Homework Assignment 9

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Problem 1. Verify that the forward Euler scheme (9.29) has first order accuracy on a smooth solution $u = u(x)$ of problem (9.30).

Solution. Suppose we have the problem $Lu = f$, as defined in 9.30 i.e.

$$Lu = \begin{cases} \frac{du}{dx} - G(x,u), \ 0 < x \leq 1 \\ 0 \end{cases} \quad \text{and } f = \begin{cases} 0, \ 0 < x \leq 1 \\ a \end{cases} .$$

The forward Euler scheme $L_h u^{(h)} = f^{(h)}$ is given by

$$L_h u^{(h)} = \begin{cases} \frac{u_{n+1} - u_n}{h} - G(x_n, u_n), \ n = 0, 1, \ldots N-1 \\ u_0 \end{cases} \quad \text{and } f^{(h)} = \begin{cases} 0, \ n = 0, 1, \ldots N-1 \\ a \end{cases} .$$

Let $[u]_h$ denote the discretized solution to $Lu = f$. This scheme has first order accuracy if $\left\| L_h[u]_h - L_h u^{(h)} \right\| \leq Ch$ where C is a constant that does not depend on h.

Note that the Taylor series expansion of $u(x+h)$ centered at x is given by

$$u(x+h) = u(x) + u'(x)h + \frac{u''(\xi)h^2}{2}$$

for $x \leq \xi \leq x + h$. This implies that

$$u'(x) = \frac{u(x+h) - u(x)}{h} - \frac{u''(\xi)h}{2}$$

or that

$$u'(x) - G(x,u) = \frac{u(x+h) - u(x)}{h} - \frac{u''(\xi)h}{2} - G(x,u).$$

As $u'(x) - G(x,u) = 0$ is the exact solution to $Lu = f$, we know that the discretized exact solution is given by

$$u'(x) - G(x,u) = \frac{u(x_{n+1}) - u(x_n)}{h} - \frac{u''(\xi(x_n))h}{2} - G(x_n, u_n) = 0$$

where $\xi(x_n)$ depends on the node x_n. But under the forward Euler scheme, $L_h[u]_h = \frac{u_{n+1} - u_n}{h} - G(x_n, u_n)$ so that

$$u'(x) - G(x,u) = L_h[u]_h - \frac{u''(\xi(x_n))h}{2} = 0$$

i.e.

$$u'(x) - G(x,u) = L_h[u]_h - L_h u^{(h)} = \frac{u''(\xi(x_n))h}{2}$$

since $L_h u^{(h)} = 0$. If $|u''(x)| \leq M$ for $x \in [0,1]$, then the above implies that

$$\left\| L_h[u]_h - L_h u^{(h)} \right\| = \left\| \frac{u''(\xi(x_n))h}{2} \right\| \leq \frac{M}{2} h.$$

As $M/2$ does not depend on h, we have shown $\left\| L_h[u]_h - L_h u^{(h)} \right\| \leq Ch$ where $C = M/2$ and that the forward Euler scheme has first order of accuracy. $\square$

Problem 2. Verify that the Crank-Nicolson scheme (9.33) has second order accuracy on a smooth solution $u = u(x)$ of problem (9.30).

Solution. Suppose we have the problem $Lu = f$, as defined in 9.30 i.e.

$$Lu = \begin{cases} \frac{du}{dx} - G(x,u), \ 0 < x \leq 1 \\ 0 \end{cases} \quad \text{and } f = \begin{cases} 0, \ o < x \leq 1 \\ a \end{cases} .$$

The Crank-Nicolson scheme $L_h u^{(h)} = f^{(h)}$ is given by

$$L_h u^{(h)} = \begin{cases} \frac{u_{n+1} - u_n}{h} - \frac{1}{2}[G(x_n, u_n) + G(x_{n+1}, u_{n+1})], \ n = 0, \ldots N-1 \\ u_0 \end{cases}$$

and

$$f^{(h)} = \begin{cases} 0, \ n = 0, \ldots N-1 \\ a \end{cases} .$$

Let $[u]_h$ denote the discretized solution to $Lu = f$. This scheme has second order accuracy if $\left\| L_h[u]_h - L_h u^{(h)} \right\| \leq Ch^2$ where C is a constant that does not depend on h.

