

Homework Assignment 2

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Problem 2.10. Solve the Cauchy problem for the Klein-Gordon equation

$$\begin{aligned}u_{tt} - c^2 u_{xx} + a^2 u &= 0, & -\infty < x < \infty, \quad t > 0, \\u(x, 0) &= f(x) & \text{for } -\infty < x < \infty, \\ \left[\frac{\partial u}{\partial t} \right]_{t=0} &= g(x) & \text{for } -\infty < x < \infty.\end{aligned}$$

Solution. Consider the function $u(x, y)$. The Fourier transform of u with respect to x is defined as

$$\mathcal{F}\{u(x, y)\} = U(k, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} u(x, y) dx. \quad (1)$$

From this definition and the Leibniz integral rule, we can see by induction that

$$\begin{aligned}\mathcal{F}\left\{\frac{\partial^n}{\partial y^n}[u(x, y)]\right\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^n}{\partial y^n}[u(x, y)] e^{-ikx} dx \\&= \frac{d^n}{dy^n} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, y) e^{-ikx} dx \right] \\&= \frac{d^n}{dy^n} [\mathcal{F}\{u(x, y)\}].\end{aligned} \quad (2)$$

Similarly, we see from definition (1) and previous theorems regarding the Fourier transform that

$$\begin{aligned}\mathcal{F}\left\{\frac{\partial^n}{\partial x^n}[u(x, y)]\right\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^n}{\partial x^n}[u(x, y)] e^{-ikx} dx \\&= (ik)^n \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, y) e^{-ikx} dx \right] \\&= (ik)^n \mathcal{F}\{u(x, y)\}.\end{aligned} \quad (3)$$

Now, applying the Fourier transform to the first equation, we see that

$$\begin{aligned}\mathcal{F}\{u_{tt} - c^2 u_{xx} + a^2 u\} &= \frac{d^2}{dt^2} [U(k, t)] - c^2 (ik)^2 U(k, t) + a^2 U(k, t) \\&= \frac{d^2}{dt^2} [U(k, t)] + (c^2 k^2 + a^2) U(k, t).\end{aligned}$$

Thus, setting $\omega^2 = c^2 k^2 + a^2$, we see that

$$\frac{d^2}{dt^2} [U(k, t)] + \omega^2 U(k, t) = 0.$$

This is a second-order linear homogeneous ordinary differential equation, the solution to which we readily see is given by

$$U(k, t) = c_1 e^{-i\omega t} + c_2 e^{i\omega t}. \quad (4)$$

Applying the Fourier transform to the last two equations yields

$$\mathcal{F} \{u(x, 0)\} = U(k, 0) = F(k) = \mathcal{F} \{f(x)\}$$

and

$$\mathcal{F} \left\{ \left[\frac{\partial u}{\partial t} \right]_{t=0} \right\} = \frac{d}{dt} [U(k, t)]_{t=0} = G(k) = \mathcal{F} \{g(x)\}.$$

Using (5), we see that the first equation reduces to

$$c_1 + c_2 = F(k).$$

Taking the derivative of $U(k, t)$ with respect to t yields

$$\frac{d}{dt} [U(k, t)] = -i\omega c_1 e^{-i\omega t} + i\omega c_2 e^{i\omega t}$$

and evaluating when $t = 0$ produces a second equation

$$i\omega(c_2 - c_1) = G(k).$$

This results in a system of two equations in two unknowns; the solution of which is given by

$$c_1 = \frac{\omega F(k) + iG(k)}{2\omega}, \quad c_2 = \frac{\omega F(k) - iG(k)}{2\omega}.$$

Therefore, (5) becomes

$$U(k, t) = \left(\frac{\omega F(k) + iG(k)}{2\omega} \right) e^{-i\omega t} + \left(\frac{\omega F(k) - iG(k)}{2\omega} \right) e^{i\omega t}.$$

Taking the Inverse Fourier transform yields that the solution to original differential equation is given by

$$\begin{aligned} u(x, t) &= \mathcal{F}^{-1} \left\{ \left(\frac{\omega F(k) + iG(k)}{2\omega} \right) e^{-i\omega t} + \left(\frac{\omega F(k) - iG(k)}{2\omega} \right) e^{i\omega t} \right\} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\left(\frac{\omega F(k) + iG(k)}{2\omega} \right) e^{-i\omega t} + \left(\frac{\omega F(k) - iG(k)}{2\omega} \right) e^{i\omega t} \right] e^{ikx} dk \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{\omega F(k) + iG(k)}{2\omega} \right) e^{i(kx - \omega t)} + \left(\frac{\omega F(k) - iG(k)}{2\omega} \right) e^{i(kx + \omega t)} dk. \end{aligned}$$

