

Exam 2

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Problem 1. Find the inverse Laplace transform of the function

$$\bar{f}(s) = \frac{s}{(s-a)(s^2+b^2)}$$

for $a, b > 0$, by using the following three different approaches:

- i. Using partial fraction decomposition,
- ii. Applying the Convolution Theorem,
- iii. Applying Heaviside's Expansion Theorem.

Solution. We will now find the inverse Laplace transform of $\bar{f}(s)$ using the respective approaches listed above:

- i. From the partial fractions method, we see that

$$\bar{f}(s) = \frac{s}{(s-a)(s^2+b^2)} = \frac{c_0}{s-a} + \frac{d_1s+d_0}{s^2+b^2}.$$

Combining the rational fractions on the right side under a common denominator and equating the coefficients in the numerator we arrive at the following system of equations

$$\begin{aligned}c_0 + d_1 &= 0 \\d_0 - ad_1 &= 0 \\c_0b^2 - ad_0 &= 0.\end{aligned}$$

Solving this system, we see that $c_0 = \frac{a}{a^2+b^2}$, $d_1 = -\frac{a}{a^2+b^2}$, and $d_0 = \frac{b^2}{a^2+b^2}$. Thus, we have that

$$\bar{f}(s) = \frac{1}{a^2+b^2} \left[\frac{a}{s-a} - \frac{as}{s^2+b^2} + \frac{b^2}{s^2+b^2} \right].$$

From our table of Laplace transforms, we know that

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} &= e^{at} \\ \mathcal{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\} &= \cos bt \\ \mathcal{L}^{-1}\left\{\frac{b}{s^2+b^2}\right\} &= \sin bt.\end{aligned}$$

Therefore, the inverse Laplace transform of $\bar{f}(s)$ is

$$\begin{aligned}f(t) = \mathcal{L}^{-1}\{\bar{f}(s)\} &= \frac{1}{a^2+b^2} \left[a\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} - a\mathcal{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\} + b\mathcal{L}^{-1}\left\{\frac{b}{s^2+b^2}\right\} \right] \\ &= \frac{1}{a^2+b^2} [ae^{at} - a\cos bt + b\sin bt].\end{aligned}$$

ii. The Convolution Theorem states that if $\bar{f}(s) = \bar{g}(s)\bar{h}(s)$, then

$$f(t) = \mathcal{L}^{-1}\{\bar{f}(s)\} = \mathcal{L}^{-1}\{\bar{g}(s)\bar{h}(s)\} = (g * h)(t)$$

where

$$(g * h)(t) = \int_0^t g(t-\tau)h(\tau)d\tau.$$

Now, suppose that $\bar{f}(s) = \bar{g}(s)\bar{h}(s)$, where $\bar{g}(s) = \frac{1}{s-a}$ and $\bar{h}(s) = \frac{s}{s^2+b^2}$.

From our table of Laplace transforms we know that $g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$ and $h(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\} = \cos bt$.

Thus, by the Convolution Theorem, we have that

$$f(t) = \mathcal{L}^{-1}\{\bar{f}(s)\} = \mathcal{L}^{-1}\{\bar{g}(s)\bar{h}(s)\} = \int_0^t g(t-\tau)h(\tau)d\tau.$$

Therefore, using a computer algebra system, we see that

$$\begin{aligned}f(t) &= \int_0^t g(t-\tau)h(\tau)d\tau \\ &= \int_0^t e^{a(t-\tau)} \cos b\tau d\tau \\ &= e^{at} \int_0^t e^{-a\tau} \cos b\tau d\tau \\ &= \frac{1}{a^2+b^2} [ae^{at} - a\cos bt + b\sin bt].\end{aligned}$$

- iii. Heaviside's Expansion Theorem states that if $\bar{f}(s) = \frac{\bar{p}(s)}{\bar{q}(s)}$, where $\bar{p}(s)$ and $\bar{q}(s)$ are polynomials in s and the degree of \bar{q} is higher than that of \bar{p} , then

$$f(t) = \mathcal{L}^{-1} \{ \bar{f}(s) \} = \sum_{k=1}^n \frac{\bar{p}(\alpha_k)}{\bar{q}'(\alpha_k)} e^{t\alpha_k}$$

where α_k are the distinct root of $\bar{q}(s) = 0$.

For $\bar{f}(s) = \frac{s}{(s-a)(s^2+b^2)}$, we identify $\bar{p}(s) = s$ and $\bar{q}(s) = (s-a)(s^2+b^2)$. Since \bar{p} and \bar{q} are polynomials in s with the degree of \bar{q} greater than that of the degree of \bar{p} , the assumptions of Heaviside's Expansion Theorem are satisfied.

