Homework Assignment 4

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Problem 1. Find the dual of the following linear programs:

a. Maximize $f(\boldsymbol{x}) = \boldsymbol{c}^\mathsf{T}\boldsymbol{x}$ subject to $A\boldsymbol{x} = \boldsymbol{b}$.

b. Maximize $2x_1 + 5x_2 + x_3$ subject to $\begin{cases} 2x_1 - x_2 + 7x_3 \leq 6 \\ x_1 + 3x_2 + 4x_3 \leq 9 \\ 3x_1 + 6x_2 + x_3 \leq 3 \\ x_1, x_2, x_3 \geq 0. \end{cases}$ via the symmetric form of duality.

Solution. a. Note that for this problem, the variable $\boldsymbol{x}$ is unconstrained in sign. After making the substitution $\boldsymbol{x} = \boldsymbol{x_1} - \boldsymbol{x_2}$ with $\boldsymbol{x_1}, \boldsymbol{x_2} \geq \boldsymbol{0}$, this problem in standard form is then stated as

$$\begin{aligned} \text{minimize} \quad & -\boldsymbol{c}^\mathsf{T}(\boldsymbol{x_1} - \boldsymbol{x_2}) \\ \text{subject to} \quad & A(\boldsymbol{x_1} - \boldsymbol{x_2}) = \boldsymbol{b} \\ & \boldsymbol{x_1}, \boldsymbol{x_2} \geq \boldsymbol{0}. \end{aligned}$$

The realization that the equality $A(\boldsymbol{x_1} - \boldsymbol{x_2}) = \boldsymbol{b}$ can be represented as the system of inequalities

$$\begin{aligned} A(\boldsymbol{x_1} - \boldsymbol{x_2}) &\geq \boldsymbol{b} \\ -A(\boldsymbol{x_1} - \boldsymbol{x_2}) &\geq -\boldsymbol{b} \end{aligned}$$

yields that the standard form of the LP is equivalent to:

$$\begin{aligned} \text{minimize} \quad & -\boldsymbol{c}^\mathsf{T}\boldsymbol{x_1} + \boldsymbol{c}^\mathsf{T}\boldsymbol{x_2} \\ \text{subject to} \quad & A\boldsymbol{x_1} - A\boldsymbol{x_2} \geq \boldsymbol{b} \\ & -A\boldsymbol{x_1} + A\boldsymbol{x_2} \geq -\boldsymbol{b} \\ & \boldsymbol{x_1}, \boldsymbol{x_2} \geq \boldsymbol{0}. \end{aligned}$$

But this can be stated as

$$\text{minimize} \quad \begin{bmatrix} -\boldsymbol{c} \\ \boldsymbol{c} \end{bmatrix}^\mathsf{T} \begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \end{bmatrix}$$

$$\text{subject to} \quad \begin{bmatrix} A & -A \\ -A & A \end{bmatrix} \begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \end{bmatrix} \geq \begin{bmatrix} \boldsymbol{b} \\ -\boldsymbol{b} \end{bmatrix}$$

$$\boldsymbol{x_1}, \boldsymbol{x_2} \geq \boldsymbol{0}$$

or, more succinctly,

$$\begin{array}{rl} \text{minimize} & \boldsymbol{C}^\top \boldsymbol{X} \\ \text{subject to} & \mathscr{A}\boldsymbol{X} \geq \boldsymbol{B} \\ & \boldsymbol{X} \geq \boldsymbol{0} \end{array} \tag{1}$$

where

$$\mathscr{A} = \begin{bmatrix} A & -A \\ -A & A \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} -\boldsymbol{c} \\ \boldsymbol{c} \end{bmatrix}, \quad \boldsymbol{X} = \begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{b} \\ -\boldsymbol{b} \end{bmatrix}. \tag{2}$$

