Homework Assignment 9

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Problem 8.1. Find the Mellin transform of each of the following functions:

a. $f(x) = H(a - x)$, $a > 0$,

b. $f(x) = x^m e^{-nx}$, $m, n > 0$,

c. $f(x) = \dfrac{1}{x^2 + 1}$.

Solution. The Mellin transform of the function $f(x)$ is defined to be

$$\mathscr{M}\left\{f(x)\right\} = \tilde{f}(p) = \int_0^\infty x^{p-1} f(x) dx.$$

a. Recall that the Heaviside function H is defined as

$$H(a - x) = \begin{cases} 1 & \text{if } x < a \\ 0 & \text{if } x > a \end{cases}.$$

Therefore, from the definition of the Mellin transform, we have that for $f(x) = H(a-x)$ with $a > 0$,

$$\begin{aligned} \tilde{f}(p) = \mathscr{M}\left\{f(x)\right\} &= \int_0^\infty x^{p-1} H(a - x) dx \\ &= \int_0^a x^{p-1} dx \\ &= \frac{a^p}{p}. \end{aligned}$$

b. Let $f(x) = x^m g(x)$ where $g(x) = e^{-nx}$ with $m, n > 0$ and let $\tilde{g}(p) = \mathscr{M}\left\{g(x)\right\}$. By the shifting property of the Mellin transform, we have that

$$\tilde{f}(p) = \mathscr{M}\left\{f(x)\right\} = \mathscr{M}\left\{x^m g(x)\right\} = \tilde{g}(p + m).$$

From our table of Mellin transforms, we know that

$$\tilde{g}(p) = \mathscr{M}\left\{g(x)\right\} = \frac{\Gamma(p)}{n^p}$$

where $\Re\{p\} > 0$.

Therefore,

$$\tilde{f}(p) = \mathscr{M}\left\{f(x)\right\} = \tilde{g}(p+m) = \frac{\Gamma(p+m)}{n^{p+m}}$$

where $\Re\{p+m\} > 0$.

c. From our table of Mellin transforms, we see that

$$\mathscr{M}\left\{\frac{1}{(x^a+1)^s}\right\} = \frac{\Gamma(p/a)\Gamma(s-p/a)}{a\Gamma(s)}.$$

Therefore, for $f(x) = \frac{1}{x^2+1}$, identifying $a = 2$ and $s = 1$, we have that

$$\begin{aligned}\tilde{f}(p) = \mathscr{M}\left\{f(x)\right\} = \mathscr{M}\left\{\frac{1}{x^2+1}\right\} &= \frac{\Gamma(p/2)\Gamma(1-p/2)}{2\Gamma(1)} \\ &= \frac{\Gamma(p/2)\Gamma(1-p/2)}{2}\end{aligned}$$

where $\Re\{p/2\} > 0$ and $\Re\{1-p/2\} > 0$.

$\square$

Problem 8.4. Show that

$$\mathscr{M}\left\{\frac{1}{(1+ax)^n}\right\} = \frac{\Gamma(p)\Gamma(n-p)}{a^p\Gamma(n)}.$$

Solution. Let $f(x) = \dfrac{1}{(1+x)^n}$ where $n > 0$. Recall that the Beta function

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

satisfies the property that

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

From the definition of the Gamma function, we see that

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt.$$

Thus, we have from the definition of the Mellin transform that

$$\begin{aligned}\mathscr{M}\left\{f(x)\right\} &= \int_0^\infty \frac{x^{p-1}}{(1+x)^n}dx \\ &= \int_0^\infty \frac{x^{p-1}}{(1+x)^{n-p+p}}dx \\ &= \frac{\Gamma(p)\Gamma(n-p)}{\Gamma(n)}.\end{aligned}$$

Therefore, by the scaling property of the Mellin transform,

$$\begin{aligned}\mathscr{M}\left\{\frac{1}{(1+ax)^n}\right\} = \mathscr{M}\left\{f(ax)\right\} &= \frac{\mathscr{M}\left\{f(x)\right\}}{a^p} \\ &= \frac{\Gamma(p)\Gamma(n-p)}{a^p\Gamma(n)}.\end{aligned}$$

$\square$

Problem 8.10. Show that the integral equation

$$f(x) = h(x) + \int_0^\infty f(\xi) g\left(\frac{x}{\xi}\right) \frac{d\xi}{\xi}$$

has the formal solution

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{-p}\tilde{h}(p)}{1-\tilde{g}(p)} dp.$$

Solution. Define the convolution of the functions $f(x)$ and $g(x)$ as

$$(f * g)(x) = \int_0^\infty f(\xi) g\left(\frac{x}{\xi}\right) \frac{d\xi}{\xi}.$$

Then the integral equation becomes

$$f(x) = h(x) + (f * g)(x)$$

Now, let $\tilde{f}(p), \tilde{g}(p)$, and $\tilde{h}(p)$ denote the Mellin transforms of $f(x), g(x)$, and $h(x)$, respectively. Taking the Mellin transform of the integral equation shows that

$$\begin{aligned} \tilde{f}(p) &= \mathscr{M}\{h(x) + (f * g)(x)\} \\ &= \tilde{h}(p) + \mathscr{M}\{(f * g)(x)\} \\ &= \tilde{h}(p) + \tilde{f}(p)\tilde{g}(p) \end{aligned}$$

where we have used the Convolution Type Theorem which states that

$$\mathscr{M}\{(f * g)(x)\} = \tilde{f}(p)\tilde{g}(p).$$

Thus, after taking the Mellin transform, the integral equation becomes an algebraic one in the variable p. Solving for $\tilde{f}(p)$ shows that

$$\tilde{f}(p) = \frac{\tilde{h}(p)}{1-\tilde{g}(p)}.$$

Therefore, the formal solution to the integral equation is

$$f(x) = \mathscr{M}^{-1}\left\{\tilde{f}(p)\right\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{-p}\tilde{h}(p)}{1-\tilde{g}(p)} dp.$$

$\square$

Problem 8.12. Assuming that

$$\mathscr{M}\left\{f\left(re^{i\theta}\right)\right\} = \int_0^\infty r^{p-1} f\left(re^{i\theta}\right) dr$$

where p is real. Putting $\xi = re^{i\theta}$ and $\mathscr{M}\left\{f(\xi)\right\} = F(p)$, show that

a. $\mathscr{M}\left\{f(re^{i\theta})\right\} = e^{-ip\theta}F(p)$.

