Homework Assignment 4

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Problem 1. Find the first three terms in the asymptotic expansions of $x \to 0^+$ of the following integrals:

$$\int_x^1 \cos(xt)dt, \qquad \int_0^{1/x} e^{-t^2} dt.$$

Solution. If the function $f(t,x)$ possesses the asymptotic expansion

$$f(t,x) \sim \sum_{n=0}^{\infty} f_n(t)(x-x_0)^{\alpha n} \qquad \text{as } x \to x_0$$

for some $\alpha > 0$, uniformly for $a \le t \le b$, then the asymptotic expansion of the integral

$$I(x) = \int_a^b f(t,x)dt$$

as $x \to x_0$ is given by

$$I(x) \sim \sum_{n=0}^{\infty} (x-x_0)^{\alpha n} \int_a^b f_n(t)dt \qquad \text{as } x \to x_0.$$

We begin with finding the first three terms of the asymptotic expansion of the integral

$$I_1(x) = \int_x^1 \cos(xt)dt \qquad \text{as } x \to 0^+.$$

Note that $f(t,x) = \cos(xt)$ has the following asymptotic expansion as $x \to 0^+$:

$$f(t,x) = \cos(xt) \sim 1 - \frac{t^2x^2}{2} + \frac{t^4x^4}{24}.$$

This expansion converges uniformly for all $x \le t \le 1$ as $x \to 0^+$. Therefore, we have that the first three terms of the asymptotic expansion of $I_1(x)$ as $x \to 0^+$ are given by

$$I_1(x) \sim \int_x^1 dt - \frac{x^2}{2}\int_x^1 t^2 dt + \frac{x^4}{24}\int_x^1 t^4 dt = (1-x) - \frac{x^2}{2}\left[\frac{1-x^3}{3}\right] + \frac{x^4}{24}\left[\frac{1-x^5}{5}\right].$$

Similar to what was shown above, we have that if

$$f(t,x) \sim f_0(t) \qquad \text{as } x \to x_0$$

uniformly for $a \leq t \leq b$, then the asymptotic expansion of the integral is given by

$$I(x) = \int_a^b f(t,x)dt \sim \int_a^b f_0(t)dt \qquad \text{as } x \to x_0.$$

Let us continue by finding the first three terms of the asymptotic expansion of the integral

$$I_2(x) = \int_0^{1/x} e^{-t^2} dt \qquad \text{as } x \to 0^+.$$

Note that $f(t,x) = e^{-t^2}$ has the following asymptotic expansion as $x \to 0^+$:

$$f(t,x) = e^{-t^2} \sim 1 - t^2 + \frac{t^4}{2}.$$

This expansion converges uniformly for all finite points, so it converges uniformly for $0 \leq t \leq 1/x$ as $x \to 0^+$. Therefore, we may integrate the expansion term by term and we have that the first three terms of the asymptotic expansion of $I_2(x)$ as $x \to 0^+$ are given by

$$I_2(x) \sim \int_0^{1/x} dt - \int_0^{1/x} t^2 dt + \frac{1}{2}\int_0^{1/x} t^4 dt = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{10x^5}.$$

$\square$

Problem 2. Find the full asymptotic behavior as $x \to 0^+$ of the following integral:

$$\int_0^1 \frac{e^{-t}}{1 + x^2 t^3} dt$$

Solution. Note that the function $f(t,x) = e^{-t}/(1 + x^2 t^3)$ has the asymptotic expansion

$$f(t,x) = \frac{e^{-t}}{1 + x^2 t^3} \sim e^{-t} \sum_{n=0}^{\infty} \left[(-1)^n t^{3n}\right] x^{2n} \qquad \text{as } x \to 0^+.$$

Note that this asymptotic expansion converges uniformly for $0 \le x \le t < 1 - \epsilon$ for all $\epsilon > 0$. To see this, we note that for $0 < m < n$, we have that

$$\left| \sum_{k=m+1}^{n} (-1)^k (x^2 t^3)^k \right| < \sum_{k=m+1}^{n} (1 - \epsilon)^{5k}.$$

Since $(1-\epsilon)^5 < 1$, we have that its geometric series converges and we can make it as small as we wish. Thus, by the Cauchy criterion we have uniform convergence for $0 \le x \le t < 1 - \epsilon$ for all $\epsilon > 0$.

