Homework Assignment 4

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Problem 4.56. Suppose that on each play of the game a gambler either wins 1 with probability p or loses 1 with probability $1 - p$. The gambler continues betting until she or he is either up n or down m. What is the probability that the gambler quits a winner?

Solution. $\square$

Problem 4.59. For the gambler's ruin problem of Section 4.5.1, let M_i denote the mean number of games that must be played until the gambler either goes broke or reaches a fortune of N, given that he starts with i for $i = 0, 1, \ldots, N$. Show that M_i satisfies

$$M_0 = M_N = 0; \quad M_i = 1 + pM_{i+1} + qM_{i-1}, \quad i = 1, \ldots, N-1.$$

Solve these equations to obtain

$$M_i = \begin{cases} i(N-i) & \text{if } p = 1/2 \\ \dfrac{i}{q-p} - \dfrac{N}{q-p}\dfrac{1-(q/p)^i}{1-(q/p)^N} & \text{if } p \neq 1/2 \end{cases}.$$

Solution. It is clear that if M_i is the mean number of games that must be played until the gambler either goes broke or reaches a fortune of N given that he starts with i for $i = 0, 1, \ldots, N$, then $M_0 = M_N = 0$ since if the gambler starts with either 0 or N the process ends, i.e. no games will be played.

So suppose that $i = 1, \ldots, N-1$ and let X_n denote the number of games that will be played and let $Y = \{0, 1\}$ indicate whether the initial game is won or lost. Assuming that the initial start for the gambler is i, we have that $M_i = E[X_n \mid X_0 = i]$. Conditioning on the initial outcome of the game, i.e. that the gambler either wins or loses, we have that

$$E[X_n \mid X_0 = i] = pE[X_n \mid X_0 = i, Y = 1] + qE[X_n \mid X_0 = i, Y = 0].$$

If the gambler starts with fortune i and the outcome of the game is a win, then the number of games that will be played is 1 plus the expected number of games to be played given that the gambler starts with fortune $i+1$. Similarly, if the outcome of the game is a loss, the number of games to be played is 1 plus the expected number of games to be played given that the gambler starts with fortune $i-1$. Thus,

$$\begin{aligned} M_i = E[X_n \mid X_0 = i] &= p(1 + E[X_n \mid X_0 = i+1]) + q(1 + E[X_n \mid X_0 = i-1]) \\ &= p + q + pE[X_n \mid X_0 = i+1] + qE[X_n \mid X_0 = i-1] \\ &= 1 + pM_{i+1} + qM_{i-1}. \end{aligned}$$

Note that $p + q = 1$ so that $M_i = 1 + pM_{i+1} + qM_{i-1}$ is equivalent to

$$pM_i + qM_i = pM_{i+1} + qM_{i-1} + 1$$

for $i = 1, \ldots N-1$. Hence, we have that

$$M_{i+1} - M_i = \frac{q}{p}(M_i - M_{i-1}) - \frac{1}{p}.$$

Since $M_0 = 0$, we easily see that

$$\begin{aligned}
M_2 - M_1 &= \frac{q}{p}(M_1 - M_0) - \frac{1}{p} = \frac{q}{p}M_1 - \frac{1}{p} \\
M_3 - M_2 &= \frac{q}{p}(M_2 - M_1) - \frac{1}{p} = \left(\frac{q}{p}\right)^2 M_1 - \frac{q}{p^2} - \frac{1}{p} \\
&\vdots \\
M_{i+1} - M_i &= \frac{q}{p}(M_i - M_{i-1}) - \frac{1}{p} = \left(\frac{q}{p}\right)^i M_1 - \frac{1}{q}\sum_{k=1}^{i}\left(\frac{q}{p}\right)^k \\
&\vdots \\
M_N - M_{N-1} &= \frac{q}{p}(M_{N-1} - M_{N-2}) - \frac{1}{p} = \left(\frac{q}{p}\right)^{N-1} M_1 - \frac{1}{q}\sum_{k=1}^{N-1}\left(\frac{q}{p}\right)^k
\end{aligned}$$

Adding the first i equations shows that

$$\begin{aligned}
M_i - M_1 &= \sum_{j=1}^{i-1}\left[\left(\frac{q}{p}\right)^j M_1 - \frac{1}{q}\sum_{k=1}^{j}\left(\frac{q}{p}\right)^k\right] \\
&= M_1\sum_{j=1}^{i-1}\left(\frac{q}{p}\right)^j - \frac{1}{q}\sum_{j=1}^{i-1}\sum_{k=1}^{j}\left(\frac{q}{p}\right)^k. \qquad (1)
\end{aligned}$$

These sums are finite geometric progressions, so it is easy to find their closed forms. Thus, if $p \neq 1/2$, then

$$\begin{aligned}
M_i &= M_1 - M_1\left[\frac{-q + p(q/p)^i}{p-q}\right] + \frac{1}{q}\sum_{j=1}^{i}\frac{q(-1+(q/p)^j)}{p-q} \\
&= M_1 - M_1\left[\frac{-q + p(q/p)^i}{p-q}\right] - \frac{-iq + p(-1+i+(q/p)^i)}{(p-q)^2}
\end{aligned}$$

