

Homework Assignment 4

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Problem 1. Find the first three terms in the asymptotic expansions of $x \rightarrow 0^+$ of the following integrals:

$$\int_x^1 \cos(xt) dt, \quad \int_0^{1/x} e^{-t^2} dt.$$

Solution. If the function $f(t, x)$ possesses the asymptotic expansion

$$f(t, x) \sim \sum_{n=0}^{\infty} f_n(t)(x - x_0)^{\alpha n} \quad \text{as } x \rightarrow x_0$$

for some $\alpha > 0$, uniformly for $a \leq t \leq b$, then the asymptotic expansion of the integral

$$I(x) = \int_a^b f(t, x) dt$$

as $x \rightarrow x_0$ is given by

$$I(x) \sim \sum_{n=0}^{\infty} (x - x_0)^{\alpha n} \int_a^b f_n(t) dt \quad \text{as } x \rightarrow x_0.$$

We begin with finding the first three terms of the asymptotic expansion of the integral

$$I_1(x) = \int_x^1 \cos(xt) dt \quad \text{as } x \rightarrow 0^+.$$

Note that $f(t, x) = \cos(xt)$ has the following asymptotic expansion as $x \rightarrow 0^+$:

$$f(t, x) = \cos(xt) \sim 1 - \frac{t^2 x^2}{2} + \frac{t^4 x^4}{24}.$$

This expansion converges uniformly for all $x \leq t \leq 1$ as $x \rightarrow 0^+$. Therefore, we have that the first three terms of the asymptotic expansion of $I_1(x)$ as $x \rightarrow 0^+$ are given by

$$I_1(x) \sim \int_x^1 dt - \frac{x^2}{2} \int_x^1 t^2 dt + \frac{x^4}{24} \int_x^1 t^4 dt = (1 - x) - \frac{x^2}{2} \left[\frac{1 - x^3}{3} \right] + \frac{x^4}{24} \left[\frac{1 - x^5}{5} \right].$$

Similar to what was shown above, we have that if

$$f(t, x) \sim f_0(t) \quad \text{as } x \rightarrow x_0$$

uniformly for $a \leq t \leq b$, then the asymptotic expansion of the integral is given by

$$I(x) = \int_a^b f(t, x) dt \sim \int_a^b f_0(t) dt \quad \text{as } x \rightarrow x_0.$$

Let us continue by finding the first three terms of the asymptotic expansion of the integral

$$I_2(x) = \int_0^{1/x} e^{-t^2} dt \quad \text{as } x \rightarrow 0^+.$$

Note that $f(t, x) = e^{-t^2}$ has the following asymptotic expansion as $x \rightarrow 0^+$:

$$f(t, x) = e^{-t^2} \sim 1 - t^2 + \frac{t^4}{2}.$$

This expansion converges uniformly for all finite points, so it converges uniformly for $0 \leq t \leq 1/x$ as $x \rightarrow 0^+$. Therefore, we may integrate the expansion term by term and we have that the first three terms of the asymptotic expansion of $I_2(x)$ as $x \rightarrow 0^+$ are given by

$$I_2(x) \sim \int_0^{1/x} dt - \int_0^{1/x} t^2 dt + \frac{1}{2} \int_0^{1/x} t^4 dt = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{10x^5}.$$

□

Problem 2. Find the full asymptotic behavior as $x \rightarrow 0^+$ of the following integral:

$$\int_0^1 \frac{e^{-t}}{1+x^2t^3} dt$$

Solution. Note that the function $f(t, x) = e^{-t}/(1+x^2t^3)$ has the asymptotic expansion

$$f(t, x) = \frac{e^{-t}}{1+x^2t^3} \sim e^{-t} \sum_{n=0}^{\infty} [(-1)^n t^{3n}] x^{2n} \quad \text{as } x \rightarrow 0^+.$$

Note that this asymptotic expansion converges uniformly for $0 \leq x \leq t < 1 - \epsilon$ for all $\epsilon > 0$. To see this, we note that for $0 < m < n$, we have that

$$\left| \sum_{k=m+1}^n (-1)^k (x^2t^3)^k \right| < \sum_{k=m+1}^n (1-\epsilon)^{5k}.$$

Since $(1-\epsilon)^5 < 1$, we have that its geometric series converges and we can make it as small as we wish. Thus, by the Cauchy criterion we have uniform convergence for $0 \leq x \leq t < 1 - \epsilon$ for all $\epsilon > 0$.