Note that $\bar{q}'(s) = s(3s-2a) + b^2$ and $\alpha_1 = a$, $\alpha_2 = bi$, and $\alpha_3 = -bi$ are the roots of $\bar{q}(s)$.

Therefore, by the Heaviside's Expansion Theorem, we have that

$$\begin{aligned} f(t) &= \mathcal{L}^{-1} \{ \bar{f}(s) \} = \sum_{k=1}^n \frac{\bar{p}(\alpha_k)}{\bar{q}'(\alpha_k)} e^{t\alpha_k} \\ &= \frac{a}{a^2+b^2} e^{at} - \frac{bi}{2bi(a-ib)} e^{bit} - \frac{bi}{2bi(a+ib)} e^{-bit} \\ &= \frac{1}{a^2+b^2} \left[ae^{at} - \frac{a+ib}{2} e^{bit} - \frac{a-ib}{2} e^{-bit} \right] \\ &= \frac{1}{a^2+b^2} [ae^{at} - a \cos bt + b \sin bt] . \end{aligned}$$

□

Problem 2. a. Evaluate the improper definite integral

$$\int_{-\infty}^{\infty} \frac{\cos tx}{x^2 + a^2} dx$$

where $a, t > 0$.

b. Show that

$$\int_0^{\infty} \frac{\sin \pi tx}{x(1+x^2)} dx = \frac{\pi}{2}(1 - e^{-\pi t})$$

where $t > 0$.

Solution. a. Suppose that

$$f(t) = \int_{-\infty}^{\infty} \frac{\cos tx}{x^2 + a^2} dx.$$

In order to evaluate this integral, we take the Laplace transform of $f(t)$ with respect to t . Now, due to uniform convergence, we have that

$$\begin{aligned} \bar{f}(s) = \mathcal{L}\{f(t)\} &= \mathcal{L}\left\{\int_{-\infty}^{\infty} \frac{\cos tx}{x^2 + a^2} dx\right\} = \int_{-\infty}^{\infty} \mathcal{L}\left\{\frac{\cos tx}{x^2 + a^2}\right\} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{x^2 + a^2} \mathcal{L}\{\cos tx\} dx \\ &= \int_{-\infty}^{\infty} \frac{s}{(x^2 + a^2)(x^2 + s^2)} dx. \end{aligned}$$

Using the method of partial fraction decomposition, we see that this last integral becomes

$$\begin{aligned} \bar{f}(s) &= \int_{-\infty}^{\infty} \frac{s dx}{(x^2 + a^2)(x^2 + s^2)} \\ &= \frac{s}{s^2 - a^2} \int_{-\infty}^{\infty} \frac{1}{x^2 + a^2} - \frac{1}{x^2 + s^2} dx. \end{aligned}$$

Thus, we see that

$$\begin{aligned} \bar{f}(s) &= \frac{s}{s^2 - a^2} \int_{-\infty}^{\infty} \frac{1}{x^2 + a^2} - \frac{1}{x^2 + s^2} dx \\ &= \frac{s}{s^2 - a^2} \left[\tan^{-1} \frac{x}{a} \Big|_{-\infty}^{\infty} - \tan^{-1} \frac{x}{s} \Big|_{-\infty}^{\infty} \right] \\ &= \frac{s}{s^2 - a^2} \left[\frac{\pi}{a} - \frac{\pi}{s} \right] \\ &= \frac{\pi}{a} \left[\frac{s}{s^2 - a^2} - \frac{a}{s^2 - a^2} \right]. \end{aligned}$$

Using the table of Laplace transforms, we know that $\mathcal{L}^{-1} \left\{ \frac{s}{s^2 - a^2} \right\} = \cosh at$ and $\mathcal{L}^{-1} \left\{ \frac{a}{s^2 - a^2} \right\} = \sinh at$. Therefore, we have that

