Homework Assignment 3

Matthew Tiger

February 25, 2017

Problem 2.20. Apply the Fourier cosine transform to find the solution $u(x, y)$ of the problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0, && 0 < x < \infty, \quad 0 < y < \infty \\ u(x, 0) &= H(a - x), && x < a \\ u_x(0, y) &= 0, && 0 < x, y < \infty. \end{aligned}$$

Solution. Consider the function $u(x, y)$. The Fourier cosine transform of u with respect to x is defined as

$$\mathscr{F}_c \left\{u(x, y)\right\} = U_c(k, y) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x, y) \cos(kx) dx.$$

From this definition we see using the Leibniz integral rule that

$$\begin{aligned} \mathscr{F}_c \left\{\frac{\partial^n u(x, y)}{\partial y^n}\right\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^n u(x, y)}{\partial y^n} \cos(kx) dx \\ &= \frac{d^n}{dy^n} \left[\sqrt{\frac{2}{\pi}} \int_0^\infty u(x, y) \cos(kx) dx\right] \\ &= \frac{d^n}{dy^n} \left[\mathscr{F}_c \left\{u(x, y)\right\}\right]. \end{aligned}$$

The transforms of the partials of u with respect to x are not as easy to characterize. Nevertheless, we see from the properties of the Fourier cosine transform that

$$\mathscr{F}_c \left\{\frac{\partial u(x, y)}{\partial x}\right\} = k \mathscr{F}_s \left\{u(x, y)\right\} - \sqrt{\frac{2}{\pi}} u(0, y)$$

and

$$\mathscr{F}_c \left\{\frac{\partial^2 u(x, y)}{\partial x^2}\right\} = -k^2 \mathscr{F}_c \left\{u(x, y)\right\} - \sqrt{\frac{2}{\pi}} u_x(0, y)$$

Let $U_c(x, y) = \mathscr{F}_c \left\{u(x, y)\right\}$. Then, applying the Fourier cosine transform to the first differential equation shows that

$$\mathscr{F}_c \left\{u_{xx} + u_{yy}\right\} = -k^2 U_c(k, y) - \sqrt{\frac{2}{\pi}} u_x(0, y) + \frac{d^2}{dy^2} \left[U_c(k, y)\right] = 0 = \mathscr{F}_c \left\{0\right\}.$$

From the third equation we see that $u_x(0, y) = 0$ for all $0 < x, y < \infty$ which implies that the above equation reduces to

$$\frac{d^2}{dy^2}\left[U_c(k, y)\right] - k^2 U_c(k, y) = 0.$$

This is a second-order linear homogeneous differential equation, the solution to which is readily seen to be

$$U_c(k, y) = c_1 e^{-ky} + c_2 e^{ky}.$$

However, since $U_c(k, y) \to 0$ as $k \to \infty$, we must have that $c_2 = 0$. Thus, the solution to the previous differential equation is given by

$$U_c(k, y) = c_1 e^{-ky}. \tag{1}$$

We now apply the Fourier cosine transform to the second differential equation yielding

$$\mathscr{F}_c\left\{u(x, 0)\right\} = U_c(k, 0) = \mathscr{F}_c\left\{H(a - x)\right\}.$$

Using the form (1) of the solution to the transformed differential equation and a table of Fourier cosine transforms we see that

$$U_c(k, 0) = c_1 = \mathscr{F}_c\left\{H(a - x)\right\} = \sqrt{\frac{2}{\pi}}\left(\frac{\sin ak}{k}\right).$$

Thus, the solution to the transformed differential equation with the boundary conditions listed above is given by

$$U_c(k, y) = \mathscr{F}_c\left\{H(a - x)\right\} e^{-ky} = \sqrt{\frac{2}{\pi}}\left(\frac{\sin ak}{k}\right) e^{-ky}.$$

Therefore, taking the inverse Fourier cosine transform to both sides shows that the solution to the original differential equation is given by

$$\begin{aligned} u(x, y) = \mathscr{F}_c^{-1}\left\{U_c(k, y)\right\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{\sin ak}{k}\right) e^{-ky} \cos kx dk \\ &= \frac{2}{\pi} \int_0^\infty \left(\frac{\sin ak}{k}\right) e^{-ky} \cos kx dk. \end{aligned}$$

$\square$

Problem 2.23. Use the Parseval formula to evaluate the following integrals with $a > 0$ and $b > 0$:

a. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2}$,

c. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2 ax}{x^2} dx$.

