Homework Assignment 9

Matthew Tiger

November 9, 2015

Problem 1. Verify that the forward Euler scheme (9.29) has first order accuracy on a smooth solution $u = u(x)$ of problem (9.30).

Solution. Suppose we have the problem $Lu = f$, as defined in 9.30 i.e.

$$Lu = \begin{cases} \frac{du}{dx} - G(x,u), \ 0 < x \leq 1 \\ 0 \end{cases} \quad \text{and } f = \begin{cases} 0, \ 0 < x \leq 1 \\ a \end{cases}.$$

The forward Euler scheme $L_h u^{(h)} = f^{(h)}$ is given by

$$L_h u^{(h)} = \begin{cases} \frac{u_{n+1}-u_n}{h} - G(x_n, u_n), \ n = 0, 1, \dots N-1 \\ u_0 \end{cases} \quad \text{and } f^{(h)} = \begin{cases} 0, \ n = 0, 1, \dots N-1 \\ a \end{cases}.$$

Let $[u]_h$ denote the discretized solution to $Lu = f$. This scheme has first order accuracy if $\left\| L_h[u]_h - L_h u^{(h)} \right\| \leq Ch$ where C is a constant that does not depend on h.

Note that the Taylor series expansion of $u(x+h)$ centered at x is given by

$$u(x+h) = u(x) + u'(x)h + \frac{u''(\xi)h^2}{2}$$

for $x \leq \xi \leq x+h$. This implies that

$$u'(x) = \frac{u(x+h) - u(x)}{h} - \frac{u''(\xi)h}{2}$$

or that

$$u'(x) - G(x,u) = \frac{u(x+h) - u(x)}{h} - \frac{u''(\xi)h}{2} - G(x,u).$$

As $u'(x) - G(x,u) = 0$ is the exact solution to $Lu = f$, we know that the discretized exact solution is given by

$$u'(x) - G(x,u) = \frac{u(x_{n+1}) - u(x_n)}{h} - \frac{u''(\xi(x_n))h}{2} - G(x_n, u_n) = 0$$

where $\xi(x_n)$ depends on the node x_n. But under the forward Euler scheme, $L_h[u]_h = \frac{u_{n+1}-u_n}{h} - G(x_n, u_n)$ so that

$$u'(x) - G(x,u) = L_h[u]_h - \frac{u''(\xi(x_n))h}{2} = 0$$

i.e.

$$u'(x) - G(x,u) = L_h[u]_h - L_h u^{(h)} = \frac{u''(\xi(x_n))h}{2}$$

since $L_h u^{(h)} = 0$. If $|u''(x)| \le M$ for $x \in [0,1]$, then the above implies that

$$\left\| L_h[u]_h - L_h u^{(h)} \right\| = \left\| \frac{u''(\xi(x_n))h}{2} \right\| \le \frac{M}{2}h.$$

As $M/2$ does not depend on h, we have shown $\left\| L_h[u]_h - L_h u^{(h)} \right\| \le Ch$ where $C = M/2$ and that the forward Euler scheme has first order of accuracy. □

Problem 2. Verify that the Crank-Nicolson scheme (9.33) has second order accuracy on a smooth solution $u = u(x)$ of problem (9.30).

Solution. Suppose we have the problem $Lu = f$, as defined in 9.30 i.e.

$$Lu = \begin{cases} \frac{du}{dx} - G(x,u), \ 0 < x \le 1 \\ 0 \end{cases} \quad \text{and } f = \begin{cases} 0, \ o < x \le 1 \\ a \end{cases}.$$

The Crank-Nicolson scheme $L_h u^{(h)} = f^{(h)}$ is given by

$$L_h u^{(h)} = \begin{cases} \frac{u_{n+1} - u_n}{h} - \frac{1}{2}[G(x_n, u_n) + G(x_{n+1}, u_{n+1})], \ n = 0, \dots N-1 \\ u_0 \end{cases}$$

and

$$f^{(h)} = \begin{cases} 0, \ n = 0, \dots N-1 \\ a \end{cases}.$$

Let $[u]_h$ denote the discretized solution to $Lu = f$. This scheme has second order accuracy if $\left\| L_h[u]_h - L_h u^{(h)} \right\| \le Ch^2$ where C is a constant that does not depend on h.

