Homework Assignment 2

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Problem 1. Convert the following linear programming problem to *standard form*:

$$\begin{array}{rll} \text{maximize} & 2x_1 + x_2 & \\ \text{subject to} & 0 \leq x_1 & \leq 2 \\ & x_1 + x_2 & \leq 3 \\ & x_1 + 2x_2 & \leq 5 \\ & x_2 \geq 0 & \end{array}$$

Solution. In order to convert this linear programming problem into standard form, we must transform the objective from *maximize* to *minimize* and the constraints must be transformed from linear inequalities into linear equations.

Our first step will be to rewrite the objective function as a minimization problem and write each constraint as a linear inequality as so:

$$\begin{array}{rll} \text{minimize} & -2x_1 - x_2 & \\ \text{subject to} & x_1 & \leq 2 \\ & x_1 + x_2 & \leq 3 \\ & x_1 + 2x_2 & \leq 5 \\ & x_1 \geq 0, x_2 \geq 0 & \end{array}$$

We can then introduce three slack variables x_3, x_4, x_5 to turn the linear inequalities into linear equations:

$$\begin{array}{rllllll} \text{minimize} & -2x_1 & -x_2 & & & & \\ \text{subject to} & x_1 & +x_2 & +x_3 & & & = 2 \\ & x_1 & +x_2 & & +x_4 & & = 3 \\ & x_1 & +2x_2 & & & +x_5 & = 5 \\ & x_1 \geq 0, & x_2 \geq 0, & x_3 \geq 0, & x_4 \geq 0, & x_5 \geq 0 & \end{array}$$

As the above linear programming problem is written as

$$\begin{array}{rl} \text{minimize} & \boldsymbol{c}^\intercal \boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{array}$$

where

$$\boldsymbol{c}^\mathsf{T} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}^\mathsf{T}, \quad A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

the linear programming problem is in standard form and we are done. $\square$

Problem 2. Solve the system $Ax = b$ where

$$A = \begin{bmatrix} 2 & -1 & 2 & -1 & 3 \\ 1 & 2 & 3 & 1 & 0 \\ 1 & 0 & -2 & 0 & -5 \end{bmatrix}, \quad b = \begin{bmatrix} 14 \\ 5 \\ -10 \end{bmatrix}.$$

If possible, generate a non-basic feasible solution of the system from which you derive next a basic feasible one.

Solution. $\square$

Problem 3. Does every linear programming problem in standard form have a nonempty feasible set? If "yes", provide a proof. If "no", provide a counter-example.

Does every linear programming problem in standard form (assuming a nonempty feasible set) have an optimal solution? If "yes", provide a proof. If "no", provide a counter-example.

Solution. □

Problem 4. a. Solve the following linear program graphically:

$$\begin{array}{ll} \text{maximize} & 2x_1 + 5x_2 \\ \text{subject to} & 0 \leq x_1 \leq 4 \\ & 0 \leq x_2 \leq 6 \\ & x_1 + x_2 \leq 8 \end{array}$$

b. Solve the linear program in (b) the same way Example 15.15 was solved in class. Compute only the vertices that lead to the optimal vertex found at (a).

Solution. □