Homework Assignment 2

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February 20, 2017

Problem 2.10. Solve the Cauchy problem for the Klein-Gordon equation

$$\begin{aligned} u_{tt} - c^2 u_{xx} + a^2 u &= 0, \qquad -\infty < x < \infty, \quad t > 0, \\ u(x,0) &= f(x) \qquad \text{for } -\infty < x < \infty, \\ \left[\frac{\partial u}{\partial t}\right]_{t=0} &= g(x) \qquad \text{for } -\infty < x < \infty. \end{aligned}$$

Solution. Consider the function $u(x,y)$. The Fourier transform of u with respect to x is defined as

$$\mathscr{F}\left\{u(x,y)\right\} = U(k,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} u(x,y)dx. \tag{1}$$

From this definition and the Leibniz integral rule, we can see by induction that

$$\begin{aligned} \mathscr{F}\left\{\frac{\partial^n}{\partial y^n}\left[u(x,y)\right]\right\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^n}{\partial y^n}\left[u(x,y)\right] e^{-ikx} dx \\ &= \frac{d^n}{dy^n}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,y) e^{-ikx} dx\right] \\ &= \frac{d^n}{dy^n}\left[\mathscr{F}\left\{u(x,y)\right\}\right]. \end{aligned} \tag{2}$$

Similarly, we see from definition (1) and previous theorems regarding the Fourier transform that

$$\begin{aligned} \mathscr{F}\left\{\frac{\partial^n}{\partial x^n}\left[u(x,y)\right]\right\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^n}{\partial x^n}\left[u(x,y)\right] e^{-ikx} dx \\ &= (ik)^n \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,y) e^{-ikx} dx\right] \\ &= (ik)^n \mathscr{F}\left\{u(x,y)\right\}. \end{aligned} \tag{3}$$

Now, applying the Fourier transform to the first equation, we see that

$$\begin{aligned} \mathscr{F}\left\{u_{tt} - c^2 u_{xx} + a^2 u\right\} &= \frac{d^2}{dt^2}\left[U(k,t)\right] - c^2 (ik)^2 U(k,t) + a^2 U(k,t) \\ &= \frac{d^2}{dt^2}\left[U(k,t)\right] + \left(c^2 k^2 + a^2\right) U(k,t). \end{aligned}$$

Thus, setting $\omega^2 = c^2k^2 + a^2$, we see that

$$\frac{d^2}{dt^2}\left[U(k,t)\right] + \omega^2 U(k,t) = 0.$$

This is a second-order linear homogeneous ordinary differential equation, the solution to which we readily see is given by

$$U(k,t) = c_1 e^{-i\omega t} + c_2 e^{i\omega t}. \tag{4}$$

Applying the Fourier transform to the last two equations yields

$$\mathscr{F}\left\{u(x,0)\right\} = U(k,0) = F(k) = \mathscr{F}\left\{f(x)\right\}$$

and

$$\mathscr{F}\left\{\left[\frac{\partial u}{\partial t}\right]_{t=0}\right\} = \frac{d}{dt}\left[U(k,t)\right]_{t=0} = G(k) = \mathscr{F}\left\{g(x)\right\}.$$

Using (5), we see that the first equation reduces to

$$c_1 + c_2 = F(k).$$

Taking the derivative of $U(k,t)$ with respect to t yields

$$\frac{d}{dt}\left[U(k,t)\right] = -i\omega c_1 e^{-i\omega t} + i\omega c_2 e^{i\omega t}$$

and evaluating when $t = 0$ produces a second equation

$$i\omega(c_2 - c_1) = G(k).$$

This results in a system of two equations in two unknowns; the solution of which is given by

$$c_1 = \frac{\omega F(k) + iG(k)}{2\omega}, \qquad c_2 = \frac{\omega F(k) - iG(k)}{2\omega}.$$

Therefore, (5) becomes

$$U(k,t) = \left(\frac{\omega F(k) + iG(k)}{2\omega}\right) e^{-i\omega t} + \left(\frac{\omega F(k) - iG(k)}{2\omega}\right) e^{i\omega t}.$$

