

Exam 1

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March 12, 2017

Problem 1. Find the Fourier Transforms of the following functions:

- a. $f(x) = x^2 e^{-a|x|}$, $a > 0$,
- b. $f(x) = \left(1 - \frac{|x|}{2}\right) H\left(1 - \frac{|x|}{2}\right)$.

Solution. Recall that if $f(x) \in L^1(\mathbb{R})$, then the Fourier Transform of f is defined to be

$$\mathcal{F}\{f(x)\} = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx. \quad (1)$$

- a. If $f(x) = x^2 e^{-a|x|}$, $a > 0$, we see from (1) that, by definition, the Fourier Transform of f is given by

$$\begin{aligned} \mathcal{F}\{f(x)\} = F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-a|x|} e^{-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^0 x^2 e^{-(-a+ik)x} dx + \int_0^{\infty} x^2 e^{-(a+ik)x} dx \right]. \end{aligned}$$

Now, we see by integration by parts that

$$\begin{aligned} \int_c^d x^2 e^{-(b+ik)x} dx &= -\frac{x^2}{b+ik} e^{-(b+ik)x} \Big|_c^d + \frac{2}{b+ik} \int_c^d x e^{-(b+ik)x} dx \\ &= -\frac{x^2}{b+ik} e^{-(b+ik)x} \Big|_c^d + \frac{2}{b+ik} \left[-\frac{x}{b+ik} e^{-(b+ik)x} \Big|_c^d + \frac{1}{b+ik} \int_c^d e^{-(b+ik)x} dx \right] \\ &= -\frac{x^2}{b+ik} e^{-(b+ik)x} \Big|_c^d + \frac{2}{b+ik} \left[-\frac{x}{b+ik} e^{-(b+ik)x} \Big|_c^d - \frac{1}{(b+ik)^2} e^{-(b+ik)x} \Big|_c^d \right]. \end{aligned}$$

Thus,

$$\int_{-\infty}^0 x^2 e^{-(-a+ik)x} dx = -\frac{2}{(-a+ik)^3}$$

and

$$\int_0^\infty x^2 e^{-(a+ik)x} dx = \frac{2}{(a+ik)^3}.$$

Therefore,

$$\begin{aligned} \mathcal{F}\{f(x)\} = F(k) &= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^0 x^2 e^{-(-a+ik)x} dx + \int_0^\infty x^2 e^{-(a+ik)x} dx \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[-\frac{2}{(-a+ik)^3} + \frac{2}{(a+ik)^3} \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{4a(a^2 - 3k)}{(a^2 + k^2)^3} \right] \\ &= \sqrt{\frac{2}{\pi}} \left[\frac{2a(a^2 - 3k)}{(a^2 + k^2)^3} \right]. \end{aligned}$$

b. Recall that the Heaviside function H is defined as

$$H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Thus,

$$H\left(1 - \frac{|x|}{2}\right) = \begin{cases} 1 & \text{if } |x| < 2 \\ 0 & \text{if } |x| > 2. \end{cases}$$

If $f(x) = \left(1 - \frac{|x|}{2}\right) H\left(1 - \frac{|x|}{2}\right)$, we see from (1) that, by definition, the Fourier Transform of f is given by

$$\begin{aligned} \mathcal{F}\{f(x)\} = F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \left(1 - \frac{|x|}{2}\right) H\left(1 - \frac{|x|}{2}\right) e^{-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-2}^2 \left(1 - \frac{|x|}{2}\right) e^{-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \left[\int_{-2}^0 \left(1 + \frac{x}{2}\right) e^{-ikx} dx + \int_0^2 \left(1 - \frac{x}{2}\right) e^{-ikx} dx \right]. \end{aligned}$$

