Homework Assignment 8

Matthew Tiger

May 1, 2016

Problem 1. Write at least two necessary conditions and at least two sufficient conditions for a function $f : \mathbb{R}^n \to \mathbb{R}$ to be concave.

Solution. By definition, a function $f : \mathbb{R}^n \to \mathbb{R}$ is concave over the convex set $\Omega \subset \mathbb{R}^n$ if $-f$ is convex over Ω. This definition will allow us to obtain results for concave functions by replacing f with $-f$ in previously obtained results concerning convex functions.

Using this definition and Theorem 22.2, we see that the condition that if for all $\alpha \in (0, 1)$ and for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, we have that

$$f(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \geq \alpha f(\boldsymbol{x}) + (1-\alpha)f(\boldsymbol{y})$$

is a necessary and sufficient condition for f to be concave on the convex set Ω.

Further, if the function f is $\mathcal{C}^1$-smooth, we see from the above definition and Theorem 22.4 that the condition that if for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, we have that

$$f(\boldsymbol{x}) \leq f(\boldsymbol{y}) + Df(\boldsymbol{x})(\boldsymbol{x} - \boldsymbol{y})$$

is a necessary and sufficient condition for f to be concave on the open convex set Ω.

Going one last step further, if the function f is $\mathcal{C}^2$-smooth, we see from the above definition and Theorem 22.5 that the condition that if for all $\boldsymbol{x} \in \Omega$, the Hessian matrix $\boldsymbol{F}(\boldsymbol{x})$ of f at $\boldsymbol{x}$ is a negative semi-definite matrix is a necessary and sufficient condition for f to be concave on the open convex set Ω. □

Problem 2. Let $S \subset \mathbb{R}^n$ be a convex set and let $\boldsymbol{x}^* \in S$. Prove that a vector $\boldsymbol{d} \in \mathbb{R}^n$ is a feasible direction at $\boldsymbol{x}^*$ (relative to S) if and only there exists $t_0 > 0$ such that $\boldsymbol{x}^* + t_0\boldsymbol{d} \in S$ with $\boldsymbol{d} \neq \boldsymbol{0}$.

Solution. Suppose first that the vector $\boldsymbol{d} \in \mathbb{R}^n$ is a feasible direction at $\boldsymbol{x}^*$ (relative to S). By definition, the vector $\boldsymbol{d}$ is a feasible direction at $\boldsymbol{x}^* \in S$ if there exists $t_0 > 0$ such that $\boldsymbol{x}^* + t\boldsymbol{d} \in S$ for all $t \in [0, t_0]$ with $\boldsymbol{d} \neq \boldsymbol{0}$. Thus, choosing $t = t_0$, we have by the above definition that there exists $t_0 > 0$ such that $\boldsymbol{x}^* + t_0\boldsymbol{d} \in S$ with $\boldsymbol{d} \neq \boldsymbol{0}$, proving the first implication.

Now suppose that there exists $t_0 > 0$ such that $\boldsymbol{x}^* + t_0\boldsymbol{d} \in S$ with $\boldsymbol{d} \neq \boldsymbol{0}$. Since S is convex and $\boldsymbol{x}^* \in S$, any convex combination of $\boldsymbol{x}^*$ and $\boldsymbol{x}^* + t_0\boldsymbol{d}$ will also be in S, i.e. for all $\alpha \in [0, 1]$, we have that

$$\alpha\boldsymbol{x}^* + (1-\alpha)(\boldsymbol{x}^* + t_0\boldsymbol{d}) = \boldsymbol{x}^* + (1-\alpha)t_0\boldsymbol{d} \in S.$$

Since $t_0 > 0$, we have that the following two sets are equal:

$$\{(1-\alpha)t_0 \mid 0 \leq \alpha \leq 1\} = \{t \mid 0 \leq t \leq t_0\}.$$

Thus, if $\boldsymbol{x}^* + (1-\alpha)t_0\boldsymbol{d} \in S$ for all $\alpha \in [0, 1]$, then $\boldsymbol{x}^* + t\boldsymbol{d} \in S$ for all $t \in [0, t_0]$. Therefore, if $\boldsymbol{x}^* \in S$ with S a convex set and there exists $t_0 > 0$ such that $\boldsymbol{x}^* + t_0\boldsymbol{d} \in S$ with $\boldsymbol{d} \neq \boldsymbol{0}$, then $\boldsymbol{x}^* + t\boldsymbol{d} \in S$ for all $t \in [0, t_0]$, i.e. the vector $\boldsymbol{d}$ is a feasible direction at $\boldsymbol{x}^*$ (relative to S). $\square$