□

Problem 2.12. Solve the equation

$$\begin{aligned} u_{tt} + u_{xxxx} &= 0, & -\infty < x < \infty, & \quad t > 0 \\ u(x, 0) &= f(x), & u_t(x, 0) &= 0 & \quad \text{for } -\infty < x < \infty. \end{aligned}$$

Solution. Using (2) and (3), we begin by applying the Fourier transform to the first differential equation yielding

$$\mathcal{F}\{u_{tt} + u_{xxxx}\} = \frac{d^2}{dt^2} [U(k, t)] + k^4 U(k, t) = 0 = \mathcal{F}\{0\}.$$

This is a second-order linear homogeneous differential equation, the solution to which we readily see is given by

$$U(k, t) = c_1 \cos(k^2 t) + c_2 \sin(k^2 t). \quad (5)$$

Applying the Fourier transform to the last two equations yields

$$\mathcal{F}\{u(x, 0)\} = U(k, 0) = F(k) = \mathcal{F}\{f(x)\}$$

and

$$\mathcal{F}\{u_t(x, 0)\} = \mathcal{F}\left\{\left[\frac{\partial u}{\partial t}\right]_{t=0}\right\} = \frac{d}{dt} [U(k, t)] \Big|_{t=0} = 0 = \mathcal{F}\{0\}.$$

Using the form (5) of the solution we see from the second equation that

$$U(k, 0) = c_1 = F(k).$$

Similarly, we also see from the last equation that

$$\frac{d}{dt} [U(k, t)] = -c_1 k^2 \sin(k^2 t) + c_2 k^2 \cos(k^2 t)$$

which implies that

$$\frac{d}{dt} [U(k, t)] \Big|_{t=0} = c_2 k^2 = 0.$$

Since this equation must hold for all k , this implies that $c_2 = 0$. Thus, the solution (5) becomes

$$U(k, t) = F(k) \cos(k^2 t).$$

From the Convolution Theorem, we see that

$$\begin{aligned} u(x, t) &= \mathcal{F}^{-1}\{F(k) \cos(k^2 t)\} = (f * g)(x) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - \xi) g(\xi) d\xi \end{aligned}$$

where $g(x) = \mathcal{F}^{-1} \{ \cos(k^2 t) \}$.

Note that it can be shown that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik(x \pm kt)} dk = \frac{1}{2} \left(\frac{(1 \pm i)e^{\mp ix^2/4t}}{\sqrt{t}} \right).$$

Thus, using the definition of the complex exponential, we have that

$$\begin{aligned} g(x) &= \mathcal{F}^{-1} \{ \cos(k^2 t) \} \\ &= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik(x+kt)} + e^{ik(x-kt)} dk \\ &= \frac{1}{4\sqrt{t}} \left[(1+i)e^{-ix^2/4t} + (1-i)e^{ix^2/4t} \right] \\ &= \frac{1}{2\sqrt{t}} \left[\frac{e^{ix^2/4t} + e^{-ix^2/4t}}{2} + \frac{e^{ix^2/4t} - e^{-ix^2/4t}}{2i} \right] \\ &= \frac{\cos\left(\frac{x^2}{4t}\right) + \sin\left(\frac{x^2}{4t}\right)}{2\sqrt{t}}. \end{aligned}$$

Therefore, the solution to the original differential equation is given by

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - \xi) g(\xi) d\xi \\ &= \frac{1}{2\sqrt{2\pi t}} \int_{-\infty}^{\infty} f(x - \xi) \left(\cos\left(\frac{\xi^2}{4t}\right) + \sin\left(\frac{\xi^2}{4t}\right) \right) d\xi. \end{aligned}$$

□

Problem 2.14. Obtain the Fourier cosine transforms of the following functions:

- a. xe^{-ax} , $a > 0$.

Solution. Recall that the definition of the Fourier cosine transform of a function $f(x)$ is given by

$$\mathcal{F}_c\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos kx f(x) dx.$$

- a. From the definition of the Fourier cosine transform we have that

$$\mathcal{F}_c\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}} \int_0^\infty xe^{-ax} \cos kx dx.$$

Using the definition of the complex exponential, we see that

$$\begin{aligned} \mathcal{F}_c\{xe^{-ax}\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty xe^{-ax} \left[\frac{e^{-ikx} + e^{ikx}}{2} \right] dx \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty x [e^{-(a+ik)x} + e^{-(a-ik)x}] dx. \end{aligned}$$

Now, for $w = a \pm ik$ with $a > 0$, we see using integration by parts with $u = x$ and $dv = e^{-wx} dx$ that

$$\int_0^\infty xe^{-wx} dx = -\frac{xe^{-wx}}{w} \Big|_0^\infty + \frac{1}{w} \int_0^\infty e^{-wx} dx.$$

