Homework Assignment 3

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Problem 4.28. Every time that the team wins a game, it wins its next game with probability 0.8; every time it loses a game, it wins its next game with probability 0.3. If the team wins a game, then it has dinner together with probability 0.7, whereas if the team loses then it has dinner together with probability 0.2. What proportion of games result in a team dinner?

Solution. Let $\{X_n : n \geq 0\}$ be the stochastic process of the outcomes of the team's games where if $X_n = 1$, then the team won game n and if $X_n = 0$, then the team lost game n. Since the outcome of each game is dependent only upon the previous game, this stochastic process may be modeled as a Markov chain with state space $\mathcal{M} = \{0, 1\}$.

From the assumptions of the model, if the team wins a game, the probability it wins the next game is 0.8 so that

$$P_{11} = P\{X_{n+1} = 1 \mid X_n = 1\} = 0.8$$

which implies that

$$P_{10} = P\{X_{n+1} = 1 \mid X_n = 0\} = 0.2.$$

Similarly, from the assumptions of the model, if the team loses a game, it wins its next game with probability 0.3 so that

$$P_{01} = P\{X_{n+1} = 1 \mid X_n = 0\} = 0.3$$

which implies that

$$P_{00} = P\{X_{n+1} = 0 \mid X_n = 0\} = 0.7.$$

Thus, the trasition matrix $\mathbf{P}$ of the Markov chain is given by

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} = \begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix}.$$

Note that state 0 communicates with state 1 and that the expected number of transitions this Markov chain must make before transitioning from state 0 to state 1 or vice versa is finite. This implies that this Markov chain is irreducible and ergodoic. Therefore, if π_i

represents the long-run proportion that the Markov chain spends in state i, then the long-run proportions of the Markov chain satisfy the following equations:

$$\mathbf{P}^{\intercal}\begin{bmatrix}\pi_0\\ \pi_1\end{bmatrix} = \begin{bmatrix}\pi_0\\ \pi_1\end{bmatrix},$$
$$\sum_{j\in\mathcal{M}} \pi_j = 1.$$

Solving the matrix equation we see that $\pi_1 = 3\pi_0/2$. Thus, by the second equation we have that $\pi_0 = 2/5$ and $\pi_1 = 3/5$. and the Markov chain will spend 2/5 of its time in state 0 and 3/5 of its time in state 1. Therefore, the team will win 3/5 of its games and lose 2/5 of its games after playing a large number of games.

Since the team will have dinner with probability 0.7 if the team wins and the team will have dinner with probability 0.2 if the team loses, we have that the proportion of games that result in a team dinner is given by 20% of its losses plus 70% of its wins, i.e.

$$0.2\pi_0 + 0.7\pi_1 = \frac{1}{2}.$$

Therefore, half of the team's games will result in a team dinner. $\square$

Problem 4.29. An organization has N employees where N is a large number. Each employee has one of three possible job classifications and changes classifications (independently) according to a Markov chain with transition probabilities

$$\mathbf{P} = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$$

What percentage of employees are in each classification?

Solution. Let $\mathcal{M} = \{0, 1, 2\}$ be the states of this Markov process. If N is large, the percentage of employees in classification $0, 1, 2$ are given by the long-run proportions of the stated Markov chain π_0, π_1, π_2, respectively. It is clear that this Markov chain is ergodic and irreducible. Thus, the long-run proportions of the Markov chain satisfy the following equations:

$$\mathbf{P}^{\intercal} \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \end{bmatrix},$$
$$\sum_{j \in \mathcal{M}} \pi_j = 1.$$

Thus, we must solve the system of equations

$$\begin{aligned} 0.2\pi_0 + 0.6\pi_1 + 0.4\pi_2 &= \pi_1 \\ 0.1\pi_0 + 0.2\pi_1 + 0.5\pi_2 &= \pi_2 \\ \pi_0 + \pi_1 + \pi_2 &= 1. \end{aligned}$$

Using a computer algebra system we see that the solution to the system is given by

$$\begin{aligned} \pi_0 &= \frac{6}{17} \approx 0.352941 \\ \pi_1 &= \frac{7}{17} \approx 0.411765 \\ \pi_2 &= \frac{4}{17} \approx 0.235294. \end{aligned}$$

Therefore, if N is large, approximately 35.29% of employees are in classification 0, 41.18% of employees are in classification 1, and 23.53% of employees are in classification 2. $\square$

Problem 4.30. Three out of every four trucks on the road are followed by a car, while only one out of every five cars is followed by a truck. What fraction of vehicles on the road are trucks?

Solution. □

Problem 4.33. A professor continually gives exams to her students. She can give three possible types of exams, and her class is graded as either having done well or badly. Let p_i denote the probability that the class does well on a type i exam, and suppose that $p_1 = 0.3$, $p_2 = 0.6$, and $p_3 = 0.9$. If the class does well on an exam, then the next exam is equally likely to be any of the three types. If the class does badly, then the next exam is always type 1. What proportion of exams are type i?

Solution. □

Problem 4.35. Consider a Markov chain with states $0, 1, 2, 3, 4$. Suppose $P_{04} = 1$; and suppose that when the chain is in state i, with $i > 0$, the next state is equally likely to be any of the states $0, 1, \ldots, i-1$. Find the limiting probabilities of this Markov chain.

Solution. $\square$