Homework Assignment 7

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November 6, 2016

Problem 6.5.2. Let $\Sigma = \{(a_1, a_2, a_3, \dots) \mid a_i \in \{0, 1\}\}$, the sequence space of zeroes and ones with the metric defined previously. Let C be the Cantor set and define $f : \Sigma \to C$ by

$$f((a_1, a_2, a_3, \dots)) = .b_1 b_2 b_3 \dots \quad \text{where } b_i = 0 \text{ if } a_i = 0 \text{ and } b_i = 2 \text{ if } a_i = 1$$

giving the ternary expansion of a real number in $[0, 1]$. Show that f defines a homeomorphism between Σ and C, the Cantor set.

Solution. Note that $f : \Sigma \to C$ is a homemorphism if f is a continuous bijection with continuous inverse.

We begin by showing that f is a bijection. Suppose that $x_1 = (a_{11}, a_{12}, a_{13}, \dots) \in \Sigma$ and $x_2 = (a_{21}, a_{22}, a_{23}, \dots) \in \Sigma$ with $x_1 \neq x_2$. Then $a_{1k} \neq a_{2k}$ for some $k \in \mathbb{Z}^+$. Since $x_1, x_2 \in \Sigma$, this implies that if $a_{1k} = 1$ then $a_{2k} = 0$ and if $a_{1k} = 0$ then $a_{2k} = 1$. Now, we see from the definition of f that

$$f(x_1) = .b_{11} b_{12} b_{13} \dots b_{1k} \dots \neq .b_{21} b_{22} b_{23} \dots b_{2k} \dots = f(x_2)$$

since if $a_{1k} = 0$ then $b_{1k} = 0 \neq 2 = b_{2k}$ and if $a_{1k} = 1$ then $b_{1k} = 2 \neq 0 = b_{2k}$. Thus, $f(x_1) \neq f(x_2)$ and f is injective.

Now let $y = .b_1 b_2 b_3 \dots \in C$ be the ternary expansion of a real number in $[0, 1]$. Then $b_i \in \{0, 2\}$ for all $i \in \mathbb{Z}^+$. Take $x = (a_1, a_2, a_3, \dots)$ where $a_i = 0$ if $b_i = 0$ and $a_i = 1$ if $b_i = 2$. Then $x \in \Sigma$ and we see from the definition of f that

$$f(x) = f((a_1, a_2, a_3, \dots)) = .b_1 b_2 b_3 \dots = y$$

so that f is surjective, making f a bijection.

To show that f is continuous, we must show that if the distance between two points is small in the metric space Σ, then the distance between their mapped points in C is small, i.e. if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $d(x_1, x_2) < \delta$, then $d(f(x_1), f(x_2)) < \varepsilon$. So, suppose that $x_1 = (a_{11}, a_{12}, a_{13}, \dots) \in \Sigma$ and $x_2 = (a_{21}, a_{22}, a_{23}, \dots) \in \Sigma$. Then,

$$f(x_k) = .b_{k1} b_{k2} b_{k3} \dots = \sum_{n=1}^{\infty} \frac{b_{kn}}{3^n} \in C$$

for $k = 1, 2$. Let $S = \{k \in \mathbb{Z}^+ \mid a_{1k} \neq a_{2k}\}$. Then

$$d(x_1, x_2) = \sum_{n=1}^{\infty} \frac{|a_{1n} - a_{2n}|}{2^n} = \sum_{k \in S} \frac{1}{2^k}.$$

Similarly, if $a_{1k} \neq a_{2k}$, then $b_{1k} \neq b_{2k}$ so that

$$d(f(x_1), f(x_2)) = |f(x_1) - f(x_2)| = \sum_{n=1}^{\infty} \frac{|b_{1n} - b_{2n}|}{2^n} = \sum_{k \in S} \frac{2}{3^k}.$$

Choose $\delta = \varepsilon/2 > 0$. Then we have that

$$d(x_1, x_2) = \sum_{k \in S} \frac{1}{2^k} < \delta = \frac{\varepsilon}{2}$$

which implies that

$$d(f(x_1), f(x_2)) = \sum_{k \in S} \frac{2}{3^k} < \sum_{k \in S} \frac{2}{2^k} < \varepsilon.$$

Therefore, f is continuous.

