Homework Assignment 2

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Problem 2.10. Solve the Cauchy problem for the Klein-Gordon equation

$$\begin{aligned} u_{tt} - c^2 u_{xx} + a^2 u = 0, &\qquad -\infty < x < \infty, \quad t > 0, \\ u(x,0) = f(x) &\qquad \text{for } -\infty < x < \infty, \\ \left[\frac{\partial u}{\partial t}\right]_{t=0} = g(x) &\qquad \text{for } -\infty < x < \infty. \end{aligned}$$

Solution. $\square$

Problem 2.12. Solve the equation

$$\begin{gathered} u_{tt} + u_{xxxx} = 0, \qquad -\infty < x < \infty, \quad t > 0 \\ u(x,0) = f(x), \quad u_t(x,0) = 0 \qquad \text{for } -\infty < x < \infty. \end{gathered}$$

Solution. □

Problem 2.14. Obtain the Fourier cosine transforms of the following functions:

a. xe^{-ax}, $a > 0$.

Solution. Recall that the definition of the Fourier cosine transform of a function $f(x)$ is given by

$$\mathscr{F}_c\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos kx f(x) dx.$$

a. From the definition of the Fourier cosine transform we have that

$$\mathscr{F}_c\left\{xe^{-ax}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty xe^{-ax}\cos kx dx.$$

Using the definition of the complex exponential, we see that

$$\begin{aligned}\mathscr{F}_c\left\{xe^{-ax}\right\} &= \sqrt{\frac{2}{\pi}}\int_0^\infty xe^{-ax}\left[\frac{e^{-ikx}+e^{ikx}}{2}\right]dx \\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty x\left[e^{-(a+ik)x}+e^{-(a-ik)x}\right]dx.\end{aligned}$$

Now, for $w = a \pm ik$, we see using integration by parts with $u = x$ and $dv = e^{-wx}dx$ that

$$\int_0^\infty xe^{-wx}dx = -\frac{xe^{-wx}}{w}\bigg|_0^\infty + \frac{1}{w}\int_0^\infty e^{-wx}dx.$$

Note that

$$\lim_{x\to\infty}\left|e^{-wx}\right| = \lim_{x\to\infty}\left|e^{-(a\pm ik)x}\right| = \lim_{x\to\infty}\left|e^{-ax}\right|\left|e^{\mp ikx}\right| \leq \lim_{x\to\infty}\left|e^{-ax}\right| = 0.$$

This implies that $\lim_{x\to\infty} e^{-wx} = 0$. Thus,

$$\begin{aligned}\int_0^\infty xe^{-wx}dx &= -\frac{xe^{-wx}}{w}\bigg|_0^\infty + \frac{1}{w}\int_0^\infty e^{-wx}dx \\ &= -\frac{1}{w^2}\left[e^{-wx}\big|_0^\infty\right] \\ &= \frac{1}{w^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathscr{F}_c\left\{xe^{-ax}\right\} &= \frac{1}{\sqrt{2\pi}}\left[\int_0^\infty xe^{-(a+ik)x}dx + \int_0^\infty xe^{-(a-ik)x}dx\right] \\ &= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{(a+ik)^2}+\frac{1}{(a-ik)^2}\right] \\ &= \sqrt{\frac{2}{\pi}}\frac{a^2-k^2}{(a^2+k^2)^2}.\end{aligned}$$

□

Problem 2.15. Find the Fourier sine transform of the following functions:

a. xe^{-ax}, $a > 0$.

b. $\frac{1}{x}e^{-ax}$, $a > 0$.

Solution. $\square$

Problem 2.20. Apply the Fourier cosine transform to find the solution $u(x, y)$ of the problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \qquad && 0 < x < \infty, \quad 0 < y < \infty \\ u(x, 0) &= H(a - x), \qquad && x < a \\ u_x(0, y) &= 0, \qquad && 0 < x, \quad y < \infty. \end{aligned}$$

Solution. □

Problem 2.22. Solve the diffusion equation in the semi-infinite line

$$u_t = \kappa u_x x, \qquad 0 \leq x < \infty, \quad t > 0,$$

with the boundary and initial data

$$\begin{aligned} u(0,t) &= 0 && \text{for } t > 0, \\ u(x,t) &\to 0 && \text{as } x \to \infty \text{ for } t > 0, \\ u(x,0) &= f(x) && \text{for } 0 < x < \infty. \end{aligned}$$

Solution. □