Problem 4.5. A Markov chain $\{X_n : n \geq 0\}$ with states 0,1,2, has the transition probability matrix

$$\boldsymbol{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

If $P\{X_0 = 0\} = P\{X_0 = 1\} = 1/4$, find $E[X_3]$.

Solution. If $\{X_n : n \geq 0\}$ is a Markov chain with state space $\mathcal{M} = \{0, 1, 2\}$, then we know that $\{X_n : n \geq 0\}$ is a stochastic process and that this stochastic process has the property that

$$P\{X_n = j \mid X_{n-1} = i, X_{n-2} = i_{n-2}, \ldots, X_0 = i_0\} = P\{X_n = j \mid X_{n-1} = i\}$$

for any time $n \in \mathbb{Z}^+$, i.e. the probability that X_n is in state j depends only on the probability that X_{n-1} is in state i. We denote $P\{X_n = j \mid X_{n-1} = i\}$ by P_{ij} and $P\{X_n = j \mid X_0 = i\}$ by P_{ij}^n.

We wish to find

$$E[X_3] = \sum_{j=0}^{2} xP\{X_3 = j\}.$$

If we know the probability that $X_0 = i$ for all states $i \in \mathcal{M}$, then we can condition $P\{X_3 = j\}$ on the probability that X_0 is in state $i \in \mathcal{M}$ for all states, i.e.

$$\begin{aligned} P\{X_3 = j\} &= \sum_{i \in \mathcal{M}} P\{X_3 = j \mid X_0 = i\}P\{X_0 = i\} \\ &= \sum_{i \in \mathcal{M}} P_{ij}^3 P\{X_0 = i\} \qquad (1) \end{aligned}$$

By assumption, we know that $P\{X_0 = 0\} = P\{X_0 = 1\} = 1/4$. Since $\{X_n : n \geq 0\}$ is a stochastic process, X_0 is a random variable so that $P\{X_0 = 2\} > 0$ and in particular

$$P\{X_0 = 2\} = 1 - \sum_{i \in \mathcal{M}, i \neq 2} P\{X_0 = i\} = \frac{1}{2}.$$

With this, we are able to compute $P\{X_0 = i\}$ for all $i \in \mathcal{M}$ and use these probabilities to find (1).

Lastly, in order to compute (1), we need to compute P_{ij}^3. Note that the transition matrix gives the probability of transitioning from state i to state j i.e. $\boldsymbol{P} = (P_{ij})$. Let $\boldsymbol{P}^{(n)}$ be the matrix of n-step transition probabilities P_{ij}^n. By the Chapman-Kolmogorov equations, we have that $\boldsymbol{P}^{(n)} = \boldsymbol{P}^n$ so that the n-step transition probability matrix can be found through multiplication of the transition matrix $\boldsymbol{P}$. Thus,

$$\boldsymbol{P}^{(3)} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}^3 = \begin{bmatrix} \frac{13}{36} & \frac{11}{54} & \frac{47}{108} \\ \frac{4}{9} & \frac{4}{27} & \frac{11}{27} \\ \frac{5}{12} & \frac{2}{9} & \frac{13}{36} \end{bmatrix} \qquad (2)$$

and P^3_{ij} is the ij-th entry of $\boldsymbol{P}^{(3)}$.

Using the transition matrix (2) and equation (1), we thus have that

$$\begin{aligned} P\{X_3 = 0\} &= \sum_{i \in \mathcal{M}} P^3_{i0} P\{X_0 = i\} \\ &= \frac{13}{36} \cdot \frac{1}{4} + \frac{4}{9} \cdot \frac{1}{4} + \frac{5}{12} \cdot \frac{1}{2} \\ &= \frac{59}{144} \end{aligned}$$

$$\begin{aligned} P\{X_3 = 1\} &= \sum_{i \in \mathcal{M}} P^3_{i1} P\{X_0 = i\} \\ &= \frac{11}{54} \cdot \frac{1}{4} + \frac{4}{27} \cdot \frac{1}{4} + \frac{2}{9} \cdot \frac{1}{2} \\ &= \frac{43}{216} \end{aligned}$$

$$\begin{aligned} P\{X_3 = 2\} &= \sum_{i \in \mathcal{M}} P^3_{i2} P\{X_0 = i\} \\ &= \frac{47}{108} \cdot \frac{1}{4} + \frac{11}{27} \cdot \frac{1}{4} + \frac{13}{36} \cdot \frac{1}{2} \\ &= \frac{169}{432}. \end{aligned}$$

