Homework Assignment 7

Matthew Tiger

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Problem 4.28. Using the Laplace transform, evaluate the following integrals:

a. $f(t) = \displaystyle\int_0^\infty \frac{\sin tx}{\sqrt{x}} dx$,

e. $f(t) = \displaystyle\int_0^\infty e^{-tx^2} dx$, $0 < t$.

Solution. a. We begin by taking the Laplace transform of $f(t)$. Doing so yields

$$\begin{aligned}
\bar{f}(s) = \mathscr{L}\left\{f(t)\right\} &= \mathscr{L}\left\{\int_0^\infty \frac{\sin tx}{\sqrt{x}} dx\right\} \\
&= \int_0^\infty \mathscr{L}\left\{\frac{\sin tx}{\sqrt{x}}\right\} dx \\
&= \int_0^\infty \frac{\sqrt{x}}{s^2 + x^2} dx.
\end{aligned}$$

Using a computer algebra system, we see that this integral evaluates to

$$\begin{aligned}
\bar{f}(s) &= \int_0^\infty \frac{\sqrt{x}}{s^2 + x^2} dx \\
&= \frac{\pi}{\sqrt{2s}}.
\end{aligned}$$

From our table of Laplace transforms, we see that

$$\mathscr{L}^{-1}\left\{\frac{\Gamma(a+1)}{s^{a+1}}\right\} = t^a.$$

In particular, for $a = -1/2$, we see that

$$\mathscr{L}^{-1}\left\{\frac{\Gamma(1/2)}{s^{-1/2}}\right\} = \mathscr{L}^{-1}\left\{\frac{\sqrt{\pi}}{s^{-1/2}}\right\} = t^{-1/2}.$$

Therefore, the evaluation of the original integral is

$$\begin{aligned} f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} &= \mathscr{L}^{-1}\left\{\frac{\pi}{\sqrt{2s}}\right\} \\ &= \sqrt{\frac{\pi}{2}}\mathscr{L}^{-1}\left\{\frac{\sqrt{\pi}}{s^{-1/2}}\right\} \\ &= \sqrt{\frac{\pi}{2t}}. \end{aligned}$$

e. Applying the Laplace transform to $f(t)$ yields

$$\begin{aligned} \bar{f}(s) = \mathscr{L}\left\{f(t)\right\} &= \mathscr{L}\left\{\int_0^\infty e^{-tx^2}dx\right\} \\ &= \int_0^\infty \mathscr{L}\left\{e^{-tx^2}\right\}dx \\ &= \int_0^\infty \frac{1}{s+x^2}dx \end{aligned}$$

Using a computer algebra system, we see that

$$\begin{aligned} \bar{f}(s) &= \int_0^\infty \frac{1}{s+x^2}dx \\ &= \frac{\pi}{2\sqrt{s}}. \end{aligned}$$

Therefore, using previous arguments, we see that the evaluation of the original integral is

$$\begin{aligned} f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} &= \mathscr{L}^{-1}\left\{\frac{\pi}{2\sqrt{s}}\right\} \\ &= \sqrt{\frac{\pi}{4}}\mathscr{L}^{-1}\left\{\frac{\sqrt{\pi}}{s^{-1/2}}\right\} \\ &= \sqrt{\frac{\pi}{4t}}. \end{aligned}$$

$\square$

Problem 4.29. Show that

b. $I(a) = \int_0^\infty e^{-ax}\left(\frac{\sin qx - \sin px}{x}\right)dx = \tan^{-1}\left(\frac{q}{a}\right) - \tan^{-1}\left(\frac{p}{a}\right)$

Solution. b. Let $f(x) = \sin qx - \sin px$ and $g(x) = \frac{f(x)}{x}$.

