Homework Assignment 1

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Problem 1.1.2. Use Example 1.1.3 for affine maps to find the solutions to the difference equations:

i. $x_{n+1} - \frac{x_n}{3} = 2,\ x_0 = 2.$

ii. $x_{n+1} + 3x_n = 4,\ x_0 = -1.$

Solution. Consider the affine map $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = ax + b$. Define the sequence $x_{n+1} = f(x_n) = ax_n + b$ where $x_0 \in \mathbb{R}$ is given. As was shown in the reading, the closed form solution to the above recurrence relation is given by

$$x_n = \left(x_0 - \frac{b}{1-a}\right) a^n + \frac{b}{1-a}. \tag{1}$$

Thus, the solutions to the provided difference equations can be solved by rewriting the equation in the form of an affine map, identifying a, b, and x_0, and using the closed solution (1).

i. For the difference equation $x_{n+1} - \frac{x_n}{3} = 2,\ x_0 = 2$, we readily see by rewriting the equation that $a = 1/3$ and $b = 2$ with $x_0 = 2$ given. Therefore, using (1), the solution to the difference equation is

$$\begin{aligned} x_n &= \left(x_0 - \frac{b}{1-a}\right) a^n + \frac{b}{1-a} \\ &= \left(2 - \frac{2}{1-1/3}\right)\left(\frac{1}{3}\right)^n + \frac{2}{1-1/3} \\ &= 3 - 3^{-n} \end{aligned}$$

ii. For the difference equation $x_{n+1} + 3x_n = 4,\ x_0 = -1$, we readily see by rewriting the equation that $a = -3$ and $b = 4$ with $x_0 = -1$ given. Therefore, using (1), the solution to the difference equation is

$$\begin{aligned} x_n &= \left(x_0 - \frac{b}{1-a}\right) a^n + \frac{b}{1-a} \\ &= \left(-1 - \frac{4}{1-(-3)}\right)(-3)^n + \frac{4}{1-(-3)} \\ &= 1 - 2(-3)^n. \end{aligned}$$

$\square$

Problem 1.1.3. A *logistic difference equation* is one of the form $x_{n+1} = \mu x_n(1 - x_n)$ for some fixed $\mu \in \mathbb{R}$. Find exact (closed form) solutions to the following logistic difference equations:

i. $x_{n+1} = 2x_n(1 - x_n)$. Hint: Use the substitution $x_n = (1 - y_n)/2$ to transform the equation into a simpler equation that is easily solved.

ii. $x_{n+1} = 4x_n(1 - x_n)$. Hint: Set $x_n = \sin^2(\theta_n)$ and simplify to get an equation that is easily solved.

Solution. i. Let $x_n = (1 - y_n)/2$ for $n \in \mathbb{N}$ with x_0 given. Substituting this expression into the original difference equation yields the new difference equation

$$\begin{aligned} \frac{1 - y_{n+1}}{2} &= 2\left(\frac{1 - y_n}{2}\right)\left[1 - \left(\frac{1 - y_n}{2}\right)\right] \\ &= (1 - y_n)\left(\frac{1 + y_n}{2}\right) \\ &= \frac{1 - y_n^2}{2}. \end{aligned}$$

This new difference equation reduces to $y_{n+1} = y_n^2$ for $n \in \mathbb{N}$, the solution of which is readily seen to be $y_{n+1} = y_0^{2^{n+1}}$. Making the substitution $y_n = 1 - 2x_n$ shows that, for $n \in \mathbb{N}$, the solution to the original difference equation is given by

$$x_{n+1} = \frac{1 - (1 - 2x_0)^{2^{n+1}}}{2}.$$

ii. Let $x_n = \sin^2(\theta_n)$ for $n \in \mathbb{N}$ with x_0 given. We may assume without loss of generality that $\theta_n \in [0, \pi)$ for if the angle θ_n isn't in the stated range, we can find an integer k such that $\theta_n + k\pi \in [0, \pi)$ and $\sin^2(\theta_n) = \sin^2(\theta_n + k\pi)$. We then declare the sum $\theta_n + k\pi$ to be the new angle θ_n. Substituting the above expression for x_n into the original difference equation yields the new difference equation

$$\begin{aligned} \sin^2(\theta_{n+1}) &= 4\sin^2(\theta_n)\left(1 - \sin^2(\theta_n)\right) \\ &= (2\sin(\theta_n)\cos(\theta_n))^2 \\ &= \sin^2(2\theta_n). \end{aligned}$$