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\cos tx}{x^2 + a^2} dx = f(t) &= \mathcal{L}^{-1} \{ \bar{f}(s) \} = \mathcal{L}^{-1} \left\{ \frac{\pi}{a} \left[\frac{s}{s^2 - a^2} - \frac{a}{s^2 - a^2} \right] \right\} \\ &= \frac{\pi}{a} \left[\mathcal{L}^{-1} \left\{ \frac{s}{s^2 - a^2} \right\} - \mathcal{L}^{-1} \left\{ \frac{a}{s^2 - a^2} \right\} \right] \\ &= \frac{\pi}{a} [\cosh at - \sinh at] \\ &= \frac{\pi}{a} e^{-at}. \end{aligned}$$

b. Suppose that

$$f(t) = \int_0^{\infty} \frac{\sin \pi tx}{x(1+x^2)} dx.$$

In order to evaluate this integral, we take the Laplace transform of $f(t)$ with respect to t . Now, due to uniform convergence, we have that

$$\begin{aligned} \bar{f}(s) = \mathcal{L} \{ f(t) \} &= \mathcal{L} \left\{ \int_0^{\infty} \frac{\sin \pi tx}{x(1+x^2)} dx \right\} = \int_0^{\infty} \mathcal{L} \left\{ \frac{\sin \pi tx}{x(1+x^2)} \right\} dx \\ &= \int_0^{\infty} \frac{1}{x(1+x^2)} \mathcal{L} \{ \sin \pi tx \} dx \\ &= \int_0^{\infty} \frac{\pi}{(x^2 + 1)(\pi^2 x^2 + s^2)} dx. \end{aligned}$$

Using a computer algebra system, we see that this last integral reduces to

$$\begin{aligned} \bar{f}(s) &= \int_0^{\infty} \frac{\pi}{(x^2 + 1)(\pi^2 x^2 + s^2)} dx \\ &= \frac{\pi^2}{2s(\pi + s)} \\ &= \frac{\pi}{2} \left[\frac{1}{s} - \frac{1}{s + \pi} \right]. \end{aligned}$$

Therefore, from our table of Laplace transforms, we have that

$$\begin{aligned} \int_0^{\infty} \frac{\sin \pi tx}{x(1+x^2)} dx = f(t) &= \mathcal{L}^{-1} \{ \bar{f}(s) \} = \frac{\pi}{2} \left[\mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} - \mathcal{L}^{-1} \left\{ \frac{1}{s + \pi} \right\} \right] \\ &= \frac{\pi}{2} (1 - e^{-\pi t}). \end{aligned}$$

□

Problem 3. Apply the Laplace transform to solve the following Initial Value Problems:

- a. $y'' + 2ay' + (a^2 + 4)y = f(t)$
 $y(0) = 1, \quad y'(0) = -a.$
- b. $u_{tt} = c^2 u_{xx} + \sin x, \quad 0 < x < \pi, \quad t > 0$
 $u(0, t) = u(\pi, t) = 1, \quad u(x, 0) = u_t(x, 0) = 0.$

Solution. Recall that if $\bar{y}(s) = \mathcal{L}\{y(t)\}$, then the Laplace transform of the n -th derivative of $y(t)$ is given by

$$\mathcal{L}\{y^{(n)}(t)\} = s^n \bar{y}(s) - \sum_{k=0}^{n-1} s^{n-1-k} y^{(k)}(0). \quad (1)$$

- a. Suppose that $Ly \equiv y''(t) + 2ay'(t) + (a^2 + 4)y(t)$. Using (1), application of the Laplace transform to $Ly = f(t)$ yields that

$$\mathcal{L}\{Ly\} = (s^2 + 2as + a^2 + 4)\bar{y}(s) - 2ay(0) - sy(0) - y'(0) = \bar{f}(s) = \mathcal{L}\{f(t)\}.$$

From the initial data, we see that this reduces to

$$(s^2 + 2as + a^2 + 4)\bar{y}(s) - (s + a) = \bar{f}(s).$$

Solving for $\bar{y}(s)$ yields

$$\bar{y}(s) = \frac{\bar{f}(s) + s + a}{s^2 + 2as + a^2 + 4} = \frac{\bar{f}(s) + s + a}{(s + a + 2i)(s + a - 2i)}.$$

Note that from our table of Laplace transforms that

$$\mathcal{L}^{-1}\left\{\frac{a - b}{(s - a)(s - b)}\right\} = e^{at} - e^{bt}$$

and

$$\mathcal{L}^{-1}\left\{\frac{s}{(s - a)(s - b)}\right\} = \frac{ae^{at} - be^{bt}}{a - b}.$$

Therefore, the solution to the original differential equation is given by

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{\bar{f}(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{\bar{f}(s) + s + a}{(s + a + 2i)(s + a - 2i)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{\bar{f}(s)}{(s + a + 2i)(s + a - 2i)}\right\} + \frac{e^{-(2i+a)t}}{4} [(2 - i + ia)e^{4it} + 2 + i - ia]. \end{aligned}$$

