Homework Assignment 7

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Problem 3.2. For those processes in Problem 3.1 that are causal.

Solution. From problem 3.1, the following processes are causal:

a. $X_t + 0.2X_{t-1} - 0.48X_{t-2} = Z_t$

b. $X_t + 0.6X_{t-1} = Z_t + 1.2Z_{t-1}$

c. $X_t + 1.8X_{t-1} + 0.81X_{t-2} = Z_t$.

$\square$

Problem 3.4. Compute the ACF and PACF of the AR(2) process

$$X_t = 0.8X_{t-2} + Z_t, \quad \{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$$

Solution. This process is equivalently written as

$$X_t - 0X_{t-1} - 0.8X_{t-2} = Z_t.$$

Thus, $\phi_0 = 1$, $\phi_1 = 0$, $\phi_2 = 0.8$, and $\phi_k = 0$ for $k > 2$ and $\theta_0 = 1$, $\theta_k = 0$ for $k > 0$. The ACF, $\rho(h)$, is defined as

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)}$$

where $\gamma(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|}$ and $\psi_0 = 1$, $\psi_j = \theta_j + \sum_{k=1}^{p} \phi_k \psi_{j-k}$ for $j > 1$ and $p = 2$.

Using the coefficients ϕ_k and θ_k, we see that

$$\psi_j = \phi_1 \psi_{j-1} + \phi_2 \psi_{j-2} = 0.8\psi_{j-2}$$

where $\psi_j = 0$ if $j < 0$.

If $j = 2k + 1$ for $k \geq 0$, then $\psi_j = 0$ and if $j = 2k$ for $k \geq 0$, then $\psi_j = 0.8^k$. These two formulations are easily proved via induction.

Now,

$$\begin{aligned} \gamma(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|} &= \sigma^2 \left(\sum_{k=0}^{\infty} \psi_{2k} \psi_{2k+|h|} + \sum_{j=0}^{\infty} \psi_{2k+1} \psi_{2k+1+|h|} \right) \\ &= \sigma^2 \sum_{k=0}^{\infty} \psi_{2k} \psi_{2k+|h|} \end{aligned}$$

since $\psi_j = 0$ for even j.

If $h = 2l+1$, then $\gamma(h) = \sigma^2 \sum_{k=0}^{\infty} \psi_{2k}\psi_{2k+|2l+1|} = 0$ since $\psi_{2k+|2l+1|} = 0$. If $h = 2l$, then

$$\gamma(h) = \sigma^2 \sum_{k=0}^{\infty} \psi_{2k}\psi_{2(k+|l|)+1} = \sigma^2 (0.8)^{|l|} \sum_{k=0}^{\infty} (0.8^2)^k = \frac{\sigma^2(0.8^{|l|})}{0.36}$$

Therefore,

$$\rho(h) = \begin{cases} 0 & \text{if } h = 2l+1 \\ (0.8)^{|l|} & \text{if } h = 2l \end{cases}$$

For an AR(p) process, for the PACF function $\alpha(h)$,

$$\alpha(p) = \phi_p \quad \text{and} \quad \alpha(h) = 0 \text{ if } h > p.$$

Thus, we need only compute $\alpha(1)$. Now, $\alpha(1) = \gamma(1)/\gamma(0) = 0$. Therefore,

$$\alpha(h) = \begin{cases} 1 & \text{if } h = 0 \\ 0.8 & \text{if } h = 2 \\ 0 & \text{otherwise} \end{cases}.$$

□

Problem 3.6. Show that the two MA(1) processes

$$X_t = Z_t + \theta Z_{t-1}, \quad \{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$$
$$Y_t = \widetilde{Z}_t + \theta \widetilde{Z}_{t-1}, \quad \{\widetilde{Z}_t\} \sim \mathrm{WN}(0, \sigma^2)$$

where $0 < |\theta| < 1$, have the same autocovariance functions.

Solution. Note that for an ARMA(p, q) the autocovariance function $\gamma(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|}$ where $\psi_0 = 1$, $\psi_j = \theta_j + \sum_{k=1}^{p} \phi_k \psi_{j-k}$ for $j > 1$ and $\psi_j = 0$ for $j < 0$.

For X_t, we have

$$\psi_j = \begin{cases} 1 & \text{if } j = 0 \\ \theta & \text{if } j = 1 \\ 0 & \text{if } j > 1 \end{cases}$$

and for Y_t, we have

$$\widetilde{\psi}_j = \begin{cases} 1 & \text{if } j = 0 \\ \frac{1}{\theta} & \text{if } j = 1 \\ 0 & \text{if } j > 1 \end{cases}$$

Thus,

$$\gamma_X(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|} = \sigma^2 \sum_{j=0}^{1} \psi_j \psi_{j+|h|} = \sigma^2 (\psi_{|h|} + \theta \psi_{1+|h|})$$

and

$$\gamma_Y(h) = \sigma^2\theta^2 \sum_{j=0}^{\infty} \widetilde{\psi}_j \widetilde{\psi}_{j+|h|} = \sigma^2\theta^2 \sum_{j=0}^{1} \widetilde{\psi}_j \widetilde{\psi}_{j+|h|} = \sigma^2\theta^2 \left(\psi_{|h|} + \frac{1}{\theta}\psi_{1+|h|} \right).$$

Explicitly,

$$\gamma_X(h) = \begin{cases} \sigma^2(1+\theta^2) & \text{if } h = 0 \\ \sigma^2\theta & \text{if } |h| = 1 \\ 0 & \text{if } |h| > 1 \end{cases}$$

and

$$\gamma_X(h) = \begin{cases} \sigma^2\theta^2\left(1+\frac{1}{\theta^2}\right) & \text{if } h = 0 \\ \sigma^2\theta^2\left(\frac{1}{\theta}\right) & \text{if } |h| = 1 \\ 0 & \text{if } |h| > 1 \end{cases}$$

$\square$

Problem 3.10.

Solution. $\square$