Homework Assignment 3

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Problem 2.20. Apply the Fourier cosine transform to find the solution $u(x, y)$ of the problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0, && 0 < x < \infty, \quad 0 < y < \infty \\ u(x, 0) &= H(a - x), && x < a \\ u_x(0, y) &= 0, && 0 < x, y < \infty. \end{aligned}$$

Solution. Consider the function $u(x, y)$. The Fourier cosine transform of u with respect to x is defined as

$$\mathscr{F}_c \{u(x, y)\} = U_c(k, y) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x, y) \cos(kx) dx.$$

From this definition we see using the Leibniz integral rule that

$$\begin{aligned} \mathscr{F}_c \left\{ \frac{\partial^n u(x, y)}{\partial y^n} \right\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^n u(x, y)}{\partial y^n} \cos(kx) dx \\ &= \frac{d^n}{dy^n} \left[\sqrt{\frac{2}{\pi}} \int_0^\infty u(x, y) \cos(kx) dx \right] \\ &= \frac{d^n}{dy^n} \left[\mathscr{F}_c \{u(x, y)\} \right]. \end{aligned}$$

The transforms of the partials of u with respect to x are not as easy to characterize. Nevertheless, we see from the properties of the Fourier cosine transform that

$$\mathscr{F}_c \left\{ \frac{\partial u(x, y)}{\partial x} \right\} = k \mathscr{F}_s \{u(x, y)\} - \sqrt{\frac{2}{\pi}} u(0, y)$$

and

$$\mathscr{F}_c \left\{ \frac{\partial^2 u(x, y)}{\partial x^2} \right\} = -k^2 \mathscr{F}_c \{u(x, y)\} - \sqrt{\frac{2}{\pi}} u_x(0, y)$$

Let $U_c(x, y) = \mathscr{F}_c \{u(x, y)\}$. Then, applying the Fourier cosine transform to the first differential equation shows that

$$\mathscr{F}_c \{u_{xx} + u_{yy}\} = -k^2 U_c(k, y) - \sqrt{\frac{2}{\pi}} u_x(0, y) + \frac{d^2}{dy^2} [U_c(k, y)] = 0 = \mathscr{F}_c \{0\}.$$

From the third equation we see that $u_x(0, y) = 0$ for all $0 < x, y < \infty$ which implies that the above equation reduces to

$$\frac{d^2}{dy^2}\left[U_c(k, y)\right] - k^2 U_c(k, y) = 0.$$

This is a second-order linear homogeneous differential equation, the solution to which is readily seen to be

$$U_c(k, y) = c_1 e^{-ky} + c_2 e^{ky}.$$

However, since $U_c(k, y) \to 0$ as $k \to \infty$, we must have that $c_2 = 0$. Thus, the solution to the previous differential equation is given by

$$U_c(k, y) = c_1 e^{-ky}. \tag{1}$$

We now apply the Fourier cosine transform to the second differential equation yielding

$$\mathscr{F}_c\left\{u(x, 0)\right\} = U_c(k, 0) = \mathscr{F}_c\left\{H(a - x)\right\}.$$

Using the form (1) of the solution to the transformed differential equation and a table of Fourier cosine transforms we see that

$$U_c(k, 0) = c_1 = \mathscr{F}_c\left\{H(a - x)\right\} = \sqrt{\frac{2}{\pi}}\left(\frac{\sin ak}{k}\right).$$

Thus, the solution to the transformed differential equation with the boundary conditions listed above is given by

$$U_c(k, y) = \mathscr{F}_c\left\{H(a - x)\right\} e^{-ky} = \sqrt{\frac{2}{\pi}}\left(\frac{\sin ak}{k}\right) e^{-ky}.$$

Therefore, taking the inverse Fourier cosine transform to both sides shows that the solution to the original differential equation is given by

$$\begin{aligned} u(x, y) = \mathscr{F}_c^{-1}\left\{U_c(k, y)\right\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{\sin ak}{k}\right) e^{-ky} \cos kx dk \\ &= \frac{2}{\pi} \int_0^\infty \left(\frac{\sin ak}{k}\right) e^{-ky} \cos kx dk. \end{aligned}$$

$\square$

Problem 2.23. Use the Parseval formula to evaluate the following integrals with $a > 0$ and $b > 0$:

a. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2}$,

c. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2 ax}{x^2} dx$.

