

# Homework Assignment 7

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**Problem 1.** State all of the KKT conditions for ( $N$ -max). More precisely state all of the following results for ( $N$ -max): KKT-FONC, KKT-FOSC, KKT-SONC, KKT-SOSC.

*Solution.* For the following theorems, we assume ( $N$ -max) has the following form

$$\begin{aligned} (N\text{-max}) \quad & \text{maximize} && f(\mathbf{x}) \\ & \text{subject to} && \mathbf{h}(\mathbf{x}) = \mathbf{0} \\ & && \mathbf{g}(\mathbf{x}) \leq \mathbf{0} \end{aligned}$$

where  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\mathbf{h} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , and  $\mathbf{g} : \mathbb{R}^n \rightarrow \mathbb{R}^p$  with  $m \leq n$ . Additionally, define the following Lagrangian function to be  $\mathbf{L}(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) := -f(\mathbf{x}) + \boldsymbol{\lambda}^\top \mathbf{h}(\mathbf{x}) + \boldsymbol{\mu}^\top \mathbf{g}(\mathbf{x})$ .

**Theorem 1** (KKT-FONC for ( $N$ -max)). Let  $f, \mathbf{g}, \mathbf{h} \in C^1$  and let  $\mathbf{x}^*$  be a regular point and local maximizer for the problem ( $N$ -max). Then, there exist  $\boldsymbol{\lambda}^* \in \mathbb{R}^m$  and  $\boldsymbol{\mu}^* \in \mathbb{R}^p$  such that:

- i.  $\boldsymbol{\mu}^* \geq \mathbf{0}$ .
- ii.  $D_{\mathbf{x}}\mathbf{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) = -Df(\mathbf{x}^*) + \boldsymbol{\lambda}^{*\top} D\mathbf{h}(\mathbf{x}^*) + \boldsymbol{\mu}^{*\top} D\mathbf{g}(\mathbf{x}^*) = \mathbf{0}^\top$ .
- iii.  $\boldsymbol{\mu}^{*\top} \mathbf{g}(\mathbf{x}^*) = 0$ .

Note that there are no explicit first-order conditions that are sufficient in general to show optimality.

**Theorem 2** (KKT-SONC for ( $N$ -max)). Let  $f, \mathbf{g}, \mathbf{h} \in C^2$  and let  $\mathbf{x}^*$  be a regular point and local maximizer for the problem ( $N$ -max). Then, there exist  $\boldsymbol{\lambda}^* \in \mathbb{R}^m$  and  $\boldsymbol{\mu}^* \in \mathbb{R}^p$  such that:

- i.  $\boldsymbol{\mu}^* \geq \mathbf{0}$ ,  $D_{\mathbf{x}}\mathbf{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) = \mathbf{0}^\top$ ,  $\boldsymbol{\mu}^{*\top} \mathbf{g}(\mathbf{x}^*) = 0$ .
- ii. For all  $\mathbf{y} \in T(\mathbf{x}^*) = \{\mathbf{y} \mid D\mathbf{h}(\mathbf{x}^*)\mathbf{y} = \mathbf{0}, D\mathbf{g}_j(\mathbf{x}^*)\mathbf{y} = 0, j \in J(\mathbf{x}^*)\}$ , we have that  $\mathbf{y}^\top D_{\mathbf{x}}^2 \mathbf{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) \mathbf{y} \leq 0$ .

**Theorem 3** (KKT-SOSC for ( $N$ -max)). Let  $f, \mathbf{g}, \mathbf{h} \in C^2$  and suppose there exists a feasible point  $\mathbf{x}^*$  and vectors  $\boldsymbol{\lambda}^* \in \mathbb{R}^m$  and  $\boldsymbol{\mu}^* \in \mathbb{R}^p$  such that:

- i.  $\boldsymbol{\mu}^* \geq \mathbf{0}$ ,  $D_{\mathbf{x}}\mathbf{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) = \mathbf{0}^\top$ ,  $\boldsymbol{\mu}^{*\top} \mathbf{g}(\mathbf{x}^*) = 0$ .

ii. For all

$$\mathbf{y} \in \tilde{T}(\mathbf{x}^*, \boldsymbol{\mu}^*) = \{\mathbf{y} \mid D\mathbf{h}(\mathbf{x}^*)\mathbf{y} = \mathbf{0}, Dg_i(\mathbf{x}^*)\mathbf{y} = 0, \text{ for } i \in \{i \mid g_i(\mathbf{x}^*) = 0, \mu_i^* > 0\}\},$$

with  $\mathbf{y} \neq \mathbf{0}$ , we have that  $\mathbf{y}^\top D_x^2 \mathbf{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) \mathbf{y} < 0$ .

