Homework Assignment 4

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Problem 1. Find the dual of the following linear programs:

a. Maximize $f(\boldsymbol{x}) = \boldsymbol{c}^\mathsf{T}\boldsymbol{x}$ subject to $A\boldsymbol{x} = \boldsymbol{b}$.

b. Maximize $2x_1 + 5x_2 + x_3$ subject to $\begin{cases} 2x_1 - x_2 + 7x_3 \leq 6 \\ x_1 + 3x_2 + 4x_3 \leq 9 \\ 3x_1 + 6x_2 + x_3 \leq 3 \\ x_1, x_2, x_3 \geq 0. \end{cases}$ via the symmetric form of duality.

Solution.   a. Note that for this problem, the variable $\boldsymbol{x}$ is unconstrained in sign. After making the substitution $\boldsymbol{x} = \boldsymbol{x_1} - \boldsymbol{x_2}$ with $\boldsymbol{x_1}, \boldsymbol{x_2} \geq \boldsymbol{0}$, this problem in standard form is then stated as

$$\begin{aligned} \text{minimize} \quad & -\boldsymbol{c}^\mathsf{T}(\boldsymbol{x_1} - \boldsymbol{x_2}) \\ \text{subject to} \quad & A(\boldsymbol{x_1} - \boldsymbol{x_2}) = \boldsymbol{b} \\ & \boldsymbol{x_1}, \boldsymbol{x_2} \geq \boldsymbol{0}. \end{aligned}$$

The realization that the equality $A(\boldsymbol{x_1} - \boldsymbol{x_2}) = \boldsymbol{b}$ can be represented as the system of inequalities

$$\begin{aligned} A(\boldsymbol{x_1} - \boldsymbol{x_2}) &\geq \boldsymbol{b} \\ -A(\boldsymbol{x_1} - \boldsymbol{x_2}) &\geq -\boldsymbol{b} \end{aligned}$$

yields that the standard form of the LP is equivalent to:

$$\begin{aligned} \text{minimize} \quad & -\boldsymbol{c}^\mathsf{T}\boldsymbol{x_1} + \boldsymbol{c}^\mathsf{T}\boldsymbol{x_2} \\ \text{subject to} \quad & A\boldsymbol{x_1} - A\boldsymbol{x_2} \geq \boldsymbol{b} \\ & -A\boldsymbol{x_1} + A\boldsymbol{x_2} \geq -\boldsymbol{b} \\ & \boldsymbol{x_1}, \boldsymbol{x_2} \geq \boldsymbol{0}. \end{aligned}$$

But this can be stated as

$$\text{minimize} \quad \begin{bmatrix} -\boldsymbol{c} \\ \boldsymbol{c} \end{bmatrix}^\mathsf{T} \begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \end{bmatrix}$$

$$\text{subject to} \quad \begin{bmatrix} A & -A \\ -A & A \end{bmatrix} \begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \end{bmatrix} \geq \begin{bmatrix} \boldsymbol{b} \\ -\boldsymbol{b} \end{bmatrix}$$

$$\boldsymbol{x_1}, \boldsymbol{x_2} \geq \boldsymbol{0}$$

or, more succinctly,

$$\begin{aligned}
\text{minimize} \quad & \boldsymbol{C}^\mathsf{T}\boldsymbol{X} \\
\text{subject to} \quad & \mathscr{A}\boldsymbol{X} \geq \boldsymbol{B} \\
& \boldsymbol{X} \geq \boldsymbol{0}
\end{aligned} \tag{1}$$

where

$$\mathscr{A} = \begin{bmatrix} A & -A \\ -A & A \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} -\boldsymbol{c} \\ \boldsymbol{c} \end{bmatrix}, \quad \boldsymbol{X} = \begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{b} \\ -\boldsymbol{b} \end{bmatrix}. \tag{2}$$

By definition, the dual of the primal problem (1) is

$$\begin{aligned}
\text{maximize} \quad & \boldsymbol{B}^\mathsf{T}\boldsymbol{\Lambda} \\
\text{subject to} \quad & \mathscr{A}^\mathsf{T}\boldsymbol{\Lambda} \leq \boldsymbol{C} \\
& \boldsymbol{\Lambda}^\mathsf{T} = \begin{bmatrix} \boldsymbol{\lambda_1}^\mathsf{T} \boldsymbol{\lambda_2}^\mathsf{T} \end{bmatrix} \geq \boldsymbol{0}^\mathsf{T}.
\end{aligned} \tag{3}$$

