Homework Assignment 1

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Problem 3.7. Suppose $p(x, y, z)$, the joint probability mass function of the random variables X, Y, and Z, is given by

$$\begin{aligned} p(1,1,1) &= \tfrac{1}{8}, & p(2,1,1) &= \tfrac{1}{4}, \\ p(1,1,2) &= \tfrac{1}{8}, & p(2,1,2) &= \tfrac{3}{16}, \\ p(1,2,1) &= \tfrac{1}{16}, & p(2,2,1) &= 0, \\ p(1,2,2) &= 0, & p(2,2,2) &= \tfrac{1}{4}. \end{aligned}$$

What is $E[X|Y = 2]$? What is $E[X|Y = 2, Z = 1]$?

Solution. Recall that the conditional probability mass function of X given that $Y = y$ is given by

$$p_{X|Y}(x|y) = P\{X = x|Y = y\} = \frac{P\{X = x, Y = y\}}{P\{Y = y\}}.$$

As a natural extension, we have that the conditional expectation of X given that $Y = y$ is given by

$$E[X|Y = y] = \sum_x xP\{X = x|Y = y\} = \sum_x xp_{X|Y}(x|y).$$

Thus, in order to find the conditional expectation of X given that $Y = 2$, i.e. $E[X|Y = 2]$, we first need to determine $p_{X|Y}(x|2)$. We note from the above joint probability mass function that

$$P\{Y = 2\} = \sum_{x,z} p(x, 2, z) = p(1, 2, 1) + p(2, 2, 1) + p(1, 2, 2) + p(2, 2, 2) = \frac{5}{16}.$$

Similarly, we have from the above joint probability mass function that

$$P\{X = x, Y = 2\} = \sum_z p(x, 2, z) = p(x, 2, 1) + p(x, 2, 2).$$

Thus, the conditional probability mass function of X given that $Y = 2$ is given by

$$p_{X|Y}(x|2) = \frac{P\{X = x, Y = 2\}}{P\{Y = 2\}} = \begin{cases} \frac{p(1,2,1)+p(1,2,2)}{5/16} = \frac{1}{5} & \text{if } x = 1 \\ \frac{p(1,2,1)+p(1,2,2)}{5/16} = \frac{4}{5} & \text{if } x = 2. \end{cases}$$

Using $p_{X|Y}(x|2)$, we readily see that

$$E[X|Y = 2] = \sum_x x p_{X|Y}(x|2) = 1 \cdot p_{X|Y}(1|2) + 2 \cdot p_{X|Y}(2|2) = \frac{9}{5}.$$

In order to find the conditional expectation of X given that $Y = 2$ and $Z = 1$, i.e. $E[X|Y = 2, Z = 1]$, we proceed in a similar manner as previously by first finding $p_{X|Y,Z}(x|2, 1)$. We note from the above joint probability mass function that

$$P\{Y = 2, Z = 1\} = \sum_x p(x, 2, 1) = p(1, 2, 1) + p(2, 2, 1) = \frac{1}{16}$$

Similarly, we have from the above joint probability mass function that

$$P\{X = x, Y = 2, Z = 1\} = p(x, 2, 1).$$

Thus, the conditional probability mass function of X given that $Y = 2$ and $Z = 1$ is given by

$$p_{X|Y,Z}(x|2, 1) = \frac{P\{X = x, Y = 2, Z = 1\}}{P\{Y = 2, Z = 1\}} = \begin{cases} \frac{p(1,2,1)}{1/16} = 1 & \text{if } x = 1 \\ \frac{p(2,2,1)}{1/16} = 0 & \text{if } x = 2. \end{cases}$$

Using $p_{X|Y,Z}(x|2, 1)$, we readily see that

$$E[X|Y = 2, Z = 1] = \sum_x x p_{X|Y,Z}(x|2, 1) = 1 \cdot p_{X|Y,Z}(1|2, 1) + 2 \cdot p_{X|Y,Z}(2|2, 1) = 1.$$

$\square$

Problem 3.8. An unbiased die is successively rolled. Let X and Y denote, respectively, the number of rolls necessary to obtain a six and a five. Find:

a. $E[X]$,

b. $E[X|Y = 1]$,

c. $E[X|Y = 5]$.