Therefore,

$$\begin{aligned} E[X_3] &= \sum_{j=0}^{2} x P\{X_3 = j\} \\ &= P\{X_3 = 1\} + 2P\{X_3 = 2\} \\ &= \frac{43}{216} + 2 \cdot \frac{169}{432} = \frac{53}{54}. \end{aligned}$$

□

Problem 4.6. Let the transition probability matrix of a two-state Markov chain be given, as in Example 4.2, by

$$\boldsymbol{P} = \begin{Vmatrix} p & 1-p \\ 1-p & p \end{Vmatrix}.$$

Show by mathematical induction that

$$\boldsymbol{P}^{(n)} = \begin{Vmatrix} \frac{1}{2} + \frac{1}{2}(2p-1)^n & \frac{1}{2} - \frac{1}{2}(2p-1)^n \\ \frac{1}{2} - \frac{1}{2}(2p-1)^n & \frac{1}{2} + \frac{1}{2}(2p-1)^n \end{Vmatrix}. \tag{3}$$

Solution. Recall by the Chapman-Kolmogorov equations that the n-step transition matrix of a Markov chain can be found through repeated multiplication of its initial transition matrix i.e. $\boldsymbol{P}^{(n)} = \boldsymbol{P}^n$.

In order to show that equation (3) is true by induction, we must first show that the equation holds if $n = 1$. Note that by our definition of the transition matrix $\boldsymbol{P}$ we have that

$$\boldsymbol{P}^{(1)} = \begin{Vmatrix} \frac{1}{2} + \frac{1}{2}(2p-1)^1 & \frac{1}{2} - \frac{1}{2}(2p-1)^1 \\ \frac{1}{2} - \frac{1}{2}(2p-1)^1 & \frac{1}{2} + \frac{1}{2}(2p-1)^1 \end{Vmatrix} = \begin{Vmatrix} p & 1-p \\ 1-p & p \end{Vmatrix} = \boldsymbol{P}^1$$

and we have established the initial step of the induction.

Now suppose that (3) holds for n. As mentioned, by the Chapman-Kolmogorov equations, we have that

$$\boldsymbol{P}^{(n+1)} = \boldsymbol{P}^{n+1} = \boldsymbol{P}\boldsymbol{P}^n.$$

Thus, by our supposition, we have that

$$\begin{aligned}
\boldsymbol{P}^{(n+1)} &= \boldsymbol{P}\boldsymbol{P}^n \\
&= \begin{Vmatrix} p & 1-p \\ 1-p & p \end{Vmatrix} \begin{Vmatrix} \frac{1}{2} + \frac{1}{2}(2p-1)^n & \frac{1}{2} - \frac{1}{2}(2p-1)^n \\ \frac{1}{2} - \frac{1}{2}(2p-1)^n & \frac{1}{2} + \frac{1}{2}(2p-1)^n \end{Vmatrix} \\
&= \begin{Vmatrix} \frac{p}{2} + \frac{p}{2}(2p-1)^n + (1-p)\left(\frac{1}{2} - \frac{1}{2}(2p-1)^n\right) & \frac{p}{2} - \frac{p}{2}(2p-1)^n + (1-p)\left(\frac{1}{2} + \frac{1}{2}(2p-1)^n\right) \\ (1-p)\left(\frac{1}{2} + \frac{1}{2}(2p-1)^n\right) + \frac{p}{2} - \frac{p}{2}(2p-1)^n & (1-p)\left(\frac{1}{2} - \frac{1}{2}(2p-1)^n\right) + \frac{p}{2} + \frac{p}{2}(2p-1)^n \end{Vmatrix} \\
&= \begin{Vmatrix} p(2p-1)^n + \frac{1}{2} - \frac{1}{2}(2p-1)^n & -p(2p-1)^n + \frac{1}{2} + \frac{1}{2}(2p-1)^n \\ \frac{1}{2} + \frac{1}{2}(2p-1)^n - p(2p-1)^n & \frac{1}{2} - \frac{1}{2}(2p-1)^n + p(2p-1)^n \end{Vmatrix} \\
&= \begin{Vmatrix} \frac{1}{2} + \left(p - \frac{1}{2}\right)(2p-1)^n & \frac{1}{2} - \left(p - \frac{1}{2}\right)(2p-1)^n \\ \frac{1}{2} - \left(p - \frac{1}{2}\right)(2p-1)^n & \frac{1}{2} + \left(p - \frac{1}{2}\right)(2p-1)^n \end{Vmatrix} \\
&= \begin{Vmatrix} \frac{1}{2} + \frac{1}{2}(2p-1)^{n+1} & \frac{1}{2} - \frac{1}{2}(2p-1)^{n+1} \\ \frac{1}{2} - \frac{1}{2}(2p-1)^{n+1} & \frac{1}{2} + \frac{1}{2}(2p-1)^{n+1} \end{Vmatrix}
\end{aligned}$$