Solution. Suppose that $f \in L^2(\mathbb{R})$ and that $F(k) = \mathscr{F}\{f(x)\}$. Then Parseval's relation states that

$$\int_{-\infty}^{\infty} f(x)\overline{f(x)}dx = \int_{-\infty}^{\infty} F(k)\overline{F(k)}dk.$$

a. Let $f(x) = \dfrac{1}{x^2+a^2}$. Then from our table of Fourier transforms we see that

$$\mathscr{F}\{f(x)\} = F(k) = \sqrt{\frac{\pi}{2}}\left(\frac{e^{-a|k|}}{a}\right).$$

From Parseval's relation, we see that

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = \int_{-\infty}^{\infty} f(x)\overline{f(x)}dx = \int_{-\infty}^{\infty} F(k)\overline{F(k)}dk = \frac{\pi}{2a^2}\int_{-\infty}^{\infty} e^{-2a|k|}dk.$$

Therefore, we have that

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} &= \frac{\pi}{2a^2}\int_{-\infty}^{\infty} e^{-2a|k|}dk \\ &= \frac{\pi}{a^2}\int_{0}^{\infty} e^{-2ak}dk \\ &= \frac{\pi}{a^2}\left[-\frac{e^{-2ak}}{2a}\bigg|_0^\infty\right] \\ &= \frac{\pi}{2a^3}.\end{aligned}$$

c. Let $f(x) = \dfrac{\sin ax}{x}$. Then from our table of Fourier transforms we see that

$$\mathscr{F}\{f(x)\} = F(k) = \sqrt{\frac{\pi}{2}}H(a-|k|).$$

From Parseval's relation, we see that

$$\int_{-\infty}^{\infty} \frac{\sin^2 ax}{x} dx = \int_{-\infty}^{\infty} f(x)\overline{f(x)}dx = \int_{-\infty}^{\infty} F(k)\overline{F(k)}dk = \frac{\pi}{2}\int_{-\infty}^{\infty} H(a-|k|)^2 dk.$$

Therefore, we have using the definition of the Heaviside function that

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\sin^2 ax}{x} dx &= \frac{\pi}{2} \int_{-\infty}^{\infty} H(a - |k|)^2 dk \\ &= \frac{\pi}{2} \int_{-a}^{a} dk \\ &= a\pi.\end{aligned}$$

□

Problem 2.47. Apply the Fourier transform to solve the equation

$$u_{xxxx} + u_{yy} = 0, \qquad -\infty < x < \infty,\ 0 \leq y$$

satisfying the conditions

$$u(x,0) = f(x), \quad u_y(x,0) = 0, \qquad \text{for } -\infty < x < \infty$$

where $u(x,y)$ and its partial derivatives vanish as $|x| \to \infty$.

Solution. We begin by applying the Fourier transform to the system of differential equations. Using the properties of the Fourier transform with respect to x, we see that

$$\begin{aligned}
&\frac{d^2}{dy^2}\left[U(k,y)\right] + k^4 U(k,y) = 0 \\
&U(k,0) = F(k) \\
&\frac{d}{dy}\left[U(k,y)\right]\bigg|_{y=0} = 0, \qquad -\infty < k < \infty,\ 0 \leq y.
\end{aligned}$$

The first equation of the transformed system is a second-order linear homogeneous ordinary differential equation. Its solution is given by

$$U(k,y) = c_1 \cos(k^2 y) + c_2 \sin(k^2 y).$$

From this general solution, we see from the second equation that

$$U(k,0) = c_1 = F(k).$$

Similarly, using the general solution, we see from the third equation that

$$\frac{d}{dy}\left[U(k,y)\right] = -c_1 k^2 \sin(k^2 y) + c_2 k^2 \cos(k^2 y)$$

which implies that

$$\frac{d}{dy}\left[U(k,y)\right]\bigg|_{y=0} = c_2 k^2 = 0.$$

Since this must hold for all k, we must have have that $c_2 = 0$.

