Homework Assignment 5

Matthew Tiger

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Problem 1. Use the method of stationary phase to find the leading behavior of the following integral as $x \to +\infty$:

$$I(x) = \int_0^1 e^{ixt^2} \cosh t^2 dt.$$

Solution. We begin by noting that the integral $I(x)$ is a generalized Fourier integral which can be written as

$$I(x) = \int_0^1 f(t) e^{ix\psi(t)} dt$$

where $f(t) = \cosh t^2$ and $\psi(t) = t^2$. The leading asymptotic behavior of such integrals as $x \to +\infty$ may be found, in general, using integration by parts. However, this method may fail at *stationary points*, i.e. any point on the interval of definition such that $\psi'(t) = 0$. For the integral $I(x)$ we note that $t = 0$ is a stationary point. Thus, we proceed by writing $I(x)$ as follows:

$$I(x) = I_1(x) + I_2(x) = \int_0^\varepsilon f(t) e^{ix\psi(t)} dt + \int_\varepsilon^1 f(t) e^{ix\psi(t)} dt$$

for some $\varepsilon > 0$. Since $I_2(x)$ does not have any stationary points and the function $f(t) = \cosh t^2 \in L^1$ over the interval $[0, 1]$, i.e. we have that $\int_0^1 |f(t)|\, dt < +\infty$, integration by parts works on $I_2(x)$ and by the Riemann-Lebesgue lemma, $I_2(x) \to 0$ as $x \to +\infty$. Thus, as $x \to +\infty$,

$$I(x) \sim I_1(x) = \int_0^\varepsilon f(t) e^{ix\psi(t)} dt = \int_0^\varepsilon \cosh t^2 e^{ixt^2}\, dt.$$

We continue by replacing $f(t)$ with $f(0) = \cosh 0 = 1$ and ε with ∞, since these are the parts that contribute the most to the integral, introducing error terms that vanish as $x \to +\infty$ so that

$$I(x) \sim \int_0^\infty e^{ixt^2} dt$$

Making the substitution

$$t = e^{i\pi/4} \left[\frac{u}{x} \right]^{1/2}$$

yields that

$$\int_0^\infty e^{ixt^2}dt = e^{i\pi/4}\left[\frac{1}{x}\right]^{1/2}\frac{\Gamma(1/2)}{2} = \frac{e^{i\pi/4}}{2}\sqrt{\frac{\pi}{x}}.$$

Therefore, as $x \to +\infty$,

$$I(x) \sim \int_0^\infty e^{ixt^2}dt = \frac{e^{i\pi/4}}{2}\sqrt{\frac{\pi}{x}}.$$

$\square$

Problem 2. Use second-order perturbation theory to find approximations to the roots of the following equation:

$$x^3 + \varepsilon x^2 - x = 0.$$

Solution. If we assume that the roots of the above equation are functions of ε, then the roots x_i for $i = 0, 1, 2$ of the equation are of the form

$$x_i(\varepsilon) = \sum_{k=0}^{\infty} a_{i_k} \varepsilon^k.$$

Second-order perturbation theory prescribes that the roots are of the form

$$x_i(\varepsilon) = a_{i_0} + a_{i_1}\varepsilon + a_{i_2}\varepsilon^2 + O(\varepsilon^3)$$

where we disregard terms of order ε^3 or greater. Substituting $\varepsilon = 0$ into the equation yields the new equation $x^{+3} - x = 0$, the roots of which are $-1, 0,$ and 1 which we will say correspond to the coefficients $a_{0_0} = -1, a_{1_0} = 0,$ and $a_{2_0} = 1$.

In order to find the values of the coefficients a_{i_k} for $k \geq 1$, we substitute the expression $x_i(\varepsilon) = a_{i_0} + a_{i_1}\varepsilon + a_{i_2}\varepsilon^2 + O(\varepsilon^3)$ into the original equation yielding

$$a_{i_0}^3 - a_{i_0} + (a_{i_0}^2 - a_{i_1} + 3a_{i_0}^2 a_{i_1})\varepsilon + (2a_{i_0}a_{i_1} + 3a_{i_0}a_{i_1}^2 - a_{i_2} + 3a_{i_0}^2 a_{i_2})\varepsilon^2 = O(\varepsilon^3).$$

Since ε is variable we must have that the coefficients of ε in the above equation are 0. This yields two equations for each root:

$$a_{i_0}^2 - a_{i_1} + 3a_{i_0}^2 a_{i_1} = 0$$
$$2a_{i_0}a_{i_1} + 3a_{i_0}a_{i_1}^2 - a_{i_2} + 3a_{i_0}^2 a_{i_2} = 0.$$

For the root x_0, we have that $a_{0_0} = -1$ and the two equations become

$$(-1)^2 - a_{0_1} + 3(-1)^2 a_{0_1} = 0$$
$$-2a_{0_1} - 3a_{0_1}^2 - a_{0_2} + 3(-1)^2 a_{0_2} = 0.$$

The first equation yields that $a_{0_1} = -1/2$ and substituting into the second equation yields that $a_{0_2} = -1/8$. Thus, $x_0 = -1 + (-1/2)\varepsilon + (-1/8)\varepsilon^2 + O(\varepsilon^3)$.

For the root $x_1 = 0$, we see that $a_{1_0} = 0$ and consequently from the equations that $a_{1_1} = 0$ and $a_{1_2} = 0$. Thus, $x_1 = 0 + 0\varepsilon + 0\varepsilon^2 + O(\varepsilon^3)$.