Then  $\mathbf{x}^*$  is a strict local maximizer for the problem ( $N$ -max).

□

**Problem 2.** Find local minimizers for

$$\begin{aligned} (N\text{-min}) \quad & \text{minimize} \quad x_1^2 + 6x_1x_2 - 4x_1 - 2x_2 \\ & \text{subject to} \quad x_1^2 + 2x_2 \leq 1 \\ & \quad \quad \quad 2x_1 - 2x_2 \leq 1. \end{aligned}$$

*Solution.* We begin by rewriting the above problem as follows:

$$\begin{aligned} (N\text{-min}) \quad & \text{minimize} \quad f(\mathbf{x}) = x_1^2 + 6x_1x_2 - 4x_1 - 2x_2 \\ & \text{subject to} \quad g_1(\mathbf{x}) = x_1^2 + 2x_2 - 1 \leq 0 \\ & \quad \quad \quad g_2(\mathbf{x}) = 2x_1 - 2x_2 - 1 \leq 0. \end{aligned}$$

We proceed by using the KKT-FONC to determine the possible local minimizers for this problem. The Lagrangian associated to this problem is given by

$$\begin{aligned} L(\mathbf{x}, \boldsymbol{\mu}) &= f(\mathbf{x}) + \boldsymbol{\mu}^\top \mathbf{g}(\mathbf{x}) \\ &= f(\mathbf{x}) + \mu_1 g_1(\mathbf{x}) + \mu_2 g_2(\mathbf{x}) \\ &= x_1^2 + 6x_1x_2 - 4x_1 - 2x_2 + \mu_1(x_1^2 + 2x_2 - 1) + \mu_2(2x_1 - 2x_2 - 1). \end{aligned}$$

This implies that

$$D_x L(\mathbf{x}, \boldsymbol{\mu}) = \begin{bmatrix} 2x_1 + 6x_2 - 4 + 2\mu_1 x_1 + 2\mu_2 \\ 6x_1 - 2 + 2\mu_1 - 2\mu_2 \end{bmatrix}^\top = \begin{bmatrix} 0 \\ 0 \end{bmatrix}^\top. \quad (1)$$

Thus, the KKT-FONC are then written as

- i.  $\mu_1, \mu_2 \geq 0$ .
- ii.  $2x_1 + 6x_2 - 4 + 2\mu_1 x_1 + 2\mu_2 = 0$ .
- iii.  $6x_1 - 2 + 2\mu_1 - 2\mu_2 = 0$ .
- iv.  $\mu_1 g_1(\mathbf{x}) + \mu_2 g_2(\mathbf{x}) = \mu_1(x_1^2 + 2x_2 - 1) + \mu_2(2x_1 - 2x_2 - 1) = 0$ .
- v.  $g_1(\mathbf{x}) = x_1^2 + 2x_2 - 1 \leq 0$ .
- vi.  $g_2(\mathbf{x}) = 2x_1 - 2x_2 - 1 \leq 0$ .

Solving the system (1) for  $x_1, x_2$  yields that

$$\begin{aligned} x_1 &= \frac{\mu_2 - \mu_1 + 1}{3} \\ x_2 &= \frac{\mu_1^2 - \mu_1\mu_2 - 4\mu_2 + 5}{9} \end{aligned} \quad (2)$$

with  $\mu_1, \mu_2 \geq 0$ . Using these representations of  $x_1, x_2$  we see that condition iv. yields three possible solutions in terms of  $\mu_1, \mu_2$ :

$$\begin{aligned} \text{Case 1 : } \mu_2 &= \frac{13 + 12\mu_1 + 6\mu_1^2 - \sqrt{169 + 200\mu_1 + 388\mu_1^2}}{2(14 + 3\mu_1)} \\ \text{Case 2 : } \mu_2 &= \frac{13 + 12\mu_1 + 6\mu_1^2 + \sqrt{169 + 200\mu_1 + 388\mu_1^2}}{2(14 + 3\mu_1)} \\ \text{Case 3 : } \mu_1 &= -\frac{14}{3}, \mu_2 = -\frac{3220}{789} \end{aligned}$$

We readily see that Case 3 cannot happen in light of condition i.

