# Homework Assignment 9

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**Problem 8.2.2.** If $Sf(x)$ is the Schwarzian derivative of $f(x)$ with $f \in C^3$ and $F(x) = \dfrac{f''(x)}{f'(x)}$, show that $Sf(x) = F'(x) - (F(x))^2/2$.

*Solution.* Recall that the Schwarzian derivative of $f$ is given by

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2.$$

We readily see from the definition of $F(x)$ that

$$\begin{aligned} F'(x) - \frac{1}{2}\left[F(x)\right]^2 &= \frac{f'(x)f'''(x) - f''(x)^2}{f'(x)^2} - \frac{1}{2}\left[\frac{f''(x)}{f'(x)}\right]^2 \\ &= \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2 \\ &= Sf(x) \end{aligned}$$

and we are done. $\square$

**Problem 8.2.5.** i. Show that if $p$ is a polynomial of degree $n$ having $n$ distinct fixed points, and negative Schwarzian derivative, then not all of the fixed points can be attracting.

ii. On the other hand, show that the logistic maps $L_\mu : \mathbb{R} \to \mathbb{R}$ for $\mu > 2 + \sqrt{5}$ have negative Schwarzian derivative but have no attracting periodic orbits.

*Solution.* i. Suppose to the contrary that $p$ is a polynomial of degree $n$ with $n$ distinct fixed points and negative Schwarzian derivative but all of its fixed points are attracting. Let $x_1, \ldots, x_n$ denote these attracting fixed points.

Since $p$ is a polynomial, it is continuous, which implies that for each attracting fixed point $x_k$, its immediate basin of attraction $W_k$ is an open interval. Note that these fixed points are distinct and attracting so that the immediate basins of attraction of two fixed points $x_k$ and $x_j$ with $k \neq j$ are mutually exclusive, i.e. $W_j \cap W_k = \emptyset$ for any $k \neq j$.

Since $p \in C^3$ with negative Schwarzian derivative, we have by Singer's theorem that for every fixed point $x_k$, either $W_k$ is an unbounded interval, or the orbit of some critical point of $p$ is attracted to the orbit of $x_k$ under $f$.

ii.

$\square$

**Problem 8.2.6.**

*Solution.* □

**Problem 8.2.10.**

*Solution.* □

**Problem 10.3.4.**

*Solution.* □

**Problem 10.3.6.**

*Solution.* □

**Problem 10.3.7.**

*Solution.* $\square$