Homework Assignment 5

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Problem 1. Find the 1, 2, ∞ norms of the matrices

$$\boldsymbol{T} := \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \qquad \boldsymbol{I} := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \boldsymbol{J} := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then find their p-condition numbers $\text{cond}_p(\boldsymbol{T})$ for $p = 1, 2, \infty$.

Solution.

- **1-Norm** Note that if $\boldsymbol{X}$ is a 2×2 matrix then $||\boldsymbol{X}||_1 = \max_{1\leq j\leq 2} \sum_{i=1}^{2} |x_{ij}|$ where x_{ij} is the entry in the i -th row and the j -th column of $\boldsymbol{X}$. So,

  $$||\boldsymbol{X}||_1 = \max_{1\leq j\leq 2} \{|x_{1j}|+|x_{2j}|\} = \max\{|x_{11}|+|x_{21}|, |x_{12}|+|x_{22}|\}$$

  If $\boldsymbol{T} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, then $\boldsymbol{T}^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$. So,

  $$||\boldsymbol{T}||_1 = \max\{|2|+|-1|, |-1|+|2|\} = \max\{3, 3\} = 3$$

  and

  $$||\boldsymbol{T}^{-1}||_1 = \max\{|2/3|+|1/3|, |1/3|+|2/3|\} = \max\{1, 1\} = 1.$$

  Thus, $||\boldsymbol{T}||_1 = 3$ and $\text{cond}_p(\boldsymbol{T}) = ||\boldsymbol{T}||_1 ||\boldsymbol{T}^{-1}||_1 = 3 \cdot 1 = 3$.

  If $\boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $\boldsymbol{I}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \boldsymbol{I}$. So,

  $$||\boldsymbol{I}||_1 = \max\{|1|+|0|, |0|+|1|\} = \max\{1, 1\} = 1.$$

  Thus, $||\boldsymbol{I}||_1 = 1$ and $\text{cond}_p(\boldsymbol{I}) = ||\boldsymbol{I}||_1 ||\boldsymbol{I}^{-1}||_1 = 1 \cdot 1 = 1$.

  If $\boldsymbol{J} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $\boldsymbol{J}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. So,

  $$||\boldsymbol{J}||_1 = \max\{|1|+|0|, |1|+|1|\} = \max\{1, 2\} = 2$$

  and

  $$||\boldsymbol{J}^{-1}||_1 = \max\{|1|+|0|, |-1|+|1|\} = \max\{1, 2\} = 2.$$

  Thus, $||\boldsymbol{J}||_1 = 2$ and $\text{cond}_p(\boldsymbol{J}) = ||\boldsymbol{J}||_1 ||\boldsymbol{J}^{-1}||_1 = 2 \cdot 2 = 4$.

- **2-Norm** Note that if $\boldsymbol{X}$ is a 2×2 matrix then $||\boldsymbol{X}||_2 = \sqrt{\lambda_{\max}(\boldsymbol{X}^*\boldsymbol{X})}$ where $\lambda_{\max}(\boldsymbol{X}^*\boldsymbol{X})$ is the largest eigenvalue of the matrix product $\boldsymbol{X}^*\boldsymbol{X}$. Since all of our matrices our real, we can use the following Matlab code to find the 2-norm of each matrix:

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norm = sqrt(max(eig(X' * X)))
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  where we define the matrix X beforehand.
  If $\boldsymbol{T} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, then $\boldsymbol{T}^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$. Using the code above, $||\boldsymbol{T}||_2 = 3$ and $||\boldsymbol{T}^{-1}||_2 = 1$. Thus, $||\boldsymbol{T}||_2 = 3$ and $\text{cond}_p(\boldsymbol{T}) = ||\boldsymbol{T}||_2 ||\boldsymbol{T}^{-1}||_2 = 3 \cdot 1 = 3$.
  If $\boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $\boldsymbol{I}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \boldsymbol{I}$. Using the code above, $||\boldsymbol{I}||_2 = 1$. Thus, $||\boldsymbol{I}||_2 = 1$ and $\text{cond}_p(\boldsymbol{I}) = ||\boldsymbol{I}||_2 ||\boldsymbol{I}^{-1}||_2 = 1 \cdot 1 = 1$.
  If $\boldsymbol{J} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $\boldsymbol{J}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. Using the code above, $||\boldsymbol{J}||_2 = 1.61803$ and $||\boldsymbol{J}^{-1}||_2 = 1.61803$. Thus, $||\boldsymbol{J}||_2 = 1.61803$ and $\text{cond}_p(\boldsymbol{J}) = ||\boldsymbol{J}||_2 ||\boldsymbol{J}^{-1}||_2 = 1.61803 \cdot 1.61803 = 2.61802$.

- **∞-norm** Note that if $\boldsymbol{X}$ is a 2×2 matrix then $||\boldsymbol{X}||_\infty = \max_{1 \le i \le 2} \sum_{j=1}^{2} |x_{ij}|$ where x_{ij} is the entry in the i -th row and the j -th column of $\boldsymbol{X}$. So,

$$||\boldsymbol{X}||_\infty = \max_{1 \le i \le 2} \{|x_{i1}| + |x_{i2}|\} = \max\{|x_{11}| + |x_{12}|, |x_{21}| + |x_{22}|\}$$

  If $\boldsymbol{T} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, then $\boldsymbol{T}^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$. So,

$$||\boldsymbol{T}||_\infty = \max\{|2| + |-1|, |-1| + |2|\} = \max\{3, 3\} = 3$$

  and

$$||\boldsymbol{T}^{-1}||_\infty = \max\{|2/3| + |1/3|, |1/3| + |2/3|\} = \max\{1, 1\} = 1.$$

  Thus, $||\boldsymbol{T}||_\infty = 3$ and $\text{cond}_p(\boldsymbol{T}) = ||\boldsymbol{T}||_\infty ||\boldsymbol{T}^{-1}||_\infty = 3 \cdot 1 = 3$.
  If $\boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $\boldsymbol{I}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \boldsymbol{I}$. So,

$$||\boldsymbol{I}||_\infty = \max\{|1| + |0|, |0| + |1|\} = \max\{1, 1\} = 1.$$

  Thus, $||\boldsymbol{I}||_\infty = 1$ and $\text{cond}_p(\boldsymbol{I}) = ||\boldsymbol{I}||_\infty ||\boldsymbol{I}^{-1}||_\infty = 1 \cdot 1 = 1$.
  If $\boldsymbol{J} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $\boldsymbol{J}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. So,

$$||\boldsymbol{J}||_\infty = \max\{|1| + |1|, |0| + |1|\} = \max\{2, 1\} = 2$$

  and

$$||\boldsymbol{J}^{-1}||_\infty = \max\{|1| + |-1|, |0| + |1|\} = \max\{2, 1\} = 2.$$

  Thus, $||\boldsymbol{J}||_\infty = 2$ and $\text{cond}_p(\boldsymbol{J}) = ||\boldsymbol{J}||_\infty ||\boldsymbol{J}^{-1}||_\infty = 2 \cdot 2 = 4$.

$\square$