Knowing that for $x, y \in [0, \pi)$ we have that $\sin^2(x) = \sin^2(y)$ if and only if $x = y$, the new difference equation reduces to $\theta_{n+1} = 2\theta_n$ for $n \in \mathbb{N}$ where it is implicitly understood that θ_{n+1} will be mapped to the corresponding angle between 0 and π if $2\theta_n \geq \pi$. Using the closed form solution for difference equations in the form of linear maps, the solution to the reduced difference equation is given by $\theta_{n+1} = 2^{n+1}\theta_0$ for $n \in \mathbb{N}$. Making the substitution $\theta_n = \sin^{-1}(\sqrt{x_n})$ shows that, for $n \in \mathbb{N}$, the solution to the original difference equation is given by

$$x_{n+1} = \sin^2(2^{n+1}\sin^{-1}(\sqrt{x_0})).$$

$\square$

Problem 1.1.4. You borrow $\$P$ at $r\%$ per annum and pay off $\$M$ at the end of each subsequent month. Write down a difference equation for the amount owing $A(n)$ at the end of each month (so $A(0) = P$). Solve the equation to find a closed form for $A(n)$. If $P = 100,000$, $M = 1,000$, and $r = 4$, after how long will the loan be paid off?

Solution. Let $A(n)$ be the amount owed on the loan at the end of month n. If the principal amount of the loan is $\$P$, then $A(0) = P$. If the annual interest rate is $r\%$, then the monthly interest rate is $r/12\%$. Assuming each month a payment of $\$M$ is made on the loan, a difference equation representing the amount owed on the loan at the end of month n is given by

$$\begin{aligned} A(n+1) &= A(n) + A(n)\left[\frac{r}{12(100)}\right] - M \\ &= \left[1 + \frac{r}{12(100)}\right] A(n) - M \end{aligned}$$

for $n \in \mathbb{N}$.

Using the closed form solution for difference equations in the form of affine maps, the solution to the difference equation is given by

$$\begin{aligned} A(n) &= \left(A(0) + \frac{M}{1 - \left(1 + \frac{r}{12(100)}\right)}\right)\left(1 + \frac{r}{12(100)}\right)^n - \frac{M}{1 - \left(1 + \frac{r}{12(100)}\right)} \\ &= \left(P - \frac{1200M}{r}\right)\left(1 + \frac{r}{1200}\right)^n + \frac{1200M}{r}. \end{aligned}$$

The loan will be paid off after $k \in \mathbb{R}$ months when $A(k) = 0$ from which we can gather that the loan will be paid off after $n = \lceil k \rceil$ full months. Solving

$$A(k) = \left(100000 - \frac{1200(1000)}{4}\right)\left(1 + \frac{4}{1200}\right)^n + \frac{1200(1000)}{4} = 0$$

shows that $k = 121.842$. Therefore, the loan will be paid off in full after 122 months. □

Problem 1.1.7. Let $f(x) = x^2+bx+c$. Give conditions on b and c for $f : [0, 1] \to [0, 1]$ to be a dynamical system. Hint: Recall that the maximum and minimum values of a continuous function defined on a closed interval $[a, b]$ occur either at the end points or at the critical points of the function.

Solution. The function $f(x) = x^2 + bx + c$ for $f : [0, 1] \to [0, 1]$ is a dynamical system if the image of the function is contained in its domain, i.e. if $f([0, 1]) \subseteq [0, 1]$. The minimum and maximum values of a continuous function occur either at the end points of the domain or at the critical points of the function. Thus, for the continuous function f, if we ensure that the evaluation of f at $x = 0$, $x = 1$, and the critical points of f are contained in $[0, 1]$ then the image of f will necessarily be contained in $[0, 1]$ and f will be a dynamical system.

At the end points of the domain we have that $f(0) = c$ and $f(1) = b + c + 1$. Thus, in order for f to be a dynamical system, we must have that $c \in [0, 1]$ and $b + c \in [-1, 0]$.