Using the corresponding definitions found in (2), we see that after some algebraic manipulation the dual problem (6) can be written as

$$\begin{aligned}
\text{maximize} \quad & \boldsymbol{b}^\mathsf{T}(\boldsymbol{\lambda_1} - \boldsymbol{\lambda_2}) \\
\text{subject to} \quad & A^\mathsf{T}(\boldsymbol{\lambda_1} - \boldsymbol{\lambda_2}) \leq -\boldsymbol{c} \\
& A^\mathsf{T}(\boldsymbol{\lambda_1} - \boldsymbol{\lambda_1}) \geq -\boldsymbol{c} \\
& \boldsymbol{\lambda_1}, \boldsymbol{\lambda_2} \geq \boldsymbol{0}.
\end{aligned}$$

Noting that the system of inequalities can be written as an equality and making the substitution $\boldsymbol{\lambda} = (\boldsymbol{\lambda_1} - \boldsymbol{\lambda_2})$ where $\boldsymbol{\lambda}$ is free, we see that the dual of the problem

$$\begin{aligned}
\text{maximize} \quad & \boldsymbol{c}^\mathsf{T}\boldsymbol{x} \\
\text{subject to} \quad & A\boldsymbol{x} = \boldsymbol{b}
\end{aligned}$$

is

$$\begin{aligned}
\text{minimize} \quad & -\boldsymbol{b}^\mathsf{T}\boldsymbol{\lambda} \\
\text{subject to} \quad & A^\mathsf{T}\boldsymbol{\lambda} = -\boldsymbol{c}.
\end{aligned}$$

b. Note that the linear program

$$\begin{aligned}
\text{maximize} \quad & 2x_1 + 5x_2 + x_3 \\
\text{subject to} \quad & 2x_1 - x_2 + 7x_3 \leq 6 \\
& x_1 + 3x_2 + 4x_3 \leq 9 \\
& 3x_1 + 6x_2 + x_3 \leq 3 \\
& x_1, x_2, x_3 \geq 0.
\end{aligned} \tag{4}$$

can be written as

$$\begin{aligned}
\text{maximize} \quad & \boldsymbol{c}^\mathsf{T}\boldsymbol{x} \\
\text{subject to} \quad & A\boldsymbol{x} \leq \boldsymbol{b} \\
& \boldsymbol{x} \geq \boldsymbol{0}
\end{aligned}$$

where

$$A = \begin{bmatrix} 2 & -1 & 7 \\ 1 & 3 & 4 \\ 3 & 6 & 1 \end{bmatrix}, \quad \boldsymbol{c} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 6 \\ 9 \\ 3 \end{bmatrix}.$$

Some algebraic manipulations allows us to write the above problem as

$$\begin{aligned} \text{minimize} \quad & -\boldsymbol{c}^\mathsf{T}\boldsymbol{x} \\ \text{subject to} \quad & -A\boldsymbol{x} \geq -\boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned} \tag{5}$$

By definition, the symmetric dual to the primal problem (5) is

$$\begin{aligned} \text{maximize} \quad & -\boldsymbol{b}^\mathsf{T}\boldsymbol{\lambda} \\ \text{subject to} \quad & -A^\mathsf{T}\boldsymbol{\lambda} \leq -\boldsymbol{c} \\ & \boldsymbol{\lambda} = [\lambda_1, \lambda_2, \lambda_3]^\mathsf{T} \geq \boldsymbol{0}. \end{aligned}$$

Therefore, the dual to the primal problem (5) can be written as

$$\begin{aligned} \text{maximize} \quad & -6\lambda_1 - 9\lambda_2 - 3\lambda_3 \\ \text{subject to} \quad & -2\lambda_1 - \lambda_2 - 3\lambda_3 \leq -2 \\ & \lambda_1 - 3\lambda_2 - 6\lambda_3 \leq -5 \\ & -7\lambda_1 - 4\lambda_2 - \lambda_3 \leq -1 \\ & \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{aligned}$$

and the dual to the original primal problem (4) is

$$\begin{aligned} \text{minimize} \quad & 6\lambda_1 + 9\lambda_2 + 3\lambda_3 \\ \text{subject to} \quad & 2\lambda_1 + \lambda_2 + 3\lambda_3 \geq 2 \\ & -\lambda_1 + 3\lambda_2 + 6\lambda_3 \geq 5 \\ & 7\lambda_1 + 4\lambda_2 + \lambda_3 \geq 1 \\ & \lambda_1, \lambda_2, \lambda_3 \geq 0. \end{aligned}$$

$\square$

Problem 2.   a. Prove (via the symmetric form of duality) that the dual of the dual problem in an asymmetric form of duality is the primal (standard) problem.

   b. Prove the weak duality proposition for the symmetric form of duality.

   c. Prove that the primal problem is infeasible if and only if the dual problem is unbounded.

