Homework Assignment 7

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Problem 1. Write at least two necessary conditions and at least two sufficient conditions for a function $f : \mathbb{R}^n \to \mathbb{R}$ to be concave.

Solution. By definition, a function $f : \mathbb{R}^n \to \mathbb{R}$ is concave over the convex set $\Omega \subset \mathbb{R}^n$ if $-f$ is convex over Ω. This definition will allow us to obtain results for concave functions by replacing f with $-f$ in previously obtained results concerning convex functions.

Using this definition and Theorem 22.2, we see that the condition that if for all $\alpha \in (0, 1)$ and for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, we have that

$$f(\alpha \boldsymbol{x} + (1 - \alpha)\boldsymbol{y}) \geq \alpha f(\boldsymbol{x}) + (1 - \alpha) f(\boldsymbol{y})$$

is a necessary and sufficient condition for f to be concave on the convex set Ω.

Further, if the function f is $\mathcal{C}^1$-smooth, we see from the above definition and Theorem 22.4 that the condition that if for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, we have that

$$f(\boldsymbol{x}) \leq f(\boldsymbol{y}) + Df(\boldsymbol{x})(\boldsymbol{x} - \boldsymbol{y})$$

is a necessary and sufficient condition for f to be concave on the open convex set Ω.

Going one last step further, if the function f is $\mathcal{C}^2$-smooth, we see from the above definition and Theorem 22.5 that the condition that if for all $\boldsymbol{x} \in \Omega$, the Hessian matrix $\boldsymbol{F}(\boldsymbol{x})$ of f at $\boldsymbol{x}$ is a negative semi-definite matrix is a necessary and sufficient condition for f to be concave on the open convex set Ω. $\square$

Problem 2. Let $S \subset \mathbb{R}^n$ be a convex set and let $\boldsymbol{x}^* \in S$. Prove that a vector $\boldsymbol{d} \in \mathbb{R}^n$ is a feasible direction at $\boldsymbol{x}^*$ (relative to S) if and only there exists $t_0 > 0$ such that $\boldsymbol{x}^* + t_0\boldsymbol{d} \in S$ with $\boldsymbol{d} \neq \mathbf{0}$.

Solution. □

Problem 3. Recall that

$$\max\{\alpha, \beta\} := \begin{cases} \alpha & \text{if } \alpha \geq \beta \\ \beta & \text{if } \alpha < \beta \end{cases}.$$

Given two convex functions $f_1 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $f_2 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, prove that for $\boldsymbol{x} \in \mathbb{R}^n$, the function

$$f(\boldsymbol{x}) := \max\{f_1(\boldsymbol{x}), f_2(\boldsymbol{x})\}$$

is convex.

Solution. □

Problem 4. Consider the pair of linear programming problems in asymmetric duality:

$$
\begin{array}{lll}
(P_a) & \text{minimize} & f(\boldsymbol{x}) = \boldsymbol{c}^\mathsf{T}\boldsymbol{x} \\
 & \text{subject to} & A\boldsymbol{x} = \boldsymbol{b} \\
 & & \boldsymbol{x} \geq \boldsymbol{0}
\end{array}
\qquad
\begin{array}{lll}
(D_a) & \text{maximize} & F(\boldsymbol{\lambda}) = \boldsymbol{\lambda}^\mathsf{T}\boldsymbol{b} \\
 & \text{subject to} & \boldsymbol{\lambda}^\mathsf{T}A \leq \boldsymbol{c}^\mathsf{T}
\end{array}
$$

a. Prove that (D_a) is a convex programming problem.

b. Write the KKT conditions for (D_a).

c. Suppose that $\boldsymbol{x}^*$ is feasible for (P_a) and $\boldsymbol{\lambda}^*$ is feasible for (D_a). Use the KKT conditions to prove that if $(\boldsymbol{c}^\mathsf{T} - \boldsymbol{\lambda}^{*\mathsf{T}}A)\boldsymbol{x}^* = 0$, then $\boldsymbol{\lambda}^*$ is optimal for (D_a).

Solution. □