Per the discussion in Problem 1, using this uniformly convergent asymptotic expansion, we have that as $x \rightarrow 0^+$

$$\int_0^1 \frac{e^{-t}}{1+x^2t^3} dt \sim \sum_{n=0}^{\infty} (-1)^n x^{2n} \int_0^1 e^{-t} t^{3n} dt = \sum_{n=0}^{\infty} (-1)^n x^{2n} [\Gamma(3n+1) - \Gamma(3n+1, 1)]$$

where $\Gamma(a, k) = \int_k^{\infty} t^{a-1} e^{-t} dt$. □

Problem 3. Find the full asymptotic expansion of $\int_0^x \text{Bi}(t)dt$ as $x \rightarrow +\infty$.

Solution. Note that for $x \rightarrow +\infty$, the integral above can be written as

$$\int_0^x \text{Bi}(t)dt = \int_0^1 \text{Bi}(t)dt + \int_1^x \text{Bi}(t)dt \quad (1)$$

Thus, the asymptotic expansion of the integral depends only on the second integral on the right. The Airy function $\text{Bi}(t)$ satisfies the differential equation $y'' = ty$. Using this differential equation and integrating the integral on the right by parts we see that

$$\begin{aligned} \int_1^x \text{Bi}(t)dt &= \int_1^x \frac{1}{t} \text{Bi}''(t)dt \\ &= \frac{1}{x} \text{Bi}'(x) - \text{Bi}'(1) + \int_1^x \frac{1}{t^2} \text{Bi}'(t)dt. \end{aligned}$$

Note that it is clear that as $x \rightarrow +\infty$ the following relations hold

$$\begin{aligned} \text{Bi}'(1) &\ll \frac{1}{x} \text{Bi}'(x) \\ \int_0^1 \text{Bi}(t)dt &\ll \int_0^x \text{Bi}(t)dt. \end{aligned}$$

Thus, from equation (1) and the above relations, we have that as $x \rightarrow +\infty$

$$\int_0^x \text{Bi}(t)dt \sim \frac{1}{x} \text{Bi}'(x) + \int_1^x \frac{1}{t^2} \text{Bi}'(t)dt. \quad (2)$$

However, upon further investigation we see that as $x \rightarrow +\infty$

$$\int_1^x \frac{1}{t^2} \text{Bi}'(t)dt \ll \frac{1}{x} \text{Bi}'(x). \quad (3)$$

To see that this is true, we integrate the integral on the left by parts which yields

$$f(x) = \int_1^x \frac{1}{t^2} \text{Bi}'(t)dt = x^{-2} \text{Bi}(x) - \text{Bi}(1) + 2 \int_1^x t^{-3} \text{Bi}(t)dt.$$

In comparing the function $f(x)$ with the function $g(x) = x^{-1} \text{Bi}'(x)$ as $x \rightarrow +\infty$, we see that

$$\lim_{x \rightarrow +\infty} \frac{f(x)}{g(x)} = \frac{+\infty}{+\infty}$$

an indeterminate form. Thus, applying L'Hôpital's rule, we see that derivatives of $f(x)$ and $g(x)$ are

$$\begin{aligned} f'(x) &= -2x^{-3} \text{Bi}(x) + x^{-2} \text{Bi}'(x) + 2 [x^{-3} \text{Bi}(x) - \text{Bi}(1)] \\ &= x^{-2} \text{Bi}'(x) - 2\text{Bi}(1) \\ g'(x) &= -x^{-2} \text{Bi}'(x) + x^{-1} \text{Bi}''(x) \end{aligned}$$

and that

$$\lim_{x \rightarrow +\infty} \frac{f'(x)}{g'(x)} = \frac{x^{-2}\text{Bi}'(x) - 2\text{Bi}(1)}{-x^{-2}\text{Bi}'(x) + x^{-1}\text{Bi}''(x)} = \frac{1}{1 + \frac{x^{-1}\text{Bi}''(x)}{x^{-2}\text{Bi}'(x)}} = 0$$

Therefore, we must have that relation (3) is true and that relation (2) reduces to

$$\int_0^x \text{Bi}(t)dt \sim \frac{1}{x}\text{Bi}'(x) \quad (x \rightarrow +\infty).$$

Note that the asymptotic expansion of $\text{Bi}(x)$ as $x \rightarrow +\infty$ is given by

$$\text{Bi}(x) \sim \pi^{-1/2}x^{-1/4} \exp\left(\frac{2x^{3/2}}{3}\right) \sum_{n=0}^{\infty} c_n x^{-3n/2}$$

where

$$c_n = \frac{1}{2\pi} \left(\frac{3}{4}\right)^n \frac{\Gamma(n+5/6)\Gamma(n+1/6)}{n!}.$$