From the problem $Lu = f$, we see that $\frac{du}{dx} = G(x, u(x))$ implies that

$$\begin{aligned} \frac{d^2u}{dx^2} = \frac{d}{dx}\left[G(x,u(x))\right] &= \frac{\partial G(x,u(x))}{\partial x} + \frac{\partial G(x,u(x))}{\partial u}\frac{du}{dx} \\ &= \frac{\partial G(x,u(x))}{\partial x} + \frac{\partial G(x,u(x))}{\partial u} G(x,u(x)) \end{aligned}$$

The Taylor expansion of $u(x+h)$ centered at x is given by

$$u(x+h) = u(x) + u'(x)h + \frac{u''(x)h^2}{2} + \frac{u^{(3)}(\xi_1)h^3}{6}$$

for $x \leq \xi_1 \leq x + h$. This implies that

$$u'(x) - G(x,u(x)) = -G(x,u(x)) + \frac{u(x+h) - u(x)}{h} - \frac{u''(x)h}{2} - \frac{u^{(3)}(\xi_1)h^2}{6} = 0$$

Since $u''(x) = \frac{\partial G(x,u(x))}{\partial x} + \frac{\partial G(x,u(x))}{\partial u} G(x, u(x))$ from our earlier calculation, we have that

$$\begin{aligned} u'(x) - G(x, u(x)) &= \frac{u(x+h) - u(x)}{h} - \left[G(x, u(x)) + \frac{h}{2} \left(\frac{\partial G}{\partial x} + \frac{\partial G}{\partial u} G(x, u(x)) \right) \right] \\ &= \frac{u^{(3)}(\xi_1) h^2}{6}. \end{aligned} \tag{1}$$

We now wish to show that we can replace the expression $\frac{\partial G(x,u(x))}{\partial x} + \frac{\partial G(x,u(x))}{\partial u} G(x, u(x))$ with $\frac{1}{2}[G(x, u(x)) + G(x + h, u(x) + hG(x, u(x)))]$ by expressing the Taylor expansion of $G(x + h, u(x) + hG(x, u(x))) = G(x_1, y_1)$ centered at $(x, u(x))$. This Taylor expansion is given by

$$\begin{aligned} G(x_1, y_1) = G(x, u(x)) &+ h \left[\frac{\partial G(x, u(x))}{\partial x} + \frac{\partial G(x, u(x))}{\partial u} G(x, u(x)) \right] \\ &+ h^2 \frac{1}{2} \left[\frac{\partial^2 G(x, u(x))}{\partial x^2} + 2 \frac{\partial^2 G(x, u(x))}{\partial x \partial u} G(x, u(x)) + \frac{\partial^2 G(x, u(x))}{\partial u^2} G(x, u(x))^2 \right] \Bigg|_{x=\xi_2} \\ &= G(x, u(x)) + h \left[\frac{\partial G(x, u(x))}{\partial x} + \frac{\partial G(x, u(x))}{\partial u} G(x, u(x)) \right] + Rh^2 \end{aligned}$$

where $x \le \xi_2 \le x+h$ and $u(x) \le u(\xi_2) \le u(x)+hG(x, u(x))$ and R is the additional constant of the second order term. From the above identity we can see that

$$\frac{1}{2} [G(x, u) + G(x + h, u + hG(x, u))] = G(x, u) + \frac{h}{2} \left[\frac{\partial G(x, u)}{\partial x} + \frac{\partial G(x, u)}{\partial u} G(x, u) \right] + \frac{R}{2} h^2$$

where we have replaced $u(x)$ with u to shorten the expression. Note that from this identity it is clear that

$$G(x, u) + \frac{h}{2} \left(\frac{\partial G(x, u)}{\partial x} + \frac{\partial G(x, u)}{\partial u} G(x, u) \right) = \frac{1}{2} [G(x, u) + G(x + h, u + hG(x, u))] - \frac{R}{2} h^2 \tag{2}$$

and replacing (2) in (1) yields the exact solution to the problem $Lu = f$ as

$$\begin{aligned} u'(x) - G(x, u(x)) &= \frac{u(x+h) - u(x)}{h} - \frac{1}{2} [G(x, u) + G(x + h, u + hG(x, u))] \\ &= \left[\frac{u^{(3)}(\xi_1)}{6} + \frac{R}{2} \right] h^2. \end{aligned} \tag{3}$$

