Homework Assignment 3

Matthew Tiger

September 25, 2016

Problem 1.5.1. Find the fixed points of the following maps and use the appropriate theorems to determine whether they are asymptotically stable, semi-stable, or unstable:

i. $f(x) = \frac{x^3}{2} + \frac{x}{2}$,

ii. $f(x) = \arctan(x)$,

iii. $f(x) = x^3 + x^2 + x$,

iv. $f(x) = x^3 - x^2 + x$,

v. $f(x) = \begin{cases} 3x/4 & x \leq 1/2 \\ 3(1-x)/4 & x > 1/2 \end{cases}$.

Solution. Note that a point $x = c$ is a fixed point of f if c is a solution to the equation $g(x) = f(x) - x = 0$. If $x = c$ is a fixed point, then the behavior of the derivatives of f at the point $x = c$ will allow us to classify the stability of the fixed point.

i. The solutions to the equation

$$\begin{aligned} g(x) &= f(x) - x \\ &= \frac{x^3}{2} + \frac{x}{2} - x \\ &= \frac{x^3}{2} - \frac{x}{2} - x = 0 \end{aligned}$$

are given by $x = -1$, $x = 0$, and $x = 1$. Note that $f'(x) = 3x^2/2 + 1/2$.

For the fixed point $x = -1$, we see that $|f'(-1)| = 2 > 1$ so that $x = -1$ is a hyperbolic fixed point and by theorem 1.4.4, this fixed point is unstable.

For the fixed point $x = 0$, we see that $|f'(0)| = 1/2 < 1$ so that $x = 0$ is a hyperbolic fixed point and by theorem 1.4.4, this fixed point is stable.

For the fixed point $x = 1$, we see that $|f'(1)| = 2 > 1$ so that $x = 1$ is a hyperbolic fixed point and by theorem 1.4.4, this fixed point is unstable.

ii. Note that for any $x \in \mathbb{R}$, we have that $-\pi/2 < \arctan(x) < \pi/2$. Thus, if $|x| > \pi/2$, then $|\arctan(x)| < \pi/2 < |x|$ so that for any such x we have that $\arctan(x) \neq x$, i.e. $f(x) = \arctan(x)$ has no fixed points for $|x| > \pi/2$.

Since $f(x)$ is continuous on the interval $[-\pi/2, \pi/2]$, we know that $f(x)$ must have a fixed point on this interval. By the Mean Value Theorem, we know that if $x > 0$, then

$$0 < \frac{x}{x^2+1} < \arctan(x).$$

It can be shown that for $g(x) = \arctan(x) - x$, if $x > 0$, then $g'(x) < 0$. This implies that the function $g(x)$ is monotonically decreasing and that $g(x) < g(0) = 0$, i.e. $\arctan(x) < x$. Combining, we see that

$$0 < \arctan(x) < x.$$

From this inequality, we gather that if $x \in (0, \pi/2)$, we have that $\arctan(x) > 0$ and that

$$0 < f^n(x) < f^{n-1}(x) < \cdots < f(x) < x,$$

i.e. the iterates of f are monotonically decreasing and bounded below. Thus, the limit converges to the infimum, i.e. $\lim f^n(x) = 0$. Therefore, we must have $x = 0$ is a fixed point if $x \in (0, \pi/2)$.

Using a similar inequality, we can show that if $x \in (-\pi/2, 0)$, then the iterates of f form a monotonically increasing sequence that is bounded above. Thus, the limit in this case converges to the supremum, i.e. $\lim f^n(x) = 0$ and $x = 0$ is a fixed point if $x \in (-\pi/2, 0)$. Therefore, $x = 0$ is the only fixed point of $f(x) = \arctan(x)$.

Note that

$$f'(x) = 1/(x^2+1), \quad f''(x) = -2x/(1+x^2)^2, \quad f'''(x) = 8x^2/(1+x^2)^3 - 2/(1+x^2)^2.$$

Thus, for the fixed point $x = 0$, we see that $f'(0) = 1$, $f''(0) = 0$, and $f'''(0) = -2$. Therefore, according to theorem 1.5.3 (iii), this fixed point is non-hyperbolic and stable.

iii. The solutions to the equation

$$\begin{aligned} g(x) &= f(x) - x \\ &= x^3 + x^2 + x - x \\ &= x^2(x+1) = 0 \end{aligned}$$

are given by $x = -1$ and $x = 0$. Note that $f'(x) = 3x^2 + 2x + 1$, $f''(x) = 6x + 2$, and $f'''(x) = 6$.