Thus, the solution to the transformed system is given by

$$U(k,y) = F(k)\cos(k^2 y).$$

Therefore, the solution to the original differential equation is

$$u(x,y) = \mathscr{F}^{-1}\left\{U(k,y)\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)\cos(k^2 y) e^{ikx} dk.$$

$\square$

Problem 2.48. The transverse vibration of a thin membrane of great extent satisfies the wave equation

$$c^2(u_{xx} + u_{yy}) = u_{tt}, \qquad -\infty < x, y < \infty,\ 0 < t,$$

with the initial and boundary conditions

$$\begin{aligned} &u(x, y, t) \to 0 \quad \text{as} \quad |x| \to \infty,\ |y| \to \infty \qquad \text{for all } t \geq 0, \\ &u(x, y, 0) = f(x, y), \quad u_t(x, y, 0) = 0 \qquad \text{for all } x, y. \end{aligned}$$

Solve the differential equation.

Solution. Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and suppose that $u(\boldsymbol{x}, t)$ is given. The Fourier transform of $u(\boldsymbol{x}, t)$ with respect to $\boldsymbol{x}$ is defined to be

$$\mathscr{F}\{u(\boldsymbol{x}, t)\} = U(\boldsymbol{k}, t) = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} u(\boldsymbol{x}, t) e^{-i\boldsymbol{x}\cdot\boldsymbol{k}} d\boldsymbol{x} \tag{2}$$

where $\boldsymbol{k} \in \mathbb{R}^n$.

In order to investigate the Fourier transform of partials of $u(\boldsymbol{x}, t)$ with respect to a given component of $\boldsymbol{x}$, define the Fourier transform of $u(\boldsymbol{x}, t)$ with respect to x_j as the following

$$\mathscr{F}_{[x_j]}\{u(\boldsymbol{x}, t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(\boldsymbol{x}, t) e^{-ix_j k_j} dx_j.$$

Further, we will also use the function $\pi_j : \mathbb{R}^n \to \mathbb{R}^{n-1}$ defined as

$$\pi_j(\boldsymbol{x}) := (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$$

to aid in our description of the Fourier transform of partials of $u(\boldsymbol{x}, t)$. Now from definition (2) and Leibniz's integral rule we see that

$$\begin{aligned} \mathscr{F}\left\{\frac{\partial^n u(\boldsymbol{x}, t)}{\partial t^n}\right\} &= \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \frac{\partial^n}{\partial t^n}\left[u(\boldsymbol{x}, t)\right] e^{-i\boldsymbol{x}\cdot\boldsymbol{k}} d\boldsymbol{x} \\ &= \frac{d^n}{dt^n}\left[\frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} u(\boldsymbol{x}, t) e^{-i\boldsymbol{x}\cdot\boldsymbol{k}} d\boldsymbol{x}\right] \\ &= \frac{d^n}{dt^n}\left[\mathscr{F}\{u(\boldsymbol{x}, t)\}\right]. \end{aligned}$$

Similarly, from definition (2) and previous results about the Fourier transform, we see that

$$\begin{aligned} \mathscr{F}\left\{\frac{\partial^n u(\boldsymbol{x}, t)}{\partial x_j^n}\right\} &= \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\partial^n}{\partial x_j^n}\left[u(\boldsymbol{x}, t)\right] e^{-ix_1k_1} \cdots e^{-ix_nk_n} dx_1 \ldots dx_n \\ &= \frac{1}{(2\pi)^{(n-1)/2}} \int_{-\infty}^{\infty} \mathscr{F}_{[x_j]}\left\{\frac{\partial^n}{\partial x_j^n}\left[u(\boldsymbol{x}, t)\right]\right\} e^{-i\pi_j(\boldsymbol{x})\cdot\pi_j(\boldsymbol{k})} d\pi_j(\boldsymbol{x}) \\ &= \frac{(ik_j)^n}{(2\pi)^{(n-1)/2}} \int_{-\infty}^{\infty} \mathscr{F}_{[x_j]}\{u(\boldsymbol{x}, t)\} e^{-i\pi_j(\boldsymbol{x})\cdot\pi_j(\boldsymbol{k})} d\pi_j(\boldsymbol{x}) \\ &= (ik_j)^n \mathscr{F}\{u(\boldsymbol{x}, t)\}. \end{aligned}$$