and the equation holds for $n + 1$. Therefore, (3) is true by induction. $\square$

Problem 4.8. Suppose that coin 1 has probability 0.7 of coming up heads and coin 2 has probability 0.6 of coming up heads. If the coin flipped today comes up heads, then we select coin 1 to flip tomorrow and if it comes up tails then we select coin 2 to flip tomorrow. If the coin initially flipped is equally likely to be coin 1 or coin 2, then what is the probability that the coin flipped on the third day after the initial flip is coin 1? Suppose that the coin flipped on Monday comes up heads. What is the probability that the coin flipped on Friday of the same week also comes up heads?

Solution. □

Problem 4.14. Specify the classes of the following Markov chains and determine whether they are transient or recurrent:

$$\boldsymbol{P_1} = \begin{Vmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{Vmatrix} \qquad \boldsymbol{P_2} = \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{Vmatrix}$$

$$\boldsymbol{P_3} = \begin{Vmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{Vmatrix} \quad \boldsymbol{P_4} = \begin{Vmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{Vmatrix}$$

Solution. In order to determine the equivalence classes of a Markov chain with state space $\mathcal{M}$, we must partition the state space by the relation communicates, i.e. states i and j are in the same equivalence class if there are $m, n \in \mathbb{N}$ with $P_{ij}^m > 0$ and $P_{ji}^n > 0$. From there, we classify the classes as transient or recurrent depending on whether $\sum_{n=1}^{\infty} P_{ii}^n$ is finite or infinite, respectively.

We begin with the Markov chain defined by the transition matrix $\boldsymbol{P_1}$ with state space $\mathcal{M}_1 = \{0, 1, 2\}$. Note that from the definition of $\boldsymbol{P_1}$, we readily see that $P_{01} = 1/2 > 0$ and that $P_{10} = 1/2 > 0$ so that state 0 communicates with state 1. Similarly, we see that $P_{02} = 1/2 > 0$ and that $P_{20} = 1/2 > 0$ so that state 0 communicates with state 2. Since every state communicates with itself by definition, we have that state 0 communicates with all $i \in \mathcal{M}_1$. Therefore the only equivalence class of the Markov chain is $\mathcal{M}_1$ itself and this Markov chain is irreducible. From a previous result, since $\mathcal{M}_1$ is finite and the Markov chain is irreducible, every state is recurrent.

