

Homework Assignment 9

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Problem 8.2.2. If $Sf(x)$ is the Schwarzian derivative of $f(x)$ with $f \in C^3$ and $F(x) = \frac{f''(x)}{f'(x)}$, show that $Sf(x) = F'(x) - (F(x))^2/2$.

Solution. Recall that the Schwarzian derivative of f is given by

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left[\frac{f''(x)}{f'(x)} \right]^2.$$

We readily see from the definition of $F(x)$ that

$$\begin{aligned} F'(x) - \frac{1}{2} [F(x)]^2 &= \frac{f'(x)f'''(x) - f''(x)^2}{f'(x)^2} - \frac{1}{2} \left[\frac{f''(x)}{f'(x)} \right]^2 \\ &= \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left[\frac{f''(x)}{f'(x)} \right]^2 \\ &= Sf(x) \end{aligned}$$

and we are done. □

Problem 8.2.5. i. Show that if p is a polynomial of degree n having n distinct fixed points, and negative Schwarzian derivative, then not all of the fixed points can be attracting.

ii. On the other hand, show that the logistic maps $L_\mu : \mathbb{R} \rightarrow \mathbb{R}$ for $\mu > 2 + \sqrt{5}$ have negative Schwarzian derivative but have no attracting periodic orbits.

Solution. i. Suppose to the contrary that p is a polynomial of degree n with n distinct fixed points and negative Schwarzian derivative but all of its fixed points are attracting. Let x_1, \dots, x_n denote these attracting fixed points.

Since p is a polynomial, it is continuous, which implies that for each attracting fixed point x_k , its immediate basin of attraction W_k is an open interval. Note that these fixed points are distinct and attracting so that the immediate basins of attraction of two fixed points x_j and x_k with $j \neq k$ are mutually exclusive, i.e. $W_j \cap W_k = \emptyset$ for any $j \neq k$.

Since $p \in C^3$ with negative Schwarzian derivative, we have by Singer's theorem that for every fixed point x_k , either W_k is an unbounded interval, or the orbit of some critical point of p is attracted to the orbit of x_k under f .

ii.

□

Problem 8.2.6.*Solution.*

Problem 8.2.10.*Solution.*

Problem 10.3.4. The Sierpinski carpet is a 2-dimensional version of the Cantor set and the Menger sponge. Start with the unit square $[0, 1] \times [0, 1]$ partitioned into nine equal squares. Remove the “open middle third” square $(1/3, 2/3) \times (1/3, 2/3)$. From each of the remaining eight squares of side length $1/3$, remove the open middle third squares and continue indefinitely.

- i. Show that the resulting area removed is equal to one square unit.
- ii. Show that the box counting dimension of the Sierpinski carpet is $\log 8 / \log 3$.
- iii. Show that the Sierpinski carpet has no interior, i.e. contains no open balls in \mathbb{R}^2 .

Solution. i. On the first iteration, one square of dimension $1/3 \times 1/3$ is removed. Each subsequent iteration removes 8 times as many squares that were removed in the previous iteration where the length of each square to be removed is $1/3$ of the length of the previous squares removed. Therefore, A_n , the area removed at each iteration n for $n = 1, 2, \dots$, is $A_n = 8^{n-1}/3^{2n}$ and the total area removed, A , is given by

$$A = \sum_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} \frac{8^{n-1}}{3^{2n}} = \frac{1}{8} \sum_{n=1}^{\infty} \left(\frac{8}{9}\right)^n = 1.$$

- ii. Let K is a non-empty subset of \mathbb{R}^n and let $N_\delta(K)$ be the minimum number of boxes of equal length $\delta > 0$ needed to cover the set K . Then the box-counting dimension of K is given by

$$\dim(K) = \lim_{\delta \rightarrow 0^+} \frac{\log N_\delta(K)}{\log 1/\delta}.$$

Now let $K \subseteq \mathbb{R}^2$ be the Sierpinski carpet. We will construct a sequence For the first iteration, we see that we would need 8 squares with side-length $1/3$ in order to completely cover the Sierpinski carpet. The next iteration would require 64 squares of side-length $1/9$ in order to completely cover the Sierpinski carpet. In general, for the n -th iteration, we would require $N_{\delta_n}(K) = 8^n$ squares of length $\delta_n = 3^{-n}$. Therefore,

$$\dim(K) = \lim_{\delta \rightarrow 0^+} \frac{\log N_\delta(K)}{\log 1/\delta} = \lim_{n \rightarrow \infty} \frac{\log N_{\delta_n}(K)}{\log 1/\delta_n} = \lim_{n \rightarrow \infty} \frac{\log 8^n}{\log 3^n} = \frac{\log 8}{\log 3}.$$

- iii. Suppose to the contrary that the interior of K , the Sierpinski carpet, is non-empty, i.e. there is some point $x \in K$ such that a square centered at x is contained completely in K . Since this square is completely within the Sierpinski carpet, it contains a square defined by the coordinates $\left(\frac{j}{3^n}, \frac{k}{3^n}\right), \left(\frac{j}{3^n}, \frac{k+1}{3^n}\right), \left(\frac{j+1}{3^n}, \frac{k}{3^n}\right), \left(\frac{j+1}{3^n}, \frac{k+1}{3^n}\right)$ for some $0 < j, k < 3^n$ that are not multiples of 3. However, after the n -th iteration this square will be removed from K , contradicting the fact that the square centered at x is completely inside K . Therefore, the interior of the Sierpinski carpet is empty. □

Problem 10.3.6.*Solution.*

Problem 10.3.7.*Solution.*