Homework Assignment 5

Matthew Tiger

October 10, 2016

Problem 2.6.9. i. Use the results of section 2.6 to show that the logistic map $L_4(x) = 4x(1-x)$ cannot have a super-attracting cycle.

ii. Find a point $x_0 \in (0, 1)$ which is not a periodic point for L_4.

Solution. i. Suppose that $k > 1$ and x_k is a period k point so that $\{x_1, x_2, \ldots, x_k\}$ is a k-cycle with $L_4^i(x_1) = x_i$ for $0 < i < k$ and $L_4^k(x_1) = x_1$. This cycle will be super-attracting if

$$\prod_{i=1}^{k} L_4'(x_i) = 0.$$

Note that $L_4'(x) = 4 - 8x = 0$ only if $x = 1/2$. Thus, the cycle will be super attracting if and only if $x_i = 1/2$ for some $i = 1, \ldots k$. Note that the point $x = 1/2$ does not generate a cycle since $L_4(1/2) = 1$ and $L_4^n(1/2) = 0$ for $n > 1$ so $x_1 \neq 1/2$.

We will now demonstrate that there is no point $x \in [0, 1]$ such that $L_4^n(x) = 1/2$ for $n > 0$. It has been shown previously that

$$L_4^n(x) = \sin^2\left(2^n \sin^{-1}\left(\sqrt{x}\right)\right) = \sin^2(\theta)$$

for some $\theta \in (0, \pi/2]$. Note that for $\theta_1, \theta_2 \in (0, \pi/2]$, we have that $\sin^2(\theta_1) = 1/2$ if and only if $\theta_1 = \pi/4$ and $\sin^2(\theta_2) = \pi/4$ if and only if $\theta_2 = \sin^{-1}(\sqrt{\pi}/2) > 1$. However, since $\theta_2 > 1$, there is no $\theta \in (0, \pi/2]$ such that $\sin^2(\theta) = \theta_2$.

So there is no $x \in [0, 1]$ such that $L_4^n(x) = \theta_2$ for any $n > 0$ and hence no $n > 0$ such that $L_4^n(x) = 1/2$. Thus, $x_i = L_4^i(x_1) \neq 1/2$ for any $i > 0$ so that $L_4'(x_i) \neq 0$. Therefore, L_4 has no super-attracting cycle.

ii. As was shown previously, $x = 1/2$ is such that $L_4(x) = 1$ and $L_4^n(x) = 0 \neq 1/2$ for $n > 1$. Therefore $x = 1/2$ is not a periodic point of L_4.

$\square$

Problem 2.8.3. Show that

$$\left\{\frac{\mu}{1+\mu^3}, \frac{\mu^2}{1+\mu^3}, \frac{\mu^3}{1+\mu^3}\right\}$$

is a 3-cycle for T_μ when $\mu \geq (1+\sqrt{5})/2$.

Solution. The tent map is defined as

$$T_\mu(x) := \begin{cases} \mu x & 0 \leq x \leq 1/2 \\ \mu(1-x) & 1/2 < x \leq 1 \end{cases}.$$

Now suppose that $\mu \geq (1+\sqrt{5})/2 > 1$ and let $x_1 = \frac{\mu}{1+\mu^3}$. We will now show that

$$T_\mu(x_1) = \frac{\mu^2}{1+\mu^3} = x_2, \quad T_\mu^2(x_1) = T_\mu(x_2) = \frac{\mu^3}{1+\mu^3} = x_3, \quad T_\mu^3(x_1) = T_\mu(x_3) = \frac{\mu}{1+\mu^3} = x_1$$

demonstrating that $\{x_1, x_2, x_3\}$ is a 3-cycle.

Note that $\mu/(1+\mu^3)$ is monotonically decreasing if $\mu \geq (1+\sqrt{5})/2$. Thus,

$$0 \leq \frac{\mu}{1+\mu^3} \leq \frac{\frac{1+\sqrt{5}}{2}}{1+\left(\frac{1+\sqrt{5}}{2}\right)^3} = \frac{-1+\sqrt{5}}{4} \leq \frac{1}{2}.$$

Hence,

$$T_\mu(x_1) = \mu\left(\frac{\mu}{1+\mu^3}\right) = \frac{\mu^2}{1+\mu^3} = x_2.$$

Similarly, we see that $\mu^2/(1+\mu^3)$ is monotonically decreasing if $\mu \geq (1+\sqrt{5})/2$ so

$$0 \leq \frac{\mu^2}{1+\mu^3} \leq \frac{\left(\frac{1+\sqrt{5}}{2}\right)^2}{1+\left(\frac{1+\sqrt{5}}{2}\right)^3} = \frac{1}{2}.$$