Hence, deduce

b. $\mathscr{M}^{-1}\left\{F(p)\cos p\theta\right\} = \Re\{f(re^{i\theta})\}$,

c. $\mathscr{M}^{-1}\left\{F(p)\sin p\theta\right\} = -\Im\{f(re^{i\theta})\}$.

Solution. a. If $\xi = re^{i\theta}$, then $r = \xi e^{-i\theta}$ and $dr = e^{-i\theta}d\xi$. Now, from our assumption, we see that

$$\begin{aligned}\mathscr{M}\left\{f\left(re^{i\theta}\right)\right\} &= \int_0^\infty r^{p-1} f\left(re^{i\theta}\right) dr \\ &= \int_0^\infty \left(\xi e^{-i\theta}\right)^{p-1} f\left(\xi\right) e^{-i\theta} d\xi \\ &= e^{-ip\theta}\int_0^\infty \xi^{p-1} f\left(\xi\right) d\xi \\ &= e^{-ip\theta}F(p),\end{aligned}$$

as desired.

b. As shown above, we have that

$$\mathscr{M}\left\{f(re^{i\theta})\right\} = e^{-ip\theta}F(p).$$

From the definition of the complex exponential, this implies that

$$\mathscr{M}\left\{f(re^{i\theta})\right\} = e^{-ip\theta}F(p) = F(p)\cos p\theta - iF(p)\sin p\theta.$$

Thus, applying the inverse Mellin transform, we see that

$$f(re^{i\theta}) = \mathscr{M}^{-1}\left\{F(p)\cos p\theta\right\} - i\mathscr{M}^{-1}\left\{F(p)\sin p\theta\right\}. \tag{1}$$

Therefore, we have that

$$\mathscr{M}^{-1}\left\{F(p)\cos p\theta\right\} = \Re\{f(re^{i\theta})\}.$$

c. Similarly, from (1), we have that

$$\mathscr{M}^{-1}\left\{F(p)\sin p\theta\right\} = -\Im\{f(re^{i\theta})\}.$$

□

Problem 8.14. Use $\mathscr{M}^{-1}\left\{\dfrac{\pi}{\sin p\pi}\right\} = \dfrac{1}{1+x} = f(x)$ and exercise 8.12 to show that

a. $\mathscr{M}^{-1}\left\{\dfrac{\pi\cos p\theta}{\sin p\pi}\right\} = \dfrac{1+r\cos\theta}{1+2r\cos\theta+r^2}$,

b. $\mathscr{M}^{-1}\left\{\dfrac{\pi\sin p\theta}{\sin p\pi}\right\} = \dfrac{r\sin\theta}{1+2r\cos\theta+r^2}$.

Solution. Suppose that $F(p) = \dfrac{\pi}{\sin p\pi}$. Then from our assumption, we have that

$$\mathscr{M}\left\{f(x)\right\} = \mathscr{M}\left\{\frac{1}{1+x}\right\} = \frac{\pi}{\sin p\pi} = F(p).$$

Now suppose that $x = re^{i\theta}$. Then, from the complex exponential and other properties of complex numbers and trigonometric functions, we have that

$$\begin{aligned} f(re^{i\theta}) = \frac{1}{1+re^{i\theta}} &= \frac{1}{1+r\cos\theta+ir\sin\theta}\left[\frac{1+r\cos\theta-ir\sin\theta}{1+r\cos\theta-ir\sin\theta}\right] \\ &= \left[\frac{1+r\cos\theta}{(1+r\cos\theta)^2+r^2\sin^2\theta}\right] - i\left[\frac{r\sin\theta}{(1+r\cos\theta)^2+r^2\sin^2\theta}\right] \\ &= \left[\frac{1+r\cos\theta}{1+2r\cos\theta+r^2}\right] - i\left[\frac{r\sin\theta}{1+2r\cos\theta+r^2}\right] \end{aligned}$$

Thus, we have that

$$\Re\{f(re^{i\theta})\} = \frac{1+r\cos\theta}{1+2r\cos\theta+r^2} \tag{2a}$$

$$\Re\{f(re^{i\theta})\} = -\frac{r\sin\theta}{1+2r\cos\theta+r^2} \tag{2b}$$

a. Therefore, we have from (2a) and exercise 8.12.b that

$$\mathscr{M}^{-1}\left\{\frac{\pi\cos p\theta}{\sin p\pi}\right\} = \mathscr{M}^{-1}\left\{F(p)\cos p\theta\right\} = \Re\{f(re^{i\theta})\} = \frac{1+r\cos\theta}{1+2r\cos\theta+r^2}.$$

b. Similarly, we see from (2b) and exercise 8.12.c that

$$\mathscr{M}^{-1}\left\{\frac{\pi\sin p\theta}{\sin p\pi}\right\} = \mathscr{M}^{-1}\left\{F(p)\sin p\theta\right\} = -\Im\{f(re^{i\theta})\} = \frac{r\sin\theta}{1+2r\cos\theta+r^2}.$$

$\square$

Problem 8.21.

Solution. □