Homework Assignment 3

Matthew Tiger

September 19, 2015

Problem 1. Let A and B be square $n \times n$ matrices and $x \in \mathbb{R}^{n \times 1}$ be a column vector. Count the number of multiplications needed to compute $(AB)x$ versus $A(Bx)$. Which one is better for large values of n?

Solution. To determine the number of multiplications necessary to compute the product $(AB)x$ we must find out how many multiplications it takes to compute $AB = C$ and how many multiplications it takes to compute Cx. Since A and B are square $n \times n$ matrices, each entry in their product will require n multiplications. Since AB has n^2 entries, it will take n^3 multiplications to compute $AB = C$. Now, C is a square $n \times n$ matrix and x is a $n \times 1$ matrix, so each entry in the product Cx will require n multiplications. Since there are n entries in Cx, the product will require n^2 multiplications. Therefore, $n^3 + n^2$ multiplications are necessary to compute $(AB)x$.

Similarly the number of multiplications necessary to compute $A(Bx)$ is determined by the number of multiplications necessary to compute $Bx = D$ and AD. Since Bx has n entries and it takes n multiplications to compute each entry, it takes n^2 multiplications to compute Bx. For similar reasons it takes n^2 multiplications to compute AD. Therefore, it takes $2n^2$ multiplications to compute $A(Bx)$.

Clearly, it is better to compute $A(Bx)$ for large values of n if the goal is to reduce the number of multiplications necessary to compute the product. $\square$

Problem 2. Let A and B be square $n \times n$ upper triangular matrices. Show that $C = AB$ is also upper triangular. How many multiplications are needed to compute C?

Solution. Note that a matrix $A = (a_{ij})$ is upper triangular if every entry of the matrix below the main diagonal is 0, i.e. if $a_{ij} = 0$ when $i > j$.

If $C = AB$, then it is clear that by definition $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$. Thus, the matrix product C is upper triangular if $c_{ij} = 0$ when $i > j$. So, let's consider c_{ij} for which $1 \leq j < i \leq n$. Then

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{j} a_{ik}b_{kj} + \sum_{k=j+1}^{n} a_{ik}b_{kj} \tag{1}$$

Now for $1 \leq k \leq j < i$, the entry $a_{ik} = 0$ since A is upper triangular showing that the left sum in (1) is 0 and for $j < j + 1 \leq k \leq n$, the entry $b_{kj} = 0$ since B is upper triangular showing that the right sum in (1) is 0. Therefore, $c_{ij} = 0$ for $i > j$ and C is an upper triangular matrix.

To compute the number of multiplications necessary to compute this product, we must determine what the other entries c_{ij} are when $i \leq j$.

When $1 \leq i \leq j \leq n$,

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{i-1} a_{ik}b_{kj} + \sum_{k=i}^{j} a_{ik}b_{kj} + \sum_{k=j+1}^{n} a_{ik}b_{kj} = \sum_{k=i}^{j} a_{ik}b_{kj}$$

since $a_{ik} = 0$ for $1 \leq k \leq i-1$ due to the fact that A is upper triangular and $b_{kj} = 0$ for $j+1 \leq k \leq n$ due to the fact that B is upper triangular. Thus, to compute the entry c_{ij} when $i \leq j$, there are $j - i + 1$ multiplications necessary to compute the entry and 0 multiplications are necessary when $i > j$.

If $x(c_{ij})$ is the number of multiplications necessary to calculate the entry c_{ij}, then X, the number of computations necessary to calculate the product C, using the above, is given by

$$\begin{aligned}
X = \sum_{j=1}^{n}\sum_{i=1}^{n} x(c_{ij}) &= \sum_{j=1}^{n}\sum_{i=1}^{j} x(c_{ij}) \\
&= \sum_{j=1}^{n}\sum_{i=1}^{j} j - i + 1 \\
&= \frac{1}{2}\sum_{j=1}^{n} 2j^2 - j(j+1) + 2j \\
&= \frac{1}{2}\sum_{j=1}^{n} j^2 + j = \frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{4} = \frac{n(n+1)(n+2)}{6}
\end{aligned}$$

Therefore, the number of multiplications necessary to compute the product of two $n \times n$ upper triangular matrices is, in general, $n(n+1)(n+2)/6$. $\square$

Problem 3. Let A be a square $n \times n$ upper triangular matrix. Show that A^{-1} is also upper triangular.

Solution. Suppose a matrix A is an upper triangular matrix. Then

$$\begin{aligned}
A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix} &= \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & \dots & a_{1n} \\ 0 & 0 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \\
&= \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & \dots & a_{1n} \\ 0 & 0 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \right) \\
&= \Lambda(I + U)
\end{aligned}$$

where Λ is a diagonal matrix consisting of the main diagonal of A, I is the identity matrix, and U is an upper triangular nilpotent matrix.

Using this definition of A, we can see easily that $A^{-1} = (I+U)^{-1}\Lambda^{-1}$. Let $B = I - U + U^2 + \cdots + (-1)^n U^n$. Then it is easy to see that B commutes with $I+U$ and

$$\begin{aligned}
(I+U)B &= (I+U)(I - U + U^2 + \cdots + (-1)^n U^n) \\
&= (I+U)\sum_{i=0}^{n}(-1)^i U^i \\
&= \sum_{i=0}^{n}(-1)^i U^i + \sum_{i=0}^{n}(-1)^i U^{i+1} \\
&= I + \sum_{i=1}^{n}(-1)^i U^i + \sum_{i=1}^{n}(-1)^{i-1} U^i + U^{n+1} \\
&= I + \sum_{i=1}^{n}\left((-1)^i + 1^i\right) U^i + U^{n+1} = I + U^{n+1}. \qquad (2)
\end{aligned}$$

Now $(I+U)B = I + U^{n+1} = I$ since U is a nilpotent matrix. Thus, $(I+U)^{-1} = B = I - U + U^2 + \cdots + (-1)^n U^n$. Note that since the sum of two upper triangular matrices is an upper triangular matrix and the product of two upper triangular matrices is an upper triangular matrix, $(I+U)^{-1} = I - U + U^2 + \cdots + (-1)^n U^n$ is an upper triangular matrix since I and U are upper triangular matrices.

Therefore, $A^{-1} = (I+U)^{-1}\Lambda^{-1}$ must be upper triangular since Λ^{-1} is an upper triangular matrix. $\square$