From the problem $Lu = f$, we see that $\frac{du}{dx} = G(x, u(x))$ implies that

$$\begin{aligned} \frac{d^2u}{dx^2} = \frac{d}{dx}\left[G(x,u(x))\right] &= \frac{\partial G(x,u(x))}{\partial x} + \frac{\partial G(x,u(x))}{\partial u}\frac{du}{dx} \\ &= \frac{\partial G(x,u(x))}{\partial x} + \frac{\partial G(x,u(x))}{\partial u}G(x,u(x)) \end{aligned}$$

The Taylor expansion of $u(x+h)$ centered at x is given by

$$u(x+h) = u(x) + u'(x)h + \frac{u''(x)h^2}{2} + \frac{u^{(3)}(\xi_1)h^3}{6}$$

for $x \le \xi_1 \le x + h$. This implies that

$$u'(x) - G(x,u(x)) = -G(x,u(x)) + \frac{u(x+h) - u(x)}{h} - \frac{u''(x)h}{2} - \frac{u^{(3)}(\xi_1)h^2}{6} = 0$$

Since $u''(x) = \frac{\partial G(x,u(x))}{\partial x} + \frac{\partial G(x,u(x))}{\partial u} G(x, u(x))$ from our earlier calculation, we have that

$$
\begin{aligned}
u'(x) - G(x, u(x)) &= \frac{u(x+h) - u(x)}{h} - \left[G(x, u(x)) + \frac{h}{2} \left(\frac{\partial G}{\partial x} + \frac{\partial G}{\partial u} G(x, u(x)) \right) \right] \qquad (1) \\
&= \frac{u^{(3)}(\xi_1) h^2}{6}.
\end{aligned}
$$

We now wish to show that we can replace the expression $\frac{\partial G(x,u(x))}{\partial x} + \frac{\partial G(x,u(x))}{\partial u} G(x, u(x))$ with $\frac{1}{2}[G(x, u(x)) + G(x + h, u(x) + hG(x, u(x)))]$ by expressing the Taylor expansion of $G(x + h, u(x) + hG(x, u(x))) = G(x_1, y_1)$ centered at $(x, u(x))$. This Taylor expansion is given by

$$
\begin{aligned}
G(x_1, y_1) = G(x, u(x)) &+ h \left[\frac{\partial G(x, u(x))}{\partial x} + \frac{\partial G(x, u(x))}{\partial u} G(x, u(x)) \right] \\
&+ h^2 \frac{1}{2} \left[\frac{\partial^2 G(x, u(x))}{\partial x^2} + 2 \frac{\partial^2 G(x, u(x))}{\partial x \partial u} G(x, u(x)) + \frac{\partial^2 G(x, u(x))}{\partial u^2} G(x, u(x))^2 \right] \Bigg|_{x=\xi_2} \\
&= G(x, u(x)) + \frac{\partial G(x, u(x))}{\partial x} h + \frac{\partial G(x, u(x))}{\partial u} h G(x, u(x)) + R h^2
\end{aligned}
$$

where $x \leq \xi_2 \leq x+h$ and $u(x) \leq u(\xi_2) \leq u(x)+hG(x, u(x))$ and R is the additional constant of the second order term. From the above identity we can see that

$$
\frac{1}{2} \left[G(x, u) + G(x + h, u + hG(x, u)) \right] = G(x, u) + \frac{h}{2} \left[\frac{\partial G(x, u)}{\partial x} + \frac{\partial G(x, u)}{\partial u} G(x, u) \right] + \frac{R}{2} h^2
$$

where we have replaced $u(x)$ with u to shorten the expression. Note that from this identity it is clear that

$$
G(x, u) + \frac{h}{2} \left(\frac{\partial G(x, u)}{\partial x} + \frac{\partial G(x, u)}{\partial u} G(x, u) \right) = \frac{1}{2} \left[G(x, u) + G(x + h, u + hG(x, u)) \right] - \frac{R}{2} h^2 \qquad (2)
$$

and replacing (2) in (1) yields the exact solution to the problem $Lu = f$ as

$$
\begin{aligned}
u'(x) - G(x, u(x)) &= \frac{u(x+h) - u(x)}{h} - \frac{1}{2} \left[G(x, u) + G(x + h, u + hG(x, u)) \right] \qquad (3) \\
&= \left[\frac{u^{(3)}(\xi_1)}{6} + \frac{R}{2} \right] h^2.
\end{aligned}
$$

