Homework Assignment 1

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Problem 1. To be comprehensive, the second derivative test for two-variable functions $f = f(x, y)$ studied in Calculus III should contain (among others) the cases:

a. $D(a, b) > 0$ and $f_{xx}(a, b) = 0$,

b. $D(a, b) = 0$ and $f_{xx}(a, b) = 0$.

Why aren't these cases considered? Explain.

Solution. Throughout, we assume that $f : S \subset \mathbb{R}^2 \to \mathbb{R}$ and that $f \in C^2(S)$ so that $f_{xy}(a, b) = f_{yx}(a, b)$. Therefore,

$$\begin{aligned} D(a, b) &= f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)f_{yx}(a, b) \\ &= f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2. \end{aligned}$$

a. To illustrate that this case can never happen, suppose to the contrary that $D(a, b) > 0$ and $f_{xx}(a, b) = 0$. Since $D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2$, we see that $0 < D(a, b) = -f_{xy}(a, b)^2$ which is a contradiction since $f_{xy}(a, b)^2 > 0$. Therefore, this case cannot happen.

b. Now suppose that $D(a, b) = 0$ and $f_{xx}(a, b) = 0$. As $D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2$, it is true under our supposition that $f_{xy}(a, b)^2 = 0$, i.e. $f_{xy}(a, b) = 0$. We cannot conclusively state whether the point is a local extrema or saddle point as the function could be increasing or decreasing in the direction of x or y.

   To illustrate, take as an example $f_1(x, y) = -x^4 - y^4$ and $f_2(x, y) = x^4 + y^4$. Note that f_1 and f_2 both satisfy $D(a, b) = 0$ and $f_{xx}(a, b) = 0$ for the point $(a, b) = (0, 0)$. However, upon further inspection f_1 obtains a local maximum at $(0, 0)$, yet f_2 obtains a local minimum at $(0, 0)$. Thus, two different results occur for two different functions in the case where $D(a, b) = 0$ and $f_{xx}(a, b) = 0$ and we conclude that the test is inconclusive in such cases.

□

Problem 2. Recall that

- (a, b) is called an *absolute maximum* of $f = f(x, y)$ on a domain $D \subset \mathbb{R}^2$ if $f(x, y) \leq f(a, b)$ for every $(x, y) \in D$.
- (The Extreme Value Theorem) If f is continuous and D is closed and bounded, then f attains both an absolute maximum value and an absolute minimum value.

a. Describe in steps (and in words) how one finds absolute extrema for a two-variable function $f = f(x, y)$ on a closed bounded $D \subset \mathbb{R}^2$.

b. Apply your procedure derived in (a) to find absolute extrema for $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$ over the rectangle $D := \{(x, y) \mid -2 \leq x \leq 3, 0 \leq y \leq 2\}$.

Solution. a. The steps below outline the process to obtain the absolute extreme for a two-variable, continuous function $f = f(x, y)$ on a closed bounded $D \subset \mathbb{R}^2$.

I. First, identify the critical points of the function, i.e. find the points (x_i, y_i) such that

$$\nabla f(x_i, y_i) = \langle f_x(x_i, y_i), f_y(x_i, y_i) \rangle = \langle 0, 0 \rangle$$

or such that $f_x(x_i, y_i)$ or $f_y(x_i, y_i)$ do not exist.

II. Suppose that S_f is the set of critical points obtained in step I. Then $P = S_f \cap D$ is the set of possible points at which the function f obtains its absolute minimum and maximum on the closed bounded domain D.

III. Note that our function satisfies the assumptions of The Extreme Value Theorem and as a result, using the set P obtained in step II, $\max f(P)$ is the absolute maximum of the function f and $\min f(P)$ is the absolute minimum of the function f.

b. Let $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$ where $f : D = \{(x, y) \mid -2 \leq x \leq 3, 0 \leq y \leq 2\} \to \mathbb{R}^2$. Then

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \langle 2x(3x + 5) + y^2, 2y(x + 1) \rangle.$$

Note that $f_y(x, y) = 0$ if $x = -1$ or $y = 0$ as the real numbers form a field and thus form an integral domain. Also note that $f_x(x, y) = 0$ if $x = -1$ and $y = \pm 2$ or $x = -5/3$ and $y = 0$ or $x = 0$ and $y = 0$. Thus, $\nabla f(x, y) = \langle 0, 0 \rangle$ if $(x, y) \in \{(-5/3, 0), (-1, -2), (-1, 2), (0, 0)\} = S_f$. Since the partial derivatives of f exist everywhere, the set S_f contains every critical point of the function f.

