

# Homework Assignment 2

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**Problem 4.5.** A Markov chain  $\{X_n : n \geq 0\}$  with states 0,1,2, has the transition probability matrix

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

If  $P\{X_0 = 0\} = P\{X_0 = 1\} = 1/4$ , find  $E[X_3]$ .

*Solution.* If  $\{X_n : n \geq 0\}$  is a Markov chain with state space  $\mathcal{M} = \{0, 1, 2\}$ , then we know that  $\{X_n : n \geq 0\}$  is a stochastic process and that this stochastic process has the property that

$$P\{X_n = j \mid X_{n-1} = i, X_{n-2} = i_{n-2}, \dots, X_0 = i_0\} = P\{X_n = j \mid X_{n-1} = i\}$$

for any time  $n \in \mathbb{Z}^+$ , i.e. the probability that  $X_n$  is in state  $j$  depends only on the probability that  $X_{n-1}$  is in state  $i$ . We denote  $P\{X_n = j \mid X_{n-1} = i\}$  by  $P_{ij}$  and  $P\{X_n = j \mid X_0 = i\}$  by  $P_{ij}^n$ .

We wish to find

$$E[X_3] = \sum_{j=0}^2 x P\{X_3 = j\}.$$

If we know the probability that  $X_0 = i$  for all states  $i \in \mathcal{M}$ , then we can condition  $P\{X_3 = j\}$  on the probability that  $X_0$  is in state  $i \in \mathcal{M}$  for all states, i.e.

$$\begin{aligned} P\{X_3 = j\} &= \sum_{i \in \mathcal{M}} P\{X_3 = j \mid X_0 = i\} P\{X_0 = i\} \\ &= \sum_{i \in \mathcal{M}} P_{ij}^3 P\{X_0 = i\} \end{aligned} \tag{1}$$

By assumption, we know that  $P\{X_0 = 0\} = P\{X_0 = 1\} = 1/4$ . Since  $\{X_n : n \geq 0\}$  is a stochastic process,  $X_0$  is a random variable so that  $P\{X_0 = 2\} > 0$  and in particular

$$P\{X_0 = 2\} = 1 - \sum_{i \in \mathcal{M}, i \neq 2} P\{X_0 = i\} = \frac{1}{2}.$$

With this, we are able to compute  $P\{X_0 = i\}$  for all  $i \in \mathcal{M}$  and use these probabilities to find (1).

Lastly, in order to compute (1), we need to compute  $P_{ij}^3$ . Note that the transition matrix gives the probability of transitioning from state  $i$  to state  $j$  i.e.  $\mathbf{P} = (P_{ij})$ . Let  $\mathbf{P}^{(n)}$  be the matrix of  $n$ -step transition probabilities  $P_{ij}^n$ . By the Chapman-Kolmogorov equations, we have that  $\mathbf{P}^{(n)} = \mathbf{P}^n$  so that the  $n$ -step transition probability matrix can be found through multiplication of the transition matrix  $\mathbf{P}$ . Thus,

$$\mathbf{P}^{(3)} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}^3 = \begin{bmatrix} \frac{13}{36} & \frac{11}{54} & \frac{47}{108} \\ \frac{4}{9} & \frac{4}{27} & \frac{11}{27} \\ \frac{5}{12} & \frac{2}{9} & \frac{13}{36} \end{bmatrix} \quad (2)$$

and  $P_{ij}^3$  is the  $ij$ -th entry of  $\mathbf{P}^{(3)}$ .

Using the transition matrix (2) and equation (1), we thus have that

$$\begin{aligned} P\{X_3 = 0\} &= \sum_{i \in \mathcal{M}} P_{i0}^3 P\{X_0 = i\} \\ &= \frac{13}{36} \cdot \frac{1}{4} + \frac{4}{9} \cdot \frac{1}{4} + \frac{5}{12} \cdot \frac{1}{2} \\ &= \frac{11}{36} \end{aligned}$$

$$\begin{aligned} P\{X_3 = 1\} &= \sum_{i \in \mathcal{M}} P_{i1}^3 P\{X_0 = i\} \\ &= \frac{11}{54} \cdot \frac{1}{4} + \frac{4}{27} \cdot \frac{1}{4} + \frac{2}{9} \cdot \frac{1}{2} \\ &= \frac{21}{216} \end{aligned}$$

$$\begin{aligned} P\{X_3 = 2\} &= \sum_{i \in \mathcal{M}} P_{i2}^3 P\{X_0 = i\} \\ &= \frac{47}{108} \cdot \frac{1}{4} + \frac{11}{27} \cdot \frac{1}{4} + \frac{13}{36} \cdot \frac{1}{2} \\ &= \frac{65}{108}. \end{aligned}$$

Therefore,

$$\begin{aligned} E[X_3] &= \sum_{j=0}^2 j P\{X_3 = j\} \\ &= P\{X_3 = 1\} + 2P\{X_3 = 2\} \\ &= \frac{21}{216} + 2 \cdot \frac{65}{108} = \frac{281}{216}. \end{aligned}$$

□

**Problem 4.6.** Let the transition probability matrix of a two-state Markov chain be given, as in Example 4.2, by

$$\mathbf{P} = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}.$$

Show by mathematical induction that

$$\mathbf{P}^{(n)} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}(2p-1)^n & \frac{1}{2} - \frac{1}{2}(2p-1)^n \\ \frac{1}{2} - \frac{1}{2}(2p-1)^n & \frac{1}{2} + \frac{1}{2}(2p-1)^n \end{bmatrix}.$$

*Solution.*

□

**Problem 4.8.** Suppose that coin 1 has probability 0.7 of coming up heads and coin 2 has probability 0.6 of coming up heads. If the coin flipped today comes up heads, then we select coin 1 to flip tomorrow and if it comes up tails then we select coin 2 to flip tomorrow. If the coin initially flipped is equally likely to be coin 1 or coin 2, then what is the probability that the coin flipped on the third day after the initial flip is coin 1? Suppose that the coin flipped on Monday comes up heads. What is the probability that the coin flipped on Friday of the same week also comes up heads?

*Solution.*

□

**Problem 4.14.** Specify the classes of the following Markov chains and determine whether they are transient or recurrent:

$$P_1 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \quad P_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad P_4 = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

*Solution.*

□

**Problem 4.16.** Show that if state  $i$  is recurrent and state  $i$  does not communicate with state  $j$ , then  $P_{ij} = 0$ . This implies that once a process enters a recurrent class of states it can never leave that class. For this reason, a recurrent class is often referred to as a *closed* class.

*Solution.*

□