Homework Assignment 12

Matthew Tiger

December 7, 2015

Problem 1. Use the methods of this section to approximate the solution to

$$y'' + y = 3x^2, \quad y(0) = 0, y(2) = 3.5$$

For basis functions, take $n = 2$ and $\phi_1(x) = x(x-2), \phi_2(x) = x^2(x-2)$.

Solution. Note that $u(x) = \frac{7}{4}x$ satisfies the boundary conditions of the problem, i.e. $u(0) = 0$ and $u(2) = 3.5$. Therefore our approximation to the differential equation is given by a linear combination of the basis functions $\phi_1(x), \phi_2(x)$ and $u(x)$. So the approximation is given by

$$y_2 = u(x) + a_1\phi_1(x) + a_2\phi_2(x) \tag{1}$$

to the solution of the original differential equation which also satisfies the boundary conditions.

We wish to find values of the coefficients a_1, a_2 such that

$$\int_0^2 \left(y_2'' + y_2 - 3x^2\right)\phi_i(x)dx = 0 \quad \text{for } i = 1, 2. \tag{2}$$

Using our definition of the approximation found in (1), we carry out the computations in eqrefsystem with MATLAB to arrive at the following system of equations

$$\begin{bmatrix} 8/5 & 8/5 \\ 8/5 & 64/21 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 37/15 \\ 18/5 \end{bmatrix}.$$

The solution to this system yields that the coefficients are given by $a_1 = 173/228$ and $a_2 = 119/152$.

Therefore, the approximation to the solution to the original differential equation is given by

$$y_2(x) = \frac{7}{4}x + \frac{173}{228}x(x-2) + \frac{119}{152}x^2(x-2).$$

□

Problem 3. Use the methods of this section to approximate the solution to

$$y'' + y^2 = x, \quad y(0) = y(1) = 0.$$

Let $n = 1$ and for a basis function take $\phi_1(x) = x\sin(\pi x)$.

Solution. Note that $\phi_1(x) = x\sin(\pi x)$ satisfies the boundary conditions, i.e. $\phi_1(0) = \phi_1(1) = 0$. Therefore our approximation to the differential equation is given by a linear combination of the basis function $\phi_1(x)$. Thus, the approximation is

$$y_1 = a_1 x\sin(\pi x).$$

We wish to find the value of a_1 such that

$$\int_0^1 \left(y_1'' + y_1^2 - x\right) x\sin(\pi x)dx = 0.$$

Using MATLAB, we see that the above equation reduces to the following quadratic equation

$$\left(\frac{2\pi^2}{3\pi^3} - \frac{40}{9\pi^3}\right) a_1^2 + \left(\frac{-\pi^2}{6} - \frac{1}{4}\right) a_1 + \frac{4-\pi^2}{\pi^3} = 0.$$

Choosing the solution to the equation $a_1 = -0.099539826159056$ give us that the approximate solution is

$$y_1 = -0.099539826159056\sin(\pi x).$$

□

Problem 5. The solution to

$$((x+1)y')' - (x+2)y = \frac{x+2}{e-1}, \quad y(0) = 0, y(1) = 1.$$

is $y = \frac{e^x - 1}{e-1}$. Use the methods of this section to compute approximate solutions $y_5(x)$, $y_{10}(x)$, and $y_{15}(x)$ and compare these approximations to the actual solution.

Solution. Note the above differential equation can be represented in the form

$$(p(x)y')' + q(x)y' + r(x)y = f(x), \quad y(a) = A, y(b) = B,$$

where $p(x) = x+1$, $q(x) = 0$, $r(x) = -(x+2)$, $f(x) = \frac{x+2}{e-1}$, and the boundary conditions are appropriately defined. Using the methods from the reading, we see that an approximation to the solution $y(x)$ of the differential equation is given by

$$y_n(x) = \sum_{j=1}^{n} a_j\phi_j(x) + l(x)$$

where the basis functions $\phi_j = sin(j\pi x)$ satisfy $\phi_j(a) = \phi_j(b) = 0$ and $l(x) = A\frac{b-x}{b-a} + B\frac{x-a}{b-a} = x$. Note that $y_n(x)$ satisfies the boundary conditions. The coefficients a_j are chosen so that they satisfy the following system of equations

$$\sum_{j=1}^{n} a_j \int_a^b \left[p(x)\phi_j'(x)\phi_i(x) - q(x)\phi_j'(x)\phi_i(x) - r(x)\phi_j(x)\phi_i(x)\right] dx$$
$$= -\int_a^b \left[f(x) - Ll(x)\right]\phi_i(x)dx \quad \text{for } i = 1, \dots n, \tag{3}$$

where L is the differential operator defining our differential equation.

