. The fixed points of f_{-6} are the solutions to

$$f_{-6}(z) - z = z^2 - z - 6 = 0.$$

Thus, the fixed points of f_{-6} are $z_0 = 3$ and $z_1 = -2$. The period 2 points are the solutions to

$$f_{-6}^2(z) - z = (z^2 - 6)^2 - z - 6 = 0$$

that are also not fixed points of f_{-6}. Factoring $f_{-6}^2(z) - z$, we see that

$$f_{-6}^2(z) - z = (z - 3)(z + 2)(z^2 + z - 5).$$

Thus, the period 2 points of f_{-6} are the solutions to $z^2 + z - 5 = 0$, i.e. the period 2 points of f_{-6} are

$$z_2 = \frac{-1 - \sqrt{21}}{2}, \quad z_3 = \frac{-1 + \sqrt{21}}{2}.$$

b. Recall that for a polynomial $p(z)$ with $\deg(p) > 1$, the filled-in Julia set of $p(z)$ is the set of all points that do not converge to ∞ under iteration of p.

Note that $2\sqrt{2}$ is an eventual fixed point of f_{-6}. We see that $f_{-6}^2(2\sqrt{2}) = -2$ so that $f_{-6}^k(2\sqrt{2}) = -2$ for $k > 2$. This implies that $2\sqrt{2}$ does not converge to ∞ under iteration of f_{-6} so that $2\sqrt{2}$ is in the filled-in Julia set of f_{-6}, i.e. $2\sqrt{2} \in K_{-6}$.

For reasons similar to those listed above, we see that -3 is an eventual fixed point of f_{-6}, i.e. $f_{-6}(-3) = 3$, so that $-3 \in K_{-6}$.

c. For a polynomial $p(z)$ with $\deg(p) > 1$, the Julia set of $p(z)$ is the boundary of the basin of attraction of ∞.

Since all of the points listed do not converge to ∞ under iteration of f_{-6}, we see that none of the listed points belong to the Julia set of f_{-6}.

d. The definition of the Mandelbrot set is the set of all $c \in \mathbb{C}$ such that the orbit of 0 is bounded under iteration by f_c. For f_{-6}, we see that $f_{-6}(0) = -6$ and $f_{-6}^2(0) = 30$. By a previously proven escape criterion, since $|30| > 2$ and $|30| > |-6|$, we have that $f_{-6}^n(30) \to \infty$ as $n \to \infty$. This implies that the orbit of 0 is not bounded under iteration by f_{-6}. Therefore, we must have that $-6 \notin \mathcal{M}$.

□

Problem 3. Let $f(z) = z^2 + c$. Find the values of c so that $z = i$ is a period 2 point. Find the fixed points in each case and determine their stability. Is $c \in \mathcal{M}$?

Solution. $\square$

Problem 4. Show that the function $H(z) = \dfrac{z-i}{z+i}$ gives a conjugacy between the Newton map N_{f_1} of $f_1(z) = z^2 + 1$ and the function $f_0(z) = z^2$. Deduce the Julia set of N_{f_1} and show that it is chaotic on its Julia set.

Solution. $\square$

Problem 5. Let $p(z)$ be a polynomial of degree $d > 1$ with Newton function

$$N_p(z) = z - \frac{p(z)}{p'(z)}.$$

a. If $p(\alpha) = 0$ and $p'(\alpha) \neq 0$, show that α is a fixed point of multiplicity two for N_p, i.e. there is a rational function $k(z) = m(z)/n(z)$ with $n(\alpha) \neq 0$ and $N_p(z) - \alpha = (z - \alpha)^2 k(z)$.

b. If $p(\alpha) = 0$, $p'(\alpha) \neq 0$, and $p''(\alpha) = 0$, show that α is a fixed point of multiplicity three for N_p.

Solution. □

Problem 6. a. Show that for $p_\alpha(z) = z(z-1)(z-\alpha)$, the Newton function N_{p_α} has a critical point where $z = (\alpha+1)/3$.

b. For what values of α does p_α satisfy $p(\alpha) = 0$, $p'(\alpha) \neq= 0$, and $p''(\alpha) = 0$?

Solution. □

Problem 7. Let $0 < \mu < \lambda < 1$ and let $h : [0,1] \to [0,1]$ be a homeomorphism with $h \circ L_\mu(x) = L_\lambda \circ h(x)$ for all $x \in [0,1]$.

a. Show that h is orientation-preserving.

b. Show that $h(x) + h(1-x) = 1$ for all $x \in [0,1]$. Deduce that $h(1/2) = 1/2$.

c. Show that $h(\mu/4) = \lambda/4$ and $h(x) > x$ for $0 < x < 1/2$ and $h(x) < x$ for $1/2 < x < 1$.

Solution. □

Problem 8. Prove that if $f_c(z) = z^2 + c$ has an attracting periodic point, then $c \in \mathcal{M}$, the Mandelbrot set.

Solution. $\square$