Homework Assignment 6

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Problem 1. a. Where is the assumption "$\boldsymbol{x}^*$ is regular" essential in the proof of the results of section: Lagrange Multipliers?

b. In the example on page 49 (Example 20.8 in *An Introduction to Optimization*) explain in what way is (P_0) equivalent to (P_1).

c. State the SOSC Theorem on p. 51 (Theorem 20.5 p. 474 in the book) for $\boldsymbol{x}^*$ a local maximizer.

Solution. a. The assumption that $\boldsymbol{x}^*$ is regular is essential in the proof of the Lagrange Multipliers Theorem in applying the results of Theorem 20.1, i.e. assuming that $\boldsymbol{y} \in T(\boldsymbol{x}^*)$ if and only if there exists a differentiable curve in S passing through $\boldsymbol{x}^*$ with derivative $\boldsymbol{y}$ at $\boldsymbol{x}^*$.

b. The two problems to consider are:

$$(P_0) \quad \begin{array}{rl} \text{maximize} & \frac{\boldsymbol{x}^\top Q\boldsymbol{x}}{\boldsymbol{x}^\top P\boldsymbol{x}} \\ \text{subject to} & Q = Q^\top \geq 0 \\ & P = P^\top > 0. \end{array} \qquad (P_1) \quad \begin{array}{rl} \text{maximize} & \boldsymbol{x}^\top Q\boldsymbol{x} \\ \text{subject to} & \boldsymbol{x}^\top P\boldsymbol{x} = 1. \end{array}$$

Note that if P is positive semi-definite and Q is positive definite, then $\boldsymbol{x}^\top Q\boldsymbol{x} \geq 0$ and $\boldsymbol{x}^\top P\boldsymbol{x} > 0$ for every $\boldsymbol{x}$. Consequently

$$\frac{\boldsymbol{x}^\top Q\boldsymbol{x}}{\boldsymbol{x}^\top P\boldsymbol{x}} \geq 0$$

for every $\boldsymbol{x}$. From problem (P_0) we see that if $\boldsymbol{x}$ is a solution to the problem, then so is $t\boldsymbol{x}$ for any $t \neq 0$. Note that

$$\frac{(t\boldsymbol{x})^\top Q(t\boldsymbol{x})}{(t\boldsymbol{x})^\top P(t\boldsymbol{x})} = \frac{t^2\boldsymbol{x}^\top Q\boldsymbol{x}}{t^2\boldsymbol{x}^\top P\boldsymbol{x}} = \frac{\boldsymbol{x}^\top Q\boldsymbol{x}}{\boldsymbol{x}^\top P\boldsymbol{x}}$$

showing that the above remark is true. Adding the additional constraint to problem (P_0) that $\boldsymbol{x}^\top P\boldsymbol{x} = 1$ removes the multiplicity of the solutions and transforms the original problem into problem (P_1). To see this, if the constraint $\boldsymbol{x}^\top P\boldsymbol{x} = 1$ is satisfied then for any non-zero scalar multiple of $\boldsymbol{x}^*$ we have that

$$(t\boldsymbol{x})^\top P(t\boldsymbol{x}) = t^2\boldsymbol{x}^\top P\boldsymbol{x} = \boldsymbol{x}^\top P\boldsymbol{x}$$

Since $\boldsymbol{x}^\top P\boldsymbol{x} > 0$ we must have that $t = 1$ removing the multiplicity of the solutions and the problems are equivalent.

c.

Theorem 1 (***Second-Order Sufficient Conditions***)**.** Suppose that $f, \boldsymbol{h} \in \mathcal{C}^2$ with $f : \mathbb{R}^n \to \mathbb{R}$ and $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^m$. Let $l(\boldsymbol{x}, \boldsymbol{\lambda}) = f(\boldsymbol{x}) + \lambda_1 h_1(\boldsymbol{x}) + \lambda_2 h_2(\boldsymbol{x}) + \dots \lambda_m h_m(\boldsymbol{x})$ be the Lagrangian function. Let

