Homework Assignment 9

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Problem 8.1. Find the Mellin transform of each of the following functions:

a. $f(x) = H(a - x)$, $a > 0$,

b. $f(x) = x^m e^{-nx}$, $m, n > 0$,

c. $f(x) = \dfrac{1}{x^2+1}$.

Solution. The Mellin transform of the function $f(x)$ is defined to be

$$\mathscr{M}\{f(x)\} = \tilde{f}(p) = \int_0^\infty x^{p-1} f(x) dx.$$

a. Recall that the Heaviside function H is defined as

$$H(a-x) = \begin{cases} 1 & \text{if } x < a \\ 0 & \text{if } x > a \end{cases}.$$

Therefore, from the definition of the Mellin transform, we have that for $f(x) = H(a-x)$ with $a > 0$,

$$\begin{aligned}\tilde{f}(p) = \mathscr{M}\{f(x)\} &= \int_0^\infty x^{p-1} H(a-x) dx \\ &= \int_0^a x^{p-1} dx \\ &= \frac{a^p}{p}.\end{aligned}$$

b. Let $f(x) = x^m g(x)$ where $g(x) = e^{-nx}$ with $m, n > 0$ and let $\tilde{g}(p) = \mathscr{M}\{g(x)\}$. By the shifting property of the Mellin transform, we have that

$$\tilde{f}(p) = \mathscr{M}\{f(x)\} = \mathscr{M}\{x^m g(x)\} = \tilde{g}(p+m).$$

From our table of Mellin transforms, we know that

$$\tilde{g}(p) = \mathscr{M}\{g(x)\} = \frac{\Gamma(p)}{n^p}$$

where $\Re\{p\} > 0$.

Therefore,

$$\tilde{f}(p) = \mathscr{M}\left\{f(x)\right\} = \tilde{g}(p+m) = \frac{\Gamma(p+m)}{n^{p+m}}$$

where $\Re\{p+m\} > 0$.

c. From our table of Mellin transforms, we see that

$$\mathscr{M}\left\{\frac{1}{(x^a+1)^s}\right\} = \frac{\Gamma(p/a)\Gamma(s-p/a)}{a\Gamma(s)}.$$

Therefore, for $f(x) = \frac{1}{x^2+1}$, identifying $a = 2$ and $s = 1$, we have that

$$\begin{aligned}\tilde{f}(p) = \mathscr{M}\left\{f(x)\right\} = \mathscr{M}\left\{\frac{1}{x^2+1}\right\} &= \frac{\Gamma(p/2)\Gamma(1-p/2)}{2\Gamma(1)} \\ &= \frac{\Gamma(p/2)\Gamma(1-p/2)}{2}.\end{aligned}$$

$\square$

Problem 8.4.

Solution. □

Problem 8.10.

Solution. □

Problem 8.12.

Solution. $\square$

Problem 8.14.

Solution. □

Problem 8.21.

Solution. □