Assuming Case 1 is true and using the representations of  $x_1, x_2$  in (2), we see that  $g_1(\mathbf{x}) < 0$  for  $\mu_1, \mu_2 \geq 0$  implying that this constraint is inactive and that  $\mu_1 = 0$ . This implies that  $\mu_2 = 0$  which in turn implies that  $x_1 = 1/3, x_2 = 5/9$ . However,  $g_1(x_1, x_2) = 2/9 \not\leq 0$  violating condition v. Thus, Case 1 cannot happen.

Assuming Case 2 is true and using the representations of  $x_1, x_2$  in (2), we again see that  $g_1(\mathbf{x}) < 0$  for  $\mu_1, \mu_2 \geq 0$  implying that this constraint is inactive and that  $\mu_1 = 0$ . This implies that  $\mu_2 = 13/14$  which in turn implies that  $x_1 = 9/14, x_2 = 1/7$ . These values of  $x_1, x_2$  satisfy conditions v. and vi.

Therefore, the only vector  $\mathbf{x}^*$  that satisfies conditions i. - vi., i.e. the only possible local minimizer for this problem is

$$\mathbf{x}^* = \begin{bmatrix} 9/14 \\ 1/7 \end{bmatrix}$$

with associated KKT multiplier

$$\boldsymbol{\mu}^* = \begin{bmatrix} 0 \\ 13/14 \end{bmatrix}.$$

To verify whether or not this vector is a strict local minimizer, we check the KKT-SOSC. For the vectors  $\mathbf{x}^*$  and  $\boldsymbol{\mu}^*$  defined above, we see that  $\{i \mid g_i(\mathbf{x}^*) = 0, \mu_i^* > 0\} = \{2\}$  and that

$$\begin{aligned} \tilde{T}(\mathbf{x}^*, \boldsymbol{\mu}^*) &= \{\mathbf{y} \in \mathbb{R}^2 \mid Dg_2(\mathbf{x}^*)\mathbf{y} = 0\} \\ &= \{\mathbf{y} \in \mathbb{R}^2 \mid [2, -2]\mathbf{y} = 0\} \\ &= \{\mathbf{y} = [y_1, y_2]^\top \in \mathbb{R}^2 \mid y_1 = y_2\}. \end{aligned}$$

Further, we have, for these vectors, that

$$D_x^2 L(\mathbf{x}^*, \boldsymbol{\mu}^*) = \begin{bmatrix} 2 + 2\mu_1 & 6 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 6 & 0 \end{bmatrix}.$$

Combining we see that for  $\mathbf{0} \neq \mathbf{y} \in \tilde{T}(\mathbf{x}^*, \boldsymbol{\mu}^*)$ , we have that

$$\mathbf{y}^\top D_x^2 L(\mathbf{x}^*, \boldsymbol{\mu}^*) \mathbf{y} = \begin{bmatrix} y_1 \\ y_1 \end{bmatrix}^\top \begin{bmatrix} 2 & 6 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_1 \end{bmatrix} = 14y_1^2 > 0$$

for  $y_1 \neq 0$ . Therefore,  $\mathbf{x}^* = [9/14, 1/7]^\top$  is a strict local minimizer. □

**Problem 3.** Consider the problem of optimizing

$$\begin{array}{ll} (N) & \text{minimize (maximize)} \quad (x_1 - 2)^2 + (x_2 - 1)^2 \\ & \text{subject to} \quad \begin{array}{ll} x_2 - x_1^2 & \geq 0 \\ 2 - x_1 - x_2 & \geq 0 \\ x_1 & \geq 0. \end{array} \end{array}$$

Let  $x^* = [0, 0]$ .

- a. Does  $x^*$  satisfy the KKT-FONC for minimization or maximization? What are the KKT multipliers?
- b. Does  $x^*$  satisfy the KKT-SOSC? Justify your answer.

*Solution.*

□

**Problem 4.** Consider the problem with equality constraint

$$\begin{array}{ll}\text{minimize} & f(\mathbf{x}) \\ \text{subject to} & \mathbf{h}(\mathbf{x}) = \mathbf{0}.\end{array}$$

We can convert the above into the equivalent optimization problem

$$\begin{array}{ll}\text{minimize} & f(\mathbf{x}) \\ \text{subject to} & \frac{1}{2} \|\mathbf{h}(\mathbf{x})\|^2 \leq 0.\end{array}$$

Write down the KKT condition for the equivalent problem and explain why the KKT theorem cannot be applied in this case.

*Solution.*

□