Homework Assignment 7

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Problem 4.28. Using the Laplace transform, evaluate the following integrals:

a. $f(t) = \int_0^\infty \frac{\sin tx}{\sqrt{x}} dx,$

e. $f(t) = \int_0^\infty e^{-tx^2} dx$, $0 < t$.

Solution. a. We begin by taking the Laplace transform of $f(t)$. Doing so yields

$$\begin{aligned}
\bar{f}(s) = \mathscr{L}\{f(t)\} &= \mathscr{L}\left\{\int_0^\infty \frac{\sin tx}{\sqrt{x}} dx\right\} \\
&= \int_0^\infty \mathscr{L}\left\{\frac{\sin tx}{\sqrt{x}}\right\} dx \\
&= \int_0^\infty \frac{\sqrt{x}}{s^2 + x^2} dx.
\end{aligned}$$

Using a computer algebra system, we see that this integral evaluates to

$$\begin{aligned}
\bar{f}(s) &= \int_0^\infty \frac{\sqrt{x}}{s^2 + x^2} dx \\
&= \frac{\pi}{\sqrt{2s}}.
\end{aligned}$$

From our table of Laplace transforms, we see that

$$\mathscr{L}^{-1}\left\{\frac{\Gamma(a+1)}{s^{a+1}}\right\} = t^a.$$

In particular, for $a = -1/2$, we see that

$$\mathscr{L}^{-1}\left\{\frac{\Gamma(1/2)}{s^{-1/2}}\right\} = \mathscr{L}^{-1}\left\{\frac{\sqrt{\pi}}{s^{-1/2}}\right\} = t^{-1/2}.$$

Therefore, the evaluation of the original integral is

$$\begin{aligned} f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} &= \mathscr{L}^{-1}\left\{\frac{\pi}{\sqrt{2s}}\right\} \\ &= \sqrt{\frac{\pi}{2}}\mathscr{L}^{-1}\left\{\frac{\sqrt{\pi}}{s^{-1/2}}\right\} \\ &= \sqrt{\frac{\pi}{2t}}. \end{aligned}$$

e. Applying the Laplace transform to $f(t)$ yields

$$\begin{aligned} \bar{f}(s) = \mathscr{L}\left\{f(t)\right\} &= \mathscr{L}\left\{\int_0^\infty e^{-tx^2}dx\right\} \\ &= \int_0^\infty \mathscr{L}\left\{e^{-tx^2}\right\}dx \\ &= \int_0^\infty \frac{1}{s+x^2}dx \end{aligned}$$

Using a computer algebra system, we see that

$$\begin{aligned} \bar{f}(s) &= \int_0^\infty \frac{1}{s+x^2}dx \\ &= \frac{\pi}{2\sqrt{s}}. \end{aligned}$$

Therefore, using previous arguments, we see that the evaluation of the original integral is

$$\begin{aligned} f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} &= \mathscr{L}^{-1}\left\{\frac{\pi}{2\sqrt{s}}\right\} \\ &= \sqrt{\frac{\pi}{4}}\mathscr{L}^{-1}\left\{\frac{\sqrt{\pi}}{s^{-1/2}}\right\} \\ &= \sqrt{\frac{\pi}{4t}}. \end{aligned}$$

$\square$

Problem 4.29. Show that

b. $I(a) = \int_0^\infty e^{-ax} \left(\frac{\sin qx - \sin px}{x} \right) dx = \tan^{-1}\left(\frac{q}{a}\right) - \tan^{-1}\left(\frac{p}{a}\right)$

Solution. b. Let $f(x) = \sin qx - \sin px$ and $g(x) = \frac{f(x)}{x}$.

From the definition of the Laplace transform, we see that this integral is the Laplace transform of $\frac{f(x)}{x}$ with respect to x in the variable a, i.e.

$$I(a) = \int_0^\infty e^{-ax} \left(\frac{\sin qx - \sin px}{x} \right) dx = \mathscr{L}\left\{ \frac{f(x)}{x} \right\} = \bar{g}(a).$$

From a previous result, we know that

$$I(a) = \mathscr{L}\left\{ \frac{f(x)}{x} \right\} = \int_a^\infty \bar{f}(a) da$$

where $\bar{f}(a) = \mathscr{L}\{f(x)\}$. Our table of Laplace transforms shows that

$$\begin{aligned} \bar{f}(a) = \mathscr{L}\{f(x)\} &= \mathscr{L}\{\sin qx - \sin px\} \\ &= \frac{q}{a^2+q^2} - \frac{p}{a^2+p^2}. \end{aligned}$$

