

# Homework Assignment 1

Matthew Tiger

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**Problem 1.1.2.** Use Example 1.1.3 for affine maps to find the solutions to the difference equations:

i.  $x_{n+1} - \frac{x_n}{3} = 2, x_0 = 2.$

ii.  $x_{n+1} + 3x_n = 4, x_0 = -1.$

*Solution.* Consider the affine map  $f : \mathbb{R} \rightarrow \mathbb{R}$  with  $f(x) = ax + b$ . Define the sequence  $x_{n+1} = f(x_n) = ax_n + b$  where  $x_0 \in \mathbb{R}$  is given. As was shown in the reading, the closed form solution to the above recurrence relation is given by

$$x_n = \left( x_0 - \frac{b}{1-a} \right) a^n + \frac{b}{1-a}. \quad (1)$$

Thus, the solutions to the provided difference equations can be solved by rewriting the equation in the form of an affine map, identifying  $a, b$ , and  $x_0$ , and using the closed solution (1).

- i. For the difference equation  $x_{n+1} - \frac{x_n}{3} = 2, x_0 = 2$ , we readily see by rewriting the equation that  $a = 1/3$  and  $b = 2$  with  $x_0 = 2$  given. Therefore, using (1), the solution to the difference equation is

$$\begin{aligned} x_n &= \left( x_0 - \frac{b}{1-a} \right) a^n + \frac{b}{1-a} \\ &= \left( 2 - \frac{2}{1-1/3} \right) \left( \frac{1}{3} \right)^n + \frac{2}{1-1/3} \\ &= 3 - 3^{-n} \end{aligned}$$

- ii. For the difference equation  $x_{n+1} + 3x_n = 4, x_0 = -1$ , we readily see by rewriting the equation that  $a = -3$  and  $b = 4$  with  $x_0 = -1$  given. Therefore, using (1), the solution to the difference equation is

$$\begin{aligned} x_n &= \left( x_0 - \frac{b}{1-a} \right) a^n + \frac{b}{1-a} \\ &= \left( -1 - \frac{4}{1-(-3)} \right) (-3)^n + \frac{4}{1-(-3)} \\ &= 1 - 2(-3)^n. \end{aligned}$$

□

**Problem 1.1.3.** A *logistic difference equation* is one of the form  $x_{n+1} = \mu x_n(1 - x_n)$  for some fixed  $\mu \in \mathbb{R}$ . Find exact (closed form) solutions to the following logistic difference equations:

- i.  $x_{n+1} = 2x_n(1 - x_n)$ . Hint: Use the substitution  $x_n = (1 - y_n)/2$  to transform the equation into a simpler equation that is easily solved.
- ii.  $x_{n+1} = 4x_n(1 - x_n)$ . Hint: Set  $x_n = \sin^2(\theta_n)$  and simplify to get an equation that is easily solved.

*Solution.* i. Let  $x_n = (1 - y_n)/2$  for  $n \in \mathbb{N}$  with  $x_0$  given. Substituting this expression into the original difference equation yields the new difference equation

$$\begin{aligned} \frac{1 - y_{n+1}}{2} &= 2 \left( \frac{1 - y_n}{2} \right) \left[ 1 - \left( \frac{1 - y_n}{2} \right) \right] \\ &= (1 - y_n) \left( \frac{1 + y_n}{2} \right) \\ &= \frac{1 - y_n^2}{2}. \end{aligned}$$

This new difference equation reduces to  $y_{n+1} = y_n^2$  for  $n \in \mathbb{N}$ , the solution of which is readily seen to be  $y_{n+1} = y_0^{2^{n+1}}$ . Making the substitution  $y_n = 1 - 2x_n$  shows that, for  $n \in \mathbb{N}$ , the solution to the original difference equation is given by

$$x_{n+1} = \frac{1 - (1 - 2x_0)^{2^{n+1}}}{2}.$$

- ii. Let  $x_n = \sin^2(\theta_n)$  for  $n \in \mathbb{N}$  with  $x_0$  given. We may assume without loss of generality that  $\theta_n \in [0, \pi)$  for if the angle  $\theta_n$  isn't in the stated range, we can find an integer  $k$  such that  $\theta_n + k\pi \in [0, \pi)$  and  $\sin^2(\theta_n) = \sin^2(\theta_n + k\pi)$ . We then declare the sum  $\theta_n + k\pi$  to be the new angle  $\theta_n$ . Substituting the above expression for  $x_n$  into the original difference equation yields the new difference equation

$$\begin{aligned} \sin^2(\theta_{n+1}) &= 4 \sin^2(\theta_n) (1 - \sin^2(\theta_n)) \\ &= (2 \sin(\theta_n) \cos(\theta_n))^2 \\ &= \sin^2(2\theta_n). \end{aligned}$$

Knowing that for  $x, y \in [0, \pi)$  we have that  $\sin^2(x) = \sin^2(y)$  if and only if  $x = y$ , the new difference equation reduces to  $\theta_{n+1} = 2\theta_n$  for  $n \in \mathbb{N}$  where it is implicitly understood that  $\theta_{n+1}$  will be mapped to the corresponding angle between 0 and  $\pi$  if  $2\theta_n \geq \pi$ . Using the closed form solution for difference equations in the form of linear maps, the solution to the reduced difference equation is given by  $\theta_{n+1} = 2^{n+1}\theta_0$  for  $n \in \mathbb{N}$ . Making the substitution  $\theta_n = \sin^{-1}(\sqrt{x_n})$  shows that, for  $n \in \mathbb{N}$ , the solution to the original difference equation is given by

$$x_{n+1} = \sin^2(2^{n+1} \sin^{-1}(\sqrt{x_0})).$$

□

**Problem 1.1.4.** You borrow  $\$P$  at  $r\%$  per annum and pay off  $\$M$  at the end of each subsequent month. Write down a difference equation for the amount owing  $A(n)$  at the end of each month (so  $A(0) = P$ ). Solve the equation to find a closed form for  $A(n)$ . If  $P = 100,000$ ,  $M = 1,000$ , and  $r = 4$ , after how long will the loan be paid off?