For the fixed point $x = -1$, we see that $|f'(-1)| = 2 > 1$ so that $x = -1$ is a hyperbolic fixed point and by theorem 1.4.4, this fixed point is unstable.

For the fixed point $x = 0$, we see that $f'(0) = 1$ so that $x = 0$ is a non-hyperbolic fixed point. Since $f''(0) = 2 > 0$, we have by theorem 1.5.3 (i)(a) that this fixed point is one-sided stable to the left of $x = 0$.

iv. The solutions to the equation

$$\begin{aligned} g(x) &= f(x) - x \\ &= x^3 - x^2 + x - x \\ &= x^2(x-1) = 0 \end{aligned}$$

are given by $x = 1$ and $x = 0$. Note that $f'(x) = 3x^2 - 2x + 1$, $f''(x) = 6x - 2$, and $f'''(x) = 6$.

For the fixed point $x = 1$, we see that $|f'(1)| = 2 > 1$ so that $x = 1$ is a hyperbolic fixed point and by theorem 1.4.4, this fixed point is unstable.

For the fixed point $x = 0$, we see that $f'(0) = 1$ so that $x = 0$ is a non-hyperbolic fixed point. Since $f''(0) = -2 < 0$, we have by theorem 1.5.3 (i)(b) that this fixed point is one-sided stable to the right of $x = 0$.

v. If $x \leq 1/2$, then

$$f(x) - x = \frac{3x}{4} - x = -\frac{x}{4} = 0$$

if $x = 0$. Since $x = 0 \leq 1/2$, we have that $x = 0$ is a fixed point of $f(x)$.

If $x > 1/2$, then

$$f(x) - x = \frac{3(1-x)}{4} - x = \frac{3 - 7x}{4} = 0$$

if $x = 3/7$. Since $3/7 < 1/2$, we have that $x = 3/7$ is not a fixed point of $f(x)$.

If $x \leq 1/2$, then $f'(x) = 3/4$. Thus, for the fixed point $x = 0$, we see that $|f'(0)| < 1$ and $x = 0$ is a non-hyperbolic stable fixed point by theorem 1.4.4.

$\square$

Problem 1.5.2. Consider the family of quadratic maps $f_c(x) = x^2 + c$ where $x \in \mathbb{R}$.

i. Use the theorems of section 1.5 to determine the stability of the hyperbolic fixed points of the the family of maps for all possible values of c.

ii. Find any values of c such that f_c has a non-hyperbolic fixed point and determine the stability of these fixed points.

Solution. As was shown in problem 1.2.1, we know that $f_c : \mathbb{R} \to \mathbb{R}$ with $f_c(x) = x^2 + c$ has two fixed points given by

$$x_1 = \frac{1 - \sqrt{1-4c}}{2}, \qquad x_2 = \frac{1 + \sqrt{1-4c}}{2} \tag{1}$$

provided that $c \leq 1/4$.

i. Suppose that $c \leq 1/4$. Then the fixed points of f_c are provided by (1). Recall that a fixed point $x = a$ is a hyperbolic fixed point of a function g if $|g'(a)| \neq 1$. In particular, $x = a$ will be asymptotically stable if $|g'(a)| < 1$ and unstable if $|g'(a)| > 1$.

   We begin by assuming the fixed point of the function f_c has the form x_1. Then x_1 will be a stable hyperbolic fixed point if

$$|f_c'(x_1)| = \left|1 - \sqrt{1-4c}\right| < 1. \tag{2}$$

   However, this is only true if $-3/4 < c < 1/4$. Thus, x_1 will be an asymptotically stable hyperbolic fixed point if $-3/4 < c < 1/4$. Similarly, by reversing the inequality in (2), we can easily see that the fixed point x_1 will be an unstable hyperbolic fixed point if $c < -3/4$.

   Now, assuming that the fixed point of f_c has the form x_2, then the fixed point x_2 will be a stable hyperbolic fixed point if

$$|f_c'(x_2)| = \left|1 + \sqrt{1-4c}\right| < 1.$$

   However, this has no real solutions if $c \leq 1/4$. On the other hand, we can see that

$$|f_c'(x_2)| = \left|1 + \sqrt{1-4c}\right| > 1$$

   if $c < 1/4$. Therefore, every hyperbolic fixed point of f_c of the form x_2 is unstable.

ii. A fixed point $x = a$ is a non-hyperbolic fixed point of a function g if $|g(a)| = 1$.