Problem 3. Recall that

$$\max\{\alpha, \beta\} := \begin{cases} \alpha & \text{if } \alpha \geq \beta \\ \beta & \text{if } \alpha < \beta \end{cases}.$$

Given two convex functions $f_1 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $f_2 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, prove that for $\boldsymbol{x} \in \mathbb{R}^n$, the function

$$f(\boldsymbol{x}) := \max\{f_1(\boldsymbol{x}), f_2(\boldsymbol{x})\}$$

is convex.

Solution. Note that it is clear that the set $\mathbb{R}^n$ is convex. Therefore, the function $f(\boldsymbol{x}) := \max\{f_1(\boldsymbol{x}), f_2(\boldsymbol{x})\}$ is convex if for all $\alpha \in (0,1)$ and for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ we have that

$$f(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \leq \alpha f(\boldsymbol{x}) + (1-\alpha) f(\boldsymbol{y}). \tag{1}$$

If either $f(\boldsymbol{x}) = +\infty$ or $f(\boldsymbol{y}) = +\infty$, then for all $\alpha \in (0,1)$ and for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ we see that $\alpha f(\boldsymbol{x}) + (1-\alpha) f(\boldsymbol{y}) = +\infty$ and inequality (1) holds showing the convexity of f in this case.

Now suppose that both $f(\boldsymbol{x})$ and $f(\boldsymbol{y})$ are finite. Without loss of generality, we may assume that at the point $\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}$, we have that the max of f_1 and f_2 at that point occurs at f_1, i.e.

$$\begin{aligned} f(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) &= \max\{f_1(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}), f_2(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y})\} \\ &= f_1(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}). \end{aligned}$$

Then, due to the convexity of f_1, we have that for all $\alpha \in (0,1)$ and for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$,

$$f_1(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \leq \alpha f_1(\boldsymbol{x}) + (1-\alpha) f_1(\boldsymbol{y}).$$

From the above definition of max, we readily see that

$$\begin{aligned} \alpha f_1(\boldsymbol{x}) + (1-\alpha) f_1(\boldsymbol{y}) &\leq \alpha \max\{f_1(\boldsymbol{x}), f_2(\boldsymbol{x})\} + (1-\alpha)\max\{f_1(\boldsymbol{y}), f_2(\boldsymbol{y})\} \\ &= \alpha f(\boldsymbol{x}) + (1-\alpha) f(\boldsymbol{y}). \end{aligned}$$

Therefore, combining, we have that for all $\alpha \in (0,1)$ and for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$,

$$f(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) = f_1(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \leq \alpha f(\boldsymbol{x}) + (1-\alpha) f(\boldsymbol{y})$$

showing that inequality (1) holds and that the function f is convex. $\square$

Problem 4. Consider the pair of linear programming problems in asymmetric duality:

$$\begin{array}{lll} (P_a) & \text{minimize} & f(\boldsymbol{x}) = \boldsymbol{c}^\mathsf{T}\boldsymbol{x} \\ & \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \\ & & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \qquad \begin{array}{lll} (D_a) & \text{maximize} & F(\boldsymbol{\lambda}) = \boldsymbol{\lambda}^\mathsf{T}\boldsymbol{b} \\ & \text{subject to} & \boldsymbol{\lambda}^\mathsf{T}A \leq \boldsymbol{c}^\mathsf{T} \end{array}$$

a. Prove that (D_a) is a convex programming problem.

b. Write the KKT conditions for (D_a).

c. Suppose that $\boldsymbol{x}^*$ is feasible for (P_a) and $\boldsymbol{\lambda}^*$ is feasible for (D_a). Use the KKT conditions to prove that if $(\boldsymbol{c}^\mathsf{T} - \boldsymbol{\lambda}^{*\mathsf{T}}A)\boldsymbol{x}^* = 0$, then $\boldsymbol{\lambda}^*$ is optimal for (D_a).

Solution. □