Proceeding in the same way above we see for the root x_2, we have that $a_{2_0} = 1$ and the above two equations become

$$(1)^2 - a_{2_1} + 3(1)^2 a_{2_1} = 0$$
$$2a_{2_1} + 3a_{2_1}^2 - a_{2_2} + 3(1)^2 a_{2_2} = 0.$$

The first equation yields that $a_{2_1} = -1/2$ and substituting into the second equation yields that $a_{2_2} = 1/8$. Therefore, $x_2 = 1 + (-1/2)\varepsilon + (1/8)\varepsilon^2 + O(\varepsilon^3)$ and we have found second-order approximations for all of the roots of the original equation.

$\square$

Problem 3. Analyze in the limit $\varepsilon \to 0$ the roots of the polynomial

$$\varepsilon x^8 - \varepsilon^2 x^6 + x - 2 = 0.$$

Solution. When $\varepsilon = 0$, the unperturbed equation only has one root while the original equation has eight roots implying that this is a singular perturbation problem.

The root of the unperturbed problem is given by $x_0 = 2$. To find the other roots, we employ the method of dominant balance to the original equation and find a consistent balance in order to understand the order of magnitude of the roots as $\varepsilon \to 0$.

As there are four terms in the original equation, there are 6 possible dominant balances to consider:

i. Suppose $\varepsilon x^8 \sim \varepsilon^2 x^6$ as $\varepsilon \to 0$ is the dominant balance. Then $x = O(\varepsilon^{1/2})$ as $\varepsilon \to 0$. However, $\varepsilon x^8 \ll 2$ as $\varepsilon \to 0$ violating the assumption that εx^8 and $\varepsilon^2 x^6$ are the dominant terms showing that the original balance is inconsistent.

ii. Suppose $\varepsilon x^8 \sim x$ as $\varepsilon \to 0$ is the dominant balance. Then $x = O(\varepsilon^{-1/7})$ as $\varepsilon \to 0$. Since $2 \ll x$ and $\varepsilon^2 x^6 = O(\varepsilon^{8/7})$ is such that $\varepsilon^2 x^6 \ll x$, this dominant balance is consistent.

iii. Suppose $\varepsilon x^8 \sim 2$ as $\varepsilon \to 0$ is the dominant balance. Then $x = O(\varepsilon^{-1/8})$ as $\varepsilon \to 0$. However, $2 \ll x$ as $\varepsilon \to 0$ violating the assumption that εx^8 and 2 are the dominant terms showing that the original balance is inconsistent.

iv. Suppose $\varepsilon^2 x^6 \sim x$ as $\varepsilon \to 0$ is the dominant balance. Then $x = O(\varepsilon^{-2/5})$ as $\varepsilon \to 0$. However, $\varepsilon x^8 = O(\varepsilon^{-11/5})$ is such that $x \ll \varepsilon x^8$ violating the assumption that $\varepsilon^2 x^6$ and x are the dominant terms showing that the original balance is inconsistent.

v. Suppose $\varepsilon^2 x^6 \sim 2$ as $\varepsilon \to 0$ is the dominant balance. Then $x = O(\varepsilon^{-1/3})$ as $\varepsilon \to 0$. However, $\varepsilon x^8 = O(\varepsilon^{-5/3})$ is such that $2 \ll \varepsilon x^8$ violating the assumption that $\varepsilon^2 x^6$ and 2 are the dominant terms showing that the original balance is inconsistent.

vi. Suppose $x \sim 2$ as $\varepsilon \to 0$ is the dominant balance. This balance is consistent since $x = O(1)$ and we recover the original root with this equation.

The above analysis suggests that, from the only consistent dominant balance that provides additional information, the magnitudes of the missing roots are $O(\varepsilon^{-1/7})$ as $\varepsilon \to 0$. Making the scale transformation $x = \varepsilon^{-1/7} y$ and substituting into the original equation yields

$$y^8 - \varepsilon^{9/7} y^6 + y - 2\varepsilon^{1/7} = 0 \tag{1}$$

The unperturbed problem now does not vanish and the problem is now a regular perturbation problem. Thus, a perturbation expansion exists for the roots y in terms of powers of $\varepsilon^{1/7}$, i.e. the roots to (1) are of the form

$$y = \sum_{n=0}^{\infty} y_n \left(\varepsilon^{1/7}\right)^n.$$

$\square$

Problem 4. Solve perturbatively

$$
\begin{cases}
y'' = (\sin x)y \\
y(0) = 1 \\
y'(0) = 1
\end{cases}
.
$$

Is the resulting perturbation series uniformly valid for $0 \le x \le \infty$? Why?

Solution. □

Problem 5. Find leading-order uniform asymptotic approximations to the solution of the following equation in the limit $\varepsilon \to 0^+$:

$$\varepsilon y'' + (x^2 + 1)y' - x^3 y = 0$$
$$y(0) = 1, \ y(1) = 1.$$

Solution. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Problem 6. Obtain a uniform approximation accurate to order ε^2 as $\varepsilon \to 0^+$ for the problem

$$\varepsilon y'' + (1+x)^2 y' + y = 0$$
$$y(0) = 1, \ y(1) = 1.$$

Solution.   □

Problem 7. For what real values of the constant α does the singular perturbation problem

$$\varepsilon y''(x) + y'(x) - x^\alpha y(x) = 0$$
$$y(0) = 1, \ y(1) = 1.$$

have a solution with a boundary layer near $x = 0$ as $\varepsilon \to 0^+$?

Solution.   □