Homework Assignment 9

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Problem 8.2.2. If $Sf(x)$ is the Schwarzian derivative of $f(x)$ with $f \in C^3$ and $F(x) = \dfrac{f''(x)}{f'(x)}$, show that $Sf(x) = F'(x) - (F(x))^2/2$.

Solution. Recall that the Schwarzian derivative of f is given by

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2.$$

We readily see from the definition of $F(x)$ that

$$\begin{aligned}
F'(x) - \frac{1}{2}\left[F(x)\right]^2 &= \frac{f'(x)f'''(x) - f''(x)^2}{f'(x)^2} - \frac{1}{2}\left[\frac{f''(x)}{f'(x)}\right]^2 \\
&= \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2 \\
&= Sf(x)
\end{aligned}$$

and we are done. $\square$

Problem 8.2.5. i. Show that if p is a polynomial of degree n having n distinct fixed points, and negative Schwarzian derivative, then not all of the fixed points can be attracting.

ii. On the other hand, show that the logistic maps $L_\mu : \mathbb{R} \to \mathbb{R}$ for $\mu > 2 + \sqrt{5}$ have negative Schwarzian derivative but have no attracting periodic orbits.

Solution. i. Suppose to the contrary that p is a polynomial of degree n with n distinct fixed points and negative Schwarzian derivative but all of its fixed points are attracting. Let $x_1, \ldots, x_n$ denote these attracting fixed points.

Since p is a polynomial, it is continuous, which implies that for each attracting fixed point x_k, its immediate basin of attraction W_k is an open interval. Note that these fixed points are distinct and attracting so that the immediate basins of attraction of two fixed points x_j and x_k with $j \neq k$ are mutually exclusive, i.e. $W_j \cap W_k = \emptyset$ for any $j \neq k$.

Since $p \in C^3$ with negative Schwarzian derivative, we have by Singer's theorem that for every fixed point x_k, either W_k is an unbounded interval, or the orbit of some critical point of p is attracted to the orbit of x_k under f.

From the above remarks, we see that p can have at most 2 fixed points with unbounded basins of attraction of the form $(-\infty, a]$ or $[a, \infty)$. Without loss of generality, assume that $x_1 < \cdots < x_n$ and that the fixed points x_1 and x_n have unbounded basins of attraction, i.e. $W_1 = (-\infty, a_1]$ and $W_n = [a_n, \infty)$ for some $a_1, a_n \in \mathbb{R}$.

Consider the fixed points $x_{k-1} < x_k < x_{k+1}$ of p. Then since all of these fixed points are attracting and $Sp(x) < 0$, we will have that p has a critical point in (x_{k-1}, x_{k+1}). Thus, for $k = 2, \ldots, n-1$, we have that p has a critical point in (x_{k-1}, x_{k+1}) and since $x_k \in (x_{k-1}, x_{k+1})$, we see that $W_k \cap (x_{k-1}, x_{k+1}) \neq \emptyset$.

ii. Suppose that $\mu > 2 + \sqrt{5}$ and consider $L_\mu : \mathbb{R} \to \mathbb{R}$ where $L_\mu(x) = \mu x(1-x)$. We readily see that $L'_\mu(x) = \mu(1-2x)$, $L''_\mu(x) = -2\mu$, and $L'''_\mu(x) = 0$ so that

$$
\begin{aligned}
SL_\mu(x) &= \frac{L'''_\mu(x)}{L'_\mu(x)} - \frac{3}{2}\left[\frac{L''_\mu(x)}{L'_\mu(x)}\right]^2 \\
&= -\frac{3}{2}\left[\frac{4\mu^2}{\mu^2(1-2x)^2}\right] \\
&= -\frac{6}{(1-2x)^2}.
\end{aligned}
$$

Therefore, we see that $SL_\mu(x) < 0$. Suppose that $\{c_0, \ldots, c_{n-1}\}$ is an n-cycle of $L_\mu(x)$ and that $x = c_0$ is the period n point that generates the cycle. If $\mu > 2 + \sqrt{5}$, then $x = 1/2$ is not a periodic point of L_μ since $L^2_\mu(1/2) \neq 1/2$ and $L^{n+1}_\mu(1/2) < L^n_\mu(1/2) < 0$ for all $n > 2$. Then we see that

$$
\left|L^n_\mu(c_0)'\right| = \left|L'_\mu(c_0) \cdots L'_\mu(c_{n-1})\right| = \mu^n \left|(1-2c_0) \cdots (1-2c_{n-1})\right| \geq \mu^n > 1
$$

so that L_μ has no attracting periodic orbit.