Thus, we see that

$$I(a) = \int_a^\infty \bar{f}(a) da = \int_a^\infty \frac{q}{a^2+q^2} - \frac{p}{a^2+p^2} da.$$

Recall that

$$\int \frac{t}{a^2+t^2} da = \tan^{-1}\left(\frac{a}{t}\right) + C.$$

Therefore, we have that

$$\begin{aligned} I(a) &= \int_a^\infty \frac{q}{a^2+q^2} - \frac{p}{a^2+p^2} da \\ &= \left[\frac{\pi}{2} - \tan^{-1}\left(\frac{a}{q}\right) \right] - \left[\frac{\pi}{2} - \tan^{-1}\left(\frac{a}{p}\right) \right] \\ &= \tan^{-1}\left(\frac{q}{a}\right) - \tan^{-1}\left(\frac{p}{a}\right). \end{aligned}$$

□

Problem 4.32.

Solution. □

Problem 4.35. Using the Laplace transform, solve the following difference equations:

a. $\Delta u_n - 2u_n = 0$, $u_0 = 1$

b. $\Delta^2 u_n - 2u_{n+1} + 3u_n = 0$, $u_0 = 0$, $u_1 = 1$.

Solution. Define $S_n(t) = H(t-n) - H(t-n-1)$ for $n \le t < n+1$ and define

$$u(t) = \sum_{n=0}^{\infty} u_n S_n(t).$$

It follows that for $n \le t < n+1$ we have that $u(t) = u_n$.

By a previous theorem, if $\bar{u}(s) = \mathscr{L}\{u(t)\}$, then

$$\mathscr{L}\{u(t+1)\} = e^s \left[\bar{u}(s) - u_0 \bar{S}_0(s)\right]$$

where $\bar{S}_0 = \frac{1}{s}(1 - e^{-s})$. It then follows that

$$\mathscr{L}\{u(t+2)\} = e^{2s} \left[\bar{u}(s) - (u_0 + u_1 e^{-s})\bar{S}_0(s)\right].$$

a. Note that this difference equation is equivalent to

$$\Delta u_n - 2u_n = u_{n+1} - 3u_n = 0.$$

Applying the Laplace transform to the difference equation yields that

$$\mathscr{L}\{u_{n+1} - 3u_n\} = e^s \left[\bar{u}(s) - u_0 \bar{S}_0(s)\right] - 3\bar{u}(s) = 0 = \mathscr{L}\{0\}.$$

In light of the initial data, this equation becomes

$$e^s \left[\bar{u}(s) - \bar{S}_0(s)\right] - 3\bar{u}(s) = 0.$$

Thus, we see that

$$\bar{u}(s) = \frac{e^s \bar{S}_0(s)}{e^s - 3}.$$

Therefore, from a previous result, we see that the solution to the original difference equation is

$$u(t) = \mathscr{L}^{-1}\{\bar{u}(s)\} = \mathscr{L}^{-1}\left\{\frac{e^s \bar{S}_0(s)}{e^s - 3}\right\} = 3^n$$

b. Note that this difference equation is equivalent to

$$\Delta^2 u_n - 2u_{n+1} + 3u_n = u_{n+2} - 4u_{n+1} + 4u_n = 0.$$

Applying the Laplace transform to the difference equation yields that

$$\mathscr{L}\{u_{n+2} - 4u_{n+1} + 4u_n\} = e^{2s} \left[\bar{u}(s) - (u_0 + u_1 e^{-s})\bar{S}_0(s)\right] - 4e^s \left[\bar{u}(s) - u_0 \bar{S}_0(s)\right] + 4\bar{u}(s) = 0.$$

In light of the initial data, this equation becomes

$$e^{2s}\left[\bar{u}(s) - e^{-s}\bar{S}_0(s)\right] - 4e^s\bar{u}(s) + 4\bar{u}(s) = 0.$$

Thus, we see that

$$\bar{u}(s) = \frac{e^s\bar{S}_0(s)}{(e^s - 2)^2}.$$

From a previous result, we know that

$$\mathscr{L}\left\{na^n\right\} = \frac{ae^s\bar{S}_0(s)}{(e^s - a)^2}$$

Therefore, we see that the solution to the original difference equation is

$$u(t) = \mathscr{L}^{-1}\left\{\bar{u}(s)\right\} = \mathscr{L}^{-1}\left\{\frac{e^s\bar{S}_0(s)}{(e^s - 2)^2}\right\} = n2^{n-1}.$$

