

Homework Assignment 7

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Problem 4.28. Using the Laplace transform, evaluate the following integrals:

a. $f(t) = \int_0^\infty \frac{\sin tx}{\sqrt{x}} dx,$

e. $f(t) = \int_0^\infty e^{-tx^2} dx, 0 < t.$

Solution. a. We begin by taking the Laplace transform of $f(t)$. Doing so yields

$$\begin{aligned}\bar{f}(s) &= \mathcal{L}\{f(t)\} = \mathcal{L}\left\{\int_0^\infty \frac{\sin tx}{\sqrt{x}} dx\right\} \\ &= \int_0^\infty \mathcal{L}\left\{\frac{\sin tx}{\sqrt{x}}\right\} dx \\ &= \int_0^\infty \frac{\sqrt{x}}{s^2 + x^2} dx.\end{aligned}$$

Using a computer algebra system, we see that this integral evaluates to

$$\begin{aligned}\bar{f}(s) &= \int_0^\infty \frac{\sqrt{x}}{s^2 + x^2} dx \\ &= \frac{\pi}{\sqrt{2s}}.\end{aligned}$$

From our table of Laplace transforms, we see that

$$\mathcal{L}^{-1}\left\{\frac{\Gamma(a+1)}{s^{a+1}}\right\} = t^a.$$

In particular, for $a = -1/2$, we see that

$$\mathcal{L}^{-1}\left\{\frac{\Gamma(1/2)}{s^{-1/2}}\right\} = \mathcal{L}^{-1}\left\{\frac{\sqrt{\pi}}{s^{-1/2}}\right\} = t^{-1/2}.$$

Therefore, the evaluation of the original integral is

$$\begin{aligned} f(t) &= \mathcal{L}^{-1} \{ \bar{f}(s) \} = \mathcal{L}^{-1} \left\{ \frac{\pi}{\sqrt{2s}} \right\} \\ &= \sqrt{\frac{\pi}{2}} \mathcal{L}^{-1} \left\{ \frac{\sqrt{\pi}}{s^{-1/2}} \right\} \\ &= \sqrt{\frac{\pi}{2t}}. \end{aligned}$$

e. Applying the Laplace transform to $f(t)$ yields

$$\begin{aligned} \bar{f}(s) &= \mathcal{L} \{ f(t) \} = \mathcal{L} \left\{ \int_0^\infty e^{-tx^2} dx \right\} \\ &= \int_0^\infty \mathcal{L} \{ e^{-tx^2} \} dx \\ &= \int_0^\infty \frac{1}{s+x^2} dx \end{aligned}$$

Using a computer algebra system, we see that

$$\begin{aligned} \bar{f}(s) &= \int_0^\infty \frac{1}{s+x^2} dx \\ &= \frac{\pi}{2\sqrt{s}}. \end{aligned}$$

Therefore, using previous arguments, we see that the evaluation of the original integral is

$$\begin{aligned} f(t) &= \mathcal{L}^{-1} \{ \bar{f}(s) \} = \mathcal{L}^{-1} \left\{ \frac{\pi}{2\sqrt{s}} \right\} \\ &= \sqrt{\frac{\pi}{4}} \mathcal{L}^{-1} \left\{ \frac{\sqrt{\pi}}{s^{-1/2}} \right\} \\ &= \sqrt{\frac{\pi}{4t}}. \end{aligned}$$

□

Problem 4.29. Show that

$$\text{b. } I(a) = \int_0^\infty e^{-ax} \left(\frac{\sin qx - \sin px}{x} \right) dx = \tan^{-1} \left(\frac{q}{a} \right) - \tan^{-1} \left(\frac{p}{a} \right)$$

Solution. b. Let $f(x) = \sin qx - \sin px$ and $g(x) = \frac{f(x)}{x}$.

