# Homework Assignment 3

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**Problem 1.5.1.** Find the fixed points of the following maps and use the appropriate theorems to determine whether they are asymptotically stable, semi-stable, or unstable:

i. $f(x) = \frac{x^3}{2} + \frac{x}{2}$,

ii. $f(x) = \arctan(x)$,

iii. $f(x) = x^3 + x^2 + x$,

iv. $f(x) = x^3 - x^2 + x$,

v. $f(x) = \begin{cases} 3x/4 & x \leq 1/2 \\ 3(1-x)/4 & x > 1/2 \end{cases}$.

*Solution.* Note that a point $x = c$ is a fixed point of $f$ if $c$ is a solution to the equation $g(x) = f(x) - x = 0$. If $x = c$ is a fixed point, then the behavior of the derivatives of $f$ at the point $x = c$ will allow us to classify the stability of the fixed point.

i. The solutions to the equation

$$\begin{aligned} g(x) &= f(x) - x \\ &= \frac{x^3}{2} + \frac{x}{2} - x \\ &= \frac{x^3}{2} - \frac{x}{2} - x = 0 \end{aligned}$$

are given by $x = -1$, $x = 0$, and $x = 1$. Note that $f'(x) = 3x^2/2 + 1/2$.

For the fixed point $x = -1$, we see that $|f'(-1)| = 2 > 1$ so that $x = -1$ is a hyperbolic fixed point and by theorem 1.4.4, this fixed point is unstable.

For the fixed point $x = 0$, we see that $|f'(0)| = 1/2 < 1$ so that $x = 0$ is a hyperbolic fixed point and by theorem 1.4.4, this fixed point is stable.

For the fixed point $x = 1$, we see that $|f'(1)| = 2 > 1$ so that $x = 1$ is a hyperbolic fixed point and by theorem 1.4.4, this fixed point is unstable.

ii. Note that for any $x \in \mathbb{R}$, we have that $-\pi/2 < \arctan(x) < \pi/2$. Thus, if $|x| > \pi/2$, then $|\arctan(x)| < \pi/2 < |x|$ so that for any such $x$ we have that $\arctan(x) \neq x$, i.e. $f(x) = \arctan(x)$ has no fixed points.

Since $f(x)$ is continuous on the interval $[-\pi/2, \pi/2]$, we know that $f(x)$ must have a fixed point on this interval. Note that if $x > 0$, we can show that $0 < x/(x^2+1) < f(x) = \arctan(x) < x$. From this identity, we can show that

$$0 < f^{n+1}(x) < f^n(x) < \cdots < x,$$

i.e. the iterates of $f$ are monotonically decreasing and bounded below. Thus, the limit converges to the infimum, i.e. $\lim f^n(x) = 0$. Therefore, we must have $x = 0$ is a fixed point.

Using a similar inequality, we can show that the iterates of $f$ form a monotonically increasing sequence that is bounded above. Thus, the limit converges to the supremum, i.e. $\lim f^n(x) = 0$ and $x = 0$ is a fixed point. Therefore, $x = 0$ is the only fixed point of $f(x) = \arctan(x)$.

Note that

$$f'(x) = 1/(x^2+1), \quad f''(x) = -2x/(1+x^2)^2, \quad f'''(x) = 8x^2/(1+x^2)^3 - 2/(1+x^2)^2.$$

Thus, for the fixed point $x = 0$, we see that $f'(0) = 1$, $f''(0) = 0$, and $f'''(0) = -2$. Therefore, according to theorem 1.5.3 (iii), this fixed point is non-hyperbolic and stable.

iii. The solutions to the equation

$$\begin{aligned} g(x) &= f(x) - x \\ &= x^3 + x^2 + x - x \\ &= x^2(x+1) = 0 \end{aligned}$$

are given by $x = -1$ and $x = 0$. Note that $f'(x) = 3x^2 + 2x + 1$, $f''(x) = 6x + 2$, and $f'''(x) = 6$.

For the fixed point $x = -1$, we see that $|f'(-1)| = 2 > 1$ so that $x = -1$ is a hyperbolic fixed point and by theorem 1.4.4, this fixed point is unstable.