Solution. Suppose that $f \in L^2(\mathbb{R})$ and that $F(k) = \mathscr{F}\{f(x)\}$. Then Parseval's relation states that

$$\int_{-\infty}^{\infty} f(x)\overline{f(x)}dx = \int_{-\infty}^{\infty} F(k)\overline{F(k)}dk.$$

a. Let $f(x) = \dfrac{1}{x^2+a^2}$. Then from our table of Fourier transforms we see that

$$\mathscr{F}\{f(x)\} = F(k) = \sqrt{\frac{\pi}{2}}\left(\frac{e^{-a|k|}}{a}\right).$$

From Parseval's relation, we see that

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = \int_{-\infty}^{\infty} f(x)\overline{f(x)}dx = \int_{-\infty}^{\infty} F(k)\overline{F(k)}dk = \frac{\pi}{2a^2}\int_{-\infty}^{\infty} e^{-2a|k|}dk.$$

Therefore, we have that

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} &= \frac{\pi}{2a^2}\int_{-\infty}^{\infty} e^{-2a|k|}dk \\
&= \frac{\pi}{a^2}\int_{0}^{\infty} e^{-2ak}dk \\
&= \frac{\pi}{a^2}\left[-\frac{e^{-2ak}}{2a}\bigg|_0^{\infty}\right] \\
&= \frac{\pi}{2a^3}.
\end{aligned}$$

c. Let $f(x) = \dfrac{\sin ax}{x}$. Then from our table of Fourier transforms we see that

$$\mathscr{F}\{f(x)\} = F(k) = \sqrt{\frac{\pi}{2}}H(a-|k|).$$

From Parseval's relation, we see that

$$\int_{-\infty}^{\infty} \frac{\sin^2 ax}{x}dx = \int_{-\infty}^{\infty} f(x)\overline{f(x)}dx = \int_{-\infty}^{\infty} F(k)\overline{F(k)}dk = \frac{\pi}{2}\int_{-\infty}^{\infty} H(a-|k|)^2dk.$$

Therefore, we have using the definition of the Heaviside function that

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\sin^2 ax}{x} dx &= \frac{\pi}{2} \int_{-\infty}^{\infty} H(a - |k|)^2 dk \\ &= \frac{\pi}{2} \int_{-a}^{a} dk \\ &= a\pi.\end{aligned}$$

□

Problem 2.47. Apply the Fourier transform to solve the equation

$$u_{xxxx} + u_{yy} = 0, \qquad -\infty < x < \infty,\ 0 \leq y$$

satisfying the conditions

$$u(x,0) = f(x), \quad u_y(x,0) = 0, \qquad \text{for } -\infty < x < \infty$$

where $u(x,y)$ and its partial derivatives vanish as $|x| \to \infty$.

Solution. We begin by applying the Fourier transform to the system of differential equations. Using the properties of the Fourier transform with respect to x, we see that

$$\begin{aligned}
&\frac{d^2}{dy^2}\left[U(k,y)\right] + k^4 U(k,y) = 0 \\
&U(k,0) = F(k) \\
&\frac{d}{dy}\left[U(k,y)\right]\bigg|_{y=0} = 0, \qquad -\infty < k < \infty,\ 0 \leq y.
\end{aligned}$$

The first equation of the transformed system is a second-order linear homogeneous ordinary differential equation. Its solution is given by

$$U(k,y) = c_1 \cos(k^2 y) + c_2 \sin(k^2 y).$$

From this general solution, we see from the second equation that

$$U(k,0) = c_1 = F(k).$$

Similarly, using the general solution, we see from the third equation that

$$\frac{d}{dy}\left[U(k,y)\right] = -c_1 k^2 \sin(k^2 y) + c_2 k^2 \cos(k^2 y)$$

which implies that

$$\frac{d}{dy}\left[U(k,y)\right]\bigg|_{y=0} = c_2 k^2 = 0.$$

Since this must hold for all k, we must have have that $c_2 = 0$.

Thus, the solution to the transformed system is given by

$$U(k,y) = F(k)\cos(k^2 y).$$

Therefore, the solution to the original differential equation is

$$u(x,y) = \mathscr{F}^{-1}\left\{U(k,y)\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)\cos(k^2 y) e^{ikx} dk.$$

□

Problem 2.48. The transverse vibration of a thin membrane of great extent satisfies the wave equation

$$c^2(u_{xx} + u_{yy}) = u_{tt}, \qquad -\infty < x, y < \infty,\ 0 < t,$$

with the initial and boundary conditions

$$\begin{aligned} &u(x, y, t) \to 0 \quad \text{as} \quad |x| \to \infty,\ |y| \to \infty \qquad \text{for all } t \geq 0, \\ &u(x, y, 0) = f(x, y), \quad u_t(x, y, 0) = 0 \qquad \text{for all } x, y. \end{aligned}$$

Solution. □

Problem 2.54. Solve the following equations

a. $u_{xxxx} - u_{yy} + 2u = f(x, y)$,

b. $u_{xx} + 2u_{yy} + 3u_x - 4u = f(x, y)$,

where $f(x, y)$ is a given function.

Solution. □