Conditioning and Error

Unit 4

Spectral radius and norms

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$$Av = \lambda v \;\; \rightarrow \;\; \|\lambda v\| = \|Av\| \quad \rightarrow \;\; |\lambda|\|v\| = \|\lambda v\| = \|Av\| \leq \|A\|\|v\|$$

Therefore, $|\lambda| \leq \|A\|$, and $\rho(A) \leq \|A\|$.

Question: why does the norm need to be subordinate?

Spectral radius of symmetric matrix

For any Hermitian A

$$\|A\|_2 = \rho(A), \quad \|A^{-1}\|_2 = 1/\mu, \ \text{ where } \ \mu := \min\{|\lambda|, \ \ \lambda \in \sigma(A)\}$$

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A^{-1} is also Hermitian with $\rho(A^{-1}) = 1/\mu$. Thus $\|A^{-1}\|_2 = 1/\mu$.

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- $\|A\|_2 \leq \|A\|_F \leq \sqrt{n} \|A\|_2$ *without proof*

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• The spectral norm is usually the choice for analyzing properties of the condition number. The 1- and ∞- norms are used in computations.

Examples

- Almost singular

$$\begin{pmatrix} 1 & 1 \\ 1 & 1-10^{-6} \end{pmatrix}$$

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- Hilbert matrix $H(n)$, $h_{ij} := \frac{1}{i+j+1}$.

$$H(3) = \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix}$$

$$cond_2(H(10)) \approx 10^{13}, \quad cond_1(A) \approx cond_\infty(A) \approx 3 \times 10^{13}$$

HW: produce a picture similar to Fig. 4.1 for Hilbert matrix

Linear systems, example

$$x_1 + x_2 = 20$$
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We can still find the exact analytic solution easily
$x_1 = 30, \quad x_2 = -10.$

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- a significant change in the solution would be $\frac{\|\delta x\|}{\|x\|} = \frac{20}{10} = 2$
- We have $\frac{\|\delta x\|}{\|x\|} \approx 2 \times 10^{6} \frac{\|\delta A\|}{\|A\|} \approx$

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Main theorem I

If A and $A + \delta A$ are both invertible, then

$$\|\delta x\| < \|x\| \quad \to \quad \frac{\|\delta x\|}{\|x\|} \leq 2cond(A)\frac{\|\delta A\|}{\|A\|},$$

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$\frac{\|\delta x\|}{\|x+\delta x\|} \leq \|A^{-1}\|\|\delta A\| \ \rightarrow \ \frac{\|\delta x\|}{\|x+\delta x\|} \leq \|A^{-1}\|\|A\|\frac{\|\delta A\|}{\|A\|}$

and the second inequality follows.

HW: prove the first inequality

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Once again, a very small change in b seems to yield a significant change in the solution. Can we quantify this?

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