

Homework Assignment 5

Matthew Tiger

September 29, 2015

Problem 2.7. Show, using the geometric series $1/(1-x) = \sum_{j=0}^{\infty} x^j$ for $|x| < 1$, that $1/(1-\phi z) = -\sum_{j=1}^{\infty} \phi^{-j} z^{-j}$ for $|\phi| > 1$ and $|z| \geq 1$.

Solution. If $|\phi| > 1$ and $|z| \geq 1$, then $|\phi z| = |\phi||z| > 1$ and $1/|\phi z| = 1/(|\phi||z|) < 1$. Now,

$$\sum_{j=0}^{\infty} \phi^{-j} z^{-j} = \sum_{j=0}^{\infty} (\phi z)^{-j} = \sum_{j=0}^{\infty} \left(\frac{1}{\phi z} \right)^j.$$

Note that this is a geometric series where $|\frac{1}{\phi z}| < 1$. So the series converges and

$$\sum_{j=0}^{\infty} \left(\frac{1}{\phi z} \right)^j = \frac{1}{1 - \frac{1}{\phi z}} = \frac{\phi z}{\phi z - 1}.$$

This implies that

$$\frac{\phi z}{\phi z - 1} = \sum_{j=0}^{\infty} \left(\frac{1}{\phi z} \right)^j = 1 + \sum_{j=1}^{\infty} \left(\frac{1}{\phi z} \right)^j$$

so that

$$\sum_{j=1}^{\infty} \left(\frac{1}{\phi z} \right)^j = \frac{\phi z}{\phi z - 1} - 1 = \frac{\phi z - (\phi z - 1)}{\phi z - 1} = \frac{1}{\phi z - 1}.$$

From this identity it is clear that

$$-\sum_{j=1}^{\infty} \phi^{-j} z^{-j} = -\sum_{j=1}^{\infty} \left(\frac{1}{\phi z} \right)^j = \frac{1}{-(\phi z - 1)} = \frac{1}{1 - \phi z}$$

and we are done. □

Problem 2.

Solution. □

Problem 3. Prove that if $|\theta| < 1$, then the ARMA(1, 1) process is invertible.

Solution. Note that the ARMA(1, 1) process is invertible if Z_t is expressed in terms of X_s for $s \leq t$. Let $\phi(z) = 1 - \phi z$ and $\theta(z) = 1 + \theta z$ and suppose that $|\theta| < 1$. Then the ARMA(1, 1) equation can be written as $\phi(B)X_t = \theta(B)Z_t$ where B is the back-shift operator. Note that the power series expansion of $1/\theta(z)$ is given by

$$\frac{1}{\theta(z)} = \frac{1}{1 + \theta z} = \sum_{j=0}^{\infty} (-\theta)^j z^j$$

which converges since $|\theta| < 1$. From this it is clear that $Z_t = (1/\theta(B))\phi(B)X_t$ and, evaluating $(1/\theta(B))\phi(B)$, we see that

$$\begin{aligned} \frac{1}{\theta(B)}\phi(B) &= (1 - \theta B + (-\theta)^2 B^2 + \dots)(1 - \phi B) \\ &= 1 + (-\theta)B + (-\theta)^2 B^2 + \dots - \phi B + (-1)^2 \phi \theta B^2 + (-1)^3 \phi \theta^2 B^3 + \dots \\ &= 1 + -(\theta + \phi)B + (-1)^2(\theta^2 + \phi\theta)B^2 + (-1)^3(\theta^3 + \phi\theta^2)B^3 + \dots \\ &= 1 - (\theta + \phi) \sum_{j=1}^{\infty} (-\theta)^{j-1} B^j. \end{aligned}$$

Using this derivation we notice that

$$\begin{aligned} Z_t &= \frac{1}{\theta(B)}\phi(B)X_t \\ &= X_t - (\theta + \phi) \sum_{j=1}^{\infty} (-\theta)^{j-1} B^j X_t \\ &= X_t - (\theta + \phi) \sum_{j=1}^{\infty} (-\theta)^{j-1} X_{t-j}. \end{aligned}$$

But this shows that Z_t is expressed in terms of X_s where $s \leq t$, showing that when $|\theta| < 1$, the ARMA(1, 1) process is invertible. \square

Problem 4. Prove that if $|\theta| > 1$, then the ARMA(1, 1) process is non-invertible.

Solution. Note that the ARMA(1, 1) process is non-invertible if Z_t is expressed in terms of X_s for $s \geq t$. Let $\phi(z) = 1 - \phi z$ and $\theta(z) = 1 + \theta z$ and suppose that $|\theta| > 1$. Then the ARMA(1, 1) equation can be written as $\phi(B)X_t = \theta(B)Z_t$ where B is the back-shift operator. Note that the power series expansion of $1/\theta(z)$ is given by

$$\frac{1}{\theta(z)} = \frac{1}{1 + \theta z} = - \sum_{j=1}^{\infty} (-\theta)^{-j} z^{-j}.$$

Since $|\theta| > 1$, we know that $|1/\theta| < 1$ and this series converges. From this it is clear that

$Z_t = (1/\theta(B))\phi(B)X_t$ and, evaluating $(1/\theta(B))\phi(B)$, we see that

$$\begin{aligned}
\frac{1}{\theta(B)}\phi(B) &= -((- \theta)^{-1}B^{-1} + (-\theta)^{-2}B^{-2} + (-\theta)^{-3}B^{-3} + \dots)(1 - \phi B) \\
&= ((-\theta)^{-1}B^{-1} + (-\theta)^{-2}B^{-2} + (-\theta)^{-3}B^{-3} + \dots)(\phi B - 1) \\
&= -\phi\theta^{-1} + (-1)^2\phi\theta^{-2}B^{-1} + (-1)^3\phi\theta^{-3}B^{-2} + \dots \\
&\quad + (-1)^2\theta^{-1}B^{-1} + (-1)^3\theta^{-2}B^{-2} + (-1)^4\theta^{-3}B^{-3} + \dots \\
&= -\phi\theta^{-1} + (-1)^2(\phi\theta^{-2} + \theta^{-1})B^{-1} + (-1)^3(\phi\theta^{-3} + \theta^{-2})B^{-2} + \dots \\
&= -\phi\theta^{-1} + (\phi + \theta) \sum_{j=1}^{\infty} (-\theta)^{-j-1}B^{-j}
\end{aligned}$$

Using this derivation we notice that

$$\begin{aligned}
Z_t &= \frac{1}{\theta(B)}\phi(B)X_t \\
&= -\phi\theta^{-1}X_t + (\phi + \theta) \sum_{j=1}^{\infty} (-\theta)^{-j-1}B^{-j}X_t \\
&= -\phi\theta^{-1}X_t + (\phi + \theta) \sum_{j=1}^{\infty} (-\theta)^{-j-1}X_{t+j}.
\end{aligned}$$

But this shows that Z_t is expressed in terms of X_s where $s \geq t$, showing that when $|\theta| > 1$, the ARMA(1, 1) process is non-invertible. \square