Test 1

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Problem 1. a. Use the Frobenius method to find a series expansion of $x = -2$ of the general solution of the equation

$$x(x+2)y'' + (x+1)y' - 4y = 0. \tag{1}$$

b. Use your answer to part a. to find a series solution of the BVP

$$x(x+2)y'' + (x+1)y' - 4y = 0, \quad y(-2) = y(-1.5) = 1.$$

Solution. a. Note that the differential equation (1) may be written as

$$y'' + \left[\frac{x+1}{x(x+2)}\right] y' + \left[\frac{-4}{x(x+2)}\right] y = 0.$$

Since the functions

$$p_1(x) = \frac{x+1}{x(x+2)}, \quad p_0(x) = \frac{-4}{x(x+2)}$$

are not analytic at $x = -2$, but both $(x+2)p_1(x)$ and $(x+2)^2 p_0(x)$ are analytic at that point, we classify the point $x = -2$ as a regular singular point.

As such, we rewrite equation (1) as

$$y'' + \left[\frac{p(x)}{x+2}\right] y' + \left[\frac{q(x)}{(x+2)^2}\right] y = 0.$$

where $p(x)$ and $q(x)$ are defined as the following analytic functions at $x = -2$:

$$p(x) = \frac{x+1}{x}, \quad q(x) = -\frac{4(x+2)}{x}.$$

Using the following power series expansion of the function $f(x) = 1/x$ about $x = -2$,

$$\frac{1}{x} = \sum_{n=0}^{\infty} \left[\frac{-1}{2^{n+1}}\right] (x+2)^n,$$

we may write the power series expansions of the analytic functions $p(x)$ and $q(x)$ about $x = -2$ as follows:

$$\begin{aligned} p(x) &= \sum_{n=0}^{\infty} p_n (x+2)^n = \frac{1}{2} + \sum_{n=1}^{\infty} \left[\frac{-1}{2^{n+1}}\right] (x+2)^n \\ q(x) &= \sum_{n=0}^{\infty} q_n (x+2)^n = \sum_{n=1}^{\infty} \left[\frac{1}{2^{n-2}}\right] (x+2)^n. \end{aligned} \tag{2}$$

Identifying $p_0 = 1/2$ and $q_0 = 0$, the indicial polynomial associated to the differential equation (1) is

$$P(\alpha) = \alpha^2 + (p_0 - 1)\alpha + q_0 = \alpha\left(\alpha - \frac{1}{2}\right).$$

The two roots to the indicial polynomial are $\alpha_1 = 1/2$ and $\alpha_2 = 0$. Since the roots of the indicial polynomial do not differ by an integer, there exist two linearly independent solutions in Frobenius form. Therefore, the two linearly independent solutions are

$$y_1(x) = \sum_{n=0}^{\infty} a_n (x+2)^{n+\alpha_1}, \quad y_2(x) = \sum_{n=0}^{\infty} b_n (x+2)^{n+\alpha_2} \tag{3}$$

where the sequence a_n satisfies the recurrence relations

$$\begin{aligned} P(\alpha_1)a_0 &= 0 \\ P(\alpha_1 + n)a_n &= -\sum_{k=0}^{n-1} \left[(\alpha_1 + k)p_{n-k} + q_{n-k}\right] a_k, \qquad n = 1, 2, \ldots \end{aligned} \tag{4}$$

with $a_0 \neq 0$ and the sequence b_n satisfies the recurrence relations

$$\begin{aligned} P(\alpha_2)b_0 &= 0 \\ P(\alpha_2 + n)b_n &= -\sum_{k=0}^{n-1} \left[(\alpha_2 + k)p_{n-k} + q_{n-k}\right] b_k, \qquad n = 1, 2, \ldots \end{aligned} \tag{5}$$

with $b_0 \neq 0$. Thus, we need only solve the recurrence relations (4) and (5) to completely determine the linearly independent solutions (3).

