

Homework Assignment 8

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Problem 1. Write at least two necessary conditions and at least two sufficient conditions for a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ to be concave.

Solution. By definition, a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is concave over the convex set $\Omega \subset \mathbb{R}^n$ if $-f$ is convex over Ω . This definition will allow us to obtain results for concave functions by replacing f with $-f$ in previously obtained results concerning convex functions.

Using this definition and Theorem 22.2, we see that the condition that if for all $\alpha \in (0, 1)$ and for all $\mathbf{x}, \mathbf{y} \in \Omega$, we have that

$$f(\alpha \mathbf{x} + (1 - \alpha) \mathbf{y}) \geq \alpha f(\mathbf{x}) + (1 - \alpha) f(\mathbf{y})$$

is a necessary and sufficient condition for f to be concave on the convex set Ω .

Further, if the function f is \mathcal{C}^1 -smooth, we see from the above definition and Theorem 22.4 that the condition that if for all $\mathbf{x}, \mathbf{y} \in \Omega$, we have that

$$f(\mathbf{x}) \leq f(\mathbf{y}) + Df(\mathbf{x})(\mathbf{x} - \mathbf{y})$$

is a necessary and sufficient condition for f to be concave on the open convex set Ω .

Going one last step further, if the function f is \mathcal{C}^2 -smooth, we see from the above definition and Theorem 22.5 that the condition that if for all $\mathbf{x} \in \Omega$, the Hessian matrix $\mathbf{F}(\mathbf{x})$ of f at \mathbf{x} is a negative semi-definite matrix is a necessary and sufficient condition for f to be concave on the open convex set Ω . \square

Problem 2. Let $S \subset \mathbb{R}^n$ be a convex set and let $\mathbf{x}^* \in S$. Prove that a vector $\mathbf{d} \in \mathbb{R}^n$ is a feasible direction at \mathbf{x}^* (relative to S) if and only there exists $t_0 > 0$ such that $\mathbf{x}^* + t_0\mathbf{d} \in S$ with $\mathbf{d} \neq \mathbf{0}$.

Solution.

□

Problem 3. Recall that

$$\max\{\alpha, \beta\} := \begin{cases} \alpha & \text{if } \alpha \geq \beta \\ \beta & \text{if } \alpha < \beta \end{cases}.$$

Given two convex functions $f_1 : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ and $f_2 : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$, prove that for $\mathbf{x} \in \mathbb{R}^n$, the function

$$f(\mathbf{x}) := \max\{f_1(\mathbf{x}), f_2(\mathbf{x})\}$$

is convex.

Solution. Note that it is clear that the set \mathbb{R}^n is convex. Therefore, the function $f(\mathbf{x}) := \max\{f_1(\mathbf{x}), f_2(\mathbf{x})\}$ is convex if for all $\alpha \in (0, 1)$ and for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we have that

$$f(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) \leq \alpha f(\mathbf{x}) + (1 - \alpha)f(\mathbf{y}). \quad (1)$$

If either $f(\mathbf{x}) = +\infty$ or $f(\mathbf{y}) = +\infty$, then for all $\alpha \in (0, 1)$ and for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we see that $\alpha f(\mathbf{x}) + (1 - \alpha)f(\mathbf{y}) = +\infty$ and inequality (1) holds showing the convexity of f in this case.

Now suppose that both $f(\mathbf{x})$ and $f(\mathbf{y})$ are finite. Without loss of generality, we may assume that at the point $\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}$, we have that the max of f_1 and f_2 at that point occurs at f_1 , i.e.

$$\begin{aligned} f(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) &= \max\{f_1(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}), f_2(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y})\} \\ &= f_1(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}). \end{aligned}$$

Then, due to the convexity of f_1 , we have that for all $\alpha \in (0, 1)$ and for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$f_1(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) \leq \alpha f_1(\mathbf{x}) + (1 - \alpha)f_1(\mathbf{y}).$$

From the above definition of max, we readily see that

$$\begin{aligned} \alpha f_1(\mathbf{x}) + (1 - \alpha)f_1(\mathbf{y}) &\leq \alpha \max\{f_1(\mathbf{x}), f_2(\mathbf{x})\} + (1 - \alpha) \max\{f_1(\mathbf{y}), f_2(\mathbf{y})\} \\ &= \alpha f(\mathbf{x}) + (1 - \alpha)f(\mathbf{y}). \end{aligned}$$

Therefore, combining, we have that for all $\alpha \in (0, 1)$ and for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$f(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) = f_1(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) \leq \alpha f(\mathbf{x}) + (1 - \alpha)f(\mathbf{y})$$

showing that inequality (1) holds and that the function f is convex. □

Problem 4. Consider the pair of linear programming problems in asymmetric duality:

$$\begin{array}{ll} (P_a) & \text{minimize } f(\mathbf{x}) = \mathbf{c}^\top \mathbf{x} \\ & \text{subject to } A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array} \quad \begin{array}{ll} (D_a) & \text{maximize } F(\boldsymbol{\lambda}) = \boldsymbol{\lambda}^\top \mathbf{b} \\ & \text{subject to } \boldsymbol{\lambda}^\top A \leq \mathbf{c}^\top \end{array}$$

- Prove that (D_a) is a convex programming problem.
- Write the KKT conditions for (D_a) .
- Suppose that \mathbf{x}^* is feasible for (P_a) and $\boldsymbol{\lambda}^*$ is feasible for (D_a) . Use the KKT conditions to prove that if $(\mathbf{c}^\top - \boldsymbol{\lambda}^{*\top} A)\mathbf{x}^* = 0$, then $\boldsymbol{\lambda}^*$ is optimal for (D_a) .

Solution.

□