Homework Assignment 5

Matthew Tiger

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Problem 2.7. Show, using the geometric series $1/(1-x) = \sum_{j=0}^{\infty} x^j$ for $|x| < 1$, that $1/(1-\phi z) = -\sum_{j=1}^{\infty} \phi^{-j} z^{-j}$ for $|\phi| > 1$ and $|z| \geq 1$.

Solution. If $|\phi| > 1$ and $|z| \geq 1$, then $|\phi z| = |\phi||z| > 1$ and $|1/\phi z| = 1/|\phi z| < 1$. Now,

$$\sum_{j=0}^{\infty} \phi^{-j} z^{-j} = \sum_{j=0}^{\infty} (\phi z)^{-j} = \sum_{j=0}^{\infty} \left(\frac{1}{\phi z}\right)^j.$$

Note that this is a geometric series where $|\frac{1}{\phi z}| < 1$. So the series converges and

$$\sum_{j=0}^{\infty} \left(\frac{1}{\phi z}\right)^j = \frac{1}{1-\frac{1}{\phi z}} = \frac{\phi z}{\phi z - 1}.$$

This implies that

$$\frac{\phi z}{\phi z - 1} = \sum_{j=0}^{\infty} \left(\frac{1}{\phi z}\right)^j = 1 + \sum_{j=1}^{\infty} \left(\frac{1}{\phi z}\right)^j$$

so that

$$\sum_{j=1}^{\infty} \left(\frac{1}{\phi z}\right)^j = \frac{\phi z}{\phi z - 1} - 1 = \frac{\phi z - (\phi z - 1)}{\phi z - 1} = \frac{1}{\phi z - 1}.$$

From this identity it is clear that

$$-\sum_{j=1}^{\infty} \phi^{-j} z^{-j} = -\sum_{j=1}^{\infty} \left(\frac{1}{\phi z}\right)^j = \frac{1}{-(\phi z - 1)} = \frac{1}{1 - \phi z}$$

and we are done. □

Problem 2. Prove that if $|\phi| > 1$, then the unique stationary solution is given by (2.3.4).

Solution. Let $\phi(z) = 1 - \phi z$ and $\theta(z) = 1 + \theta z$ and suppose that $|\phi| > 1$. Note that the ARMA(1, 1) process is defined by the equation $\phi(B)X_t = \theta(B)Z_t$ where B is the back-shift operator. Using our definition of $\phi(z)$, we can see that the power series expansion of $1/\phi(z)$ is given by

$$\frac{1}{\phi(z)} = \frac{1}{1 - \phi z} = -\sum_{j=1}^{\infty} \phi^{-j} z^{-j}$$

which converges since $|1/\phi|< 1$. From this it is clear that $X_t = (1/\phi(B))\theta(B)Z_t$ and, evaluating $(1/\phi(B))\theta(B)$, we see that

$$\begin{aligned}
\frac{1}{\phi(B)}\theta(B) &= -(\phi^{-1}B^{-1} + \phi^{-2}B^{-2} + \phi^{-3}B^{-3} + \dots)(1 + \theta B) \\
&= (\phi^{-1}B^{-1} + \phi^{-2}B^{-2} + \phi^{-3}B^{-3} + \dots)(-1 - \theta B) \\
&= -\phi^{-1}B^{-1} - \phi^{-2}B^{-1} - \phi^{-3}B^{-2} + \dots + -\theta\phi^{-1} - \theta\phi^{-2}B^{-1} - \theta\phi^{-3}B^{-2} + \dots \\
&= -\theta\phi^{-1} + (-\theta\phi^{-2} - \phi^{-1})B^{-1} + (-\theta\phi^{-3} - \phi^{-2})B^{-2} + \dots \\
&= -\theta\phi^{-1} - (\theta + \phi)\sum_{j=1}^{\infty}\phi^{-j-1}B^{-j}.
\end{aligned}$$

Using this derivation we notice that

$$\begin{aligned}
X_t &= \frac{1}{\phi(B)}\theta(B)Z_t \\
&= -\theta\phi^{-1}Z_t - (\theta + \phi)\sum_{j=1}^{\infty}\phi^{-j-1}B^{-j}Z_t \\
&= -\theta\phi^{-1}Z_t - (\theta + \phi)\sum_{j=1}^{\infty}\phi^{-j-1}Z_{t+j} \\
&= -\theta\phi^{-1}Z_t - (\theta + \phi)\sum_{j=-\infty}^{-1}\phi^{j-1}Z_{t-j}. \qquad (1)
\end{aligned}$$

It is then clear that X_t is of the form $\sum_{j=-\infty}^{\infty}\psi_j Z_{t-j}$ where $\psi_0 = -\theta\phi^{-1}$ and $\psi_j = -(\theta + \phi)\phi^{j-1}$for $j \leq -1$. If $|\phi|> 1$, then $\sum_{j=-\infty}^{\infty}|\psi_j|= \sum_{j=-\infty}^{-1}|\psi_j|< \infty$. This fact combined with the fact that $\{Z_t\}$ is stationary shows that $\{X_t\}$ must be stationary by proposition 2.2.1. Therefore (1) is the unique stationary solution to the ARMA(1, 1) equation when $|\phi|> 1$. $\square$

Problem 3. Prove that if $|\theta|< 1$, then the ARMA(1, 1) process is invertible.

