Homework Assignment 1

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Problem 1. Solve the IVP:

$$y' = y^2 \cos(x), \quad y(0) = 2.$$

Solution. Note that this is a separable differential equation and after separating we see that

$$\begin{aligned} \frac{dy}{y^2} &= \cos(x)dx \\ \int \frac{dy}{y^2} &= \int \cos(x)dx \\ -\frac{1}{y} &= \sin(x) + c_1 \end{aligned}$$

so that

$$y = -\frac{1}{\sin(x) + c_1}$$

is the general solution to the differential equation. Using the initial value $y(0) = 2$ and solving for c_1 we see that $c_1 = -1/2$ and the solution to the IVP is given by

$$y = -\frac{1}{\sin(x) - 1/2}.$$

$\square$

Problem 2. Review solutions of first-order linear ODEs (p. 14) and solve the IVP:

$$y' - xy = x^3, \quad y(1) = \frac{1}{2}.$$

Solution. The solution to the first-order linear ODE

$$y'(x) + p_0(x)y(x) = f(x)$$

is given by

$$y(x) = \frac{c_1}{I(x)} + \frac{1}{I(x)} \int_0^x f(t)I(t)dt, \quad I(x) = \exp\left(\int_0^x p_0(t)dt\right).$$

For this problem, we set $p_0(x) = -x$ and $f(x) = x^3$ and see that

$$I(x) = \exp\left(\int_0^x p_0(t)dt\right) = \exp\left(\int_0^x -tdt\right) = \exp\left(-\frac{x^2}{2}\right).$$

Thus the general solution to the ODE $y' - xy = x^3$ is given by

$$\begin{aligned} y &= \frac{c_1}{\exp\left(-\frac{x^2}{2}\right)} + \frac{1}{\exp\left(-\frac{x^2}{2}\right)} \int_0^x t^3 \exp\left(-\frac{t^2}{2}\right) dt \\ &= \frac{c_1}{\exp\left(-\frac{x^2}{2}\right)} - \frac{\exp\left(-\frac{x^2}{2}\right)}{\exp\left(-\frac{x^2}{2}\right)}(2 + x^2) \\ &= \frac{c_1}{\exp\left(-\frac{x^2}{2}\right)} - (2 + x^2) \end{aligned}$$

Using the initial value $y(1) = \frac{1}{2}$, we see that $c_1 = \frac{7}{2}\exp\left(-\frac{1}{2}\right)$ and the solution to the IVP is

$$y = \frac{7\exp\left(-\frac{1}{2}\right)}{2\exp\left(-\frac{x^2}{2}\right)} - (2 + x^2).$$

$\square$

Problem 3. Let $Ly = y^{(4)} - 4y''' + 3y'' + 4y' - 4y$.

a. Find the general solutions of the homogeneous ODE $Ly = 0$.

b. Solve the IVP:

$$Ly = 0, \quad y(0) = 0, \quad y'(0) = -7, \quad y''(0) = 5, \quad y'''(0) = 9.$$

c. Solve the BVP:

$$Ly = 0, \quad y(0) = 1, \quad \lim_{x\to\infty} y(x) = 0.$$

Is this BVP well-posed?

d. Solve the BVP:

$$Ly = 0, \quad y(0) = 1, \quad \lim_{x\to-\infty} y(x) = 0.$$

Is this BVP well-posed?

Solution. a. The characteristic equation associated to the homogeneous ODE $Ly = 0$ is $m(x) = x^4 - 4x^3 + 3x^2 + 4x - 4$. The roots of the characteristic polynomial are $r_1 = -1$, $r_2 = 1$, $r_3 = 2$, and $r_4 = 2$.

Therefore, the general solution of the homogeneous ODE is

$$y(x) = c_1 e^{-x} + c_2 e^{x} + c_3 e^{2x} + c_4 x e^{2x}. \tag{1}$$

b. Through an abuse of notation, we note that the matrix associated to the Wronskian of this equation as function of x is given by

$$W(x) = \begin{bmatrix} e^{-x} & e^{x} & e^{2x} & xe^{2x} \\ -e^{-x} & e^{x} & 2e^{2x} & e^{2x} + 2xe^{2x} \\ e^{-x} & e^{x} & 4e^{2x} & 4e^{2x} + 4xe^{2x} \\ -e^{-x} & e^{x} & 8e^{2x} & 12e^{2x} + 8xe^{2x} \end{bmatrix}.$$

The solution to the IVP is determined by particular values of the coefficients in the general solution (1). These coefficients are found as the solution to the system of equations $W(0)\boldsymbol{c} = \boldsymbol{b}$ where

$$\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ -7 \\ 5 \\ 9 \end{bmatrix}.$$

The solution to this system is given by $\boldsymbol{c} = \langle 4, -3, -1, 2 \rangle$. Therefore, the solution to the IVP is

$$y(x) = 4e^{-x} - 3e^{x} - e^{2x} + 2xe^{2x}.$$

c. The general solution to the ODE, $y(x)$, is given by (1). The second condition that $\lim_{x\to\infty} y(x) = 0$ can not be satisfied by the general solution since $\lim_{x\to\infty} e^{ax} = \infty$ for $a > 0$. Therefore, the BVP as stated is not well-posed.

d. The general solution to the ODE, $y(x)$, is given by (1). The second condition that $\lim_{x\to-\infty} y(x) = 0$ can not be satisfied by the general solution since $\lim_{x\to-\infty} e^{ax} = \infty$ for $a < 0$. Therefore, the BVP as stated is not well-posed.

