Homework Assignment 3

Matthew Tiger

October 12, 2016

Problem 4.28. Every time that the team wins a game, it wins its next game with probability 0.8; every time it loses a game, it wins its next game with probability 0.3. If the team wins a game, then it has dinner together with probability 0.7, whereas if the team loses then it has dinner together with probability 0.2. What proportion of games result in a team dinner?

Solution. Let $\{X_n : n \geq 0\}$ be the stochastic process of the outcomes of the team's games where if $X_n = 1$, then the team won game n and if $X_n = 0$, then the team lost game n. Since the outcome of each game is dependent only upon the previous game, this stochastic process may be modeled as a Markov chain with state space $\mathcal{M} = \{0, 1\}$.

From the assumptions of the model, if the team wins a game, the probability it wins the next game is 0.8 so that

$$P_{11} = P\{X_{n+1} = 1 \mid X_n = 1\} = 0.8$$

which implies that

$$P_{10} = P\{X_{n+1} = 1 \mid X_n = 0\} = 0.2.$$

Similarly, from the assumptions of the model, if the team loses a game, it wins its next game with probability 0.3 so that

$$P_{01} = P\{X_{n+1} = 1 \mid X_n = 0\} = 0.3$$

which implies that

$$P_{00} = P\{X_{n+1} = 0 \mid X_n = 0\} = 0.7.$$

Thus, the trasition matrix $\mathbf{P}$ of the Markov chain is given by

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} = \begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix}.$$

Note that state 0 communicates with state 1 and that the expected number of transitions this Markov chain must make before transitioning from state 0 to state 1 or vice versa is finite. This implies that this Markov chain is irreducible and positive recurrent. Therefore, if π_i represents the long-run proportion that the Markov chain spends in state i, then the long-run proportions of the Markov chain satisfy the following equations:

$$\mathbf{P}^\intercal \begin{bmatrix} \pi_0 \\ \pi_1 \end{bmatrix} = \begin{bmatrix} \pi_0 \\ \pi_1 \end{bmatrix},$$
$$\sum_{j \in \mathcal{M}} \pi_j = 1.$$

Solving the matrix equation we see that $\pi_1 = 3\pi_0/2$. Thus, by the second equation we have that $\pi_0 = 2/5$ and $\pi_1 = 3/5$. and the Markov chain will spend 2/5 of its time in state 0 and 3/5 of its time in state 1. Therefore, the team will win 3/5 of its games and lose 2/5 of its games after playing a large number of games.

Since the team will have dinner with probability 0.7 if the team wins and the team will have dinner with probability 0.2 if the team loses, we have that the proportion of games that result in a team dinner is given by 20% of its losses plus 70% of its wins, i.e.

$$0.2\pi_0 + 0.7\pi_1 = \frac{1}{2}.$$

Therefore, half of the team's games will result in a team dinner. □

Problem 4.29. An organization has N employees where N is a large number. Each employee has one of three possible job classifications and changes classifications (independently) according to a Markov chain with transition probabilities

$$\mathbf{P} = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$$

What percentage of employees are in each classification?

Solution. Let $\mathcal{M} = \{0, 1, 2\}$ be the states of this Markov process. If N is large, the percentage of employees in classification $0, 1, 2$ are given by the long-run proportions of the stated Markov chain π_0, π_1, π_2, respectively. It is clear that this Markov chain is irreducible and positive recurrent. Thus, the long-run proportions of the Markov chain satisfy the following equations:

$$\mathbf{P}^\intercal \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \end{bmatrix},$$
$$\sum_{j \in \mathcal{M}} \pi_j = 1.$$

Thus, we must solve the system of equations

$$\begin{aligned} 0.2\pi_0 + 0.6\pi_1 + 0.4\pi_2 &= \pi_1 \\ 0.1\pi_0 + 0.2\pi_1 + 0.5\pi_2 &= \pi_2 \\ \pi_0 + \pi_1 + \pi_2 &= 1. \end{aligned}$$

Using a computer algebra system we see that the solution to the system is given by

$$\begin{aligned} \pi_0 &= \frac{6}{17} \approx 0.352941 \\ \pi_1 &= \frac{7}{17} \approx 0.411765 \\ \pi_2 &= \frac{4}{17} \approx 0.235294. \end{aligned}$$

Therefore, if N is large, approximately 35.29% of employees are in classification 0, 41.18% of employees are in classification 1, and 23.53% of employees are in classification 2. $\square$

Problem 4.30. Three out of every four trucks on the road are followed by a car, while only one out of every five cars is followed by a truck. What fraction of vehicles on the road are trucks?

Solution. Let $\{X_n : n \geq 0\}$ be the stochastic process representing the n-th vehicle on the road where if $X_n = 0$, then the n-th vehicle is a car and if $X_n = 1$, then the n-th vehicle is a truck. From the assumptions of the model, the probability that the n-th vehicle is either a car or truck is dependent only upon the previous vehicle so that this stochastic process may be modeled as a Markov chain with state space $\mathcal{M} = \{0, 1\}$.

