Homework Assignment 4

Matthew Tiger

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Problem 2.3. Find the ACVF of the time series $X_t = Z_t + aZ_{t-1} + bZ_{t-2}$ where $Z_t \sim WN(0, \sigma^2)$ when:

a. $a = 0.3$, $b = -0.4$, and $\sigma^2 = 1$.

b. $a = -1.2$, $b = -1.6$, and $\sigma^2 = 0.25$.

Solution. The ACVF of the time series $\{X_t\}$, $\gamma_X(h)$, is by definition:

$$\begin{aligned}
\gamma_X(h) &= \mathrm{Cov}(X_{t+h}, X_t) \\
&= \mathrm{Cov}(Z_{t+h} + aZ_{t+h-1} + bZ_{t+h-2}, Z_t + aZ_{t-1} + bZ_{t-2}) \\
&= \mathrm{Cov}(Z_{t+h}, Z_t) + a\mathrm{Cov}(Z_{t+h}, Z_{t-1}) + b\mathrm{Cov}(Z_{t+h}, Z_{t-2}) \\
&\quad + a\mathrm{Cov}(Z_{t+h-1}, Z_t) + a^2\mathrm{Cov}(Z_{t+h-1}, Z_{t-1}) + ab\mathrm{Cov}(Z_{t+h-1}, Z_{t-2}) \\
&\quad + b\mathrm{Cov}(Z_{t+h-2}, Z_t) + ab\mathrm{Cov}(Z_{t+h-2}, Z_{t-1}) + b^2\mathrm{Cov}(Z_{t+h-2}, Z_{t-2}). \qquad (1)
\end{aligned}$$

Using (1), we can see that since $Z_t \sim WN(0, \sigma^2)$,

$$\gamma_X(h) = \begin{cases} (1 + a^2 + b^2)\sigma^2 & \text{if } h = 0 \\ a(1 + b)\sigma^2 & \text{if } h = \pm 1 \\ b\sigma^2 & \text{if } h = \pm 2 \\ 0 & \text{otherwise} \end{cases}.$$

Therefore, when

a. $a = 0.3$, $b = -0.4$, and $\sigma^2 = 1$, the ACVF of $\{X_t\}$ is:

$$\begin{cases} 1.25 & \text{if } h = 0 \\ 0.18 & \text{if } h = \pm 1 \\ -0.4 & \text{if } h = \pm 2 \\ 0 & \text{otherwise} \end{cases}$$

b. $a = -1.2$, $b = -1.6$, and $\sigma^2 = 0.25$, the ACVF of $\{X_t\}$ is:

$$\begin{cases} 1.25 & \text{if } h = 0 \\ 0.18 & \text{if } h = \pm 1 \\ -0.4 & \text{if } h = \pm 2 \\ 0 & \text{otherwise} \end{cases}$$

□

Problem 2.5. Suppose that $\{X_t, t = 0, \pm 1, \dots\}$ is stationary and that $|\theta| < 1$. Show that for each fixed n the sequence

$$S_m = \sum_{j=1}^{m} \theta^j X_{n-j}$$

is convergent absolutely and in mean square as $m \to \infty$.

Solution. Let $a_j = \theta^j X_{n-j}$. Since each X_i is a random variable, each X_i maps to a real, non-infinite value so let $X = \max\{|X_i|\}$. Then to see that S_m is convergent absolutely as $m \to \infty$, notice that

$$\begin{aligned}
\sum_{j=1}^{m} |a_j| &= \sum_{j=1}^{m} |\theta^j X_{n-j}| \\
&= \sum_{j=1}^{m} |\theta|^j |X_{n-j}| \\
&\leq \sum_{j=1}^{m} X |\theta|^j = \sum_{j=1}^{m} b_j = T_m
\end{aligned}$$

Since $|\theta| < 1$, we know that as $m \to \infty$, the partial sum $\sum_{j=1}^{m} X|\theta|^j \to 0$ and it must hold that $T_m \to 0$. Thus, we know that as $m \to \infty$, $\sum_{j=1}^{m} |a_j|$ converges to some L since $|a_j| \leq b_j$ and T_m is convergent. Therefore, S_m is convergent absolutely.

To see that S_m is convergent in the mean square, it suffices to show that $\mathrm{E}(S_m - S_l)^2 \to 0$ as $m, l \to \infty$.

Without loss of generality, assume that $m > l > 0$. Notice that $S_m - S_l = \sum_{j=1}^{m} a_j - \sum_{j=1}^{n} a_j = \sum_{j=l+1}^{m} a_j$. Thus,

$$\mathrm{E}(S_m - S_l) = \mathrm{E}(\sum_{j=l+1}^{m} a_j) = \sum_{j=l+1}^{m} \mathrm{E}(a_j).$$

It is clear that $\mathrm{E}(a_j) = \mathrm{E}(\theta^j X_{n-j}) = \theta^j \mathrm{E}(X_{n-j})$. Since $\{X_t\}$ is a stationary time series, its expectation does not depend on t, so say $\mathrm{E}(X_{n-j}) = \mu_X$. Then

$$\begin{aligned}
\mathrm{E}(S_m - S_l) &= \sum_{j=l+1}^{m} \theta^j \mathrm{E}(X_{n-j}) \\
&= \mu_X \sum_{j=l+1}^{m} \theta^j \\
&= \frac{\mu_X \theta^{l+1}(1 - \theta^{m-l-1})}{1 - \theta}
\end{aligned}$$

Since $|\theta| < 1$, it is clear then that $\mathrm{E}(S_m - S_l)^2 \to 0$ as $m, l \to \infty$ showing that S_m is convergent in mean square for any n. □