Consider now the Markov chain defined by the transition matrix $\boldsymbol{P_2}$ with state space $\mathcal{M}_2 = \{0, 1, 2, 3\}$. It can be shown that $\boldsymbol{P_2}^4 = \boldsymbol{P_2}$. Thus, we must have for $m > 0$ that $\boldsymbol{P_2}^m = \boldsymbol{P_2}^n$ where $n = 1, 2, 3$. So in order to classify the states of $\mathcal{M}_2$ we need only look at $\boldsymbol{P_2}, \boldsymbol{P_2}^2$, and $\boldsymbol{P_2}^3$. Computing these powers of the initial transition matrix show that

$$\boldsymbol{P_2} = \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{Vmatrix}, \quad \boldsymbol{P_2}^2 = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{Vmatrix}, \quad \boldsymbol{P_2}^3 = \begin{Vmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}.$$

From these transition matrices we see that:

$$\begin{aligned} P_{01}^3 &= 1/2 > 0 \text{ and } P_{10}^3 = 1/2 > 0, \\ P_{02}^2 &= 1 > 0 \text{ and } P_{20} = 1/2 > 0, \\ P_{03} &= 1 > 0 \text{ and } P_{30}^2 = 1/2 > 0. \end{aligned}$$

Thus, state 0 communicates with all states $i \in \mathcal{M}_2$ and $\mathcal{M}_2$ is the only equivalence class of this Markov chain. Therefore, as mentioned above, the Markov chain is irreducible and every state is recurrent.

Now consider now the Markov chain defined by the transition matrix $\boldsymbol{P_3}$ with state space $\mathcal{M}_3 = \{0, 1, 2, 3, 4\}$. Note that

$$\boldsymbol{P_3} = \begin{Vmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{Vmatrix} = \begin{Vmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{Vmatrix},$$

a diagonal matrix. Thus, for any $n > 0$,

$$\boldsymbol{P_3}^n = \begin{Vmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{Vmatrix}^n = \begin{Vmatrix} \boldsymbol{A}^n & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B}^n \end{Vmatrix}. \tag{4}$$

Note that $P_{34} = 1/2 > 0$ and $P_{43} = 1/2 > 0$ so that state 3 communicates with state 4. We can readily see from (4) that for all $m \geq 0$ we have that $P_{i3}^m = 0$ and $P_{i4}^m = 0$ for $i = 0, 1, 2$ so that states 0, 1, and 2 do not communicate with states 3 and 4. Thus $\{3, 4\}$ forms an equivalence class of this Markov chain.

We can also see that $P_{02} = 1/2 > 0$ and $P_{20} = 1/2 > 0$ so that state 0 communicates with state 2. However, it can be shown through induction that

$$\boldsymbol{A}^n = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{2^n-1}{2^{n+1}} & \frac{1}{2^n} & \frac{2^n-1}{2^{n+1}} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \tag{5}$$

so that it is clear for all $m \geq 0$ that $P_{01}^m = 0$ and $P_{21}^m = 0$. Thus, states 0 and 2 do not communicate with state 1. Thus, the other equivalence classes are formed by $\{0, 2\}$ and $\{1\}$. In order to classify these equivalences classes of this Markov chain we examine the probabilities P_{ii}^n. Recall by a previous proposition that a state i is recurrent if

$$\sum_{n=1}^{\infty} P_{ii}^n = \infty$$

and is transient if the sum is finite. Note that $\boldsymbol{B}$ is an idempotent matrix, i.e. $\boldsymbol{B}^2 = \boldsymbol{B}$. Thus, $\boldsymbol{B}^n = \boldsymbol{B}$ and for $n > 0$, we have that $P_{33}^n = P_{44}^n = 1/2$. From (5) we see that for $n > 0$

we have that $P_{00}^n = P_{22}^n = 1/2$ while $P_{11} = 2^{-n}$. Now, we see that for $i = 0, 2, 3, 4$ we have that

$$\sum_{n=1}^{\infty} P_{ii}^n = \sum_{n=1}^{\infty} \frac{1}{2} = \infty$$

so that states $i = 0, 2, 3, 4$ are recurrent. However,

$$\sum_{n=1}^{\infty} P_{11}^n = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 < \infty$$

so that state 1 is transient. Therefore, the classes $\{0, 2\}$ and $\{3, 4\}$ are recurrent and $\{1\}$ is transient.

