Exam 1

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Problem 1. Find the Fourier Transforms of the following functions:

a. $f(x) = x^2 e^{-a|x|}$, $a > 0$,

b. $f(x) = \left(1 - \frac{|x|}{2}\right) H\left(1 - \frac{|x|}{2}\right)$.

Solution. Recall that if $f(x) \in L^1(\mathbb{R})$, then the Fourier Transform of f is defined to be

$$\mathscr{F}\left\{f(x)\right\} = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx. \tag{1}$$

a. If $f(x) = x^2 e^{-a|x|}$, $a > 0$, we see from (1) that, by definition, the Fourier Transform of f is given by

$$\begin{aligned}\mathscr{F}\left\{f(x)\right\} = F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-a|x|} e^{-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} x^2 e^{-(-a+ik)x} dx + \int_{0}^{\infty} x^2 e^{-(a+ik)x} dx\right].\end{aligned}$$

Now, we see by integration by parts that

$$\begin{aligned}\int_c^d x^2 e^{-(b+ik)x} dx &= -\frac{x^2}{b+ik} e^{-(b+ik)x}\bigg|_c^d + \frac{2}{b+ik} \int_c^d x e^{-(b+ik)x} dx \\ &= -\frac{x^2}{b+ik} e^{-(b+ik)x}\bigg|_c^d + \frac{2}{b+ik}\left[-\frac{x}{b+ik} e^{-(b+ik)x}\bigg|_c^d + \frac{1}{b+ik}\int_c^d e^{-(b+ik)x} dx\right] \\ &= -\frac{x^2}{b+ik} e^{-(b+ik)x}\bigg|_c^d + \frac{2}{b+ik}\left[-\frac{x}{b+ik} e^{-(b+ik)x}\bigg|_c^d - \frac{1}{(b+ik)^2} e^{-(b+ik)x}\bigg|_c^d\right].\end{aligned} \tag{2}$$

Thus,

$$\int_{-\infty}^{0} x^2 e^{-(-a+ik)x} dx = -\frac{2}{(-a+ik)^3}$$

and

$$\int_0^\infty x^2 e^{-(a+ik)x} dx = \frac{2}{(a+ik)^3}.$$

Therefore,

$$\begin{aligned}
\mathscr{F}\{f(x)\} = F(k) &= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^0 x^2 e^{-(-a+ik)x} dx + \int_0^\infty x^2 e^{-(a+ik)x} dx \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[-\frac{2}{(-a+ik)^3} + \frac{2}{(a+ik)^3} \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{4a(a^2-3k)}{(a^2+k^2)^3} \right] \\
&= \sqrt{\frac{2}{\pi}} \left[\frac{2a(a^2-3k)}{(a^2+k^2)^3} \right].
\end{aligned}$$

b. Recall that the Heaviside function H is defined as

$$H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Thus,

$$H\left(1 - \frac{|x|}{2}\right) = \begin{cases} 1 & \text{if } |x| < 2 \\ 0 & \text{if } |x| > 2. \end{cases}$$

If $f(x) = \left(1 - \frac{|x|}{2}\right) H\left(1 - \frac{|x|}{2}\right)$, we see from (1) that, by definition, the Fourier Transform of f is given by

$$\begin{aligned}
\mathscr{F}\{f(x)\} = F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(1 - \frac{|x|}{2}\right) H\left(1 - \frac{|x|}{2}\right) e^{-ikx} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} \left(1 - \frac{|x|}{2}\right) e^{-ikx} dx \\
&= \frac{1}{\sqrt{2\pi}} \left[\int_{-2}^{0} \left(1 + \frac{x}{2}\right) e^{-ikx} dx + \int_0^2 \left(1 - \frac{x}{2}\right) e^{-ikx} dx \right].
\end{aligned}$$