Taking the Inverse Fourier transform yields that the solution to the original differential equation is given by

$$\begin{aligned}
u(x,t) &= \mathscr{F}^{-1}\left\{\left(\frac{\omega F(k) + iG(k)}{2\omega}\right) e^{-i\omega t} + \left(\frac{\omega F(k) - iG(k)}{2\omega}\right) e^{i\omega t}\right\} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\left(\frac{\omega F(k) + iG(k)}{2\omega}\right) e^{-i\omega t} + \left(\frac{\omega F(k) - iG(k)}{2\omega}\right) e^{i\omega t}\right] e^{ikx} dk \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{\omega F(k) + iG(k)}{2\omega}\right) e^{i(kx - \omega t)} + \left(\frac{\omega F(k) - iG(k)}{2\omega}\right) e^{i(kx + \omega t)} dk.
\end{aligned}$$

□

Problem 2.12. Solve the equation

$$\begin{gathered} u_{tt} + u_{xxxx} = 0, \qquad -\infty < x < \infty, \quad t > 0 \\ u(x,0) = f(x), \quad u_t(x,0) = 0 \qquad \text{for } -\infty < x < \infty. \end{gathered}$$

Solution. Using (2) and (3), we begin by applying the Fourier transform to the first differential equation yielding

$$\mathscr{F}\left\{u_{tt} + u_{xxxx}\right\} = \frac{d^2}{dt^2}\left[U(k,t)\right] + k^4 U(k,t) = 0 = \mathscr{F}\left\{0\right\}.$$

This is a second-order linear homogeneous differential equation, the solution to which we readily see is given by

$$U(k,t) = c_1 \cos(k^2 t) + c_2 \sin(k^2 t). \tag{5}$$

Applying the Fourier transform to the boundary conditions yields

$$\mathscr{F}\left\{u(x,0)\right\} = U(k,0) = F(k) = \mathscr{F}\left\{f(x)\right\}$$

and

$$\mathscr{F}\left\{u_t(x,0)\right\} = \mathscr{F}\left\{\left[\frac{\partial u}{\partial t}\right]_{t=0}\right\} = \frac{d}{dt}\left[U(k,t)\right]\bigg|_{t=0} = 0 = \mathscr{F}\left\{0\right\}.$$

Using the form (5) of the solution we see from the first boundary condition that

$$U(k,0) = c_1 = F(k).$$

Similarly, we also see from the second boundary condition that

$$\frac{d}{dt}\left[U(k,t)\right] = -c_1 k^2 \sin(k^2 t) + c_2 k^2 \cos(k^2 t)$$

which implies that

$$\frac{d}{dt}\left[U(k,t)\right]\bigg|_{t=0} = c_2 k^2 = 0.$$

Since this equation must hold for all k, this implies that $c_2 = 0$. Thus, the solution (5) becomes

$$U(k,t) = F(k)\cos(k^2 t).$$

From the Convolution Theorem, we see that

$$\begin{aligned} u(x,t) = \mathscr{F}^{-1}\left\{F(k)\cos(k^2 t)\right\} &= (f * g)(x) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\xi) g(\xi) d\xi \end{aligned}$$

where $g(x) = \mathscr{F}^{-1}\left\{\cos(k^2 t)\right\}$.

Note that it can be shown that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik(x \pm kt)} dk = \frac{1}{2}\left(\frac{(1 \pm i)e^{\mp ix^2/4t}}{\sqrt{t}}\right).$$

Thus, using the definition of the complex exponential, we have that

$$\begin{aligned}
g(x) &= \mathscr{F}^{-1}\left\{\cos(k^2 t)\right\} \\
&= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik(x+kt)} + e^{ik(x-kt)} dk \\
&= \frac{1}{4\sqrt{t}}\left[(1+i)e^{-ix^2/4t} + (1-i)e^{ix^2/4t}\right] \\
&= \frac{1}{2\sqrt{t}}\left[\frac{e^{ix^2/4t} + e^{-ix^2/4t}}{2} + \frac{e^{ix^2/4t} - e^{-ix^2/4t}}{2i}\right] \\
&= \frac{\cos\left(\frac{x^2}{4t}\right) + \sin\left(\frac{x^2}{4t}\right)}{2\sqrt{t}}.
\end{aligned}$$

