Homework Assignment 9

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Problem 2.20. In the innovations algorithm, show that for each $n \geq 2$, the innovation $X_n - \hat{X}_n$ is uncorrelated with $X_1, X_2, \ldots, X_{n-1}$. Conclude that $X_n - \hat{X}_n$ is uncorrelated with the innovations $X_1 - \hat{X}_1, X_2 - \hat{X}_2, \ldots, X_{n-1} - \hat{X}_{n-1}$

Solution. □

Problem 2.21. Let X_1, X_2, X_3, X_4, X_5 be observations from the MA(1) model.

$$X_t = Z_t + \theta Z_{t-1}, \{Z_t\} \sim \mathrm{WN}(0, \sigma^2).$$

a. Find the best linear estimate of the missing value X_3 in terms of X_1 and X_2.

b. Find the best linear estimate of the missing value X_3 in terms of X_4 and X_5.

c. Find the best linear estimate of the missing value X_3 in terms of X_1, X_2, X_4, and X_5.

d. Compute the mean squared errors for each of the estimates in (a), (b), and (c).

Solution. If Y and $W_n, \ldots, W_1$ are random variables, then for $\boldsymbol{W} = (W_n, \ldots, W_1)$ and $\boldsymbol{\mu}_W = (\mathrm{E}(W_n), \ldots, \mathrm{E}(W_1))^\intercal$, the best linear predictor of Y in terms of $\{1, W_n, \ldots, W_1\}$ is

$$P(Y|\boldsymbol{W}) = \mathrm{E}(Y) + \boldsymbol{a}^\intercal(\boldsymbol{W} - \boldsymbol{\mu}_{\boldsymbol{W}})$$

where $\boldsymbol{a}$ is the solution of $\Gamma\boldsymbol{a} = \gamma$ for $\Gamma = \mathrm{Cov}(\boldsymbol{W}, \boldsymbol{W})$ and $\gamma = \mathrm{Cov}(Y, \boldsymbol{W})$.

Also, note for an MA(1) process, the autocovariance function is defined as

$$\gamma_X(h) = \begin{cases} \sigma^2(1+\theta^2) & \text{if } h = 0 \\ \sigma^2\theta & \text{if } |h| = 1 \\ 0 & \text{if } |h| > 1 \end{cases}$$

a. Using the above, set $Y = X_3$ and $W = (X_2, X_1)^\intercal$. Then

$$\Gamma = \mathrm{Cov}(\boldsymbol{W}, \boldsymbol{W}) = \begin{bmatrix} \gamma_X(0) & \gamma_X(1) \\ \gamma_X(1) & \gamma_X(0) \end{bmatrix} = \sigma^2 \begin{bmatrix} 1+\theta^2 & \theta \\ \theta & 1+\theta^2 \end{bmatrix}$$

and

$$\gamma = \begin{bmatrix} \gamma_X(1) \\ \gamma_X(2) \end{bmatrix} = \sigma^2 \begin{bmatrix} \theta \\ 0 \end{bmatrix}.$$

The solution to the system of equations $\Gamma \boldsymbol{a} = \gamma$ is

$$\boldsymbol{a} = \frac{\theta}{1+\theta^2+\theta^4} \begin{bmatrix} 1+\theta^2 \\ -\theta \end{bmatrix}.$$

Therefore, the best predictor of X_3 is

$$\begin{aligned} P(X_3 | \boldsymbol{W}) &= \mathrm{E}(X_3) + \boldsymbol{a}^\intercal (\boldsymbol{W} - \boldsymbol{\mu}_W) \\ &= \frac{\theta}{1+\theta^2+\theta^4}((1+\theta^2)X_2 - \theta X_1) \end{aligned}$$

b. Using the above, set $Y = X_3$ and $W = (X_5, X_4)^\intercal$. Then

$$\Gamma = \mathrm{Cov}(\boldsymbol{W}, \boldsymbol{W}) = \begin{bmatrix} \gamma_X(0) & \gamma_X(1) \\ \gamma_X(1) & \gamma_X(0) \end{bmatrix} = \sigma^2 \begin{bmatrix} 1+\theta^2 & \theta \\ \theta & 1+\theta^2 \end{bmatrix}$$

and

$$\gamma = \begin{bmatrix} \gamma_X(2) \\ \gamma_X(1) \end{bmatrix} = \sigma^2 \begin{bmatrix} 0 \\ \theta \end{bmatrix}.$$