From the definition of the Laplace transform, we see that this integral is the Laplace transform of $\frac{f(x)}{x}$ with respect to x in the variable a , i.e.

$$I(a) = \int_0^\infty e^{-ax} \left(\frac{\sin qx - \sin px}{x} \right) dx = \mathcal{L} \left\{ \frac{f(x)}{x} \right\} = \bar{g}(a).$$

From a previous result, we know that

$$I(a) = \mathcal{L} \left\{ \frac{f(x)}{x} \right\} = \int_a^\infty \bar{f}(a) da$$

where $\bar{f}(a) = \mathcal{L} \{f(x)\}$. Our table of Laplace transforms shows that

$$\begin{aligned} \bar{f}(a) &= \mathcal{L} \{f(x)\} = \mathcal{L} \{\sin qx - \sin px\} \\ &= \frac{q}{a^2 + q^2} - \frac{p}{a^2 + p^2}. \end{aligned}$$

Thus, we see that

$$I(a) = \int_a^\infty \bar{f}(a) da = \int_a^\infty \frac{q}{a^2 + q^2} - \frac{p}{a^2 + p^2} da.$$

Recall that

$$\int \frac{t}{a^2 + t^2} da = \tan^{-1} \left(\frac{a}{t} \right) + C.$$

Therefore, we have that

$$\begin{aligned} I(a) &= \int_a^\infty \frac{q}{a^2 + q^2} - \frac{p}{a^2 + p^2} da \\ &= \left[\frac{\pi}{2} - \tan^{-1} \left(\frac{a}{q} \right) \right] - \left[\frac{\pi}{2} - \tan^{-1} \left(\frac{a}{p} \right) \right] \\ &= \tan^{-1} \left(\frac{q}{a} \right) - \tan^{-1} \left(\frac{p}{a} \right). \end{aligned}$$

□

Problem 4.32.*Solution.*

Problem 4.35. Using the Laplace transform, solve the following difference equations:

a. $\Delta u_n - 2u_n = 0, u_0 = 1$

b. $\Delta^2 u_n - 2u_{n+1} + 3u_n = 0, u_0 = 0, u_1 = 1.$

Solution. Define $S_n(t) = H(t - n) - H(t - n - 1)$ for $n \leq t < n + 1$ and define

$$u(t) = \sum_{n=0}^{\infty} u_n S_n(t).$$

It follows that for $n \leq t < n + 1$ we have that $u(t) = u_n$.

By a previous theorem, if $\bar{u}(s) = \mathcal{L}\{u(t)\}$, then

$$\mathcal{L}\{u(t+1)\} = e^s [\bar{u}(s) - u_0 \bar{S}_0(s)]$$

where $\bar{S}_0 = \frac{1}{s}(1 - e^{-s})$. It then follows that

$$\mathcal{L}\{u(t+2)\} = e^{2s} [\bar{u}(s) - (u_0 + u_1 e^{-s}) \bar{S}_0(s)].$$

a. Note that this difference equation is equivalent to

$$\Delta u_n - 2u_n = u_{n+1} - 3u_n = 0.$$

Applying the Laplace transform to the difference equation yields that

$$\mathcal{L}\{u_{n+1} - 3u_n\} = e^s [\bar{u}(s) - u_0 \bar{S}_0(s)] - 3\bar{u}(s) = 0 = \mathcal{L}\{0\}.$$

In light of the initial data, this equation becomes

$$e^s [\bar{u}(s) - \bar{S}_0(s)] - 3\bar{u}(s) = 0.$$

Thus, we see that

$$\bar{u}(s) = \frac{e^s \bar{S}_0(s)}{e^s - 3}.$$

Therefore, from a previous result, we see that the solution to the original difference equation is

$$u(t) = \mathcal{L}^{-1}\{\bar{u}(s)\} = \mathcal{L}^{-1}\left\{\frac{e^s \bar{S}_0(s)}{e^s - 3}\right\} = 3^n$$

b. Note that this difference equation is equivalent to

$$\Delta^2 u_n - 2u_{n+1} + 3u_n = u_{n+2} - 4u_{n+1} + 4u_n = 0.$$

Applying the Laplace transform to the difference equation yields that

$$\mathcal{L}\{u_{n+2} - 4u_{n+1} + 4u_n\} = e^{2s} [\bar{u}(s) - (u_0 + u_1 e^{-s}) \bar{S}_0(s)] - 4e^s [\bar{u}(s) - u_0 \bar{S}_0(s)] + 4\bar{u}(s) = 0.$$

