Homework Assignment 2

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Problem 1. Convert the following linear programming problem to *standard form*:

$$\begin{aligned}
\text{maximize} \quad & 2x_1 + x_2 \\
\text{subject to} \quad & 0 \le x_1 \le 2 \\
& x_1 + x_2 \le 3 \\
& x_1 + 2x_2 \le 5 \\
& x_2 \ge 0
\end{aligned}$$

Solution. In order to convert this linear programming problem into standard form, we must transform the objective from *maximize* to *minimize* and the constraints must be transformed from linear inequalities into linear equations.

Our first step will be to rewrite the objective function as a minimization problem and write each constraint as a linear inequality as so:

$$\begin{aligned}
\text{minimize} \quad & -2x_1 - x_2 \\
\text{subject to} \quad & x_1 \le 2 \\
& x_1 + x_2 \le 3 \\
& x_1 + 2x_2 \le 5 \\
& x_1 \ge 0, x_2 \ge 0
\end{aligned}$$

We can then introduce three slack variables x_3, x_4, x_5 to turn the linear inequalities into linear equations:

$$\begin{aligned}
\text{minimize} \quad & -2x_1 \quad -x_2 \\
\text{subject to} \quad & x_1 \quad +x_2 \quad +x_3 \qquad\qquad\qquad = 2 \\
& x_1 \quad +x_2 \qquad\quad +x_4 \qquad\quad = 3 \\
& x_1 \quad +2x_2 \qquad\qquad\quad +x_5 = 5 \\
& x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0, \quad x_4 \ge 0, \quad x_5 \ge 0
\end{aligned}$$

As the above linear programming problem is written as

$$\begin{aligned}
\text{minimize} \quad & \boldsymbol{c}^\mathsf{T}\boldsymbol{x} \\
\text{subject to} \quad & A\boldsymbol{x} = \boldsymbol{b} \\
& \boldsymbol{x} \ge 0
\end{aligned}$$

where

$$\boldsymbol{c}^{\mathsf{T}} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}^{\mathsf{T}}, \quad A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

with $\boldsymbol{x} \geq 0$ and $\boldsymbol{b} \geq 0$ the linear programming problem is in standard form and we are done. $\qquad\square$

Problem 2. Solve the system $A\boldsymbol{x} = \boldsymbol{b}$ where

$$A = \begin{bmatrix} 2 & -1 & 2 & -1 & 3 \\ 1 & 2 & 3 & 1 & 0 \\ 1 & 0 & -2 & 0 & -5 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 14 \\ 5 \\ -10 \end{bmatrix}.$$

If possible, generate a non-basic feasible solution of the system from which you derive next a basic feasible one.

Solution. In order to solve the system $Ax = b$, we must perform row operations on the augmented matrix to reduce it to reduced row form. We perform these operations below:

$$\begin{bmatrix} 2 & -1 & 2 & -1 & 3 & | & 14 \\ 1 & 2 & 3 & 1 & 0 & | & 5 \\ 1 & 0 & -2 & 0 & -5 & | & -10 \end{bmatrix} \xrightarrow[\substack{[2]-(1/2)[1] \\ [3]-(1/2)[1]}]{(1/2)[1]} \begin{bmatrix} 1 & -1/2 & 1 & -1/2 & 3/2 & | & 7 \\ 0 & 5/2 & 2 & 3/2 & -3/2 & | & -2 \\ 0 & 1/2 & -3 & 1/2 & -13/2 & | & -17 \end{bmatrix} \xrightarrow{(2/5)[2]}$$

$$\begin{bmatrix} 1 & -1/2 & 1 & -1/2 & 3/2 & | & 7 \\ 0 & 1 & 4/5 & 3/5 & -3/5 & | & -4/5 \\ 0 & 1/2 & -3 & 1/2 & -13/2 & | & -17 \end{bmatrix} \xrightarrow[\substack{[3]-(1/2)[2]}]{[1]+(1/2)[2]} \begin{bmatrix} 1 & 0 & 7/5 & -1/5 & 6/5 & | & 33/5 \\ 0 & 1 & 4/5 & 3/5 & -3/5 & | & -4/5 \\ 0 & 0 & -17/5 & 1/5 & -31/5 & | & -83/5 \end{bmatrix}$$

$$\xrightarrow{(-5/17)[3]} \begin{bmatrix} 1 & 0 & 7/5 & -1/5 & 6/5 & | & 33/5 \\ 0 & 1 & 4/5 & 3/5 & -3/5 & | & -4/5 \\ 0 & 0 & 1 & -1/17 & 31/17 & | & 83/17 \end{bmatrix} \xrightarrow[\substack{[2]-(4/5)[3]}]{[1]-(7/5)[3]} \begin{bmatrix} 1 & 0 & 0 & -2/17 & -23/17 & | & -4/17 \\ 0 & 1 & 0 & 11/17 & -35/17 & | & -80/17 \\ 0 & 0 & 1 & -1/17 & 31/17 & | & 83/17 \end{bmatrix}.$$

Using the above row-reduced augmented matrix, we see that the solution to the system $A\boldsymbol{x} = \boldsymbol{b}$ is given by

$$\boldsymbol{x} = \begin{bmatrix} -4/17 \\ -80/17 \\ 83/17 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2/17 \\ -11/17 \\ 1/17 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 23/17 \\ 35/17 \\ -31/17 \\ 0 \\ 1 \end{bmatrix} \tag{1}$$

where $s \in \mathbb{R}$ and $t \in \mathbb{R}$.

