Homework Assignment 2

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Problem 1. Convert the following linear programming problem to *standard form*:

$$\begin{array}{rll} \text{maximize} & 2x_1 + x_2 & \\ \text{subject to} & 0 \leq x_1 & \leq 2 \\ & x_1 + x_2 & \leq 3 \\ & x_1 + 2x_2 & \leq 5 \\ & x_2 \geq 0 & \end{array}$$

Solution. In order to convert this linear programming problem into standard form, we must transform the objective from *maximize* to *minimize* and the constraints must be transformed from linear inequalities into linear equations.

Our first step will be to rewrite the objective function as a minimization problem and write each constraint as a linear inequality as so:

$$\begin{array}{rll} \text{minimize} & -2x_1 - x_2 & \\ \text{subject to} & x_1 & \leq 2 \\ & x_1 + x_2 & \leq 3 \\ & x_1 + 2x_2 & \leq 5 \\ & x_1 \geq 0, x_2 \geq 0 & \end{array}$$

We can then introduce three slack variables x_3, x_4, x_5 to turn the linear inequalities into linear equations:

$$\begin{array}{rllllll} \text{minimize} & -2x_1 & -x_2 & & & & \\ \text{subject to} & x_1 & +x_2 & +x_3 & & & = 2 \\ & x_1 & +x_2 & & +x_4 & & = 3 \\ & x_1 & +2x_2 & & & +x_5 & = 5 \\ & x_1 \geq 0, & x_2 \geq 0, & x_3 \geq 0, & x_4 \geq 0, & x_5 \geq 0 & \end{array}$$

As the above linear programming problem is written as

$$\begin{array}{rl} \text{minimize} & \boldsymbol{c}^\intercal \boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{array}$$

where

$$\boldsymbol{c}^{\mathsf{T}} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}^{\mathsf{T}}, \quad A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

with $\boldsymbol{x} \geq 0$ and $\boldsymbol{b} \geq 0$ the linear programming problem is in standard form and we are done. $\square$

Problem 2. Solve the system $A\boldsymbol{x} = \boldsymbol{b}$ where

$$A = \begin{bmatrix} 2 & -1 & 2 & -1 & 3 \\ 1 & 2 & 3 & 1 & 0 \\ 1 & 0 & -2 & 0 & -5 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 14 \\ 5 \\ -10 \end{bmatrix}.$$

If possible, generate a non-basic feasible solution of the system from which you derive next a basic feasible one.

Solution. In order to solve the system $Ax = b$, we must perform row operations on the augmented matrix to reduce it to reduced row form. We perform these operations below:

$$\left[\begin{array}{ccccc|c} 2 & -1 & 2 & -1 & 3 & 14 \\ 1 & 2 & 3 & 1 & 0 & 5 \\ 1 & 0 & -2 & 0 & -5 & -10 \end{array}\right] \xrightarrow[\substack{[2]-(1/2)[1] \\ [3]-(1/2)[1]}]{(1/2)[1]} \left[\begin{array}{ccccc|c} 1 & -1/2 & 1 & -1/2 & 3/2 & 7 \\ 0 & 5/2 & 2 & 3/2 & -3/2 & -2 \\ 0 & 1/2 & -3 & 1/2 & -13/2 & -17 \end{array}\right] \xrightarrow{(2/5)[2]}$$

$$\left[\begin{array}{ccccc|c} 1 & -1/2 & 1 & -1/2 & 3/2 & 7 \\ 0 & 1 & 4/5 & 3/5 & -3/5 & -4/5 \\ 0 & 1/2 & -3 & 1/2 & -13/2 & -17 \end{array}\right] \xrightarrow[{[3]-(1/2)[2]}]{[1]+(1/2)[2]} \left[\begin{array}{ccccc|c} 1 & 0 & 7/5 & -1/5 & 6/5 & 33/5 \\ 0 & 1 & 4/5 & 3/5 & -3/5 & -4/5 \\ 0 & 0 & -17/5 & 1/5 & -31/5 & -83/5 \end{array}\right]$$

$$\xrightarrow{(-5/17)[3]} \left[\begin{array}{ccccc|c} 1 & 0 & 7/5 & -1/5 & 6/5 & 33/5 \\ 0 & 1 & 4/5 & 3/5 & -3/5 & -4/5 \\ 0 & 0 & 1 & -1/17 & 31/17 & 83/17 \end{array}\right] \xrightarrow[{[2]-(4/5)[3]}]{[1]-(7/5)[3]} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -2/17 & -23/17 & -4/17 \\ 0 & 1 & 0 & 11/17 & -35/17 & -80/17 \\ 0 & 0 & 1 & -1/17 & 31/17 & 83/17 \end{array}\right].$$