In light of the initial data, this equation becomes

$$e^{2s} [\bar{u}(s) - e^{-s} \bar{S}_0(s)] - 4e^s \bar{u}(s) + 4\bar{u}(s) = 0.$$

Thus, we see that

$$\bar{u}(s) = \frac{e^s \bar{S}_0(s)}{(e^s - 2)^2}.$$

From a previous result, we know that

$$\mathcal{L} \{na^n\} = \frac{ae^s \bar{S}_0(s)}{(e^s - a)^2}$$

Therefore, we see that the solution to the original difference equation is

$$u(t) = \mathcal{L}^{-1} \{\bar{u}(s)\} = \mathcal{L}^{-1} \left\{ \frac{e^s \bar{S}_0(s)}{(e^s - 2)^2} \right\} = n2^{n-1}.$$

□

Problem 4.36. Show that the solution of the difference equation

$$u_{n+2} + 4u_{n+1} + u_n = 0$$

with $u_0 = 0$ and $u_1 = 1$, is

$$u_n = \frac{1}{2\sqrt{3}} \left[\left(\sqrt{3} - 2 \right)^n + (-1)^{n+1} \left(2 + \sqrt{3} \right)^n \right]$$

Solution. Applying the Laplace transform to the difference equation yields that

$$\mathcal{L} \{u_{n+2} + 4u_{n+1} + u_n\} = e^{2s} [\bar{u}(s) - (u_0 + u_1 e^{-s})\bar{S}_0(s)] + 4e^s [\bar{u}(s) - u_0\bar{S}_0(s)] + \bar{u}(s) = 0.$$

In light of the initial data, this equation becomes

$$e^{2s} [\bar{u}(s) - e^{-s}\bar{S}_0(s)] + 4e^s \bar{u}(s) + \bar{u}(s) = 0.$$

Thus, we see that

$$\bar{u}(s) = \frac{e^s \bar{S}_0(s)}{e^{2s} + 4e^s + 1} = \frac{e^s \bar{S}_0(s)}{(e^s - \alpha_1)(e^s - \alpha_2)},$$

where $\alpha_1 = -2 - \sqrt{3}$ and $\alpha_2 = -2 + \sqrt{3}$. From the method of partial fraction decomposition, we then see that

$$\begin{aligned} \bar{u}(s) &= \frac{e^s \bar{S}_0(s)}{(e^s - \alpha_1)(e^s - \alpha_2)} \\ &= \frac{e^s \bar{S}_0(s)}{\alpha_2 - \alpha_1} \left(\frac{1}{e^s - \alpha_2} - \frac{1}{e^s - \alpha_1} \right). \end{aligned}$$

From a previous result, we know that

$$\mathcal{L}^{-1} \left\{ \frac{e^s \bar{S}_0(s)}{e^s - a} \right\} = a^n.$$

Therefore, the solution to the original difference equation is

$$\begin{aligned} u(t) = \mathcal{L}^{-1} \{ \bar{u}(s) \} &= \mathcal{L}^{-1} \left\{ \frac{e^s \bar{S}_0(s)}{\alpha_2 - \alpha_1} \left(\frac{1}{e^s - \alpha_2} - \frac{1}{e^s - \alpha_1} \right) \right\} \\ &= \frac{1}{\alpha_2 - \alpha_1} \left[\mathcal{L}^{-1} \left\{ \frac{e^s \bar{S}_0(s)}{e^s - \alpha_2} \right\} - \mathcal{L}^{-1} \left\{ \frac{e^s \bar{S}_0(s)}{e^s - \alpha_1} \right\} \right] \\ &= \frac{\alpha_2^n - \alpha_1^n}{\alpha_2 - \alpha_1} \\ &= \frac{1}{2\sqrt{3}} \left[\left(\sqrt{3} - 2 \right)^n + (-1)^{n+1} \left(2 + \sqrt{3} \right)^n \right]. \end{aligned}$$

□

Problem 4.37. Show that the solution of the differential difference equation

$$\frac{d}{dt}u(t) - u(t-1) = 2, \quad u(0) = 0$$

is

$$u(t) = 2 \left[t - \frac{(t-1)^2}{2!} + \frac{(t-2)^3}{3!} + \cdots + \frac{(t-n)^{n+1}}{(n+1)!} \right], \quad t > n$$