Now, define $\boldsymbol{x} = (x_1, x_2) = (x, y) \in \mathbb{R}^2$. Then the system of differential equations of the function $u(\boldsymbol{x}, t) = u(x, y, t)$ becomes

$$c^2 \left(u_{x_1x_1} + u_{x_2x_2}\right) - u_{tt} = 0, \qquad -\infty < x_1, x_2 < \infty,\ 0 < t,$$

with the initial and boundary conditions

$$\begin{aligned} &u(\boldsymbol{x}, t) \to 0 \quad \text{as} \quad |x_1| \to \infty,\ |x_2| \to \infty \qquad \text{for all } t \geq 0, \\ &u(\boldsymbol{x}, 0) = f(\boldsymbol{x}), \quad \frac{\partial}{\partial t}\left[u(\boldsymbol{x}, 0)\right] = 0 \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^2. \end{aligned}$$

Applying the Fourier transform to the left-hand side of the first equation yields

$$\begin{aligned} \mathscr{F}\left\{c^2 \left(u_{x_1x_1} + u_{x_2x_2}\right) - u_{tt}\right\} &= -c^2k_1^2U(\boldsymbol{k}, t) - c^2k_2^2U(\boldsymbol{k}, t) - \frac{d^2}{dt^2}\left[U(\boldsymbol{k}, t)\right] \\ &= -\frac{d^2}{dt^2}\left[U(\boldsymbol{k}, t)\right] - c^2 \left\|\boldsymbol{k}\right\|^2 U(\boldsymbol{k}, t) \end{aligned}$$

which implies that the transformed first equation becomes

$$\frac{d^2}{dt^2}\left[U(\boldsymbol{k}, t)\right] + c^2 \left\|\boldsymbol{k}\right\|^2 U(\boldsymbol{k}, t) = 0.$$

Similarly, we deduce that the transformed initial and boundary conditions become

$$\begin{aligned} &\mathscr{F}\left\{u(\boldsymbol{x}, t)\right\} = U(\boldsymbol{k}, t) \to 0 \quad \text{as} \quad |k_1| \to \infty,\ |k_2| \to \infty \qquad \text{for all } t \geq 0, \\ &\mathscr{F}\left\{u(\boldsymbol{x}, 0)\right\} = U(\boldsymbol{k}, 0) = F(\boldsymbol{k}) = \mathscr{F}\left\{f(\boldsymbol{x})\right\}, \\ &\mathscr{F}\left\{\frac{\partial}{\partial t}\left[u(\boldsymbol{x}, 0)\right]\right\} = \frac{d}{dt}\left[U(\boldsymbol{k}, 0)\right] = 0 \qquad \text{for all } \boldsymbol{k} \in \mathbb{R}^2. \end{aligned}$$

We see that the transformed first equation is a second-order linear homogeneous ordinary differential equation, from which we readily see the solution is

$$U(\boldsymbol{k}, t) = c_1 \cos(c \left\|\boldsymbol{k}\right\| t) + c_2 \sin(c \left\|\boldsymbol{k}\right\| t).$$

Using this solution, we see from the transformed boundary condition that

$$U(\boldsymbol{k}, 0) = c_1 = F(\boldsymbol{k}).$$

Also from this solution, we see from the transformed initial condition that

$$\frac{d}{dt}\left[U(\boldsymbol{k}, t)\right] = -c_1 \left(c \left\|\boldsymbol{k}\right\|\right) \sin(c \left\|\boldsymbol{k}\right\| t) + c_2(c \left\|\boldsymbol{k}\right\|) \cos(c \left\|\boldsymbol{k}\right\| t)$$

which implies that

$$\frac{d}{dt}\left[U(\boldsymbol{k}, 0)\right] = c_2(c \left\|\boldsymbol{k}\right\|) = 0.$$

Since this holds for all $\boldsymbol{k} \in \mathbb{R}^2$, we must have that $c_2 = 0$. Thus, the solution to the transformed system of differential equations is

$$U(\boldsymbol{k}, t) = F(\boldsymbol{k}) \cos(c \|\boldsymbol{k}\| t).$$

Therefore, from the definition of the inverse Fourier transform, the solution to the original system of differential equations is given by

$$u(\boldsymbol{x}, t) = \mathscr{F}^{-1} \{U(\boldsymbol{k}, t)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\boldsymbol{k}) \cos(c \|\boldsymbol{k}\| t) e^{i\boldsymbol{x} \cdot \boldsymbol{k}} d\boldsymbol{k}.$$

$\square$

Problem 2.54. Solve the following equations

a. $u_{xxxx} - u_{yy} + 2u = f(x, y)$,

b. $u_{xx} + 2u_{yy} + 3u_x - 4u = f(x, y)$,

where $f(x, y)$ is a given function.

Solution. □