Using our basis functions $\phi_j(x) = \sin(j\pi x)$, we see that $\phi_j'(x) = j\pi cos(j\pi x)$. Additionally, we also see that $f(x) - Ll(x) = \frac{x+2}{e-1} + x^2 + 2x - 1$. From theseidentities and using our definitions of $p(x)$, $q(x)$, and $r(x)$, we see that the system in (3) simplifies to the following system

$$\sum_{j=1}^{n} a_j \int_0^1 \left[(x+1)ij\pi^2 \cos(j\pi x)\cos(i\pi x) + (x+2)\sin(j\pi x)\sin(i\pi x)\right] dx$$
$$= -\int_0^1 \left[\frac{x+2}{e-1} + x^2 + 2x - 1\right] \sin(i\pi x)dx \quad \text{for } i = 1, \ldots, n. \tag{4}$$

We can now find an approximation to the solution $y(x)$ in the form

$$y_n(x) = x + \sum_{j=1}^{n} a_j \sin(j\pi x) \tag{5}$$

where the coefficients a_j are determined by the solution to the system found in (4).

After creating a MATLAB implementation of the solution to the above system of equations we see that $y_5(x) = x + \sum_{j=1}^{5} a_j \sin(j\pi x)$ where the coefficients are given by

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} -0.126756792130151 \\ 0.007873907784233 \\ -0.005123612817319 \\ 0.001007418513753 \\ -0.001164503231156 \end{bmatrix}$$

.

Similarly, $y_{10}(x) = x + \sum_{j=1}^{10} a_j \sin(j\pi x)$ where the coefficients are given by

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \end{bmatrix} = \begin{bmatrix} -0.126740377319384 \\ 0.007866963591612 \\ -0.005111897265922 \\ 0.001003683279394 \\ -0.001111840960052 \\ 0.000300043256596 \\ -0.000405543093287 \\ 0.000129499026931 \\ -0.000189239173159 \\ 0.000082228427550 \end{bmatrix}$$

and $y_{15}(x) = x + \sum_{j=1}^{15} a_j \sin(j\pi x)$ where the coefficients are given by

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \\ a_{11} \\ a_{12} \\ a_{13} \\ a_{14} \\ a_{15} \end{bmatrix} = \begin{bmatrix} -0.126740669728242 \\ 0.007864339443911 \\ -0.005112288932438 \\ 0.001001882459024 \\ -0.001112248945762 \\ 0.000298067482786 \\ -0.000406198853140 \\ 0.000126040763869 \\ -0.000191339879818 \\ 0.000064709821496 \\ -0.000104942850428 \\ 0.000037631839019 \\ -0.000063854388930 \\ 0.000023917094979 \\ -0.000043873909881 \end{bmatrix}.$$

We now compare these approximations to the actual solution for each of the above derived approximations. The following tables evaluate the approximations and the exact solution at the points 0.25, 0.50, and 0.75 and compare the absolute difference between these values and the relative error of the approximations.

x	$y(x)$	$y_5(x)$	$\lvert y(x) - y_5(x)\rvert$	$(\lvert y(x) - y_5(x)\rvert / \lvert y(x)\rvert) * 100$
0.25	0.165296	0.165443	1.476306e-04	0.089312
0.50	0.377540	0.377202	3.383513e-04	0.089619
0.75	0.650067	0.649695	3.719995e-04	0.057224

Table 1: Comparison of approximation y_5 to solution y.

x	$y(x)$	$y_{10}(x)$	$\lvert y(x) - y_{10}(x)\rvert$	$(\lvert y(x) - y_{10}(x)\rvert / \lvert y(x)\rvert) * 100$
0.25	0.165296	0.165354	5.847486e-05	0.035375
0.50	0.377540	0.377475	6.468589e-05	0.017133
0.75	0.650067	0.650056	1.163722e-05	0.001790

Table 2: Comparison of approximation y_{10} to solution y.

Note that as n increases the value $|y(x) - y_n(x)|$ decreases. Moreover, this rate of decrease appears to be uniform over the entire interval of definition.

□

x	$y(x)$	$y_{15}(x)$	$\lvert y(x) - y_{15}(x)\rvert$	$(\lvert y(x) - y_{15}(x)\rvert / \lvert y(x)\rvert) * 100$
0.25	0.165296	0.165313	1.714344e-05	0.010371
0.50	0.377540	0.377559	1.852280e-05	0.004906
0.75	0.650067	0.650099	3.119950e-05	0.004799

Table 3: Comparison of approximation y_{15} to solution y.