Solution.   a. Suppose that we have the following primal problem in standard form

$$(P_a) \quad \begin{aligned} \text{minimize} \quad & \boldsymbol{c}^\mathsf{T} \boldsymbol{x} \\ \text{subject to} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned}$$

The dual of this problem via an asymmetric form of duality is given by

$$(D_a) \quad \begin{aligned} \text{maximize} \quad & \boldsymbol{b}^\mathsf{T} \boldsymbol{\lambda} \\ \text{subject to} \quad & A^\mathsf{T} \boldsymbol{\lambda} \leq \boldsymbol{c}. \end{aligned}$$

where $\boldsymbol{\lambda}$ is free. Making the substitution $\boldsymbol{\lambda} = \boldsymbol{\lambda_1} - \boldsymbol{\lambda_2}$ and performing some algebraic manipulation, we can transform this dual problem into the following equivalent form

$$(D_a) \quad \begin{aligned} \text{minimize} \quad & -\boldsymbol{b}^\mathsf{T} (\boldsymbol{\lambda_1} - \boldsymbol{\lambda_2}) \\ \text{subject to} \quad & -A^\mathsf{T} (\boldsymbol{\lambda_1} - \boldsymbol{\lambda_2}) \geq -\boldsymbol{c} \\ & \boldsymbol{\lambda_1}, \boldsymbol{\lambda_2} \geq \boldsymbol{0}. \end{aligned}$$

From some more algebraic manipulation, we rewrite the dual problem into the following form

$$(D_a) \quad \begin{aligned} \text{minimize} \quad & \boldsymbol{B}^\mathsf{T} \boldsymbol{\Lambda} \\ \text{subject to} \quad & \mathscr{A}^\mathsf{T} \boldsymbol{\Lambda} \geq -\boldsymbol{c} \\ & \boldsymbol{\Lambda} \geq \boldsymbol{0} \end{aligned} \qquad (6)$$

where

$$\mathscr{A} = \begin{bmatrix} -A \\ A \end{bmatrix}, \quad \boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{\lambda_1} \\ \boldsymbol{\lambda_2} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} -\boldsymbol{b} \\ \boldsymbol{b} \end{bmatrix}.$$

Therefore, by definition, the dual via the symmetric form of duality of the dual problem (6) is

$$(P_s) \quad \begin{aligned} \text{maximize} \quad & -\boldsymbol{c}^\mathsf{T} \boldsymbol{x} \\ \text{subject to} \quad & \mathscr{A} \boldsymbol{x} \leq \boldsymbol{B} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

or equivalently

$$(P_s) \quad \begin{aligned} \text{maximize} \quad & -\boldsymbol{c}^\mathsf{T} \boldsymbol{x} \\ \text{subject to} \quad & -A\boldsymbol{x} \leq -\boldsymbol{b} \\ & A\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned} \qquad (7)$$

By rewriting the objective from *maximize* to *minimize* and combining the inequalities into one equality in (7), we see that the dual via the symmetric form of duality of the dual in (6) is

$$(P_s) \quad \begin{array}{ll} \text{minimize} & \boldsymbol{c}^\mathsf{T}\boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{array}$$

which is equivalent to the standard form of the primal problem (P_a) and we are done.

b. Suppose that we have the following primal problem in symmetric form and dual problem obtained in a symmetric form:

$$(P_s) \quad \begin{array}{ll} \text{minimize} & \boldsymbol{c}^\mathsf{T}\boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \quad \Longrightarrow \quad (D_s) \quad \begin{array}{ll} \text{maximize} & \boldsymbol{\lambda}^\mathsf{T}\boldsymbol{b} \\ \text{subject to} & \boldsymbol{\lambda}^\mathsf{T}A \leq \boldsymbol{c}^\mathsf{T} \\ & \boldsymbol{\lambda} \geq \boldsymbol{0}. \end{array}$$

We wish to show that if $\boldsymbol{x}$ and $\boldsymbol{\lambda}$ are feasible solutions to the primal and dual problem, respectively, then $\boldsymbol{\lambda}^\mathsf{T}\boldsymbol{b} \leq \boldsymbol{c}^\mathsf{T}\boldsymbol{x}$.

Since $\boldsymbol{x}$ is a feasible solution to the primal problem, we know that

$$\boldsymbol{\lambda}^\mathsf{T}\boldsymbol{b} \leq \boldsymbol{\lambda}^\mathsf{T}(A\boldsymbol{x}).$$

We also know, for $\boldsymbol{x} \geq \boldsymbol{0}$, that since $\boldsymbol{\lambda}$ is a feasible solution to the dual problem,

$$\boldsymbol{\lambda}^\mathsf{T}A \leq \boldsymbol{c}^\mathsf{T} \implies \boldsymbol{\lambda}^\mathsf{T}A\boldsymbol{x} \leq \boldsymbol{c}^\mathsf{T}\boldsymbol{x}.$$

Therefore, by combining the two obtained inequalities, we have that

$$\boldsymbol{\lambda}^\mathsf{T}\boldsymbol{b} \leq \boldsymbol{\lambda}^\mathsf{T}(A\boldsymbol{x}) \leq \boldsymbol{c}^\mathsf{T}\boldsymbol{x}$$

proving the weak duality proposition.

