Homework Assignment 2

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Problem 1. Let $A = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$. With `MATLAB` , compute $\det A$.

Solution. Entering the following into the `MATLAB` console gives:

```
>> A = [4 1 0 0; 1 4 1 0; 0 1 4 1; 0 0 1 4];
>> det(A)

ans =

   209
```

Therefore, $\det A = 209$. $\square$

Problem 2. Determine the rank of $A = \begin{bmatrix} 2 & 1 & 3 & 3 \\ 1 & 0 & -1 & 0 \\ a & 2 & 1 & 1 \\ 4 & 3 & 2 & 4 \end{bmatrix}$ as a function of a. Check with `MATLAB` and the function `rank` for different values of a.

Solution. We know that A has full rank, i.e. $\text{rank}(A) = 4$, if $\det A \neq 0$. Running the following in the `MATLAB` console shows:

```
>> syms a;
>> A = [2 1 3 3; 1 0 -1 0; a 2 1 1; 4 3 2 4];
>> det(A)

ans =

5*a
```

so that $\det A = 5a$. It is clear that A has full rank if $a \neq 0$. When $a = 0$, we can see using the `MATLAB` console that

```
>> A = [2 1 3 3; 1 0 -1 0; 0 2 1 1; 4 3 2 4];
>> rank(A)

ans =

     3.
```

This is because the Reduced Row Echelon form of the matrix has 3 leading entries as `MATLAB` can verify:

```
>> A = [2 1 3 3; 1 0 -1 0; 0 2 1 1; 4 3 2 4];
>> rref(A)

ans =

    1.0000         0         0    0.5556
         0    1.0000         0    0.2222
         0         0    1.0000    0.5556
         0         0         0         0
```

Therefore,

$$\operatorname{rank}(A) = \begin{cases} 4 & a \neq 0 \\ 3 & a = 0 \end{cases}$$

□

Problem 3. Let $A = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$. Define $B = A - 2I$.

1. Compute by hand B^k for any $k \in \mathbb{N}$.
2. Find A^n for $n \in \mathbb{N}$.
3. Compute $\left(I + \frac{B}{2}\right)\left(I - \frac{B}{2} + \frac{B^2}{4}\right)$
4. Find A^{-n} for $n \in \mathbb{N}$.

Solution. 1. Note that $B = A - 2I = \begin{bmatrix} 0 & 4 & 6 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ is a nilpotent matrix since it is an upper triangular matrix where each entry along the diagonal is a 0. Thus, $B^k = 0$ for $k \geq 3$ and we need only compute B^2 to find all B^k for $k \in \mathbb{N}$. It is easy to see that

$$B^2 = \begin{bmatrix} 0 & 4 & 6 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 4 & 6 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 12 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and we are done.

2. It is clear that $A = B + 2I$, so that $A^n = (B + 2I)^n$. Since B and $2I$ commute, we can use the binomial theorem to find $(B + 2I)^n$. Thus,

$$\begin{aligned} A^n &= (B + 2I)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} B^{n-k}(2I)^k \\ &= \sum_{k=0}^{n} 2^k \binom{n}{k} B^{n-k}, \end{aligned}$$

where we take special note that $B^{n-k} = 0$ if $n - k \geq 3$

3. It is easy to see that

$$\left(I + \frac{B}{2}\right)\left(I - \frac{B}{2} + \frac{B^2}{4}\right) = \left(I - \frac{B}{2} + \frac{B^2}{4}\right)\left(I + \frac{B}{2}\right)$$

and

$$\left(I + \frac{B}{2}\right)\left(I - \frac{B}{2} + \frac{B^2}{4}\right) = \left(I - \frac{B}{2} + \frac{B^2}{4} + \frac{B}{2} - \frac{B^2}{4} + \frac{B^3}{8}\right) = I,$$

since $B^3 = 0$. Thus, $\left(I + \frac{B}{2}\right)^{-1} = \left(I - \frac{B}{2} + \frac{B^2}{4}\right)$. Since $\frac{1}{2}A = \left(I + \frac{B}{2}\right)$, $2A^{-1} = \left(I - \frac{B}{2} + \frac{B^2}{4}\right)$. Therefore,

$$\begin{aligned} A^{-1} &= \left(\frac{I}{2} - \frac{B}{4} + \frac{B^2}{8}\right) \\ &= \begin{bmatrix} 1/2 & -1 & -3/4 \\ 0 & 1/2 & -3/4 \\ 0 & 0 & 1/2 \end{bmatrix}. \end{aligned}$$

4. Note that $A^{-n} = (A^{-1})^n$. Using the same technique above, define

$$C = A^{-1} - \frac{1}{2}I$$

Then C is a nilpotent matrix and due to the reasons above, we need only calculate C^2, where

$$\begin{aligned} C^2 &= \begin{bmatrix} 0 & -1 & -3/4 \\ 0 & 0 & -3/4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & -3/4 \\ 0 & 0 & -3/4 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 3/4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Since $A^{-1} = (C + \frac{1}{2}I)$ and we know that C and $\frac{1}{2}I$ commute,

$$\begin{aligned}
(A^{-1})^n &= \left(C - \frac{1}{2}I\right)^n \\
&= \sum_{k=0}^{n} \binom{n}{k} C^{n-k} \left(-\frac{1}{2}I\right)^k \\
&= \sum_{k=0}^{n} \frac{(-1)^k}{2^k} \binom{n}{k} C^{n-k},
\end{aligned}$$

where we take special note that $C^{n-k} = 0$ if $n - k \geq 3$.

$\square$