The solution to the system of equations $\Gamma \boldsymbol{a} = \gamma$ is

$$\boldsymbol{a} = \frac{\theta}{1+\theta^2+\theta^4} \begin{bmatrix} -\theta \\ 1+\theta^2 \end{bmatrix}.$$

Therefore, the best predictor of X_3 is

$$\begin{aligned} P(X_3 | \boldsymbol{W}) &= \mathrm{E}(X_3) + \boldsymbol{a}^\intercal (\boldsymbol{W} - \boldsymbol{\mu}_W) \\ &= \frac{\theta}{1+\theta^2+\theta^4}(-\theta X_5 + (1+\theta^2)X_4) \end{aligned}$$

c. Using the above, set $Y = X_3$ and $W = (X_5, X_4, X_2, X_1)^\intercal$. Then

$$\begin{aligned} \Gamma = \mathrm{Cov}(\boldsymbol{W}, \boldsymbol{W}) &= \begin{bmatrix} \gamma_X(0) & \gamma_X(1) & \gamma_X(3) & \gamma_X(4) \\ \gamma_X(1) & \gamma_X(0) & \gamma_X(2) & \gamma_X(3) \\ \gamma_X(3) & \gamma_X(2) & \gamma_X(0) & \gamma_X(1) \\ \gamma_X(4) & \gamma_X(3) & \gamma_X(1) & \gamma_X(0) \end{bmatrix} \\ &= \sigma^2 \begin{bmatrix} 1+\theta^2 & \theta & 0 & 0 \\ \theta & 1+\theta^2 & 0 & 0 \\ 0 & 0 & 1+\theta^2 & \theta \\ 0 & 0 & \theta & 1+\theta^2 \end{bmatrix} \end{aligned}$$

and

$$\gamma = \begin{bmatrix} \gamma_X(2) \\ \gamma_X(1) \\ \gamma_X(1) \\ \gamma_X(2) \end{bmatrix} = \sigma^2 \begin{bmatrix} 0 \\ \theta \\ \theta \\ 0 \end{bmatrix}.$$

The solution to the system of equations $\Gamma \boldsymbol{a} = \gamma$ is

$$\boldsymbol{a} = \frac{\theta}{1+\theta^2+\theta^4}\begin{bmatrix} -\theta \\ 1+\theta^2 \\ 1+\theta^2 \\ -\theta \end{bmatrix}.$$

Therefore, the best predictor of X_3 is

$$\begin{aligned} P(X_3|\boldsymbol{W}) &= \mathrm{E}(X_3) + \boldsymbol{a}^\intercal(\boldsymbol{W} - \boldsymbol{\mu}_W) \\ &= \frac{\theta}{1+\theta^2+\theta^4}(-\theta X_5 + (1+\theta^2)X_4 + (1+\theta^2)X_2 - \theta X_1) \end{aligned}$$

d. The mean squared error of the predictor in terms of the known random variables is $\mathrm{E}\left[(Y - P(Y|\boldsymbol{W}))^2\right] = \mathrm{Var}(Y) - \boldsymbol{a}^\intercal\gamma$.

Therefore, the mean squared error for:

(a) is $\mathrm{E}\left[(X_3 - P(X_3|\boldsymbol{W}))^2\right] = \frac{-\sigma^2\theta^2(1+\theta^2)}{1+\theta^2+\theta^4}$

(b) is $\mathrm{E}\left[(X_3 - P(X_3|\boldsymbol{W}))^2\right] = \frac{-\sigma^2\theta^2(1+\theta^2)}{1+\theta^2+\theta^4}$

(c) is $\mathrm{E}\left[(X_3 - P(X_3|\boldsymbol{W}))^2\right] = \frac{-2\sigma^2\theta^2(1+\theta^2)}{1+\theta^2+\theta^4}$

$\square$

Problem 2.22. Repeat parts (a)-(d) of Problem 2.21 assuming now that the observations X_1, X_2, X_3, X_4, X_5 are from the causal AR(1) model

$$X_t = \phi X_{t-1} + Z_t, \{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$$

Solution. If Y and $W_n, \ldots, W_1$ are random variables, then for $\boldsymbol{W} = (W_n, \ldots, W_1)$ and $\boldsymbol{\mu}_W = (\mathrm{E}(W_n), \ldots, \mathrm{E}(W_1))^\intercal$, the best linear predictor of Y in terms of $\{1, W_n, \ldots, W_1\}$ is

$$P(Y|\boldsymbol{W}) = \mathrm{E}(Y) + \boldsymbol{a}^\intercal(\boldsymbol{W} - \boldsymbol{\mu_W})$$

where $\boldsymbol{a}$ is the solution of $\Gamma\boldsymbol{a} = \gamma$ for $\Gamma = \mathrm{Cov}(\boldsymbol{W}, \boldsymbol{W})$ and $\gamma = \mathrm{Cov}(Y, \boldsymbol{W})$.

