Example 5.1

$$\phi(x) = \frac{1}{2}(x + \frac{2}{x})$$

- Find all fixed points of ϕ, i.e., solve $\phi(x) = x$
- Run a simple fixed-point iteration scheme, i.e.

$$x_{k+1} = \phi(x_k), \quad \text{starting with some} \quad x_0$$

- Determine the numerical order of convergence, i.e.,

$$|e_{n+1}| \sim c|e_n|^p, \text{when} \quad n \to \infty$$

- For what values of x_0 does it seem to converge?
- What equation did we actually solve? $x^2 = 2$

Basic convergence results

Let $\phi : I \to I$ be continuous. x^* is a fixed-point if

$$x^* = \phi(x^*)$$

Basic fixed-point iteration

$$x_0 \in I, \quad x_{k+1} = \phi(x_k), \quad k = 0, 1, \ldots$$

If $\{x_n\}$ converges, its limit is a fixed-point. Well-posed?

Contractive mappings

If $\phi : I \to I$ is a contractive mapping then it has a unique fixed point, and $\{x_n\}$ converges to it.

Contractive mapping: $|\phi(x) - \phi(y)| \leq \lambda |x - y|, \quad 0 \leq \lambda < 1$

- contractive $\to$ continuous

- $x_n = x_0 + (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1})$ **HW: prove this using abs convergence of series and geometric series**

- uniqueness

$|\phi'(x)| < 1 \quad \to \quad$ contractiveness

Error analysis

$e_n = x_n - x^*$ is the n-th term error, and p is the order of convergence if

$$|e_{n+1}| \sim c|e_n|^p, \text{when} \quad n \to \infty$$

- Typical approach

$$x_{n+1} - x^* = \phi(x_n) - \phi(x^*) = \phi'(\xi_n)(x_n - x^*)$$

- Order of convergence approach

$e_{n+1} = \phi(x_n) - \phi(x^*) = \phi(x^* + e_n) - \phi(x^*)$ followed by Taylor expansion

Order of convergence approach

- Example 5.1: determine the order of convergence analytically, $\phi(x) = \frac{1}{2}(x + \frac{2}{x})$

- In general, if we know enough derivatives

$$e_{n+1} = e_n\phi'(x^*) + \frac{1}{2}e_n^2\phi''(x^*) + \cdots + \frac{1}{(r-1)!}e_n^{r-1}\phi^{(r-1)}(x^*) + \frac{1}{r!}e_n^r\phi^{(r)}(\xi_n)$$

- The more derivatives of ϕ at the fixed-point vanish, the higher the order of convergence is

Newton's method

$$f(x) = 0 \quad \to \quad \phi(x) = x - \frac{f(x)}{f'(x)}, \quad f(x^*) = 0 \quad \to \quad \phi(x^*) = x^*$$

In fact,

$$f(x) = 0 \quad \to \quad \phi(x) = x - \frac{f(x)}{f'(x)}, \quad f(x^*) = 0 \quad \to \quad \phi(x^*) = x^*$$

Error analysis

$$e_{n+1} = e_n \phi'(x^*) + \frac{1}{2} e_n^2 \phi''(x^*) + \cdots + \frac{1}{(r-1)!} e_n^{r-1} \phi^{(r-1)}(x^*) + \frac{1}{r!} e_n^r \phi^{(r)}(\xi_n)$$

$$\phi'(x^*) = 0, \quad \phi''(x^*) \neq 0, \quad \to \quad p = 2$$