Homework Assignment 4

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Problem 3.1. Find the Laplace transforms of the following functions:

b. $f(t) = (1-2t)e^{-2t}$

c. $f(t) = t\cos at$

d. $f(t) = t^{3/2}$

g. $f(t) = (t-3)^2 H(t-3)$

Solution. Recall that the Laplace transform of the function $f(t)$ defined for $t > 0$ is given by

$$\mathscr{L}\left\{f(t)\right\} = \bar{f}(s) = \int_0^\infty f(t)e^{-st}dt. \tag{1}$$

b. Let $g(t) = 1-2t$. Then $f(t) = (1-2t)e^{-2t} = g(t)e^{-2t}$. From the definition of the Laplace transform, we have that

$$\begin{aligned}\mathscr{L}\left\{g(t)\right\} = \bar{g}(s) &= \int_0^\infty (1-2t)e^{-st}dt \\ &= \int_0^\infty t^0 e^{-st}dt - 2\int_0^\infty t^1 e^{-st}dt \\ &= \mathscr{L}\left\{t^0\right\} - 2\mathscr{L}\left\{t^1\right\}.\end{aligned}$$

From a previous theorem, we know for $n \in \mathbb{N}$ that

$$\mathscr{L}\left\{t^n\right\} = \int_0^\infty t^n e^{-st}dt = \frac{n!}{s^{n+1}}.$$

Thus,

$$\bar{g}(s) = \mathscr{L}\left\{t^0\right\} - 2\mathscr{L}\left\{t^1\right\} = \frac{1}{s} - \frac{2}{s^2} = \frac{s-2}{s^2}.$$

From Heaviside's First Shifting Theorem, we know that for $\bar{g}(s) = \mathscr{L}\left\{g(t)\right\}$ that

$$\mathscr{L}\left\{g(t)e^{-at}\right\} = \bar{g}(s+a).$$

Therefore, the Laplace transform of $f(t) = (1-2t)e^{-2t} = g(t)e^{-2t}$ is

$$\mathscr{L}\left\{f(t)\right\} = \mathscr{L}\left\{g(t)e^{-2t}\right\} = \bar{g}(s+2) = \frac{s}{(s+2)^2}.$$

c. From the definition of the complex exponential, we have that $f(t) = t\cos at = \frac{t}{2}\left(e^{-iat} + e^{iat}\right)$. From the definition of the Laplace transform, we have that

$$\begin{aligned}\mathscr{L}\left\{f(t)\right\} = \bar{f}(s) &= \int_0^\infty \frac{t}{2}\left(e^{-iat} + e^{iat}\right)e^{-st}dt \\ &= \frac{1}{2}\left[\int_0^\infty te^{-(s+ia)t}dt + \int_0^\infty te^{-(s-ia)t}dt\right].\end{aligned}$$

We readily see by integrating by parts using $u = t$ and $dv = e^{-(s\pm ia)t}dt$ that

$$\begin{aligned}\int_0^\infty te^{-(s\pm ia)t}dt &= -\frac{t}{s\pm ia}e^{-(s\pm ia)t}\Big|_0^\infty + \frac{1}{s\pm ia}\int_0^\infty e^{-(s\pm ia)t}dt \\ &= -\frac{1}{(s\pm ia)^2}e^{-(s\pm ia)t}\Big|_0^\infty \\ &= \frac{1}{(s\pm ia)^2}.\end{aligned}$$

Therefore, the Laplace transform of $f(t)$ is given by

$$\begin{aligned}\mathscr{L}\left\{f(t)\right\} = \bar{f}(s) &= \frac{1}{2}\left[\int_0^\infty te^{-(s+ia)t}dt + \int_0^\infty te^{-(s-ia)t}dt\right] \\ &= \frac{1}{2}\left[\frac{1}{(s+ia)^2} + \frac{1}{(s-ia)^2}\right] \\ &= \frac{s^2 - a^2}{(s+ia)^2(s-ia)^2} \\ &= \frac{s^2-a^2}{(s^2+a^2)^2}.\end{aligned}$$

d. By definition, the Laplace transform of $f(t)$ is given by

$$\mathscr{L}\left\{f(t)\right\} = \bar{f}(s) = \int_0^\infty t^{3/2}e^{-st}dt.$$

