Homework Assignment 4

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Problem 1. Find the dual of the following linear programs:

a. Maximize $f(\boldsymbol{x}) = \boldsymbol{c}^\mathsf{T}\boldsymbol{x}$ subject to $A\boldsymbol{x} = \boldsymbol{b}$.

b. Maximize $2x_1 + 5x_2 + x_3$ subject to $\begin{cases} 2x_1 - x_2 + 7x_3 \leq 6 \\ x_1 + 3x_2 + 4x_3 \leq 9 \\ 3x_1 + 6x_2 + x_3 \leq 3 \\ x_1, x_2, x_3 \geq 0. \end{cases}$ via the symmetric form of duality.

Solution. a. Note that for this problem, the variable $\boldsymbol{x}$ is unconstrained in sign. After making the substitution $\boldsymbol{x} = \boldsymbol{x_1} - \boldsymbol{x_2}$ with $\boldsymbol{x_1}, \boldsymbol{x_2} \geq \boldsymbol{0}$, this problem in standard form is then stated as

$$\begin{array}{rl} \text{minimize} & -\boldsymbol{c}^\mathsf{T}(\boldsymbol{x_1} - \boldsymbol{x_2}) \\ \text{subject to} & A(\boldsymbol{x_1} - \boldsymbol{x_2}) = \boldsymbol{b} \\ & \boldsymbol{x_1}, \boldsymbol{x_2} \geq \boldsymbol{0}. \end{array}$$

The realization that the equality $A(\boldsymbol{x_1} - \boldsymbol{x_2}) = \boldsymbol{b}$ can be represented as the system of inequalities

$$\begin{aligned} A(\boldsymbol{x_1} - \boldsymbol{x_2}) &\geq \boldsymbol{b} \\ -A(\boldsymbol{x_1} - \boldsymbol{x_2}) &\geq -\boldsymbol{b} \end{aligned}$$

yields that the standard form of the LP is equivalent to:

$$\begin{array}{rl} \text{minimize} & -\boldsymbol{c}^\mathsf{T}\boldsymbol{x_1} + \boldsymbol{c}^\mathsf{T}\boldsymbol{x_2} \\ \text{subject to} & A\boldsymbol{x_1} - A\boldsymbol{x_2} \geq \boldsymbol{b} \\ & -A\boldsymbol{x_1} + A\boldsymbol{x_2} \geq -\boldsymbol{b} \\ & \boldsymbol{x_1}, \boldsymbol{x_2} \geq \boldsymbol{0}. \end{array}$$

But this can be stated as

$$\text{minimize} \quad \begin{bmatrix} -\boldsymbol{c} \\ \boldsymbol{c} \end{bmatrix}^\mathsf{T} \begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \end{bmatrix}$$

$$\text{subject to} \quad \begin{bmatrix} A & -A \\ -A & A \end{bmatrix} \begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \end{bmatrix} \geq \begin{bmatrix} \boldsymbol{b} \\ -\boldsymbol{b} \end{bmatrix}$$

$$\boldsymbol{x_1}, \boldsymbol{x_2} \geq \boldsymbol{0}$$

or, more succinctly,

$$\begin{array}{rl} \text{minimize} & \boldsymbol{C}^\top \boldsymbol{X} \\ \text{subject to} & \mathscr{A}\boldsymbol{X} \geq \boldsymbol{B} \\ & \boldsymbol{X} \geq \boldsymbol{0} \end{array} \tag{1}$$

where

$$\mathscr{A} = \begin{bmatrix} A & -A \\ -A & A \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} -\boldsymbol{c} \\ \boldsymbol{c} \end{bmatrix}, \quad \boldsymbol{X} = \begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{b} \\ -\boldsymbol{b} \end{bmatrix}. \tag{2}$$

By definition, the dual of the primal problem (1) is

$$\begin{array}{rl} \text{maximize} & \boldsymbol{B}^\top \boldsymbol{\Lambda} \\ \text{subject to} & \mathscr{A}^\top \boldsymbol{\Lambda} \leq \boldsymbol{C} \\ & \boldsymbol{\Lambda}^\top = \left[\boldsymbol{\lambda_1}^\top \boldsymbol{\lambda_2}^\top\right] \geq \boldsymbol{0}^\top. \end{array} \tag{3}$$