$$\boldsymbol{F}(\boldsymbol{x}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(\boldsymbol{x}) & \frac{\partial^2 f}{\partial x_2 \partial x_1}(\boldsymbol{x}) & \dots & \frac{\partial^2 f}{\partial x_n \partial x_1}(\boldsymbol{x}) \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(\boldsymbol{x}) & \frac{\partial^2 f}{\partial x_2^2}(\boldsymbol{x}) & \dots & \frac{\partial^2 f}{\partial x_n \partial x_2}(\boldsymbol{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(\boldsymbol{x}) & \frac{\partial^2 f}{\partial x_2 \partial x_n}(\boldsymbol{x}) & \dots & \frac{\partial^2 f}{\partial x_n^2}(\boldsymbol{x}) \end{bmatrix}$$

be the Hessian matrix of f at $\boldsymbol{x}$ and

$$\boldsymbol{H_k}(\boldsymbol{x}) = \begin{bmatrix} \frac{\partial^2 h_k}{\partial x_1^2}(\boldsymbol{x}) & \frac{\partial^2 h_k}{\partial x_2 \partial x_1}(\boldsymbol{x}) & \dots & \frac{\partial^2 h_k}{\partial x_n \partial x_1}(\boldsymbol{x}) \\ \frac{\partial^2 h_k}{\partial x_1 \partial x_2}(\boldsymbol{x}) & \frac{\partial^2 h_k}{\partial x_2^2}(\boldsymbol{x}) & \dots & \frac{\partial^2 h_k}{\partial x_n \partial x_2}(\boldsymbol{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 h_k}{\partial x_1 \partial x_n}(\boldsymbol{x}) & \frac{\partial^2 h_k}{\partial x_2 \partial x_n}(\boldsymbol{x}) & \dots & \frac{\partial^2 h_k}{\partial x_n^2}(\boldsymbol{x}) \end{bmatrix}$$

be the Hessian matrix of h_k at $\boldsymbol{x}$ for $k = 1, \dots, m$. Define

$$\boldsymbol{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = \boldsymbol{F}(\boldsymbol{x}) + \lambda_1 \boldsymbol{H_1}(\boldsymbol{x}) + \dots + \lambda_m \boldsymbol{H_m}(\boldsymbol{x})$$

to be the Hessian Matrix of $l(\boldsymbol{x}, \boldsymbol{\lambda})$ with respect to $\boldsymbol{x}$.

Suppose there exists a point $\boldsymbol{x}^* \in \mathbb{R}^n$ and $\boldsymbol{\lambda^*} \in \mathbb{R}^m$ such that

- $Df(\boldsymbol{x}^*) + \boldsymbol{\lambda}^{*\top} D\boldsymbol{h}(\boldsymbol{x}^*) = \boldsymbol{0}^\top$.
- For all $\boldsymbol{y} \in T(\boldsymbol{x}^*)$, $\boldsymbol{y} \neq \boldsymbol{0}$, we have that $\boldsymbol{y}^\top \boldsymbol{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)\boldsymbol{y} < 0$, i.e. $\boldsymbol{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)$ is negative definite on $T(\boldsymbol{x}^*)$.

Then $\boldsymbol{x}^*$ is a strict local maximizer of f subject to $\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}$.

$\square$

Problem 2. Find local extremizers for the following optimization problem:

$$\begin{array}{rl} \text{maximize} & x_1x_2 \\ \text{subject to} & x_1^2 + 4x_2^2 = 1. \end{array}$$

Solution. Lagrange's Theorem prescribes how to find the local extremizers for the optimization problem. Let $f(\boldsymbol{x}) = x_1x_2$ and $h(\boldsymbol{x}) = x_1^2 + 4x_2^2 - 1$. Note that we then have that

$$\nabla f(\boldsymbol{x})^\mathsf{T} = \begin{bmatrix} x_2 & x_1 \end{bmatrix} \quad \text{and} \quad \nabla h(\boldsymbol{x})^\mathsf{T} = \begin{bmatrix} 2x_1 & 8x_2 \end{bmatrix}.$$