$\square$

Problem 4. Read §1.6 and then solve the ODEs:

$$xy' + 2y = x^2\sqrt{y}, \quad y' = \frac{4x^3 - 6xy^2 - 2xy}{x^2 + 6x^2y - 3y^2}, \quad y' + y^2 + (2x+1)y + 1 + x + x^2 = 0.$$

Solution. We begin with the differential equation

$$xy' + 2y = x^2\sqrt{y}.$$

Note that this equation can be rewritten as

$$y' = \left(-\frac{2}{x}\right) y + xy^{1/2}, \tag{2}$$

which is a Bernoulli equation with $P = 1/2$. Dividing (2) by $y^{1/2}$ and making the substitution $u(x) = y(x)^{1-1/2}$ yields the new linear differential equation

$$u'(x) = -\left(\frac{1}{x}\right) u(x) + \frac{x}{2}.$$

The solution to this linear equation is $u(x) = x^2/6 + c_1/x$ suggesting that

$$y(x) = u(x)^2 = \left(\frac{x^2}{6} + \frac{c_1}{x}\right)^2$$

is the solution to (2).

Let us next investigate

$$y' = \frac{4x^3 - 6xy^2 - 2xy}{x^2 + 6x^2y - 3y^2}.$$

Note that this equation can be written as

$$-(4x^3 - 6xy^2 - 2xy) + (x^2 + 6x^2y - 3y^2)y'(x) = 0.$$

Identifying $M(x,y) = -(4x^3 - 6xy^2 - 2xy)$ and $N(x,y) = (x^2 + 6x^2y - 3y^2)$, we notice that

$$\frac{\partial M(x,y)}{\partial y} = 12xy + 2x = \frac{\partial N(x,y)}{\partial x}$$

making this equation exact. The solution to the exact differential equation is then $f(x,y) = c_1$ where $f_x = M(x,y)$ and $f_y = N(x,y)$. Thus,

$$f(x,y) = \int f_x(x,y)dx = -\int (4x^3 - 6xy^2 - 2xy)dx = -x^4 + 3x^2y^2 + x^2y + h(y). \tag{3}$$

In order to find out what $h(y)$ is, we take the partial derivative of (3) and compare it with $N(x,y)$. Doing so, we see that

$$f_y(x,y) = x^2 + 6x^2y + h'(y) = x^2 + 6x^2y - 3y^2 = N(x,y)$$

implying that $h'(y) = -3y^2$ and that $h(y) = -y^3$. Therefore, the solution to the differential equation is

$$f(x, y) = -x^4 + 3x^2y^2 + x^2y - y^3 = c_1.$$

Finally let us investigate the differential equation

$$y' + y^2 + (2x + 1)y + 1 + x + x^2 = 0.$$

This equation can be rewritten as

$$y' = -y^2 - (2x + 1)y - (1 + x + x^2) \tag{4}$$

which is a Riccati equation. The procedure to find the solution of such equations is to produce a particular solution $y_p(x)$ to the equation and then find the general solution which will be in the form $y(x) = y_p(x) + u(x)$ by using this formula in the original equation. Note that $y_p(x) = -x$ is a particular solution of (4). Thus the general solution is of the form $y(x) = -x + u(x)$.

Making this substitution reveals the following Bernoulli equation in $u(x)$:

$$u'(x) = -u(x) - u(x)^2$$

The solution to this differential equation is $u(x) = -(e^{c_1}/(-e^x + e^{c_1}))$. Therefore, the general solution to (4) is

$$y(x) = -x - \frac{e^{c_1}}{-e^x + e^{c_1}}.$$

$\square$

Problem 5. a. Use mathematical induction to prove Leibnitz's differentiation rule:

$$D^k(fg) = \sum_{j=0}^{k} \binom{k}{j} (D^j f)(D^{k-j} g).$$

Here $f = f(x)$ and $g = g(x)$ are k-times differentiable functions and $D^k = \frac{d^k}{dx^k}$.

b. Consider the constant-coefficient ODE

$$D^n y + p_{n-1} D^{n-1} y + \cdots + p_1 D y + p_0 y = 0, \tag{5}$$

where $p_0, p_1, \ldots, p_{n-1}$ are real numbers. Let r be a double root of the characteristic polynomial $P(z) = z^n + p_{n-1} z^{n-1} + \cdots + p_1 z + p_0$. Use Leibnitz's rule to show that the function xe^{rx} is a solution of (5).

c. Let r be a triple root of the characteristic polynomial $P(z)$ from part (b). Use Leibnitz's rule to show that the function $x^2 e^{rx}$ is then also a solution of (5).

d. Let r be a real number. Show that the functions e^{rx}, xe^{rx}, and $x^2 e^{rx}$ are linearly independent on $\mathbb{R}$.

Solution. □

Problem 6. Use the formula for the derivative of a determinant from the lectures, other properties of determinants, and the linear ODE (1.3.1) to verify identity (1.3.4) in the textbook.

Solution. $\square$