Homework Assignment 1

Matthew Tiger

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Problem 1. Solve the IVP:

$$y' = y^2 \cos(x), \quad y(0) = 2.$$

Solution. Note that this is a separable differential equation and after separating we see that

$$\begin{aligned}
\frac{dy}{y^2} &= \cos(x)dx \\
\int \frac{dy}{y^2} &= \int \cos(x)dx \\
-\frac{1}{y} &= \sin(x) + c_1
\end{aligned}$$

so that

$$y = -\frac{1}{\sin(x) + c_1}$$

is the general solution to the differential equation. Using the initial value $y(0) = 2$ and solving for c_1 we see that $c_1 = -1/2$ and the solution to the IVP is given by

$$y = -\frac{1}{\sin(x) - 1/2}.$$

□

Problem 2. Review solutions of first-order linear ODEs (p. 14) and solve the IVP:'

$$y' - xy = x^3, \quad y(1) = \frac{1}{2}.$$

Solution. $\square$

Problem 3. Let $Ly = y^{(4)} - 4y''' + 3y'' + 4y' - 4y$.

a. Find the general solutions of the homogeneous ODE $Ly = 0$.

b. Solve the IVP:

$$Ly = 0, \quad y(0) = 0, \quad y'(0) = -7, \quad y''(0) = 5, \quad y'''(0) = 9.$$

c. Solve the BVP:

$$Ly = 0, \quad y(0) = 1, \quad \lim_{x\to\infty} y(x) = 0.$$

Is this BVP well-posed?

d. Solve the BVP:

$$Ly = 0, \quad y(0) = 1, \quad \lim_{x\to-\infty} y(x) = 0.$$

Is this BVP well-posed?

Solution. □

Problem 4. Read §1.6 and then solve the ODEs:

$$xy' + 2y = x^2\sqrt{y}, \quad y' = \frac{4x^3 - 6xy^2 - 2xy}{x^2 + 6x^2y - 3y^2}, \quad y' + y^2 + (2x+1)y + 1 + x + x^2 = 0.$$

Solution. □

Problem 5. a. Use mathematical induction to prove Leibnitz's differentiation rule:

$$D^k(fg) = \sum_{j=0}^{k} \binom{k}{j} (D^j f)(D^{k-j} g).$$

Here $f = f(x)$ and $g = g(x)$ are k-times differentiable functions and $D^k = \frac{d^k}{dx^k}$.

b. Consider the constant-coefficient ODE

$$D^n y + p_{n-1} D^{n-1} y + \cdots + p_1 Dy + p_0 y = 0, \tag{1}$$

where $p_0, p_1, \ldots, p_{n-1}$ are real numbers. Let r be a double root of the characteristic polynomial $P(z) = z^n + p_{n-1} z^{n-1} + \cdots + p_1 z + p_0$. Use Leibnitz's rule to show that the function xe^{rx} is a solution of (1).

c. Let r be a triple root of the characteristic polynomial $P(z)$ from part (b). Use Leibnitz's rule to show that the function $x^2 e^{rx}$ is then also a solution of (1).

d. Let r be a real number. Show that the functions e^{rx}, xe^{rx}, and $x^2 e^{rx}$ are linearly independent on $\mathbb{R}$.

Solution. □

Problem 6. Use the formula for the derivative of a determinant from the lectures, other properties of determinants, and the linear ODE (1.3.1) to verify identity (1.3.4) in the textbook.

Solution. □