Homework Assignment 5

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Problem 3.23. Show that:

a. $\mathscr{L}\left\{t\cos(at)e^{-bt}\right\} = \dfrac{(s+b)^2 - a^2}{\left[(s+b)^2 + a^2\right]^2}$.

Solution. a. Let $f(t) = t\cos(at)$ and suppose that $\bar{f}(s) = \mathscr{L}\{f(t)\}$.

As shown previously, we know that

$$\bar{f}(s) = \mathscr{L}\{f(t)\} = \mathscr{L}\{t\cos(at)\} = \frac{s^2 - a^2}{(s^2 + a^2)^2}.$$

Therefore, by Heaviside's First Shifting Theorem,

$$\mathscr{L}\left\{t\cos(at)e^{-bt}\right\} = \mathscr{L}\left\{f(t)e^{-bt}\right\} = \bar{f}(s+b) = \frac{(s+b)^2 - a^2}{\left[(s+b)^2 + a^2\right]^2},$$

and we are done.

□

Problem 3.24. Suppose that $\mathscr{L}\left\{f(t)\right\} = \bar{f}(s)$ and $\mathscr{L}\left\{g(x,t)\right\} = \bar{h}(s)\exp(-x\bar{h}(s))$. Prove that:

a. $\mathscr{L}\left\{\int_0^\infty g(x,t)f(x)dx\right\} = \bar{h}(s)\bar{f}(\bar{h}(s))$.

Solution. a. From the definition of the Laplace transform, we have that

$$\mathscr{L}\left\{\int_0^\infty g(x,t)f(x)dx\right\} = \int_0^\infty \left[\int_0^\infty g(x,t)f(x)dx\right] e^{-st}dt.$$

Interchanging the order of integration yields that

$$\begin{aligned}\mathscr{L}\left\{\int_0^\infty g(x,t)f(x)dx\right\} &= \int_0^\infty \left[\int_0^\infty g(x,t)f(x)dx\right] e^{-st}dt \\ &= \int_0^\infty f(x)\left[\int_0^\infty g(x,t)e^{-st}dt\right]dx \\ &= \int_0^\infty f(x)\mathscr{L}\left\{g(x,t)\right\}dx.\end{aligned}$$

From the relation $\mathscr{L}\left\{g(x,t)\right\} = \bar{h}(s)\exp(-x\bar{h}(s))$, we thus see that

$$\begin{aligned}\mathscr{L}\left\{\int_0^\infty g(x,t)f(x)dx\right\} &= \int_0^\infty f(x)\mathscr{L}\left\{g(x,t)\right\}dx \\ &= \int_0^\infty f(x)\bar{h}(s)\exp(-x\bar{h}(s))dx.\end{aligned}$$

Using the definition of the Laplace transform, we see that

$$\bar{f}(\bar{h}(s)) = \int_0^\infty f(t)\exp(-\bar{h}(s)t)dt.$$

Therefore,

$$\begin{aligned}\mathscr{L}\left\{\int_0^\infty g(x,t)f(x)dx\right\} &= \int_0^\infty f(x)\bar{h}(s)\exp(-x\bar{h}(s))dx \\ &= \bar{h}(s)\int_0^\infty f(x)\exp(-x\bar{h}(s))dx \\ &= \bar{h}(s)\bar{f}(\bar{h}(s)).\end{aligned}$$

and we are done.

□

Problem 3.27. Use the Initial Value Theorem to find $f(0)$ and $f'(0)$ from the following functions:

a. $\bar{f}(s) = \dfrac{s}{s^2 - 5s + 12}$,

c. $\bar{f}(s) = \dfrac{e^{-sa}}{s^2 + 3s + 5}$, $a > 0$.

Solution. The Initial Value Theorem states that if $f(t)$ and its derivatives exist as $t \to 0$, then

$$\text{i.}\quad \lim_{s\to\infty} s\bar{f}(s) = f(0) \tag{1a}$$

$$\text{ii.}\quad \lim_{s\to\infty} [s^2\bar{f}(s) - sf(0)] = f'(0). \tag{1b}$$

a. If $\bar{f}(s) = \dfrac{s}{s^2 - 5s + 12}$, then (1a) of the Initial Value Theorem shows that

$$f(0) = \lim_{s\to\infty} s\bar{f}(s) = \lim_{s\to\infty} \frac{s^2}{s^2 - 5s + 12} = 1.$$

This implies from (1b) of the Initial Value Theorem that

$$\begin{aligned} f'(0) = \lim_{s\to\infty} [s^2\bar{f}(s) - sf(0)] &= \lim_{s\to\infty} \frac{s^3}{s^2 - 5s + 12} - s \\ &= \lim_{s\to\infty} \frac{s^3 - (s^3 - 5s^2 + 12s)}{s^2 - 5s + 12} \\ &= \lim_{s\to\infty} \frac{5s^2 - 12s}{s^2 - 5s + 12} \\ &= 5. \end{aligned}$$

c. Suppose that $p(s)$ and $q(s)$ are both polynomials in s and that $a > 0$. Then from L'Hospital's rule we have that

$$\lim_{s\to\infty} \frac{p(s)e^{-sa}}{q(s)} = \lim_{s\to\infty} \frac{p(s)}{e^{sa}q(s)} = 0. \tag{2}$$

If $\bar{f}(s) = \dfrac{e^{-sa}}{s^2 + 3s + 5}$ where $a > 0$, then (1a) of the Initial Value Theorem in combination with (2) shows that

$$f(0) = \lim_{s\to\infty} s\bar{f}(s) = \lim_{s\to\infty} \frac{se^{-sa}}{s^2 + 3s + 5} = 0.$$

Using this result, we have from (1b) of the Initial Value Theorem in combination with (2) that

$$f'(0) = \lim_{s\to\infty} [s^2\bar{f}(s) - sf(0)] = \lim_{s\to\infty} \frac{s^2e^{-sa}}{s^2 + 3s + 5} = 0.$$

□

Problem 3.28.

Solution. □

Problem 3.29.

Solution. □

Problem 3.32.

Solution. □

Problem 3.34.

Solution. □

Problem 4.1.

Solution. □