The sequence defining the solution $y_1(x)$ associated to the root $\alpha_1 = 1/2$ satisfies recurrence relation (4). Since $P(\alpha_1) = 0$, the first equation of the recurrence relation (4) is satisfied and using the sequences defining the analytic functions $p(x)$ and $q(x)$, we have that the other equation becomes

$$\begin{aligned} P(n + 1/2)a_n &= -\sum_{k=0}^{n-1} \left[-\frac{(k+1/2)}{2^{n-k+1}} + \frac{1}{2^{n-k-2}}\right] a_k \\ &= \sum_{k=0}^{n-1} \left[\frac{2k - 15}{2^{n-k+2}}\right] a_k, \qquad n = 1, 2, \ldots. \end{aligned} \tag{6}$$

We can prove through induction that the above relation satisfies the formula

$$a_n = \frac{4n^2 - 4n - 15}{8n^2 + 4n} a_{n-1}, \qquad n = 1, 2, \ldots.$$

To see this we can note that

$$a_1 = \frac{4 - 4 - 15}{8 + 4} a_0 = -\frac{5}{4} a_0$$

and have established that the formula holds for $n = 1$. Now suppose the formula holds for general $n > 1$. Using our supposition, we see from relation (6) that

$$\begin{aligned} P(n + 1 + 1/2) a_{n+1} &= \sum_{k=0}^{n} \left[\frac{2k - 15}{2^{n-k+3}} \right] a_k \\ &= \frac{1}{2} \sum_{k=0}^{n-1} \left[\frac{2k - 15}{2^{n-k+2}} \right] a_k + \frac{2n - 15}{8} a_n \\ &= \left[\frac{P(n + 1/2)}{2} + \frac{2n - 15}{8} \right] a_n. \end{aligned}$$

Performing some algebra on this expression we see that

$$a_{n+1} = \frac{4(n+1)^2 - 4(n+1) - 15}{8(n+1)^2 + 4(n+1)} a_n$$

and the formula holds for $n + 1$ completing the proof. Mathematica reports that the solution to this recurrence relation is

$$a_n = \left[-\frac{\Gamma(2)}{\Gamma(-1/2)} \frac{(2n+3)\Gamma(n - 3/2)}{2^{n+1}\Gamma(n+1)} \right] a_0 = \frac{1}{\sqrt{2\pi}} \left[\frac{(2n+3)\Gamma(n - 3/2)}{2^{n+1}\Gamma(n+1)} \right] a_0$$

Therefore, using (3), the solution to the differential equation (1) associated to the root $\alpha_1 = 1/2$ is

$$\begin{aligned} y_1(x) &= a_0 \sum_{n=0}^{\infty} \left[\frac{(2n+3)\Gamma(n - 3/2)}{\sqrt{2\pi} 2^{n+1}\Gamma(n+1)} \right] (x+2)^{n+1/2} \\ &= a_0 \left[\frac{-(x+1)\sqrt{-x(x+2)}}{\sqrt{2}} \right] \end{aligned} \tag{7}$$

which has radius of convergence 2 centered at $x = -2$.

We now look to identify the solution $y_2(x)$. The sequence defining the solution $y_2(x)$ associated to the root $\alpha_2 = 0$ satisfies recurrence relation (5). Since $P(\alpha_2) = 0$, the first equation of the recurrence relation (5) is satisfied and using the sequences defining the analytic functions $p(x)$ and $q(x)$, we have that the other equation becomes

$$\begin{aligned} P(n) b_n &= -\sum_{k=0}^{n-1} \left[-\frac{k}{2^{n-k+1}} + \frac{1}{2^{n-k-2}} \right] b_k \\ &= \sum_{k=0}^{n-1} \left[\frac{k - 8}{2^{n-k+1}} \right] b_k, \qquad n = 1, 2, \ldots. \end{aligned} \tag{8}$$

We can prove through induction that the above relation satisfies the formula

$$b_n = \frac{-n^2+2n+3}{-2n^2+n} b_{n-1}, \qquad n = 1, 2, \ldots.$$

To see this we can note that

$$b_1 = \frac{-1+2+3}{-2+1} b_0 = -4b_0$$

and have established that the formula holds for $n = 1$. Now suppose the formula holds for general $n > 1$. Using our supposition, we see from relation (8) that

$$\begin{aligned} P(n+1)b_{n+1} &= \sum_{k=0}^{n} \left[\frac{k-8}{2^{n-k+2}} \right] b_k \\ &= \frac{1}{2} \sum_{k=0}^{n-1} \left[\frac{k-8}{2^{n-k+1}} \right] b_k + \frac{n-8}{4} b_n \\ &= \left[\frac{P(n)}{2} + \frac{n-8}{4} \right] b_n. \end{aligned}$$

Performing some algebra on this expression we see that

$$b_{n+1} = \frac{-(n+1)^2 + 2(n+1) + 3}{-2(n+1)^2 + (n+1)} b_n$$

and the formula holds for $n+1$ completing the proof.