   We first investigate fixed points of the form x_1. Assuming the fixed point of f_c is of the form x_1, then x_1 is non-hyperbolic if

$$|f_c'(x_1)| = \left|1 - \sqrt{1-4c}\right| = 1$$

   from which we see that $1 - \sqrt{1-4c} = 1$ if $c = 1/4$ and that $1 - \sqrt{1-4c} = -1$ if $c = -3/4$. Thus, x_1 is a non-hyperbolic fixed point if $c = 1/4$ or $c = -3/4$.

In the case that $c = 1/4$, then $f_c'(x_1) = 1$ and $f_c''(x_1) = 2$. Thus, since $f_c''(x_2) > 0$, applying theorem 1.5.3 (i) (a), we see that this fixed point is one-sided stable to the left of x_1. On the other hand, if $c = -3/4$, then $f_c'(x_1) = -1$ with $f_c''(x_1) = 2$ and $f_c'''(x_1) = 0$. Since $f_c'(x_1) = -1$, the Schwarzian derivative of f_c is given by

$$Sf_c(x) = -f_c'''(x) - \frac{3\left(f_c''(x)\right)^2}{2} = -6.$$

Note that $Sf_c(x_1) < 0$, so applying theorem 1.5.7 (i) we find that the fixed point x_1 is asymptotically stable if $c = -3/4$.

We now investigate fixed points of the form x_2. Assuming the fixed point of f_c is of the form x_2, then

$$|f_c'(x_2)| = \left|1 + \sqrt{1-4c}\right| = 1$$

only if $c = 1/4$. Thus, x_2 is a non-hyperbolic fixed point if $c = 1/4$.

In this case, we see that $f_c'(x_2) = 1$ and $f_c''(x_2) = 2$. Thus, since $f_c''(x_2) > 0$, applying theorem 1.5.3 (i) (a), we see that this fixed point is one-sided stable to the left of x_2 if $c = 1/4$.

□

Problem 1.5.3. i. Show that $f(x) = -2x^3 + 2x^2 + x$ has two non-hyperbolic fixed points and determine their stability.

ii. If $x = 0$ and $x = 1$ are non-hyperbolic fixed points for $f : \mathbb{R} \to \mathbb{R}$ for $f(x) = ax^3 + bx^2 + cx + d$, find all possible values of a, b, c, and d.

iii. Write down the function $f(x)$ in each case of (ii) above and determine the stability of the fixed points.

Solution. i. The fixed points of $f(x)$ are the roots of the function

$$\begin{aligned} g(x) &= f(x) - x \\ &= -2x^3 + 2x^2 + x - x \\ &= -2x^2(x-1). \end{aligned}$$

From the factorization of $g(x)$, we clearly see that its roots are given by $x = 0$ and $x = 1$. Note that $f'(x) = -6x^2 + 4x + 1$. From this we see that $f'(0) = 1$ and $f'(1) = -1$, implying that both fixed points are non-hyperbolic fixed points.

Using $f''(x) = -12x + 4$, we see that $f''(0) = 4 > 0$ so that by theorem 1.5.3 (i) the fixed point $x = 0$ is one-sided stable to the left of $x = 0$.

To determine the stability of $x = 1$, we note that the Schwarzian derivative of $f(x)$ when $f'(x) = -1$ is given by $Sf(x) = -f'''(x) - 3f''(x)^2/2$. Thus, $f'(1) = -1$ and $Sf(1) = 12 - 96 < 0$ so that by theorem 1.5.7, the fixed point $x = 1$ is asymptotically stable.

ii. Suppose that $f(x) = ax^3 + bx^2 + cx + d$. We know that if $x = 0$ and $x = 1$ are fixed points of $f(x)$ then

$$\begin{aligned} f(0) &= d &&= 0 \\ f(1) &= a + b + c &&= 1. \end{aligned} \tag{3}$$

The fixed points $x = 0$ and $x = 1$ are non-hyperbolic fixed points if $|f'(0)| = |f'(1)| = 1$. Using $f(x)$, we see that $f'(x) = 3ax^2 + 2bx + c$. Thus, the fixed points are non-hyperbolic if

$$\begin{aligned} |f'(0)| &= |c| &&= 1 \\ |f'(1)| &= |3a + 2b + c| &&= 1. \end{aligned} \tag{4}$$

From (3), we see that $a = 1 - b - c$ and substituting into (4) we have that $|f'(1)|$ reduces to

$$|f'(1)| = |3(1 - b - c) + 2b + c| = |3 - b - 2c| = 1. \tag{5}$$

Note that (4) tells us there are two cases to consider, the case that $c = 1$ and the case that $c = -1$.