Now, we see by integration by parts that

$$\begin{aligned} \int_c^d \left(1 \pm \frac{x}{2}\right) e^{-ikx} dx &= \frac{i}{k} e^{-ikx} \Big|_c^d \pm \frac{1}{2} \int_c^d x e^{-ikx} dx \\ &= \frac{i}{k} e^{-ikx} \Big|_c^d \pm \frac{1}{2} \left[\frac{ix}{k} e^{-ikx} \Big|_c^d - \frac{i}{k} \int_c^d e^{-ikx} dx \right] \\ &= \frac{i}{k} e^{-ikx} \Big|_c^d \pm \frac{1}{2} \left[\frac{ix}{k} e^{-ikx} \Big|_c^d + \frac{e^{-ikx}}{k^2} \Big|_c^d \right]. \end{aligned}$$

Thus,

$$\int_{-2}^0 \left(1 + \frac{x}{2}\right) e^{-ikx} dx = \frac{1 - e^{2ik} + 2ik}{2k^2}$$

and

$$\int_0^2 \left(1 - \frac{x}{2}\right) e^{-ikx} dx = \frac{1 - e^{-2ik} - 2ik}{2k^2}$$

Therefore, using the definition of the complex exponential and various trigonometric identities, we have that

$$\begin{aligned} \mathcal{F}\{f(x)\} = F(k) &= \frac{1}{\sqrt{2\pi}} \left[\int_{-2}^0 \left(1 + \frac{x}{2}\right) e^{-ikx} dx + \int_0^2 \left(1 - \frac{x}{2}\right) e^{-ikx} dx \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{1 - e^{2ik} + 2ik}{2k^2} + \frac{1 - e^{-2ik} - 2ik}{2k^2} \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{k^2} - \frac{e^{-2ik} + e^{2ik}}{2k^2} \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{1 - \cos 2k}{k^2} \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{1 - (\cos^2 k - \sin^2 k)}{k^2} \right] \\ &= \frac{2 \sin^2 k}{\sqrt{2\pi} k^2}. \end{aligned}$$

□

Problem 2. Find the Laplace Transforms of the following functions:

a. $f(t) = \int_0^t \frac{\sin ax}{x} dx,$

b. $f(t) = tH(t - a).$

Solution. Recall that if $f(t) \in L^1(\mathbb{R})$, then the Laplace Transform of f is defined to be

$$\mathcal{L}\{f(t)\} = \bar{f}(s) = \int_0^\infty f(t)e^{-st}dt, \quad \operatorname{Re} s > 0. \quad (2)$$

Note that it can be shown that the Laplace transform satisfies the important property

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} [\mathcal{L}\{f(t)\}]. \quad (3)$$

a. Let $f(t) = \int_0^t \frac{\sin ax}{x} dx$. Then $f(0) = 0$ and $f'(t) = \frac{\sin at}{t}$ so that $tf'(t) = \sin at$. This implies that

$$\mathcal{L}\{tf'(t)\} = \mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}.$$

The Laplace transform satisfies the following property that relates a function's derivative to its Laplace Transform:

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

This combined with (3) shows that

$$\mathcal{L}\{tf'(t)\} = -\frac{d}{ds} [\mathcal{L}\{f'(t)\}] = -\frac{d}{ds} [s\mathcal{L}\{f(t)\} - f(0)] = \frac{a}{s^2 + a^2},$$

or that

$$\frac{d}{ds} [s\mathcal{L}\{f(t)\}] = -\frac{a}{s^2 + a^2}.$$

Integrating both sides of the above equation yields that

$$s\mathcal{L}\{f(t)\} = -\int \frac{a}{s^2 + a^2} ds = -\tan^{-1}(s/a) + C.$$

In order to determine the constant of integration, we use the Initial Value Theorem which states that

$$\lim_{s \rightarrow \infty} s\mathcal{L}\{f(t)\} = \lim_{t \rightarrow 0} f(t) = f(0).$$