Therefore, the solution to the original differential equation is given by

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\xi) g(\xi) d\xi \\
&= \frac{1}{2\sqrt{2\pi t}} \int_{-\infty}^{\infty} f(x-\xi)\left(\cos\left(\frac{\xi^2}{4t}\right) + \sin\left(\frac{\xi^2}{4t}\right)\right) d\xi.
\end{aligned}$$

□

Problem 2.14. Obtain the Fourier cosine transforms of the following functions:

a. xe^{-ax}, $a > 0$.

Solution. Recall that the definition of the Fourier cosine transform of a function $f(x)$ is given by

$$\mathscr{F}_c\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos kx f(x) dx.$$

a. From the definition of the Fourier cosine transform we have that

$$\mathscr{F}_c\left\{xe^{-ax}\right\} = \sqrt{\frac{2}{\pi}} \int_0^\infty xe^{-ax} \cos kx dx.$$

Using the definition of the complex exponential, we see that

$$\begin{aligned}\mathscr{F}_c\left\{xe^{-ax}\right\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty xe^{-ax} \left[\frac{e^{-ikx} + e^{ikx}}{2}\right] dx \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty x\left[e^{-(a+ik)x} + e^{-(a-ik)x}\right] dx.\end{aligned}$$

Now, for $w = a \pm ik$ with $a > 0$, we see using integration by parts with $u = x$ and $dv = e^{-wx}dx$ that

$$\int_0^\infty xe^{-wx} dx = -\frac{xe^{-wx}}{w}\bigg|_0^\infty + \frac{1}{w}\int_0^\infty e^{-wx} dx.$$

Note that

$$\lim_{x\to\infty} \left|e^{-wx}\right| = \lim_{x\to\infty} \left|e^{-(a\pm ik)x}\right| = \lim_{x\to\infty} \left|e^{-ax}\right| \left|e^{\mp ikx}\right| \leq \lim_{x\to\infty} \left|e^{-ax}\right| = 0.$$

This implies that $\lim_{x\to\infty} e^{-wx} = 0$. Thus,

$$\begin{aligned}\int_0^\infty xe^{-wx} dx &= -\frac{xe^{-wx}}{w}\bigg|_0^\infty + \frac{1}{w}\int_0^\infty e^{-wx} dx \\ &= -\frac{1}{w^2}\left[e^{-wx}\Big|_0^\infty\right] \\ &= \frac{1}{w^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathscr{F}_c\left\{xe^{-ax}\right\} &= \frac{1}{\sqrt{2\pi}}\left[\int_0^\infty xe^{-(a+ik)x}dx + \int_0^\infty xe^{-(a-ik)x}dx\right] \\ &= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{(a+ik)^2} + \frac{1}{(a-ik)^2}\right] \\ &= \sqrt{\frac{2}{\pi}} \frac{a^2 - k^2}{(a^2+k^2)^2}.\end{aligned}$$

□

Problem 2.15. Find the Fourier sine transform of the following functions:

a. xe^{-ax}, $a > 0$.

b. $\dfrac{e^{-ax}}{x}$, $a > 0$.

Solution. Recall that the definition of the Fourier sine transform of a function $f(x)$ is given by

$$\mathscr{F}_s\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \sin kx f(x) dx.$$

a. From the definition of the Fourier sine transform we have that

$$\mathscr{F}_s\left\{xe^{-ax}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty xe^{-ax}\sin kx dx.$$

Using the definition of the complex exponential, we see that

$$\begin{aligned}\mathscr{F}_s\left\{xe^{-ax}\right\} &= \sqrt{\frac{2}{\pi}}\int_0^\infty xe^{-ax}\left[\frac{e^{ikx}-e^{-ikx}}{2i}\right]dx \\ &= -\frac{i}{\sqrt{2\pi}}\int_0^\infty x\left[e^{-(a-ik)x}-e^{-(a+ik)x}\right]dx.\end{aligned}$$