Note that $b_3 = 0$ which implies that $b_n = 0$ for $n \geq 3$ and that

$$b_n = \begin{cases} b_1 = -4b_0 \\ b_2 = 2b_0 \\ b_n = 0 & \text{for } n \geq 3 \end{cases}.$$

Therefore, using (3), the solution to the differential equation (1) associated to the root $\alpha_2 = 0$ is

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n (x+2)^n \\ &= b_0 \left[1 - 4(x+2) + 2(x+2)^2 \right] \\ &= b_0 \left[2x^2 + 4x + 1 \right]. \end{aligned} \tag{9}$$

The general solution to the differential equation is then

$$y(x) = a_0 \left[\frac{-(x+1)\sqrt{-x(x+2)}}{\sqrt{2}} \right] + b_0 \left[2x^2 + 4x + 1 \right]. \tag{10}$$

b. Note that (10) is the general solution to the BVP. So, for $-2 \leq x \leq -3/2$,

$$y(x) = a_0 \left[\frac{-(x+1)\sqrt{-x(x+2)}}{\sqrt{2}} \right] + b_0 \left[2x^2 + 4x + 1\right]$$

satisfies the differential equation. From the boundary conditions $y(-2) = y(-3/2) = 1$ we see that

$$\begin{aligned} y(-2) &= b_0 = 1 \\ y(-3/2) &= \frac{a_0\sqrt{3}}{4\sqrt{2}} - \frac{b_0}{2} = 1 \end{aligned}$$

from which we readily see that $a_0 = 2\sqrt{6}$ and $b_0 = 1$. Therefore, the solution to the BVP is

$$y(x) = 2\sqrt{6} \left[\frac{-(x+1)\sqrt{-x(x+2)}}{\sqrt{2}} \right] + \left[2x^2 + 4x + 1\right].$$

□

Problem 2. a. Transform the equation $x(x+2)y'' + (x+1)y' - 4y = 0$ to the form

$$\ddot{y} + t^{-1}p(t)\dot{y} + t^{-2}q(t)y = 0 \tag{11}$$

and use the result to determine whether the point at ∞ is an ordinary, regular singular, or irregular singular point for the original equation.

b. Apply an appropriate method to equation (11) to obtain two series that represent linearly independent solutions of the original equation as $x \to +\infty$.

Solution. a. In order to investigate the point at $+\infty$, we map the point at $+\infty$ into 0 and identify the point at 0 in the resulting equation. We can complete the mapping by making the following transformations

$$\begin{aligned} x &= \frac{1}{t} \\ y' &= -t^2\dot{y} \\ y'' &= t^4\ddot{y} + 2t^3\dot{y}. \end{aligned}$$

Thus, the differential equation (1) becomes

$$\begin{aligned} Ly &= x(x+2)y'' + (x+1)y' - 4y \\ &= \left(\frac{2t+1}{t^2}\right)\left(t^4\ddot{y} + 2t^3\dot{y}\right) - t(t+1)\dot{y} - 4y \\ &= t^2(2t+1)\ddot{y} + t(3t+1)\dot{y} - 4y. \end{aligned}$$

We can write this differential equation in the form

$$t^2(2t+1)\ddot{y} + t(3t+1)\dot{y} - 4y = \ddot{y} + \left[\frac{3t+1}{t(2t+1)}\right]\dot{y} + \left[-\frac{4}{t^2(2t+1)}\right]y = 0. \tag{12}$$

Identifying $p(t) = (3t+1)/(2t+1)$ and $q(t) = -4/(2t+1)$, we see that the equation is written as

$$\ddot{y} + \left[\frac{3t+1}{t(2t+1)}\right]\dot{y} + \left[-\frac{4}{t^2(2t+1)}\right]y = \ddot{y} + \left[\frac{p(t)}{t}\right]\dot{y} + \left[\frac{q(t)}{t^2}\right]y = 0.$$