From the definition of the Laplace transform, we see that this integral is the Laplace transform of $\frac{f(x)}{x}$ with respect to x in the variable a, i.e.

$$I(a) = \int_0^\infty e^{-ax}\left(\frac{\sin qx - \sin px}{x}\right)dx = \mathscr{L}\left\{\frac{f(x)}{x}\right\} = \bar{g}(a).$$

From a previous result, we know that

$$I(a) = \mathscr{L}\left\{\frac{f(x)}{x}\right\} = \int_a^\infty \bar{f}(a)da$$

where $\bar{f}(a) = \mathscr{L}\{f(x)\}$. Our table of Laplace transforms shows that

$$\begin{aligned}\bar{f}(a) = \mathscr{L}\{f(x)\} &= \mathscr{L}\{\sin qx - \sin px\}\\ &= \frac{q}{a^2+q^2} - \frac{p}{a^2+p^2}.\end{aligned}$$

Thus, we see that

$$I(a) = \int_a^\infty \bar{f}(a)da = \int_a^\infty \frac{q}{a^2+q^2} - \frac{p}{a^2+p^2}da.$$

Recall that

$$\int \frac{t}{a^2+t^2}da = \tan^{-1}\left(\frac{a}{t}\right) + C.$$

Therefore, we have that

$$\begin{aligned}I(a) &= \int_a^\infty \frac{q}{a^2+q^2} - \frac{p}{a^2+p^2}da\\ &= \left[\frac{\pi}{2} - \tan^{-1}\left(\frac{a}{q}\right)\right] - \left[\frac{\pi}{2} - \tan^{-1}\left(\frac{a}{p}\right)\right]\\ &= \tan^{-1}\left(\frac{q}{a}\right) - \tan^{-1}\left(\frac{p}{a}\right).\end{aligned}$$

$\square$

Problem 4.32. A uniform horizontal beam of length $2l$ is clamped at the end $x = 0$ and freely supported at $x = 2l$. It carries a distributed load of constant value W in $l/2 < x < 3l/2$ and zero elsewhere. Obtain the deflection of the beam which satisfies the boundary value problem

$$\begin{aligned} &EI\frac{d^4}{dx^4}\left[y(x)\right] = W\left[H\left(x-\frac{l}{2}\right) - H\left(x - \frac{3l}{2}\right)\right], \qquad 0 < x < 2l \\ &y(0) = y'(0) = 0, \quad y''(2l) = y'''(2l) = 0. \end{aligned}$$

Solution. We apply the Laplace transform to this equation to transform it from an ordinary differential equation to an algebraic one. Doing so yields

$$EI\left[s^4\bar{y}(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)\right] = W\left[\frac{e^{-\frac{l}{2}s} - e^{-\frac{3l}{2}s}}{s}\right].$$

In light of the initial data, this equation becomes

$$EI\left[s^4\bar{y}(s) - sy''(0) - y'''(0)\right] = W\left[\frac{e^{-\frac{l}{2}s} - e^{-\frac{3l}{2}s}}{s}\right],$$

or, equivalently,

$$EI\bar{y}(s) = W\left[\frac{e^{-\frac{l}{2}s} - e^{-\frac{3l}{2}s}}{s^5}\right] + EI\left[\frac{y''(0)}{s^3} + \frac{y'''(0)}{s^4}\right]$$

From the Convolution Theorem and previous properties of the Laplace transform, we know that

$$\begin{aligned} \mathscr{L}^{-1}\left\{\frac{e^{-as}}{s^5}\right\} &= \mathscr{L}^{-1}\left\{\frac{1}{s^4}\frac{e^{-as}}{s}\right\} \\ &= \int_0^t H(\tau - a)\frac{(t-\tau)^3}{6}d\tau \\ &= \frac{(t-a)^4}{24}H\left(t-a\right). \end{aligned}$$

Thus, by taking the inverse Laplace transform, we see that

$$y(x) = \frac{W}{24EI}\left[\left(t-\frac{l}{2}\right)^4 H\left(t-\frac{l}{2}\right) - \left(t-\frac{3l}{2}\right)^4 H\left(t-\frac{3l}{2}\right)\right] + \frac{y''(0)x^2}{2} + \frac{y''(0)x^3}{6}.$$

Now, using a computer algebra system and this form of $y(x)$, we see that the boundary conditions give us a system of equations

$$\begin{aligned} y''(0) + 2y'''(0)l + \frac{l^2W}{EI} &= 0 \\ y'''(0) + \frac{lW}{EI} &= 0 \end{aligned}$$

Solving this system we see that $y''(0) = \frac{l^2 W}{EI}$ and $y'''(0) = -\frac{lW}{EI}$.