Now, for $w = a \pm ik$ with $a > 0$, we see using integration by parts with $u = x$ and $dv = e^{-wx}dx$ that

$$\int_0^\infty xe^{-wx}dx = -\frac{xe^{-wx}}{w}\bigg|_0^\infty + \frac{1}{w}\int_0^\infty e^{-wx}dx.$$

Note that

$$\lim_{x\to\infty}\left|e^{-wx}\right| = \lim_{x\to\infty}\left|e^{-(a\pm ik)x}\right| = \lim_{x\to\infty}\left|e^{-ax}\right|\left|e^{\mp ikx}\right| \le \lim_{x\to\infty}\left|e^{-ax}\right| = 0.$$

This implies that $\lim_{x\to\infty} e^{-wx} = 0$. Thus,

$$\begin{aligned}\int_0^\infty xe^{-wx}dx &= -\frac{xe^{-wx}}{w}\bigg|_0^\infty + \frac{1}{w}\int_0^\infty e^{-wx}dx \\ &= -\frac{1}{w^2}\left[e^{-wx}\big|_0^\infty\right] \\ &= \frac{1}{w^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathscr{F}_s\left\{xe^{-ax}\right\} &= -\frac{i}{\sqrt{2\pi}}\left[\int_0^\infty xe^{-(a-ik)x}dx - \int_0^\infty xe^{-(a+ik)x}dx\right] \\ &= -\frac{i}{\sqrt{2\pi}}\left[\frac{1}{(a-ik)^2}-\frac{1}{(a+ik)^2}\right] \\ &= \sqrt{\frac{2}{\pi}}\frac{2ak}{\left(a^2+k^2\right)^2}.\end{aligned}$$

b. From our table of Fourier sine transforms, we see for $a > 0$ that

$$\mathscr{F}_s\left\{e^{-ax}\right\} = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin(kx)dx = \sqrt{\frac{2}{\pi}} \left(\frac{k}{k^2 + a^2}\right).$$

Thus, we must have that both sides are equal after integrating with respect to a from a to ∞, i.e.

$$\sqrt{\frac{2}{\pi}} \int_a^\infty \left[\int_0^\infty e^{-ax} \sin(kx)dx\right] da = \sqrt{\frac{2}{\pi}} \int_a^\infty \left(\frac{k}{k^2 + a^2}\right) da.$$

For the integral on the left, since the integrand is continuous on the domain of integration, we can interchange the order of integration and we see that

$$\begin{aligned}\sqrt{\frac{2}{\pi}} \int_a^\infty \left[\int_0^\infty e^{-ax} \sin(kx)dx\right] da &= \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(kx) \left[\int_a^\infty e^{-ax} da\right] dx \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \sin(kx)dx \\ &= \mathscr{F}_s\left\{\frac{e^{-ax}}{x}\right\}.\end{aligned}$$

Therefore,

$$\mathscr{F}_s\left\{\frac{e^{-ax}}{x}\right\} = \sqrt{\frac{2}{\pi}} \int_a^\infty \left(\frac{k}{k^2 + a^2}\right) da.$$

Note, the indefinite integral evaluates to $\tan^{-1}\left(\frac{a}{k}\right)$ and we therefore see that

$$\begin{aligned}\mathscr{F}_s\left\{\frac{e^{-ax}}{x}\right\} &= \sqrt{\frac{2}{\pi}} \int_a^\infty \left(\frac{k}{k^2 + a^2}\right) da \\ &= \sqrt{\frac{2}{\pi}} \left(\tan^{-1}\left(\frac{a}{k}\right)\Big|_a^\infty\right) \\ &= \sqrt{\frac{2}{\pi}} \left(\frac{\pi}{2} - \tan^{-1}\left(\frac{a}{k}\right)\right)\end{aligned}$$

$\square$

Problem 2.20. Apply the Fourier cosine transform to find the solution $u(x, y)$ of the problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \qquad 0 < x < \infty, \quad 0 < y < \infty \\ u(x, 0) &= H(a - x), \qquad x < a \\ u_x(0, y) &= 0, \qquad 0 < x, y < \infty. \end{aligned}$$