Using the corresponding definitions found in (2), we see that after some algebraic manipulation the dual problem (6) can be written as

$$\begin{array}{rl} \text{maximize} & \boldsymbol{b}^\top (\boldsymbol{\lambda_1} - \boldsymbol{\lambda_2}) \\ \text{subject to} & A^\top (\boldsymbol{\lambda_1} - \boldsymbol{\lambda_2}) \leq -\boldsymbol{c} \\ & A^\top (\boldsymbol{\lambda_1} - \boldsymbol{\lambda_1}) \geq -\boldsymbol{c} \\ & \boldsymbol{\lambda_1}, \boldsymbol{\lambda_2} \geq \boldsymbol{0}. \end{array}$$

Noting that the system of inequalities can be written as an equality and making the substitution $\boldsymbol{\lambda} = (\boldsymbol{\lambda_1} - \boldsymbol{\lambda_2})$ where $\boldsymbol{\lambda}$ is free, we see that the dual of the problem

$$\begin{array}{rl} \text{maximize} & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \end{array}$$

is

$$\begin{array}{rl} \text{minimize} & -\boldsymbol{b}^\top \boldsymbol{\lambda} \\ \text{subject to} & A^\top \boldsymbol{\lambda} = -\boldsymbol{c}. \end{array}$$

b. Note that the linear program

$$\begin{array}{rl} \text{maximize} & 2x_1 + 5x_2 + x_3 \\ \text{subject to} & 2x_1 - x_2 + 7x_3 \leq 6 \\ & x_1 + 3x_2 + 4x_3 \leq 9 \\ & 3x_1 + 6x_2 + x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0. \end{array} \tag{4}$$

can be written as

$$\begin{array}{rl} \text{maximize} & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{array}$$

where

$$A = \begin{bmatrix} 2 & -1 & 7 \\ 1 & 3 & 4 \\ 3 & 6 & 1 \end{bmatrix}, \quad \boldsymbol{c} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 6 \\ 9 \\ 3 \end{bmatrix}.$$

Some algebraic manipulations allows us to write the above problem as

$$\begin{aligned} \text{minimize} \quad & -\boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} \quad & -A\boldsymbol{x} \geq -\boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned} \tag{5}$$

By definition, the symmetric dual to the primal problem (5) is

$$\begin{aligned} \text{maximize} \quad & -\boldsymbol{b}^\top \boldsymbol{\lambda} \\ \text{subject to} \quad & -A^\top \boldsymbol{\lambda} \leq -\boldsymbol{c} \\ & \boldsymbol{\lambda} = [\lambda_1, \lambda_2, \lambda_3]^\top \geq \boldsymbol{0}. \end{aligned}$$

Therefore, the dual to the primal problem (5) can be written as

$$\begin{array}{rrl} \text{maximize} & -6\lambda_1 - 9\lambda_2 - 3\lambda_3 & \\ \text{subject to} & -2\lambda_1 - \lambda_2 - 3\lambda_3 & \leq -2 \\ & \lambda_1 - 3\lambda_2 - 6\lambda_3 & \leq -5 \\ & -7\lambda_1 - 4\lambda_2 - \lambda_3 & \leq -1 \\ & \lambda_1, \lambda_2, \lambda_3 \geq 0 & \end{array}$$

and the dual to the original primal problem (4) is

$$\begin{array}{rrl} \text{minimize} & 6\lambda_1 + 9\lambda_2 + 3\lambda_3 & \\ \text{subject to} & 2\lambda_1 + \lambda_2 + 3\lambda_3 & \geq 2 \\ & -\lambda_1 + 3\lambda_2 + 6\lambda_3 & \geq 5 \\ & 7\lambda_1 + 4\lambda_2 + \lambda_3 & \geq 1 \\ & \lambda_1, \lambda_2, \lambda_3 \geq 0. & \end{array}$$

□

Problem 2. a. Prove (via the symmetric form of duality) that the dual of the dual problem in an asymmetric form of duality is the primal (standard) problem.

b. Prove the weak duality proposition for the symmetric form of duality.

c. Prove that the primal problem is infeasible if and only if the dual problem is unbounded.

Solution. a. Suppose that we have the following primal problem in standard form

$$\begin{array}{rrl} (P_a) & \text{minimize} & \boldsymbol{c}^\top \boldsymbol{x} \\ & \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \\ & & \boldsymbol{x} \geq \boldsymbol{0}. \end{array}$$

The dual of this problem via an asymmetric form of duality is given by

$$\begin{array}{rrl} (D_a) & \text{maximize} & \boldsymbol{b}^\top \boldsymbol{\lambda} \\ & \text{subject to} & A^\top \boldsymbol{\lambda} \leq \boldsymbol{c}. \end{array}$$

where $\boldsymbol{\lambda}$ is free. Making the substitution $\boldsymbol{\lambda} = \boldsymbol{\lambda_1} - \boldsymbol{\lambda_2}$ and performing some algebraic manipulation, we can transform this dual problem into the following equivalent form

$$\begin{array}{rrl} (D_a) & \text{minimize} & -\boldsymbol{b}^\top (\boldsymbol{\lambda_1} - \boldsymbol{\lambda_2}) \\ & \text{subject to} & -A^\top (\boldsymbol{\lambda_1} - \boldsymbol{\lambda_2}) \geq -\boldsymbol{c} \\ & & \boldsymbol{\lambda_1}, \boldsymbol{\lambda_2} \geq \boldsymbol{0}. \end{array}$$

