Homework Assignment 1

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February 3, 2017

Problem 2.1. Find the Fourier transforms of each of the following functions:

c. $f(x) = \delta^{(n)}(x)$,

f. $f(x) = x \exp\left(-\dfrac{ax^2}{2}\right), a > 0,$

g. $f(x) = x^2 \exp\left(-\dfrac{x^2}{2}\right).$

Solution. Recall that, by definition, we have that for a function $f(x) \in L^1(\mathbb{R})$, its Fourier transform is given by

$$\mathscr{F}\{f(x)\} = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx \tag{1}$$

where $k \in \mathbb{R}$.

c. The Dirac delta function $\delta(x)$ is defined such that for any good function $g(x)$ we have that

$$\int_{-\infty}^{\infty} \delta(x) g(x) dx = g(0).$$

A good function is defined as a function in C^∞ that decays sufficiently rapidly. Since it is clear that $\delta(x) \to 0$ as $|x| \to \infty$, we have by a previous theorem that

$$\mathscr{F}\{\delta'(x)\} = ik\mathscr{F}\{\delta(x)\}. \tag{2}$$

By (1) and the definition of the Dirac delta function, we see that

$$\mathscr{F}\{\delta(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \delta(x) dx = \frac{1}{\sqrt{2\pi}}.$$

Thus, using (2), we can easily see by induction for $n > 1$ that

$$\mathscr{F}\left\{\delta^{(n)}(x)\right\} = ik\mathscr{F}\left\{\delta^{(n-1)}(x)\right\} = \cdots = \frac{(ik)^n}{\sqrt{2\pi}}.$$

f. From (1), we see that

$$\begin{aligned}
\mathscr{F}\left\{f(x)\right\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp(-ikx) \exp\left(-\frac{ax^2}{2}\right) dx \\
&= \frac{\exp\left(\frac{(ik)^2}{2a}\right)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp\left(-\frac{ax^2}{2} - ikx - \frac{(ik)^2}{2a}\right) dx \\
&= \frac{\exp\left(-\frac{k^2}{2a}\right)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp\left(-\frac{a}{2}\left(x + \frac{ik}{a}\right)^2\right) dx.
\end{aligned}$$

Making the substitution $u = x + ik/a$, where $du = dx$, we have that

$$\begin{aligned}
\mathscr{F}\left\{f(x)\right\} &= \frac{\exp\left(-\frac{k^2}{2a}\right)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(u - \frac{ik}{a}\right) \exp\left(-\frac{au^2}{2}\right) du \\
&= \frac{\exp\left(-\frac{k^2}{2a}\right)}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} u \exp\left(-\frac{au^2}{2}\right) du - \frac{ik}{a} \int_{-\infty}^{\infty} \exp\left(-\frac{au^2}{2}\right) du\right]. \qquad (3)
\end{aligned}$$

Since the function $g(x) = u \exp\left(-\frac{au^2}{2}\right)$ is odd, we know that

$$\int_{-\infty}^{\infty} u \exp\left(-\frac{au^2}{2}\right) du = 0.$$

Using the formula for the general Gaussian integral we have that

$$\int_{-\infty}^{\infty} \exp\left(-\frac{au^2}{2}\right) = \frac{\sqrt{2\pi}}{\sqrt{a}}$$

when $a > 0$.

Combining, we see from (3) that the Fourier transform of $f(x) = x \exp\left(-\frac{ax^2}{2}\right)$ for $a > 0$ is

$$\begin{aligned}
\mathscr{F}\left\{f(x)\right\} &= \frac{\exp\left(-\frac{k^2}{2a}\right)}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} u \exp\left(-\frac{au^2}{2}\right) du - \frac{ik}{a} \int_{-\infty}^{\infty} \exp\left(-\frac{au^2}{2}\right) du\right] \\
&= \frac{\exp\left(-\frac{k^2}{2a}\right)}{\sqrt{2\pi}} \left(-\frac{ik}{a}\right) \left(\frac{\sqrt{2\pi}}{\sqrt{a}}\right) \\
&= -\frac{ik \exp\left(-\frac{k^2}{2a}\right)}{a\sqrt{a}}.
\end{aligned}$$

