Matrices and Linear Systems Continued

Unit 3

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- Work through Ex. 3.1 (1,2,4), Ex. 3.2, and Ex. 3.4. Do not submit.

The QR method

Based on several theoretical results: Eigenvalues of similar matrices, Schur's factorization, Gauss-Schmidt process.

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QR algorithm produces QR-factorization $A = QR$, where Q is unitary and R is upper triangular.

Gauss-Schmidt process

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where $\quad \mathbf{a_i'} = \mathbf{a_i} - \sum_{k=1}^{i-1} (\mathbf{q_k^T} \mathbf{a_i}) \mathbf{q_k}, \quad i = 1, 2, \ldots, n$

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Entries of R $\quad r_{ii} = ||\mathbf{a_i'}||, \quad r_{ij} = \mathbf{q_i^T} \mathbf{a_j}, \quad i = 1, \ldots, n, \quad j = i+1, \ldots, n$

hw due next meeting

1. Create a random triadiagonal symmetric matrix of size n

2. Write your own single iteration QR code for triadiagonal symmetric matrices.

3. Test running time of your code vs the Matlab qr for $n = 3, \ldots$ as as far as you can go hoping to outrun Matlab.

4. Write you own code to iterate your QR for triadiagonal symmetric matrices. Use several reasonable thresholds for stopping.

5. Test running time of your code vs the Matlab eig for $n = 3, \ldots$ as as far as you can go hoping to outrun Matlab.

Operator norm

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \|Ax\|$$

$$\|AB\| \leq \|A\| \|B\|$$

Note that it requires a vector norm. We use the following

1. $\|x\|_1 = \sum_{j=1}^{n} |x_j|$
2. $\|x\|_2 = \sqrt{\sum_{j=1}^{n} x_j^2}$
3. $\|x\|_\infty = \max_{1 \leq j \leq n} |x_j|$

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What are the subordinate matrix norms?

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Ex. 4.1 Please write up and turn in