Solution. Applying the Laplace transform to the differential difference equation yields that

$$s\bar{u}(s) - u(0) - e^{-s} [\bar{u}(s) - u(0)\bar{S}_0(s)] = \frac{2}{s}$$

In light of the initial data, this equation reduces to

$$s\bar{u}(s) - e^{-s}\bar{u}(s) = \frac{2}{s},$$

or, equivalently,

$$\bar{u}(s) = \frac{2}{s(s - e^{-s})} = \frac{2}{s^2} \left(1 - \frac{e^{-s}}{s} \right)^{-1}$$

Expanding the right term in terms of its power series we see that

$$\begin{aligned} \bar{u}(s) &= \frac{2}{s^2} \left(1 - \frac{e^{-s}}{s} \right)^{-1} \\ &= \frac{2}{s^2} \sum_{n=0}^{\infty} \frac{e^{-ns}}{s^n} \\ &= 2 \sum_{n=0}^{\infty} \frac{e^{-ns}}{s^{n+2}}. \end{aligned}$$

Recall that

$$\mathcal{L}^{-1} \left\{ \frac{e^{-as}}{s^n} \right\} = \frac{(t-a)^{n-1}}{\Gamma(n)} H(t-a).$$

Therefore, the solution to the original differential difference equation is

$$\begin{aligned} u(t) &= \mathcal{L}^{-1} \left\{ 2 \sum_{n=0}^{\infty} \frac{e^{-ns}}{s^{n+2}} \right\} \\ &= 2 \sum_{n=0}^{\infty} \mathcal{L}^{-1} \left\{ \frac{e^{-ns}}{s^{n+2}} \right\} \\ &= 2 \sum_{n=0}^{\infty} \frac{(t-n)^{n+1}}{\Gamma(n+2)} H(t-n) \\ &= 2 \sum_{n=0}^{\infty} \frac{(t-n)^{n+1}}{(n+1)!} H(t-n). \end{aligned}$$

□

Problem 4.40. Solve the telegraph equation

$$\begin{aligned} u_{tt} - c^2 u_{xx} + 2au_t &= 0, & -\infty < x < \infty, & \quad 0 < t \\ u(x, 0) &= 0, & u_t(x, 0) &= g(x). \end{aligned}$$

Solution. We begin by applying the Laplace transform to the equation. Doing so yields

$$s^2 \bar{u}(x, s) - su(x, 0) - u_t(x, 0) - c^2 \frac{d^2}{dx^2} [\bar{u}(x, s)] + 2as\bar{u}(x, s) - 2au(x, 0) = 0.$$

Using the initial data, this equation reduces to

$$s^2 \bar{u}(x, s) - g(x) - c^2 \frac{d^2}{dx^2} [\bar{u}(x, s)] + 2as\bar{u}(x, s) = 0,$$

or, equivalently,

$$\frac{d^2}{dx^2} [\bar{u}(x, s)] - \frac{s^2 + 2as}{c^2} \bar{u}(x, s) = -\frac{g(x)}{c^2}.$$

Now, applying the Fourier transform to this equation yields

$$k^2 \bar{U}(k, s) + \frac{s^2 + 2as}{c^2} \bar{U}(k, s) = \frac{G(k)}{c^2}.$$

Solving the resulting algebraic equation for $\bar{U}(k, s)$ shows that

$$\bar{U}(k, s) = \frac{G(k)}{s^2 + 2as + k^2 c^2} = \frac{G(k)}{(s + a)^2 + k^2 c^2 - a^2}.$$

From our table of Laplace transforms, we see that

$$\mathcal{L} \{e^{at} \sin bt\} = \frac{b}{(s - a)^2 + b^2}.$$

Thus, the inverse Laplace transform of the above equation is

$$\begin{aligned} U(k, t) &= \mathcal{L}^{-1} \{ \bar{U}(k, s) \} = \mathcal{L}^{-1} \left\{ \frac{G(k)}{(s + a)^2 + k^2 c^2 - a^2} \right\} \\ &= G(k) \frac{e^{-at} \sin(t\sqrt{k^2 c^2 - a^2})}{\sqrt{k^2 c^2 - a^2}}. \end{aligned}$$