Let $u = st$, then $du/s = dt$ and

$$\begin{aligned}\mathscr{L}\left\{f(t)\right\} = \bar{f}(s) &= \frac{1}{s}\int_0^\infty \left(\frac{u}{s}\right)^{3/2} e^{-u}dt \\ &= \frac{1}{s^{5/2}}\int_0^\infty u^{3/2}e^{-u}dt.\end{aligned}$$

Recall that the definition of the Gamma function is given by

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}du.$$

Therefore, the Laplace transform of $f(t) = t^{3/2}$ is

$$\begin{aligned}\mathscr{L}\{f(t)\} = \bar{f}(s) &= \frac{1}{s^{5/2}} \int_0^\infty u^{5/2-1} e^{-u} dt \\ &= \frac{\Gamma\left(\frac{5}{2}\right)}{s^{5/2}}.\end{aligned}$$

g. Let $g(t) = t^2$ and suppose that $\mathscr{L}\{g(t)\} = \bar{g}(s)$. Then Heaviside's Second Shifting Theorem shows that

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{g(t-3)H(t-3)\} = e^{-3s}\bar{g}(s).$$

As shown previously, we know for $n \in \mathbb{N}$ that

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}.$$

Therefore, the Laplace transform of $f(t)$ is

$$\mathscr{L}\{f(t)\} = \bar{f}(s) = e^{-3s}\bar{g}(s) = \frac{2e^{-3s}}{s^3}.$$

$\square$

Problem 3.3. The following is a result relating the Laplace transform of a function's derivative to the Laplace transform of that function:

$$\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - f(0). \tag{2}$$

Use the result to find

a. $\mathscr{L}\{\cos at\}$,

b. $\mathscr{L}\{\sin at\}$.

Solution. a. Let $f(t) = \cos at$. Then $f'(t) = -a\sin at$ and from (2) we have

$$-a\mathscr{L}\{\sin at\} = s\mathscr{L}\{\cos at\} - 1. \tag{3}$$

Now let $g(t) = \sin at$. Then $g'(t) = a\cos at$ and applying (2) to $g(t)$ yields

$$a\mathscr{L}\{\cos at\} = s\mathscr{L}\{\sin at\}.$$

Therefore, from (3) we have that

$$-a\left(\frac{a}{s}\mathscr{L}\{\cos at\}\right) = s\mathscr{L}\{\cos at\} - 1$$

which implies that

$$\mathscr{L}\{\cos at\} = \frac{s}{s^2 + a^2}.$$

b. Let $f(t) = \sin at$. Then $f'(t) = a\cos at$ and from (2) we have

$$a\mathscr{L}\{\cos at\} = s\mathscr{L}\{\sin at\}. \tag{4}$$

Now let $g(t) = \cos at$. Then $g'(t) = -a\sin at$ and applying (2) to $g(t)$ yields

$$-a\mathscr{L}\{\sin at\} = s\mathscr{L}\{\cos at\} - 1$$

which implies that

$$\mathscr{L}\{\cos at\} = \frac{1}{s} - \frac{a}{s}\mathscr{L}\{\sin at\}.$$

Therefore, from (4) we have that

$$a\left(\frac{1}{s} - \frac{a}{s}\mathscr{L}\{\sin at\}\right) = s\mathscr{L}\{\sin at\}$$

which implies that

$$\mathscr{L}\{\sin at\} = \frac{a}{s^2 + a^2}.$$

□

Problem 3.6. Show that
$$\mathscr{L}\left\{\int_0^t \frac{f(u)}{u}du\right\} = \frac{1}{s}\int_s^\infty \bar{f}(x)dx.$$

