Homework Assignment 7

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Problem 3.2. For those processes in Problem 3.1 that are causal, compute and graph their ACF and PACF using the program ITSM.

Solution. From problem 3.1, the following processes are causal:

a. $X_t - (-0.2)X_{t-1} - 0.48X_{t-2} = Z_t$

b. $X_t - (-0.6)X_{t-1} = Z_t + 1.2Z_{t-1}$

c. $X_t - (-1.8)X_{t-1} - (-0.8)1X_{t-2} = Z_t$.

I am unable to copy the graphs from the program and save the image data.
To create one such graph, open itsm.exe and follow the below steps.

a. Create a new univariate project.

b. Go to `Model > Specify`.

c. Specify the AR order and MA order and enter in the coefficients to the above processes.

d. Go to `Model > ACF/PACF > Model` to display the graph.

These instructions can be repeated for each model to create the three graphs □

Problem 3.4. Compute the ACF and PACF of the AR(2) process

$$X_t = 0.8X_{t-2} + Z_t, \quad \{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$$

Solution. This process is equivalently written as

$$X_t - 0X_{t-1} - 0.8X_{t-2} = Z_t.$$

Thus, $\phi_0 = 1$, $\phi_1 = 0$, $\phi_2 = 0.8$, and $\phi_k = 0$ for $k > 2$ and $\theta_0 = 1$, $\theta_k = 0$ for $k > 0$. The ACF, $\rho(h)$, is defined as

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)}$$

where $\gamma(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|}$ and $\psi_0 = 1$, $\psi_j = \theta_j + \sum_{k=1}^{p} \phi_k \psi_{j-k}$ for $j > 1$ and $p = 2$.

Using the coefficients ϕ_k and θ_k, we see that

$$\psi_j = \phi_1 \psi_{j-1} + \phi_2 \psi_{j-2} = 0.8\psi_{j-2}$$

where $\psi_j = 0$ if $j < 0$.

If $j = 2k+1$ for $k \geq 0$, then $\psi_j = 0$ and if $j = 2k$ for $k \geq 0$, then $\psi_j = 0.8^k$. These two formulations are easily proved via induction.

Now,

$$\gamma(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|} = \sigma^2 \left(\sum_{k=0}^{\infty} \psi_{2k} \psi_{2k+|h|} + \sum_{j=0}^{\infty} \psi_{2k+1} \psi_{2k+1+|h|} \right)$$
$$= \sigma^2 \sum_{k=0}^{\infty} \psi_{2k} \psi_{2k+|h|}$$

since $\psi_j = 0$ for even j.

If $h = 2l+1$, then $\gamma(h) = \sigma^2 \sum_{k=0}^{\infty} \psi_{2k} \psi_{2k+|2l+1|} = 0$ since $\psi_{2k+|2l+1|} = 0$. If $h = 2l$, then

$$\gamma(h) = \sigma^2 \sum_{k=0}^{\infty} \psi_{2k} \psi_{2(k+|l|)+1} = \sigma^2 (0.8)^{|l|} \sum_{k=0}^{\infty} (0.8^2)^k = \frac{\sigma^2 (0.8^{|l|})}{0.36}$$

Therefore,

$$\rho(h) = \begin{cases} 0 & \text{if } h = 2l+1 \\ (0.8)^{|l|} & \text{if } h = 2l \end{cases}$$

For an AR(p) process, for the PACF function $\alpha(h)$,

$$\alpha(p) = \phi_p \quad \text{and} \quad \alpha(h) = 0 \text{ if } h > p.$$

Thus, we need only compute $\alpha(1)$. Now, $\alpha(1) = \gamma(1)/\gamma(0) = 0$. Therefore,

$$\alpha(h) = \begin{cases} 1 & \text{if } h = 0 \\ 0.8 & \text{if } h = 2 \\ 0 & \text{otherwise} \end{cases}.$$

□

Problem 3.6. Show that the two MA(1) processes

$$X_t = Z_t + \theta Z_{t-1}, \quad \{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$$
$$Y_t = \widetilde{Z}_t + \theta \widetilde{Z}_{t-1}, \quad \{\widetilde{Z}_t\} \sim \mathrm{WN}(0, \sigma^2)$$

where $0 < |\theta| < 1$, have the same autocovariance functions.

Solution. Note that for an ARMA(p,q) the autocovariance function $\gamma(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|}$ where $\psi_0 = 1$, $\psi_j = \theta_j + \sum_{k=1}^{p} \phi_k \psi_{j-k}$ for $j > 1$ and $\psi_j = 0$ for $j < 0$.

For X_t, we have

$$\psi_j = \begin{cases} 1 & \text{if } j = 0 \\ \theta & \text{if } j = 1 \\ 0 & \text{if } j > 1 \end{cases}$$

and for Y_t, we have

$$\widetilde{\psi}_j = \begin{cases} 1 & \text{if } j = 0 \\ \frac{1}{\theta} & \text{if } j = 1 \\ 0 & \text{if } j > 1 \end{cases}$$

Thus,

$$\gamma_X(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|} = \sigma^2 \sum_{j=0}^{1} \psi_j \psi_{j+|h|} = \sigma^2(\psi_{|h|} + \theta\psi_{1+|h|})$$

and

$$\gamma_Y(h) = \sigma^2\theta^2 \sum_{j=0}^{\infty} \widetilde{\psi}_j \widetilde{\psi}_{j+|h|} = \sigma^2\theta^2 \sum_{j=0}^{1} \widetilde{\psi}_j \widetilde{\psi}_{j+|h|} = \sigma^2\theta^2 \left(\psi_{|h|} + \frac{1}{\theta}\psi_{1+|h|} \right).$$

Explicitly,

$$\gamma_X(h) = \begin{cases} \sigma^2(1+\theta^2) & \text{if } h = 0 \\ \sigma^2\theta & \text{if } |h| = 1 \\ 0 & \text{if } |h| > 1 \end{cases}$$

and

$$\gamma_X(h) = \begin{cases} \sigma^2\theta^2 \left(1 + \frac{1}{\theta^2}\right) & \text{if } h = 0 \\ \sigma^2\theta^2 \left(\frac{1}{\theta}\right) & \text{if } |h| = 1 \\ 0 & \text{if } |h| > 1 \end{cases}.$$

Therefore, these two processes autocovariance functions are the same. □

Problem 3.10. As defined in the book.

Solution. We want to fit the `strikes.tsm` data to the mean corrected model $Y_t - \phi Y_{t-1} = Z_t$ where $Y_t = X_t - \mu$ and X_t is the data point from `strikes.tsm` at time t. Note that this is an AR(1) model. As such we know that the AR(1) autocovariance function is given by

$$\gamma_Y(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|} = \sigma^2 \phi^{|h|} \sum_{j=0}^{\infty} \phi^j = \frac{\sigma^2 \phi^{|h|}}{1 - \phi^2}$$

for $|\phi| < 1$.

From the ITSM tool, we know that the sample variance is $\hat{\gamma}(0) = 676789$. Using the fact that the sample ACF at lag 1 is $\hat{\rho}(1) = .7323$, we know that $\hat{\gamma}(1) = \hat{\rho}(1)\hat{\gamma}(0) = 495612.5847$.

Equating $\gamma_Y(h)$ to $\hat{\gamma}(h)$ at lags 0 and 1 gives us two equations to solve for the unknown parameters ϕ and σ^2.

Writing out the equations and choosing ϕ so that $|\phi| < 1$ gives $phi = 0.5646$ and $\sigma^2 = 461050$. These parameters define the given model and we are done. □