Now, we see by integration by parts that

$$\begin{aligned}
\int_c^d \left(1 \pm \frac{x}{2}\right) e^{-ikx} dx &= \frac{i}{k} e^{-ikx} \bigg|_c^d \pm \frac{1}{2} \int_c^d x e^{-ikx} dx \\
&= \frac{i}{k} e^{-ikx} \bigg|_c^d \pm \frac{1}{2} \left[\frac{ix}{k} e^{-ikx} \bigg|_c^d - \frac{i}{k} \int_c^d e^{-ikx} dx \right] \\
&= \frac{i}{k} e^{-ikx} \bigg|_c^d \pm \frac{1}{2} \left[\frac{ix}{k} e^{-ikx} \bigg|_c^d + \frac{e^{-ikx}}{k^2} \bigg|_c^d \right].
\end{aligned}$$

Thus,

$$\int_{-2}^{0} \left(1 + \frac{x}{2}\right) e^{-ikx} dx = \frac{1 - e^{2ik} + 2ik}{2k^2}$$

and

$$\int_{0}^{2} \left(1 - \frac{x}{2}\right) e^{-ikx} dx = \frac{1 - e^{-2ik} - 2ik}{2k^2}$$

Therefore, using the definition of the complex exponential and various trigonometric identities, we have that

$$\begin{aligned}
\mathscr{F}\left\{f(x)\right\} = F(k) &= \frac{1}{\sqrt{2\pi}} \left[\int_{-2}^{0} \left(1 + \frac{x}{2}\right) e^{-ikx} dx + \int_{0}^{2} \left(1 - \frac{x}{2}\right) e^{-ikx} dx\right] \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{1 - e^{2ik} + 2ik}{2k^2} + \frac{1 - e^{-2ik} - 2ik}{2k^2}\right] \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{k^2} - \frac{e^{-2ik} + e^{2ik}}{2k^2}\right] \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{1 - \cos 2k}{k^2}\right] \\
&= \frac{1}{\sqrt{2\pi}} \left[\frac{1 - (\cos^2 k - \sin^2 k)}{k^2}\right] \\
&= \frac{2\sin^2 k}{\sqrt{2\pi} k^2}.
\end{aligned}$$

□

Problem 2. Find the Laplace Transforms of the following functions:

a. $f(t) = \displaystyle\int_0^t \frac{\sin ax}{x} dx$,

b. $f(t) = tH(t-a)$.

Solution. Recall that if $f(t) \in L^1(\mathbb{R})$, then the Laplace Transform of f is defined to be

$$\mathscr{L}\{f(t)\} = \bar{f}(s) = \int_0^\infty f(t)e^{-st}dt, \quad \text{Re } s > 0. \tag{3}$$

Note that it can be shown that the Laplace transform satisfies the important property

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\left[\mathscr{L}\{f(t)\}\right]. \tag{4}$$

a. Let $f(t) = \displaystyle\int_0^t \frac{\sin ax}{x} dx$. Then $f(0) = 0$ and $f'(t) = \dfrac{\sin at}{t}$ so that $tf'(t) = \sin at$. This implies that

$$\mathscr{L}\{tf'(t)\} = \mathscr{L}\{\sin at\} = \frac{a}{s^2 + a^2}.$$

The Laplace transform satisfies the following property that relates a function's derivative to its Laplace Transform:

$$\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - f(0).$$

This combined with (4) shows that

$$\mathscr{L}\{tf'(t)\} = -\frac{d}{ds}\left[\mathscr{L}\{f'(t)\}\right] = -\frac{d}{ds}\left[s\mathscr{L}\{f(t)\} - f(0)\right] = \frac{a}{s^2 + a^2},$$

or that

$$\frac{d}{ds}\left[s\mathscr{L}\{f(t)\}\right] = -\frac{a}{s^2 + a^2}.$$

Integrating both sides of the above equation yields that

$$s\mathscr{L}\{f(t)\} = -\int \frac{a}{s^2 + a^2} ds = -\tan^{-1}(s/a) + C.$$

In order to determine the constant of integration, we use the Initial Value Theorem which states that

$$\lim_{s\to\infty} s\mathscr{L}\{f(t)\} = \lim_{t\to 0} f(t) = f(0).$$

Since $f(0) = 0$, this implies that

$$\lim_{s\to\infty} s\mathscr{L}\{f(t)\} = \lim_{s\to\infty}\left[-\tan^{-1}(s/a) + C\right] = 0$$