Therefore, the solution to the original equation is

$$y(x) = \frac{W}{24EI}\left[\left(t - \frac{l}{2}\right)^4 H\left(t - \frac{l}{2}\right) - \left(t - \frac{3l}{2}\right)^4 H\left(t - \frac{3l}{2}\right)\right] + \frac{l^2 W x^2}{2EI} - \frac{lWx^3}{6EI}.$$

$\square$

Problem 4.35. Using the Laplace transform, solve the following difference equations:

a. $\Delta u_n - 2u_n = 0$, $u_0 = 1$

b. $\Delta^2 u_n - 2u_{n+1} + 3u_n = 0$, $u_0 = 0$, $u_1 = 1$.

Solution. Define $S_n(t) = H(t-n) - H(t-n-1)$ for $n \leq t < n+1$ and define

$$u(t) = \sum_{n=0}^{\infty} u_n S_n(t).$$

It follows that for $n \leq t < n+1$ we have that $u(t) = u_n$.

By a previous theorem, if $\bar{u}(s) = \mathscr{L}\left\{u(t)\right\}$, then

$$\mathscr{L}\left\{u(t+1)\right\} = e^s\left[\bar{u}(s) - u_0\bar{S}_0(s)\right]$$

where $\bar{S}_0 = \frac{1}{s}\left(1 - e^{-s}\right)$. It then follows that

$$\mathscr{L}\left\{u(t+2)\right\} = e^{2s}\left[\bar{u}(s) - (u_0 + u_1 e^{-s})\bar{S}_0(s)\right].$$

a. Note that this difference equation is equivalent to

$$\Delta u_n - 2u_n = u_{n+1} - 3u_n = 0.$$

Applying the Laplace transform to the difference equation yields that

$$\mathscr{L}\left\{u_{n+1} - 3u_n\right\} = e^s\left[\bar{u}(s) - u_0\bar{S}_0(s)\right] - 3\bar{u}(s) = 0 = \mathscr{L}\left\{0\right\}.$$

In light of the initial data, this equation becomes

$$e^s\left[\bar{u}(s) - \bar{S}_0(s)\right] - 3\bar{u}(s) = 0.$$

Thus, we see that

$$\bar{u}(s) = \frac{e^s\bar{S}_0(s)}{e^s - 3}.$$

Therefore, from a previous result, we see that the solution to the original difference equation is

$$u(t) = \mathscr{L}^{-1}\left\{\bar{u}(s)\right\} = \mathscr{L}^{-1}\left\{\frac{e^s\bar{S}_0(s)}{e^s - 3}\right\} = 3^n$$

b. Note that this difference equation is equivalent to

$$\Delta^2 u_n - 2u_{n+1} + 3u_n = u_{n+2} - 4u_{n+1} + 4u_n = 0.$$

Applying the Laplace transform to the difference equation yields that

$$\mathscr{L}\left\{u_{n+2} - 4u_{n+1} + 4u_n\right\} = e^{2s}\left[\bar{u}(s) - (u_0 + u_1 e^{-s})\bar{S}_0(s)\right] - 4e^s\left[\bar{u}(s) - u_0\bar{S}_0(s)\right] + 4\bar{u}(s) = 0.$$

In light of the initial data, this equation becomes

$$e^{2s}\left[\bar{u}(s) - e^{-s}\bar{S}_0(s)\right] - 4e^{s}\bar{u}(s) + 4\bar{u}(s) = 0.$$

Thus, we see that

$$\bar{u}(s) = \frac{e^{s}\bar{S}_0(s)}{\left(e^{s}-2\right)^2}.$$

From a previous result, we know that

$$\mathscr{L}\left\{na^n\right\} = \frac{ae^{s}\bar{S}_0(s)}{\left(e^{s}-a\right)^2}$$

