Exam 1

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Problem 1. You pay into an annuity a sum of $\$P$ dollars. This annuity pays you $\$\alpha$ per year, compounded monthly. The interest is $r\%$ and is calculated as simple interest on the remaining balance at the end of each month. If $A(n)$ is the amount remaining at the end of the n-th month, with $A(0) = P$, write down $A(n+1)$ in terms of $A(n)$ and deduce a closed form solution for $A(n)$.

If $P = \$100,000$, $\alpha = \$500$, and the interest rate is 4% per month, how long will the annuity last?

Solution. $\square$

Problem 2. Let $g_\mu(x) = \mu x \dfrac{(1-x)}{(1+x)}$, for $\mu > 0$.

a) Show that g_μ has a maximum at $x = \sqrt{2} - 1$ and the maximum value is $\mu(3 - 2\sqrt{2})$.

b) Deduce that g_μ is a dynamical system on $[0, 1]$ for $0 \leq \mu \leq 3 + 2\sqrt{2}$, i.e. $g_\mu([0, 1]) \subseteq [0, 1]$.

c) Find the fixed points of g_μ for $mu \geq 1$.

d) Find g_μ' and determine whether the fixed points are attracting or repelling.

e) Use a graphing utility to graph g_μ^2 and g_μ^3 and estimate when a period 2 point is created.

Solution. $\square$

Problem 3. Consider the family of functions $f_\lambda(x) = x^3 - \lambda x$ for some parameter $\lambda \in \mathbb{R}$.

a) Find all fixed points and determine their nature and where they are created as λ varies.

b) Find where a 2-cycle is created and give the graph of where this happens. Determine the stability of the hyperbolic 2-cycles.

c) Use a graphing utility to find an approximate value of λ where the 3-cycle is created. Give the graph of this situation.

Solution. $\square$

Problem 4. Let f be a 4-times continuously differentiable function. Its Newton function is $N_f(x) = x - f(x)/f'(x)$. Suppose that c is a zero of f. If $Sf(x)$ is the Schwarzian derivative of f, show that

$$N_f'''(c) = 2Sf(c)$$

Solution. □

Problem 5. Let $f : [0,1] \to [0,1]$ be continuous on $[0,1]$ and differentiable on $(0, 1)$ with $|f'(x)| < 1$ for all $x \in (0,1)$.

a) Prove that f has a unique fixed point p in $[0,1]$.

b) Prove that f cannot have a point of period 2 in $[a,b]$.

c) Prove that $f^n(x) \to p$ as $n \to \infty$ for all $x \in (0,1)$.

Solution. □

Problem 6. Let $f(x) = ax^3 + bx + c$ where a and b satisfy $a/b > 0$. Denote by N_f the corresponding Newton function.

a) Show that N_f has a unique fixed point.

b) Show that N_f cannot have any period 2 points.

c) Why does it follow that N_f has no points of period n for $n > 2$?

Solution. □

Problem 7. a) Show that the function $f(x) = -1/(x+1)$ has the property that $f^3(x) = x$ for all $x \neq -1, 0$.

b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined on a set I, with $f^3(x) = x$ for all $x \in I$. Set $g(x) = f^2(x)$. Show that $g^3(x) = x$ for all $x \in I$. Deduce a function different from that in a) that has this property.

c) In general, show that such a function cannot have a 2-cycle.

d) Deduce that a function $f : \mathbb{R} \to \mathbb{R}$ with the property $f^3(x) = x$ cannot be continuous.

e) Show that the inverse of f must exist.

f) If $f'(x)$ exists for all $x \in I$, show that the 3-cycles are non-hyperbolic where f is not the identity map.

g) Suppose that $f(x) = \dfrac{ax+b}{cx+d}$ satisfies $f^3(x) = x$. Show that if f is not the identity map and $a \neq d$, then $a^2 + bc + ad + d^2 = 0$.

   i) Use this to find other functions with the property $f^3(x) = x$.

   ii) Deduce that if $ad - bc > 0$, then such a function cannot have any fixed points.

Solution. $\square$