$\square$

Problem 4.36. Show that the solution of the difference equation

$$u_{n+2} + 4u_{n+1} + u_n = 0$$

with $u_0 = 0$ and $u_1 = 1$, is

$$u_n = \frac{1}{2\sqrt{3}} \left[\left(\sqrt{3} - 2\right)^n + (-1)^{n+1} \left(2 + \sqrt{3}\right)^n \right]$$

Solution. Applying the Laplace transform to the difference equation yields that

$$\mathscr{L}\{u_{n+2} + 4u_{n+1} + u_n\} = e^{2s}\left[\bar{u}(s) - (u_0 + u_1 e^{-s})\bar{S}_0(s)\right] + 4e^s\left[\bar{u}(s) - u_0\bar{S}_0(s)\right] + \bar{u}(s) = 0.$$

In light of the initial data, this equation becomes

$$e^{2s}\left[\bar{u}(s) - e^{-s}\bar{S}_0(s)\right] + 4e^s\bar{u}(s) + \bar{u}(s) = 0.$$

Thus, we see that

$$\bar{u}(s) = \frac{e^s\bar{S}_0(s)}{e^{2s} + 4e^s + 1} = \frac{e^s\bar{S}_0(s)}{(e^s - \alpha_1)(e^s - \alpha_2)},$$

where $\alpha_1 = -2 - \sqrt{3}$ and $\alpha_2 = -2 + \sqrt{3}$. From the method of partial fraction decomposition, we then see that

$$\begin{aligned} \bar{u}(s) &= \frac{e^s\bar{S}_0(s)}{(e^s - \alpha_1)(e^s - \alpha_2)} \\ &= \frac{e^s\bar{S}_0(s)}{\alpha_2 - \alpha_1}\left(\frac{1}{e^s - \alpha_2} - \frac{1}{e^s - \alpha_1}\right). \end{aligned}$$

From a previous result, we know that

$$\mathscr{L}^{-1}\left\{\frac{e^s\bar{S}_0(s)}{e^s - a}\right\} = a^n.$$

Therefore, the solution to the original difference equation is

$$\begin{aligned} u(t) = \mathscr{L}^{-1}\{\bar{u}(s)\} &= \mathscr{L}^{-1}\left\{\frac{e^s\bar{S}_0(s)}{\alpha_2 - \alpha_1}\left(\frac{1}{e^s - \alpha_2} - \frac{1}{e^s - \alpha_1}\right)\right\} \\ &= \frac{1}{\alpha_2 - \alpha_1}\left[\mathscr{L}^{-1}\left\{\frac{e^s\bar{S}_0(s)}{e^s - \alpha_2}\right\} - \mathscr{L}^{-1}\left\{\frac{e^s\bar{S}_0(s)}{e^s - \alpha_1}\right\}\right] \\ &= \frac{\alpha_2^n - \alpha_1^n}{\alpha_2 - \alpha_1} \\ &= \frac{1}{2\sqrt{3}}\left[\left(\sqrt{3} - 2\right)^n + (-1)^{n+1}\left(2 + \sqrt{3}\right)^n\right]. \end{aligned}$$

□

Problem 4.37. Show that the solution of the differential difference equation

$$\frac{d}{dt}u(t) - u(t-1) = 2, \qquad u(0) = 0$$

is

$$u(t) = 2\left[t - \frac{(t-1)^2}{2!} + \frac{(t-2)^3}{3!} + \cdots + \frac{(t-n)^{n+1}}{(n+1)!}\right], \quad t > n$$

Solution. Applying the Laplace transform to the differential difference equation yields that

$$s\bar{u}(s) - u(0) - e^{-s}\left[\bar{u}(s) - u(0)\bar{S}_0(s)\right] = \frac{2}{s}$$

In light of the initial data, this equation reduces to

$$s\bar{u}(s) - e^{-s}\bar{u}(s) = \frac{2}{s},$$

or, equivalently,

$$\bar{u}(s) = \frac{2}{s(s-e^s)} = \frac{2}{s^2}\left(1 - \frac{e^{-s}}{s}\right)^{-1}$$

