

# Homework Assignment 5

Matthew Tiger

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**Problem 3.23.** Show that:

$$\text{a. } \mathcal{L} \{t \cos(at)e^{-bt}\} = \frac{(s+b)^2 - a^2}{[(s+b)^2 + a^2]^2}.$$

*Solution.*     a. Let  $f(t) = t \cos(at)$  and suppose that  $\bar{f}(s) = \mathcal{L} \{f(t)\}$ .

As shown previously, we know that

$$\bar{f}(s) = \mathcal{L} \{f(t)\} = \mathcal{L} \{t \cos(at)\} = \frac{s^2 - a^2}{(s^2 + a^2)^2}.$$

Therefore, by Heaviside's First Shifting Theorem,

$$\mathcal{L} \{t \cos(at)e^{-bt}\} = \mathcal{L} \{f(t)e^{-bt}\} = \bar{f}(s+b) = \frac{(s+b)^2 - a^2}{[(s+b)^2 + a^2]^2},$$

and we are done.

□

**Problem 3.24.** Suppose that  $\mathcal{L}\{f(t)\} = \bar{f}(s)$  and  $\mathcal{L}\{g(x, t)\} = \bar{h}(s) \exp(-x\bar{h}(s))$ . Prove that:

a.  $\mathcal{L}\left\{\int_0^\infty g(x, t)f(x)dx\right\} = \bar{h}(s)\bar{f}(\bar{h}(s)).$

*Solution.* a. From the definition of the Laplace transform, we have that

$$\mathcal{L}\left\{\int_0^\infty g(x, t)f(x)dx\right\} = \int_0^\infty \left[\int_0^\infty g(x, t)f(x)dx\right] e^{-st}dt.$$

Interchanging the order of integration yields that

$$\begin{aligned}\mathcal{L}\left\{\int_0^\infty g(x, t)f(x)dx\right\} &= \int_0^\infty \left[\int_0^\infty g(x, t)f(x)dx\right] e^{-st}dt \\ &= \int_0^\infty f(x) \left[\int_0^\infty g(x, t)e^{-st}dt\right] dx \\ &= \int_0^\infty f(x)\mathcal{L}\{g(x, t)\} dx.\end{aligned}$$

From the relation  $\mathcal{L}\{g(x, t)\} = \bar{h}(s) \exp(-x\bar{h}(s))$ , we thus see that

$$\begin{aligned}\mathcal{L}\left\{\int_0^\infty g(x, t)f(x)dx\right\} &= \int_0^\infty f(x)\mathcal{L}\{g(x, t)\} dx \\ &= \int_0^\infty f(x)\bar{h}(s) \exp(-x\bar{h}(s))dx.\end{aligned}$$

Using the definition of the Laplace transform, we see that

$$\bar{f}(\bar{h}(s)) = \int_0^\infty f(t) \exp(-\bar{h}(s)t)dt.$$

Therefore,

$$\begin{aligned}\mathcal{L}\left\{\int_0^\infty g(x, t)f(x)dx\right\} &= \int_0^\infty f(x)\bar{h}(s) \exp(-x\bar{h}(s))dx \\ &= \bar{h}(s) \int_0^\infty f(x) \exp(-x\bar{h}(s))dx \\ &= \bar{h}(s)\bar{f}(\bar{h}(s)).\end{aligned}$$

and we are done. □

**Problem 3.27.** Use the Initial Value Theorem to find  $f(0)$  and  $f'(0)$  from the following functions:

a.  $\bar{f}(s) = \frac{s}{s^2 - 5s + 12},$

c.  $\bar{f}(s) = \frac{e^{-sa}}{s^2 + 3s + 5}, a > 0.$

*Solution.* The Initial Value Theorem states that if  $f(t)$  and its derivatives exist as  $t \rightarrow 0$ , then

$$\text{i. } \lim_{s \rightarrow \infty} s\bar{f}(s) = f(0) \quad (1a)$$

$$\text{ii. } \lim_{s \rightarrow \infty} [s^2\bar{f}(s) - sf(0)] = f'(0). \quad (1b)$$

a. If  $\bar{f}(s) = \frac{s}{s^2 - 5s + 12}$ , then (1a) of the Initial Value Theorem shows that

$$f(0) = \lim_{s \rightarrow \infty} s\bar{f}(s) = \lim_{s \rightarrow \infty} \frac{s^2}{s^2 - 5s + 12} = 1.$$