The only critical point of the function f is found when $f'(x) = 0$ or when $x = -b/2$. Thus, we require that $f(-b/2) = -b^2/4+c \in [0, 1]$. This reduces to requiring that $4c-4 \leq b^2 \leq 4c$. Thus, when $b \in \mathbb{R}$, we must have that $b \in [-2\sqrt{c}, 2\sqrt{c}]$.

Combining all of these inequalities shows that in order for the image of f to be contained in the domain of f, we must have that $c \in [0, 1]$ and $b \in [-2\sqrt{c}, -c]$, i.e. the function $f(x) = x^2+bx+c$ for $f : [0, 1] \to [0, 1]$ is a dynamical system if $0 \leq c \leq 1$ and $-2\sqrt{c} \leq b \leq -c$. $\square$

Problem 1.2.1. Give conditions on b and c for the map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + bx + c$ to have a fixed point. Use these conditions to show that $f_c(x) = x^2 + c$ has a fixed point provided $c \leq 1/4$.

Solution. Let $g(x) = f(x) - x = x^2 + (b-1)x + c$ for $g : \mathbb{R} \to \mathbb{R}$. From our definition of g, it is clear that the roots of the function g are the fixed points of the function f. Note that $g(x) = 0$ if

$$x = \frac{-b + 1 \pm \sqrt{(b-1)^2 - 4c}}{2}. \tag{2}$$

However, in order for x to be a root of $g(x)$, we must have that $x \in \mathbb{R}$, i.e. we must have that $(b-1)^2 - 4c \geq 0$. Thus, x is a fixed point of the function f if x is of the form (2) and for $b, c \in \mathbb{R}$ we have that $c \leq (b-1)^2/4$.

Take the function $f_c(x) = x^2 + c$ for $f_c : \mathbb{R} \to \mathbb{R}$. Note that f_c has the same form as the function f if $b = 0$. Thus, according to the conditions described above, we see that f_c has a fixed point if $c \leq (0-1)^2/4 = 1/4$.

□

Problem 1.2.6. Consider the eventual fixed points of the logistic map $L_\mu : [0,1] \to [0,1]$, $L_\mu(x) = \mu x(1-x)$ for $0 < \mu < 4$.

i. Show that there are no eventual fixed points associated with the fixed point $x = 0$, other than $x = 1$.

ii. Show that for $1 < \mu \leq 2$, the only eventual fixed point associated with the fixed point $x = 1 - 1/\mu$ is $x = 1/\mu$.

iii. Show that there are additional eventual fixed points associated with $x = 1 - 1/\mu$ when $2 < \mu < 3$.

iv. Investigate the eventual fixed points of the logistic map when $\mu = 5/2$.

Solution. i. It is clear that $x = 1$ is an eventual fixed point since $x = 0$ is a fixed point and $L_\mu(1) = 0$. This is the only eventual fixed point associated to $x = 0$ since no point in the interval $(0,1)$ maps to either 0 or 1 under L_μ, i.e. for $y \in (0,1)$, the equations $L_\mu(y) = 0$ and $L_\mu(y) = 1$ have no solutions. Therefore, since no $y \in D_{L_\mu} = [0,1]$ besides $y = 0$ and $y = 1$ maps to 0 or 1, there are no other eventual fixed points associated to $x = 0$.

ii. Let $1 < \mu \leq 2$. It is clear that $x = 1/\mu$ is an eventual fixed point since $x = 1 - 1/\mu$ is a fixed point and $L_\mu(1/\mu) = 1 - 1/\mu$. We will now demonstrate that this is the only eventual fixed point associated to $x = 1 - 1/\mu$. Note that for $x \in [0,1]$, the only solution to $L_\mu(x) = 1 - 1/\mu$ is $x = 1/\mu$. Therefore, in order for a point $x \in [0,1]$ to be an eventual fixed point associated to $x = 1 - 1/\mu$, we must have that x either maps to $1/\mu$ or eventually maps to $1/\mu$, i.e. for $x \in [0,1]$ we must have that $L_\mu(x) = 1/\mu$ has a solution. However, if $1 < \mu \leq 2$, then $L_\mu(x) = 1/\mu$ has no real solutions for $x \in [0,1]$ and so there are no other eventual fixed points associated to $x = 1 - 1/\mu$.