or that $C = \frac{\pi}{2}$. Therefore, we have that

$$s\mathscr{L}\left\{f(t)\right\} = -\tan^{-1}(s/a) + \frac{\pi}{2} = \tan^{-1}(a/s)$$

so that

$$\mathscr{L}\left\{f(t)\right\} = \frac{\tan^{-1}(a/s)}{s}.$$

b. Let $f(t) = tH(t-a)$ and assume that $a > 0$. By property (4) we see that

$$\mathscr{L}\left\{tH(t-a)\right\} = -\frac{d}{ds}\left[\mathscr{L}\left\{H(t-a)\right\}\right].$$

From our table of Laplace Transforms, we have that

$$\mathscr{L}\left\{H(t-a)\right\} = \frac{e^{-as}}{s},$$

assuming that $a > 0$. Therefore,

$$\begin{aligned}\mathscr{L}\left\{f(t)\right\} = -\frac{d}{ds}\left[\mathscr{L}\left\{H(t-a)\right\}\right] &= -\frac{d}{ds}\left[s^{-1}e^{-as}\right] \\ &= -\left[-s^{-2}e^{-as} - as^{-1}e^{-as}\right] \\ &= \frac{(1+as)e^{-as}}{s^2}.\end{aligned}$$

$\square$

Problem 3. Solve the following integral equations:

a. $\displaystyle\int_0^\infty f(x)\sin kx dx = \begin{cases} 1-k & k<1 \\ 0 & k>1 \end{cases},$

b. $\displaystyle\int_{-\infty}^\infty \frac{f(t)}{(x-t)^2+4}dt = \frac{1}{x^2+9}.$

Solution. a. Recall that the definition of the Fourier Sine Transform is given by

$$\mathscr{F}_s\{f(x)\} = F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin kx dx.$$

Thus, we see that

$$\int_0^\infty f(x)\sin kx dx = \sqrt{\frac{\pi}{2}}\mathscr{F}_s\{f(x)\} = \sqrt{\frac{\pi}{2}}F_s(k).$$

Let $G_s(k) = \begin{cases} 1-k & k<1 \\ 0 & k>1 \end{cases}$. Then the above integral equation becomes

$$F_s(k) = \sqrt{\frac{2}{\pi}}G_s(k).$$

Thus, applying the inverse Fourier Sine Transform, we have that

$$f(x) = \mathscr{F}_s^{-1}\{F_s(k)\} = \sqrt{\frac{2}{\pi}}\mathscr{F}_s^{-1}\{G_s(k)\}$$

where the inverse Fourier Sine Transform is defined as

$$g(x) = \mathscr{F}_s^{-1}\{G_s(k)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty G_s(k)\sin kx dk. \tag{5}$$

Therefore, the solution to the integral equation is

$$\begin{aligned} f(x) = \sqrt{\frac{2}{\pi}}\mathscr{F}_s^{-1}\{G_s(k)\} &= \frac{2}{\pi}\int_0^\infty G_s(k)\sin kx dk \\ &= \frac{2}{\pi}\int_0^1 (1-k)\sin kx dk \\ &= \frac{2}{\pi}\left[\frac{1-\cos x}{x} - \frac{\sin x - x\cos x}{x^2}\right] \\ &= \frac{2}{\pi}\left[\frac{x-\sin x}{x^2}\right]. \end{aligned}$$

b. Recall that the convolution of two functions f and g is defined such that

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi.$$

Let $g(x) = \dfrac{1}{x^2+2^2}$. Then we see that

$$\int_{-\infty}^{\infty} \frac{f(t)}{(x-t)^2+4}dt = \int_{-\infty}^{\infty} f(t)g(x-t)dt = \sqrt{2\pi}(g * f)(x) = \sqrt{2\pi}(f * g)(x).$$

Now, let $h(x) = \dfrac{1}{x^2+3^2}$. Then in light of the above remarks, the integral equation becomes

$$\int_{-\infty}^{\infty} \frac{f(t)}{(x-t)^2+4}dt = \sqrt{2\pi}(f * g)(x) = h(x) = \frac{1}{x^2+9}.$$