If three out of every four trucks is followed by a car as per the model, then

$$P_{10} = P\{X_{n+1} = 0 \mid X_n = 1\} = \frac{3}{4}$$

and thus,

$$P_{11} = P\{X_{n+1} = 1 \mid X_n = 1\} = \frac{1}{4}.$$

Similarly, if one out of every five cars is followed by a truck as per the model, then

$$P_{01} = P\{X_{n+1} = 1 \mid X_n = 0\} = \frac{1}{5}$$

and thus,

$$P_{00} = P\{X_{n+1} = 0 \mid X_n = 0\} = \frac{4}{5}.$$

Therefore, the transition matrix $\mathbf{P}$ of this Markov chain is given by

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} = \begin{bmatrix} 4/5 & 1/5 \\ 3/4 & 1/4 \end{bmatrix}.$$

If there are a large number of vehicles on the road, then the long-run proportions associated to this Markov chain π_0 and π_1 will represent the fraction of vehicles on the road that are cars and trucks, respectively. Note that this Markov chain is irreducible and positive recurrent. Therefore, π_0 and π_1 satisfy the following equations:

$$\mathbf{P}^\intercal \begin{bmatrix} \pi_0 \\ \pi_1 \end{bmatrix} = \begin{bmatrix} \pi_0 \\ \pi_1 \end{bmatrix},$$
$$\sum_{j \in \mathcal{M}} \pi_j = 1.$$

Thus, to find the long-run proportions, we must solve the system of equations:

$$\begin{array}{cccc} \frac{1}{5}\pi_0 & +\frac{1}{4}\pi_1 & = \pi_1 \\ \pi_0 & + \ \pi_1 & = 1 \end{array}.$$

Using a computer algebra system, we see that the solution of the system is given by

$$\pi_0 = \frac{15}{19} \approx 0.7895$$
$$\pi_1 = \frac{4}{19} \approx 0.2105.$$

Therefore, if the number of vehicles on the road is large, approximately 21.05% of the vehicles will be trucks. □

Problem 4.33. A professor continually gives exams to her students. She can give three possible types of exams, and her class is graded as either having done well or badly. Let p_i denote the probability that the class does well on a type i exam, and suppose that $p_1 = 0.3$, $p_2 = 0.6$, and $p_3 = 0.9$. If the class does well on an exam, then the next exam is equally likely to be any of the three types. If the class does badly, then the next exam is always type 1. What proportion of exams are type i?

Solution. Let $\{X_n : n \geq 0\}$ be the stochastic process of test types assigned on the n-th test where if $X_n = i$, then the n-test is of type i. The assumptions of the problem imply that the probability that the next test is of a certain type depends only on the type of the previous test so that this stochastic process may be modeled as a Markov chain with state space $\mathcal{M} = \{1, 2, 3\}$.

Since the probability that the next test is of type j is dependent upon the knowledge that the current test is of type i and also whether or not the class does well on the current test, we can find P_{ij} by conditioning on the stochastic process $\{S_n : n \geq 0\}$ where we define $S_n = 1$ to indicate that the class does well on test n and $S_n = 0$ to indicate that the class does not do well on test n. Thus,

$$\begin{aligned} P_{ij} &= P\{X_{n+1} = j \mid X_n = i\} \\ &= \sum_k P\{X_{n+1} = j \mid X_n = i, S_n = k\} P\{S_n = k\}. \end{aligned}$$

Note that if the current test is of type i then the class does well on the exam with probability p_i, i.e. $P\{S_n = 1\} = p_i$ and hence $P\{S_n = 0\} = 1 - p_i$. We now examine $P\{X_{n+1} = j \mid X_n = i, S_n = k\}$ for $k = 0, 1$. It is clear that

$$P\{X_{n+1} = j \mid X_n = i, S_n = k\} = \frac{P\{X_{n+1} = j, X_n = i, S_n = k\}}{P\{X_n = i, S_n = k\}}.$$

If $k = 0$, then the class does not do well on the current exam so that $P\{X_n = i, S_n = 0\} = 1 - p_i$. If the class does not do well on the exam, then the next test they receive is of type 1. Thus,

$$P\{X_{n+1} = j, X_n = i, S_n = 0\} = \begin{cases} 1 - p_i & \text{if } j = 1 \\ 0 & \text{if } j \neq 1 \end{cases}.$$

Similarly, if $k = 1$, then the class does well on the exam so that $P\{X_n = i, S_n = 1\} = p_i$. If the class does not do well on the exam, then the next test is of type 1, 2, 3 and occurs with equal probability. Thus,

$$P\{X_{n+1} = j, X_n = i, S_n = 1\} = \frac{p_i}{3}.$$

Using these relations, we see that

$$P\{X_{n+1} = j \mid X_n = i, S_n = k\} = \begin{cases} \frac{1-p_i}{1-p_i}(1 - p_i) & \text{if } k = 0 \text{ and } j = 1 \\ 0 & \text{if } k = 0 \text{ and } j \neq 1 \\ \frac{p_i}{3p_i}(p_i) & \text{if } k = 1 \end{cases}.$$