Let $\omega(k) = \sqrt{k^2 c^2 - a^2}$. Then the above equation becomes

$$\begin{aligned} U(k, t) &= G(k) \frac{e^{-at} \sin(t\sqrt{k^2 c^2 - a^2})}{\sqrt{k^2 c^2 - a^2}} \\ &= G(k) \frac{e^{-at} \sin(\omega(k)t)}{\omega(k)} \\ &= e^{-at} G(k) F(k, t), \end{aligned}$$

where

$$F(k, t) = \frac{\sin(\omega(k)t)}{\omega(k)}.$$

Therefore, according to the Convolution Theorem, the solution to the original equation is

$$\begin{aligned} u(x, t) &= \mathcal{F}^{-1}\{U(k, t)\} = e^{-at} \mathcal{F}^{-1}\{G(k)F(k, t)\} \\ &= e^{-at} [g(x) * f(x, t)] \\ &= \frac{e^{-at}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\xi) f(x - \xi, t) d\xi. \end{aligned}$$

□

Problem 4.43. Solve the diffusion equation

$$\begin{aligned} u_t &= k u_{xx}, & -a < x < a, & \quad 0 < t \\ u(x, 0) &= 1, & -a < x < a \\ u(-a, t) &= u(a, t) = 0, & \quad 0 < t. \end{aligned}$$

Solution. We begin by applying the Laplace transform to the differential equation. Doing so yields

$$s\bar{u}(x, s) - u(x, 0) = k \frac{d^2}{dx^2} [\bar{u}(x, s)].$$

Using the initial data, this equation reduces to

$$s\bar{u}(x, s) - 1 = k \frac{d^2}{dx^2} [\bar{u}(x, s)],$$

or, equivalently,

$$\frac{d^2}{dx^2} [\bar{u}(x, s)] - \frac{s}{k} \bar{u}(x, s) = -\frac{1}{k}.$$

The homogeneous solution to this linear second-order ordinary differential equation is readily seen to be

$$\begin{aligned} \bar{u}_h(x, s) &= b_1 \exp\left(-\sqrt{\frac{s}{k}}x\right) + b_2 \exp\left(\sqrt{\frac{s}{k}}x\right) \\ &= c_1 \cosh\left(\sqrt{\frac{s}{k}}x\right) + c_2 \sinh\left(\sqrt{\frac{s}{k}}x\right) \end{aligned}$$

By inspection, we see that

$$\bar{u}_p(x, s) = -\frac{1}{s}$$

is a particular solution of the transformed equation. Thus, the solution to the transformed equation is

$$\bar{u}(x, s) = \bar{u}_h(x, s) + \bar{u}_p(x, s) = c_1 \cosh\left(\sqrt{\frac{s}{k}}x\right) + c_2 \sinh\left(\sqrt{\frac{s}{k}}x\right) - \frac{1}{s}.$$

Note that the transformed boundary conditions are

$$\bar{u}(a, s) = \bar{u}(-a, s) = 0.$$

Using the above solution in conjunction with the transformed boundary conditions leads us to the following system of equations

$$\begin{aligned} \bar{u}(-a, s) &= c_1 \cosh\left(-a\sqrt{\frac{s}{k}}\right) + c_2 \sinh\left(-a\sqrt{\frac{s}{k}}\right) - \frac{1}{s} = 0 \\ \bar{u}(a, s) &= c_1 \cosh\left(a\sqrt{\frac{s}{k}}\right) + c_2 \sinh\left(a\sqrt{\frac{s}{k}}\right) - \frac{1}{s} = 0. \end{aligned}$$

Solving for c_1 and c_2 we see that

$$c_1 = \frac{1}{s \cosh\left(\sqrt{\frac{s}{k}}a\right)}$$

$$c_2 = 0.$$

Thus, the solution to the transformed equation adhering to the transformed boundary conditions is

$$\bar{u}(x, s) = \left(\frac{\cosh\left(\sqrt{\frac{s}{k}}x\right)}{s \cosh\left(\sqrt{\frac{s}{k}}a\right)} \right) - \frac{1}{s}.$$