Since $f(0) = 0$, this implies that

$$\lim_{s \rightarrow \infty} s\mathcal{L}\{f(t)\} = \lim_{s \rightarrow \infty} [-\tan^{-1}(s/a) + C] = 0$$

or that $C = \frac{\pi}{2}$. Therefore, we have that

$$s\mathcal{L}\{f(t)\} = -\tan^{-1}(s/a) + \frac{\pi}{2} = \tan^{-1}(a/s)$$

so that

$$\mathcal{L}\{f(t)\} = \frac{\tan^{-1}(a/s)}{s}.$$

b. Let $f(t) = tH(t-a)$ and assume that $a > 0$. By property (3) we see that

$$\mathcal{L}\{tH(t-a)\} = -\frac{d}{ds} [\mathcal{L}\{H(t-a)\}].$$

From our table of Laplace Transforms, we have that

$$\mathcal{L}\{H(t-a)\} = \frac{e^{-as}}{s},$$

assuming that $a > 0$. Therefore,

$$\begin{aligned} \mathcal{L}\{f(t)\} &= -\frac{d}{ds} [\mathcal{L}\{H(t-a)\}] = -\frac{d}{ds} [s^{-1}e^{-as}] \\ &= -[-s^{-2}e^{-as} - as^{-1}e^{-as}] \\ &= \frac{(1+as)e^{-as}}{s^2}. \end{aligned}$$

□

Problem 3. Solve the following integral equations:

a. $\int_0^\infty f(x) \sin kx dx = \begin{cases} 1 - k & k < 1 \\ 0 & k > 1 \end{cases},$

b. $\int_{-\infty}^\infty \frac{f(t)}{(x-t)^2 + 4} dt = \frac{1}{x^2 + 9}.$

Solution. a. Recall that the definition of the Fourier Sine Transform is given by

$$\mathcal{F}_s \{f(x)\} = F_s(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin kx dx.$$

Thus, we see that

$$\int_0^\infty f(x) \sin kx dx = \sqrt{\frac{\pi}{2}} \mathcal{F}_s \{f(x)\} = \sqrt{\frac{\pi}{2}} F_s(k).$$

Let $G_s(k) = \begin{cases} 1 - k & k < 1 \\ 0 & k > 1 \end{cases}$. Then the above integral equation becomes

$$F_s(k) = \sqrt{\frac{2}{\pi}} G_s(k).$$

Thus, we have that

$$f(x) = \mathcal{F}_s^{-1} \{F_s(k)\} = \sqrt{\frac{2}{\pi}} \mathcal{F}_s^{-1} \{G_s(k)\}$$

where

$$\mathcal{F}_s^{-1} \{G_s(k)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty G_s(k) \sin kx dk.$$

Therefore, the solution to the integral equation is

$$\begin{aligned} f(x) &= \sqrt{\frac{2}{\pi}} \mathcal{F}_s^{-1} \{G_s(k)\} = \frac{2}{\pi} \int_0^\infty G_s(k) \sin kx dk \\ &= \frac{2}{\pi} \int_0^1 (1 - k) \sin kx dk \\ &= \frac{2}{\pi} \left[\frac{1 - \cos x}{x} - \frac{\sin x - x \cos x}{x^2} \right] \\ &= \frac{2}{\pi} \left[\frac{x - \sin x}{x^2} \right]. \end{aligned}$$

b. Recall that the convolution of two functions f and g is defined such that

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - \xi)g(\xi)d\xi.$$

Let $g(x) = \frac{1}{x^2 + 2^2}$. Then we see that

$$\int_{-\infty}^{\infty} \frac{f(t)}{(x - t)^2 + 4} dt = \int_{-\infty}^{\infty} f(t)g(x - t)dt = \sqrt{2\pi}(g * f)(x) = \sqrt{2\pi}(f * g)(x).$$

Now, let $h(x) = \frac{1}{x^2 + 3^2}$. Then in light of the above remarks, the integral equation becomes

$$\int_{-\infty}^{\infty} \frac{f(t)}{(x - t)^2 + 4} dt = \sqrt{2\pi}(f * g)(x) = h(x) = \frac{1}{x^2 + 9}.$$