- b. Let $u(x, t)$ be a function in x and t . The Laplace transform of $u(x, t)$ with respect to t is given by

$$\mathcal{L}\{u(x, t)\} = \bar{u}(x, s) = \int_0^\infty u(x, t)e^{-st}dt.$$

From this definition, we see from previous theorems that

$$\mathcal{L}\left\{\frac{\partial^n}{\partial t^n}[u(x, t)]\right\} = s^n \bar{u}(x, s) - \sum_{k=0}^{n-1} s^{n-1-k} \frac{\partial^k}{\partial t^k}[u(x, 0)]$$

Similarly, we see from the Leibniz integral rule that

$$\mathcal{L}\left\{\frac{\partial^n}{\partial x^n}[u(x, t)]\right\} = \frac{d^n}{dx^n}[\bar{u}(x, s)].$$

Applying the Laplace transform with respect to t to the differential equation yields that

$$\mathcal{L}\{u_{tt} - c^2 u_{xx}\} = s^2 \bar{u}(x, s) - su(x, 0) - u_t(x, 0) - c^2 \frac{d^2 \bar{u}(x, s)}{dx^2} = \frac{\sin x}{s} = \mathcal{L}\{\sin x\}.$$

In light of the initial data, this equation reduces to

$$s^2 \bar{u}(x, s) - c^2 \frac{d^2 \bar{u}(x, s)}{dx^2} = \frac{\sin x}{s},$$

or, equivalently,

$$\frac{d^2 \bar{u}(x, s)}{dx^2} - \left(\frac{s}{c}\right)^2 \bar{u}(x, s) = -\frac{\sin x}{sc^2}.$$

The homogeneous solution to the above differential equation is easily seen to be

$$\bar{u}_h(x, s) = c_1 \exp\left(-\frac{xs}{c}\right) + c_2 \exp\left(\frac{xs}{c}\right)$$

From the method of undetermined coefficients, assuming the particular solution of the equation is of the form $\bar{u}_p(x, s) = A \sin x$ for some unknown A , the particular solution of the transformed equation is given by

$$\bar{u}_p(x, s) = \frac{\sin x}{s(s^2 + c^2)}.$$

Therefore, the general solution to the transformed equation is given by

$$\bar{u}(x, s) = \bar{u}_h(x, s) + \bar{u}_p(x, s) = c_1 \exp\left(-\frac{xs}{c}\right) + c_2 \exp\left(\frac{xs}{c}\right) + \frac{\sin x}{s(s^2 + c^2)}.$$

Note that the transformed boundary data is given by $\bar{u}(0, s) = \bar{u}(\pi, s) = \frac{1}{s}$. Using the form of the solution to the transformed equation listed above, we see that in light of the transformed boundary data that

$$\begin{aligned} c_1 + c_2 &= \frac{1}{s} \\ c_1 \exp\left(-\frac{\pi s}{c}\right) + c_2 \exp\left(\frac{\pi s}{c}\right) &= \frac{1}{s} \end{aligned}$$

After solving the above system, we therefore see that the solution to the transformed equation is given by

$$\begin{aligned} \bar{u}(x, s) &= c_1 \exp\left(-\frac{xs}{c}\right) + c_2 \exp\left(\frac{xs}{c}\right) + \frac{\sin x}{s(s^2 + c^2)} \\ &= \frac{\exp\left(\frac{\pi s}{c}\right) \exp\left(-\frac{xs}{c}\right)}{s(1 + \exp\left(\frac{\pi s}{c}\right))} + \frac{\exp\left(\frac{xs}{c}\right)}{s(1 + \exp\left(\frac{\pi s}{c}\right))} + \frac{\sin x}{s(s^2 + c^2)} \end{aligned}$$

Therefore, the solution to the original differential equation is given by

$$u(x, t) = \mathcal{L}^{-1}\{\bar{u}(x, s)\} = \mathcal{L}^{-1}\left\{\frac{\exp\left(\frac{\pi s}{c}\right) \exp\left(-\frac{xs}{c}\right)}{s(1 + \exp\left(\frac{\pi s}{c}\right))} + \frac{\exp\left(\frac{xs}{c}\right)}{s(1 + \exp\left(\frac{\pi s}{c}\right))} + \frac{\sin x}{s(s^2 + c^2)}\right\}.$$

□

Problem 4. Apply the Laplace transform to solve the following wave equation

$$\begin{aligned}\frac{\partial^2 u(x, t)}{\partial t^2} - c^2 \frac{\partial^2 u(x, t)}{\partial x^2} &= f(t), \\ u(0, t) &= 0, \quad t > 0, \\ u(x, 0) &= 0, \quad \frac{\partial}{\partial t} [u(x, 0)] = 0, \quad x > 0.\end{aligned}$$