If $c = 1$, then (5) tells us that

$$|3 - b - 2c| = |1 - b| = 1$$

from which we gather that $b = 0$ or $b = 2$. If $b = 0$, then using (3), we see that $a = 0$. On the other hand, if $b = 2$, we see that we must have that $a = -2$.

If $c = -1$, then (5) tells us that

$$|3 - b - 2c| = |5 - b| = 1$$

from which we gather that $b = 4$ or $b = 6$. If $b = 4$, then using (3), we see that $a = -2$. On the other hand, if $b = 6$, we see that we must have that $a = -4$.

iii. The previous remarks allow us to explicitly write out the function $f : \mathbb{R} \to \mathbb{R}$ for $f(x) = ax^3 + bx^2 + cx + d$ when $f(x)$ has two non-hyperbolic fixed points $x = 0$ and $x = 1$. We write the four possibilities as follows:

$$\begin{aligned} f_1(x) &= x \\ f_2(x) &= -2x^3 + 2x^2 + x \\ f_3(x) &= -2x^3 + 4x^2 - x \\ f_4(x) &= -4x^3 + 6x^2 - x. \end{aligned}$$

We now evaluate the stability of the fixed points $x = 0$ and $x = 1$ for $f_1(x)$. From the definition of stability, we have that a point c is stable for f if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $x \in I = \mathbb{R}$ and $|x - c| < \delta$, then $|f^n(x) - c| < \epsilon$ for every positive integer n. Note that $f_1^n(x) = x$ for every positive integer n. It is clear from the definition of stability that by taking $\delta = \epsilon$, the fixed points $x = 0$ and $x = 1$ are stable. They are not however asymptotically stable since these are not attracting fixed points. This is clear because there is no neighborhood of either fixed points such that the iterates of f_1 for points in that neighborhood converge to $x = 0$ or $x = 1$. Intuitively, this is because every point of $f_1(x)$ is a fixed point.

We will now evaluate the stability of $x = 0$ and $x = 1$ for the remaining three functions. Recall that the Schwarzian derivative of a function $g(x)$ when $g'(x) = -1$ is given by

$$Sg(x) = -g'''(x) - \frac{3g''(x)^2}{2}.$$

Now, if $f(x) = ax^3 + bx^2 + cx$, then

$$\begin{aligned} f'(x) &= 3ax^2 + 2bx + c \\ f''(x) &= 6ax + 2b \\ f'''(x) &= 6a. \end{aligned}$$

Thus, for $x = 0$, we can see that

$$\begin{array}{rclrclrcl} f_2'(0) &=& 1 & f_3'(0) &=& -1 & f_4'(0) &=& -1 \\ f_2''(0) &=& 4 & f_3''(0) &=& 8 & f_4''(0) &=& 12 \\ f_2'''(0) &=& -12 & f_3'''(0) &=& -12 & f_4'''(0) &=& -24 \end{array}$$

We see that for f_2, we have that $f_2'(0) = 1$ and $f_2'(0) > 0$ so by theorem 1.5.3 we have that the fixed point $x = 0$ is one-sided stable to the left of $x = 0$. For f_3 and f_4, we have that $f_3'(0) = f_4'(0) = -1$ as well as $Sf_3(0) = -84 < 0$ and $Sf_4(0) = -192 < 0$. Thus, by theorem 1.5.7, $x = 0$ is an asymptotically stable fixed point for f_3 and f_4.