Therefore, we see that the solution to the original difference equation is

$$u(t) = \mathscr{L}^{-1}\left\{\bar{u}(s)\right\} = \mathscr{L}^{-1}\left\{\frac{e^{s}\bar{S}_0(s)}{\left(e^{s}-2\right)^2}\right\} = n2^{n-1}.$$

$\square$

Problem 4.36. Show that the solution of the difference equation

$$u_{n+2} + 4u_{n+1} + u_n = 0$$

with $u_0 = 0$ and $u_1 = 1$, is

$$u_n = \frac{1}{2\sqrt{3}} \left[\left(\sqrt{3} - 2 \right)^n + (-1)^{n+1} \left(2 + \sqrt{3} \right)^n \right]$$

Solution. Applying the Laplace transform to the difference equation yields that

$$\mathscr{L}\left\{u_{n+2} + 4u_{n+1} + u_n\right\} = e^{2s}\left[\bar{u}(s) - (u_0 + u_1 e^{-s})\bar{S}_0(s)\right] + 4e^s\left[\bar{u}(s) - u_0\bar{S}_0(s)\right] + \bar{u}(s) = 0.$$

In light of the initial data, this equation becomes

$$e^{2s}\left[\bar{u}(s) - e^{-s}\bar{S}_0(s)\right] + 4e^s\bar{u}(s) + \bar{u}(s) = 0.$$

Thus, we see that

$$\bar{u}(s) = \frac{e^s\bar{S}_0(s)}{e^{2s} + 4e^s + 1} = \frac{e^s\bar{S}_0(s)}{(e^s - \alpha_1)(e^s - \alpha_2)},$$

where $\alpha_1 = -2 - \sqrt{3}$ and $\alpha_2 = -2 + \sqrt{3}$. From the method of partial fraction decomposition, we then see that

$$\begin{aligned} \bar{u}(s) &= \frac{e^s\bar{S}_0(s)}{(e^s - \alpha_1)(e^s - \alpha_2)} \\ &= \frac{e^s\bar{S}_0(s)}{\alpha_2 - \alpha_1}\left(\frac{1}{e^s - \alpha_2} - \frac{1}{e^s - \alpha_1}\right). \end{aligned}$$

From a previous result, we know that

$$\mathscr{L}^{-1}\left\{\frac{e^s\bar{S}_0(s)}{e^s - a}\right\} = a^n.$$

Therefore, the solution to the original difference equation is

$$\begin{aligned} u(t) = \mathscr{L}^{-1}\left\{\bar{u}(s)\right\} &= \mathscr{L}^{-1}\left\{\frac{e^s\bar{S}_0(s)}{\alpha_2 - \alpha_1}\left(\frac{1}{e^s - \alpha_2} - \frac{1}{e^s - \alpha_1}\right)\right\} \\ &= \frac{1}{\alpha_2 - \alpha_1}\left[\mathscr{L}^{-1}\left\{\frac{e^s\bar{S}_0(s)}{e^s - \alpha_2}\right\} - \mathscr{L}^{-1}\left\{\frac{e^s\bar{S}_0(s)}{e^s - \alpha_1}\right\}\right] \\ &= \frac{\alpha_2^n - \alpha_1^n}{\alpha_2 - \alpha_1} \\ &= \frac{1}{2\sqrt{3}}\left[\left(\sqrt{3} - 2\right)^n + (-1)^{n+1}\left(2 + \sqrt{3}\right)^n\right]. \end{aligned}$$

□

Problem 4.37. Show that the solution of the differential difference equation

$$\frac{d}{dt}u(t) - u(t-1) = 2, \qquad u(0) = 0$$

is

$$u(t) = 2\left[t - \frac{(t-1)^2}{2!} + \frac{(t-2)^3}{3!} + \cdots + \frac{(t-n)^{n+1}}{(n+1)!}\right], \quad t > n$$

Solution. Applying the Laplace transform to the differential difference equation yields that

$$s\bar{u}(s) - u(0) - e^{-s}\left[\bar{u}(s) - u(0)\bar{S}_0(s)\right] = \frac{2}{s}$$