Thus,

$$T_\mu(x_2) = \mu\left(\frac{\mu^2}{1+\mu^3}\right) = \frac{\mu^3}{1+\mu^3} = x_3.$$

Lastly, if $\mu \geq (1+\sqrt{5})/2$ then $\mu^3/(1+\mu^3)$ is monotonically increasing so that

$$\frac{1}{2} \leq \frac{1+\sqrt{5}}{4} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^3}{1+\left(\frac{1+\sqrt{5}}{2}\right)^3} \leq \frac{\mu^3}{1+\mu^3} \leq 1$$

Therefore,

$$T_\mu(x_3) = \mu\left(1 - \frac{\mu^3}{1+\mu^3}\right) = \frac{\mu}{1+\mu^3} = x_1$$

and $\{x_1, x_2, x_3\}$ is a 3-cycle.

$\square$

Problem 3.2.5. Show that the map $f(x) = (x - 1/x)/2$, $x \neq 0$, has no fixed points but it has period 2-points. Find the 2-cycle, and by looking at the graph of $f^3(x)$, check to see whether or not it has a 3-cycle. Why does this not contradict Sharkovskys Theorem?

Solution. The function f will have a fixed point if and only if the function $g(x) = f(x) - x$ has real roots. We see that

$$g(x) = f(x) - x = \frac{x^2 - 1}{2x} - x = -\frac{x^2 + 1}{2x} = 0$$

if and only if $x^2 + 1 = 0$. Thus, g has no real roots and f has no fixed points.

If $h(x) = f^2(x) - x$ has real solutions, then these solutions give rise to a 2-cycle of f. Thus,

$$h(x) = f(f(x)) - x = \frac{\left(\frac{x^2-1}{2x}\right)^2 - 1}{2\left(\frac{x^2-1}{2x}\right)} - x = -\frac{-3x^4 - 2x^2 + 1}{-4x^3 + 4x} = 0$$

if and only if $-3x^4 - 2x^2 + 1 = 0$, the real solutions of which are given by $3^{-1/2}$ and $-3^{-1/2}$. Hence, $\{3^{-1/2}, -3^{-1/2}\}$ is a 2-cycle of f.

The graphs of $f^3(x)$ and $y = x$ are shown in Figure 1. From these graphs we see that the graph of $f^3(x)$ crosses the line $y = x$ at 6 points. Since f has a 2-cycle, 2 of these points arise from the fact that the solutions of $f^2(x) = x$ also satisfy $f^3(x) = x$. However, of the four remaining points, the graph of $f^3(x)$ does not cross the graph $y = x$ at exactly 3 points so a 3-cycle does not arise for $f(x)$.

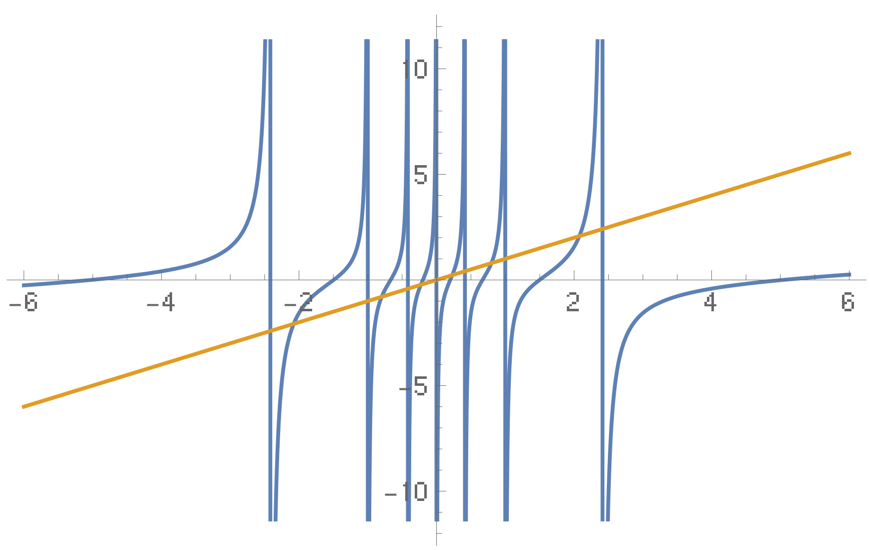

Figure 1: The graphs of $f^3(x)$ (blue) and $y = x$ (orange).

Note that the domain of f is given by $(-\infty, 0) \cup (0, \infty)$. Since the domain of f is not an interval, it does not satisfy the assumptions of Sharkovsky's Theorem and thus the theorem does not apply.

□

Problem 3.2.6. A map $f : [1,7] \to [1,7]$ is defined so that $f(1) = 4$, $f(2) = 7$, $f(3) = 6$, $f(4) = 5$, $f(5) = 3$, $f(6) = 2$, $f(7) = 1$, and the corresponding points are joined so the map is continuous and piece-wise linear. Show that f has a 7-cycle but no 5-cycle.