Similarly, for $x = 1$, we can see that

$$\begin{array}{rlrlrl} f_2'(1) & = -1 & f_3'(1) & = \ \ 1 & f_4'(1) & = -1 \\ f_2''(1) & = -8 & f_3''(1) & = -4 & f_4''(1) & = -12 \\ f_2'''(1) & = -12 & f_3'''(1) & = -12 & f_4'''(1) & = -24 \end{array}$$

We see that for f_3, we have that $f_3'(1) = 1$ and $f_3''(1) < 0$ so by theorem 1.5.3 we have that the fixed point $x = 1$ is one-sided stable to the right of $x = 1$. For f_2 and f_4, we have that $f_2'(1) = f_4'(1) = -1$ as well as $Sf_2(1) = -84 < 0$ and $Sf_4(1) = -192 < 0$. Thus, by theorem 1.5.7, $x = 1$ is an asymptotically stable fixed point for f_2 and f_4.

$\square$

Problem 1.5.6. Find the Schwarzian derivative of both $f(x) = e^x$ and $g(x) = \sin(x)$ and show that they are always negative.

Solution. Recall that the Schwarzian derivative of a function $h(x)$ is given by

$$Sh(x) = \frac{h'''(x)}{h'(x)} - \frac{3}{2}\left[\frac{h''(x)}{h'(x)}\right]^2$$

and this derivative exists if $h'''(x)$ exists and $h'(x) \neq 0$.

Suppose that $f(x) = e^x$. Then we know that $f^{(n)}(x) = e^x = f(x)$ for any positive integer n. Therefore,

$$Sf(x) = \frac{e^x}{e^x} - \frac{3}{2}\left[\frac{e^x}{e^x}\right]^2 = 1 - \frac{3}{2} = -\frac{1}{2} < 0$$

and we are done.

Now suppose that $g(x) = \sin(x)$. The successive derivatives of g are given by

$$\begin{aligned} g'(x) &= \cos(x) \\ g''(x) &= -\sin(x) \\ g'''(x) &= -\cos(x). \end{aligned}$$

Computing the Schwarzian derivative of $g(x)$, we see that

$$\begin{aligned} Sg(x) &= -\frac{\cos(x)}{\cos(x)} - \frac{3}{2}\left[-\frac{\sin(x)}{\cos(x)}\right]^2 \\ &= -1 - \frac{3\tan^2(x)}{2}. \end{aligned}$$

Since $\tan^2(x) \geq 0$ for any $x \in \mathbb{R}$, we have that $1 + (3/2)\tan^2(x) \geq 1$ so that

$$Sg(x) = -1 - \frac{3\tan^2(x)}{2} \leq -1 < 0$$

and we are done. $\square$

Problem 1.5.9. Let $f(x)$ be a polynomial such that $f(c) = c$. (Recall that a polynomial $p(x)$ has $(x-c)^2$ as a factor if and only if both $p(c) = 0$ and $p'(c) = 0$.)

i. If $f'(c) = 1$, show that $(x-c)^2$ is a factor of $g(x) = f(x) - x$.

ii. If $|f'(c)| = 1$, show that $(x-c)^2$ is a factor of $h(x) = f^2(x) - x$.

iii. Show in the case that $f'(c) = -1$, we actually have that $(x-c)^3$ is a factor of $h(x) = f^2(x) - x$.

iv. Check that (iii) holds for the non-hyperbolic fixed point $x = 2/3$ of the logistic map $L_3(x) = 3x(1-x)$.

v. Check that (i), (ii), (iii) hold for the non-hyperbolic fixed points of the polynomial $f(x) = -2x^3 + 2x^2 + x$.

Solution. i. Suppose that f is a polynomial, c is a fixed point of f, and that $f'(c) = 1$. Let $g(x) = f(x) - x$, a polynomial. Note that $(x-c)^2$ will be a factor of $g(x)$ if $g(c) = g'(c) = 0$. Since $f(c) = c$, it is clear that $g(c) = f(c) - c = 0$. Note that $g'(x) = f'(x) - 1$. From our supposition, $f'(c) = 1$ so that $g'(c) = 0$. Therefore, $(x-c)^2$ is a factor of $g(x) = f(x) - x$.

ii. Suppose that f is a polynomial, c is a fixed point of f, and that $|f'(c)| = 1$. Let $h(x) = f^2(x) - x$, a polynomial. Note that $(x-c)^2$ will be a factor of $h(x)$ if $h(c) = h'(c) = 0$. Since $f(c) = c$, it is clear that

$$f^2(c) = f(f(c)) = f(c) = c$$

and that $h(c) = f^2(c) - c = 0$. Note that $h'(x) = f'(f(x))f'(x) - 1$. Thus, since c is a fixed point,

$$\begin{aligned} h'(c) &= f'(f(c))f'(c) - 1 \\ &= f'(c)^2 - 1 \\ &= |f'(c)| - 1. \end{aligned}$$