Therefore, the transition matrix $\mathbf{P}$ of the Markov chain is given by

$$\mathbf{P} = \begin{bmatrix} \frac{p_1}{3} + 1 - p_1 & \frac{p_1}{3} & \frac{p_1}{3} \\ \frac{p_2}{3} + 1 - p_2 & \frac{p_2}{3} & \frac{p_2}{3} \\ \frac{p_3}{3} + 1 - p_2 & \frac{p_3}{3} & \frac{p_3}{3} \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & \frac{1}{10} & \frac{1}{10} \\ \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{3}{10} & \frac{3}{10} \end{bmatrix}.$$

It is clear that this Markov chain is irreducible and positive recurrent. Thus, the long-run proportions of the Markov chain satisfy the following equations:

$$\mathbf{P}^\intercal \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix},$$
$$\sum_{j \in \mathcal{M}} \pi_j = 1.$$

Thus, in order to determine the long-run proportions we must solve the system of equations

$$\begin{array}{ccccccc} (1/10)\pi_1 & + & (1/5)\pi_2 & + & (3/10)\pi_3 & = & \pi_2 \\ (1/10)\pi_1 & + & (1/5)\pi_2 & + & (3/10)\pi_3 & = & \pi_3 \\ \pi_1 & + & \pi_2 & + & \pi_3 & = & 1 \end{array}.$$

Using a computer algebra system, we see that the solution to the above system is given by

$$\pi_1 = \frac{5}{7}, \quad \pi_2 = \frac{1}{7}, \quad \pi_3 = \frac{1}{7}.$$

Therefore, 5/7 of the exams are of type 1, and the rest of the exams are split equally between types 2 and 3. $\square$

Problem 4.35. Consider a Markov chain with states $0, 1, 2, 3, 4$. Suppose $P_{04} = 1$; and suppose that when the chain is in state i, with $i > 0$, the next state is equally likely to be any of the states $0, 1, \ldots, i-1$. Find the limiting probabilities of this Markov chain.

Solution. In order to find the limiting probabilities we must first find the transition matrix of this Markov chain. Note that if $P_{04} = 1$, then $P_{0j} = 0$ for $0 \leq j < 4$. Now suppose that $i > 0$. If the next state is equally likely to be any of the i number of states $0, 1, \ldots, i-1$ given that the chain is currently in state i, then the probability of transitioning from i to one of the listed states is $1/i$ and 0 otherwise. Thus,

$$P_{ij} = \begin{cases} \frac{1}{i} & \text{if } j < i \\ 0 & \text{if } j \geq i \end{cases}.$$

Therefore, the transition matrix $\mathbf{P}$ of this Markov chain is given by

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \end{bmatrix}.$$

Since we can see that $P_{ij}^n > 0$ for $n = 5$, every state communicates with every other state and this Markov chain is irreducible. We can also see that the expected number of transitions each state must make before it returns to that respective state is finite so that this Markov chain is positive recurrent. Further, since $P_{ij}^n > 0$ for $n > 5$, this Markov chain is aperiodic. Therefore, the long-run proportions give the limiting probabilities of this Markov chain and the limiting probabilities π_i must satisfy

$$\mathbf{P}^\intercal \begin{bmatrix}\pi_0 & \pi_1 & \pi_2 & \pi_3 & \pi_4\end{bmatrix}^\intercal = \begin{bmatrix}\pi_0 & \pi_1 & \pi_2 & \pi_3 & \pi_4\end{bmatrix}^\intercal,$$
$$\sum_{j \in \mathcal{M}} \pi_j = 1.$$

Thus, in order to find the limiting probabilities we must solve the system of equations

$$\begin{array}{ccccccccccc} & & & & (1/2)\pi_2 & + & (1/3)\pi_3 & + & (1/4)\pi_4 & = & \pi_1 \\ & & & & & & (1/3)\pi_3 & + & (1/4)\pi_4 & = & \pi_2 \\ & & & & & & & & (1/4)\pi_4 & = & \pi_3 \\ \pi_0 & & & & & & & & & = & \pi_4 \\ \pi_0 & + & \pi_1 & + & \pi_2 & + & \pi_3 & + & \pi_4 & = & 1 \end{array}.$$

Using a computer algebra system, we see that the solution to the above system, and hence the limiting probabilities of this Markov chain, are given by

$$\begin{bmatrix}\pi_0 & \pi_1 & \pi_2 & \pi_3 & \pi_4\end{bmatrix}^\intercal = \begin{bmatrix}12/37 & 6/37 & 4/37 & 3/37 & 12/37\end{bmatrix}^\intercal.$$

$\square$