This implies from (1b) of the Initial Value Theorem that

$$\begin{aligned} f'(0) &= \lim_{s \rightarrow \infty} [s^2\bar{f}(s) - sf(0)] = \lim_{s \rightarrow \infty} \frac{s^3}{s^2 - 5s + 12} - s \\ &= \lim_{s \rightarrow \infty} \frac{s^3 - (s^3 - 5s^2 + 12s)}{s^2 - 5s + 12} \\ &= \lim_{s \rightarrow \infty} \frac{5s^2 - 12s}{s^2 - 5s + 12} \\ &= 5. \end{aligned}$$

c. Suppose that  $p(s)$  and  $q(s)$  are both polynomials in  $s$  and that  $a > 0$ . Then from L'Hospital's rule we have that

$$\lim_{s \rightarrow \infty} \frac{p(s)e^{-sa}}{q(s)} = \lim_{s \rightarrow \infty} \frac{p(s)}{e^{sa}q(s)} = 0. \quad (2)$$

If  $\bar{f}(s) = \frac{e^{-sa}}{s^2 + 3s + 5}$  where  $a > 0$ , then (1a) of the Initial Value Theorem in combination with (2) shows that

$$f(0) = \lim_{s \rightarrow \infty} s\bar{f}(s) = \lim_{s \rightarrow \infty} \frac{se^{-sa}}{s^2 + 3s + 5} = 0.$$

Using this result, we have from (1b) of the Initial Value Theorem in combination with (2) that

$$f'(0) = \lim_{s \rightarrow \infty} [s^2\bar{f}(s) - sf(0)] = \lim_{s \rightarrow \infty} \frac{s^2e^{-sa}}{s^2 + 3s + 5} = 0.$$

□

**Problem 3.28.** Use the Final Value Theorem to find  $\lim_{t \rightarrow \infty} f(t)$  if it exists from the following functions:

a.  $\bar{f}(s) = \frac{1}{s(s^2 + as + b)},$

d.  $\bar{f}(s) = \frac{3}{(s^2 + 4)^2}.$

*Solution.* The Final Value Theorem states that if  $\bar{f}(s) = \frac{\bar{p}(s)}{\bar{q}(s)}$  where  $\bar{p}(s)$  and  $\bar{q}(s)$  are polynomials in  $s$  and the degree of  $\bar{p}(s)$  is less than that of  $\bar{q}(s)$ , and if all roots of  $\bar{q}(s)$  have negative real parts with the possible exception of the root  $s = 0$ , then

$$\lim_{s \rightarrow 0} s\bar{f}(s) = \lim_{t \rightarrow \infty} f(t), \quad (3)$$

if the limit exists.

- a. Suppose that  $\bar{f}(s) = \frac{1}{s(s^2 + as + b)} = \frac{\bar{p}(s)}{\bar{q}(s)}$ . Note that the roots of  $\bar{q}(s)$  are at  $s = 0$  and  $s = \frac{1}{2}(-a \pm \sqrt{a^2 - 4b})$ .

If  $a \leq 0$ , then the assumptions of the Final Value Theorem are not satisfied and thus cannot be applied. However, if  $a > 0$ , then the assumptions are satisfied and from (3) we see that

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} s\bar{f}(s) = \frac{s}{s(s^2 + as + b)} = \frac{1}{b}.$$

- d. Suppose that  $\bar{f}(s) = \frac{3}{(s^2 + 4)^2} = \frac{\bar{p}(s)}{\bar{q}(s)}$ . Note that the roots of  $\bar{q}(s)$  are  $s = \pm 2i$  each with multiplicity 2. Since the real parts of these roots are not negative, the Final Value Theorem cannot be applied.

□

**Problem 3.29.***Solution.*

**Problem 3.32.***Solution.*

**Problem 3.34.***Solution.*

**Problem 4.1.***Solution.*