c. We wish to show that the primal problem is infeasible if and only if the dual problem is unbounded. This is equivalent to showing that the dual problem is bounded if and only if the primal problem is feasible. The primal problem in symmetric form and the dual problem obtained in a symmetric form are presented below:

$$(P_s) \quad \begin{array}{ll} \text{minimize} & \boldsymbol{c}^\mathsf{T}\boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array} \quad \Longrightarrow \quad (D_s) \quad \begin{array}{ll} \text{maximize} & \boldsymbol{\lambda}^\mathsf{T}\boldsymbol{b} \\ \text{subject to} & \boldsymbol{\lambda}^\mathsf{T}A \leq \boldsymbol{c}^\mathsf{T} \\ & \boldsymbol{\lambda} \geq \boldsymbol{0}. \end{array}$$

Suppose first that the primal problem is feasible. Then there exists $\boldsymbol{x_0} \geq \boldsymbol{0}$ with $A\boldsymbol{x_0} \geq \boldsymbol{b}$. By the Weak Duality proposition, we know that for every feasible $\boldsymbol{\lambda}$ that $\boldsymbol{\lambda}^\mathsf{T}\boldsymbol{b} \leq \boldsymbol{c}^\mathsf{T}\boldsymbol{x_0}$. Thus, $\max\{\boldsymbol{\lambda}^\mathsf{T}\boldsymbol{b} \mid \boldsymbol{\lambda}^\mathsf{T}A \leq \boldsymbol{c}^\mathsf{T}, \boldsymbol{\lambda} \geq \boldsymbol{0}\} \leq \boldsymbol{c}^\mathsf{T}\boldsymbol{x_0}$ and the dual problem is bounded above.

Now suppose that the dual problem is bounded above. Then the dual problem is feasible and has a feasible solution. As such, the dual problem has a basic feasible solution.

The simplex method will then either generate an optimal basic feasible solution or terminate because the problem is unbounded. As the second case can never happen, this implies that the dual problem has an optimal basic feasible solution. By the Duality Theorem, the primal problem must also have an optimal solution implying that the primal problem is feasible. Therefore, if the dual problem is bounded, the primal problem is feasible and we are done.

□

Problem 3. Prove the Duality Theorem for the symmetric case.

Solution. The Duality Theorem is presented below:

Theorem 1 (***Duality Theorem***). If the primal problem, in symmetric form, has an optimal solution, then so does the dual, and the optimal values of their respective objective functions are equal.

Proof. We define the symmetric form for the primal problem below:

$$(P_s) \quad \begin{aligned} \text{minimize} \quad & \boldsymbol{c}^\mathsf{T}\boldsymbol{x} \\ \text{subject to} \quad & A\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

Note that we can transform the symmetric problem to an equivalent asymmetric form by adding excess variables in the constraints:

$$(P_a) \quad \begin{aligned} \text{minimize} \quad & \boldsymbol{C}^\mathsf{T}\boldsymbol{X} \\ \text{subject to} \quad & \mathscr{A}\boldsymbol{X} = \boldsymbol{b} \\ & \boldsymbol{X} \geq \boldsymbol{0} \end{aligned}$$

where

$$\mathscr{A} = \begin{bmatrix} A^\mathsf{T} \\ -I \end{bmatrix}^\mathsf{T}, \quad \boldsymbol{X} = \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{0} \end{bmatrix}.$$

We wish to show that $\boldsymbol{x}$ is optimal in (P_s) if and only if $\boldsymbol{X} = \begin{bmatrix} \boldsymbol{x} \\ A\boldsymbol{x} - \boldsymbol{b} \end{bmatrix}$ is optimal in (P_a).

Suppose first that $\boldsymbol{x}$ is optimal in (P_s). We begin by showing that $\boldsymbol{X}$ is feasible, i.e. $\boldsymbol{X} \geq \boldsymbol{0}$. Since $\boldsymbol{x}$ is a feasible solution of (P_s), we have that $\boldsymbol{x} \geq \boldsymbol{0}$ and since $\boldsymbol{x}$ satisfies $A\boldsymbol{x} \geq \boldsymbol{b}$, we have that $A\boldsymbol{x} - \boldsymbol{b} \geq \boldsymbol{0}$ showing indeed that $\boldsymbol{X} \geq \boldsymbol{0}$. Further, $\boldsymbol{X}$ is a feasible solution since

$$\mathscr{A}\boldsymbol{X} = \begin{bmatrix} A & -I \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ A\boldsymbol{x} - \boldsymbol{b} \end{bmatrix} = A\boldsymbol{x} - (A\boldsymbol{x} - \boldsymbol{b}) = \boldsymbol{b}.$$

Lastly, since $\boldsymbol{x}$ is optimal in (P_s), we know that $\boldsymbol{X}$ is optimal in (P_a) since

$$\boldsymbol{C}^\mathsf{T}\boldsymbol{X} = \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{0} \end{bmatrix}^\mathsf{T} \begin{bmatrix} \boldsymbol{x} \\ A\boldsymbol{x} - \boldsymbol{b} \end{bmatrix} = \boldsymbol{c}^\mathsf{T}\boldsymbol{x}, \tag{8}$$

i.e. the minimum for both problems is the same.

