Homework Assignment 7

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Problem 4.28. Using the Laplace transform, evaluate the following integrals:

a. $f(t) = \displaystyle\int_0^\infty \frac{\sin tx}{\sqrt{x}} dx$,

e. $f(t) = \displaystyle\int_0^\infty e^{-tx^2} dx$, $0 < t$.

Solution. a. We begin by taking the Laplace transform of $f(t)$. Doing so yields

$$\begin{aligned}\bar{f}(s) = \mathscr{L}\left\{f(t)\right\} &= \mathscr{L}\left\{\int_0^\infty \frac{\sin tx}{\sqrt{x}} dx\right\} \\ &= \int_0^\infty \mathscr{L}\left\{\frac{\sin tx}{\sqrt{x}}\right\} dx \\ &= \int_0^\infty \frac{\sqrt{x}}{s^2 + x^2} dx.\end{aligned}$$

Using a computer algebra system, we see that this integral evaluates to

$$\begin{aligned}\bar{f}(s) &= \int_0^\infty \frac{\sqrt{x}}{s^2 + x^2} dx \\ &= \frac{\pi}{\sqrt{2s}}.\end{aligned}$$

From our table of Laplace transforms, we see that

$$\mathscr{L}^{-1}\left\{\frac{\Gamma(a+1)}{s^{a+1}}\right\} = t^a.$$

In particular, for $a = -1/2$, we see that

$$\mathscr{L}^{-1}\left\{\frac{\Gamma(1/2)}{s^{-1/2}}\right\} = \mathscr{L}^{-1}\left\{\frac{\sqrt{\pi}}{s^{-1/2}}\right\} = t^{-1/2}.$$

Therefore, the evaluation of the original integral is

$$\begin{aligned} f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} &= \mathscr{L}^{-1}\left\{\frac{\pi}{\sqrt{2s}}\right\} \\ &= \sqrt{\frac{\pi}{2}}\mathscr{L}^{-1}\left\{\frac{\sqrt{\pi}}{s^{-1/2}}\right\} \\ &= \sqrt{\frac{\pi}{2t}}. \end{aligned}$$

e. Applying the Laplace transform to $f(t)$ yields

$$\begin{aligned} \bar{f}(s) = \mathscr{L}\left\{f(t)\right\} &= \mathscr{L}\left\{\int_0^\infty e^{-tx^2}dx\right\} \\ &= \int_0^\infty \mathscr{L}\left\{e^{-tx^2}\right\}dx \\ &= \int_0^\infty \frac{1}{s+x^2}dx \end{aligned}$$

Using a computer algebra system, we see that

$$\begin{aligned} \bar{f}(s) &= \int_0^\infty \frac{1}{s+x^2}dx \\ &= \frac{\pi}{2\sqrt{s}}. \end{aligned}$$

Therefore, using previous arguments, we see that the evaluation of the original integral is

$$\begin{aligned} f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} &= \mathscr{L}^{-1}\left\{\frac{\pi}{2\sqrt{s}}\right\} \\ &= \sqrt{\frac{\pi}{4}}\mathscr{L}^{-1}\left\{\frac{\sqrt{\pi}}{s^{-1/2}}\right\} \\ &= \sqrt{\frac{\pi}{4t}}. \end{aligned}$$

□

Problem 4.29. Show that

b. $I(a) = \int_0^\infty e^{-ax}\left(\frac{\sin qx - \sin px}{x}\right) dx = \tan^{-1}\left(\frac{q}{a}\right) - \tan^{-1}\left(\frac{p}{a}\right)$

Solution. b. Let $f(x) = \sin qx - \sin px$ and $g(x) = \frac{f(x)}{x}$.

From the definition of the Laplace transform, we see that this integral is the Laplace transform of $\frac{f(x)}{x}$ with respect to x in the variable a, i.e.

$$I(a) = \int_0^\infty e^{-ax}\left(\frac{\sin qx - \sin px}{x}\right) dx = \mathscr{L}\left\{\frac{f(x)}{x}\right\} = \bar{g}(a).$$

From a previous result, we know that

$$I(a) = \mathscr{L}\left\{\frac{f(x)}{x}\right\} = \int_a^\infty \bar{f}(a) da$$

where $\bar{f}(a) = \mathscr{L}\{f(x)\}$. Our table of Laplace transforms shows that

$$\begin{aligned}\bar{f}(a) = \mathscr{L}\{f(x)\} &= \mathscr{L}\{\sin qx - \sin px\}\\ &= \frac{q}{a^2+q^2} - \frac{p}{a^2+p^2}.\end{aligned}$$

Thus, we see that

$$I(a) = \int_a^\infty \bar{f}(a) da = \int_a^\infty \frac{q}{a^2+q^2} - \frac{p}{a^2+p^2} da.$$

Recall that

$$\int \frac{t}{a^2+t^2} da = \tan^{-1}\left(\frac{a}{t}\right) + C.$$

Therefore, we have that

$$\begin{aligned}I(a) &= \int_a^\infty \frac{q}{a^2+q^2} - \frac{p}{a^2+p^2} da\\ &= \left[\frac{\pi}{2} - \tan^{-1}\left(\frac{a}{q}\right)\right] - \left[\frac{\pi}{2} - \tan^{-1}\left(\frac{a}{p}\right)\right]\\ &= \tan^{-1}\left(\frac{q}{a}\right) - \tan^{-1}\left(\frac{p}{a}\right).\end{aligned}$$

□

Problem 4.32.

Solution. □

Problem 4.35.

Solution. □

Problem 4.36.

Solution. □

Problem 4.37.

Solution. □

Problem 4.40.

Solution. □

Problem 4.43.

Solution. □

Problem 4.50.

Solution. □