g. From (1), we see that

$$\begin{aligned}\mathscr{F}\left\{f(x)\right\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^2\exp(-ikx)\exp\left(-\frac{x^2}{2}\right)dx\\ &= \frac{\exp\left(\frac{(ik)^2}{2}\right)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^2\exp\left(-\frac{x^2}{2}-ikx-\frac{(ik)^2}{2}\right)dx\\ &= \frac{\exp\left(-\frac{k^2}{2}\right)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^2\exp\left(-\frac{1}{2}\left(x+ik\right)^2\right)dx.\end{aligned}$$

Making the substitution $u = x + ik$, where $du = dx$, we have that

$$\begin{aligned}\mathscr{F}\left\{f(x)\right\} &= \frac{\exp\left(-\frac{k^2}{2}\right)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (u-ik)^2\exp\left(-\frac{u^2}{2}\right)du\\ &= \frac{\exp\left(-\frac{k^2}{2}\right)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \left(u^2-2iku-k^2\right)\exp\left(-\frac{u^2}{2}\right)du. \qquad (4)\end{aligned}$$

After distributing the exponential term to the polynomial in (4) and splitting the integral using the operator's linearity, the first integral may be computed by parts by setting $w = u$ and $dv = -u\exp\left(-\frac{u^2}{2}\right)du$ as so:

$$\int_{-\infty}^{\infty} u^2\exp\left(-\frac{u^2}{2}\right)du = -\left[u\exp\left(-\frac{u^2}{2}\right)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\exp\left(-\frac{u^2}{2}\right)du\right] = \sqrt{2\pi}$$

where the last equality follows from the formula for the general Gaussian integral.

The other two resulting integrals in (4) are calculated very similarly to their corresponding integrals in 2.1.f, i.e.

$$\begin{aligned}2ik\int_{-\infty}^{\infty} u\exp\left(-\frac{u^2}{2}\right)du &= 0,\\ k^2\int_{-\infty}^{\infty}\exp\left(-\frac{u^2}{2}\right)du &= k^2\sqrt{2\pi}.\end{aligned}$$

Therefore, we have that the Fourier transform of $f(x) = x^2\exp\left(-\frac{x^2}{2}\right)$ is given by

$$\begin{aligned}\mathscr{F}\left\{f(x)\right\} &= \frac{\exp\left(-\frac{k^2}{2}\right)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \left(u^2-2iku-k^2\right)\exp\left(-\frac{u^2}{2}\right)du\\ &= \frac{\exp\left(-\frac{k^2}{2}\right)}{\sqrt{2\pi}}\left[\sqrt{2\pi}-k^2\sqrt{2\pi}\right]\\ &= (1-k^2)\exp\left(-\frac{k^2}{2}\right).\end{aligned}$$

□

Problem 2.2. Show that

a. $\mathscr{F}\left\{\delta(x-ct)+\delta(x+ct)\right\} = \sqrt{\dfrac{2}{\pi}}\cos(kct)$,

b. $\mathscr{F}\left\{H\left(ct-|x|\right)\right\} = \mathscr{F}\left\{\chi_{[-ct,ct]}(x)\right\} = \sqrt{\dfrac{2}{\pi}}\dfrac{\sin(kct)}{k}$.

