Test 2

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Problem 1. Approximate the roots of $\varepsilon x^3 + x^2 + x - 2 = 0$, $\varepsilon \to 0^+$, with precision $O(\varepsilon^2)$.

Solution. When $\varepsilon = 0$, the unperturbed equation only has two roots while the original equation has three roots implying that this is a singular perturbation problem.

To find the other root of this equation we must employ the method of dominant balance on the original equation. There are six possible two-term balances to consider as $\varepsilon \to 0^+$:

a. Suppose that $\varepsilon x^3 \sim x^2$ is the dominant balance and that $x \ll x^2$, $1 \ll x^2$. If $\varepsilon x^3 \sim x^2$ as $\varepsilon \to 0^+$, then $x = O(\varepsilon^{-1})$, which is consistent with the assumptions that $x \ll x^2$ and $1 \ll x^2$ and the balance itself is consistent.

b. Suppose that $\varepsilon x^3 \sim x$ is the dominant balance and that $x^2 \ll x$, $1 \ll x$. If $\varepsilon x^3 \sim x$ as $\varepsilon \to 0^+$, then $x = O(\varepsilon^{-1/2})$. However, if $x = O(\varepsilon^{-1/2})$, then

$$\frac{x^2}{x} = \frac{\left(\varepsilon^{-1/2}\right)^2}{\varepsilon^{-1/2}} = \varepsilon^{-1/2}$$

which implies that as $\varepsilon \to 0^+$ the assumption that $x^2 \ll x$ is false and that this balance is inconsistent.

c. Suppose that $\varepsilon x^3 \sim 2$ is the dominant balance and that $x^2 \ll 1$, $x \ll 1$. If $\varepsilon x^3 \sim 2$ as $\varepsilon \to 0^+$, then $x = O(\varepsilon^{-1/3})$, which implies that as $\varepsilon \to 0^+$ the assumption that $x \ll 1$ is false and that this balance is inconsistent.

d. Suppose that $x^2 \sim x$ is the dominant balance and that $\varepsilon x^3 \ll x$, $1 \ll x$. If $x^2 \sim x$ as $\varepsilon \to 0^+$, then $x \sim 1$ and $x = O(1)$, which is consistent with the assumptions that $\varepsilon x^3 \ll 1$ and $x^2 \ll 1$ and the balance is consistent. Using this balance will recover the root $x = 1$ from the unperturbed equation and the root can be expanded with a perturbation series in ε in the normal way.

e. Suppose that $x^2 \sim 2$ is the dominant balance and that $\varepsilon x^3 \ll 1$, $x \ll 1$. If $x^2 \sim 2$ as $\varepsilon \to 0^+$, then $x = O(1)$, which implies that as $\varepsilon \to 0^+$ the assumption that $x \ll 1$ is false and that this balance is inconsistent.

f. Suppose that $x \sim -2$ is the dominant balance and that $\varepsilon x^3 \ll 1$, $x^2 \ll 1$. If $x \sim -2$ as $\varepsilon \to 0^+$, then $x = O(1)$, which is consistent with the assumptions that $\varepsilon x^3 \ll 1$ and $x^2 \ll 1$ and the balance is consistent. Using this balance will recover the root $x = -2$ from the unperturbed equation and the root can be expanded with a perturbation series in ε in the normal way.

Assuming the balance in case a., we see that the roots of the equation are of order ε^{-1}. Making the transformation $x = \varepsilon^{-1}y$ we see that the original equation becomes

$$y^3 + y^2 + \varepsilon y - 2\varepsilon^2 = 0. \tag{1}$$

Suppose that the roots of the equation (1) can be expressed in terms of ε, i.e.

$$y = \sum_{n=0}^{\infty} a_n \varepsilon^n.$$

Suppose $y = a_0 + a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3 + O(\varepsilon^4)$. Substituting y into (1) and equating coefficients of ε, we see that the following equations must be satisfied:

$$\begin{aligned}
&a_0^2 + a_0^3 &&= 0\\
&a_0 + 2a_0a_1 + 3a_0^2a_1 &&= 0\\
&-2 + a_1 + a_1^2 + 3a_0a_1^2 + 2a_0a_2 + 3a_0^2a_2 &&= 0\\
&a_1^3 + a_2 + 2a_1a_2 + 6a_0a_1a_2 + 2a_0a_3 + 3a_0^2a_3 &&= 0
\end{aligned}$$

When $a_0 = -1$, we see from the second equation that $a_1 = 1$, from the third equation that $a_2 = 3$, and from the fourth equation that $a_3 = 8$. Thus, the root x_1 to the original equation is given by

$$x_1 = \varepsilon^{-1}y = \varepsilon^{-1}\left(-1 + \varepsilon + 3\varepsilon^2 + 8\varepsilon^3 + O(\varepsilon^4)\right) = -\varepsilon^{-1} + 1 + 3\varepsilon + 8\varepsilon^2 + O(\varepsilon^3).$$

accurate to precision $O(\varepsilon^2)$.

When $a_0 = 0$, we have two possibilities from the above system of non-linear equations, either $a_1 = -2$, $a_2 = -8/3$, $a_3 = 0$ or $a_1 = 1$, $a_2 = -1/3$, $a_3 = 0$. Thus, the second and third roots x_2, x_3 to the original equation are given by

$$x_2 = \varepsilon^{-1}y = \varepsilon^{-1}\left(-2\varepsilon - \frac{8}{3}\varepsilon^2 + O(\varepsilon^4)\right) = -2 - \frac{8}{3}\varepsilon + O(\varepsilon^3).$$

and

$$x_3 = \varepsilon^{-1}y = \varepsilon^{-1}\left(\varepsilon - \frac{1}{3}\varepsilon^2 + O(\varepsilon^4)\right) = 1 - \frac{1}{3}\varepsilon + O(\varepsilon^3).$$

both accurate to precision $O(\varepsilon^2)$. Note that when $a_0 = 0$ we have recovered the roots to the unperturbed equation and their actual order is ε.

Therefore, the three roots to the equation $\varepsilon x^3 + x^2 + x - 2 = 0$, $\varepsilon \to 0^+$ are given by

$$\begin{aligned}
x_1 &= -\varepsilon^{-1} + 1 + 3\varepsilon + 8\varepsilon^2 + O(\varepsilon^3)\\
x_2 &= -2 - \frac{8}{3}\varepsilon + O(\varepsilon^3)\\
x_3 &= 1 - \frac{1}{3}\varepsilon + O(\varepsilon^3)
\end{aligned}$$

all accurate to precision $O(\varepsilon^2)$. □