MATH 635 Final Assessment

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Problem 1. Provide a rigorous proof of the case $x_0 = a$ in the Fundamental Lemma of the Calculus of Variations:

Theorem 1 (Fundamental Lemma of the Calculus of Variations)**.** *Suppose $M(x)$ is a continuous function defined on the interval $a \leq x \leq b$. Suppose further that for every continuous function $\zeta(x)$,*

$$\int_a^b M(x)\zeta(x)dx = 0.$$

Then

$$M(x) = 0 \text{ for all } x \in [a,b].$$

Solution. Suppose to the contrary that $M(x) \neq 0$ at the point $x_0 = a$. In that case either $M(a) > 0$ or $M(a) < 0$. Let us first assume that $M(a) > 0$. Due to the continuity of $M(x)$ there is some neighborhood of a where the function is positive, i.e. there is some $\delta > 0$ such that if $|x - a| < \delta$ then

$$|M(x) - M(a)| < \frac{M(a)}{2} \quad \text{for } x \in [a,b].$$

Thus, $0 < M(a)/2 < M(x)$ for $x \in [a, a+\delta)$. Choose the function $\zeta(x)$ to be the linear spline interpolating the points $(a, 3M(a)/2)$ and $(a+\delta, 0)$ with support on $[a, a+\delta)$, i.e.

$$\zeta(x) := \begin{cases} \frac{-3M(a)}{2\delta}(x - (a+\delta)) & \text{if } a \leq x < a+\delta \\ 0 & \text{if } a+\delta \leq x \leq b. \end{cases}$$

Clearly $\zeta(x)$ is continuous and positive on the interval $[a, a+\delta)$. Thus,

$$\int_a^b M(x)\zeta(x)dx = \int_a^{a+\delta} M(x)\zeta(x)dx > \frac{M(a)}{2}\int_a^{a+\delta} \zeta(x)dx > 0.$$

However, by our supposition

$$\int_a^b M(x)\zeta(x)dx = 0,$$

a contradiction. Therefore, if $M(a) > 0$, the function $M(x) \equiv 0$ on the interval $[a,b]$.

If $M(a) < 0$, then we can repeat the argument above replacing $M(x)$ with $-M(x)$. To demonstrate, let us investigate the case when $M(a) < 0$. Due to the continuity of $M(x)$ there is some neighborhood of a where $-M(x)$ is positive, i.e. there is some $\delta > 0$ such that if $|x - a| < \delta$ then

$$|-M(x) + M(a)| < \frac{-M(a)}{2} \quad \text{for } x \in [a,b].$$

Thus, $0 < -M(a)/2 < -M(x)$ for $x \in [a, a+\delta)$. Choose the function $\zeta(x)$ to be the linear spline interpolating the points $(a, -3M(a)/2)$ and $(a+\delta, 0)$ with support on $[a, a+\delta)$, i.e.

$$\zeta(x) := \begin{cases} \frac{3M(a)}{2\delta}(x - (a+\delta)) & \text{if } a \leq x < a+\delta \\ 0 & \text{if } a+\delta \leq x \leq b. \end{cases}$$

Clearly $\zeta(x)$ is continuous and positive on the interval $[a, a+\delta)$. Thus,

$$\int_a^b -M(x)\zeta(x)dx = \int_a^{a+\delta} -M(x)\zeta(x)dx > \frac{-M(a)}{2}\int_a^{a+\delta} \zeta(x)dx > 0.$$

However, by our supposition

$$\int_a^b M(x)\zeta(x)dx = 0,$$

a contradiction. Therefore, if $M(a) < 0$, the function $M(x) \equiv 0$ on the interval $[a, b]$ and we have proven both cases. $\square$