Note that

$$\lim_{x \rightarrow \infty} |e^{-wx}| = \lim_{x \rightarrow \infty} |e^{-(a \pm ik)x}| = \lim_{x \rightarrow \infty} |e^{-ax}| |e^{\mp ikx}| \leq \lim_{x \rightarrow \infty} |e^{-ax}| = 0.$$

This implies that $\lim_{x \rightarrow \infty} e^{-wx} = 0$. Thus,

$$\begin{aligned} \int_0^\infty xe^{-wx} dx &= -\frac{xe^{-wx}}{w} \Big|_0^\infty + \frac{1}{w} \int_0^\infty e^{-wx} dx \\ &= -\frac{1}{w^2} [e^{-wx}]_0^\infty \\ &= \frac{1}{w^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{F}_c\{xe^{-ax}\} &= \frac{1}{\sqrt{2\pi}} \left[\int_0^\infty xe^{-(a+ik)x} dx + \int_0^\infty xe^{-(a-ik)x} dx \right] \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{(a+ik)^2} + \frac{1}{(a-ik)^2} \right] \\ &= \sqrt{\frac{2}{\pi}} \frac{a^2 - k^2}{(a^2 + k^2)^2}. \end{aligned}$$

□

Problem 2.15. Find the Fourier sine transform of the following functions:

a. xe^{-ax} , $a > 0$.

b. $\frac{1}{x}e^{-ax}$, $a > 0$.

Solution. Recall that the definition of the Fourier sine transform of a function $f(x)$ is given by

$$\mathcal{F}_s\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin kx f(x) dx.$$

a. From the definition of the Fourier cosine transform we have that

$$\mathcal{F}_s\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}} \int_0^\infty xe^{-ax} \sin kx dx.$$

Using the definition of the complex exponential, we see that

$$\begin{aligned} \mathcal{F}_s\{xe^{-ax}\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty xe^{-ax} \left[\frac{e^{ikx} - e^{-ikx}}{2i} \right] dx \\ &= -\frac{i}{\sqrt{2\pi}} \int_0^\infty x [e^{-(a-ik)x} - e^{-(a+ik)x}] dx. \end{aligned}$$

Now, for $w = a \pm ik$ with $a > 0$, we see using integration by parts with $u = x$ and $dv = e^{-wx} dx$ that

$$\int_0^\infty xe^{-wx} dx = -\frac{xe^{-wx}}{w} \Big|_0^\infty + \frac{1}{w} \int_0^\infty e^{-wx} dx.$$

Note that

$$\lim_{x \rightarrow \infty} |e^{-wx}| = \lim_{x \rightarrow \infty} |e^{-(a \pm ik)x}| = \lim_{x \rightarrow \infty} |e^{-ax}| |e^{\mp ikx}| \leq \lim_{x \rightarrow \infty} |e^{-ax}| = 0.$$

This implies that $\lim_{x \rightarrow \infty} e^{-wx} = 0$. Thus,

$$\begin{aligned} \int_0^\infty xe^{-wx} dx &= -\frac{xe^{-wx}}{w} \Big|_0^\infty + \frac{1}{w} \int_0^\infty e^{-wx} dx \\ &= -\frac{1}{w^2} [e^{-wx}]_0^\infty \\ &= \frac{1}{w^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{F}_s\{xe^{-ax}\} &= -\frac{i}{\sqrt{2\pi}} \left[\int_0^\infty xe^{-(a-ik)x} dx - \int_0^\infty xe^{-(a+ik)x} dx \right] \\ &= -\frac{i}{\sqrt{2\pi}} \left[\frac{1}{(a-ik)^2} - \frac{1}{(a+ik)^2} \right] \\ &= \sqrt{\frac{2}{\pi}} \frac{2ak}{(a^2 + k^2)^2}. \end{aligned}$$



Problem 2.20. Apply the Fourier cosine transform to find the solution $u(x, y)$ of the problem

$$\begin{aligned}u_{xx} + u_{yy} &= 0, & 0 < x < \infty, \quad 0 < y < \infty \\u(x, 0) &= H(a - x), & x < a \\u_x(0, y) &= 0, & 0 < x, \quad y < \infty.\end{aligned}$$

Solution.

□

Problem 2.22. Solve the diffusion equation in the semi-infinite line

$$u_t = \kappa u_{xx}, \quad 0 \leq x < \infty, \quad t > 0,$$

with the boundary and initial data

$$\begin{aligned} u(0, t) &= 0 && \text{for } t > 0, \\ u(x, t) &\rightarrow 0 && \text{as } x \rightarrow \infty \text{ for } t > 0, \\ u(x, 0) &= f(x) && \text{for } 0 < x < \infty. \end{aligned}$$

Solution.

□