Solution. From the definition of the Laplace transform we see that
$$\mathscr{L}\left\{\int_0^t \frac{f(u)}{u}du\right\} = \int_0^\infty e^{-st}\left[\int_0^t \frac{f(u)}{u}du\right]dt.$$
Interchanging the order of integration from u to t where $0 \le t < \infty$, we see that $u \le t < \infty$ as $0 \le u < \infty$ and
$$\begin{aligned}\mathscr{L}\left\{\int_0^t \frac{f(u)}{u}du\right\} &= \int_0^\infty e^{-st}\left[\int_0^t \frac{f(u)}{u}du\right]dt \\ &= \int_0^\infty \frac{f(u)}{u}\left[\int_u^\infty e^{-st}dt\right]du \\ &= \frac{1}{s}\int_0^\infty \frac{f(u)}{u}e^{-su}du.\end{aligned}$$

We note that $\dfrac{d}{ds}\left[\dfrac{e^{-su}}{u}\right] = -e^{-su}$ so that in particular we have that
$$-\int_s^\infty e^{-su}ds = -\left.\frac{e^{-su}}{u}\right|_s^\infty = -\frac{e^{-su}}{u}$$
or that
$$\int_s^\infty e^{-su}ds = \frac{e^{-su}}{u}.$$
Thus,
$$\begin{aligned}\mathscr{L}\left\{\int_0^t \frac{f(u)}{u}du\right\} &= \frac{1}{s}\int_0^\infty \frac{f(u)}{u}e^{-su}du \\ &= \frac{1}{s}\int_0^\infty f(u)\left[\int_s^\infty e^{-su}ds\right]du.\end{aligned}$$

Interchanging the order of integration yet again from s to u where $s \le s < \infty$ as $0 \le u < \infty$, we see that the integration limits remain unchanged and therefore that
$$\begin{aligned}\mathscr{L}\left\{\int_0^t \frac{f(u)}{u}du\right\} &= \frac{1}{s}\int_0^\infty f(u)\left[\int_s^\infty e^{-su}ds\right]du \\ &= \frac{1}{s}\int_s^\infty\left[\int_0^\infty f(u)e^{-su}du\right]ds \\ &= \frac{1}{s}\int_s^\infty \bar{f}(s)ds \\ &= \frac{1}{s}\int_s^\infty \bar{f}(x)dx,\end{aligned}$$

and we are done.

□

Problem 3.7. Obtain the inverse Laplace transforms of the following functions:

b. $\bar{f}(s) = \dfrac{1}{s^2(s^2+c^2)}$.

Solution. b. Let $\bar{f}(s) = \bar{g}(s)\bar{h}(s)$ where $\bar{g}(s) = \dfrac{1}{s^2}$ and $\bar{h}(s) = \frac{1}{s^2+c^2}$.

Using previous results, we know that

$$g(t) = \mathscr{L}^{-1}\{\bar{g}(s)\} = \mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

and

$$h(t) = \mathscr{L}^{-1}\left\{\bar{h}(s)\right\} = \mathscr{L}^{-1}\left\{\frac{1}{c}\left(\frac{c}{s^2+c^2}\right)\right\} = \frac{\sin ct}{c}.$$

Now, by the Convolution Theorem for the Laplace transform, we have that

$$f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} = \mathscr{L}^{-1}\left\{\bar{g}(s)\bar{h}(s)\right\} = (g * h)(t)$$

where

$$(g * h)(s) = \int_0^t g(t-\tau)h(\tau)d\tau.$$

Therefore,

$$\begin{aligned}
f(t) = (g * h)(t) &= \frac{1}{c}\int_0^t (t-\tau)\sin c\tau d\tau \\
&= \frac{t}{c}\int_0^t \sin c\tau d\tau - \frac{1}{c}\int_0^t \tau \sin c\tau d\tau \\
&= \frac{t}{c}\left[\frac{1}{c} - \frac{\cos ct}{c}\right] - \frac{1}{c}\left[-\frac{t\cos ct}{c} + \frac{\sin ct}{c^2}\right] \\
&= \frac{t}{c^2} - \frac{\sin ct}{c^3}.
\end{aligned}$$

$\square$

Problem 3.8. Use the Convolution Theorem to fin the inverse Laplace transforms of the following functions:

a. $\bar{f}(s) = \dfrac{s^2}{(s^2+a^2)^2}$.