Thus, we see that as $x \rightarrow +\infty$,

$$\begin{aligned} \text{Bi}'(x) &\sim \pi^{-1/2} \exp\left(\frac{2x^{3/2}}{3}\right) \left[\left(x^{3/2} - \frac{1}{4}\right) x^{-5/4} \sum_{n=0}^{\infty} c_n x^{-3n/2} + x^{-1/4} \sum_{n=0}^{\infty} \frac{-3nc_n}{2} x^{-3n/2-1} \right] \\ &= \pi^{-1/2} \exp\left(\frac{2x^{3/2}}{3}\right) \left(x^{3/2} + \frac{3}{4}\right) \sum_{n=0}^{\infty} \left(1 - \frac{3n}{2}\right) c_n x^{-3n/2-5/4}. \end{aligned}$$

Therefore, we can readily see that the full asymptotic behavior as $x \rightarrow +\infty$ of the integral of the problem is given by

$$\int_0^x \text{Bi}(t)dt \sim \frac{\text{Bi}'(x)}{x} \sim \pi^{-1/2} \exp\left(\frac{2x^{3/2}}{3}\right) \left(x^{3/2} + \frac{3}{4}\right) \sum_{n=0}^{\infty} \left(1 - \frac{3n}{2}\right) c_n x^{-3n/2-9/4}.$$

□

Problem 4. Find the first five terms in the asymptotic expansion as $x \rightarrow +\infty$ of the integral

$$\int_0^{\pi/4} e^{-xt^2} \sqrt{\tan t} dt$$

- by using a suitable change of variables and then applying Watson's lemma.
- by applying Laplace's method directly to the given integral.

Solution. a. Watson's lemma provides a formula for an asymptotic expansion as $x \rightarrow +\infty$ for integrals of the form

$$I(x) = \int_0^b f(s) e^{-xs} ds \quad b > 0 \quad (4)$$

where the function $f(s)$ is continuous on the interval $0 \leq s \leq b$ and has the asymptotic expansion

$$f(s) \sim s^\alpha \sum_{n=0}^{\infty} a_n s^{\beta n} \quad (s \rightarrow 0^+)$$

with $\alpha > -1$ and $\beta > 0$. Given these assumptions, Watson's lemma states that

$$I(x) \sim \sum_{n=0}^{\infty} \frac{a_n \Gamma(\alpha + \beta n + 1)}{x^{\alpha + \beta n + 1}} \quad (x \rightarrow +\infty). \quad (5)$$

For the integral

$$I(x) = \int_0^{\pi/4} e^{-xt^2} \sqrt{\tan t} dt,$$

we proceed by making the change of variables $s = t^2$. The integral then becomes

$$I(x) = \int_0^{\sqrt{\pi}/2} 2^{-1} s^{-1/2} \sqrt{\tan s^{1/2}} e^{-xs} ds.$$

Identifying the function $f(s) = 2^{-1} s^{-1/2} \sqrt{\tan s^{1/2}}$, we see that the above integral is of the form (4) with $f(s)$ being continuous on $0 \leq s \leq \sqrt{\pi}/2$. Further, the function $f(s)$ has the following asymptotic expansion

$$f(s) \sim \frac{1}{2} s^{-1/4} + \frac{1}{12} s^{3/4} + \frac{19}{720} s^{7/4} + \frac{55}{6048} s^{11/4} + \frac{11813}{3628800} s^{15/4} \quad (s \rightarrow 0^+).$$

Therefore, identifying $\alpha = -1/4$ and $\beta = 1$, we see that by Watson's lemma the first five terms in the asymptotic expansion of $I(x)$ as $x \rightarrow +\infty$ is given by

$$I(x) \sim \frac{\Gamma(\frac{3}{4})}{2} x^{-3/4} + \frac{\Gamma(\frac{7}{4})}{12} x^{-7/4} + \frac{19\Gamma(\frac{11}{4})}{720} x^{-11/4} + \frac{55\Gamma(\frac{15}{4})}{6048} x^{-15/4} + \frac{11813\Gamma(\frac{19}{4})}{3628800} x^{-19/4}.$$

b. Laplace's method states that, as $x \rightarrow +\infty$, for an integral of the form

$$I(x) = \int_a^b f(t)e^{x\phi(t)} dt$$

where $f(t)$ and $\phi(t)$ are real continuous functions, the integral $I(x)$ is asymptotic to the integral of $f(t)e^{x\phi(t)}$ over some small neighborhood of the point where $\phi(t)$ obtains its maximum over the interval $[a, b]$.