Lastly, consider the Markov chain defined by the transition matrix $\boldsymbol{P_4}$ with state space $\mathcal{M}_4 = \{0, 1, 2, 3, 4\}$. It can be shown that $\boldsymbol{P_4}$ is a lower-triangular block matrix and that by induction we have that

$$\boldsymbol{P_4}^n = \begin{Vmatrix} \frac{3}{5}\left(4 + (-1)^n \left(\frac{1}{4}\right)^n\right) & -\frac{3}{5}\left(4 + (-1)^n \left(\frac{1}{4}\right)^n\right) & 0 & 0 & 0 \\ \frac{2}{5}\left(6 + (-1)^{n+1} \left(\frac{1}{4}\right)^n\right) & \frac{2}{5}\left(-6 + \left(-\frac{1}{4}\right)^n\right) & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 - \left(\frac{2}{3}\right)^n & \left(\frac{2}{3}\right)^n & 0 \\ \frac{12}{5}\left(1 + (-1)^{n+1} \left(\frac{1}{4}\right)^n\right) & \frac{12}{5}\left(-1 + \left(-\frac{1}{4}\right)^n\right) & 0 & 0 & 0 \end{Vmatrix}. \tag{6}$$

Thus, from the initial transition matrix we see that $P_{01} = 3/4 > 0$ and $P_{10} = 1/2 > 0$ so that state 0 communicates with state 1. From (6) we see that for all $m \geq 0$ we have that $P_{0i}^m = 0$ and $P_{1i}^m = 0$ for $i = 2, 3, 4$ so that states 0 and 1 do not communicate with state 2,3, nor 4. Thus, $\{0, 1\}$ forms an equivalence class. We also see from (6) that for all $m \geq 0$ we have that $P_{2i}^m = 0$ for $i = 0, 1, 3, 4$ so that state 2 does not communicate with states 0, 1, 3, nor 4 and $\{2\}$ forms an equivalence class. Similarly, we see from (6) that for all $m \geq 0$ we have that $P_{i3}^m = 0$ and $P_{j4}^m = 0$ for $i = 0, 1, 2, 4$ and $j = 0, 1, 2, 3$. Thus state 3 does not communicate with any other states and state 4 does not communicate with any other states. Thus, the last equivalence classes are formed by $\{3\}$ and $\{4\}$. Finally, from (6) we can see that

$$\sum_{n=1}^{\infty} P_{00}^n = \sum_{n=1}^{\infty} \frac{3}{5}\left(4 + (-1)^n \left(\frac{1}{4}\right)^n\right) = \infty$$
$$\sum_{n=1}^{\infty} P_{22}^n = \sum_{n=1}^{\infty} 1 = \infty$$

and the equivalence classes $\{0, 1\}$ and $\{2\}$ are recurrent while

$$\sum_{n=1}^{\infty} P_{33}^n = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = 2 < \infty$$
$$\sum_{n=1}^{\infty} P_{44}^n = 0 < \infty$$

so the equivalence classes $\{3\}$ and $\{4\}$ are transient.

$\square$

Problem 4.16. Show that if state i is recurrent and state i does not communicate with state j, then $P_{ij} = 0$. This implies that once a process enters a recurrent class of states it can never leave that class. For this reason, a recurrent class is often referred to as a *closed* class.

Solution. Suppose that state i is recurrent and that state i does not communicate with state j. If state i does not communicate with state j, then either state j is not accessible from state i or state i is not accessible from state j, i.e. for all $m \in \mathbb{N}$, either $P_{ij}^m = 0$ or $P_{ji}^m = 0$.

If $P_{ij}^m = 0$ for all $m \in \mathbb{N}$, then it is true in particular for $m = 1$ so that $P_{ij} = 0$ and we are done.

Now, suppose to the contrary that state i is not accessible from state j, i.e. $P_{ji}^m = 0$ for all $m \in \mathbb{N}$, but $P_{ij} > 0$ with state i a recurrent state. If $P_{ij} > 0$, then there is a non-zero probability of entering state j from state i. However, if $P_{ji}^m = 0$ for all $m \in \mathbb{N}$, then once entering state j from state i we will never re-enter state i. However, this contradicts the assumption that state i is recurrent, i.e. that it occurs with probability 1 that starting from state i we will eventually transition to state i. Therefore, in either case, if state i is recurrent and state i does not communicate with state j, then $P_{ij} = 0$. $\square$