Solution. Let $\bar{f}(s) = \bar{g}(s)^2$ where $\bar{g}(s) = \dfrac{s}{s^2+a^2}$. Then we know that

$$g(t) = \mathscr{L}^{-1}\left\{\bar{g}(s)\right\} = \cos at.$$

From the Convolution Theorem, we then have that

$$f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} = \mathscr{L}^{-1}\left\{\bar{g}(s)\bar{g}(s)\right\} = (g * g)(t).$$

Therefore,

$$\begin{aligned} f(t) = (g*g)(t) &= \int_0^t \cos a(t-\tau)\cos a\tau d\tau \\ &= \cos at \int_0^t \cos^2 a\tau d\tau + \sin at \int_0^t \sin a\tau \cos a\tau d\tau \\ &= \cos at \left[\frac{2at + \sin 2at}{4a}\right] + \sin at \left[\frac{\sin^2 at}{2a}\right] \\ &= \frac{at\cos at + \sin at}{2a}. \end{aligned}$$

□

Problem 3.10. Show that

a. $\mathscr{L}\left\{\frac{1}{t}(\sin at - at\cos at)\right\} = \tan^{-1}\left(\frac{a}{s}\right) - \frac{as}{s^2+a^2}$,

b. $\mathscr{L}\left\{\int_0^t \frac{1}{\tau}(\sin a\tau - a\tau\cos a\tau)d\tau\right\} = \frac{1}{s}\left[\tan^{-1}\left(\frac{a}{s}\right) - \frac{as}{s^2+a^2}\right]$.

Solution. a. If $\bar{f}(s) = \mathscr{L}\{f(t)\}$, then from a previous theorem we have that

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)ds.$$

Thus, we have that

$$\begin{aligned}\mathscr{L}\left\{\frac{1}{t}(\sin at - at\cos at)\right\} &= \int_s^\infty \mathscr{L}\{\sin at - at\cos at\}\,ds \\ &= \int_s^\infty \mathscr{L}\{\sin at\} - a\mathscr{L}\{t\cos at\}\,ds.\end{aligned}$$

From our table of Laplace transforms, we know that

$$\mathscr{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

and

$$\mathscr{L}\{t\cos at\} = \frac{s^2-a^2}{(s^2+a^2)^2}.$$

This implies that

$$\begin{aligned}\mathscr{L}\left\{\frac{1}{t}(\sin at - at\cos at)\right\} &= \int_s^\infty \mathscr{L}\{\sin at\} - a\mathscr{L}\{t\cos at\}\,ds \\ &= \int_s^\infty \frac{a}{s^2+a^2} - a\left(\frac{s^2-a^2}{(s^2+a^2)^2}\right)ds \\ &= \tan^{-1}\left(\frac{s}{a}\right)\Big|_s^\infty + \frac{as}{s^2+a^2}\Big|_s^\infty \\ &= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) - \frac{as}{s^2+a^2} \\ &= \tan^{-1}\left(\frac{a}{s}\right) - \frac{as}{s^2+a^2}.\end{aligned}$$

b. From the theorem regarding the Laplace transform of an integral, if $\bar{f}(s) = \mathscr{L}\{f(t)\}$, then

$$\mathscr{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{\bar{f}(s)}{s}. \tag{5}$$

Suppose that $f(t) = \dfrac{1}{t}(\sin at - at\cos at)$. Then we have shown previously that

$$\bar{f}(s) = \mathscr{L}\left\{f(t)\right\} = \tan^{-1}\left(\frac{a}{s}\right) - \frac{as}{s^2 + a^2}.$$

Therefore, we have by (5) that

$$\begin{aligned}
\mathscr{L}\left\{\int_0^t \frac{1}{\tau}(\sin a\tau - a\tau\cos a\tau)d\tau\right\} &= \mathscr{L}\left\{\int_0^t f(\tau)d\tau\right\} \\
&= \frac{\bar{f}(s)}{s} \\
&= \frac{1}{s}\left[\tan^{-1}\left(\frac{a}{s}\right) - \frac{as}{s^2 + a^2}\right].
\end{aligned}$$