Per the discussion in Problem 1, using this uniformly convergent asymptotic expansion, we have that as $x \to 0^+$

$$\int_0^1 \frac{e^{-t}}{1 + x^2 t^3} dt \sim \sum_{n=0}^{\infty} (-1)^n x^{2n} \int_0^1 e^{-t} t^{3n} dt = \sum_{n=0}^{\infty} (-1)^n x^{2n} \left[\Gamma(3n+1) - \Gamma(3n+1, 1)\right]$$

where $\Gamma(a,k) = \int_k^\infty t^{a-1} e^{-t} dt$. □

Problem 3. Find the full asymptotic expansion of $\int_0^x \mathrm{Bi}(t)dt$ as $x \to +\infty$.

Solution. Note that for $x \to +\infty$, the integral above can be written as

$$\int_0^x \mathrm{Bi}(t)dt = \int_0^1 \mathrm{Bi}(t)dt + \int_1^x \mathrm{Bi}(t)dt \tag{1}$$

Thus, the asymptotic expansion of the integral depends only on the second integral on the right. The Airy function $\mathrm{Bi}(t)$ satisfies the differential equation $y'' = ty$. Using this differential equation and integrating the integral on the right by parts we see that

$$\begin{aligned}\int_1^x \mathrm{Bi}(t)dt &= \int_1^x \frac{1}{t}\mathrm{Bi}''(t)dt \\ &= \frac{1}{x}\mathrm{Bi}'(x) - \mathrm{Bi}'(1) + \int_1^x \frac{1}{t^2}\mathrm{Bi}'(t)dt.\end{aligned}$$

Note that it is clear that as $x \to +\infty$ the following relations hold

$$\begin{gathered}\mathrm{Bi}'(1) \ll \frac{1}{x}\mathrm{Bi}'(x) \\ \int_0^1 \mathrm{Bi}(t)dt \ll \int_0^x \mathrm{Bi}(t)dt.\end{gathered}$$

Thus, from equation (1) and the above relations, we have that as $x \to +\infty$

$$\int_0^x \mathrm{Bi}(t)dt \sim \frac{1}{x}\mathrm{Bi}'(x) + \int_1^x \frac{1}{t^2}\mathrm{Bi}'(t)dt. \tag{2}$$

However, upon further investigation we see that as $x \to +\infty$

$$\int_1^x \frac{1}{t^2}\mathrm{Bi}'(t)dt \ll \frac{1}{x}\mathrm{Bi}'(x). \tag{3}$$

To see that this is true, we integrate the integral on the left by parts which yields

$$f(x) = \int_1^x \frac{1}{t^2}\mathrm{Bi}'(t)dt = x^{-2}\mathrm{Bi}(x) - \mathrm{Bi}(1) + 2\int_1^x t^{-3}\mathrm{Bi}(t)dt.$$

In comparing the function $f(x)$ with the function $g(x) = x^{-1}\mathrm{Bi}'(x)$ as $x \to +\infty$, we see that

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \frac{+\infty}{+\infty}$$

an indeterminate form. Thus, applying L'Hôpital's rule, we see that derivatives of $f(x)$ and $g(x)$ are

$$\begin{aligned}f'(x) &= -2x^{-3}\mathrm{Bi}(x) + x^{-2}\mathrm{Bi}'(x) + 2\left[x^{-3}\mathrm{Bi}(x) - \mathrm{Bi}(1)\right] \\ &= x^{-2}\mathrm{Bi}'(x) - 2\mathrm{Bi}(1) \\ g'(x) &= -x^{-2}\mathrm{Bi}'(x) + x^{-1}\mathrm{Bi}''(x)\end{aligned}$$

and that

$$\lim_{x\to+\infty}\frac{f'(x)}{g'(x)} = \frac{x^{-2}\mathrm{Bi}'(x) - 2\mathrm{Bi}(1)}{-x^{-2}\mathrm{Bi}'(x) + x^{-1}\mathrm{Bi}''(x)} = \frac{1}{1 + \frac{x^{-1}\mathrm{Bi}''(x)}{x^{-2}\mathrm{Bi}'(x)}} = 0$$

Therefore, we must have that relation (3) is true and that relation (2) reduces to

$$\int_0^x \mathrm{Bi}(t)dt \sim \frac{1}{x}\mathrm{Bi}'(x) \qquad (x \to +\infty).$$