Solution. Suppose that $Lu \equiv \frac{\partial^2 u(x, t)}{\partial t^2} - c^2 \frac{\partial^2 u(x, t)}{\partial x^2}$. Then applying the Laplace transform to the equation $Lu = f(t)$ yields

$$\mathcal{L}\{Lu\} = s^2 \bar{u}(x, s) - su(x, 0) - \frac{\partial}{\partial t} [u(x, 0)] - c^2 \frac{d^2 \bar{u}(x, s)}{dx^2} = \bar{f}(s) = \mathcal{L}\{f(t)\}.$$

In light of the initial data, this equation reduces to

$$s^2 \bar{u}(x, s) - c^2 \frac{d^2 \bar{u}(x, s)}{dx^2} = \bar{f}(s),$$

or, equivalently,

$$\frac{d^2 \bar{u}(x, s)}{dx^2} - \left(\frac{s}{c}\right)^2 \bar{u}(x, s) = -\frac{\bar{f}(s)}{c^2}.$$

The homogeneous solution to the above differential equation is easily seen to be

$$\bar{u}_h(x, s) = c_1 \exp\left(-\frac{xs}{c}\right) + c_2 \exp\left(\frac{xs}{c}\right).$$

By inspection, we see that

$$\bar{u}_p(x, s) = \frac{\bar{f}(s)}{s^2}$$

is a particular solution of the transformed equation. Thus, the general solution to the transformed equation is

$$\bar{u}(x, s) = \bar{u}_h(x, s) + \bar{u}_p(x, s) = c_1 \exp\left(-\frac{xs}{c}\right) + c_2 \exp\left(\frac{xs}{c}\right) + \frac{\bar{f}(s)}{s^2}.$$

Note that we must have that $\bar{u}(x, s) \rightarrow 0$ as $s \rightarrow \infty$. For this reason, we must have that $c_2 = 0$. The transformed boundary data states that $\bar{u}(0, t) = 0$. Using the above solution, this implies that $c_1 = -\bar{f}(s)/s^2$. Thus, the solution to the transformed differential equation is

$$\bar{u}(x, s) = \bar{u}_h(x, s) + \bar{u}_p(x, s) = \frac{\bar{f}(s)}{s^2} \left[1 - \exp\left(-\frac{xs}{c}\right)\right]$$

We arrive at the solution to the original differential equation by taking the inverse Laplace transform of the above equation. Note from our table of Laplace transforms that $\mathcal{L}^{-1}\{1/s^2\} = t$ and from Heaviside's Second Shifting Theorem that

$$\mathcal{L}^{-1}\left\{\frac{\exp\left(-\frac{xs}{c}\right)}{s^2}\right\} = \left(t - \frac{x}{c}\right) H\left(t - \frac{x}{c}\right).$$

Now let $g(t) = t$ and $h(t) = \left(t - \frac{x}{c}\right) H\left(t - \frac{x}{c}\right)$. Then from our previous remarks, we have that

$$\begin{aligned}\bar{u}(x, s) &= \frac{\bar{f}(s)}{s^2} \left[1 - \exp\left(-\frac{xs}{c}\right)\right] \\ &= \frac{\bar{f}(s)}{s^2} - \frac{\bar{f}(s) \exp\left(-\frac{xs}{c}\right)}{s^2} \\ &= \bar{f}(s)\bar{g}(s) - \bar{f}(s)\bar{h}(s).\end{aligned}$$

Therefore, by the Convolution Theorem and the above results, the solution to the original differential equation is

$$\begin{aligned}u(x, t) &= \mathcal{L}^{-1}\{\bar{u}(x, s)\} = \mathcal{L}^{-1}\{\bar{f}(s)\bar{g}(s) - \bar{f}(s)\bar{h}(s)\} \\ &= (f * g)(t) - (f * h)(t) \\ &= \int_0^t \tau f(t - \tau) d\tau - \int_0^t \left(\tau - \frac{x}{c}\right) H\left(\tau - \frac{x}{c}\right) f(t - \tau) d\tau.\end{aligned}$$