Solution. The definition of f shows that $f^7(1) = 1$ with $f^n(1) \neq 1$ for $0 < n < 7$. Thus, 1 is a period 7 point of f and $\{1, 4, 5, 3, 6, 2, 7\}$ is a 7-cycle of f.

Let $I_k = [k, k+1]$ for $k = 1, \ldots, 6$. Note that f has one fixed point $c \in I_4$. Suppose to the contrary that $x_1 \neq c$ is a period 5 point of f and $\{x_1, x_2, x_3, x_4, x_5\}$ is a 5-cycle.

Now, suppose that $x_1 \in I_1$. Then the definition of f tells us that

$$f(x_1) \in \bigcup_{k=4}^{6} I_k.$$

This then implies that

$$f^2(x_1) \in \bigcup_{k=1}^{4} I_k, \quad f^3(x_1) \in \bigcup_{k=3}^{6} I_k, \quad f^4(x_1) \in \bigcup_{k=1}^{5} I_k, \quad \text{and} \quad f^5(x_1) \in \bigcup_{k=2}^{6} I_k = [2,7].$$

But if $f^5(x_1) \in [2,7]$, then $f^5(x_1) \neq x_1$ since $x_1 \in [1,2]$ and $f^5(2) = 5$.

Using reasoning similar to that used above, we see for $k = 2, 3, 5, 6$ that

$$f^5(I_2) = \bigcup_{k=3}^{6} I_k, \quad f^5(I_3) = \bigcup_{k=4}^{6} I_k, \quad f^5(I_5) = \bigcup_{k=1}^{4} I_k, \quad \text{and} \quad f^5(I_6) = \bigcup_{k=1}^{5} I_k.$$

Thus, for $k = 2, 3, 5, 6$, we have that if $x_1 \in I_k$ and $x_1 \neq k, k+1$, then $x_1 \notin f^5(I_k)$ and $f^5(x_1) \neq x_1$. Similarly, if $x_1 = k, k+1$, we see from the definition of f that $f^5(x_1) \neq x_1$.

Thus, if x_1 is a period 5 point, then $x_1 \in I_4$ and $f(x_1) \in I_3 \cup I_4$. However, if $f(x_1) \in I_3$, then $f^5(x_1) \in I_1$ so that $f^5(x_1) \neq x_1$ violating the assumption that x_1 is a period 5 point. Thus, we must have that $f(x_1) \in I_4$. This in turn implies that $f^2(x_1) = I_3 \cup I_4$. However, if $f^2(x_1) \in I_3$, then $f^5(x_1) \in I_6$ so that $f^5(x_1) \neq x_1$ again violating the assumption that x_1 is a period 5 point. So we must have that $f^2(x_1) \in I_4$. We can similarly show that we also have that $f^3(x_1), f^4(x_1) \in I_4$. Note that if $x \in I_4$, then $f(x) = -2x + 13$ with fixed point $c = 13/3$. Thus, we see that

$$\begin{aligned}
f^2(x) &= 4x - 13 \\
f^3(x) &= -8x + 39 \\
f^4(x) &= 16x - 65 \\
f^5(x) &= -32x + 143.
\end{aligned}$$

Hence, $f^5(x) - x = -32x + 143 - x = 0$ if and only if $x = 13/3 = c$, a contradiction. Therefore, f has no 5-cycle. $\square$

Problem 3.2.10. Let $f : \mathbb{R} \to \mathbb{R}$. Write down all the possibilities for a 4-cycle $\{a, b, c, d\}$ with $a < b < c < d$ for f (e.g. $f(a) = c$, $f(c) = d$, $f(d) = b$, and $f(b) = a$). Indicate which are mirror images, and which give rise to a 3-cycle.

Solution. Note that if $x = a, b, c, d$, then $f(x) \neq x$ otherwise x would be a fixed point and would not generate a 4-cycle. So, first consider that a generates the 4-cycle and $f(a) = b$. Then $f(b) \neq a$ otherwise $\{a, b\}$ would be a 2-cycle of f. Thus, either $f(b) = c$ of $f(b) = d$. If $f(b) = c$, then $f(c) \neq a$ otherwise $\{a, b, c\}$ would be a 3-cycle and $f(c) \neq b$ otherwise $\{b, c\}$ would be a 2-cycle. So, $f(c) = d$ and thus $f(d) = a$ if the set of points $\{a, b, c, d\}$ generates a 4-cycle. If, on the other hand $f(b) = d$, then $f(d) \neq a$ otherwise $\{a, b, d\}$ be a 3-cycle and $f(d) \neq b$ otherwise $\{b, d\}$ would be a 2-cycle. So $f(d) = c$ and $f(c) = a$ if the set of points $\{a, b, c, d\}$ generates a 4-cycle.

