Basic Ideas

Unit 1

content

1. What methods are "computational"

2. Analytical methods vs numerical methods

3. Computational experiments

4. Computational abstract algebra, computational topology, computational algebraic geometry

5. Journals

what methods are computational

Given a mathematical model of a phenomenon, e.g.

- definite integral – a model of ????

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- spline (piecewise polynomial) – a model of ????

what methods are computational

Given a mathematical model of a phenomenon, e.g.

- definite integral – a model of ????
- differential equation – a model of ????
- spline (piecewise polynomial) – a model of ????

Analytical solutions remain the best (if known).

analytical solutions

- definite integral – analytical solution ????

analytical solutions

- definite integral – analytical solution ????
- differential equation – analytical solution ????

analytical solutions

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Analytical solutions in many cases are not known.

numerical solutions

- definite integral – numerical solutions ????

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How close is the numerical solution to the analytical one?

numerical solutions

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Which algorithm is better?

numerical solutions

- definite integral – numerical solutions ????

- differential equation – numerical solutions ????

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Which algorithm is better?

Which algorithm is faster?

computational experiment: splines

- analytically deduce dimension of C^1 quadratic splines over two triangles sharing an edge

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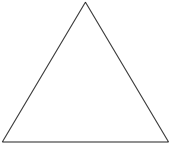

Figure: $ax^2 + by^2 + cxy + dx + ey + f$,

computational experiment: splines

- analytically deduce dimension of C^1 quadratic splines over two triangles sharing an edge

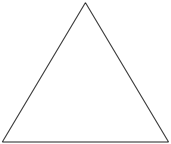

Figure: $ax^2 + by^2 + cxy + dx + ey + f, \;\; dim\ S_2^1(\Delta_1) = 6$

computational experiment: splines

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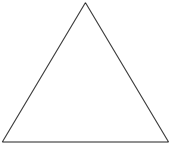

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- How about two triangles? Check numerically.

computational experiment: splines

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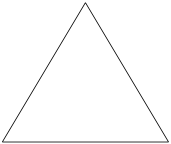


Figure: $ax^2 + by^2 + cxy + dx + ey + f$, $dim\ S_2^1(\Delta_1) = 6$

- How about two triangles? Check numerically.

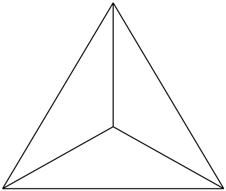
Why the computational experiment is not a proof?

computational experiment: splines

- analytically deduce dimension of C^1 quadratic splines over three triangles sharing a vertex

computational experiment: splines

• analytically deduce dimension of C^1 quadratic splines over three triangles sharing a vertex



computational experiment: splines

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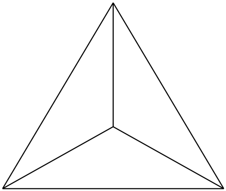

- Check numerically.

computational algebra, topology, algebraic geometry

- Link

computational algebra, topology, algebraic geometry

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computational algebra, topology, algebraic geometry

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computational algebra, topology, algebraic geometry

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journals, meetings, people

- SIAM ▸ Link

journals, meetings, people

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- Journal Ranking Link

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BREAK 10 min

content

1. Discretization (discrete vs continuous)
2. Stability
3. Well-posed vs ill-posed problems
4. Conditioning
5. Types of errors
6. Accuracy
7. Operation count
8. Loss of significant digits

discretization

continuous model $\qquad f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

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$$e(a,h) = \mathcal{O}(h)$$

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The problem of finding a solution $q = R(u)$ is well-posed, if for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any u_1, u_2 such that $q_1 = R(u_1)$, $q_2 = R(u_2)$ and $|u_1 - u_2| < \delta$ it follows that $|q_1 - q_2| < \varepsilon$.

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- Example 1: $q = \frac{1}{u}$, $0 < u < 1$

- Example 2: $q = \frac{1}{u}$, $10^{-3} < u < 1$

well-posed vs ill-posed problems

An *ill-posed problem* is a problem that either has no solutions (in the desired class),

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In this course we shall work with well posed problems. “Definitions and examples of inverse and ill-posed problems”,S. I. Kabanikhin ▸ Link See examples on p. 324

conditioning

• conditioning = sensitivity of the solution to perturbations of the input data.

low sensitivity = well conditioned

high sensitivity = ill conditioned

conditioning

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low sensitivity = well conditioned

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- a well posed problem can be ill conditioned

- Example: $q = \frac{1}{u}$, $10^{-3} < u < 1$

types of errors

- unavoidable error (input data, measurement, modeling)

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$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$

accuracy and operation count

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$$y = 1 + x + x^2 + \cdots + x^{215}$$

$$y = \frac{1-x^{216}}{1-x}$$

loss of significant digits: a famous example

$$\sqrt{x} - \sqrt{x-1}$$

Suppose we format short, and $x = 20157$. Then

$\sqrt{x} \approx 141.9753$, $\sqrt{x-1} = 141.9718$ and $\sqrt{x} - \sqrt{x-1} = 0.0035$

loss of significant digits: a famous example

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$\frac{1}{\sqrt{x}+\sqrt{x-1}}$

Try this with Matlab to see if it helps