Now, $P = S_f \cap D = \{(-5/3, 0), (-1, 2), (0, 0)\}$ and $f(P) = \{125/27, 3, 0\}$. Therefore, the absolute maximum of f is $\max f(P) = 125/27$ which occurs at the point $(-5/3, 0)$ and the absolute minimum of f is $\min f(P) = 0$ which occurs at the point $(0, 0)$.

□

Problem 3. Consider the optimization problem:

$$\begin{array}{lll} \text{Min (Max)} & f(x_1, x_2, \ldots, x_n) & \\ \text{subject to} & g_1(x_1, x_2, \ldots, x_n) & = k_1 \\ & g_2(x_1, x_2, \ldots, x_n) & = k_2 \\ & \vdots & \\ & g_m(x_1, x_2, \ldots, x_n) & = k_m \end{array}$$

a. Formulate the Lagrangean and describe how we should proceed in order to solve such a problem.

b. Find the relative extrema of $f(x, y, z) = x + 2y + 3z$ subject to $x - y + z = 1, x^2 + y^2 = 1$.

Solution. a. The Lagrangean associated to the optimization problem is the equation

$$L(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) = f(x_1, \ldots, x_n) + \sum_{i=1}^{m} \lambda_i(k_i - g_i(x_1, \ldots, x_n)).$$

Note that if the vector $(x_1, \ldots, x_n)$ minimizes (maximizes) the objective function, then there exists a vector $(\lambda_1, \ldots, \lambda_m)$ such that

$$\bigtriangledown L(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) = (0, \ldots, 0, 0, \ldots, 0). \tag{1}$$

Thus, in order to find the optimal value of the objective function we must find all vectors $(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m)$ that satisfy (1). Note that the partials of L with respect to λ_i return the original constraints. Then from that collection of vectors, test the values $(x_1, \ldots, x_n)$ in the objective function and the vector that minimizes (maximizes) the function is the optimal solution.

b. The Lagrangean associated to this problem is given by

$$L(x, y, z, \lambda_1, \lambda_2) = x + 2y + 3z + \lambda_1(1 - (x - y + z)) + \lambda_2(1 - (x^2 + y^2)) \tag{2}$$

It is straightforward to compute the partials of L with respect to the variables x, y, z and these computations are presented below:

$$\begin{array}{lll} L_x & = 1 - \lambda_1 - 2\lambda_2 x & = 0 \\ L_y & = 2 + \lambda_1 - 2\lambda_2 y & = 0 \\ L_z & = 3 - \lambda_1 & = 0 \end{array}$$

From these equations we can see that the solution vector in terms of λ_1 and λ_2 is given by

$$(x, y, z, \lambda_1, \lambda_2) = \left(-\frac{1}{\lambda_2}, \frac{5}{2\lambda_2}, z, 3, \lambda_2\right).$$

Using this vector, we solve the original constraints for the possible values of z and λ_2 and see that

$$\begin{array}{lll} \boldsymbol{v_1} & = (x_1, y_1, z_1, \lambda_{11}, \lambda_{12}) & = \left(-\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}, 1 + \frac{7}{\sqrt{29}}, 3, \frac{\sqrt{29}}{2}\right) \\ \boldsymbol{v_2} & = (x_2, y_2, z_2, \lambda_{21}, \lambda_{22}) & = \left(\frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}, 1 - \frac{7}{\sqrt{29}}, 3, -\frac{\sqrt{29}}{2}\right) \end{array} \tag{3}$$

are the possible values of $x, y, z, \lambda_1, \lambda_2$ that satisfy (1). Using $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ we see that

$$\begin{aligned} f(x_1, y_1, z_1) &= 3 + \sqrt{29} &\approx 8.38516 \\ f(x_2, y_2, z_2) &= 3 - \sqrt{29} &\approx -2.38516 \end{aligned}$$

so that $3+\sqrt{29}$ is a relative maximum at $\left(-\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}, 1+\frac{7}{\sqrt{29}}\right)$ and $3-\sqrt{29}$ is a relative minimum at $\left(\frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}, 1-\frac{7}{\sqrt{29}}\right)$.