If $[u]_h$ is the discretized solution of the problem $Lu = f$, then discretizing the exact solution shows that for $x + h = x_{n+1}$ and $G(x + h, u(x) + hG(x, u(x))) = G(x_{n+1}, u_n + hG(x_n, u_n)) = G(x_{n+1}, u_{n+1})$ we have from (3) that

$$\begin{aligned} u'(x) - G(x, u(x)) &= \frac{u_{n+1} - u_n}{h} - \frac{1}{2} \left[\frac{1}{2} [G(x_n, u_n) + G(x_{n+1}, u_{n+1})] \right] \\ &= L_h[u]_h - L_h u^{(h)} = \left[\frac{u^{(3)}(\xi_1)}{6} + \frac{R}{2} \right] h^2. \end{aligned}$$

since $L_h u^{(h)} = 0$. Assuming all second order partials and mixed partials of $G(x, u(x))$ are bounded and that the third derivative of our function $u(x)$ is bounded, it is clear that $\left\|L_h[u]_h - L_h u^{(h)}\right\| \leq Ch^2$ where C does not depend on h showing that this scheme has second order of accuracy. □

Problem 3. Create a difference scheme that is not consistent.

Solution. Suppose we have the Cauchy problem $Lu = f$, as defined in 9.30 i.e.

$$Lu = \begin{cases} \frac{du}{dx} - G(x,u), \ 0 < x \leq 1 \\ 0 \end{cases} \quad \text{and } f = \begin{cases} 0, \ 0 < x \leq 1 \\ a \end{cases}.$$

Define a variant of the forward Euler scheme $L_h u^{(h)} = f^{(h)}$ as folows

$$L_h u^{(h)} = \begin{cases} \frac{u_{n+1}-u_n}{h} - G(x_n, u_n) + 1, \ n = 0, 1, \dots N-1 \\ u_0 \end{cases} \quad \text{and } f^{(h)} = \begin{cases} 0, \ n = 0, 1, \dots N-1 \\ a \end{cases}.$$

Then it is clear that this scheme is inconsistent as the the residual will always be constant and never vanish regardless of the grid we choose. To see this, look at the Taylor expansion of $u(x+h)$ centered at x:

$$u(x+h) = u(x) + u'(x)h + \frac{u''(\xi)h^2}{2}$$

for $x \leq \xi \leq x + h$, which implies that

$$u'(x) - G(x,u) = \frac{u(x+h) - u(x)}{h} - G(x,u) - \frac{u''(\xi)h}{2} = 0.$$

From this exact solution it is clear that for our discretized solution of the problem $Lu = f$ we have that

$$u'(x) - G(x,u) = L_h[u]_h - 1 = \frac{u''(\xi)h}{2}$$

which implies that

$$u'(x) - G(x,u) = L_h[u]_h - L_h u^{(h)} = 1 + \frac{u''(\xi)h}{2}.$$

From the above identity it is clear that $\left\|L_h[u]_h - L_h u^{(h)}\right\|$ does not vanish as $h \to 0$ showing that this scheme cannot be consistent. □

Problem 4. Prove that the scheme

$$4\frac{u_{n+1} - u_{n-1}}{2h} - 3\frac{u_{n+1} - u_n}{h} + u_n = 0, \quad n = 1, 2, \dots, N-1$$

with initial conditions $u_0 = 1$ and $u_1 = e^{-h}$ is consistent for the problem

$$\frac{du}{dx} + u = 0, \quad 0 \leq x \leq 1$$

with initial condition $u(0) = 1$.

Solution. If $[u]_h$ is the discretized solution to the problem $Lu = f$ as defined above, then the scheme $L_h u^{(h)} = f^{(h)}$ is consistent if $\left\| L_h[u]_h - L_h u^{(h)} \right\| \to 0$ as $h \to 0$.