□

Problem 5. Solve the following integral equation by the Laplace transform

$$f(t) = t \cos at + a \int_0^t f(\tau) \sin a(t - \tau) d\tau.$$

Solution. Let $g(t) = t \cos at$ and $h(t) = \sin at$. Then from the definition of the convolution, the integral equation is equivalent to

$$f(t) = g(t) + a(h * f)(t).$$

From the Convolution Theorem, application of the Laplace transform to this equation yields that

$$\mathcal{L}\{f\}(s) = \bar{f}(s) = \bar{g}(s) + a\bar{f}(s)\bar{h}(s) = \mathcal{L}\{g(t) + a(h * f)(t)\},$$

where $\bar{f}(s) = \mathcal{L}\{f(t)\}$, $\bar{g}(s) = \mathcal{L}\{g(t)\}$, and $\bar{h}(s) = \mathcal{L}\{h(t)\}$. The resulting equation is an algebraic one; solving for $\bar{f}(s)$ yields that

$$\bar{f}(s) = \frac{\bar{g}(s)}{1 - a\bar{h}(s)}.$$

From the table of Laplace transforms, we know that

$$\bar{g}(s) = \mathcal{L}\{t \cos at\} = \frac{s^2 - a^2}{(s^2 + a^2)^2}$$

and

$$\bar{h}(s) = \mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}.$$

Thus, we have that

$$\begin{aligned} \bar{f}(s) &= \frac{\bar{g}(s)}{1 - a\bar{h}(s)} \\ &= \frac{s^2 - a^2}{s^2(s^2 + a^2)}. \end{aligned}$$

Applying the partial fraction decomposition method to this equation, we see that it can be rewritten as

$$\bar{f}(s) = \frac{s^2 - a^2}{s^2(s^2 + a^2)} = -\frac{1}{s^2} + \frac{2}{s^2 + a^2}.$$

Therefore, by taking the inverse Laplace transform of this equation, we see that the solution to the original integral equation is

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{\bar{f}(s)\} = -\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s^2 + a^2}\right\} \\ &= -t + \frac{2 \sin at}{a}. \end{aligned}$$

□

Problem 6. Apply the Laplace transform to solve the following diffusion problem

$$\begin{aligned}\frac{\partial u(x, t)}{\partial t} &= K \frac{\partial^2 u(x, t)}{\partial x^2}, & 0 < x < 1, \quad 0 < t, \\ u(x, 0) &= 0, & 0 < x < 1, \\ u(0, t) &= f(t), \quad \frac{\partial}{\partial x} [u(1, t)] = 0, & 0 < t.\end{aligned}$$

Solution. We begin by applying the Laplace transform to the first equation in the diffusion problem. Doing so yields

$$\mathcal{L} \left\{ \frac{\partial u(x, t)}{\partial t} \right\} = s\bar{u}(x, s) - \bar{u}(x, 0) = K \frac{d^2 \bar{u}(x, s)}{dx^2} = \mathcal{L} \left\{ K \frac{\partial^2 u(x, t)}{\partial x^2} \right\}$$

From the initial data, we see that this equation reduces to

$$s\bar{u}(x, s) = K \frac{d^2 \bar{u}(x, s)}{dx^2},$$

or, equivalently,

$$\frac{d^2 \bar{u}(x, s)}{dx^2} - \frac{s}{K} \bar{u}(x, s) = 0.$$

The solution to this homogeneous equation is readily seen to be

$$\begin{aligned}\bar{u}(x, s) &= b_1 \exp \left(-x \sqrt{\frac{s}{K}} \right) + b_2 \exp \left(x \sqrt{\frac{s}{K}} \right) \\ &= c_1 \cosh \left(x \sqrt{\frac{s}{K}} \right) + c_2 \sinh \left(x \sqrt{\frac{s}{K}} \right).\end{aligned}\tag{2}$$

Note that the transformed boundary conditions are given by

$$\bar{u}(0, s) = \bar{f}(s), \quad \frac{d}{dx} [\bar{u}(x, s)] \Big|_{x=1} = 0.$$

Using (2) in conjunction with the transformed boundary data, we see that

$$c_1 = \bar{f}(s)c_2 = \bar{f}(s) \exp \left(-2\sqrt{\frac{s}{K}} \right).$$

Therefore, the solution to the transformed equation is given by

$$\bar{u}(x, s) = \bar{f}(s) \cosh \left(x \sqrt{\frac{s}{K}} \right) + \bar{f}(s) \exp \left(-2\sqrt{\frac{s}{K}} \right) \sinh \left(x \sqrt{\frac{s}{K}} \right).$$

and the solution to the original equation is given by

$$\begin{aligned}u(x, t) &= \mathcal{L}^{-1} \{ \bar{u}(x, s) \} \\ &= \mathcal{L}^{-1} \left\{ \bar{f}(s) \cosh \left(x \sqrt{\frac{s}{K}} \right) + \bar{f}(s) \exp \left(-2\sqrt{\frac{s}{K}} \right) \sinh \left(x \sqrt{\frac{s}{K}} \right) \right\}.\end{aligned}$$

□

Problem 7. Solve the following difference and differential difference equations:

a. $u_{n+2} - 7u_{n+1} + 10u_n = 0$, $u_0 = 1$, $u_1 = 2$.

b. $\frac{du}{dt} - 2u(t-1) = 0$, $u(0) = 1$.