The other direction is trivial since if $\boldsymbol{X}$ is an optimal solution to (P_a), then $\boldsymbol{x}$ is clearly a feasible solution to (P_s) since $\boldsymbol{X} \geq \boldsymbol{0}$ and from equation (8), the objectives are equal so the problems obtain their optimum at the same point $\boldsymbol{x}$.

Also note that the dual of (P_s) is equivalent to the dual of (P_a). Note that the dual of the primal in the symmetric form and asymmetric form are presented below:

$$(D_s) \quad \begin{aligned} \text{maximize} \quad & \boldsymbol{\lambda}^\mathsf{T}\boldsymbol{b} \\ \text{subject to} \quad & \boldsymbol{\lambda}^\mathsf{T}A \leq \boldsymbol{c}^\mathsf{T} \\ & \boldsymbol{\lambda} \geq \boldsymbol{0}. \end{aligned} \qquad\qquad (D_a) \quad \begin{aligned} \text{maximize} \quad & \boldsymbol{\lambda}^\mathsf{T}\boldsymbol{b} \\ \text{subject to} \quad & \boldsymbol{\lambda}^\mathsf{T}\mathscr{A} \leq \boldsymbol{C}^\mathsf{T} \end{aligned}$$

To see that these two problems are equivalent, one merely needs to examine the constraint $\boldsymbol{\lambda}^\mathsf{T}\mathscr{A} \leq \boldsymbol{C}^\mathsf{T}$. Note that

$$\boldsymbol{\lambda}^\mathsf{T}\mathscr{A} \leq \boldsymbol{C}^\mathsf{T} \implies \begin{bmatrix} \boldsymbol{\lambda}^\mathsf{T}A & -\boldsymbol{\lambda}^\mathsf{T} \end{bmatrix} \leq \begin{bmatrix} \boldsymbol{c}^\mathsf{T} & \boldsymbol{0}^\mathsf{T} \end{bmatrix}$$

from which we deduce the two inequalities $\boldsymbol{\lambda}^\mathsf{T}A \leq \boldsymbol{c}^\mathsf{T}$ and $\boldsymbol{\lambda} \geq \boldsymbol{0}$, precisely the inequalities found in (D_s).

As was shown in class the theorem's results hold in the asymmetric case. As was just shown, the symmetric case can be transformed to its equivalent asymmetric case and therefore the theorem's results hold as well in the symmetric case. $\square$

Problem 4. Consider the following linear program:

$$
\begin{aligned}
\text{maximize} \quad & 2x_1 + 3x_2 \\
\text{subject to} \quad & x_1 + 2x_2 \le 4 \\
& 2x_1 + x_2 \le 5 \\
& x_1, x_2 \ge 0.
\end{aligned}
$$

a. Use the simplex method to solve the problem.

b. Write down the dual of the linear program and solve the dual.

Solution.   a. To begin, we transform the linear program into standard form

$$
\begin{aligned}
\text{minimize} \quad & -2x_1 \quad -3x_2 \\
\text{subject to} \quad & x_1 \quad +2x_2 \quad +x_3 \qquad\qquad = 4 \\
& 2x_1 \quad +x_2 \qquad\qquad +x_4 \quad = 5 \\
& x_1, \quad x_2, \quad x_3, \quad x_4 \ge 0
\end{aligned}
$$

The initial tableau associated to this problem is then:

	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{a_4}$	$\boldsymbol{b}$
	1	2	1	0	4
	2	1	0	1	5
$\boldsymbol{c}^\mathsf{T}$	-2	-3	0	0	0

Note that this tableau is in canonical form with respect to the basis $[\boldsymbol{a_3}, \boldsymbol{a_4}]$. Thus, the last row of the tableau contains the reduced cost coefficients. We proceed according to the simplex method by choosing a row index p and column index q to pivot around until the reduced cost coefficients are nonnegative.

