Homework Assignment 5

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Problem 1. a. Explain in a specific example why, when A and $\boldsymbol{b}$ have integer components, a general integer programming problem

$$\begin{array}{lll} \text{(GILP)} & \text{Minimize (Maximize)} & f(\boldsymbol{x}) = \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} \\ & \text{subject to} & A\boldsymbol{x} \quad \leq (\geq, =)\boldsymbol{b} \\ & & \boldsymbol{x} \geq (\leq)\boldsymbol{0}, \boldsymbol{x} \in \mathbb{Z}^n \end{array}$$

can be reduced (or is equivalent) to a standard integer programming problem

$$\begin{array}{lll} \text{(ILP)} & \text{Minimize (Maximize)} & f(\boldsymbol{X}) = \boldsymbol{C}^{\mathsf{T}}\boldsymbol{X} \\ & \text{subject to} & \mathscr{A}\boldsymbol{X} \quad = \boldsymbol{B} \\ & & \boldsymbol{X} \geq \boldsymbol{0}, \boldsymbol{X} \in \mathbb{Z}^n \end{array}$$

by adding variables or any of the transformations discussed in class that change $\boldsymbol{x}$ into $\boldsymbol{X}$.

More precisely, explain why (GILP) has a solution $\boldsymbol{x} \in \mathbb{Z}^n$ if and only if (ILP) has a solution $\boldsymbol{X} \in \mathbb{Z}^n$.

b. How do we solve (GILP) when A or $\boldsymbol{b}$ do not have integer components?

Solution. a. In general, the transformations needed to change the constraints in the (GILP) to the standard form (ILP) all involve operations on integers so that the requirement that the constraints all involve integer components is satisfied. As such, if either the (GILP) or (ILP) has a solution, then the other is guaranteed to have a feasible solution since the additional slack or excess variables all involve integer operations. Thus, one of the feasible solutions will be the optimum and the other problem is solved.

To illustrate this idea, take the (GILP)

$$\begin{array}{lll} \text{(GILP)} & \text{Minimize} & f(\boldsymbol{x}) = \begin{bmatrix} -4 & -3 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix}^{\mathsf{T}} \\ & \text{subject to} & \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 5 \\ 3 \end{bmatrix} \\ & & \boldsymbol{x} \geq \boldsymbol{0}, \boldsymbol{x} \in \mathbb{Z}^n \end{array}$$

The solution to (GILP) is $\boldsymbol{x}^{\mathsf{T}} = \begin{bmatrix} 4 & 3 \end{bmatrix}$.

In order to change the problem (GILP) to standard form (ILP) we need to add slack variables to the matrix A. Doing so results in a matrix $\mathscr{A}$ that still has all integer components and $\boldsymbol{b}$ remains unchanged so that it too still has all integer components. To see this, in standard form the problem becomes

$$\begin{aligned}
\text{(ILP)} \quad \text{Minimize} \quad & f(\boldsymbol{x}) = \begin{bmatrix}-4 & -3 & 0 & 0\end{bmatrix} \begin{bmatrix}x_1 & x_2 & x_3 & x_4\end{bmatrix}^\mathsf{T} \\
\text{subject to} \quad & \begin{bmatrix}2 & -1 & 1 & 0 \\ -1 & 2 & 0 & 1\end{bmatrix} \begin{bmatrix}x_1 \\ x_2 \\ x_3 \\ x_4\end{bmatrix} = \begin{bmatrix}5 \\ 3 \\ 0 \\ 0\end{bmatrix}. \\
& \boldsymbol{X} \geq \boldsymbol{0}, \boldsymbol{X} \in \mathbb{Z}^n
\end{aligned}$$

the solution of which is $\boldsymbol{x}^\mathsf{T} = \begin{bmatrix}4 & 3 & 0 & 0\end{bmatrix}$.

b. When the matrix A or the constraint column $\boldsymbol{b}$ do not have integer components, we can solve (GILP) by converting the problem to standard form and performing the simplex method in conjunction with the Gomory Cutting-Plan Method to obtain an integer solution.

□

Problem 2. Solve the shipping problem studied in MATH 111 with the replaced constraints over integers using the Gomory Cutting-Plane Method.

More precisely, solve:

$$\begin{array}{ll} \text{Maximize} & 9x_1 + 13x_2 \\ \text{subject to} & 4x_1 + 3x_2 \quad \leq 300 \\ & x_1 + 2x_2 \quad \leq 625/6 \\ & -2x_1 + x_2 \quad \leq 0 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{Z} \end{array} .$$

Solution. □

Problem 3. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $\Omega \subset \mathbb{R}^n$ be an open subset. Explain the meaning of $f \in C^1(\Omega)$. More precisely, give all the definitions needed and present some examples and results concerning $C^1(\Omega)$ functions.

Solution. $\square$