*Solution.* Let  $A(n)$  be the amount owed on the loan at the end of month  $n$ . If the principal amount of the loan is  $\$P$ , then  $A(0) = P$ . If the annual interest rate is  $r\%$ , then the monthly interest rate is  $r/12\%$ . Assuming each month a payment of  $\$M$  is made on the loan, a difference equation representing the amount owed on the loan at the end of month  $n$  is given by

$$\begin{aligned} A(n+1) &= A(n) + A(n) \left[ \frac{r}{12(100)} \right] - M \\ &= \left[ 1 + \frac{r}{12(100)} \right] A(n) - M \end{aligned}$$

for  $n \in \mathbb{N}$ .

Using the closed form solution for difference equations in the form of affine maps, the solution to the difference equation is given by

$$\begin{aligned} A(n) &= \left( A(0) + \frac{M}{1 - \left( 1 + \frac{r}{12(100)} \right)} \right) \left( 1 + \frac{r}{12(100)} \right)^n - \frac{M}{1 - \left( 1 + \frac{r}{12(100)} \right)} \\ &= \left( P - \frac{1200M}{r} \right) \left( 1 + \frac{r}{1200} \right)^n + \frac{1200M}{r}. \end{aligned}$$

The loan will be paid off after  $k \in \mathbb{R}$  months when  $A(k) = 0$  from which we can gather that the loan will be paid off after  $n = \lceil k \rceil$  full months. Solving

$$A(k) = \left( 100000 - \frac{1200(1000)}{4} \right) \left( 1 + \frac{4}{1200} \right)^k + \frac{1200(1000)}{4} = 0$$

shows that  $k = 121.842$ . Therefore, the loan will be paid off in full after 122 months.  $\square$

**Problem 1.1.7.** Let  $f(x) = x^2 + bx + c$ . Give conditions on  $b$  and  $c$  for  $f : [0, 1] \rightarrow [0, 1]$  to be a dynamical system. Hint: Recall that the maximum and minimum values of a continuous function defined on a closed interval  $[a, b]$  occur either at the end points or at the critical points of the function.

*Solution.* The function  $f(x) = x^2 + bx + c$  for  $f : [0, 1] \rightarrow [0, 1]$  is a dynamical system if the image of the function is contained in its domain, i.e. if  $f([0, 1]) \subseteq [0, 1]$ . The minimum and maximum values of a continuous function occur either at the end points of the domain or at the critical points of the function. Thus, for the continuous function  $f$ , if we ensure that the evaluation of  $f$  at  $x = 0$ ,  $x = 1$ , and the critical points of  $f$  are contained in  $[0, 1]$  then the image of  $f$  will necessarily be contained in  $[0, 1]$  and  $f$  will be a dynamical system.

At the end points of the domain we have that  $f(0) = c$  and  $f(1) = b + c + 1$ . Thus, in order for  $f$  to be a dynamical system, we must have that  $c \in [0, 1]$  and  $b + c \in [-1, 0]$ .

The only critical point of the function  $f$  is found when  $f'(x) = 0$  or when  $x = -b/2$ . Thus, we require that  $f(-b/2) = -b^2/4 + c \in [0, 1]$ . This reduces to requiring that  $4c - 4 \leq b^2 \leq 4c$ . Thus, when  $b \in \mathbb{R}$ , we must have that  $b \in [-2\sqrt{c}, 2\sqrt{c}]$ .

Combining all of these inequalities shows that in order for the image of  $f$  to be contained in the domain of  $f$ , we must have that  $c \in [0, 1]$  and  $b \in [-2\sqrt{c}, -c]$ . Therefore, for the function  $f(x) = x^2 + bx + c$  for  $f : [0, 1] \rightarrow [0, 1]$  to be a dynamical system we must have that  $0 \leq c \leq 1$  and  $-2\sqrt{c} \leq b \leq -c$ .

□

**Problem 1.2.1.** Give conditions on  $b$  and  $c$  for the map  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = x^2 + bx + c$  to have a fixed point. Use these conditions to show that  $f_c(x) = x^2 + c$  has a fixed point provided  $c \leq 1/4$ .

*Solution.*

□

**Problem 1.2.6.** Consider the eventual fixed points of the logistic map  $L_\mu : [0, 1] \rightarrow [0, 1]$ ,  $L_\mu(x) = \mu x(1 - x)$  for  $0 < \mu < 4$ .

- i. Show that there are no eventual fixed points associated with the fixed point  $x = 0$ , other than  $x = 1$ .
- ii. Show that for  $1 < \mu \leq 2$ , the only eventual fixed point associated with the fixed point  $x = 1 - 1/\mu$  is  $x = 1/\mu$ .
- iii. Show that there are additional eventual fixed points associated with  $x = 1 - 1/\mu$  when  $2 < \mu < 3$ .
- iv. Investigate the eventual fixed points of the logistic map when  $\mu = 5/2$ .

*Solution.*

□