From some more algebraic manipulation, we rewrite the dual problem into the following form

$$\begin{array}{rrl} (D_a) & \text{minimize} & \boldsymbol{B}^\top \boldsymbol{\Lambda} \\ & \text{subject to} & \mathscr{A}^\top \boldsymbol{\Lambda} \geq -\boldsymbol{c} \\ & & \boldsymbol{\Lambda} \geq \boldsymbol{0} \end{array} \tag{6}$$

where

$$\mathscr{A} = \begin{bmatrix} -A \\ A \end{bmatrix}, \quad \boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{\lambda_1} \\ \boldsymbol{\lambda_2} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} -\boldsymbol{b} \\ \boldsymbol{b} \end{bmatrix}.$$

Therefore, by definition, the dual via the symmetric form of duality of the dual problem (6) is

$$\begin{array}{rrl} (P_s) & \text{maximize} & -\boldsymbol{c}^\top \boldsymbol{x} \\ & \text{subject to} & \mathscr{A}\boldsymbol{x} \leq \boldsymbol{B} \\ & & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

or equivalently

$$\begin{array}{rrl} (P_s) & \text{maximize} & -\boldsymbol{c}^\top \boldsymbol{x} \\ & \text{subject to} & -A\boldsymbol{x} \leq -\boldsymbol{b} \\ & & A\boldsymbol{x} \leq \boldsymbol{b} \\ & & \boldsymbol{x} \geq \boldsymbol{0}. \end{array} \tag{7}$$

By rewriting the objective from *maximize* to *minimize* and combining the inequalities into one equality in (7), we see that the dual via the symmetric form of duality of the dual in (6) is

$$\begin{array}{lll} (P_s) & \text{minimize} & \boldsymbol{c}^\top \boldsymbol{x} \\ & \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \\ & & \boldsymbol{x} \geq \boldsymbol{0}. \end{array}$$

which is equivalent to the standard form of the primal problem (P_a) and we are done.

b. Suppose that we have the following primal problem in symmetric form and dual problem obtained in a symmetric form:

$$\begin{array}{lll} (P_s) & \text{minimize} & \boldsymbol{c}^\top \boldsymbol{x} \\ & \text{subject to} & A\boldsymbol{x} \geq \boldsymbol{b} \\ & & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \quad \Longrightarrow \quad \begin{array}{lll} (D_s) & \text{maximize} & \boldsymbol{\lambda}^\top \boldsymbol{b} \\ & \text{subject to} & \boldsymbol{\lambda}^\top A \leq \boldsymbol{c}^\top \\ & & \boldsymbol{\lambda} \geq \boldsymbol{0}. \end{array}$$

We wish to show that if $\boldsymbol{x}$ and $\boldsymbol{\lambda}$ are feasible solutions to the primal and dual problem, respectively, then $\boldsymbol{\lambda}^\top \boldsymbol{b} \leq \boldsymbol{c}^\top \boldsymbol{x}$.

Since $\boldsymbol{x}$ is a feasible solution to the primal problem, we know that

$$\boldsymbol{\lambda}^\top \boldsymbol{b} \leq \boldsymbol{\lambda}^\top (A\boldsymbol{x}).$$

We also know, for $\boldsymbol{x} \geq \boldsymbol{0}$, that since $\boldsymbol{\lambda}$ is a feasible solution to the dual problem,

$$\boldsymbol{\lambda}^\top A \leq \boldsymbol{c}^\top \implies \boldsymbol{\lambda}^\top A\boldsymbol{x} \leq \boldsymbol{c}^\top \boldsymbol{x}.$$

Therefore, by combining the two obtained inequalities, we have that

$$\boldsymbol{\lambda}^\top \boldsymbol{b} \leq \boldsymbol{\lambda}^\top (A\boldsymbol{x}) \leq \boldsymbol{c}^\top \boldsymbol{x}$$

proving the weak duality proposition.