The above argument extends to show that for every point $x \in \Sigma$ and every neighborhood U of x, there exists a neighborhood V of $f(x)$ such that $V \subseteq f(U)$. Explicitly, let $x = (a_1, a_2, a_3, \ldots) \in \Sigma$ and let $\varepsilon > 0$ be given. Then $B_{\varepsilon/2}(x) = \{a \in \Sigma \mid d(a, x) < \varepsilon/2\}$ is a neighborhood of x and we see that $f(B_{\varepsilon/2}(x)) = \{y \in C \mid d(y, f(x)) < \varepsilon\}$. Thus, the open ball of radius ε is a neighborhood of $f(x)$ contained in $f(B_{\varepsilon/2}(x))$.

Since f maps open sets to open sets, we have that f is an open map. Therefore, since f is a continuous bijection, we must have that its inverse is continuous or that f is a homeomorphism. $\square$

Problem 6.5.3. Let $f : I \to I$ be a transitive map with I an interval. Show that if U and V are non-empty open sets in I, then there exists $m \in \mathbb{Z}^+$ with $U \cap f^m(V) \neq \emptyset$

Solution. $\square$

Problem 6.5.4. Let $F : [0,1) \to [0,1)$ be the tripling map. Show that F is transitive and that its periodic points are dense in $[0,1)$.

Solution. Note that

$$F(x) := \begin{cases} 3x & \text{if } x \in [0, 1/3) \\ 3x - 1 & \text{if } x \in [1/3, 2/3) \\ 3x - 2 & \text{if } x \in [2/3, 1) \end{cases}$$

$\square$

Problem 7.1.2. i. Define $f_a : \mathbb{R} \to \mathbb{R}$ by $f_a(x) = ax$ for $a \in \mathbb{R}$. Show that $f_{1/2}$ and $f_{1/4}$ are conjugate via the map

$$h(x) = \begin{cases} \sqrt{x} & x \geq 0 \\ -\sqrt{-x} & x < 0 \end{cases}. \tag{1}$$

ii. More generally, show that $f_a, f_b : [0, \infty) \to [0, \infty)$ for $0 < a, b < 1$, the f_a and f_b are conjugate via the map $h(x) = x^p$ for $p > 0$ and similarly if $a, b > 1$.

iii. Discuss the cases where $a > 1$ and $0 < b < 1$. What happens when $a = 1/2$ and $b = 2$?

Solution. i. We begin by showing that $h : \mathbb{R} \to \mathbb{R}$ where h is defined as in (1) is a homeomorphism, i.e. it is a continuous bijection with continuous inverse.

It is clear from the definition of h that if $x_1 \neq x_2$ then $h(x_1) \neq h(x_2)$ due to the uniqueness of the square root operator. Thus, h is injective.

To show that h is surjective, suppose that $y \in \mathbb{R}$ and that $y_1 = |y|$. If $y \geq 0$, then $y = y_1$, otherwise $y = -y_1$. Now, if $y \geq 0$, then set $x = y_1^2 \geq 0$, otherwise set $x = -y_1^2 < 0$. Then we have from the definition of h that if $y \geq 0$, then

$$h(x) = \sqrt{y_1^2} = y_1 = y.$$

Similarly, we have that if $y < 0$, then

$$h(x) = -\sqrt{-\left(-y_1^2\right)} = -y_1 = y.$$

Therefore, h is surjective.

It is clear that h and its inverse are continuous so that h is a homemorphism.

