Homework Assignment 1

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Problem 3.7. Suppose $p(x, y, z)$, the joint probability mass function of the random variables X, Y, and Z, is given by

$$\begin{aligned} p(1,1,1) &= \tfrac{1}{8}, \quad & p(2,1,1) &= \tfrac{1}{4}, \\ p(1,1,2) &= \tfrac{1}{8}, \quad & p(2,1,2) &= \tfrac{3}{16}, \\ p(1,2,1) &= \tfrac{1}{16}, \quad & p(2,2,1) &= 0, \\ p(1,2,2) &= 0, \quad & p(2,2,2) &= \tfrac{1}{4}. \end{aligned}$$

What is $E[X|Y = 2]$? What is $E[X|Y = 2, Z = 1]$?

Solution. Recall that the conditional probability mass function of X given that $Y = y$ is given by

$$p_{X|Y}(x|y) = P\{X = x|Y = y\} = \frac{P\{X = x, Y = y\}}{P\{Y = y\}}.$$

As a natural extension, we have that the conditional expectation of X given that $Y = y$ is given by

$$E[X|Y = y] = \sum_x xP\{X = x|Y = y\} = \sum_x xp_{X|Y}(x|y).$$

Thus, in order to find the conditional expectation of X given that $Y = 2$, i.e. $E[X|Y = 2]$, we first need to determine $p_{X|Y}(x|2)$. We note from the above joint probability mass function that

$$P\{Y = 2\} = \sum_{x,z} p(x, 2, z) = p(1, 2, 1) + p(2, 2, 1) + p(1, 2, 2) + p(2, 2, 2) = \frac{5}{16}.$$

Similarly, we have from the above joint probability mass function that

$$P\{X = x, Y = 2\} = \sum_z p(x, 2, z) = p(x, 2, 1) + p(x, 2, 2).$$

Thus, the conditional probability mass function of X given that $Y = 2$ is given by

$$p_{X|Y}(x|2) = \frac{P\{X = x, Y = 2\}}{P\{Y = 2\}} = \begin{cases} \frac{p(1,2,1)+p(1,2,2)}{5/16} = \frac{1}{5} & \text{if } x = 1 \\ \frac{p(1,2,1)+p(1,2,2)}{5/16} = \frac{4}{5} & \text{if } x = 2. \end{cases}$$

Using $p_{X|Y}(x|2)$, we readily see that

$$E[X|Y = 2] = \sum_x x p_{X|Y}(x|2) = 1 \cdot p_{X|Y}(1|2) + 2 \cdot p_{X|Y}(2|2) = \frac{9}{5}.$$

In order to find the conditional expectation of X given that $Y = 2$ and $Z = 1$, i.e. $E[X|Y = 2, Z = 1]$, we proceed in a similar manner as previously by first finding $p_{X|Y,Z}(x|2,1)$. We note from the above joint probability mass function that

$$P\{Y = 2, Z = 1\} = \sum_x p(x,2,1) = p(1,2,1) + p(2,2,1) = \frac{1}{16}$$

Similarly, we have from the above joint probability mass function that

$$P\{X = x, Y = 2, Z = 1\} = p(x,2,1).$$

Thus, the conditional probability mass function of X given that $Y = 2$ and $Z = 1$ is given by

$$p_{X|Y,Z}(x|2,1) = \frac{P\{X = x, Y = 2, Z = 1\}}{P\{Y = 2, Z = 1\}} = \begin{cases} \frac{p(1,2,1)}{1/16} = 1 & \text{if } x = 1 \\ \frac{p(2,2,1)}{1/16} = 0 & \text{if } x = 2. \end{cases}$$

Using $p_{X|Y,Z}(x|2,1)$, we readily see that

$$E[X|Y = 2, Z = 1] = \sum_x x p_{X|Y,Z}(x|2,1) = 1 \cdot p_{X|Y,Z}(1|2,1) + 2 \cdot p_{X|Y,Z}(2|2,1) = 1.$$

$\square$

Problem 3.8. An unbiased die is successively rolled. Let X and Y denote, respectively, the number of rolls necessary to obtain a six and a five. Find:

a. $E[X]$,

b. $E[X|Y = 1]$,

c. $E[X|Y = 5]$.