The most negative reduced cost coefficient occurs in column $q = 2$. The smallest ratio between the right hand side and the positive coefficients of the q-th column is $4/2$ so we pivot around row $p = 1$. Thus, $\boldsymbol{a_2}$ enters the basis, $\boldsymbol{a_4}$ leaves the basis and we move from this tableau to the updated tableau below:

	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{a_4}$	$\boldsymbol{b}$
	1	②	1	0	4
	2	1	0	1	5
$\boldsymbol{c}^\mathsf{T}$	-2	-3	0	0	0

$\xrightarrow[{[2]-(1/2)[1]}]{(1/2)[1]}$
$\xrightarrow[{[3]+(3/2)[1]}]{}$

	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{a_4}$	$\boldsymbol{b}$
	1/2	1	1/2	0	2
	3/2	0	$-1/2$	1	3
$\boldsymbol{c}^\mathsf{T}$	$-1/2$	0	3/2	0	6

From this new tableau, the only negative reduced cost coefficient occurs in column $q = 1$. The smallest ratio between the right hand side and the positive coefficients of the q-th column is $3/(3/2)$ so we pivot around row $p = 2$. Thus, $\boldsymbol{a_1}$ enters the basis, $\boldsymbol{a_4}$ leaves the basis and we move from the tableau above to the updated tableau below:

	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{a_4}$	$\boldsymbol{b}$
	1/2	1	1/2	0	2
	③/2	0	$-1/2$	1	3
$\boldsymbol{c}^\mathsf{T}$	$-1/2$	0	3/2	0	6

$\xrightarrow[{[1]-(1/3)[2]}]{(2/3)[2]}$
$\xrightarrow[{[3]+(1/3)[2]}]{}$

	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{a_4}$	$\boldsymbol{b}$
	0	1	2/3	$-1/3$	1
	1	0	$-1/3$	2/3	2
$\boldsymbol{c}^\mathsf{T}$	0	0	4/3	1/3	7

As there are no negative reduced cost coefficients in the updated tableau, the current basic feasible solution is optimal, i.e. $\boldsymbol{x} = [2, 1, 0, 0]^{\mathsf{T}}$ is optimal. Therefore, the optimal solution to the original problem is $x_1 = 2$, $x_2 = 1$ with objective function value -7.

b. Note that the original linear program can be converted to the following primal problem in symmetric form:

$$
\begin{array}{lll}
\text{maximize} & 2x_1 + 3x_2 \\
\text{subject to} & x_1 + 2x_2 \le 4 \\
& 2x_1 + x_2 \le 5 \\
& x_1, x_2 \ge 0.
\end{array}
\qquad \Longrightarrow \qquad
\begin{array}{lll}
(P_s) & \text{minimize} & -2x_1 - 3x_2 \\
& \text{subject to} & -x_1 - 2x_2 \ge -4 \\
& & -2x_1 - x_2 \ge -5 \\
& & x_1, x_2 \ge 0.
\end{array}
$$

The primal problem in symmetric form can be written as

$$
\begin{array}{lll}
(P_s) & \text{minimize} & \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} \\
& \text{subject to} & A\boldsymbol{x} \ge \boldsymbol{b} \\
& & \boldsymbol{x} \ge \boldsymbol{0}
\end{array}
$$

where

$$
A = \begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \boldsymbol{c} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} -4 \\ -5 \end{bmatrix}.
$$

By definition, the dual to the primal problem (P_s) in symmetric form is

$$
\begin{array}{lll}
(D_s) & \text{maximize} & \boldsymbol{\lambda}^{\mathsf{T}}\boldsymbol{b} \\
& \text{subject to} & \boldsymbol{\lambda}^{\mathsf{T}}A \ge \boldsymbol{c}^{\mathsf{T}} \\
& & \boldsymbol{\lambda} \ge \boldsymbol{0}.
\end{array}
$$

By the Complementary Slackness Condition, we know that the optimal solutions to the primal and dual satisfy the two equations

$$
(\boldsymbol{c}^{\mathsf{T}} - \boldsymbol{\lambda}^{*\mathsf{T}}A)\boldsymbol{x}^* = 0
$$
$$
\boldsymbol{\lambda}^{*\mathsf{T}}(A\boldsymbol{x}^* - \boldsymbol{b}) = 0.
$$

As $\boldsymbol{x}^* = [2, 1]^{\mathsf{T}}$ is optimal for the primal problem (P_s) with $A\boldsymbol{x}^* - \boldsymbol{b} = \boldsymbol{0}$ and for each i, the component $x_i^* \ne 0$, we know that the second equation is satisfied and the first equation is satisfied if and only if $\boldsymbol{c}^{\mathsf{T}} - \boldsymbol{\lambda}^{*\mathsf{T}}A = 0$. Thus, $\boldsymbol{c}^{\mathsf{T}} = \boldsymbol{\lambda}^{*\mathsf{T}}A$ and since rank $A = 2$, the matrix A is invertible so that $\boldsymbol{\lambda}^{*\mathsf{T}} = \boldsymbol{c}^{\mathsf{T}}A^{-1}$. Therefore, by the Complementary Slackness Condition the optimal solution to the dual problem (D_s) is

$$
\boldsymbol{\lambda}^{*\mathsf{T}} = \boldsymbol{c}^{\mathsf{T}}A^{-1} = \begin{bmatrix} -2 & -3 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 1/3 & -2/3 \\ -2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 4/3 & 1/3 \end{bmatrix}^{\mathsf{T}}.
$$