Applying the Fourier transform to the integral equation, we see by the Convolution Theorem that

$$\mathscr{F}\left\{\sqrt{2\pi}(f * g)(x)\right\} = \sqrt{2\pi}F(k)G(k) = H(k) = \mathscr{F}\left\{h(x)\right\},$$

where $\mathscr{F}\{f(x)\} = F(k), \mathscr{F}\{g(x)\} = G(k)$, and $\mathscr{F}\{h(x)\} = H(k)$, respectively. From our table of Fourier transforms, we see that for $a > 0$ we have that

$$\mathscr{F}\left\{\frac{1}{x^2+a^2}\right\} = \sqrt{\frac{\pi}{2}}\frac{e^{-a|k|}}{a}. \tag{6}$$

Thus, from (6) we have that

$$F(k) = \frac{1}{\sqrt{2\pi}}\frac{H(k)}{G(k)} = \frac{2}{3\sqrt{2\pi}}\frac{e^{-3|k|}}{e^{-2|k|}} = \frac{2e^{-|k|}}{3\sqrt{2\pi}}.$$

Applying the inverse Fourier Transform to this equation yields that

$$f(x) = \mathscr{F}^{-1}\left\{F(k)\right\} = \frac{2}{3\sqrt{2\pi}}\mathscr{F}^{-1}\left\{e^{-|k|}\right\}.$$

But from (6), we know that

$$\mathscr{F}^{-1}\left\{e^{-|k|}\right\} = \sqrt{\frac{2}{\pi}}\frac{1}{x^2+1}.$$

Therefore, the solution to the integral equation is given by

$$\begin{aligned} f(x) = \frac{2}{3\sqrt{2\pi}}\mathscr{F}^{-1}\left\{e^{-|k|}\right\} &= \frac{2}{3\sqrt{2\pi}}\left[\sqrt{\frac{2}{\pi}}\frac{1}{x^2+1}\right] \\ &= \frac{2}{3\pi}\left[\frac{1}{x^2+1}\right]. \end{aligned}$$

$\square$

Problem 4. Show that

$$\int_0^\infty F_s(k)G_c(k)\sin kx dk = \frac{1}{2}\int_0^\infty g(\xi)\left[f(\xi+x)-f(\xi-x)\right]d\xi$$

where

$$F_s(k)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin kx dx$$

and

$$G_c(k)=\sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\cos kx dx.$$

Solution. Using the definition of $G_c(k)$, we see that

$$\int_0^\infty F_s(k)G_c(k)\sin kx dk = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(k)\sin kx\left[\int_0^\infty g(\xi)\cos k\xi d\xi\right]dk$$

Interchanging the order of integration from ξ to k shows that

$$\begin{aligned}\int_0^\infty F_s(k)G_c(k)\sin kx dk &= \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(k)\sin kx\left[\int_0^\infty g(\xi)\cos k\xi d\xi\right]dk\\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty g(\xi)\left[\int_0^\infty F_s(k)\cos k\xi\sin kx dk\right]d\xi\end{aligned}$$

Using the following trigonometric identity

$$\cos k\xi\sin kx = \frac{\sin k(\xi+x)-\sin k(\xi-x)}{2},$$

we then see that

$$\begin{aligned}\int_0^\infty F_s(k)G_c(k)\sin kx dk &= \sqrt{\frac{2}{\pi}}\int_0^\infty g(\xi)\left[\int_0^\infty F_s(k)\cos k\xi\sin kx dk\right]d\xi\\ &= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty g(\xi)\left[\int_0^\infty F_s(k)\sin k(\xi+x)dk-\int_0^\infty F_s(k)\sin k(\xi-x)dk\right]d\xi\\ &= \frac{1}{2}\int_0^\infty g(\xi)\left[f(\xi+x)-f(\xi-x)\right]d\xi,\end{aligned}$$

where the last line follows using (5), the definition of the inverse Fourier Sine Transform.

Therefore,

$$\int_0^\infty F_s(k)G_c(k)\sin kx dk = \frac{1}{2}\int_0^\infty g(\xi)\left[f(\xi+x)-f(\xi-x)\right]d\xi,$$

and we are done.