Expanding the right term in terms of its power series we see that

$$\begin{aligned}\bar{u}(s) &= \frac{2}{s^2}\left(1 - \frac{e^{-s}}{s}\right)^{-1}\\ &= \frac{2}{s^2}\sum_{n=0}^{\infty}\frac{e^{-ns}}{s^n}\\ &= 2\sum_{n=0}^{\infty}\frac{e^{-ns}}{s^{n+2}}.\end{aligned}$$

Recall that

$$\mathscr{L}^{-1}\left\{\frac{e^{-as}}{s^n}\right\} = \frac{(t-a)^{n-1}}{\Gamma(n)}H(t-a).$$

Therefore, the solution to the original differential difference equation is

$$\begin{aligned}u(t) &= \mathscr{L}^{-1}\left\{2\sum_{n=0}^{\infty}\frac{e^{-ns}}{s^{n+2}}\right\}\\ &= 2\sum_{n=0}^{\infty}\mathscr{L}^{-1}\left\{\frac{e^{-ns}}{s^{n+2}}\right\}\\ &= 2\sum_{n=0}^{\infty}\frac{(t-n)^{n+1}}{\Gamma(n+2)}H(t-n)\\ &= 2\sum_{n=0}^{\infty}\frac{(t-n)^{n+1}}{(n+1)!}H(t-n).\end{aligned}$$

□

Problem 4.40. Solve the telegraph equation

$$
\begin{aligned}
&u_{tt} - c^2 u_{xx} + 2au_t = 0, \qquad -\infty < x < \infty, \quad 0 < t \\
&u(x,0) = 0, \quad u_t(x,0) = g(x).
\end{aligned}
$$

Solution. We begin by applying the Laplace transform to the equation. Doing so yields

$$s^2\bar{u}(x,s) - su(x,0) - u_t(x,0) - c^2\frac{d^2}{dx^2}\left[\bar{u}(x,s)\right] + 2as\bar{u}(x,s) - 2au(x,0) = 0.$$

Using the initial data, this equation reduces to

$$s^2\bar{u}(x,s) - g(x) - c^2\frac{d^2}{dx^2}\left[\bar{u}(x,s)\right] + 2as\bar{u}(s) = 0,$$

or, equivalently,

$$\frac{d^2}{dx^2}\left[\bar{u}(x,s)\right] - \frac{s^2+2as}{c^2}\bar{u}(x,s) = -\frac{g(x)}{c^2}.$$

Now, applying the Fourier transform to this equation yields

$$k^2\bar{U}(k,s) + \frac{s^2+2as}{c^2}\bar{U}(k,s) = \frac{G(k)}{c^2}.$$

Solving the resulting algebraic equation for $\bar{U}(k,s)$ shows that

$$\bar{U}(k,s) = \frac{G(k)}{s^2+2as+k^2c^2} = \frac{G(k)}{(s+a)^2+k^2c^2-a^2}.$$

From our table of Laplace transforms, we see that

$$\mathscr{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2}.$$

Thus, the inverse Laplace transform of the above equation is

$$
\begin{aligned}
U(k,t) = \mathscr{L}^{-1}\left\{\bar{U}(k,s)\right\} &= \mathscr{L}^{-1}\left\{\frac{G(k)}{(s+a)^2+k^2c^2-a^2}\right\} \\
&= G(k)\frac{e^{-at}\sin\left(t\sqrt{k^2c^2-a^2}\right)}{\sqrt{k^2c^2-a^2}}.
\end{aligned}
$$

Let $\omega(k) = \sqrt{k^2c^2-a^2}$. Then the above equation becomes

$$
\begin{aligned}
U(k,t) &= G(k)\frac{e^{-at}\sin\left(t\sqrt{k^2c^2-a^2}\right)}{\sqrt{k^2c^2-a^2}} \\
&= G(k)\frac{e^{-at}\sin\left(\omega(k)t\right)}{\omega(k)} \\
&= e^{-at}G(k)F(k,t),
\end{aligned}
$$

where

$$F(k,t) = \frac{\sin\left(\omega(k)t\right)}{\omega(k)}.$$

Therefore, according to the Convolution Theorem, the solution to the original equation is

$$\begin{aligned} u(x,t) = \mathscr{F}^{-1}\left\{U(k,t)\right\} &= e^{-at}\mathscr{F}^{-1}\left\{G(k)F(k,t)\right\} \\ &= e^{-at}\left[g(x) * f(x,t)\right] \\ &= \frac{e^{-at}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\xi)f(x-\xi,t)d\xi. \end{aligned}$$

$\square$

Problem 4.43.

Solution. □

Problem 4.50.

Solution. $\square$