

Homework Assignment 1

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Problem 2.1. Find the Fourier transforms of each of the following functions:

c. $f(x) = \delta^{(n)}(x)$,

f. $f(x) = x \exp\left(-\frac{ax^2}{2}\right), a > 0$,

g. $f(x) = x^2 \exp\left(-\frac{1}{2}x^2\right)$.

Solution. Recall that, by definition, we have that for a function $f(x) \in L^1(\mathbb{R})$, its Fourier transform is given by

$$\mathcal{F}\{f(x)\} = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx \quad (1)$$

where $k \in \mathbb{R}$.

c.

f. Using (1), we see that

$$\begin{aligned} \mathcal{F}\{f(x)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp(-ikx) \exp\left(-\frac{ax^2}{2}\right) dx \\ &= \frac{\exp\left(\frac{(ik)^2}{2a}\right)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp\left(-\frac{ax^2}{2} - ikx - \frac{(ik)^2}{2a}\right) dx \\ &= \frac{\exp\left(-\frac{k^2}{2a}\right)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp\left(-\frac{a}{2} \left(x + \frac{ik}{a}\right)^2\right) dx. \end{aligned}$$

Making the substitution $u = x + ik/a$, where $du = dx$, we have that

$$\begin{aligned} \mathcal{F}\{f(x)\} &= \frac{\exp\left(-\frac{k^2}{2a}\right)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(u - \frac{ik}{a}\right) \exp\left(-\frac{au^2}{2}\right) du \\ &= \frac{\exp\left(-\frac{k^2}{2a}\right)}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} u \exp\left(-\frac{au^2}{2}\right) du - \frac{ik}{a} \int_{-\infty}^{\infty} \exp\left(-\frac{au^2}{2}\right) du \right]. \quad (2) \end{aligned}$$

Since the function $g(x) = u \exp\left(-\frac{au^2}{2}\right)$ is odd, we know that

$$\int_{-\infty}^{\infty} u \exp\left(-\frac{au^2}{2}\right) du = 0.$$

Using the formula for the general Gaussian integral we have that

$$\int_{-\infty}^{\infty} \exp\left(-\frac{au^2}{2}\right) du = \frac{\sqrt{2\pi}}{\sqrt{a}}$$

when $a > 0$.

Combining, we see from (2) that the Fourier transform of $f(x) = x \exp\left(-\frac{ax^2}{2}\right)$ for $a > 0$ is

$$\begin{aligned} \mathcal{F}\{f(x)\} &= \frac{\exp\left(-\frac{k^2}{2a}\right)}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} u \exp\left(-\frac{au^2}{2}\right) du - \frac{ik}{a} \int_{-\infty}^{\infty} \exp\left(-\frac{au^2}{2}\right) du \right] \\ &= \frac{\exp\left(-\frac{k^2}{2a}\right)}{\sqrt{2\pi}} \left(-\frac{ik}{a} \right) \left(\frac{\sqrt{2\pi}}{\sqrt{a}} \right) \\ &= -\frac{ik \exp\left(-\frac{k^2}{2a}\right)}{a\sqrt{a}}. \end{aligned}$$

g.

□

Problem 2.2. Show that

a. $\mathcal{F} \{ \delta(x - ct) + \delta(x + ct) \} = \sqrt{\frac{2}{\pi}} \cos(kct),$

b. $\mathcal{F} \{ H(ct - |x|) \} = \mathcal{F} \{ \chi_{[-ct, ct]}(x) \} = \sqrt{\frac{2}{\pi}} \frac{\sin(kct)}{k}.$

Solution.

□

Problem 2.3. Show that

a. $i \frac{d}{dk} F(k) = \mathcal{F} \{ x f(x) \}$

b. $i^n \frac{d^n}{dk^n} F(k) = \mathcal{F} \{ x^n f(x) \}$

Solution.

□

Problem 2.5. Prove the following:

c. If $f(x)$ has a finite discontinuity at a point $x = a$, then

$$\mathcal{F} \{f'(x)\} = (ik)F(k) - \frac{1}{\sqrt{2\pi}} \exp(-ika)[f]_a,$$

where $[f]_a = f(a+0) - f(a-0)$.

Generalize this result for $\mathcal{F} \{f^{(n)}(x)\}$.

Solution.

□

Problem 2.7. Prove the following results for the convolution:

c. $\frac{d}{dx} [f(x) * g(x)] = f'(x) * g(x) = f(x) * g'(x),$

d. $\int_{-\infty}^{\infty} (f * g)(x) dx = \int_{-\infty}^{\infty} f(u) du \int_{-\infty}^{\infty} g(v) dv.$

Solution.

□

Problem 2.8. Use the Fourier transform to solve the following ordinary differential equations for $-\infty < x < \infty$:

a. $y''(x) - y(x) + 2f(x) = 0$, where $f(x) = 0$ when $x < -a$ and when $x > a$ and its derivatives vanish at $x = \pm\infty$,

b. $2y''(x) + xy'(x) + y(x) = 0$.

Solution.

□

Problem 2.9. Solve the following integral equations for an unknown function $f(x)$:

a. $\int_{-\infty}^{\infty} \phi(x-t)f(t)dt = g(x),$

b. $\int_{-\infty}^{\infty} \exp(-at^2)f(x-t)dt = \exp(-at^2), a > b > 0,$

d. $\int_{-\infty}^{\infty} f(x-t)f(t)dt = \frac{b}{x^2 + b^2}.$

Solution.

□