Since $M_N = 0$, we know that

$$0 = M_1 - M_1\left[\frac{-q + p(q/p)^N}{p-q}\right] - \frac{-Nq + p(-1+N+(q/p)^N)}{(p-q)^2}.$$

Solving this equation shows that

$$M_1 = \frac{p - Np + Nq - p(q/p)^N}{p(p-q)(-1+(q/p)^N)}.$$

Therefore, if $p \neq 1/2$, we have that

$$\begin{aligned}
M_i &= \frac{i - i(q/p)^N + N(-1+(q/p)^N)}{(p-q)(-1+(q/p)^N)} \\
&= \frac{i(1-(q/p)^N)}{(q-p)(1-(q/p)^N)} - \frac{N(1-(q/p)^N)}{(q-p)(1-(q/p)^N)} \\
&= \frac{i}{q-p} - \frac{N}{q-p}\frac{1-(q/p)^i}{1-(q/p)^N}.
\end{aligned}$$

If on the other hand we have that $p = q = 1/2$, then from (1), we have that

$$\begin{aligned} M_i - M_1 &= \sum_{j=1}^{i-1} \left[\left(\frac{q}{p}\right)^j M_1 - \frac{1}{q} \sum_{k=1}^{j} \left(\frac{q}{p}\right)^k \right] \\ &= M_1(i-1) - i(i-1). \end{aligned}$$

Using the fact that $M_N = 0$, we see that $M_1 = N - 1$. Therefore, if $p = q$, then $M_i = i(N-1) - i(i-1) = i(N-i)$ and we have that

$$M_i = \begin{cases} i(N-i) & \text{if } p = 1/2 \\ \dfrac{i}{q-p} - \dfrac{N}{q-p}\dfrac{1-(q/p)^i}{1-(q/p)^N} & \text{if } p \neq 1/2 \end{cases}.$$

$\square$

Problem 4.63. For the Markov chain with states 1, 2, 3, 4 whose transition probability matrix $\mathbf{P}$ is as listed below find f_{i3} and s_{i3} for $i = 1, 2, 3$.

$$\mathbf{P} = \begin{bmatrix} 0.4 & 0.2 & 0.1 & 0.3 \\ 0.1 & 0.5 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.2 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution. From this matrix it is clear that states 1, 2, and 3 all communicate. However, state 4 communicates with neither states 1,2, nor 3. Since $P_{44}^n = 1$ for all $n > 0$, we easily see that $\sum_{n=1}^{\infty} P_{44}^n = \infty$, i.e. state 4 is a recurrent state. Note that recurrence is a property shared by equivalence classes under the relation communicates. We see that since states 1, 2, and 3 do not communicate with state 4, but all communicate with each other, these states must be transient.

Let s_{ij} denote the expected number of time periods that the Markov chain is in state j given that it started in state i and let $\mathbf{P}_T$ be the transition matrix from transient states into transient states. Since states 1, 2, and 3 are the transient states of the Markov chain, we have that

$$\mathbf{P}_T = \begin{bmatrix} 0.4 & 0.2 & 0.1 \\ 0.1 & 0.5 & 0.2 \\ 0.3 & 0.4 & 0.2 \end{bmatrix}.$$

If $\mathbf{S}$ is the matrix of values s_{ij} for $i, j = 1, 2, 3$, then $\mathbf{S} = (\mathbf{I} - \mathbf{P}_T)^{-1}$. Thus, we see that

$$\mathbf{S} = \begin{bmatrix} 0.6 & -0.2 & -0.1 \\ -0.1 & 0.5 & -0.2 \\ -0.3 & -0.4 & 0.8 \end{bmatrix}^{-1} = \begin{bmatrix} 2.20690 & 1.37931 & 0.62069 \\ 0.96552 & 3.10345 & 0.89655 \\ 1.31034 & 2.06897 & 1.93103 \end{bmatrix}.$$

Therefore, we have that

$$s_{13} = 0.62069, \quad s_{23} = 0.89655, \quad s_{33} = 1.93103.$$

If f_{ij} is the probability that the Markov chain ever transitions to state j given that it starts in state i, then

$$f_{ij} = \frac{s_{ij} - \delta_{i,j}}{s_{jj}},$$

where $\delta_{i,j}$ is the Kronecker delta such that $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise. Thus,

$$f_{13} = \frac{s_{13} - \delta_{1,3}}{s_{33}} = \frac{0.62069}{1.93103} = 0.321429$$
$$f_{23} = \frac{s_{23} - \delta_{2,3}}{s_{33}} = \frac{0.89655}{1.93103} = 0.464286$$
$$f_{33} = \frac{s_{33} - \delta_{3,3}}{s_{33}} = \frac{0.93103}{1.93103} = 0.482142.$$

□

Problem 4.64. Consider a branching process having $\mu < 1$. Show that if $X_0 = 1$, then the expected number of individuals that ever exist in this population is given by $1/(1-\mu)$. What if $X_0 = n$?