Now, we see that

$$h \circ f_{1/4}(x) = h\left(\frac{x}{4}\right) = \begin{cases} \frac{\sqrt{x}}{2} & x \geq 0 \\ -\frac{\sqrt{-x}}{2} & x < 0 \end{cases}$$

and that

$$\begin{aligned} f_{1/2} \circ h(x) &= \begin{cases} f_{1/2}\left(\sqrt{x}\right) & x \geq 0 \\ f_{1/2}\left(-\sqrt{-x}\right) & x < 0 \end{cases} \\ &= \begin{cases} \frac{\sqrt{x}}{2} & x \geq 0 \\ -\frac{\sqrt{-x}}{2} & x < 0 \end{cases} \end{aligned}$$

so that h is a conjugate map of $f_{1/4}$ and $f_{1/2}$.

ii. From the previous remarks, we see that if $p > 0$, then $h : [0, \infty) \to [0, \infty)$ with $h(x) = x^p$ is a homeomorphism. Let $f_c : [0, \infty) \to [0, \infty)$ be a function defined by $f_c(x) = cx$. Consider the maps f_a and f_b. Then we see that

$$h \circ f_a(x) = h(ax) = (ax)^p = a^p x^p$$

and that

$$f_b \circ h(x) = f_b(x^p) = bx^p.$$

Thus, if $a^p = b$, then $h \circ f_a = f_b \circ h$ so that f_a and f_b are conjugate via h. Note that for $a, b > 0$ we have that $a^p = b$ if and only if $0 < a, b < 1$ or $a, b > 1$.

iii. Suppose that $a > 1$ and $0 < b < 1$. Then for any $p > 0$, $a^p > 1$, so that $a^p > b$. Thus, f_a and f_b will not be conjugate via h.

Suppose that $a = 1/2$ and $b = 2$. Then $a^p = 1/2^p < 2 = b$ for any positive p and $f_{1/2}$ and f_2 are not conjugate via h.

$\square$

Problem 7.1.3. Prove that if $f : X \to X$ and $g : Y \to Y$ are conjugate maps of metric spaces, then f is one-to-one if and only if g is one-to-one and f is onto if and only if g is onto.

Solution. If f and g are conjugate maps of metric spaces, then there exists a map $h : X \to Y$, with h a bijection, such that $g \circ h = h \circ f$.

Suppose that f is one-to-one and that $g(y_1) = g(y_2)$. Since h is onto, there exist $x_1 \in X$ and $x_2 \in X$ such that $h(x_1) = y_1$ and $h(x_2) = y_2$. Thus, if $g(y_1) = g(y_2)$, then $g \circ h(x_1) = g \circ h(x_2)$. By the conjugacy of h, we then have that $h \circ f(x_1) = h \circ f(x_2)$ and since h and f are one-to-one, we have that $x_1 = x_2$. Due to the fact that h is a well-defined function, if $x_1 = x_2$, then $y_1 = h(x_1) = h(x_2) = y_2$ and we therefore have that g is one-to-one.

Now suppose that g is one-to-one and that $f(x_1) = f(x_2)$. Since h is well-defined, we have that $h \circ f(x_1) = h \circ f(x_2)$. By the conjugacy of h, we then have that $g \circ h(x_1) = g \circ h(x_2)$. Since g and h are one-to-one, it follows that $x_1 = x_2$ and f is therefore one-to-one.

Suppose that f is onto and let $y_2 \in Y$ be given. Since h is onto, there exists $x_2 \in X$ such that $h(x_2) = y_2$. Thus, since f is onto, there exists $x_1 \in X$ such that $f(x_1) = x_2$ which implies that $h \circ f(x_1) = y_2$. By the conjugacy of h we have that

$$g \circ h(x_1) = h \circ f(x_1) = y_2.$$

Hence, there exists $y_1 = h(x_1) \in Y$ such that $g(y_1) = y_2$. Therefore, since $y_2 \in Y$ was arbitary, we have that h is onto.

Now suppose that g is onto. Since g and h are onto, for every $y \in Y$, there exists $x_1 \in X$ such that $g \circ h(x_1) = y$. By the conjugacy of h, we then have that $h \circ f(x_1) = y$. So, for every $y \in Y$, there exists $f(x_1) \in X$ such that $h \circ f(x_1) = y$. However, since h is onto, we also have that for every $y \in Y$, there exists $x_2 \in X$ such that $h(x_2) = y$. Thus, for every $x_2 \in X$ we have that $h(x_2) = y = h(f(x_1))$ for some $x_1 \in X$. The fact that h is one-to-one then shows that $f(x_1) = x_2$ or that f is onto. □