Solution. a. By the shifting property of the Fourier transform, we know that

$$\mathscr{F}\left\{\delta(x\pm ct)\right\} = e^{\pm ikct}\mathscr{F}\left\{\delta(x)\right\}.$$

As shown previously, we also know from the definition of the Fourier transform and the definition of the Dirac delta function that $\mathscr{F}\left\{\delta(x)\right\} = 1/\sqrt{2\pi}$. Combining, we see using the linearity of the Fourier transform that

$$\begin{aligned}
\mathscr{F}\left\{\delta(x-ct)+\delta(x+ct)\right\} &= \mathscr{F}\left\{\delta(x-ct)\right\} + \mathscr{F}\left\{\delta(x+ct)\right\} \\
&= \left(e^{-ikct}+e^{ikct}\right)\mathscr{F}\left\{\delta(x)\right\} \\
&= \frac{2}{\sqrt{2\pi}}\left(\frac{e^{-ikct}+e^{ikct}}{2}\right) \\
&= \sqrt{\frac{2}{\pi}}\cos(kct)
\end{aligned}$$

where the last equality follows using the definition of the complex exponential.

b. Recall from the definitions of the Heaviside function H and the characteristic function χ that

$$H(ct-|x|) = \chi_{[-ct,ct]}(x) = \begin{cases} 1 & |x| < ct \\ 0 & |x| > ct \end{cases}.$$

Since the Fourier transform is a well-defined operation, this implies that

$$\mathscr{F}\left\{H\left(ct-|x|\right)\right\} = \mathscr{F}\left\{\chi_{[-ct,ct]}(x)\right\}.$$

Now, from the definition of the Fourier transform in (1), we see that

$$\begin{aligned}
\mathscr{F}\left\{H\left(ct-|x|\right)\right\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}H\left(ct-|x|\right)dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-ct}^{ct} e^{-ikx}dx \\
&= \frac{2}{k\sqrt{2\pi}}\left(\frac{e^{ikct}-e^{-ikct}}{2i}\right) \\
&= \sqrt{\frac{2}{\pi}}\frac{\sin(kct)}{k}
\end{aligned}$$

where again the last equality follows from the definition of the complex exponential.

□

Problem 2.3. Show that

a. $i\dfrac{d}{dk}F(k) = \mathscr{F}\left\{xf(x)\right\}$

b. $i^n\dfrac{d^n}{dk^n}F(k) = \mathscr{F}\left\{x^nf(x)\right\}$

Solution. a. Recall from the definition of the Fourier transform in (1) that

$$F(k) = \mathscr{F}\left\{f(x)\right\} = \int_{-\infty}^{\infty} e^{-ikx}f(x)dx.$$

Using the Leibniz integral rule, we have that

$$\begin{aligned}\frac{d}{dk}F(k) = \mathscr{F}\left\{f(x)\right\} &= \int_{-\infty}^{\infty} \frac{\partial}{\partial k}\left[e^{-ikx}f(x)\right]dx \\ &= -\int_{-\infty}^{\infty} ixe^{-ikx}f(x)dx \\ &= -i\mathscr{F}\left\{xf(x)\right\}.\end{aligned}$$

Of course, this implies that

$$i\frac{d}{dk}F(k) = \mathscr{F}\left\{xf(x)\right\}.$$

b. Suppose that for $n > 1$ we have that

$$i^n\frac{d^n}{dk^n}F(k) = \mathscr{F}\left\{x^nf(x)\right\}. \tag{5}$$

Then by the Leibniz integral rule, we have that

$$\begin{aligned}i^n\frac{d^{n+1}}{dk^{n+1}}F(k) = \frac{d}{dk}\left[i^n\frac{d^n}{dk^n}F(k)\right] &= \frac{d}{dk}\int_{-\infty}^{\infty} x^ne^{-ikx}f(x)dx \\ &= \int_{-\infty}^{\infty} \frac{\partial}{\partial k}\left[x^ne^{-ikx}f(x)\right]dx \\ &= -i\int_{-\infty}^{\infty} x^{n+1}e^{-ikx}f(x)dx \\ &= -i\mathscr{F}\left\{x^{n+1}f(x)\right\}.\end{aligned}$$

This implies that

$$i^{n+1}\frac{d^{n+1}}{dk^{n+1}}F(k) = \mathscr{F}\left\{x^{n+1}f(x)\right\}$$

and (5) holds by induction.