Solution. Define $S_n(t) = H(t-n) - H(t-n-1)$ for $n \leq t < n+1$ and define

$$u(t) = \sum_{n=0}^{\infty} u_n S_n(t).$$

It follows that for $n \leq t < n+1$ we have that $u(t) = u_n$.

By a previous theorem, if $\bar{u}(s) = \mathcal{L}\{u(t)\}$, then

$$\mathcal{L}\{u(t+1)\} = e^s [\bar{u}(s) - u_0 \bar{S}_0(s)]$$

where $\bar{S}_0 = \frac{1}{s}(1 - e^{-s})$. It then follows that

$$\mathcal{L}\{u(t+2)\} = e^{2s} [\bar{u}(s) - (u_0 + u_1 e^{-s}) \bar{S}_0(s)].$$

a. We begin by applying the Laplace transform to the difference equation. From the previous remarks, the transformed equation becomes

$$e^{2s} [\bar{u}(s) - (u_0 + u_1 e^{-s}) \bar{S}_0(s)] - 7e^s [\bar{u}(s) - u_0 \bar{S}_0(s)] + 10\bar{u}(s) = 0.$$

In light of the initial data, this equation becomes

$$e^{2s} [\bar{u}(s) - (1 + 2e^{-s}) \bar{S}_0(s)] - 7e^s [\bar{u}(s) - \bar{S}_0(s)] + 10\bar{u}(s) = 0.$$

Solving for $\bar{u}(s)$, we see that

$$\bar{u}(s) = \frac{e^s}{e^s - 2} \bar{S}_0(s).$$

Therefore, From a previous result, we see that the solution to the original difference equation is

$$u(t) = \mathcal{L}^{-1}\{\bar{u}(s)\} = \mathcal{L}^{-1}\left\{\frac{e^s}{e^s - 2} \bar{S}_0(s)\right\} = 2^n.$$

b. Applying the Laplace transformed to the differential difference equation yields that

$$s\bar{u}(s) - u(0) - 2e^{-s} [\bar{u}(s) - u(0) \bar{S}_0(s)] = 0$$

In light of the initial data, this equation reduces to

$$s\bar{u}(s) - 1 - 2e^{-s} [\bar{u}(s) - \bar{S}_0(s)] = 0,$$

or, equivalently,

$$\begin{aligned}\bar{u}(s) &= \frac{e^s - 2\bar{S}_0(s)}{se^s - 2} = \frac{1}{s} \left(1 - \frac{2}{s}e^{-s}\right)^{-1} - \frac{2\bar{S}_0(s)}{se^s - 2} \\ &= \frac{1}{s} \left(1 - \frac{2}{s}e^{-s}\right)^{-1} - \frac{2}{s^2e^s} \left(1 - \frac{2}{s}e^{-s}\right)^{-1} - \frac{2}{s^2e^{2s}} \left(1 - \frac{2}{s}e^{-s}\right)^{-1} \\ &= \left(\frac{1}{s} - \frac{2}{s^2e^s} - \frac{2}{s^2e^{2s}}\right) \left(1 - \frac{2}{s}e^{-s}\right)^{-1}.\end{aligned}$$

Expanding the right term in terms of its power series we see that

$$\begin{aligned}\bar{u}(s) &= \left(\frac{1}{s} - \frac{2}{s^2e^s} - \frac{2}{s^2e^{2s}}\right) \left(1 - \frac{2}{s}e^{-s}\right)^{-1} \\ &= \left(\frac{1}{s} - \frac{2}{s^2e^s} - \frac{2}{s^2e^{2s}}\right) \sum_{n=0}^{\infty} \frac{2^n e^{-ns}}{s^n} \\ &= \sum_{n=0}^{\infty} \frac{2^n e^{-ns}}{s^{n+1}} - \sum_{n=0}^{\infty} \frac{2^{n+1} e^{-(n-1)s}}{s^{n+2}} - \sum_{n=0}^{\infty} \frac{2^{n+1} e^{-(n-2)s}}{s^{n+2}}.\end{aligned}$$

Recall that

$$\mathcal{L}^{-1} \left\{ \frac{e^{-as}}{s^n} \right\} = \frac{(t-a)^{n-1}}{\Gamma(n)} H(t-a).$$