c. We wish to show that the primal problem is infeasible if and only if the dual problem is unbounded. This is equivalent to showing that the primal problem is feasible if and only if the dual problem is bounded. The primal problem in symmetric form and the dual problem obtained in a symmetric form are presented below:

$$\begin{array}{lll} (P_s) & \text{minimize} & \boldsymbol{c}^\top \boldsymbol{x} \\ & \text{subject to} & A\boldsymbol{x} \geq \boldsymbol{b} \\ & & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \quad \Longrightarrow \quad \begin{array}{lll} (D_s) & \text{maximize} & \boldsymbol{\lambda}^\top \boldsymbol{b} \\ & \text{subject to} & \boldsymbol{\lambda}^\top A \leq \boldsymbol{c}^\top \\ & & \boldsymbol{\lambda} \geq \boldsymbol{0}. \end{array}$$

Suppose first that the primal problem is feasible. Then there exists $\boldsymbol{x_0} \geq \boldsymbol{0}$ with $A\boldsymbol{x_0} \geq \boldsymbol{b}$. By the Weak Duality proposition, we know that for every feasible $\boldsymbol{\lambda}$ that $\boldsymbol{\lambda}^\top \boldsymbol{b} \leq \boldsymbol{c}^\top \boldsymbol{x_0}$. Thus, $\max\{\boldsymbol{\lambda}^\top \boldsymbol{b} \mid \boldsymbol{\lambda}^\top A \leq \boldsymbol{c}^\top, \boldsymbol{\lambda} \geq \boldsymbol{0}\} \leq \boldsymbol{c}^\top \boldsymbol{x_0}$ and the dual problem is bounded above.

Now suppose that the dual problem is bounded.

□

Problem 3. Prove the Duality Theorem for the symmetric case.

Solution. □

Problem 4. Consider the following linear program:

$$\begin{array}{rrl} \text{maximize} & 2x_1 + 3x_2 & \\ \text{subject to} & x_1 + 2x_2 & \leq 4 \\ & 2x_1 + x_2 & \leq 5 \\ & x_1, x_2 & \geq 0. \end{array}$$

a. Use the simplex method to solve the problem.

b. Write down the dual of the linear program and solve the dual.

Solution. □

Problem 5. Consider the following primal problem:

$$
\begin{array}{rllllll}
\text{maximize} & x_1 & +2x_2 & & & & \\
\text{subject to} & -2x_1 & +x_2 & +x_3 & & & = 2 \\
& -x_1 & +2x_2 & & +x_4 & & = 7 \\
& x_1 & & & & +x_5 & = 3 \\
& x_i \geq 0 & i = 1, 2, 3, 4, 5. & & & &
\end{array}
$$

a. Construct the dual problem corresponding to the primal problem above.

b. It is known that the solution to the primal above is $\boldsymbol{x}^* = [3, 5, 3, 0, 0]^\top$. Find the solution to the dual.

Solution. $\square$

Problem 6. Let A be a given matrix and $\boldsymbol{b}$ a given vector. We wish to prove the following result: There exists a vector $\boldsymbol{x}$ such that $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq \boldsymbol{0}$ if and only if for any given vector $\boldsymbol{y}$ satisfying $A^\top \boldsymbol{y} \leq \boldsymbol{0}$ we have $\boldsymbol{b}^\top \boldsymbol{y} \leq \boldsymbol{0}$. This result is known as *Farkas's transposition theorem*. Our program is based on duality theory, consisting of the parts listed below.

a. Consider the primal linear program

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{0}^\top \boldsymbol{x} \\ \text{subject to} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned}$$

Write down the dual of this problem using the notation $\boldsymbol{y}$ for the dual variable.

b. Show that the feasible set of the dual problem is guaranteed to be nonempty.

Hint: Think about an obvious feasible point.

c. Suppose that for any $\boldsymbol{y}$ satisfying $A^\top \boldsymbol{y} \leq \boldsymbol{0}$, we have $\boldsymbol{b}^\top \boldsymbol{y} \leq 0$. In this case what can you say about whether or not the dual has an optimal feasible solution.

Hint: Think about the obvious feasible point in part b.

d. Suppose that for any $\boldsymbol{y}$ satisfying $A^\top \boldsymbol{y} \leq \boldsymbol{0}$, we have $\boldsymbol{b}^\top \boldsymbol{y} \leq 0$. Use parts b and c to show that there exists $\boldsymbol{x}$ such that $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq \boldsymbol{0}$. (This proves one direction of Farkas's transposition theorem.)

e. Suppose that $\boldsymbol{x}$ satisfies $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq \boldsymbol{0}$. Let $\boldsymbol{y}$ be an arbitrary vector satisfying $A^\top \boldsymbol{y} \leq \boldsymbol{0}$. (This proves the other direction of Farkas's transposition theorem.)

Solution. □