If $[u]_h$ is the discretized solution of the problem $Lu = f$, then discretizing the exact solution shows that for $x + h = x_{n+1}$ and $G(x + h, u(x) + hG(x, u(x))) = G(x_{n+1}, u_n + hG(x_n, u_n)) = G(x_{n+1}, u_{n+1})$ we have from (3) that

$$
\begin{aligned}
u'(x) - G(x, u(x)) &= \frac{u_{n+1} - u_n}{h} - \frac{1}{2} \left[\frac{1}{2} [G(x_n, u_n) + G(x_{n+1}, u_{n+1})] \right] \\
&= L_h [u]_h - L_h u^{(h)} = \left[\frac{u^{(3)}(\xi_1)}{6} + \frac{R}{2} \right] h^2.
\end{aligned}
$$

since $L_h u^{(h)} = 0$. Assuming all second order partials and mixed partials of $G(x, u(x))$ are bounded and that the third derivative of our function $u(x)$ is bounded, it is clear that $\left\| L_h[u]_h - L_h u^{(h)} \right\| \leq Ch^2$ where C does not depend on h showing that this scheme has second order of accuracy. $\square$

Problem 3. Create a difference scheme that is not consistent.

Solution. Suppose we have the Cauchy problem $Lu = f$, as defined in 9.30 i.e.

$$Lu = \begin{cases} \frac{du}{dx} - G(x,u), \ 0 < x \leq 1 \\ 0 \end{cases} \quad \text{and } f = \begin{cases} 0, \ 0 < x \leq 1 \\ a \end{cases}.$$

Define a variant of the forward Euler scheme $L_h u^{(h)} = f^{(h)}$ as folows

$$L_h u^{(h)} = \begin{cases} \frac{u_{n+1} - u_n}{h} - G(x_n, u_n) + 1, \ n = 0, 1, \dots N-1 \\ u_0 \end{cases} \quad \text{and } f^{(h)} = \begin{cases} 0, \ n = 0, 1, \dots N-1 \\ a \end{cases}.$$

Then it is clear that this scheme is inconsistent as the the residual will always be a constant and never vanish regardless of the grid we choose. $\square$

Problem 4. Prove that the scheme

$$4\frac{u_{n+1} - u_{n-1}}{2h} - 3\frac{u_{n+1} - u_n}{h} + u_n = 0, \quad n = 1, 2, \dots, N-1$$

with initial conditions $u_0 = 1$ and $u_1 = e^{-h}$ is consistent for the problem

$$\frac{du}{dx} + u = 0, \quad 0 \leq x \leq 1$$

with initial condition $u(0) = 1$.

Solution. If $[u]_h$ is the discretized solution to the problem $Lu = f$ as defined above, then the scheme $L_h u^{(h)} = f^{(h)}$ is consistent if $\left\| L_h[u]_h - L_h u^{(h)} \right\| \to 0$ as $h \to 0$.

Note that the Taylor series expansions of $u(x+h)$ and $u(x-h)$ centered at x are given by

$$u(x+h) = u(x) + u'(x)h + \frac{u''(\xi_1)h^2}{2}$$
$$u(x-h) = u(x) - u'(x)h + \frac{u''(\xi_2)h^2}{2}$$

for $x \leq \xi_1 \leq x + h$ and $x - h \leq \xi_2 \leq x$. From these expansions we can see that