Solution. Note that the ARMA(1, 1) process is invertible if Z_t is expressed in terms of X_s for $s \leq t$. Let $\phi(z) = 1 - \phi z$ and $\theta(z) = 1 + \theta z$ and suppose that $|\theta|< 1$. Then the ARMA(1, 1) equation can be written as $\phi(B)X_t = \theta(B)Z_t$ where B is the back-shift operator. Note that the power series expansion of $1/\theta(z)$ is given by

$$\frac{1}{\theta(z)} = \frac{1}{1 + \theta z} = \sum_{j=0}^{\infty}(-\theta)^j z^j$$

which converges since $|\theta|< 1$. From this it is clear that $Z_t = (1/\theta(B))\phi(B)X_t$ and, evaluating

$(1/\theta(B))\phi(B)$, we see that

$$\begin{aligned}
\frac{1}{\theta(B)}\phi(B) &= (1-\theta B+(-\theta)^2B^2+\dots)(1-\phi B)\\
&= 1+(-\theta)B+(-\theta)^2B^2+\dots+-\phi B+(-1)^2\phi\theta B^2+(-1)^3\phi\theta^2B^3+\dots\\
&= 1+-(\theta+\phi)B+(-1)^2(\theta^2+\phi\theta)B^2+(-1)^3(\theta^3+\phi\theta^2)B^3+\dots\\
&= 1-(\theta+\phi)\sum_{j=1}^{\infty}(-\theta)^{j-1}B^j.
\end{aligned}$$

Using this derivation we notice that

$$\begin{aligned}
Z_t &= \frac{1}{\theta(B)}\phi(B)X_t\\
&= X_t-(\theta+\phi)\sum_{j=1}^{\infty}(-\theta)^{j-1}B^jX_t\\
&= X_t-(\theta+\phi)\sum_{j=1}^{\infty}(-\theta)^{j-1}X_{t-j}.
\end{aligned}$$

But this shows that Z_t is expressed in terms of X_s where $s \le t$, showing that when $|\theta|< 1$, the ARMA(1, 1) process is invertible. □

Problem 4. Prove that if $|\theta|> 1$, then the ARMA(1, 1) process is non-invertible.

Solution. Note that the ARMA(1, 1) process is non-invertible if Z_t is expressed in terms of X_s for $s \ge t$. Let $\phi(z) = 1-\phi z$ and $\theta(z) = 1+\theta z$ and suppose that $|\theta|> 1$. Then the ARMA(1, 1) equation can be written as $\phi(B)X_t = \theta(B)Z_t$ where B is the back-shift operator. Note that the power series expansion of $1/\theta(z)$ is given by

$$\frac{1}{\theta(z)} = \frac{1}{1+\theta z} = -\sum_{j=1}^{\infty}(-\theta)^{-j}z^{-j}.$$

Since $|\theta|> 1$, we know that $|1/\theta|< 1$ and this series converges. From this it is clear that $Z_t = (1/\theta(B))\phi(B)X_t$ and, evaluating $(1/\theta(B))\phi(B)$, we see that

$$\begin{aligned}
\frac{1}{\theta(B)}\phi(B) &= -((-\theta)^{-1}B^{-1}+(-\theta)^{-2}B^{-2}+(-\theta)^{-3}B^{-3}+\dots)(1-\phi B)\\
&= ((-\theta)^{-1}B^{-1}+(-\theta)^{-2}B^{-2}+(-\theta)^{-3}B^{-3}+\dots)(\phi B-1)\\
&= -\phi\theta^{-1}+(-1)^2\phi\theta^{-2}B^{-1}+(-1)^3\phi\theta^{-3}B^{-2}+\dots\\
&\quad +(-1)^2\theta^{-1}B^{-1}+(-1)^3\theta^{-2}B^{-2}+(-1)^4\theta^{-3}B^{-3}+\dots\\
&= -\phi\theta^{-1}+(-1)^2(\phi\theta^{-2}+\theta^{-1})B^{-1}+(-1)^3(\phi\theta^{-3}+\theta^{-2})B^{-2}+\dots\\
&= -\phi\theta^{-1}+(\phi+\theta)\sum_{j=1}^{\infty}(-\theta)^{-j-1}B^{-j}
\end{aligned}$$

Using this derivation we notice that

$$
\begin{aligned}
Z_t &= \frac{1}{\theta(B)}\phi(B)X_t \\
&= -\phi\theta^{-1}X_t + (\phi+\theta)\sum_{j=1}^{\infty}(-\theta)^{-j-1}B^{-j}X_t \\
&= -\phi\theta^{-1}X_t + (\phi+\theta)\sum_{j=1}^{\infty}(-\theta)^{-j-1}X_{t+j}.
\end{aligned}
$$

But this shows that Z_t is expressed in terms of X_s where $s \geq t$, showing that when $|\theta| > 1$, the ARMA(1, 1) process is non-invertible. $\square$