

Homework Assignment 5

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Problem 3.23. Show that:

a. $\mathcal{L}\{t \cos(at)e^{-bt}\} = \frac{(s+b)^2 - a^2}{[(s+b)^2 + a^2]^2}.$

Solution. a. Let $f(t) = t \cos(at)$ and suppose that $\bar{f}(s) = \mathcal{L}\{f(t)\}.$

As shown previously, we know that

$$\bar{f}(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{t \cos(at)\} = \frac{s^2 - a^2}{(s^2 + a^2)^2}.$$

Therefore, by Heaviside's First Shifting Theorem,

$$\mathcal{L}\{t \cos(at)e^{-bt}\} = \mathcal{L}\{f(t)e^{-bt}\} = \bar{f}(s+b) = \frac{(s+b)^2 - a^2}{[(s+b)^2 + a^2]^2},$$

and we are done.

□

Problem 3.24. Suppose that $\mathcal{L}\{f(t)\} = \bar{f}(s)$ and $\mathcal{L}\{g(x, t)\} = \bar{h}(s) \exp(-x\bar{h}(s))$. Prove that:

a. $\mathcal{L}\left\{\int_0^\infty g(x, t)f(x)dx\right\} = \bar{h}(s)\bar{f}(\bar{h}(s)).$

Solution. a. From the definition of the Laplace transform, we have that

$$\mathcal{L}\left\{\int_0^\infty g(x, t)f(x)dx\right\} = \int_0^\infty \left[\int_0^\infty g(x, t)f(x)dx\right] e^{-st}dt.$$

Interchanging the order of integration yields that

$$\begin{aligned}\mathcal{L}\left\{\int_0^\infty g(x, t)f(x)dx\right\} &= \int_0^\infty \left[\int_0^\infty g(x, t)f(x)dx\right] e^{-st}dt \\ &= \int_0^\infty f(x) \left[\int_0^\infty g(x, t)e^{-st}dt\right] dx \\ &= \int_0^\infty f(x)\mathcal{L}\{g(x, t)\} dx.\end{aligned}$$

From the relation $\mathcal{L}\{g(x, t)\} = \bar{h}(s) \exp(-x\bar{h}(s))$, we thus see that

$$\begin{aligned}\mathcal{L}\left\{\int_0^\infty g(x, t)f(x)dx\right\} &= \int_0^\infty f(x)\mathcal{L}\{g(x, t)\} dx \\ &= \int_0^\infty f(x)\bar{h}(s) \exp(-x\bar{h}(s))dx.\end{aligned}$$

Using the definition of the Laplace transform, we see that

$$\bar{f}(\bar{h}(s)) = \int_0^\infty f(t) \exp(-\bar{h}(s)t)dt.$$

Therefore,

$$\begin{aligned}\mathcal{L}\left\{\int_0^\infty g(x, t)f(x)dx\right\} &= \int_0^\infty f(x)\bar{h}(s) \exp(-x\bar{h}(s))dx \\ &= \bar{h}(s) \int_0^\infty f(x) \exp(-x\bar{h}(s))dx \\ &= \bar{h}(s)\bar{f}(\bar{h}(s)).\end{aligned}$$

and we are done. □

Problem 3.27.*Solution.*

Problem 3.28.*Solution.*

Problem 3.29.*Solution.*

Problem 3.32.*Solution.*

Problem 3.34.*Solution.*

Problem 4.1.*Solution.*