By definition, the dual of the primal problem (1) is

$$\begin{array}{rl} \text{maximize} & \boldsymbol{B}^\top \boldsymbol{\Lambda} \\ \text{subject to} & \mathscr{A}^\top \boldsymbol{\Lambda} \leq \boldsymbol{C} \\ & \boldsymbol{\Lambda}^\top = \begin{bmatrix} \boldsymbol{\lambda_1}^\top \boldsymbol{\lambda_2}^\top \end{bmatrix} \geq \boldsymbol{0}^\top. \end{array} \tag{3}$$

Using the corresponding definitions found in (2), we see that after some algebraic manipulation the dual problem (3) can be written as

$$\begin{array}{rl} \text{maximize} & \boldsymbol{b}^\top(\boldsymbol{\lambda_1} - \boldsymbol{\lambda_2}) \\ \text{subject to} & A^\top(\boldsymbol{\lambda_1} - \boldsymbol{\lambda_2}) \leq -\boldsymbol{c} \\ & A^\top(\boldsymbol{\lambda_1} - \boldsymbol{\lambda_1}) \geq -\boldsymbol{c} \\ & \boldsymbol{\lambda_1}, \boldsymbol{\lambda_2} \geq \boldsymbol{0}. \end{array}$$

Noting that the system of inequalities can be written as an equality and making the substitution $\boldsymbol{\lambda} = (\boldsymbol{\lambda_1} - \boldsymbol{\lambda_2})$ where $\boldsymbol{\lambda}$ is free, we see that the dual of the problem

$$\begin{array}{rl} \text{maximize} & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \end{array}$$

is

$$\begin{array}{rl} \text{minimize} & -\boldsymbol{b}^\top \boldsymbol{\lambda} \\ \text{subject to} & A^\top \boldsymbol{\lambda} = -\boldsymbol{c}. \end{array}$$

b. Note that the linear program

$$\begin{array}{rl} \text{maximize} & 2x_1 + 5x_2 + x_3 \\ \text{subject to} & 2x_1 - x_2 + 7x_3 \leq 6 \\ & x_1 + 3x_2 + 4x_3 \leq 9 \\ & 3x_1 + 6x_2 + x_3 \leq 3 \\ & x_1, x_2, x_3 \geq 0. \end{array} \tag{4}$$

can be written as

$$\begin{array}{rl} \text{maximize} & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{array}$$

where

$$A = \begin{bmatrix} 2 & -1 & 7 \\ 1 & 3 & 4 \\ 3 & 6 & 1 \end{bmatrix}, \quad \boldsymbol{c} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 6 \\ 9 \\ 3 \end{bmatrix}.$$

Some algebraic manipulations allows us to write the above problem as

$$\begin{array}{rl} \text{minimize} & -\boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} & -A\boldsymbol{x} \geq -\boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \tag{5}$$

By definition, the symmetric dual to the primal problem (5) is

$$\begin{array}{rl} \text{maximize} & -\boldsymbol{b}^\top \boldsymbol{\lambda} \\ \text{subject to} & -A^\top \boldsymbol{\lambda} \leq -\boldsymbol{c} \\ & \boldsymbol{\lambda} = [\lambda_1, \lambda_2, \lambda_3]^\top \geq \boldsymbol{0}. \end{array}$$

Therefore, the dual to the primal problem (5) can be written as

$$\begin{array}{rrl} \text{maximize} & -6\lambda_1 - 9\lambda_2 - 3\lambda_3 & \\ \text{subject to} & -2\lambda_1 - \lambda_2 - 3\lambda_3 & \leq -2 \\ & \lambda_1 - 3\lambda_2 - 6\lambda_3 & \leq -5 \\ & -7\lambda_1 - 4\lambda_2 - \lambda_3 & \leq -1 \\ & \lambda_1, \lambda_2, \lambda_3 \geq 0 & \end{array}$$

and the dual to the original primal problem (4) is

$$\begin{array}{rrl} \text{minimize} & 6\lambda_1 + 9\lambda_2 + 3\lambda_3 & \\ \text{subject to} & 2\lambda_1 + \lambda_2 + 3\lambda_3 & \geq 2 \\ & -\lambda_1 + 3\lambda_2 + 6\lambda_3 & \geq 5 \\ & 7\lambda_1 + 4\lambda_2 + \lambda_3 & \geq 1 \\ & \lambda_1, \lambda_2, \lambda_3 \geq 0. & \end{array}$$