$\square$

Problem 5. Consider the following primal problem:

$$
\begin{array}{llllll}
\text{maximize} & x_1 & +2x_2 & & & \\
\text{subject to} & -2x_1 & +x_2 & +x_3 & & = 2 \\
& -x_1 & +2x_2 & & +x_4 & = 7 \\
& x_1 & & & +x_5 & = 3 \\
& x_i \geq 0 & i = 1, 2, 3, 4, 5.
\end{array}
$$

a. Construct the dual problem corresponding to the primal problem above.

b. It is known that the solution to the primal above is $\boldsymbol{x}^* = [3, 5, 3, 0, 0]^\mathsf{T}$. Find the solution to the dual.

Solution.   a. The linear program found in this problem can be represented as

$$
\begin{array}{llllll}
\text{minimize} & -x_1 & -2x_2 & & & \\
\text{subject to} & -2x_1 & +x_2 & +x_3 & & = 2 \\
& -x_1 & +2x_2 & & +x_4 & = 7 \\
& x_1 & & & +x_5 & = 3 \\
& x_i \geq 0 & i = 1, 2, 3, 4, 5.
\end{array}
$$

$$
\implies \qquad
\begin{array}{ll}
(P_a) & \text{minimize} \quad \boldsymbol{c}^\mathsf{T}\boldsymbol{x} \\
& \text{subject to} \quad A\boldsymbol{x} = \boldsymbol{b} \\
& \qquad\qquad\quad \boldsymbol{x} \geq \boldsymbol{0}
\end{array}
$$

where

$$
A = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad
\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad
\boldsymbol{c} = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad
\boldsymbol{b} = \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix}.
$$

By definition, the dual to the primal problem (P_a) in asymmetric form is

$$
\begin{array}{ll}
(D_a) & \text{maximize} \quad \boldsymbol{\lambda}^\mathsf{T}\boldsymbol{b} \\
& \text{subject to} \quad \boldsymbol{\lambda}^\mathsf{T}A \leq \boldsymbol{c}
\end{array}
$$

b. By the Complementary Slackness Condition, we know that the optimal solutions to the primal and dual satisfy the two equations

$$
(\boldsymbol{c}^\mathsf{T} - \boldsymbol{\lambda}^{*\mathsf{T}}A)\boldsymbol{x}^* = 0
$$
$$
\boldsymbol{\lambda}^{*\mathsf{T}}(A\boldsymbol{x}^* - \boldsymbol{b}) = 0.
$$

As $\boldsymbol{x}^* = [3, 5, 3, 0, 0]^\mathsf{T}$ is optimal for the primal problem (P_a) with $A\boldsymbol{x}^* - \boldsymbol{b} = \boldsymbol{0}$ we know that the second equation is satisfied. Note that the first equation is expressed as

$$
\sum_{i=1}^{5}(c_i - \boldsymbol{\lambda}^{*\mathsf{T}}\boldsymbol{a}_i)x_i^* = 0
$$

where $\boldsymbol{a_i}$ is the i-th column of the matrix A. Since $c_i - \boldsymbol{\lambda^{*\mathsf{T}}} \boldsymbol{a_i} \geq 0$ and $\boldsymbol{x} \geq \boldsymbol{0}$, we know that the sum is 0 if and only if each summand is 0. Thus, omitting summands where $x_i = 0$, as the product will be 0, admits a linear system in $\boldsymbol{\lambda^{*\mathsf{T}}}$ written as

$$\begin{cases} \boldsymbol{\lambda^{*\mathsf{T}}} \boldsymbol{a_1} = c_1 \\ \boldsymbol{\lambda^{*\mathsf{T}}} \boldsymbol{a_2} = c_2 \\ \boldsymbol{\lambda^{*\mathsf{T}}} \boldsymbol{a_3} = c_3 \end{cases} \implies \begin{bmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} \end{bmatrix}^{\mathsf{T}} \boldsymbol{\lambda^*} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Thus, solving the above linear system will lead us to the optimal solution to the dual. Rewriting the linear system in terms of our problem yields

$$\begin{bmatrix} -2 & 1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{\mathsf{T}} \boldsymbol{\lambda^*} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} \implies \mathscr{A}^{\mathsf{T}} \boldsymbol{\lambda^*} = \boldsymbol{C}.$$

Since $\mathscr{A}$ has full rank, it is invertible and the solution to the system is

$$\boldsymbol{\lambda^*} = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}.$$

Therefore, $\boldsymbol{\lambda^{*\mathsf{T}}} = \begin{bmatrix} 0 & -1 & -2 \end{bmatrix}$ is the optimal solution to the dual problem (D_a).