Also, note for an MA(1) process, the autocovariance function is defined as

$$\gamma_X(h) = \begin{cases} \sigma^2(1+\theta^2) & \text{if } h = 0 \\ \sigma^2\theta & \text{if } |h| = 1 \\ 0 & \text{if } |h| > 1 \end{cases}$$

a. Using the above, set $Y = X_3$ and $W = (X_2, X_1)^\intercal$. Then

$$\Gamma = \mathrm{Cov}(\boldsymbol{W}, \boldsymbol{W}) = \begin{bmatrix} \gamma_X(0) & \gamma_X(1) \\ \gamma_X(1) & \gamma_X(0) \end{bmatrix} = \sigma^2 \begin{bmatrix} 1+\theta^2 & \theta \\ \theta & 1+\theta^2 \end{bmatrix}$$

and

$$\gamma = \begin{bmatrix} \gamma_X(1) \\ \gamma_X(2) \end{bmatrix} = \sigma^2 \begin{bmatrix} \theta \\ 0 \end{bmatrix}.$$

The solution to the system of equations $\Gamma \boldsymbol{a} = \gamma$ is

$$\boldsymbol{a} = \frac{\theta}{1+\theta^2+\theta^4} \begin{bmatrix} 1+\theta^2 \\ -\theta \end{bmatrix}.$$

Therefore, the best predictor of X_3 is

$$\begin{aligned} P(X_3|\boldsymbol{W}) &= \mathrm{E}(X_3) + \boldsymbol{a}^\intercal(\boldsymbol{W} - \boldsymbol{\mu}_W) \\ &= \frac{\theta}{1+\theta^2+\theta^4}((1+\theta^2)X_2 - \theta X_1) \end{aligned}$$

b. Using the above, set $Y = X_3$ and $W = (X_5, X_4)^\intercal$. Then

$$\Gamma = \mathrm{Cov}(\boldsymbol{W}, \boldsymbol{W}) = \begin{bmatrix} \gamma_X(0) & \gamma_X(1) \\ \gamma_X(1) & \gamma_X(0) \end{bmatrix} = \sigma^2 \begin{bmatrix} 1+\theta^2 & \theta \\ \theta & 1+\theta^2 \end{bmatrix}$$

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c. Using the above, set $Y = X_3$ and $W = (X_5, X_4, X_2, X_1)^\intercal$. Then

$$\begin{aligned} \Gamma = \mathrm{Cov}(\boldsymbol{W}, \boldsymbol{W}) &= \begin{bmatrix} \gamma_X(0) & \gamma_X(1) & \gamma_X(3) & \gamma_X(4) \\ \gamma_X(1) & \gamma_X(0) & \gamma_X(2) & \gamma_X(3) \\ \gamma_X(3) & \gamma_X(2) & \gamma_X(0) & \gamma_X(1) \\ \gamma_X(4) & \gamma_X(3) & \gamma_X(1) & \gamma_X(0) \end{bmatrix} \\ &= \sigma^2 \begin{bmatrix} 1+\theta^2 & \theta & 0 & 0 \\ \theta & 1+\theta^2 & 0 & 0 \\ 0 & 0 & 1+\theta^2 & \theta \\ 0 & 0 & \theta & 1+\theta^2 \end{bmatrix} \end{aligned}$$

and

$$\gamma = \begin{bmatrix} \gamma_X(2) \\ \gamma_X(1) \\ \gamma_X(1) \\ \gamma_X(2) \end{bmatrix} = \sigma^2 \begin{bmatrix} 0 \\ \theta \\ \theta \\ 0 \end{bmatrix}.$$

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(a) is $\mathrm{E}\left[(X_3 - P(X_3|\boldsymbol{W}))^2\right] = \frac{-\sigma^2\theta^2(1+\theta^2)}{1+\theta^2+\theta^4}$

(b) is $\mathrm{E}\left[(X_3 - P(X_3|\boldsymbol{W}))^2\right] = \frac{-\sigma^2\theta^2(1+\theta^2)}{1+\theta^2+\theta^4}$

(c) is $\mathrm{E}\left[(X_3 - P(X_3|\boldsymbol{W}))^2\right] = \frac{-2\sigma^2\theta^2(1+\theta^2)}{1+\theta^2+\theta^4}$

$\square$