□

Problem 2. a. Prove (via the symmetric form of duality) that the dual of the dual problem in an asymmetric form of duality is the primal (standard) problem.

b. Prove the weak duality proposition for the symmetric form of duality.

c. Prove that the primal problem is infeasible if and only if the dual problem is unbounded.

Solution. □

Problem 3. Prove the Duality Theorem for the symmetric case.

Solution. □

Problem 4. Consider the following linear program:

$$\begin{array}{rll} \text{maximize} & 2x_1 + 3x_2 & \\ \text{subject to} & x_1 + 2x_2 & \leq 4 \\ & 2x_1 + x_2 & \leq 5 \\ & x_1, x_2 & \geq 0. \end{array}$$

a. Use the simplex method to solve the problem.

b. Write down the dual of the linear program and solve the dual.

Solution. □

Problem 5. Consider the following primal problem:

$$
\begin{array}{rllllll}
\text{maximize} & x_1 & +2x_2 & & & & \\
\text{subject to} & -2x_1 & +x_2 & +x_3 & & & = 2 \\
 & -x_1 & +2x_2 & & +x_4 & & = 7 \\
 & x_1 & & & & +x_5 & = 3 \\
 & x_i \geq 0 & i = 1,2,3,4,5. & & & &
\end{array}
$$

a. Construct the dual problem corresponding to the primal problem above.

b. It is known that the solution to the primal above is $\boldsymbol{x}^* = [3, 5, 3, 0, 0]^\top$. Find the solution to the dual.

Solution. $\square$

Problem 6. Let A be a given matrix and $\boldsymbol{b}$ a given vector. We wish to prove the following result: There exists a vector $\boldsymbol{x}$ such that $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq \mathbf{0}$ if and only if for any given vector $\boldsymbol{y}$ satisfying $A^\top \boldsymbol{y} \leq \mathbf{0}$ we have $\boldsymbol{b}^\top \boldsymbol{y} \leq \mathbf{0}$. This result is known as *Farkas's transposition theorem*. Our program is based on duality theory, consisting of the parts listed below.

a. Consider the primal linear program

$$\begin{aligned} \text{minimize} \quad & \mathbf{0}^\top \boldsymbol{x} \\ \text{subject to} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \mathbf{0}. \end{aligned}$$

   Write down the dual of this problem using the notation $\boldsymbol{y}$ for the dual variable.

b. Show that the feasible set of the dual problem is guaranteed to be nonempty.

   Hint: Think about an obvious feasible point.

c. Suppose that for any $\boldsymbol{y}$ satisfying $A^\top \boldsymbol{y} \leq \mathbf{0}$, we have $\boldsymbol{b}^\top \boldsymbol{y} \leq 0$. In this case what can you say about whether or not the dual has an optimal feasible solution.

   Hint: Think about the obvious feasible point in part b.

d. Suppose that for any $\boldsymbol{y}$ satisfying $A^\top \boldsymbol{y} \leq \mathbf{0}$, we have $\boldsymbol{b}^\top \boldsymbol{y} \leq 0$. Use parts b and c to show that there exists $\boldsymbol{x}$ such that $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq \mathbf{0}$. (This proves one direction of Farkas's transposition theorem.)

e. Suppose that $\boldsymbol{x}$ satisfies $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq \mathbf{0}$. Let $\boldsymbol{y}$ be an arbitrary vector satisfying $A^\top \boldsymbol{y} \leq \mathbf{0}$. (This proves the other direction of Farkas's transposition theorem.)

Solution. □