Therefore, if $f(a) = b$, we have two possible 4-cycles

$$\{a, b, c, d\} \text{ and } \{a, b, d, c\}.$$

If $f(a) = c$, we can use similar reasoning to see that $\{a, c, b, d\}$ and $\{a, c, d, b\}$ are 4-cycles and if $f(a) = d$, then $\{a, d, b, c\}$ and $\{a, d, c, b\}$ are 4-cycles. Note, these are the only possible 4-cycles of f.

The mirror image of a 4-cycle $\{x_1, x_2, x_3, x_4\}$ is the 4-cycle such that $f(x_4) = x_3$, $f(x_3) = x_2$, $f(x_2) = x_1$, and $f(x_1) = x_4$, i.e. the 4-cycle $\{x_4, x_3, x_2, x_1\}$. Therefore, $\{a, b, c, d\}$ and $\{a, d, c, b\}$ are mirror images, $\{a, b, d, c\}$ and $\{a, c, d, b\}$ are mirror images, and lastly $\{a, c, b, d\}$ and $\{a, d, b, c\}$ are mirror images.

Proposition 3.1.7 tells us that for I, an interval, and $f : I \to I$, a continuous map, if I_1 and I_2 are closed sub intervals of I with at most one point in common and $I_2 \subset f(I_1)$ and $I_1 \cup I_2 \subset f(I_2)$, then f has a 3-cycle. Throughout, we assume that our function f is continuous. Let $I_1 = [a, b]$, $I_2 = [b, c]$, and $I_3 = [c, d]$.

If $\{a, b, c, d\}$ is a 4-cycle of f, then we see that

$$[c, d] = I_3 \subset f(I_2) = [c, d] \quad \text{and} \quad [b, d] = I_2 \cup I_3 \subset f(I_3) = [a, d]$$

so that a 3-cycle is generated by the proposition. Similarly, for the 4-cycles $\{a, b, d, c\}$, $\{a, c, d, b\}$, and $\{a, d, c, b\}$ we see under these 4-cycles that

$$\begin{aligned} [b, c] = I_2 \subset f(I_3) = [a, c] \quad &\text{and} \quad [b, d] = I_2 \cup I_3 \subset f(I_2) = [a, d], \\ [a, b] = I_1 \subset f(I_2) = [a, d] \quad &\text{and} \quad [a, c] = I_1 \cup I_2 \subset f(I_1) = [a, c], \\ [a, b] = I_1 \subset f(I_2) = [a, b] \quad &\text{and} \quad [a, c] = I_1 \cup I_2 \subset f(I_1) = [a, d], \end{aligned}$$

respectively, so that these 4-cycles give rise to 3-cycles by our proposition. Since the other 4-cycles do not meet the criteria of the proposition, they do not generate 3-cycles.

□

Problem 3.2.11. Use Sharkovsky's Theorem to prove that if $f : [a, b] \to [a, b]$ is a continuous function and $\lim_n f^n(x)$ exists for every $x \in [a, b]$, then f can have no points of period $n > 1$.

Solution. Suppose to the contrary that $f : [a, b] \to [a, b]$ is a continuous function and $\lim_n f^n(x)$ exists for every $x \in [a, b]$ but there is a point $y \in [a, b]$ of period $n > 1$. Then we have that $f^n(y) = y$ with $f^k(y) \neq y$ for $0 < k < n$. As a result, for every $m > n$, we will have that $f^m(y) = f^k(y)$ where $m \equiv k \mod n$.

Since $y \in [a, b]$ we have that $\lim_n f^n(y) = L$ exists by assumption. Thus, for all $\varepsilon > 0$, there exists an N such that if $n \geq N$, then $|f^n(y) - L| < \varepsilon$. Suppose that $N \equiv k \mod n$. This implies that if $|f^N(y) - L| < \varepsilon$ then $|f^k(y) - L| < \varepsilon$. Since for $m > 0$ we have that $N + m > N$, we require that if $|f^N(y) - L| < \varepsilon$, then $|f^{N+m}(y) - L| = |f^{k+m}(y) - L| < \varepsilon$. However, if these inequalities are true for all $\varepsilon > 0$, then this implies that $f^k(y) = f^{k+m}(y) = L$ for $m > 0$. If $k = 0$, then we have that $y = f^0(y) = f^m(y)$ for some $0 < m < n$. If on the other hand $k > 0$, we have for $m = n - k > 0$, that $f^k(y) = f^n(y) = y$.

So in either case, we can find some $0 < k < n$ such that $f^k(y) = y$. This is contradictory to the assumption that y is a point of period n and so our assumption must be false. Therefore, f has no points of period $n > 1$.

□