□

Problem 2.5. Prove the following:

c. If $f(x)$ has a finite discontinuity at a point $x = a$, then

$$\mathscr{F}\left\{f'(x)\right\} = (ik)F(k) - \frac{1}{\sqrt{2\pi}}\exp(-ika)[f]_a,$$

where $[f]_a = f(a+0) - f(a-0)$.

Generalize this result for $\mathscr{F}\left\{f^{(n)}(x)\right\}$.

Solution. $\square$

Problem 2.7. Prove the following results for the convolution:

c. $\dfrac{d}{dx}\left[f(x) * g(x)\right] = f'(x) * g(x) = f(x) * g'(x)$,

d. $\displaystyle\int_{-\infty}^{\infty} (f * g)(x)dx = \int_{-\infty}^{\infty} f(u)du \int_{-\infty}^{\infty} g(v)dv$.

Solution. For two functions functions $f, g \in L^1(\mathbb{R})$, the convolution of f and g, denoted by $(f * g)(x)$, is defined to be

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi.$$

c. Using the definition of the convolution of f and g, we have that

$$\begin{aligned}\frac{d}{dx}\left[f(x) * g(x)\right] &= \frac{d}{dx}\int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi \\ &= \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left[f(x-\xi)g(\xi)\right]d\xi \\ &= \int_{-\infty}^{\infty} f'(x-\xi)g(\xi)d\xi \\ &= f'(x) * g(x).\end{aligned}$$

Since the convolution of two functions is commutative, we have using the above that

$$\begin{aligned}\frac{d}{dx}\left[f(x) * g(x)\right] &= \frac{d}{dx}\left[g(x) * f(x)\right] \\ &= g'(x) * f(x) \\ &= f(x) * g'(x).\end{aligned}$$

d. From the definition of the convolution of f and g, we have that

$$\int_{-\infty}^{\infty} (f * g)(x)dx = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi\right]dx.$$

Since $f, g \in L^1(\mathbb{R})$, we may interchange the order of integration above. Doing so yields

$$\begin{aligned}\int_{-\infty}^{\infty} (f * g)(x)dx &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi\right]dx \\ &= \int_{-\infty}^{\infty} g(\xi)\left[\int_{-\infty}^{\infty} f(x-\xi)dx\right]d\xi.\end{aligned}$$

Making the substitution $u = x - \xi$ with $du = dx$, the above becomes

$$\begin{aligned}\int_{-\infty}^{\infty} (f * g)(x)dx &= \int_{-\infty}^{\infty} g(\xi)\left[\int_{-\infty}^{\infty} f(x-\xi)dx\right]d\xi \\ &= \int_{-\infty}^{\infty} g(\xi)\left[\int_{-\infty}^{\infty} f(u)du\right]d\xi \\ &= \int_{-\infty}^{\infty} g(\xi)d\xi \int_{-\infty}^{\infty} f(u)du.\end{aligned}$$

The variables u and ξ are arbitary; therefore

$$\int_{-\infty}^{\infty} (f * g)(x) dx = \int_{-\infty}^{\infty} f(u) du \int_{-\infty}^{\infty} g(v) dv$$

and we are done.

$\square$

Problem 2.8. Use the Fourier transform to solve the following ordinary differential equations for $-\infty < x < \infty$:

a. $y''(x) - y(x) + 2f(x) = 0$, where $f(x) = 0$ when $x < -a$ and when $x > a$ and its derivatives vanish at $x = \pm\infty$,

b. $2y''(x) + xy'(x) + y(x) = 0$.

Solution. $\square$

Problem 2.9. Solve the following integral equations for an unknown function $f(x)$:

a. $\displaystyle\int_{-\infty}^{\infty} \phi(x-t)f(t)dt = g(x),$

b. $\displaystyle\int_{-\infty}^{\infty} \exp(-at^2)f(x-t)dt = \exp(-at^2), a > b > 0,$

d. $\displaystyle\int_{-\infty}^{\infty} f(x-t)f(t)dt = \frac{b}{x^2+b^2}.$

Solution. □