Homework Assignment 8

Matthew Tiger

October 22, 2015

Problem 2.15. Suppose that $\{X_t\}$ is a stationary process satisfying the equations

$$X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + Z_t,$$

where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$ and Z_t is uncorrelated with X_s for each $s < t$. Show that the best linear predictor $P_n X_{n+1}$ of X_{n+1} in terms of $1, X_1, \ldots, X_n$, assuming $n > p$ is

$$P_n X_{n+1} = \phi_1 X_n + \cdots + \phi_p X_{n+1-p}.$$

What is the mean squared error of $P_n X_{n+1}$?

Solution. Note that $\{X_t\}$ is an AR(p) process and that $P_n X_{n+1} = \phi_1 X_n + \cdots + \phi_p X_{n+1-p}$ is the best linear predictor in terms of $1, X_1, \ldots, X_n$ if $\boldsymbol{a_n} = (\phi_1, \phi_2, \ldots, \phi_p, 0, \ldots, 0)^\intercal$ is the solution to the Yule-Walker equations $\sum_{i=1}^n \gamma(h-i) a_i = \gamma(h)$ for $h = 1, \ldots, n$. Since $\{X_t\}$ is an AR(p) process, we know that $\gamma(h) = \sigma^2 \sum_{j=0}^\infty \psi_j \psi_{j+h}$ where $\psi_j = \sum_{k=1}^p \phi_k \psi_{j-k}$ and $\psi_0 = 1$ for $h > 0$.

Thus, if $n \geq h \geq p \geq 1$,

$$\begin{aligned}
\sum_{i=1}^n \gamma(h-i) a_i &= \sum_{i=1}^p \phi_i \gamma(h-i) \\
&= \sigma^2 \sum_{i=1}^p \phi_i \sum_{j=0}^\infty \psi_j \psi_{j+h-i} \\
&= \sigma^2 \sum_{j=0}^\infty \psi_j \sum_{i=1}^p \phi_i \psi_{j+h-i} \\
&= \sigma^2 \sum_{j=0}^\infty \psi_j \psi_{j+h} = \gamma(h)
\end{aligned}$$

since $\sum_{i=1}^p \phi_i \psi_{j+h-i} = \psi_{j+h}$ and the equation holds for $n \geq h \geq p$.

If $1 \leq h < p < n$,

$$\begin{aligned}
\sum_{i=1}^{n} \gamma(h-i) a_i &= \sum_{i=1}^{h} \phi_i \gamma(h-i) + \sum_{i=h+1}^{p} \phi_i \gamma(i-h) \\
&= \sigma^2 \sum_{i=1}^{h} \phi_i \sum_{j=0}^{\infty} \psi_j \psi_{j+h-i} + \sigma^2 \sum_{i=h+1}^{p} \phi_i \sum_{j=0}^{\infty} \psi_j \psi_{j+i-h} \\
&= \sigma^2 \sum_{j=0}^{\infty} \psi_j \sum_{i=1}^{h} \phi_i \psi_{j+h-i} + \sigma^2 \sum_{j=0}^{\infty} \psi_j \sum_{i=h+1}^{p} \phi_i \psi_{j+i-h} \\
&= \sigma^2 \sum_{j=0}^{\infty} \psi_j \left(\sum_{i=1}^{h} \phi_i \psi_{j+h-i} + \sum_{i=h+1}^{p} \phi_i \psi_{j+i-h} \right) \\
&= \sigma^2 \sum_{j=0}^{\infty} \psi_j \sum_{i=1}^{p} \phi_i \psi_{j+h-i} \\
&= \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h} = \gamma(h)
\end{aligned}$$

and the equation holds for $1 \leq h < p < n$ showing that $\boldsymbol{a_n}$ is indeed the best linear predictor of X_{n+1} in terms of $1, X_1, \ldots, X_n$.

The mean squared error of $P_n X_{n+1}$ is given by

$$\gamma(0) - \sum_{i=1}^{n} a_i \gamma(i) = \sigma^2 \left(\sum_{j=0}^{\infty} \psi_j^2 - \sum_{j=0}^{\infty} \psi_j \sum_{i=1}^{p} \phi_i \psi_{j+i} \right) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \left(\psi_j - \sum_{i=1}^{p} \phi_i \psi_{j+i} \right).$$

□