Note that the Taylor series expansions of $u(x+h)$ and $u(x-h)$ centered at x are given by

$$\begin{aligned} u(x+h) &= u(x) + u'(x)h + \frac{u''(\xi_1)h^2}{2} \\ u(x-h) &= u(x) - u'(x)h + \frac{u''(\xi_2)h^2}{2} \end{aligned}$$

for $x \le \xi_1 \le x+h$ and $x-h \le \xi_2 \le x$. From these expansions we can see that

$$u'(x) = \frac{u(x+h) - u(x-h)}{2h} - \frac{1}{4}h(u''(\xi_1) - u''(\xi_2))$$

and

$$u'(x) = \frac{u(x+h) - u(x)}{h} - \frac{1}{2}hu''(\xi_3).$$

This shows that

$$u'(x) + u(x) = 4\frac{u(x+h) - u(x-h)}{2h} - 3\frac{u(x+h) - u(x)}{h} + u(x) + h\left(\frac{3}{2}u''(\xi_3) - (u''(\xi_1) - u''(\xi_2))\right)$$

so that if $[u]_h$ is the discretized solution to the problem defined above,

$$\begin{aligned} u'(x) + u(x) &= 4\frac{u(x_{n+1}) - u(x_{n-1})}{2h} - 3\frac{u(x_{n+1}) - u(x_n)}{h} + u(x_n) + h\left(\frac{3}{2}u''(\xi_3) - (u''(\xi_1) - u''(\xi_2))\right) \\ &= L_h[u]_h + h\left(\frac{3}{2}u''(\xi_3) - (u''(\xi_1) - u''(\xi_2))\right) = 0. \end{aligned}$$

Combining the above and the fact that $L_h u^{(h)} = 0$, we see that

$$\left\| L_h[u]_h - L_h u^{(h)} \right\| = h\left\| (u''(\xi_1) - u''(\xi_2)) - \frac{3}{2}u''(\xi_3) \right\|.$$

If $|u''(x)| \le M$, then $0 \le \left\| L_h[u]_h - L_h u^{(h)} \right\| \le h\left(\frac{7}{2}M\right)$ and it is then clear that $\left\| L_h[u]_h - L_h u^{(h)} \right\| \to 0$ as $h \to 0$ showing the consistency of the scheme. □

Problem 5. Prove that the scheme

$$4\frac{u_{n+1} - u_{n-1}}{2h} - 3\frac{u_{n+1} - u_n}{h} + u_n = 0, \quad n = 1, 2, \ldots, N-1$$

with initial conditions $u_0 = 1$ and $u_1 = e^{-h}$ is divergent for the problem

$$\frac{du}{dx} + u = 0, \quad 0 \le x \le 1$$

with initial condition $u(0) = 1$.

Solution. If $[u]_h$ is the discretized solution to the problem $Lu = f$ as defined above, then the scheme $L_h u^{(h)} = f^{(h)}$ is divergent if $\left\| [u]_h - u^{(h)} \right\|$ does not approach 0 as $h \to 0$.

The exact solution to the problem $Lu = f$ with the initial condition $u(0) = 1$ is $u(x) = e^{-x}$. Hence, $[u]_h = [e^{-x_0}, e^{-x_1}, \ldots, e^{-x_n}, \ldots, e^{-x_N}] = [e^0, e^{-h}, \ldots, e^{-nh}, \ldots, e^{-1}]$. The solution to the difference scheme $L_h u^{(h)} = f^{(h)}$ given by $u^{(h)}$ and can be found by finding the explicit solution to the difference equation defined in the scheme.

Note that this is a second order difference equation that can be rewritten as

$$-u_{n+1} + (3+h)u_n - 2u_{n-1} = 0.$$

The characteristic equation of this difference equation is given by $-m^2 + (3+h)m - 2 = 0$. As this characteristic equation has distinct real roots, the general solution to the difference equation is $u_n = c_1 m_1^n + c_2 m_2^n$ where $m_1 = \frac{1}{2}(-\sqrt{h^2+6h+1} + h + 3)$ and $m_2 = \frac{1}{2}(\sqrt{h^2+6h+1} + h + 3)$ are the roots of the characteristic equation. Choosing the constants so that the initial conditions are satisfied gives us the general solution as

$$\begin{aligned} u_n^{(h)} = u_0 &\left[\frac{m_2(h)}{m_2(h) - m_1(h)} m_1(h)^n - \frac{m_1(h)}{m_2(h) - m_1(h)} m_2(h)^n \right] \\ + u_1 &\left[-\frac{1}{m_2(h) - m_1(h)} m_1(h)^n + \frac{1}{m_2(h) - m_1(h)} m_2(h)^n \right]. \end{aligned}$$

Combining this general solution to the scheme and the exact solution to the problem we can clearly see that $\left\| [u]_h - u^{(h)} \right\|$ does not approach 0 as $h \to 0$ and that the scheme is divergent. □