Note that the asymptotic expansion of $\mathrm{Bi}(x)$ as $x \to +\infty$ is given by

$$\mathrm{Bi}(x) \sim \pi^{-1/2}x^{-1/4}\exp\left(\frac{2x^{3/2}}{3}\right)\sum_{n=0}^{\infty} c_n x^{-3n/2}$$

where

$$c_n = \frac{1}{2\pi}\left(\frac{3}{4}\right)^n \frac{\Gamma(n+5/6)\Gamma(n+1/6)}{n!}.$$

Thus, we readily see that as $x \to +\infty$,

$$\begin{aligned}\mathrm{Bi}'(x) &\sim \pi^{-1/2}\exp\left(\frac{2x^{3/2}}{3}\right)\left[\left(x^{3/2} - \frac{1}{4}\right)x^{-5/4}\sum_{n=0}^{\infty} c_n x^{-3n/2} + x^{-1/4}\sum_{n=0}^{\infty}\frac{-3nc_n}{2}x^{-3n/2-1}\right] \\ &= \pi^{-1/2}\exp\left(\frac{2x^{3/2}}{3}\right)\left(x^{3/2} + \frac{3}{4}\right)\sum_{n=0}^{\infty}\left(1 - \frac{3n}{2}\right)c_n x^{-3n/2-5/4}.\end{aligned}$$

Therefore, we can readily see that the full asymptotic behavior as $x \to +\infty$ of the integral of the problem is given by

$$\int_0^x \mathrm{Bi}(t)dt \sim \frac{\mathrm{Bi}'(x)}{x} \sim \pi^{-1/2}\exp\left(\frac{2x^{3/2}}{3}\right)\left(x^{3/2} + \frac{3}{4}\right)\sum_{n=0}^{\infty}\left(1 - \frac{3n}{2}\right)c_n x^{-3n/2-9/4}.$$

$\square$

Problem 4. Find the first five terms in the asymptotic expansion as $x \to +\infty$ of the integral

$$\int_0^{\pi/4} e^{-xt^2} \sqrt{\tan t}\, dt$$

a. by using a suitable change of variables and then applying Watson's lemma.

b. by applying Laplace's method directly to the given integral.

Solution. $\square$

Problem 5. Use Laplace's method of moving maxima to obtain the first two terms in the asymptotic expansion as $x \to +\infty$ of the integral

$$\int_0^\infty \exp\left[-t - \frac{x}{\sqrt{t}}\right] dt. \tag{4}$$

Solution. □

Problem 6. Let $f(x,t)$ be differentiable in x and continuous in (x,t) on $I \times J$, where I and J are intervals, and suppose that there exist functions $g(t)$ and $g_1(t)$ that are integrable on J such that for all $(x,t) \in I \times J$ we have that

$$|f(x,t)| \leq g(t) \quad \text{and} \quad |\partial_x f(x,t)| \leq g_1(t).$$

Then

$$\frac{d}{dx}\int_J f(x,t)dt = \int_J \partial_x f(x,t)dt.$$

a. Let $0 < a < b < \infty$. Use the above theorem to show that if $x \in (a,b)$, then

$$\frac{d^3}{dx^3}\int_0^\infty \exp\left[-t - \frac{x}{\sqrt{t}}\right]dt = -\int_0^\infty t^{-3/2}\exp\left[-t - \frac{x}{\sqrt{t}}\right]dt.$$

b. Use integration by parts to show that

$$\int_0^\infty \exp\left[-t - \frac{x}{\sqrt{t}}\right]dt = \frac{x}{2}\int_0^\infty t^{-3/2}\exp\left[-t - \frac{x}{\sqrt{t}}\right]dt.$$

c. Combine parts (a) and (b) to prove that integral (4) is a solution of the differential equation $xy''' + 2y = 0$ that also satisfies the initial condition $y(0) = 1$. Then use integration by parts to give an easy direct proof that the integral also satisfies the condition $y(+\infty) = 0$.

Solution. □

Problem 7. a. Find the leading behavior as $x \to +\infty$ of Laplace integrals of the form

$$\int_a^b (t-a)^\alpha g(t) e^{x\phi(t)} dt$$

where $\phi(t)$ has a maximum at $t = a$, $g(a) = 1$. Suppose further that $\alpha > -1$ and $\phi'(a) < 0$.

b. Repeat the analysis of part (a) when $\alpha > -1$ and $\phi'(a) = \phi''(a) = \cdots = \phi^{(p-1)}(a) = 0$ and $\phi^{(p)}(a) < 0$.

Solution. $\square$