□

Problem 8.2.10. Suppose that f is a function such that $Sf(x) = 0$. Show that:

i. $\dfrac{f''(x)^2}{f'(x)^3} = k$ for some $k \in \mathbb{R}$,

ii. the function $f(x)$ is of the form $f(x) = \dfrac{ax+b}{cx+d}$.

Solution. i. We readily see that if $Sf(x) = 0$, then we have that $2f'''(x)f'(x) = 3f''(x)^2$ from the definition of the Schwarzian derivative.

We will now show that for $F(x) = \dfrac{f''(x)^2}{f'(x)^3}$, we have that $F'(x) = 0$ for all $x \in \mathbb{R}$ implying that $F(x) = k$ for some $k \in \mathbb{R}$. Calculating, we see, using the identity $2f'''(x)f'(x) = 3f''(x)^2$ we obtained from our assumption, that

$$\begin{aligned} F'(x) &= \frac{f'(x)^3\left(2f''(x)f'''(x)\right) - f''(x)^2\left(3f'(x)^2f''(x)\right)}{f'(x)^6} \\ &= \frac{f''(x)f'(x)^2(2f'''(x)f'(x)) - 3f''(x)^3f'(x)^2}{f'(x)^6} \\ &= \frac{3f''(x)^3f'(x)^2 - 3f''(x)^3f'(x)^2}{f'(x)^6} \\ &= 0 \end{aligned}$$

for all $x \in \mathbb{R}$. Therefore, we must have that $F(x) = \dfrac{f''(x)^2}{f'(x)^3} = k$ for some $k \in \mathbb{R}$.

ii. Let $y(x) = f'(x)$. Then, from the equation $\dfrac{f''(x)^2}{f'(x)^3} = k$, we have that $y'(x)^2 = ky(x)^3$, i.e. $y'(x) = k_1 y(x)^{3/2}$ where $k_1 = k^{1/2}$.

Solving this differential equation, we see that $-2y(x)^{-1/2} = k_1 x + k_2$ or that

$$y(x) = \frac{4}{(k_1x + k_2)^2}$$

for some $k_2 \in \mathbb{R}$. Replacing $y(x) = f'(x)$, we readily see that

$$f(x) = \int \frac{4dx}{(k_1x + k_2)^2}$$

or that

$$f(x) = -\frac{4}{k_1(k_1x + k_2)} + k_3.$$

Simplifying, we have that

$$\begin{aligned} f(x) &= -\frac{4}{k_1(k_1x + k_2)} + \frac{k_3k_1(k_1x + k_2)}{k_1(k_1x + k_2)} \\ &= \frac{k_1^2k_3x + k_1k_2k_3 - 4}{k_1^2x + k_1k_2} \end{aligned}$$

and $f(x)$ is of the form $f(x) = \dfrac{ax+b}{cx+d}$.

$\square$

Problem 10.3.4. The Sierpinski carpet is a 2-dimensional version of the Cantor set and the Menger sponge. Start with the unit square $[0,1] \times [0,1]$ partitioned into nine equal squares. Remove the "open middle third" square $(1/3, 2/3) \times (1/3, 2/3)$. From each of the remaining eight squares of side length $1/3$, remove the open middle third squares and continue indefinitely.

i. Show that the resulting area removed is equal to one square unit.

ii. Show that the box counting dimension of the Sierpinski carpet is $\log 8/\log 3$.

iii. Show that the Sierpinski carpet has no interior, i.e. contains no open balls in $\mathbb{R}^2$.

Solution. i. On the first iteration, one square of dimension $1/3 \times 1/3$ is removed. Each subsequent iteration removes 8 times as many squares that were removed in the previous iteration where the length of each square to be removed is $1/3$ of the length of the previous squares removed. Therefore, A_n, the area removed at each iteration n for $n = 1, 2, \ldots,$ is $A_n = 8^{n-1}/3^{2n}$ and the total area removed, A, is given by

$$A = \sum_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} \frac{8^{n-1}}{3^{2n}} = \frac{1}{8} \sum_{n=1}^{\infty} \left(\frac{8}{9}\right)^n = 1.$$

ii. Let K is a non-empty subset of $\mathbb{R}^n$ and let $N_\delta(K)$ be the minimum number of boxes of equal length $\delta > 0$ needed to cover the set K. Then the box-counting dimension of K is given by

