

Homework Assignment 9

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Problem 8.1. Find the Mellin transform of each of the following functions:

- a. $f(x) = H(a - x)$, $a > 0$,
- b. $f(x) = x^m e^{-nx}$, $m, n > 0$,
- c. $f(x) = \frac{1}{x^2 + 1}$.

Solution. The Mellin transform of the function $f(x)$ is defined to be

$$\mathcal{M}\{f(x)\} = \tilde{f}(p) = \int_0^\infty x^{p-1} f(x) dx.$$

- a. Recall that the Heaviside function H is defined as

$$H(a - x) = \begin{cases} 1 & \text{if } x < a \\ 0 & \text{if } x > a \end{cases}.$$

Therefore, from the definition of the Mellin transform, we have that for $f(x) = H(a - x)$ with $a > 0$,

$$\begin{aligned} \tilde{f}(p) = \mathcal{M}\{f(x)\} &= \int_0^\infty x^{p-1} H(a - x) dx \\ &= \int_0^a x^{p-1} dx \\ &= \frac{a^p}{p}. \end{aligned}$$

- b. Let $f(x) = x^m g(x)$ where $g(x) = e^{-nx}$ with $m, n > 0$ and let $\tilde{g}(p) = \mathcal{M}\{g(x)\}$.

By the shifting property of the Mellin transform, we have that

$$\tilde{f}(p) = \mathcal{M}\{f(x)\} = \mathcal{M}\{x^m g(x)\} = \tilde{g}(p + m).$$

From our table of Mellin transforms, we know that

$$\tilde{g}(p) = \mathcal{M}\{g(x)\} = \frac{\Gamma(p)}{n^p}$$

where $\Re\{p\} > 0$.

Therefore,

$$\tilde{f}(p) = \mathcal{M}\{f(x)\} = \tilde{g}(p+m) = \frac{\Gamma(p+m)}{n^{p+m}}$$

where $\Re\{p+m\} > 0$.

c. From our table of Mellin transforms, we see that

$$\mathcal{M}\left\{\frac{1}{(x^a+1)^s}\right\} = \frac{\Gamma(p/a)\Gamma(s-p/a)}{a\Gamma(s)}.$$

Therefore, for $f(x) = \frac{1}{x^2+1}$, identifying $a = 2$ and $s = 1$, we have that

$$\begin{aligned}\tilde{f}(p) = \mathcal{M}\{f(x)\} &= \mathcal{M}\left\{\frac{1}{x^2+1}\right\} = \frac{\Gamma(p/2)\Gamma(1-p/2)}{2\Gamma(1)} \\ &= \frac{\Gamma(p/2)\Gamma(1-p/2)}{2}\end{aligned}$$

where $\Re\{p/2\} > 0$ and $\Re\{1-p/2\} > 0$.

□

Problem 8.4. Show that

$$\mathcal{M} \left\{ \frac{1}{(1+ax)^n} \right\} = \frac{\Gamma(p)\Gamma(n-p)}{a^p\Gamma(n)}.$$

Solution. Let $f(x) = \frac{1}{(1+x)^n}$ where $n > 0$. Recall that the Beta function

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$$

satisfies the property that

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

From the definition of the Gamma function, we see that

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}} dt.$$

Thus, we have from the definition of the Mellin transform that

$$\begin{aligned} \mathcal{M} \{f(x)\} &= \int_0^\infty \frac{x^{p-1}}{(1+x)^n} dx \\ &= \int_0^\infty \frac{x^{p-1}}{(1+x)^{n-p+p}} dx \\ &= \frac{\Gamma(p)\Gamma(n-p)}{\Gamma(n)}. \end{aligned}$$

Therefore, by the scaling property of the Mellin transform,

$$\begin{aligned} \mathcal{M} \left\{ \frac{1}{(1+ax)^n} \right\} &= \mathcal{M} \{f(ax)\} = \frac{\mathcal{M} \{f(x)\}}{a^p} \\ &= \frac{\Gamma(p)\Gamma(n-p)}{a^p\Gamma(n)}. \end{aligned}$$

□

Problem 8.10.*Solution.*

Problem 8.12.*Solution.*

Problem 8.14.*Solution.*

Problem 8.21.*Solution.*