□

Problem 5. Apply the Fourier Transform to solve the following initial value problem for the heat equation:

$$\begin{aligned}
&\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad -\infty < x < \infty, \\
&u(x,0) = \phi(x), \quad t > 0.
\end{aligned}$$

Solution. Consider the function $u(x,t)$. The Fourier transform of u with respect to x is defined as

$$\mathscr{F}\left\{u(x,t)\right\} = U(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} u(x,t) dx. \tag{7}$$

From this definition and the Leibniz integral rule, we can see by induction that

$$\begin{aligned}
\mathscr{F}\left\{\frac{\partial^n}{\partial t^n}\left[u(x,t)\right]\right\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^n}{\partial t^n}\left[u(x,t)\right] e^{-ikx} dx \\
&= \frac{d^n}{dt^n}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-ikx} dx\right] \\
&= \frac{d^n}{dt^n}\left[\mathscr{F}\left\{u(x,t)\right\}\right].
\end{aligned} \tag{8}$$

Similarly, we see from definition (7) and previous theorems regarding the Fourier transform that

$$\begin{aligned}
\mathscr{F}\left\{\frac{\partial^n}{\partial x^n}\left[u(x,t)\right]\right\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^n}{\partial x^n}\left[u(x,t)\right] e^{-ikx} dx \\
&= (ik)^n \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-ikx} dx\right] \\
&= (ik)^n \mathscr{F}\left\{u(x,y)\right\}.
\end{aligned} \tag{9}$$

Now, applying the Fourier transform to the first equation yields that

$$\mathscr{F}\left\{\frac{\partial u}{\partial t}\right\} = \frac{d}{dt}\left[U(k,t)\right] = -(ak)^2 U(k,t) + F(k,t) = \mathscr{F}\left\{a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t)\right\}.$$

This results in a first-order non-homogeneous linear differential equation

$$\frac{d}{dt}\left[U(k,t)\right] + (ak)^2 U(k,t) = F(k,t).$$

$\square$

Problem 6. Evaluate the following definite integrals:

a. $\displaystyle\int_0^\infty \frac{\sin ax \sin bx}{x^2}dx,$

b. $\displaystyle\int_0^\infty \frac{(a^2-x^2)^2}{(x^2+a^2)^4}dx, \quad a > 0.$

Solution. Suppose that $F_c(k) = \mathscr{F}_c\{f(x)\}$ and $G_c(k) = \mathscr{F}_c\{g(x)\}$. Then Parseval's relation derived from the Convolution Theorem for the Fourier Cosine Transform states that

$$\int_0^\infty F_c(k)G_c(k)dk = \int_0^\infty f(x)g(x)dx. \tag{10}$$

a. Let $F_c(k) = \dfrac{\sin ak}{k}$ and $G_c(k) = \dfrac{\sin bk}{k}$. Then Parseval's theorem shows that

$$\int_0^\infty \frac{\sin ak \sin bk}{k^2}dk = \int_0^\infty F_c(k)G_c(k)dk = \int_0^\infty f(x)g(x)dx$$

where $f(x) = \mathscr{F}_c^{-1}\{F_c(k)\}$ and $f(x) = \mathscr{F}_c^{-1}\{G_c(k)\}$. From our table of Fourier Cosine Transforms, we see that for $p \in \mathbb{R}$,

$$\mathscr{F}_c\{H(p-x)\} = \sqrt{\frac{2}{\pi}}\frac{\sin pk}{k}.$$

This implies that

$$\mathscr{F}_c^{-1}\left\{\frac{\sin pk}{k}\right\} = \sqrt{\frac{\pi}{2}}H(p-x).$$

Thus, we have that

$$\int_0^\infty \frac{\sin ak \sin bk}{k^2}dk = \int_0^\infty F_c(k)G_c(k)dk = \frac{\pi}{2}\int_0^\infty H(a-x)H(b-x)dx.$$