Using the above row-reduced augmented matrix, we see that the solution to the system $A\boldsymbol{x} = \boldsymbol{b}$ is given by

$$\boldsymbol{x} = \begin{bmatrix} -4/17 \\ -80/17 \\ 83/17 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2/17 \\ -11/17 \\ 1/17 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 23/17 \\ 35/17 \\ -31/17 \\ 0 \\ 1 \end{bmatrix} \tag{1}$$

where $s \in \mathbb{R}$ and $t \in \mathbb{R}$.

Suppose that the matrix A is written such that $A = [\boldsymbol{a}_i]$ for $1 \leq i \leq 5$ where $\boldsymbol{a}_i$ corresponds to the i-th column of the original matrix A. Recall that a solution $\boldsymbol{x}_0 \geq \boldsymbol{0}$ of the system $A\boldsymbol{x} = \boldsymbol{b}$ is a basic feasible solution if the columns of the matrix A associated to the nonzero components of $\boldsymbol{x}_0$ are linearly independent. Otherwise the solution is a non-basic feasible solution.

Using solution (1), we see that for $s = 0$ and $t = 82/31$, we get the corresponding feasible solution $\boldsymbol{x}_0 = [1762/527, 390/527, 1/17, 0, 82/31]^\intercal$ to the system $A\boldsymbol{x} = \boldsymbol{b}$. We know that this is a non-basic feasible solution since the vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, and $\boldsymbol{a}_5$ must be linearly dependent as the rank$(A) = 3$, i.e. the maximum number of linearly independent columns of A is 3.

The Fundamental Theorem of LP prescribes how to move from this non-basic feasible solution $\boldsymbol{x}_0$ to a basic feasible solution $\boldsymbol{x}_1$. As $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, and $\boldsymbol{a}_5$ are linearly dependent, there exists constants y_1, y_2, y_3, y_5 not all zero such that

$$y_1\boldsymbol{a_1} + y_2\boldsymbol{a_2} + y_3\boldsymbol{a_3} + y_5\boldsymbol{a_5} = \boldsymbol{0},$$

namely $y_1 = 1$, $y_2 = 35/23$, $y_3 = -31/23$, and $y_5 = 17/23$. Thus, the vector $\epsilon\boldsymbol{y} = \epsilon[y_1, y_2, y_3, 0, y_5]^\intercal$ satisfies $A[\epsilon\boldsymbol{y}] = \boldsymbol{0}$. As such, the vector $\boldsymbol{x}_0 - \epsilon\boldsymbol{y}$ satisfies $A[\boldsymbol{x}_0 - \epsilon\boldsymbol{y}] = \boldsymbol{b}$, i.e. the vector $\boldsymbol{x}_0 - \epsilon\boldsymbol{y}$ is a solution of the original system $A\boldsymbol{x} = \boldsymbol{b}$. Choose

$$\epsilon = \min\{x_i/y_i | i = 1, 2, 3, 5 \ y_i > 0\} = -23/527.$$

Then the vector $\boldsymbol{x}_1 = \boldsymbol{x}_0 - \epsilon\boldsymbol{y}$ will have 3 positive components and the rest of the components will be 0 showing that the vector $\boldsymbol{x}_1$ is a basic feasible solution. Therefore,

$$\boldsymbol{x}_1 = \boldsymbol{x}_0 - \epsilon\boldsymbol{y} = \begin{bmatrix} 105/31 \\ 25/31 \\ 0 \\ 0 \\ 83/31 \end{bmatrix}$$

is the desired basic feasible solution. $\square$

Problem 3. Does every linear programming problem in standard form have a nonempty feasible set? If "yes", provide a proof. If "no", provide a counter-example.

Does every linear programming problem in standard form (assuming a nonempty feasible set) have an optimal solution? If "yes", provide a proof. If "no", provide a counter-example.

Solution. □

Problem 4. a. Solve the following linear program graphically:

$$\begin{array}{ll} \text{maximize} & 2x_1 + 5x_2 \\ \text{subject to} & 0 \le x_1 \le 4 \\ & 0 \le x_2 \le 6 \\ & x_1 + x_2 \le 8 \end{array}$$

b. Solve the linear program in (b) the same way Example 15.15 was solved in class. Compute only the vertices that lead to the optimal vertex found at (a).

Solution. □