$\square$

Problem 6. Let A be a given matrix and $\boldsymbol{b}$ a given vector. We wish to prove the following result: There exists a vector $\boldsymbol{x}$ such that $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq \boldsymbol{0}$ if and only if for any given vector $\boldsymbol{y}$ satisfying $A^\mathsf{T}\boldsymbol{y} \leq \boldsymbol{0}$ we have $\boldsymbol{b}^\mathsf{T}\boldsymbol{y} \leq 0$. This result is known as *Farkas's transposition theorem*. Our program is based on duality theory, consisting of the parts listed below.

  a. Consider the primal linear program

$$(P_a) \quad \begin{aligned} &\text{minimize} \quad \boldsymbol{0}^\mathsf{T}\boldsymbol{x} \\ &\text{subject to} \quad A\boldsymbol{x} = \boldsymbol{b} \\ &\phantom{\text{subject to} \quad} \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned}$$

Write down the dual of this problem using the notation $\boldsymbol{y}$ for the dual variable.

  b. Show that the feasible set of the dual problem is guaranteed to be nonempty.

  Hint: Think about an obvious feasible point.

  c. Suppose that for any $\boldsymbol{y}$ satisfying $A^\mathsf{T}\boldsymbol{y} \leq \boldsymbol{0}$, we have $\boldsymbol{b}^\mathsf{T}\boldsymbol{y} \leq 0$. In this case what can you say about whether or not the dual has an optimal feasible solution.

  Hint: Think about the obvious feasible point in part b.

  d. Suppose that for any $\boldsymbol{y}$ satisfying $A^\mathsf{T}\boldsymbol{y} \leq \boldsymbol{0}$, we have $\boldsymbol{b}^\mathsf{T}\boldsymbol{y} \leq 0$. Use parts b and c to show that there exists $\boldsymbol{x}$ such that $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq \boldsymbol{0}$. (This proves one direction of Farkas's transposition theorem.)

  e. Suppose that $\boldsymbol{x}$ satisfies $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq \boldsymbol{0}$. Let $\boldsymbol{y}$ be an arbitrary vector satisfying $A^\mathsf{T}\boldsymbol{y} \leq \boldsymbol{0}$. Show that $\boldsymbol{b}^\mathsf{T}\boldsymbol{y} \leq 0$. (This proves the other direction of Farkas's transposition theorem.)

Solution.   a. The dual to the primal problem (P_a) in asymmetric form is presented below:

$$(D_a) \quad \begin{aligned} &\text{maximize} \quad \boldsymbol{b}^\mathsf{T}\boldsymbol{y} \\ &\text{subject to} \quad A^\mathsf{T}\boldsymbol{y} \leq \boldsymbol{0}. \end{aligned}$$

  b. Note that the solution $\boldsymbol{y} = \boldsymbol{0}$ obviously satisfies the constraints of the dual problem (D_a) showing that the feasible set of the dual problem is nonempty.

  c. If for any $\boldsymbol{y}$ satisfying $A^\mathsf{T}\boldsymbol{y} \leq \boldsymbol{0}$, we have that $\boldsymbol{b}^\mathsf{T}\boldsymbol{y} \leq 0$, then the dual problem (D_a) is bounded above. As shown in part b, the dual problem has a feasible solution. By the Fundamental Theorem of LP, the dual problem then has a basic feasible solution. The simplex method will then either generate an optimal basic feasible solution for this problem or terminate because the problem is unbounded. As the dual problem is bounded, the second case cannot happen and the dual problem has an optimal basic feasible solution.

  d. If for any $\boldsymbol{y}$ satisfying $A^\mathsf{T}\boldsymbol{y} \leq \boldsymbol{0}$, we have that $\boldsymbol{b}^\mathsf{T}\boldsymbol{y} \leq 0$, then the dual problem (D_a) has an optimal solution. By the Duality Theorem, the primal problem (P_a) therefore has an optimal feasible solution. Thus, a feasible solution to the primal problem (P_a) exists, i.e. there exists $\boldsymbol{x}$ such that $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq \boldsymbol{0}$.

e. Suppose that $\boldsymbol{x}$ satisfies $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq \boldsymbol{0}$ and let $\boldsymbol{y}$ be an arbitrary vector satisfying $A^{\mathsf{T}}\boldsymbol{y} \leq \boldsymbol{0}$. Then $\boldsymbol{x}$ and $\boldsymbol{y}$ are feasible solutions to the primal problem (P_a) and the dual problem (D_a), respectively. By the Weak Duality Theorem, we know that

$$\boldsymbol{b}^{\mathsf{T}}\boldsymbol{y} \leq \boldsymbol{0}^{\mathsf{T}}\boldsymbol{x} = 0$$

and we are done.

$\square$