$\square$

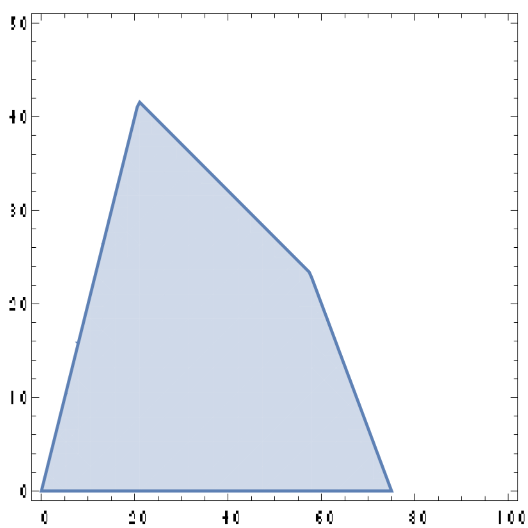
Problem 4. Solve the shipping problem studied in MATH 111 if we replace the constraint $x + 2y \leq 100$ by the constraint $x + 2y \leq 625/6$. Use Mathematica to (at least) graph the feasible set.

Solution. The linear program associated to the shipping problem with the replaced constraint is presented below:

$$\begin{array}{lll} \text{Maximize} & 13x + 9y & \\ \text{subject to} & 4x + 3y & \leq 300 \\ & x + 2y & \leq 625/6 \\ & -2x + y & \leq 0 \\ & & \\ & x \geq 0, y \geq 0 & \end{array}$$

The following Mathematica commands plot the feasible region of the linear program and find the solution to the linear program.

```
RegionPlot[4 x + 3 y ≤ 300 && x + 2 y ≤ 625 / 6 && -2 x + y ≤ 0 && x ≥ 0 && y ≥ 0,
 {x, 0, 100}, {y, 0, 50}]
```




```
Maximize[{13 x + 9 y,
  4 x + 3 y ≤ 300 && x + 2 y ≤ 625 / 6 && -2 x + y ≤ 0 && x ≥ 0 && y ≥ 0}, {x, y}]
{975, {x → 75, y → 0}}
```

As we can see, the objective function is maximized under the given constraints when $x = 75$ and $y = 0$ leading to an objective function value of 975. □

Problem 5. Suppose that f, f_1, f_2 are convex functions and $a \geq 0$. Prove that af and $f_1 + f_2$ are convex functions.

Solution. Recall that a function $f : S \to \mathbb{R}$ is convex if for all $\lambda \in [0,1]$ and $x_1, x_2 \in S$, it is true that $f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$.

Suppose that $a \geq 0$ and $f : S \to \mathbb{R}$ is a convex function. From the above definition, the function af is convex if for all $\lambda \in [0,1]$ and $x_1, x_2 \in S$ we have that

$$\begin{aligned} af(\lambda x_1 + (1-\lambda)x_2) &\leq \lambda af(x_1) + (1-\lambda)af(x_2) \\ &= a(\lambda f(x_1) + (1-\lambda) f(x_2)). \end{aligned}$$

Since $a \geq 0$, this condition is satisfied as an immediate consequence following the definition of the convexity of the function f. Therefore, for $a \geq 0$ and a convex function f, the function af is convex as well.