In light of the initial data, this equation reduces to

$$s\bar{u}(s) - e^{-s}\bar{u}(s) = \frac{2}{s},$$

or, equivalently,

$$\bar{u}(s) = \frac{2}{s(s-e^s)} = \frac{2}{s^2}\left(1 - \frac{e^{-s}}{s}\right)^{-1}$$

Expanding the right term in terms of its power series we see that

$$\begin{aligned}\bar{u}(s) &= \frac{2}{s^2}\left(1 - \frac{e^{-s}}{s}\right)^{-1} \\ &= \frac{2}{s^2}\sum_{n=0}^{\infty}\frac{e^{-ns}}{s^n} \\ &= 2\sum_{n=0}^{\infty}\frac{e^{-ns}}{s^{n+2}}.\end{aligned}$$

Recall that

$$\mathscr{L}^{-1}\left\{\frac{e^{-as}}{s^n}\right\} = \frac{(t-a)^{n-1}}{\Gamma(n)}H(t-a).$$

Therefore, the solution to the original differential difference equation is

$$\begin{aligned}u(t) &= \mathscr{L}^{-1}\left\{2\sum_{n=0}^{\infty}\frac{e^{-ns}}{s^{n+2}}\right\} \\ &= 2\sum_{n=0}^{\infty}\mathscr{L}^{-1}\left\{\frac{e^{-ns}}{s^{n+2}}\right\} \\ &= 2\sum_{n=0}^{\infty}\frac{(t-n)^{n+1}}{\Gamma(n+2)}H(t-n) \\ &= 2\sum_{n=0}^{\infty}\frac{(t-n)^{n+1}}{(n+1)!}H(t-n).\end{aligned}$$

□

Problem 4.40. Solve the telegraph equation

$$\begin{aligned}&u_{tt} - c^2 u_{xx} + 2au_t = 0, \qquad -\infty < x < \infty, \quad 0 < t\\ &u(x,0) = 0, \quad u_t(x,0) = g(x).\end{aligned}$$

Solution. We begin by applying the Laplace transform to the equation. Doing so yields

$$s^2\bar{u}(x,s) - su(x,0) - u_t(x,0) - c^2\frac{d^2}{dx^2}\left[\bar{u}(x,s)\right] + 2as\bar{u}(x,s) - 2au(x,0) = 0.$$

Using the initial data, this equation reduces to

$$s^2\bar{u}(x,s) - g(x) - c^2\frac{d^2}{dx^2}\left[\bar{u}(x,s)\right] + 2as\bar{u}(s) = 0,$$

or, equivalently,

$$\frac{d^2}{dx^2}\left[\bar{u}(x,s)\right] - \frac{s^2+2as}{c^2}\bar{u}(x,s) = -\frac{g(x)}{c^2}.$$

Now, applying the Fourier transform to this equation yields

$$k^2\bar{U}(k,s) + \frac{s^2+2as}{c^2}\bar{U}(k,s) = \frac{G(k)}{c^2}.$$

Solving the resulting algebraic equation for $\bar{U}(k,s)$ shows that

$$\bar{U}(k,s) = \frac{G(k)}{s^2+2as+k^2c^2} = \frac{G(k)}{(s+a)^2+k^2c^2-a^2}.$$

From our table of Laplace transforms, we see that

$$\mathscr{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2}.$$

Thus, the inverse Laplace transform of the above equation is

$$\begin{aligned}U(k,t) = \mathscr{L}^{-1}\left\{\bar{U}(k,s)\right\} &= \mathscr{L}^{-1}\left\{\frac{G(k)}{(s+a)^2+k^2c^2-a^2}\right\}\\ &= G(k)\frac{e^{-at}\sin\left(t\sqrt{k^2c^2-a^2}\right)}{\sqrt{k^2c^2-a^2}}.\end{aligned}$$