Therefore, the solution to the original differential difference equation is

$$\begin{aligned}u(t) &= \mathcal{L}^{-1} \left\{ \sum_{n=0}^{\infty} \frac{2^n e^{-ns}}{s^{n+1}} - \sum_{n=0}^{\infty} \frac{2^{n+1} e^{-(n-1)s}}{s^{n+2}} - \sum_{n=0}^{\infty} \frac{2^{n+1} e^{-(n-2)s}}{s^{n+2}} \right\} \\ &= \sum_{n=0}^{\infty} 2^n \mathcal{L}^{-1} \left\{ \frac{e^{-ns}}{s^{n+1}} \right\} - \sum_{n=0}^{\infty} 2^{n+1} \mathcal{L}^{-1} \left\{ \frac{e^{-(n+1)s}}{s^{n+2}} \right\} - \sum_{n=0}^{\infty} 2^{n+1} \mathcal{L}^{-1} \left\{ \frac{e^{-(n+2)s}}{s^{n+2}} \right\} \\ &= \sum_{n=0}^{\infty} 2^n \frac{(t-n)^n}{\Gamma(n+1)} H(t-n) - \sum_{n=0}^{\infty} 2^{n+1} \frac{(t-n-1)^{n+1}}{\Gamma(n+2)} H(t-n-1) \\ &\quad - \sum_{n=0}^{\infty} 2^{n+1} \frac{(t-n-2)^{n+1}}{\Gamma(n+2)} H(t-n-2).\end{aligned}$$

□

Problem 8. Solve the following heat equation by the joint Fourier and Laplace transform

$$\frac{\partial}{\partial t}u(x, t) = \frac{\partial}{\partial x^2}u(x, t), \quad -\infty < x < \infty, \quad 0 < t$$

subject to $u(x, 0) = e^{-x^2} = f(x)$.

Solution. Define the joint Fourier and Laplace transform of $u(x, t)$ by

$$\bar{U}(k, s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \left[\int_0^{\infty} e^{-st} u(x, t) dt \right] dx.$$

Applying the Laplace transform with respect to t to the differential equation yields that

$$\mathcal{L} \left\{ \frac{\partial}{\partial t} u(x, t) \right\} = s\bar{u}(x, s) - u(x, 0) = \frac{d^2}{dx^2} \bar{u}(x, s) = \mathcal{L} \left\{ \frac{\partial}{\partial x^2} u(x, t) \right\}$$

From the initial data, we see that this equation reduces to

$$s\bar{u}(x, s) - f(x) = \frac{d^2}{dx^2} \bar{u}(x, s).$$

Now, applying the Fourier transform with respect to x to the resulting equation yields that

$$\mathcal{F} \{s\bar{u}(x, s) - f(x)\} = s\bar{U}(k, s) - F(k) = -k^2 \bar{U}(k, s) = \mathcal{F} \left\{ \frac{d^2}{dx^2} \bar{u}(x, s) \right\},$$

where

$$F(k) = \frac{\exp\left(-\frac{k^2}{4}\right)}{\sqrt{2}}.$$

Thus, we see that

$$\bar{U}(k, s) = \frac{F(k)}{k^2 + s}.$$

Taking the inverse Laplace transform of this equation yields that

$$\bar{u}(k, t) = \mathcal{L}^{-1} \left\{ \frac{F(k)}{k^2 + s} \right\} = F(k) e^{-k^2 t}.$$

Note that for $G(k, t) = e^{-k^2 t}$ we have that the function's inverse Fourier transform is

$$g(x, t) = \mathcal{F}^{-1} \{G(k, t)\} = \mathcal{F}^{-1} \{e^{-k^2 t}\} = \frac{1}{\sqrt{2t}} e^{-\frac{x^2}{4t}}$$

Now, from the Convolution Theorem, we see by applying the inverse Fourier transform to the equation resulting from the inverse Laplace transform that

$$u(x, t) = \mathcal{F}^{-1} \{\bar{u}(k, t)\} = \mathcal{F}^{-1} \{F(k) e^{-k^2 t}\} = f(x) * g(x, t).$$

Therefore, the solution to the original heat equation subject to the specified initial condition is

$$\begin{aligned} u(x, t) &= f(x) * g(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) g(x - \xi, t) d\xi \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\xi^2} e^{-\frac{(x-\xi)^2}{4t}} d\xi \\ &= \frac{1}{\sqrt{1+4t}} e^{-\frac{x^2}{1+4t}}. \end{aligned}$$

□