Solution. If X_n represents the size of the n-th generation, then the sum of the sizes of all generations represents the total number of individuals that ever exist in the population. Thus, the expected number of individuals is given by $E\left[\sum_{i=0}^{\infty} X_i \mid X_0 = 1\right]$ if the size of the first generation is 1. By definition,

$$\begin{aligned}
E\left[\sum_{i=0}^{\infty} X_i \mid X_0 = 1\right] &= E\left[\lim_{n\to\infty} \sum_{i=0}^{n} X_i \mid X_0 = 1\right] \\
&= \lim_{n\to\infty} E\left[\sum_{i=0}^{n} X_i \mid X_0 = 1\right] \\
&= \lim_{n\to\infty} \sum_{i=0}^{n} E\left[X_i \mid X_0 = 1\right].
\end{aligned}$$

It was shown previously that $E\left[X_i \mid X_0 = 1\right] = \mu^i$. Therefore, if $0 \le \mu < 1$, then

$$E\left[\sum_{i=0}^{\infty} X_i \mid X_0 = 1\right] = \lim_{n\to\infty} \sum_{i=0}^{n} \mu^i = \frac{1}{1-\mu}.$$

Now suppose that $X_0 = n$. Using the previous result that $E[X_i] = \mu E[X_{i-1}]$, we have

$$E\left[X_i \mid X_0 = n\right] = n\mu^i.$$

Therefore, if $X_0 = n$ and $0 \le \mu < 1$, then

$$\begin{aligned}
E\left[\sum_{i=0}^{\infty} X_i \mid X_0 = 1\right] &= \lim_{k\to\infty} \sum_{i=0}^{k} E\left[X_i \mid X_0 = n\right] \\
&= n\left[\lim_{k\to\infty} \sum_{i=0}^{k} \mu^i\right] = \frac{n}{1-\mu}.
\end{aligned}$$

$\square$

Problem 4.66. For a branching process, calculate π_0 when

i. $P_0 = \frac{1}{4}$, $P_2 = \frac{3}{4}$.

ii. $P_0 = \frac{1}{4}$, $P_1 = \frac{1}{2}$, $P_2 = \frac{1}{4}$.

iii. $P_0 = \frac{1}{6}$, $P_1 = \frac{1}{2}$, $P_3 = \frac{1}{3}$.

Solution. Recall for a branching process that μ is the mean number of offspring of an individual such that $\mu = \sum_{n=0}^{\infty} nP_n$ where P_n is the probability that an individual will produce n offspring.

Note that π_0 is the probability that the population will eventually die out. Also note that if $\mu \leq 1$, then $\pi_0 = 1$. Otherwise, if $\mu > 1$, then π_0 is the smallest positive number satisfying the equation $\pi_0 = \sum_{n=0}^{\infty} \pi_0^n P_n$.

i. If $P_0 = \frac{1}{4}$ and $P_2 = \frac{3}{4}$, then $\mu = \frac{3}{2}$ and π_0 is the smallest positive number satisfying

$$\pi_0 = P_0 + P_2\pi_0^2.$$

Thus, π_0 is the smallest positive root of the equation

$$P_2\pi_0^2 - \pi_0 + P_0 = \frac{3}{4}\pi_0^2 - \pi_0 + \frac{1}{4} = 0.$$

Solving the above equation leads to the roots $\pi_{01} = \frac{1}{3}$ and $\pi_{02} = 1$. Therefore, since π_{01} is the smallest positive root satisfying the above equation, we have that $\pi_0 = \frac{1}{3}$.

ii. If $P_0 = \frac{1}{4}$, $P_1 = \frac{1}{2}$, $P_2 = \frac{1}{4}$, then $\mu = 1$ and therefore we must have that $\pi_0 = 1$.

iii. If $P_0 = \frac{1}{6}$, $P_1 = \frac{1}{2}$, and $P_3 = \frac{1}{3}$, then $\mu = \frac{3}{2}$ and π_0 is the smallest positive number satisfying

$$\pi_0 = P_0 + P_1\pi_0 + P_3\pi_0^3.$$

Thus, π_0 is the smallest positive root of the equation

$$P_3\pi_0^3 + (P_1 - 1)\pi_0 + P_0 = \frac{1}{3}\pi_0^3 - \frac{1}{2}\pi_0 + \frac{1}{6} = 0.$$

Solving the above equation leads to the roots $\pi_{01} = 1$, $\pi_{02} = (-1 - \sqrt{3})/2$, and $\pi_{03} = (-1 + \sqrt{3})/2$. Therefore, since π_{03} is the smallest positive root satisfying the above equation, we have that $\pi_0 = (-1 + \sqrt{3})/2$.

$\square$