Solution. Consider the function $u(x, y)$. The Fourier cosine transform of u with respect to x is defined as

$$\mathscr{F}_c\{u(x, y)\} = U_c(k, y) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x, y) \cos(kx) dx.$$

From this definition we see using the Leibniz integral rule that

$$\begin{aligned} \mathscr{F}_c\left\{\frac{\partial^n u(x, y)}{\partial y^n}\right\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^n u(x, y)}{\partial y^n} \cos(kx) dx \\ &= \frac{d^n}{dy^n}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty u(x, y) \cos(kx) dx\right] \\ &= \frac{d^n}{dy^n}\left[\mathscr{F}_c\{u(x, y)\}\right]. \end{aligned}$$

The transforms of the partials of u with respect to x are not as easy to characterize. Nevertheless, we see from the properties of the Fourier cosine transform that

$$\mathscr{F}_c\left\{\frac{\partial u(x, y)}{\partial x}\right\} = k\mathscr{F}_s\{u(x, y)\} - \sqrt{\frac{2}{\pi}} u(0, y)$$

and

$$\mathscr{F}_c\left\{\frac{\partial^2 u(x, y)}{\partial x^2}\right\} = -k^2\mathscr{F}_c\{u(x, y)\} - \sqrt{\frac{2}{\pi}} u_x(0, y)$$

Let $U_c(x, y) = \mathscr{F}_c\{u(x, y)\}$. Then, applying the Fourier cosine transform to the first differential equation shows that

$$\mathscr{F}_c\{u_{xx} + u_{yy}\} = -k^2 U_c(k, y) - \sqrt{\frac{2}{\pi}} u_x(0, y) + \frac{d^2}{dy^2}\left[U_c(k, y)\right] = 0 = \mathscr{F}_c\{0\}.$$

From the third equation we see that $u_x(0, y) = 0$ for all $0 < x, y < \infty$ which implies that the above equation reduces to

$$\frac{d^2}{dy^2}\left[U_c(k, y)\right] - k^2 U_c(k, y) = 0.$$

This is a second-order linear homogeneous differential equation, the solution to which is readily seen to be

$$U_c(k, y) = c_1 e^{-ky} + c_2 e^{ky}.$$

However, since $U_c(k,y) \to 0$ as $k \to \infty$, we must have that $c_2 = 0$. Thus, the solution to the previous differential equation is given by

$$U_c(k,y) = c_1 e^{-ky}. \tag{6}$$

We now apply the Fourier cosine transform to the second differential equation yielding

$$\mathscr{F}_c\{u(x,0)\} = U_c(k,0) = \mathscr{F}_c\{H(a-x)\}.$$

Using the form (6) of the solution to the transformed differential equation and a table of Fourier cosine transforms we see that

$$U_c(k,0) = c_1 = \mathscr{F}_c\{H(a-x)\} = \sqrt{\frac{2}{\pi}}\left(\frac{\sin ak}{k}\right).$$

Thus, the solution to the transformed differential equation with the boundary conditions listed above is given by

$$U_c(k,y) = \mathscr{F}_c\{H(a-x)\}\, e^{-ky} = \sqrt{\frac{2}{\pi}}\left(\frac{\sin ak}{k}\right) e^{-ky}.$$

Therefore, taking the inverse Fourier cosine transform to both sides shows that the solution to the original differential equation is given by

$$\begin{aligned} u(x,y) = \mathscr{F}_c^{-1}\{U_c(k,y)\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{\sin ak}{k}\right) e^{-ky} \cos kx dk \\ &= \frac{2}{\pi}\int_0^\infty \left(\frac{\sin ak}{k}\right) e^{-ky}\cos kx dk. \end{aligned}$$

□

Problem 2.22. Solve the diffusion equation in the semi-infinite line

$$u_t = \kappa u_x x, \qquad 0 \leq x < \infty, \quad t > 0,$$

with the boundary and initial data

$$\begin{aligned} u(0,t) &= 0 && \text{for } t > 0, \\ u(x,t) &\to 0 && \text{as } x \to \infty \text{ for } t > 0, \\ u(x,0) &= f(x) && \text{for } 0 < x < \infty. \end{aligned}$$

Solution. □