Solution. The experiment of rolling a die, assuming the die is six-sided, has six possible outcomes: the die lands oriented such that the side with 1, 2, 3, 4, 5, or 6 pips is face-up. Assuming the die is unbiased, each outcome occurs with probability $p = 1/6$ and each trial of rolling the die is independent of any other trial. If X and Y denote, respectively, the number of rolls necessary to obtain a six and a five, then under the given assumptions, X and Y are both geometric random variables with parameter $p = 1/6$. The probability mass function for these random variables is given by $p(n) = (1 - p)^{n-1}p = (5/6)^{n-1}(1/6)$.

a. Let Z be the random variable defined as $Z = 1$ if the result of the first roll is a six and $Z = 0$ if the result of the first roll is not a six. We may compute $E[X]$ by conditioning on the variable Z. Note that, by conditioning, we obtain

$$\begin{aligned} E[X] &= \sum_z E[X|Z = z]P\{Z = z\} \\ &= \left[\frac{1}{6}\right] E[X|Z = 1] + \left[\frac{5}{6}\right] E[X|Z = 0]. \end{aligned}$$

If $Z = 1$, then the result of the first roll is a six, so the number of rolls to obtain a six is clearly 1 and $E[X|Z = 1] = 1$. Likewise, if the result of the first roll is not a six, then the expected number of rolls to obtain a six given that the first roll is not a six is 1 more than the expected number of rolls to obtain a six so that $E[X|Z = 0] = 1+E[X]$. Therefore,

$$\begin{aligned} E[X] &= \left[\frac{1}{6}\right] E[X|Z = 1] + \left[\frac{5}{6}\right] E[X|Z = 0] \\ &= \frac{1}{6} + \left[\frac{1}{6}\right] (1 + E[X]) \end{aligned}$$

which implies that $E[X] = 6$.

b. We wish to find $E[X|Y = 1]$, i.e. the expected number of rolls to obtain a six given that the first roll is a five. Using the same reasoning as in part a, we know that the expected number of rolls to obtain a six given that the first roll is not a six (it's a five) is 1 more than the expected number of rolls to obtain a six. Therefore,

$$E[X|Y = 1] = 1 + E[X] = 7$$

where we used the result previously obtained that $E[X] = 6$.

c. In order to calculate $E[X|Y = y]$ for some $y > 1$, we first compute $p_{X|Y}(x|y)$. Suppose that $Y = y$ for some $y > 1$. From this we gather that the first $y - 1$ trials result in not rolling a five while the y-th trial results in rolling a five.

As a consequence, if $X = x$ where $x < y$ then the first x trials have only five possible outcomes with the x-th trial resulting in a success out of those five outcomes. Thus,

$$P\{X = x|Y = y\} = \frac{1}{5}\left[\frac{4}{5}\right]^{x-1},$$

i.e. for $x < y$ the conditional probability that $X = x$ given that $Y = y$ is the probability mass function of a geometric random variable with parameter $p = 1/5$.

Note that if $X = x$ where $x = y$, then

$$P\{X = x|Y = y\} = 0$$

since it cannot happen that on the y-th trial the outcome of the trial is that both a five and a six were rolled.

Finally, if $X = x$ where $x > y$, then as mentioned, the first $y - 1$ trials do not result in a five, but after the y-th trial the result obtained can in fact be a five. Thus, the first $y - 1$ failures each occur with probability $4/5$ while the y-th failure occurs with probability 1. However, after that, the failures of the trials $y+1$ through $x-1$ all occur with probability $5/6$ since it is possible for the die to roll a five during these trials. On the x-th trial the trial succeeds with probability $1/6$. Thus, if $x > y$, then

$$P\{X = x|Y = y\} = \frac{1}{6}\left[\frac{4}{5}\right]^{y-1}\left[\frac{5}{6}\right]^{x-y-1}.$$

Combining the above statements, we see that the conditional probability mass function that $X = x$ given that $Y = y$ with $y > 1$ is given by

$$p_{X|Y}(x|y) = \begin{cases} \frac{1}{5}\left[\frac{4}{5}\right]^{x-1} & \text{if } x < y \\ 0 & \text{if } x = y \\ \frac{1}{6}\left[\frac{4}{5}\right]^{y-1}\left[\frac{5}{6}\right]^{x-y-1} & \text{if } x > y \end{cases}.$$