Now suppose that $f_1 : S_1 \to \mathbb{R}$ and $f_2 : S_2 \to \mathbb{R}$ are convex functions. The function $f_1 + f_2 : S_1 \cap S_2 \to \mathbb{R}$ where $f_1 + f_2(x) := f_1(x) + f_2(x)$ is convex if for all $\lambda \in [0,1]$ and $x_1, x_2 \in S_1 \cap S_2$ it is true that

$$(f_1 + f_2)(\lambda x_1 + (1-\lambda)x_2) \leq \lambda (f_1 + f_2)(x_1) + (1-\lambda)(f_1 + f_2)(x_2).$$

Using the convexity of the functions f_1 and f_2 and the definition of the function $f_1 + f_2$, we see that for all $\lambda \in [0,1]$ and $x_1, x_2 \in S_1 \cap S_2$ we have that

$$\begin{aligned} f_1 + f_2(\lambda x_1 + (1-\lambda)x_2) &= f_1(\lambda x_1 + (1-\lambda)x_2) + f_2(\lambda x_1 + (1-\lambda)x_2) \\ &\leq \lambda f_1(x_1) + (1-\lambda) f_1(x_2) \\ &\quad + \lambda f_2(x_1) + (1-\lambda) f_2(x_2) \\ &= \lambda (f_1(x_1) + f_2(x_1)) \\ &\quad + (1-\lambda)(f_1(x_2) + f_2(x_2)) \\ &= \lambda (f_1 + f_2)(x_1) + (1-\lambda)(f_1 + f_2)(x_2). \end{aligned}$$

Therefore, if f_1 and f_2 are convex functions, the function $f_1 + f_2$ is convex as well. $\square$

Problem 6. For $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ we define its *epigraph* as the set

$$\text{epi } f = \{(x, \beta) \in \mathbb{R}^n \times \mathbb{R} | f(x) \leq \beta\} \subset \mathbb{R}^{n+1}.$$

Prove that f is convex if and only if epi f is convex.

Solution. Recall that a function $f : D \to \mathbb{R}$ is convex if for all $\lambda \in [0, 1]$ and $x_1, x_2 \in D$ we have that $f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$ and similarly that a set S is convex if for all $\lambda \in [0, 1]$ and $x_1, x_2 \in S$ we have that $\lambda x_1 + (1 - \lambda)x_2 \in S$.

Suppose first that the function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex. Then for all $\lambda \in [0, 1]$ and $x_1, x_2 \in \mathbb{R}^n$ it is true that

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2).$$

Let $y_1 = (x_1, \beta_1), y_2 = (x_2, \beta_2) \in \text{epi } f = \{(x, \beta) \in \mathbb{R}^n \times \mathbb{R} | f(x) \leq \beta\}$. Then for $x_1, x_2 \in \mathbb{R}^n$, we have that $f(x_1) \leq \beta_1$ and $f(x_2) \leq \beta_2$. Thus, using the convexity of the function f, we see that for all $\lambda \in [0, 1]$ and $y_1 = (x_1, \beta_1), y_2 = (x_2, \beta_2) \in \text{epi } f$,

$$\begin{aligned} f(\lambda x_1 + (1 - \lambda)x_2) &\leq \lambda f(x_1) + (1 - \lambda)f(x_2) \\ &\leq \lambda\beta_1 + (1 - \lambda)\beta_2 \end{aligned}$$

showing that $\lambda y_1 + (1 - \lambda)y_2 \in \text{epi } f$. Therefore, if f is convex, the set epi f is convex as well.

Now suppose that the set $\text{epi } f = \{(x, \beta) \in \mathbb{R}^n \times \mathbb{R} | f(x) \leq \beta\}$ is convex. Then for all $\lambda \in [0, 1]$ and $y_1 = (x_1, \beta_1), y_2 = (x_2, \beta_2) \in \text{epi } f$, we have that $\lambda y_1 + (1 - \lambda)y_2 = (\lambda x_1 + (1 - \lambda)x_2, \lambda\beta_1 + (1 - \lambda)\beta_2) \in \text{epi } f$, i.e.

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda\beta_1 + (1 - \lambda)\beta_2. \tag{4}$$

Note that in particular for any $x_1, x_2 \in \mathbb{R}^n$, we have that $(x_1, f(x_1)), (x_2, f(x_2)) \in \text{epi } f$. Thus, using (4) with $\beta_1 = f(x_1)$ and $\beta_2 = f(x_2)$, we have that for all $\lambda \in [0, 1]$ and $x_1, x_2 \in \mathbb{R}^n$

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2),$$

showing that f is convex. Therefore, if epi f is convex, the function f is convex as well.

□