Since for every feasible $\boldsymbol{x}$, the Jacobian is of rank 1, i.e. of full rank, every feasible point is a regular point. Using the Lagrange condition $Df(\boldsymbol{x}^*) + \boldsymbol{\lambda}^{*\mathsf{T}} D\boldsymbol{h}(\boldsymbol{x}^*) = \mathbf{0}^\mathsf{T}$, we formulate the system of equations

$$Df(\boldsymbol{x}^*) + \boldsymbol{\lambda}^{*\mathsf{T}} D\boldsymbol{h}(\boldsymbol{x}^*) = \begin{bmatrix} x_2 + 2\lambda x_1 & x_1 + 8\lambda x_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} = \mathbf{0}^\mathsf{T}$$

for $\lambda \in \mathbb{R}$. Thus, an extremizer of the optimization problem must satisfy the following system of equations

$$\begin{aligned} x_2 + 2\lambda x_1 &= 0 \\ x_1 + 8\lambda x_2 &= 0 \\ x_1^2 + 4x_2^2 &= 1. \end{aligned}$$

From the first two equations, we see that $x_2 = -2\lambda x_1$ and $x_1 = -8\lambda x_2$. These equations in conjunction show that either $x_1 = 0$, $x_2 = 0$, or $\lambda = \pm 1/4$. If $\boldsymbol{x}$ is a feasible point, then we can't have that $x_1 = 0$ nor $x_2 = 0$. Thus, we must have that $\lambda = \pm 1/4$. In that case, from the first equation, we have that $x_2 = \mp x_1/2$. Substituting this into the third equation yields that

$$x_1^2 + 4(x_1^2/4) = 2x_1^2 = 1$$

which implies that $x_1 = \pm 1/\sqrt{2}$. Thus, $x_1 = \pm 1/\sqrt{2}$ and $x_2 = \mp 1/(2\sqrt{2})$ are the only solutions that satisfy the above system. That is, the extremizers to the optimization problem are

$$\boldsymbol{x}^{(1)} = \begin{bmatrix} 1/\sqrt{2} \\ 1/(2\sqrt{2}) \end{bmatrix}, \ \boldsymbol{x}^{(2)} = \begin{bmatrix} 1/\sqrt{2} \\ -1/(2\sqrt{2}) \end{bmatrix}, \ \boldsymbol{x}^{(3)} = \begin{bmatrix} -1/\sqrt{2} \\ 1/(2\sqrt{2}) \end{bmatrix}, \ \boldsymbol{x}^{(4)} = \begin{bmatrix} -1/\sqrt{2} \\ -1/(2\sqrt{2}) \end{bmatrix}.$$

Since $f(\boldsymbol{x}^{(1)}) = f(\boldsymbol{x}^{(4)}) = 1/4$ and $f(\boldsymbol{x}^{(2)}) = f(\boldsymbol{x}^{(3)}) = -1/4$, we have that the possible maximizers of the problem are located at $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(4)}$ while the possible minimizers of the problem are located at $\boldsymbol{x}^{(2)}$ and $\boldsymbol{x}^{(3)}$. Inspecting graphically, we can see that these are the maximizers and minimizers of the optimization problem. □

Problem 3. Consider the problem

$$\begin{aligned} \text{minimize} \quad & 2x_1 + 3x_2 - 4, \quad x_1, x_2 \in \mathbb{R} \\ \text{subject to} \quad & x_1 x_2 = 6. \end{aligned}$$

a. Use Lagrange's theorem to find all possible local minimizers and maximizers.

b. Use the second-order sufficient conditions to specify which points are strict local minimizers and which are strict local maximizers.

c. Are the points in part b global minimizers or maximizers? Explain.

Solution. □

Problem 4. Consider the problem of minimizing a general quadratic function subject to a linear constraint:

$$\begin{array}{rl} \text{minimize} & \frac{1}{2}\boldsymbol{x}^\top Q\boldsymbol{x} - \boldsymbol{c}^\top \boldsymbol{x} + d \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b}, \end{array}$$

where $Q = Q^\top > 0$, $A \in \mathbb{R}^{m \times n}$ with $m < n$, $\operatorname{rank} A = m$ and d a constant. Derive a closed form solution to the problem.

Solution. □

Problem 5. Consider the discrete-time linear system $x_k = 2x_{k-1} + u_k$, $k \geq 1$, with $x_0 = 1$. Find the values of the control inputs u_1 and u_2 to minimize

$$x_2^2 + \frac{1}{2}u_1^2 + \frac{1}{3}u_2^2.$$

Solution. □