Therefore, we have that the expected value of X given that $Y = 5$ is

$$\begin{aligned} E[X|Y = 5] &= \sum_{x=1}^{\infty} x p_{X|Y}(x|5) \\ &= \frac{1}{5}\sum_{x=1}^{4} x\left[\frac{4}{5}\right]^{x-1} + \frac{1}{6}\left[\frac{4}{5}\right]^{4}\sum_{x=6}^{\infty} x\left[\frac{5}{6}\right]^{x-6} \\ &= \frac{3637}{625} \approx 5.82. \end{aligned}$$

□

Problem 3.9. Show in the discrete case that if X and Y are independent, then

$$E[X|Y = y] = E[X] \text{ for all y.}$$

Solution. Suppose that X and Y are discrete, independent random variables. Then, due to the independence of the random variables, we know that

$$\begin{aligned} P\{X = x|Y = y\} &= \frac{P\{X = x, Y = y\}}{P\{Y = y\}} \\ &= \frac{P\{X = x\}P\{Y = y\}}{P\{Y = y\}} \\ &= P\{X = x\}. \end{aligned} \tag{1}$$

Therefore, combining the definition of conditional expectation for discrete random variables and result (1), we have that for any y,

$$E[X|Y = y] = \sum_x xP\{X = x|Y = y\} = \sum_x xP\{X = x\} = E[X]$$

and we are done. $\square$

Problem 3.10. Suppose X and Y are independent continuous random variables. Show that

$$E[X|Y = y] = E[X] \text{ for all y.}$$

Solution. Suppose that X and Y are continuous, independent random variables with probability density functions $f_X(x)$ and $f_Y(y)$, respectively. The conditional probability density function of X given that $Y = y$ is given by

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

where $f(x,y)$ is the joint probability density function of X and Y. Due to the independence of the random variables X and Y, we have that $f(x,y) = f_X(x)f_Y(y)$. Thus, if X and Y are independent, then the conditional probability density function of X given that $Y = y$ is

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f_X(x)f_Y(y)}{f_Y(y)} = f_X(x). \tag{2}$$

Therefore, combining the definition of conditional expectation for continuous random variables and result (2), we have that for any y,

$$E[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx = \int_{-\infty}^{\infty} x f_X(x)dx = E[X]$$

and we are done. □

Problem 3.13. Let X be exponential with mean $1/\lambda$; that is,

$$f_X(x) = \lambda e^{-\lambda x}, \quad 0 < x < \infty.$$

Find $E[X|X > 1]$.

Solution. Suppose that X is an exponential random variable with mean $1/\lambda$. Let Y be the discrete random variable defined as

$$Y = \begin{cases} 1 & \text{if } X > 1 \\ 0 & \text{if } 0 < X \leq 1 \end{cases}$$

with probability mass function

$$p_Y(y) = \begin{cases} P\{X > 1\} & \text{if } Y = 1 \\ P\{X \leq 1\} & \text{if } Y = 0 \end{cases}$$

From these definitions of X and Y, we see that the conditional density function of X given $Y = 1$ is given by

$$f_{X|Y}(x|1) = \frac{f(x,1)}{p_Y(1)} = \begin{cases} \frac{f_X(x)}{P\{Y=1\}} & \text{if } x > 1 \\ 0 & \text{if } 0 < x \leq 1 \end{cases}$$

where we know that if $x > q$ then

$$\begin{aligned} \frac{f_X(x)}{P\{Y = 1\}} &= \frac{\lambda e^{-\lambda x}}{1 - P\{X \leq 1\}} \\ &= \frac{\lambda e^{-\lambda x}}{P\{X > 1\}} \\ &= \frac{\lambda e^{-\lambda x}}{1 - \int_0^1 \lambda e^{-\lambda x} dx} \\ &= \frac{\lambda e^{-\lambda x}}{e^{-\lambda}}. \end{aligned}$$

Thus, by definition, we now have that

$$\begin{aligned} E[X|X > 1] &= E[X|Y = 1] \\ &= \int_{-\infty}^{\infty} x f_{X|Y}(x|1) dx \\ &= \int_1^{\infty} x \frac{\lambda e^{-\lambda x}}{e^{-\lambda}} dx \\ &= \lambda e^{\lambda} \int_1^{\infty} x e^{-\lambda x} dx \\ &= \lambda e^{\lambda} \left[\frac{e^{-\lambda}(1 + \lambda)}{\lambda^2} \right] \\ &= \frac{1 + \lambda}{\lambda}. \end{aligned}$$

□

Problem 3.14. Let X be uniform over $(0, 1)$. Find $E[X|X < 1/2]$.

Solution. □