$$\dim(K) = \lim_{\delta \to 0^+} \frac{\log N_\delta(K)}{\log 1/\delta}.$$

Now let $K \subseteq \mathbb{R}^2$ be the Sierpinski carpet. We will construct a sequence For the first iteration, we see that we would need 8 squares with side-length $1/3$ in order to completely cover the Sierpinski carpet. The next iteration would require 64 squares of side-length $1/9$ in order to completely cover the Sierpinski carpet. In general, for the n-th iteration, we would require $N_{\delta_n}(K) = 8^n$ squares of length $\delta_n = 3^{-n}$. Therefore,

$$\dim(K) = \lim_{\delta \to 0^+} \frac{\log N_\delta(K)}{\log 1/\delta} = \lim_{n \to \infty} \frac{\log N_{\delta_n}(K)}{\log 1/\delta_n} = \lim_{n \to \infty} \frac{\log 8^n}{\log 3^n} = \frac{\log 8}{\log 3}.$$

iii. Suppose to the contrary that the interior of K, the Sierpinski carpet, is non-empty, i.e. there is some point $x \in K$ such that a square centered at x is contained completely in K. Since this square is completely within the Sierpinski carpet, it contains a square defined by the coordinates $\left(\frac{j}{3^n}, \frac{k}{3^n}\right), \left(\frac{j}{3^n}, \frac{k+1}{3^n}\right), \left(\frac{j+1}{3^n}, \frac{k}{3^n}\right), \left(\frac{j+1}{3^n}, \frac{k+1}{3^n}\right)$ for some $0 < j, k < 3^n$ that are not multiples of 3. However, after the n-th iteration this square will be removed from K, contradicting the fact that the square centered at x is completely inside K. Therefore, the interior of the Sierpinski carpet is empty.

$\square$

Problem 10.3.6. Find the box-counting dimension of the set $M = \{0, 1, 1/2, 1/3, 1/4, \dots\}$.

Solution. Note that $M = \{0\} \cup \bigcup_{n \in \mathbb{Z}^+} \frac{1}{n}$ is a countable set.

For any $\delta > 0$, we can cover M with countably many intervals of length δ, i.e. for each $x \in M$, we have that $\{x\} \subseteq [x - \delta/2, x + \delta/2]$ so that

$$M \subseteq \left[-\frac{\delta}{2}, \frac{\delta}{2}\right] \cup \bigcup_{n \in \mathbb{Z}^+} \left[n - \frac{\delta}{2}, n + \frac{\delta}{2}\right].$$

Thus, the minimum number of boxes needed to cover M is $N_\delta(M) = \lim_{n\to\infty} n$ and the box-counting dimension of M is given by

$$\dim(M) = \lim_{\delta \to 0^+} \frac{\log N_\delta(M)}{\log 1/\delta} = \lim_{\delta \to 0^+} \lim_{n\to\infty} \frac{\log n}{\log 1/\delta} = \lim_{n\to\infty} \frac{\log n}{\log n} = 1.$$

Note that M is a collection of points so its topological dimension is 0. Therefore, M is a fractal since its box-counting dimension is strictly greater than its topological dimension. $\square$

Problem 10.3.7. The first three steps in the construction of the fractal shown are indicated below. Determine the fractal dimension.

Solution. To begin, partition the unit square into nine equal squares and let the bottom left corner of the unit square be positioned at coordinate $(0,0)$. For the first iteration, remove the "outer middle squares", i.e. the squares of side-length $1/3$ with top left upper-coordinate in the set $\{(0,2/3),(1/3,1),(1/3,1/3),(2/3,2/3)\}$. For each subsequent iteration, remove the outer middle squares of the squares remaining from the previous iteration.

Thus, in the first iteration, in order to cover the fractal, we would require 5 squares of side-length $1/3$. In the second iteration, in order to cover the fractal, we would require 25 squares of side-length $1/9$ and for the third iteration we would require 125 squares of side-length $1/27$.

Let K be the fractal shown. In general, for the n-th iteration, we would require $N_{\delta_n}(K) = 5^n$ squares of length $\delta_n = 3^{-n}$ in order to completely cover K. Therefore, we see that the fractal dimension of K is

$$\dim(K) = \lim_{\delta \to 0^+} \frac{\log N_\delta(K)}{\log 1/\delta} = \lim_{n\to\infty} \frac{\log N_{\delta_n}(K)}{\log 1/\delta_n} = \lim_{n\to\infty} \frac{\log 5^n}{\log 3^n} = \frac{\log 5}{\log 3}.$$

□