Suppose that the matrix A is written such that $A = [\boldsymbol{a}_i]$ for $1 \leq i \leq 5$ where $\boldsymbol{a}_i$ corresponds to the i-th column of the original matrix A. Recall that a solution $\boldsymbol{x}_0 \geq \boldsymbol{0}$ of the system $A\boldsymbol{x} = \boldsymbol{b}$ is a basic feasible solution if the columns of the matrix A associated to the nonzero components of $\boldsymbol{x}_0$ are linearly independent. Otherwise the solution is a non-basic feasible solution.

Using solution (1), we see that for $s = 0$ and $t = 82/31$, we get the corresponding feasible solution $\boldsymbol{x}_0 = [1762/527, 390/527, 1/17, 0, 82/31]^\mathsf{T}$ to the system $A\boldsymbol{x} = \boldsymbol{b}$. We know that this is a non-basic feasible solution since the vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, and $\boldsymbol{a}_5$ must be linearly dependent as the $\text{rank}(A) = 3$, i.e. the maximum number of linearly independent columns of A is 3.

The Fundamental Theorem of LP prescribes how to move from this non-basic feasible solution $\boldsymbol{x}_0$ to a basic feasible solution $\boldsymbol{x}_1$. As $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, and $\boldsymbol{a}_5$ are linearly dependent, there exists constants y_1, y_2, y_3, y_5 not all zero such that

$$y_1\boldsymbol{a_1} + y_2\boldsymbol{a_2} + y_3\boldsymbol{a_3} + y_5\boldsymbol{a_5} = \boldsymbol{0},$$

namely $y_1 = 1$, $y_2 = 35/23$, $y_3 = -31/23$, and $y_5 = 17/23$. Thus, the vector $\epsilon \boldsymbol{y} = \epsilon[y_1, y_2, y_3, 0, y_5]^\mathsf{T}$ satisfies $A[\epsilon \boldsymbol{y}] = \boldsymbol{0}$. As such, the vector $\boldsymbol{x}_0 - \epsilon \boldsymbol{y}$ satisfies $A[\boldsymbol{x}_0 - \epsilon \boldsymbol{y}] = \boldsymbol{b}$, i.e. the vector $\boldsymbol{x}_0 - \epsilon \boldsymbol{y}$ is a solution of the original system $A\boldsymbol{x} = \boldsymbol{b}$. Choose

$$\epsilon = \min\{x_i/y_i | i = 1, 2, 3, 5 \ y_i > 0\} = -23/527.$$

Then the vector $\boldsymbol{x}_1 = \boldsymbol{x}_0 - \epsilon \boldsymbol{y}$ will have 3 positive components and the rest of the components will be 0 showing that the vector $\boldsymbol{x}_1$ is a basic feasible solution. Therefore,

$$\boldsymbol{x}_1 = \boldsymbol{x}_0 - \epsilon \boldsymbol{y} = \begin{bmatrix} 105/31 \\ 25/31 \\ 0 \\ 0 \\ 83/31 \end{bmatrix}$$

is the desired basic feasible solution. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Problem 3. Does every linear programming problem in standard form have a nonempty feasible set? If "yes", provide a proof. If "no", provide a counter-example.

Does every linear programming problem in standard form (assuming a nonempty feasible set) have an optimal solution? If "yes", provide a proof. If "no", provide a counter-example.

Solution. Not every linear programming problem (LP) in standard form has a nonempty feasible set. Take for instance the following LP:

$$\begin{aligned}\text{minimize} \quad & 2x_1 \\ \text{subject to} \quad & x_1 \leq -1 \\ & x_1 \geq 0\end{aligned}$$

Clearly this problem has an empty feasible set due to the contradictory constraints.

The LP in standard form is stated as:

$$\begin{aligned}\text{minimize} \quad & 2x_1 \\ \text{subject to} \quad & -x_1 - x_2 = 1 \\ & x_1 \geq 0, x_2 \geq 0\end{aligned}$$

The equation $-x_1 - x_2 = 1$ for $x_1, x_2 \geq 0$ has no solutions and the feasible set of the standard form LP is empty. Therefore, not every LP in standard form has a nonempty feasible set.

Additionally, not every LP in standard form with a nonempty feasible set has an optimal solution. Take for instance the following LP:

$$\begin{aligned}\text{minimize} \quad & -3x_1 \\ \text{subject to} \quad & x_1 \geq 1 \\ & x_1 \geq 0\end{aligned}$$

The problem can be written in standard form as so:

$$\begin{aligned}\text{minimize} \quad & -3x_1 \\ \text{subject to} \quad & x_1 - x_2 = 1 \\ & x_1 \geq 0, x_2 \geq 0\end{aligned}$$

The equation $x_1 - x_2 = 1$ for $x_1, x_2 \geq 0$ has an infinite number of solutions so that x_1 can be chosen arbitrarily large. Consequently, $-x_1$ can be made arbitrarily small and no point in the feasible set of this LP in standard form is the smallest, i.e optimal. Therefore, not every LP in standard form with a nonempty feasible set has an optimal solution. $\square$

Problem 4.   a. Solve the following linear program graphically:

$$\begin{aligned}
\text{maximize} \quad & 2x_1 + 5x_2 \\
\text{subject to} \quad & 0 \le x_1 \le 4 \\
& 0 \le x_2 \le 6 \\
& x_1 + x_2 \le 8
\end{aligned}$$

b. Solve the linear program in (b) the same way Example 15.15 was solved in class. Compute only the vertices that lead to the optimal vertex found at (a).

Solution.   □