Note that $p(t)$ and $q(t)$ are both analytic at $t = 0$. Since $t^{-1}p(t)$ and $t^{-2}q(t)$ are not analytic at $t = 0$ but both $t(t^{-1}p(t))$ and $t^2(t^{-2}q(t))$ are analytic at that point, we see that the point $t = 0$ is a regular singular point. As a result we conclude that $x = +\infty$ is also a regular singular point.

b. After identifying that the point $t = 0$ is a regular singular point of equation (12) we can expect that at least one solution to the equation is of Frobenius form. We begin by finding the power series expansions of the functions $p(t)$ and $q(t)$. Using the following power series expansion of $f(t) = 1/(2t+1)$ about $t = 0$,

$$\frac{1}{2t+1} = \sum_{n=0}^{\infty}(-2)^n t^n,$$

we see that the power series expansions of the analytic functions $p(t)$ and $q(t)$ about $t = 0$ are

$$\begin{aligned} p(t) &= \sum_{n=0}^{\infty} p_n t^n = 1 + \sum_{n=1}^{\infty} \left[3(-2)^{n-1} + (-2)^n\right] t^n = 1 + \sum_{n=1}^{\infty} \left[(-1)^{n+1} 2^{n-1}\right] t^n \\ q(t) &= \sum_{n=0}^{\infty} q_n t^n = \sum_{n=0}^{\infty} 2(-2)^{n+1} t^n = \sum_{n=0}^{\infty} (-1)^{n+1} 2^{n+2} t^n. \end{aligned} \tag{13}$$

Identifying $p_0 = 1$ and $q_0 = -4$, the indicial polynomial associated to the differential equation (12) is

$$P(\alpha) = \alpha^2 + (p_0 - 1)\alpha + q_0 = (\alpha - 2)(\alpha + 2).$$

The two roots to the indicial polynomial are $\alpha_1 = 2$ and $\alpha_2 = -2$. Since these roots differ by a positive integer, the solution to the differential equation associated to the root α_1 exists in Frobenius form and further analysis is needed to determine if the other solution to the differential equation associated to the root α_2 is also in Frobenius form. Using the sequences defining the power series expansions of $p(t)$ and $q(t)$ and setting $N = \alpha_1 - \alpha_2 = 4$, if

$$-\sum_{k=0}^{N-1} \left[(k + \alpha_2) p_{N-k} + q_{N-k}\right] a_k = 0$$

with $a_0 \neq 0$, then the solution to the differential equation associated to the root α_2 is in Frobenius form, otherwise only the solution associated to the root α_1 is in Frobenius form.

We proceed by finding $y_1(t)$, the solution to the differential equation associated to the root $\alpha_1 = 2$. This solution is in Frobenius form, i.e.

$$y_1(t) = \sum_{n=0}^{\infty} a_n t^{n+2} \tag{14}$$

where $a_0 \neq 0$ and a_n satisfies the recurrence relation

$$\begin{aligned} P(2)a_0 &= 0 \\ P(n+2)a_n &= -\sum_{k=0}^{n-1} \left[(k+2)p_{n-k} + q_{n-k}\right] a_k, \qquad n = 1, 2, \ldots \end{aligned} \tag{15}$$

and p_n, q_n are the sequences defining the power series expansions of $p(t)$ and $q(t)$. Thus, solving the recurrence relation completely identifies the first solution. The first equation of the recurrence relation is satisfied since $\alpha_1 = 2$ is a root of the indicial polynomial $P(z) = (z-2)(z+2)$. The other recurrence relation reduces to

$$\begin{aligned} P(n+2)a_n &= -\sum_{k=0}^{n-1} \left[(k+2)p_{n-k} + q_{n-k}\right] a_k \\ &= \sum_{k=0}^{n-1} (-1)^{n-k+2} 2^{n-k-1} (k+10) a_k \qquad n = 1, 2, \ldots. \end{aligned} \tag{16}$$