iii. Now suppose that $2 < \mu < 3$. Recall that $x = 1/\mu$ is an eventual fixed point. Note that

$$y = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4}{m^2}} \in [0,1]$$

satisfies $L_\mu(y) = 1/\mu$. Thus, $L_\mu^2(y) = 1 - 1/\mu$ where $1 - 1/\mu$ is a fixed point. Therefore, if $2 < \mu < 3$, then there are additional eventual fixed points associated to $x = 1 - 1/\mu$ besides $x = 1/\mu$.

iv. We proceed to describe all eventual fixed points of L_μ by first identifying all fixed points of the function. Suppose $\{x_{0_{n-1}}\}$ is the set of fixed points of L_μ where n is the number of fixed points. To find all eventual fixed points associated to the fixed point x_{0_k}, first find the pre-image of x_{0_k} minus the point x_{0_k}, i.e. find

$$L_\mu^{-1}(x_{0_k}) = \{x \neq x_{0_k} \mid L_\mu(x) = x_{0_k}\}.$$

Note that each point in $L_\mu^{-1}(x_{0_k})$ will be an eventual fixed point of x_{0_k}. Denote the set of eventual fixed points associated to x_{0_k} by x_{1_k}. If the set is empty, then there are no eventual fixed points associated to x_{0_k}. If the set is non-empty, continue the process by finding for each point in the set of eventual fixed points x_{1_k}, the set of eventual fixed points associated to the eventual fixed point x_{1_k}, i.e. $x_{2_k} = L_\mu^{-1}(x_{1_k}) = L_\mu^{-1}\left(L_\mu^{-1}(x_{0_k})\right)$. Again, every point in x_{2_k} is an eventual fixed point of x_{0_k}. Continue this process indefinitely until the pre-image x_{m_k} is empty.

Now suppose that $\mu = 5/2$. The fixed points of L_μ are found by finding the roots of $g(x) = L_\mu(x) - x = (3/2)x - (5/2)x^2$ for $g : [0,1] \to [0,1]$. It is clear that the two roots of g are given by $x = 0$ and $x = 3/5$. Thus, denote the fixed points of L_μ by $\{x_{0_0}, x_{0_1}\} = \{0, 3/5\}$.

We will now find all eventual fixed points associated to the fixed point $x_{0_0} = 0$. As was shown previously, the only eventual fixed points associated to x_{0_0} is $x = 1$. Thus, $x_{1_0} = \{1\}$ and we are done.

Now we will find all eventual fixed points associated to the fixed point $x_{0_1} = 3/5$. Solving the equation $L_\mu(x) = x_{0_1}$ shows us that the pre-image of x_{0_1} minus the point x_{0_1} is given by $L_\mu^{-1}(x_{0_1}) = \{2/5\}$, i.e. $x_{1_1} = \{2/5\}$. Continuing, we solve the equation $L_\mu(x) = 2/5$ and find that $L_\mu^{-1}(x_{1_1}) = \{1/5, 4/5\}$. Thus $x_{2_1} = \{1/5, 4/5\}$. Since, $L_\mu(x) = 4/5$ has no real solutions, there are no eventual fixed points associated to the eventual fixed point $4/5$. However, solving $L_\mu(x) = 1/5$ we see that there are real solutions associated to this eventual fixed point so $x_{3_1} = \{1/10(5 - \sqrt{17}), 1/10(5 + \sqrt{17})\}$.

Under further investigation the above sequence continues in a similar pattern. The equation $L_\mu(x) = 1/10(5+\sqrt{17})$ will have no real solutions but $L_\mu(x) = 1/10(5-\sqrt{17})$ does. Thus, we see using these solutions that $x_{4_1} = \{1/10(5 - \sqrt{5 + 4\sqrt{17}}), 1/10(5 + \sqrt{5 + 4\sqrt{17}})\}$. This process repeats indefinitely so that two more eventual fixed points are found with the set of eventual fixed points x_{k_1}. The union of all such points as described above make up the eventual fixed points associated to $x = 3/5$.

□