Now, we note from the definition of the Heaviside function that

$$H(a-x)H(b-x) = \begin{cases} 1 & \text{if } x < \min(a,b) \\ 0 & \text{if } x > \min(a,b) \end{cases}.$$

Therefore, we have that

$$\int_0^\infty \frac{\sin ak \sin bk}{k^2}dk = \frac{\pi}{2}\int_0^\infty H(a-x)H(b-x)dx = \frac{\pi}{2}\int_0^{\min(a,b)} dx = \frac{\pi}{2}\min(a,b).$$

b. Let $F_c(k) = \dfrac{a^2-k^2}{(k^2+a^2)^2}$. Then Parseval's theorem shows that

$$\int_0^\infty \frac{(a^2-k^2)^2}{(k^2+a^2)^4}dk = \int_0^\infty F_c(k)F_c(k)dk = \int_0^\infty f(x)f(x)dx$$

where $f(x) = \mathscr{F}_c^{-1}\left\{F_c(k)\right\}$. From our table of Fourier Cosine Transforms, we see that for $a > 0$,

$$\mathscr{F}_c\left\{xe^{-ax}\right\} = \sqrt{\frac{2}{\pi}}\frac{a^2-k^2}{(k^2+a^2)^2}.$$

This implies that

$$\mathscr{F}_c^{-1}\left\{\frac{a^2-k^2}{(k^2+a^2)^2}\right\} = \sqrt{\frac{\pi}{2}}.$$

Thus, we have that

$$\int_0^\infty \frac{(a^2-k^2)^2}{(k^2+a^2)^4}dk = \int_0^\infty F_c(k)F_c(k)dk = \frac{\pi}{2}\int_0^\infty x^2e^{-2ax}dx.$$

From (2), we see by setting $b = 2a$ and $k = 0$ that

$$\int_0^\infty x^2e^{-2ax}dx = \frac{1}{4a^3}.$$

Therefore, we have that

$$\int_0^\infty \frac{(a^2-k^2)^2}{(k^2+a^2)^4}dk = \frac{\pi}{2}\int_0^\infty x^2e^{-2ax}dx = \frac{\pi}{(2a)^3}.$$

$\square$

Problem 7. Use the Fourier Sine Transform to solve the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \infty$$

with the boundary data $0 < y < L$

$$\begin{aligned} &u(x, L) = 0, \quad u(x, 0) = f(x), \\ &u(0, y) = 0, \quad u(x, y) \to 0 \text{ as } x \to \infty \text{ uniformly in } y. \end{aligned}$$

Solution. Consider the function $u(x, y)$. The Fourier Sine Transform of u with respect to x is defined as

$$\mathscr{F}_s \left\{ u(x, y) \right\} = U_s(k, y) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x, y) \sin(kx) dx.$$

From this definition we see using the Leibniz integral rule that

$$\begin{aligned} \mathscr{F}_s \left\{ \frac{\partial^n u(x, y)}{\partial y^n} \right\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^n u(x, y)}{\partial y^n} \sin(kx) dx \\ &= \frac{d^n}{dy^n} \left[\sqrt{\frac{2}{\pi}} \int_0^\infty u(x, y) \sin(kx) dx \right] \\ &= \frac{d^n}{dy^n} \left[\mathscr{F}_s \left\{ u(x, y) \right\} \right]. \end{aligned}$$

The transforms of the partials of u with respect to x are not as easy to characterize. Nevertheless, we see from the properties of the Fourier Sine Transform that

$$\mathscr{F}_s \left\{ \frac{\partial u(x, y)}{\partial x} \right\} = -k \mathscr{F}_c \left\{ u(x, y) \right\}$$

and

$$\mathscr{F}_s \left\{ \frac{\partial^2 u(x, y)}{\partial x^2} \right\} = -k^2 \mathscr{F}_s \left\{ u(x, y) \right\} + k \sqrt{\frac{2}{\pi}} u(0, y).$$

Let $U_s(k, y) = \mathscr{F}_s \left\{ u(x, y) \right\}$. Then, applying the Fourier Sine Transform to the first differential equation shows that

$$\mathscr{F}_s \left\{ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right\} = \frac{d^2}{dy^2} \left[U_s(k, y) \right] - k^2 U_s(k, y) + k \sqrt{\frac{2}{\pi}} u(0, y) = 0 = \mathscr{F}_s \left\{ 0 \right\}.$$