Identifying the function $f(t) = \sqrt{\tan t}$ and $\phi(t) = -t^2$, both real and continuous on the interval $[0, \pi/4]$, we see that $\phi(t)$ obtains its maximum at the point $t = 0$ on the same interval. However the function $f(t)$ vanishes at $t = 0$. Nevertheless Laplace's method may still be used since any contribution to the integral outside of the interval $[0, \epsilon]$ is subdominant for any $\epsilon > 0$. Thus, all of the assumptions of Laplace's method are satisfied and we have that for small $\epsilon > 0$,

$$I(x) = \int_0^{\pi/4} e^{-xt^2} \sqrt{\tan t} dt \sim \int_0^\epsilon e^{-xt^2} \sqrt{\tan t} dt \quad (x \rightarrow +\infty).$$

Since $\epsilon > 0$ is small, we may replace the function $f(t)$ with the asymptotic expansion about $t = 0$

$$\sqrt{\tan t} \sim t^{1/2} + \frac{1}{6}t^{5/2} + \frac{19}{360}t^{9/2} + \frac{55}{3024}t^{13/2} + \frac{11813}{1814400}t^{17/2} \quad (t \rightarrow 0^+)$$

so that, as $x \rightarrow +\infty$, the first five terms in the asymptotic expansion of the integral are

$$\begin{aligned} I(x) &\sim \int_0^\epsilon \left[t^{1/2} + \frac{1}{6}t^{5/2} + \frac{19}{360}t^{9/2} + \frac{55}{3024}t^{13/2} + \frac{11813}{1814400}t^{17/2} \right] e^{-xt^2} dt \\ &\sim \int_0^\infty \left[t^{1/2} + \frac{1}{6}t^{5/2} + \frac{19}{360}t^{9/2} + \frac{55}{3024}t^{13/2} + \frac{11813}{1814400}t^{17/2} \right] e^{-xt^2} dt \\ &= \frac{\Gamma(\frac{3}{4})}{2} x^{-3/4} + \frac{\Gamma(\frac{7}{4})}{12} x^{-7/4} + \frac{19\Gamma(\frac{11}{4})}{720} x^{-11/4} + \frac{55\Gamma(\frac{15}{4})}{6048} x^{-15/4} + \frac{11813\Gamma(\frac{19}{4})}{3628800} x^{-19/4}. \end{aligned}$$

□

Problem 5. Use Laplace's method of moving maxima to obtain the first two terms in the asymptotic expansion as $x \rightarrow +\infty$ of the integral

$$\int_0^\infty \exp \left[-t - \frac{x}{\sqrt{t}} \right] dt. \quad (6)$$

Solution.

□

Problem 6. Let $f(x, t)$ be differentiable in x and continuous in (x, t) on $I \times J$, where I and J are intervals, and suppose that there exist functions $g(t)$ and $g_1(t)$ that are integrable on J such that for all $(x, t) \in I \times J$ we have that

$$|f(x, t)| \leq g(t) \quad \text{and} \quad |\partial_x f(x, t)| \leq g_1(t).$$

Then

$$\frac{d}{dx} \int_J f(x, t) dt = \int_J \partial_x f(x, t) dt.$$

- a. Let $0 < a < b < \infty$. Use the above theorem to show that if $x \in (a, b)$, then

$$\frac{d^3}{dx^3} \int_0^\infty \exp \left[-t - \frac{x}{\sqrt{t}} \right] dt = - \int_0^\infty t^{-3/2} \exp \left[-t - \frac{x}{\sqrt{t}} \right] dt.$$

- b. Use integration by parts to show that

$$\int_0^\infty \exp \left[-t - \frac{x}{\sqrt{t}} \right] dt = \frac{x}{2} \int_0^\infty t^{-3/2} \exp \left[-t - \frac{x}{\sqrt{t}} \right] dt.$$

- c. Combine parts (a) and (b) to prove that integral (6) is a solution of the differential equation $xy''' + 2y = 0$ that also satisfies the initial condition $y(0) = 1$. Then use integration by parts to give an easy direct proof that the integral also satisfies the condition $y(+\infty) = 0$.

Solution.

□

Problem 7. a. Find the leading behavior as $x \rightarrow +\infty$ of Laplace integrals of the form

$$\int_a^b (t-a)^\alpha g(t) e^{x\phi(t)} dt$$

where $\phi(t)$ has a maximum at $t = a$, $g(a) = 1$. Suppose further that $\alpha > -1$ and $\phi'(a) < 0$.

b. Repeat the analysis of part (a) when $\alpha > -1$ and $\phi'(a) = \phi''(a) = \dots = \phi^{(p-1)}(a) = 0$ and $\phi^{(p)}(a) < 0$.

Solution.

□