Let $\omega(k) = \sqrt{k^2c^2-a^2}$. Then the above equation becomes

$$\begin{aligned}U(k,t) &= G(k)\frac{e^{-at}\sin\left(t\sqrt{k^2c^2-a^2}\right)}{\sqrt{k^2c^2-a^2}}\\ &= G(k)\frac{e^{-at}\sin\left(\omega(k)t\right)}{\omega(k)}\\ &= e^{-at}G(k)F(k,t),\end{aligned}$$

where

$$F(k,t) = \frac{\sin\left(\omega(k)t\right)}{\omega(k)}.$$

Therefore, according to the Convolution Theorem, the solution to the original equation is

$$\begin{aligned} u(x,t) = \mathscr{F}^{-1}\left\{U(k,t)\right\} &= e^{-at}\mathscr{F}^{-1}\left\{G(k)F(k,t)\right\} \\ &= e^{-at}\left[g(x) * f(x,t)\right] \\ &= \frac{e^{-at}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\xi)f(x-\xi,t)d\xi. \end{aligned}$$

□

Problem 4.43. Solve the diffusion equation

$$\begin{aligned} &u_t = ku_{xx}, && -a < x < a, \quad 0 < t \\ &u(x,0) = 1, && -a < x < a \\ &u(-a,t) = u(a,t) = 0, && 0 < t. \end{aligned}$$

Solution. We begin by applying the Laplace transform to the differential equation. Doing so yields

$$s\bar{u}(x,s) - u(x,0) = k\frac{d^2}{dx^2}\left[\bar{u}(x,s)\right].$$

Using the initial data, this equation reduces to

$$s\bar{u}(x,s) - 1 = k\frac{d^2}{dx^2}\left[\bar{u}(x,s)\right],$$

or, equivalently,

$$\frac{d^2}{dx^2}\left[\bar{u}(x,s)\right] - \frac{s}{k}\bar{u}(x,s) = \frac{1}{k}.$$

The homogeneous solution to this linear second-order ordinary differential equation is readily seen to be

$$\begin{aligned} \bar{u}_h(x,s) &= b_1 \exp\left(-\sqrt{\frac{s}{k}}x\right) + b_2 \exp\left(\sqrt{\frac{s}{k}}x\right) \\ &= c_1 \cosh\left(\sqrt{\frac{s}{k}}x\right) + c_2 \sinh\left(\sqrt{\frac{s}{k}}x\right) \end{aligned}$$

By inspection, we see that

$$\bar{u}_p(x,s) = -\frac{1}{s}$$

is a particular solution of the transformed equation. Thus, the solution to the transformed equation is

$$\bar{u}(x,s) = \bar{u}_h(x,s) + \bar{u}_p(x,s) = c_1 \cosh\left(\sqrt{\frac{s}{k}}x\right) + c_2 \sinh\left(\sqrt{\frac{s}{k}}x\right) - \frac{1}{s}.$$

Note that the transformed boundary conditions are

$$\bar{u}(a,s) = \bar{u}(-a,s) = 0.$$

Using the above solution in conjunction with the transformed boundary conditions leads us to the following system of equations

$$\begin{aligned} \bar{u}(-a,s) &= c_1 \cosh\left(-a\sqrt{\frac{s}{k}}\right) + c_2 \sinh\left(-a\sqrt{\frac{s}{k}}\right) - \frac{1}{s} = 0 \\ \bar{u}(a,s) &= c_1 \cosh\left(a\sqrt{\frac{s}{k}}\right) + c_2 \sinh\left(a\sqrt{\frac{s}{k}}\right) - \frac{1}{s} = 0. \end{aligned}$$

Solving for c_1 and c_2 we see that

$$\begin{aligned} c_1 &= \frac{1}{s\cosh\left(\sqrt{\frac{s}{k}}a\right)} \\ c_2 &= 0. \end{aligned}$$

Thus, the solution to the transformed equation adhering to the transformed boundary conditions is

$$\bar{u}(x,s) = \left(\frac{\cosh\left(\sqrt{\frac{s}{k}}x\right)}{s\cosh\left(\sqrt{\frac{s}{k}}a\right)}\right) - \frac{1}{s}.$$