We can prove through induction that the above relation satisfies the formula

$$a_n = \frac{-2n^2 - 5n - 3}{n^2 + 4n} a_{n-1}, \qquad n = 1, 2, \ldots.$$

To see this we can note that

$$a_1 = \frac{-2 - 5 - 3}{1 + 4} a_0 = -2a_0$$

and have established that the formula holds for $n = 1$. Now suppose the formula holds for general $n > 1$. Using our supposition, we see from relation (16) that

$$\begin{aligned}
P(n+3)a_{n+1} &= \sum_{k=0}^{n} (-1)^{n-k+2} 2^{n-k-1} (k+10) a_k \\
&= -2 \sum_{k=0}^{n-1} (-1)^{n-k+2} 2^{n-k-1} (k+10) a_k - (n+10) a_n \\
&= -\left[2P(n+2) + (n+10)\right] a_n.
\end{aligned}$$

Performing some algebra on this equation we see that

$$a_{n+1} = \frac{-2(n+1)^2 - 5(n+1) - 3}{(n+1)^2 + 4(n+1)} a_{n-1}$$

and the formula holds for $n + 1$ completing the proof. Mathematica reports that the solution to this new recurrence relation is

$$\begin{aligned}
a_n &= \left[\frac{\Gamma(5)}{\Gamma(5/2)} \frac{(-2)^n (n+1)\Gamma(n+5/2)}{\Gamma(n+5)}\right] a_0 \\
&= \left[\frac{32(-2)^n (n+1)\Gamma(n+5/2)}{\sqrt{\pi}\Gamma(n+5)}\right] a_0.
\end{aligned}$$

Therefore, the solution to the differential equation associated to the root $\alpha_1 = 2$ is

$$\begin{aligned}
y_1(t) &= a_0 \sum_{n=0}^{\infty} \left[\frac{32(-2)^n (n+1)\Gamma(n+5/2)}{\sqrt{\pi}\Gamma(n+5)}\right] t^{n+2} \\
&= \begin{cases} 0 & \text{for } t = 0 \\ a_0 \left[\frac{4\left(2+4t+t^2-2\sqrt{2t+1}-2t\sqrt{2t+1}\right)}{t^2}\right] & \text{for } 0 < |t| < 1/2 \end{cases}.
\end{aligned}$$

which has radius of convergence $1/2$ centered at $t = 0$.

To find the second solution we generalize the Frobenius solution and substitute the solution into the original differential equation to obtain

$$Ly(t, \alpha) = a_0 t^{\alpha-2} P(\alpha).$$

Note that if $\alpha = \alpha_1 = 2$, then the solution is the solution already obtained to the homogeneous equation $Ly(t, \alpha) = 0$. Differentiating both sides of the equation with respect to α yields that

$$L\left[\frac{\partial}{\partial \alpha} y(t,\alpha)\bigg|_{\alpha=\alpha_1}\right] = a_0 P'(\alpha_1) t^{\alpha_2+N-2} \tag{17}$$

so that $\frac{\partial}{\partial \alpha} y(t,\alpha)\big|_{\alpha=\alpha_1}$ is a particular solution of the above differential equation. We can construct a second particular solution and subtract it from this particular solution to obtain the solution to the homogeneous equation. Note that the second particular solution has a Frobenius expansion $\sum_{n=0}^{\infty} c_n t^{n+\alpha_2}$. Using this form of the solution and substituting into (17) and equating coefficients with t^{α_2+N-2} yields the following relations

$$\begin{aligned} P(n+\alpha_2)c_n &= -\sum_{k=0}^{n-1}[(k+\alpha_2)p_{n-k} + q_{n-k}]c_k & n \neq 0, N \\ a_0 &= \frac{1}{P'(\alpha_1)} \sum_{k=0}^{N-1}[(k+\alpha_2)p_{N-k} + q_{N-k}]c_k & n = N \end{aligned} \tag{18}$$

where $c_0 \neq 0$ and $c_N \neq 0$ are arbitrary. The second solution to the homogeneous differential equation $Ly = 0$ is then

$$\begin{aligned} y_2(t) &= \sum_{n=0}^{\infty} c_n t^{n+\alpha_2} - \frac{\partial}{\partial \alpha} y(t,\alpha)\bigg|_{\alpha=\alpha_1} \\ &= \sum_{n=0}^{\infty} c_n t^{n+\alpha_2} - \log t \sum_{n=0}^{\infty} a_n t^{n+\alpha_1} - \sum_{n=0}^{\infty} \frac{\partial a_n(\alpha)}{\partial \alpha}\bigg|_{\alpha=\alpha_1} t^{n+\alpha_1} \end{aligned}$$

where a_n is the sequence of coefficients defining the first solution to the homogeneous differential equation.