Applying the Fourier transform to the integral equation, we see by the Convolution Theorem that

$$\mathcal{F} \left\{ \sqrt{2\pi}(f * g)(x) \right\} = \sqrt{2\pi}F(k)G(k) = H(k) = \mathcal{F} \{h(x)\},$$

where $\mathcal{F} \{f(x)\} = F(k)$, $\mathcal{F} \{g(x)\} = G(k)$, and $\mathcal{F} \{h(x)\} = H(k)$, respectively. From our table of Fourier transforms, we see that for $a > 0$ we have that

$$\mathcal{F} \left\{ \frac{1}{x^2 + a^2} \right\} = \sqrt{\frac{\pi}{2}} \frac{e^{-a|k|}}{a}. \quad (4)$$

Thus, from (4) we have that

$$F(k) = \frac{1}{\sqrt{2\pi}} \frac{H(k)}{G(k)} = \frac{2}{3\sqrt{2\pi}} \frac{e^{-3|k|}}{e^{-2|k|}} = \frac{2e^{-|k|}}{3\sqrt{2\pi}}.$$

Applying the Fourier Inverse Transform to this equation yields that

$$f(x) = \mathcal{F}^{-1} \{F(k)\} = \frac{2}{3\sqrt{2\pi}} \mathcal{F}^{-1} \{e^{-|k|}\}.$$

But from (4), we know that

$$\mathcal{F}^{-1} \{e^{-|k|}\} = \sqrt{\frac{2}{\pi}} \frac{1}{x^2 + 1}.$$

Therefore, the solution to the integral equation is given by

$$\begin{aligned} f(x) &= \frac{2}{3\sqrt{2\pi}} \mathcal{F}^{-1} \{e^{-|k|}\} = \frac{2}{3\sqrt{2\pi}} \left[\sqrt{\frac{2}{\pi}} \frac{1}{x^2 + 1} \right] \\ &= \frac{2}{3\pi} \left[\frac{1}{x^2 + 1} \right]. \end{aligned}$$

□

Problem 4. Show that

$$\int_0^\infty F_s(k)G_c(k) \sin kx dx = \frac{1}{2} \int_0^\infty g(\xi) [f(\xi + x) - f(\xi - x)] d\xi$$

where

$$F_s(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin kx dx$$

and

$$G_c(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \cos kx dx.$$

Solution.

□

Problem 5. Apply the Fourier Transform to solve the following initial value problem for the heat equation:

$$\begin{aligned}\frac{\partial u}{\partial t} &= a^2 \frac{\partial^2 u}{\partial x^2} + f(x, t), & -\infty < x < \infty, \\ u(x, 0) &= \phi(x), & t > 0.\end{aligned}$$

Solution.

□

Problem 6. Evaluate the following definite integrals:

a. $\int_0^\infty \frac{\sin ax \sin bx}{x^2} dx,$

b. $\int_0^\infty \frac{(a^2 - x^2)^2}{(x^2 + a^2)^4} dx, \quad a > 0.$

Solution.

□

Problem 7. Use the Fourier Sine Transform to solve the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \infty$$

with the boundary data $0 < y < L$

$$\begin{aligned} u(x, L) &= 0, & u(x, 0) &= f(x), \\ u(0, y) &= 0, & u(x, y) &\rightarrow 0 \text{ as } x \rightarrow \infty \text{ uniformly in } y. \end{aligned}$$

Solution.

□

Problem 8. Apply the Fourier Transform to solve the 3-dimensional wave problem

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \quad -\infty < x, y, z < \infty,$$

subject to the initial conditions

$$\begin{aligned} u(x, y, z, t)|_{t=0} &= 0 \\ \frac{\partial u(x, y, z, t)}{\partial t} \Big|_{t=0} &= \delta(x, y, z). \end{aligned}$$

Solution.

□

Problem 9. Show that if E is a solution of an m -th order partial differential equation

$$P(\partial)u = \sum_{k=0}^m a_k \partial^k u = \delta,$$

where δ is the Dirac delta function, then $E * f$ is the solution of the partial differential equation $P(\partial)u = f$, where $*$ is the convolution.

Solution.

□