$\square$

Problem 3.12. If $\mathscr{L}\{f(t)\} = \bar{f}(s)$, show that

ii. $\mathscr{L}^{-1}\left\{\dfrac{\bar{f}(s)}{s^2}\right\} = \displaystyle\int_0^t \left[\int_0^{t_1} f(\tau)d\tau\right] dt_1 = \int_0^t (t-\tau)f(\tau)d\tau.$

Solution. ii. Let $\bar{g}(s) = \dfrac{1}{s}$ and $\bar{h}(s) = \frac{\bar{f}(s)}{s}$. From (5) we know that

$$h(t) = \mathscr{L}^{-1}\left\{\bar{h}(s)\right\} = \mathscr{L}^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(\tau)d\tau.$$

It is easy to see that if $\bar{g}(s) = \dfrac{1}{s}$ then $g(t) = 1$. Now, by the Convolution Theorem, we have that

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{\bar{f}(s)}{s^2}\right\} = \mathscr{L}^{-1}\left\{\bar{g}(s)\bar{h}(s)\right\} = (g*h)(t) &= \int_0^t g(t-t_1)h(t_1)dt_1 \\ &= \int_0^t \int_0^{t_1} f(\tau)d\tau dt_1.\end{aligned}$$

Thus,

$$\mathscr{L}^{-1}\left\{\frac{\bar{f}(s)}{s^2}\right\} = \int_0^t \int_0^{t_1} f(\tau)d\tau dt_1. \tag{6}$$

By interchanging the order of integration from τ to t_1, where $0 \le \tau \le t_1$ as $0 \le t_1 \le t$, we see that $\tau \le t_1 \le t$ and $0 \le \tau \le t$ so that

$$\mathscr{L}^{-1}\left\{\frac{\bar{f}(s)}{s^2}\right\} = \int_0^t \int_0^{t_1} f(\tau)d\tau dt_1 = \int_0^t f(\tau)\left[\int_\tau^t dt_1\right] d\tau = \int_0^t (t-\tau)f(\tau)d\tau,$$

and we are done.

□

Problem 3.15. Show that

b. $\mathscr{L}\left\{t^n e^{at}\right\} = \dfrac{n!}{(s-a)^{n+1}}$.

Solution. b. Let $f(t) = t^n$. By Heaviside's First Shifting Theorem, we have that

$$\mathscr{L}\left\{t^n e^{at}\right\} = \mathscr{L}\left\{f(t)e^{at}\right\} = \bar{f}(s-a).$$

As shown previously,

$$\bar{f}(s) = \mathscr{L}\left\{f(t)\right\} = \frac{n!}{s^{n+1}}.$$

Therefore,

$$\mathscr{L}\left\{t^n e^{at}\right\} = \bar{f}(s-a) = \frac{n!}{(s-a)^{n+1}}.$$

□

Problem 3.18. Establish the following result:

a. $\mathscr{L}\left\{\sin^2 at\right\} = \dfrac{2a^2}{s(s^2+4a^2)}$.

Solution. a. Recall that $\cos 2\theta = 1 - 2\sin^2\theta$. Thus,

$$\mathscr{L}\left\{\sin^2 at\right\} = \mathscr{L}\left\{\frac{1}{2}(1-\cos 2at)\right\} = \frac{1}{2}\left[\mathscr{L}\{1\} - \mathscr{L}\{\cos 2at\}\right].$$

From our table of Laplace transforms, we know that

$$\mathscr{L}\{1\} = \frac{1}{s}$$

and

$$\mathscr{L}\{\cos bt\} = \frac{s}{s^2+b^2}$$

which implies that

$$\mathscr{L}\{\cos 2at\} = \frac{s}{s^2+(2a)^2} = \frac{s}{s^2+4a^2}.$$

Therefore, we have that

$$\begin{aligned}
\mathscr{L}\left\{\sin^2 at\right\} &= \frac{1}{2}\left[\mathscr{L}\{1\} - \mathscr{L}\{\cos 2at\}\right] \\
&= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4a^2}\right] \\
&= \frac{1}{2}\left[\frac{s^2+4a^2-s^2}{s(s^2+4a^2)}\right] \\
&= \frac{2a^2}{s(s^2+4a^2)}.
\end{aligned}$$

$\square$