For the fixed point $x = 0$, we see that $f'(0) = 1$ so that $x = 0$ is a non-hyperbolic fixed point. Since $f''(0) = 2 > 0$, we have by theorem 1.5.3 (i)(a) that this fixed point is one-sided stable to the left of $x = 0$.

iv. The solutions to the equation

$$\begin{aligned} g(x) &= f(x) - x \\ &= x^3 - x^2 + x - x \\ &= x^2(x-1) = 0 \end{aligned}$$

are given by $x = 1$ and $x = 0$. Note that $f'(x) = 3x^2 - 2x + 1$, $f''(x) = 6x - 2$, and $f'''(x) = 6$.

For the fixed point $x = 1$, we see that $|f'(1)| = 2 > 1$ so that $x = 1$ is a hyperbolic fixed point and by theorem 1.4.4, this fixed point is unstable.

For the fixed point $x = 0$, we see that $f'(0) = 1$ so that $x = 0$ is a non-hyperbolic fixed point. Since $f''(0) = -2 < 0$, we have by theorem 1.5.3 (i)(b) that this fixed point is one-sided stable to the right of $x = 0$.

v. If $x \leq 1/2$, then
$$f(x) - x = \frac{3x}{4} - x = -\frac{x}{4} = 0$$
if $x = 0$. Since $x = 0 \leq 1/2$, we have that $x = 0$ is a fixed point of $f(x)$.

If $x > 1/2$, then
$$f(x) - x = \frac{3(1-x)}{4} - x = \frac{3 - 7x}{4} = 0$$
if $x = 3/7$. Since $3/7 < 1/2$, we have that $x = 3/7$ is not a fixed point of $f(x)$.

If $x \leq 1/2$, then $f'(x) = 3/4$. Thus, for the fixed point $x = 0$, we see that $|f'(0)| < 1$ and $x = 0$ is a non-hyperbolic stable fixed point by theorem 1.4.4.

$\square$

**Problem 1.5.2.** Consider the family of quadratic maps $f_c(x) = x^2 + c$ where $x \in \mathbb{R}$.

i. Use the theorems of section 1.5 to determine the stability of the hyperbolic fixed points of the the family of maps for all possible values of $c$.

ii. Find any values of $c$ such that $f_c$ has a non-hyperbolic fixed point and determine the stability of these fixed points.

*Solution.* $\square$

**Problem 1.5.3.**

i. Show that $f(x) = -2x^3 + 2x^2 + x$ has two non-hyperbolic fixed points and determine their stability.

ii. If $x = 0$ and $x = 1$ are non-hyperbolic fixed points for $f : \mathbb{R} \to \mathbb{R}$ for $f(x) = ax^3 + bx^2 + cx + d$, find all possible values of $a, b, c$, and $d$.

iii. Write down the function $f(x)$ in each case of (ii) above and determine the stability of the fixed points.

*Solution.* □

**Problem 1.5.6.** Find the Schwarzian derivative of both $f(x) = e^x$ and $g(x) = \sin(x)$ and show that they are always negative.

*Solution.* $\square$

**Problem 1.5.9.** Let $f(x)$ be a polynomial such that $f(c) = c$. (Recall that a polynomial $p(x)$ has $(x - c)^2$ as a factor if and only if both $p(c) = 0$ and $p'(c) = 0$.)

i. If $f'(c) = 1$, show that $(x - c)^2$ is a factor of $g(x) = f(x) - x$.

ii. If $|f'(c)| = 1$, show that $(x - c)^2$ is a factor of $h(x) = f^2(x) - x$.

iii. Show in the case that $f'(c) = -1$, we actually have that $(x - c)^3$ is a factor of $h(x) = f^2(x) - x$.

iv. Check that (iii) holds for the non-hyperbolic fixed point $x = 2/3$ of the logistic map $L_3(x) = 3x(1 - x)$.

v. Check that (i), (ii), (iii) hold for the non-hyperbolic fixed points of the polynomial $f(x) = -2x^3 + 2x^2 + x$.

*Solution.* □