Now let

$$\bar{f}(x, s) = \left(\frac{\cosh\left(\sqrt{\frac{s}{k}}x\right)}{\cosh\left(\sqrt{\frac{s}{k}}a\right)} \right).$$

and

$$\bar{g}(s) = \frac{1}{s}$$

Then

$$\begin{aligned} \bar{u}(x, s) &= \left(\frac{\cosh\left(\sqrt{\frac{s}{k}}x\right)}{s \cosh\left(\sqrt{\frac{s}{k}}a\right)} \right) - \frac{1}{s} \\ &= \bar{g}(s)\bar{f}(x, s) - \bar{g}(s) \end{aligned}$$

and by the Convolution Theorem, the solution to the original equation is

$$\begin{aligned} u(x, t) &= \mathcal{L}^{-1}\{\bar{u}(x, s)\} = \mathcal{L}^{-1}\{\bar{g}(s)\bar{f}(x, s) - \bar{g}(s)\} \\ &= g(t) * f(x, t) - 1 \\ &= \int_0^t f(x, \tau) d\tau - 1. \end{aligned}$$

□

Problem 4.50. a. Use the joint Laplace and Fourier transform to solve the inhomogeneous diffusion problem

$$\begin{aligned} u_t - \kappa u_{xx} &= q(x, t), & -\infty < x < \infty, & \quad 0 < t \\ u(x, 0) &= f(x), & -\infty < x < \infty. \end{aligned}$$

Solution. We begin by applying the Laplace transform to the equation. Doing so yields

$$s\bar{u}(x, s) - u(x, 0) - \kappa \frac{d^2}{dx^2} [\bar{u}(x, s)] = \bar{q}(x, s).$$

Using the initial condition, we see that this equation reduces to

$$s\bar{u}(x, s) - f(x) - \kappa \frac{d^2}{dx^2} [\bar{u}(x, s)] = \bar{q}(x, s).$$

We now apply the Fourier transform to this resulting equation. Applying yields

$$s\bar{U}(k, s) - F(k) + \kappa k^2 \bar{U}(k, s) = \bar{Q}(k, s),$$

or, equivalently,

$$\bar{U}(k, s) = \frac{F(k) + \bar{Q}(k, s)}{s + \kappa k^2}.$$

Now applying the inverse Laplace transform, we have that

$$\begin{aligned} U(k, t) &= \mathcal{L}^{-1} \left\{ \frac{F(k) + \bar{Q}(k, s)}{s + \kappa k^2} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{F(k)}{s + \kappa k^2} \right\} + \mathcal{L}^{-1} \left\{ \frac{\bar{Q}(k, s)}{s + \kappa k^2} \right\} \\ &= F(k) e^{-\kappa k^2 t} + \int_0^t e^{-\kappa k^2 (t-\tau)} Q(k, \tau) d\tau. \end{aligned}$$

Let

$$G(k, t) = e^{-\kappa k^2 t}$$

Then by our table of Fourier transforms, we have that

$$g(x, t) = \mathcal{F}^{-1} \{G(k, t)\} = \frac{1}{\sqrt{2\kappa t}} e^{-\frac{x^2}{4\kappa t}}$$

Thus, using the Convolution Theorem, we see that inverse Fourier transform of the equation is

$$\begin{aligned} u(x, t) &= \mathcal{F}^{-1} \left\{ F(k) e^{-\kappa k^2 t} + \int_0^t e^{-\kappa k^2 (t-\tau)} Q(k, \tau) d\tau \right\} \\ &= f(x) * g(x, t) + \int_0^t g(x, t - \tau) * q(x, \tau) d\tau \\ &= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} f(x - \xi) g(\xi, t) d\xi + \int_0^t \left[\int_{-\infty}^{\infty} g(x - \xi, t - \tau) q(\xi, \tau) d\xi \right] d\tau \right] \\ &= \frac{1}{\sqrt{4\pi\kappa t}} \left[\int_{-\infty}^{\infty} f(x - \xi) e^{-\frac{\xi^2}{4\kappa t}} d\xi + \int_0^t \left[\int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4\kappa(t-\tau)}} q(\xi, \tau) d\xi \right] d\tau \right]. \end{aligned}$$

□