Therefore, from our supposition that $|f'(c)| = 1$, we have that $h'(c) = 0$ and $(x-c)^2$ is a factor of $h(x) = f^2(x) - x$.

iii. Suppose that f is a polynomial, c is a fixed point of f, and that $f'(c) = -1$. Let $h(x) = f^2(x) - x$, a polynomial. Note that $(x-c)^3$ will be a factor of $h(x)$ if $h(c) = h'(c) = h''(c) = 0$. By our supposition, we have that $|f'(c)| = 1$, so we have already shown previously that $h(c) = h'(c) = 0$ and all that remains is to show that $h''(c) = 0$.

Recall that $h'(x) = f'(f(x))f'(x) - 1$. Thus,

$$\begin{aligned} h''(x) &= \left[f'(f(x))\right]' f'(x) + f'(f(x)) \left[f'(x)\right]' \\ &= f''(f(x))f'(x)f'(x) + f'(f(x))f''(x) \\ &= f''(f(x)) \left(f'(x)\right)^2 + f''(x)f'(f(x)). \end{aligned}$$

Now, since $f(c) = c$ and $f'(c) = -1$, we have that

$$\begin{aligned} h''(c) &= f''(f(c))\left(f'(c)\right)^2 + f''(c)f'(f(c)) \\ &= f''(c)(-1)^2 + f''(c)(-1) \\ &= 0. \end{aligned}$$

Therefore, $(x-c)^3$ is a factor of $h(x)$.

iv. Suppose that $L_3(x) = 3x(1-x)$ and $h(x) = L_3^2(x) - x$. Thus,

$$\begin{aligned} h(x) &= 3(3x(1-x))(1-(3x(1-x))) - x \\ &= -27x^4 + 54x^3 - 36x^2 + 9x - x \\ &= -x\left(x - \frac{2}{3}\right)^3. \end{aligned} \tag{6}$$

Note that $x = 2/3$ is a fixed point of $L_3(x)$. Further, we have that $L_3'(x) = 3 - 6x$ so that $L_3'(2/3) = -1$. Thus, by the previous results we must have that $(x - 2/3)^3$ is a factor of $h(x)$. This result is in agreement with the factorization of $h(x)$ in (6).

v. Suppose that $f(x) = -2x^3 + 2x^2 + x$ and $g(x) = f(x) - x$. Thus,

$$\begin{aligned} g(x) &= -2x^3 + 2x^2 + x - x \\ &= -2x^2(x-1). \end{aligned} \tag{7}$$

Recall that $x = 0$ and $x = 1$ are the non-hyperbolic fixed points of $f(x)$. Note that $f'(x) = -6x + 4x + 1$ so that $f'(0) = 1$ and $f'(1) = -1$. By the results shown above, since $f'(0) = 1$, we must have that x^2 is a factor of $g(x)$. Similarly, since $f'(1) \neq 1$, we must have that $(x-1)^2$ is not a factor of $g(x)$. These results are in agreement with the factorization of $g(x)$ in (7).

Now suppose that $f(x) = -2x^3 + 2x^2 + x$ and $h(x) = f^2(x) - x$. Thus,

$$\begin{aligned} h(x) &= -2\left(-2x^3 + 2x^2 + x\right)^3 + 2\left(-2x^3 + 2x^2 + x\right)^2 + \left(-2x^3 + 2x^2 + x\right) - x \\ &= 16x^9 - 48x^8 + 24x^7 + 40x^6 - 28x^5 - 12x^4 + 4x^3 + 4x^2 + x \\ &= 4x^2(x-1)^3(4x^4 - 6x^2 - 4x - 1). \end{aligned} \tag{8}$$

As shown previously, the non-hyperbolic fixed points of $f(x)$ are given by $x = 0$ and $x = 1$ with $f'(0) = 1$ and $f'(1) = -1$. By the results above, since $|f'(0)| = 1$, we must have that x^2 is a factor of $h(x)$ and since $f'(1) = -1$, we must have that $(x-1)^3$ is a factor of $h(x)$. These results are in agreement with the factorization of $h(x)$ in (8).

□