From the boundary equation $u(0, y) = 0$ we see that the above equation becomes

$$\frac{d^2}{dy^2} \left[U_s(k, y) \right] - k^2 U_s(k, y) = 0.$$

This is a linear, second-order homogeneous differential equation, the solution of which we readily see is

$$U_s(k, y) = c_1 e^{-ky} + c_2 e^{ky}. \tag{11}$$

Applying the Fourier Sine Transform to the boundary equations, we see that

$$U_s(k, L) = 0, \quad U_s(k, 0) = F_s(k), \quad \text{for } 0 < k < \infty,\, 0 < y < L.$$

Using these equations and (11), the solution to the homogeneous equation, we see that

$$\begin{aligned} U_s(k, 0) &= c_1 + c_2 = F_s(k) \\ U_s(k, L) &= c_1 e^{-kL} + c_2 e^{kL} = 0. \end{aligned}$$

Solving, we see that

$$\begin{aligned} c_1 &= -\frac{e^{2kL} F_s(k)}{1 - e^{2kL}} \\ c_2 &= \frac{F_s(k)}{1 - e^{2kL}}. \end{aligned}$$

Thus, the solution to the transformed system of differential equations is

$$\begin{aligned} U_s(k, y) &= -\frac{e^{2kL} F_s(k) e^{-ky}}{1 - e^{2kL}} + \frac{F_s(k) e^{ky}}{1 - e^{2kL}} \\ &= F_s(k) \left(\frac{e^{-kL}}{e^{-kL}} \right) \left(\frac{-e^{ky} + e^{2kL - ky}}{-1 + e^{2kL}} \right) \\ &= F_s(k) \frac{\sinh k(L - y)}{\sinh kL}. \end{aligned}$$

Applying the inverse Fourier Sine Transform gives that the solution to the original system of differential equations is

$$\begin{aligned} u(x, y) &= \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(k) \frac{\sinh k(L - y)}{\sinh kL} \sin kx dk \\ &= \frac{2}{\pi} \int_0^\infty \left[\int_0^\infty f(\xi) \sin k\xi d\xi \right] \frac{\sinh k(L - y)}{\sinh kL} \sin kx dk. \end{aligned}$$

It is easy to see from the definition of the hyperbolic sine function that $\frac{\sinh k(L-y)}{\sinh kL} \sim e^{-ky}$ as $kL \to \infty$. Thus, the above problem reduces to a simpler problem in the quarter plane instead of the semi-infinite strip. Therefore, the solution to the original differential equation is

$$\begin{aligned} u(x, y) &= \frac{2}{\pi} \int_0^\infty f(\xi) d\xi \int_0^\infty \sin k\xi \sin kx e^{-ky} dk \\ &= \frac{1}{\pi} \int_0^\infty f(\xi) d\xi \int_0^\infty \left[\cos k(x - \xi) - \cos k(x + \xi) \right] e^{-ky} dk \\ &= \frac{1}{\pi} \int_0^\infty f(\xi) \left[\frac{y}{(x - \xi)^2 + y^2} - \frac{y}{(x + \xi)^2 + y^2} \right] d\xi. \end{aligned}$$

$\square$

Problem 8. Apply the Fourier Transform to solve the 3-dimensional wave problem

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \quad -\infty < x, y, z < \infty,$$

subject to the initial conditions

$$\begin{gathered} u(x,y,z,t)|_{t=0} = 0 \\ \left.\frac{\partial u(x,y,z,t)}{\partial t}\right|_{t=0} = \delta(x,y,z). \end{gathered}$$

Solution. □

Problem 9. Show that if E is a solution of an m-th order partial differential equation

$$P(\partial)u = \sum_{k=0}^{m} a_k \partial^k u = \delta,$$

where δ is the Dirac delta function, then $E * f$ is the solution of the partial differential equation $P(\partial)u = f$, where $*$ is the convolution.

Solution. $\square$