$$u'(x) = \frac{u(x+h) - u(x-h)}{2h} - \frac{1}{4}h(u''(\xi_1) - u''(\xi_2))$$

and

$$u'(x) = \frac{u(x+h) - u(x)}{h} - \frac{1}{2}hu''(\xi_3).$$

This shows that

$$u'(x)+u(x)=4\frac{u(x+h)-u(x-h)}{2h}-3\frac{u(x+h)-u(x)}{h}+u(x)+h\left(\frac{3}{2}u''(\xi_3)-(u''(\xi_1)-u''(\xi_2))\right)$$

so that if $[u]_h$ is the discretized solution to the problem defined above,

$$\begin{aligned}u'(x)+u(x)&=4\frac{u(x_{n+1})-u(x_{n-1})}{2h}-3\frac{u(x_{n+1})-u(x_n)}{h}+u(x_n)+h\left(\frac{3}{2}u''(\xi_3)-(u''(\xi_1)-u''(\xi_2))\right)\\&=L_h[u]_h+h\left(\frac{3}{2}u''(\xi_3)-(u''(\xi_1)-u''(\xi_2))\right)=0.\end{aligned}$$

Combining the above and the fact that $L_hu^{(h)}=0$, we see that

$$\left\|L_h[u]_h-L_hu^{(h)}\right\|=h\left\|(u''(\xi_1)-u''(\xi_2))-\frac{3}{2}u''(\xi_3)\right\|.$$

If $|u''(x)|\leq M$, then $0\leq\left\|L_h[u]_h-L_hu^{(h)}\right\|\leq h\left(\frac{7}{2}M\right)$ and it is then clear that $\left\|L_h[u]_h-L_hu^{(h)}\right\|\to 0$ as $h\to 0$ showing the consistency of the scheme. □

Problem 5. Prove that the scheme

$$4\frac{u_{n+1}-u_{n-1}}{2h}-3\frac{u_{n+1}-u_n}{h}+u_n=0,\quad n=1,2,\ldots,N-1$$

with initial conditions $u_0=1$ and $u_1=e^{-h}$ is divergent for the problem

$$\frac{du}{dx}+u=0,\quad 0\leq x\leq 1$$

with initial condition $u(0)=1$.

Solution. If $[u]_h$ is the discretized solution to the problem $Lu=f$ as defined above, then the scheme $L_hu^{(h)}=f^{(h)}$ is divergent if $\left\|[u]_h-u^{(h)}\right\|$ does not approach 0 as $h\to 0$.

The exact solution to the problem $Lu=f$ with the initial condition $u(0)=1$ is $u(x)=e^{-x}$. Hence, $[u]_h=[e^{-x_0},e^{-x_1},\ldots,e^{-x_n},\ldots,e^{-x_N}]=[e^0,e^{-h},\ldots,e^{-nh},\ldots,e^{-1}]$. The solution to the difference scheme $L_hu^{(h)}=f^{(h)}$ given by $u^{(h)}$ and can be found by finding the explicit solution to the difference equation defined in the scheme.

Note that this is a second order difference equation that can be rewritten as

$$-u_{n+1}+(3+h)u_n-2u_{n-1}=0.$$

The characteristic equation of this difference equation is given by $-m^2+(3+h)m-2=0$. As this characteristic equation has distinct real roots, the general solution to the difference equation is $u_n=c_1m_1^n+c_2m_2^n$ where $m_1=\frac{1}{2}(-\sqrt{h^2+6h+1}+h+3)$ and $m_2=\frac{1}{2}(\sqrt{h^2+6h+1}+h+3)$ are the roots of the characteristic equation. Choosing the constants so that the initial conditions are satisfied gives us the general solution as

$$\begin{aligned}u_n^{(h)}=u_0&\left[\frac{m_2(h)}{m_2(h)-m_1(h)}m_1(h)^n-\frac{m_1(h)}{m_2(h)-m_1(h)}m_2(h)^n\right]\\&+u_1\left[-\frac{1}{m_2(h)-m_1(h)}m_1(h)^n+\frac{1}{m_2(h)-m_1(h)}m_2(h)^n\right].\end{aligned}$$

Combining this general solution to the scheme and the exact solution to the problem we can clearly see that $\left\| [u]_h - u^{(h)} \right\|$ does not approach 0 as $h \to 0$ and that the scheme is divergent. $\square$