We begin by determining the sequence c_n. For this particular differential equation we have that $\alpha_1 = 2$, $\alpha_2 = -2$, and $N = \alpha_1 - \alpha_2 = 4$. We can see that the first equation of the recurrence relation (18) becomes

$$\begin{aligned} P(n-2)c_n &= -\sum_{k=0}^{n-1}[(k-2)p_{n-k} + q_{n-k}]c_k \\ &= \sum_{k=0}^{n-1}(-1)^{n-k+2}2^{n-k-1}(k+6)c_k & n \neq 0, 4. \end{aligned}$$

We can see from this relation that $c_1 = 2c_0$, $c_2 = c_0/2$, and $c_3 = 0$. This implies that the second equation of the recurrence relation (18) becomes

$$a_0 = \frac{1}{P'(2)} \sum_{k=0}^{4-1}[(k-2)p_{4-k} + q_{4-k}]c_k = \frac{1}{4}\sum_{k=0}^{3}(-1)^{4-k}2^{3-k}(k+6)c_k = 0.$$

Continuing from the first equation for $n > 4$, we see that the recurrence relation becomes

$$\begin{aligned}
P(n-2)c_n &= \sum_{k=0}^{n-1}(-1)^{n-k+2}2^{n-k-1}(k+6)c_k \\
&= \sum_{k=0}^{4-1}(-1)^{n-k+2}2^{n-k-1}(k+6)c_k + \sum_{k=4}^{n-1}(-1)^{n-k+2}2^{n-k-1}(k+6)c_k \\
&= \sum_{k=4}^{n-1}(-1)^{n-k+2}2^{n-k-1}(k+6)c_k
\end{aligned}$$

since $\sum_{k=0}^{4-1}(-1)^{n-k+2}2^{n-k-1}(k+6)c_k = (-1)^{n-1}2^{n-3}(24c_0 - 28c_0 + 4c_0) = 0$ from our definitions of c_k for $0 \leq k \leq 3$. Thus c_n only depends on c_4 for $n > 4$. In a very similar fashion as was shown for the sequence a_n, we see that

$$c_n = \frac{-2n^2 + 11n - 15}{n^2 - 4n} c_{n-1} \quad \text{for } n > 4.$$

Mathematica reports that the solution to this recurrence relation is

$$\begin{aligned}
c_n &= c_4 \left[\frac{\Gamma(5)}{\Gamma(5/2)} \frac{(-1)^n 2^{n-4}(n-3)\Gamma(n-4+5/2)}{\Gamma(n-4+5)} \right] \\
&= c_4 \left[\frac{32(-1)^n 2^{n-4}(n-3)\Gamma(n-4+5/2)}{\sqrt{\pi}\Gamma(n-4+5)} \right]
\end{aligned}$$

Thus, we have that $c_0, c_4 \neq 0$ and

$$c_n = \begin{cases} c_1 = 2c_0 \\ c_2 = c_0/2 \\ c_3 = 0 \\ c_n = c_4 \left[\frac{32(-1)^n 2^{n-4}(n-3)\Gamma(n-4+5/2)}{\sqrt{\pi}\Gamma(n-4+5)} \right] & \text{for } n \geq 4 \end{cases}.$$

Therefore, we have that the solution associated to the root $\alpha_2 = -2$ is

$$\begin{aligned}
y_2(t) &= c_0 t^{-2} + 2c_0 t^{-1} + \frac{c_0}{2} + c_4 \sum_{n=4}^{\infty} \left[\frac{32(-1)^n 2^{n-4}(n-3)\Gamma(n-4+5/2)}{\sqrt{\pi}\Gamma(n-4+5)} \right] t^{n-2} \\
&= c_0 \left[t^{-2} + 2t^{-1} + \frac{1}{2} \right] + c_4 \left[\frac{4(2 + 4t + t^2 - 2\sqrt{2t+1} - 2t\sqrt{2t+1})}{t^2} \right]
\end{aligned}$$

and we have found both linearly independent solutions.

□

Problem 3. + Find the first three terms in the asymptotic expansion as $x \to +\infty$ of a solution of the equation

$$y''' + \frac{y'}{x^3} = x.$$

Solution. $\square$