Solution. The experiment of rolling a die, assuming the die is six-sided, has six possible outcomes: the die lands oriented such that the side with 1, 2, 3, 4, 5, or 6 pips is face-up. Assuming the die is unbiased, each outcome occurs with probability $p = 1/6$ and each trial of rolling the die is independent of any other trial. If X and Y denote, respectively, the number of rolls necessary to obtain a six and a five, then under the given assumptions, X and Y are both geometric random variables with parameter $p = 1/6$. The probability mass function for these random variables is given by $p(n) = (1 - p)^{n-1}p = (5/6)^{n-1}(1/6)$.

For the following computations, we make use the fact that that the infinite series of a geometric sequence $a_n = q^n$ is uniformly convergent on its interval of convergence, in particular on the interval $[0, 1)$, given that $|q| < 1$. To demonstrate this, take $0 \leq q < 1$. Note that for $q < 1$, there exists $\varepsilon > 0$ such that $q < q + \varepsilon < 1$. Now let $a_n = q^n$ and $M_n = (q+\varepsilon)^n$ and note that for $n > 0$, we have that $|a_n| < M_n$ and that $\sum_{n=1}^{\infty} M_n$ converges since $q+\varepsilon$ is in the interval of convergence of the infinite series. Thus, the sequence a_n meets the criteria of Weierstrass's M-test and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} q^n$ converges uniformly on $[0, 1)$. Since the series $\sum_{n=1}^{\infty} q^n$ is uniformly convergent for $0 \leq q < 1$, we may switch the order of summation and differentiation on this series.

a. Suppose that Z is a geometric random variable with parameter p. Then, by definition, we have that the probability mass function of Z is given by $p(n) = (1 - p)^{n-1}p$ and that the expectation of Z is given by

$$E[Z] = \sum_{n=1}^{\infty} np(n) = \sum_{n=1}^{\infty} np(1 - p)^{n-1} = p \sum_{n=1}^{\infty} nq^{n-1}.$$

We know that since $0 \leq q < 1$ the power series $\sum_{n=1}^{\infty} q^n$ converges uniformly. Thus,

$$\frac{d}{dq}\left[\sum_{n=1}^{\infty} q^n\right] = \sum_{n=1}^{\infty} \frac{d}{dq}[q^n] = \sum_{n=1}^{\infty} nq^{n-1} \tag{1}$$

Using (1), we see that

$$E[Z] = p\sum_{n=1}^{\infty} nq^{n-1} = p\frac{d}{dq}\left[\sum_{n=1}^{\infty} q^n\right] = p\frac{d}{dq}\left[\frac{1}{1-q}\right] = \frac{p}{(1-q)^2}.$$

Therefore, since $q = 1 - p$ we have that $E[Z] = 1/p$.

This result shows that for the random geometric variable X with parameter $p = 1/6$, we have that $E[X] = 1/(1/6) = 6$. Therefore, we expect to have to cast the die 6 times in order to roll a six.

b.

c.

$\square$

Problem 3.9. Show in the discrete case that if X and Y are independent, then

$$E[X|Y = y] = E[X] \text{ for all y.}$$

Solution. □

Problem 3.10. Suppose X and Y are independent continuous random variables. Show that

$$E[X|Y = y] = E[X] \text{ for all y.}$$

Solution. □

Problem 3.13. Let X be exponential with mean $1/\lambda$; that is,

$$f_X(x) = \lambda e^{-\lambda x}, \quad 0 < x < \infty.$$

Find $E[X|X > 1]$.

Solution. $\square$

Problem 3.14. Let X be uniform over $(0, 1)$. Find $E[X|X < 1/2]$.

Solution. □