Homework Assignment 7

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Problem 4.28. Using the Laplace transform, evaluate the following integrals:

a. $f(t) = \displaystyle\int_0^\infty \frac{\sin tx}{\sqrt{x}} dx$,

e. $f(t) = \displaystyle\int_0^\infty e^{-tx^2} dx$, $0 < t$.

Solution. a. We begin by taking the Laplace transform of $f(t)$. Doing so yields

$$\begin{aligned}
\bar{f}(s) = \mathscr{L}\{f(t)\} &= \mathscr{L}\left\{\int_0^\infty \frac{\sin tx}{\sqrt{x}} dx\right\} \\
&= \int_0^\infty \mathscr{L}\left\{\frac{\sin tx}{\sqrt{x}}\right\} dx \\
&= \int_0^\infty \frac{\sqrt{x}}{s^2 + x^2} dx.
\end{aligned}$$

Using a computer algebra system, we see that this integral evaluates to

$$\begin{aligned}
\bar{f}(s) &= \int_0^\infty \frac{\sqrt{x}}{s^2 + x^2} dx \\
&= \frac{\pi}{\sqrt{2s}}.
\end{aligned}$$

From our table of Laplace transforms, we see that

$$\mathscr{L}^{-1}\left\{\frac{\Gamma(a+1)}{s^{a+1}}\right\} = t^a.$$

In particular, for $a = -1/2$, we see that

$$\mathscr{L}^{-1}\left\{\frac{\Gamma(1/2)}{s^{-1/2}}\right\} = \mathscr{L}^{-1}\left\{\frac{\sqrt{\pi}}{s^{-1/2}}\right\} = t^{-1/2}.$$

Therefore, the evaluation of the original integral is

$$\begin{aligned} f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} &= \mathscr{L}^{-1}\left\{\frac{\pi}{\sqrt{2s}}\right\} \\ &= \sqrt{\frac{\pi}{2}}\mathscr{L}^{-1}\left\{\frac{\sqrt{\pi}}{s^{-1/2}}\right\} \\ &= \sqrt{\frac{\pi}{2t}}. \end{aligned}$$

e. Applying the Laplace transform to $f(t)$ yields

$$\begin{aligned} \bar{f}(s) = \mathscr{L}\left\{f(t)\right\} &= \mathscr{L}\left\{\int_0^\infty e^{-tx^2}dx\right\} \\ &= \int_0^\infty \mathscr{L}\left\{e^{-tx^2}\right\}dx \\ &= \int_0^\infty \frac{1}{s+x^2}dx \end{aligned}$$

Using a computer algebra system, we see that

$$\begin{aligned} \bar{f}(s) &= \int_0^\infty \frac{1}{s+x^2}dx \\ &= \frac{\pi}{2\sqrt{s}}. \end{aligned}$$

Therefore, using previous arguments, we see that the evaluation of the original integral is

$$\begin{aligned} f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} &= \mathscr{L}^{-1}\left\{\frac{\pi}{2\sqrt{s}}\right\} \\ &= \sqrt{\frac{\pi}{4}}\mathscr{L}^{-1}\left\{\frac{\sqrt{\pi}}{s^{-1/2}}\right\} \\ &= \sqrt{\frac{\pi}{4t}}. \end{aligned}$$

$\square$

Problem 4.29.

Solution. □

Problem 4.32.

Solution. □

Problem 4.35.

Solution. □

Problem 4.36.

Solution. $\square$

Problem 4.37.

Solution. □

Problem 4.40.

Solution. □

Problem 4.43.

Solution. □

Problem 4.50.

Solution. □