Now let

$$\bar{f}(x,s) = \left(\frac{\cosh\left(\sqrt{\frac{s}{k}}x\right)}{\cosh\left(\sqrt{\frac{s}{k}}a\right)}\right).$$

and

$$\bar{g}(s) = \frac{1}{s}$$

Then

$$\begin{aligned} \bar{u}(x,s) &= \left(\frac{\cosh\left(\sqrt{\frac{s}{k}}x\right)}{s\cosh\left(\sqrt{\frac{s}{k}}a\right)}\right) - \frac{1}{s} \\ &= \bar{g}(s)\bar{f}(x,s) - \bar{g}(s) \end{aligned}$$

and by the Convolution Theorem, the solution to the original equation is

$$\begin{aligned} u(x,t) = \mathscr{L}^{-1}\left\{\bar{u}(x,s)\right\} &= \mathscr{L}^{-1}\left\{\bar{g}(s)\bar{f}(x,s) - \bar{g}(s)\right\} \\ &= g(t) * f(x,t) - 1 \\ &= \int_0^t f(x,\tau)d\tau - 1. \end{aligned}$$

□

Problem 4.50. a. Use the joint Laplace and Fourier transform to solve the inhomogeneous diffusion problem

$$u_t - \kappa u_{xx} = q(x,t), \qquad -\infty < x < \infty, \quad 0 < t$$
$$u(x,0) = f(x), \qquad -\infty < x < \infty.$$

Solution. We begin by applying the Laplace transform to the equation. Doing so yields

$$s\bar{u}(x,s) - u(x,0) - \kappa\frac{d^2}{dx^2}\left[\bar{u}(x,s)\right] = \bar{q}(x,s).$$

Using the initial condition, we see that this equation reduces to

$$s\bar{u}(x,s) - f(x) - \kappa\frac{d^2}{dx^2}\left[\bar{u}(x,s)\right] = \bar{q}(x,s).$$

We now apply the Fourier transform to this resulting equation. Applying yields

$$s\bar{U}(k,s) - F(k) + \kappa k^2\bar{U}(k,s) = \bar{Q}(k,s),$$

or, equivalently,

$$\bar{U}(k,s) = \frac{F(k) + \bar{Q}(k,s)}{s+\kappa k^2}.$$

Now applying the inverse Laplace transform, we have that

$$\begin{aligned} U(k,t) &= \mathscr{L}^{-1}\left\{\frac{F(k)+\bar{Q}(k,s)}{s+\kappa k^2}\right\} \\ &= \mathscr{L}^{-1}\left\{\frac{F(k)}{s+\kappa k^2}\right\} + \mathscr{L}^{-1}\left\{\frac{\bar{Q}(k,s)}{s+\kappa k^2}\right\} \\ &= F(k)e^{-\kappa k^2 t} + \int_0^t e^{-\kappa k^2(t-\tau)}Q(k,\tau)d\tau. \end{aligned}$$

Let

$$G(k,t) = e^{-\kappa k^2 t}$$

Then by our table of Fourier transforms, we have that

$$g(x,t) = \mathscr{F}^{-1}\left\{G(k,t)\right\} = \frac{1}{\sqrt{2\kappa t}}e^{-\frac{x^2}{4\kappa t}}$$

Thus, using the Convolution Theorem, we see that inverse Fourier transform of the equation is

$$\begin{aligned} u(x,t) &= \mathscr{F}^{-1}\left\{F(k)e^{-\kappa k^2 t} + \int_0^t e^{-\kappa k^2(t-\tau)}Q(k,\tau)d\tau\right\} \\ &= f(x) * g(x,t) + \int_0^t g(x,t-\tau) * q(x,\tau)d\tau \\ &= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} f(x-\xi)g(\xi,t)d\xi + \int_0^t \left[\int_{-\infty}^{\infty} g(x-\xi,t-\tau)q(\xi,\tau)d\xi\right]d\tau\right] \\ &= \frac{1}{\sqrt{4\pi\kappa t}}\left[\int_{-\infty}^{\infty} f(x-\xi)e^{-\frac{\xi^2}{4\kappa t}}d\xi + \int_0^t \left[\int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4\kappa(t-\tau)}}q(\xi,\tau)d\xi\right]d\tau\right]. \end{aligned}$$

□