Homework Assignment 2

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Problem 1. Use the method of variation of parameters to find the general solution of

$$y'' + 2y' + 2y = \sin x.$$

Solution. Suppose that $Ly = y'' + 2y' + 2y$. The general solution to $Ly = \sin x$ is given by $y = y_0 + y_h$ where y_0 is a particular solution of $Ly = \sin x$ and y_h is the solution to the homogeneous equation $Ly = 0$.

The characteristic equation of the equation $Ly = 0$ is $m(x) = x^2 + 2x + 2$, the roots of which are $m_1 = -1 - i$ and $m_2 = -1 + i$. As the roots of the characteristic equation are complex, the solution to $Ly = 0$ is given by

$$y_h = c_1 e^{-x} \sin x + c_2 e^{-x} \cos x. \tag{1}$$

The method of variation of parameters can be used to find a particular solution y_0. We wish to find functions $u_1(x), u_2(x)$ such that

$$y_0 = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{2}$$

satisfies $Ly_0 = \sin x$ where $y_1(x)$ and $y_2(x)$ are solutions to the homogeneous equation $Ly = 0$. If the functions $u_1(x)$ and $u_2(x)$ are solutions to the system

$$\left\{ \begin{array}{ccccc} u_1' y_1 & + & u_2' y_2 & = & 0 \\ u_1' y_1' & + & u_2' y_2' & = & \sin x \end{array} \right. \tag{3}$$

then (2) will satisfy the original differential equation $Ly = \sin x$ equation. The solution to the system (3) is

$$u_1(x) = -\int \frac{y_2(x) \sin x}{W[\{y_1, y_2\}]} dx \qquad u_2(x) = \int \frac{y_1(x) \sin x}{W[\{y_1, y_2\}]} dx \tag{4}$$

where $W[\{y_1, y_2\}]$ is the Wronskian of the functions y_1 and y_2.

Using (1), we know that $y_1(x) = e^{-x} \sin x$ and $y_2(x) = e^{-x} \cos x$ so the particular solution has the form $y_0 = u_1(x)e^{-x} \sin x + u_2(x)e^{-x} \cos x$. Further, the Wronskian of y_1 and y_2 is

$$W[\{y_1, y_2\}] = \begin{vmatrix} e^{-x} \sin x & e^{-x} \cos x \\ e^{-x} \cos x - e^{-x} \sin x & -e^{-x} \cos x - e^{-x} \sin x \end{vmatrix} = -e^{-2x}.$$

Thus, using (4), we know that

$$\begin{aligned} u_1(x) &= -\int \frac{y_2(x)\sin x}{W[\{y_1, y_2\}]}dx \\ &= \int \frac{e^{-x}\cos x \sin x}{e^{-2x}}dx \\ &= \frac{e^x}{10}\left(-2\cos 2x + \sin 2x\right) + C \end{aligned}$$

and

$$\begin{aligned} u_2(x) &= \int \frac{y_1(x)\sin x}{W[\{y_1, y_2\}]}dx \\ &= -\int \frac{e^{-x}\sin^2 x}{e^{-2x}}dx \\ &= \frac{e^x}{10}\left(-5 + \cos 2x + 2\sin 2x\right) + C. \end{aligned}$$

Therefore, a particular solution to $Ly = \sin x$ is

$$y_0(x) = \frac{1}{10}\left(-2\cos 2x + \sin 2x\right)\sin x + \frac{1}{10}\left(-5 + \cos 2x + 2\sin 2x\right)\cos x$$

and the general solution to $Ly = \sin x$ is

$$\begin{aligned} y(x) &= y_0(x) + y_h(x) \\ &= \frac{1}{10}\left(-2\cos 2x + \sin 2x\right)\sin x + \frac{1}{10}\left(-5 + \cos 2x + 2\sin 2x\right)\cos x \\ &\quad + c_1 e^{-x}\sin x + c_2 e^{-x}\cos x \end{aligned} \tag{5}$$

□

Problem 2. Find the Green function of the IVP

$$y'' + 2y' + 2y = f(x), \quad y(0) = y'(0) = 0.$$

Solution. Let $Ly = f(x)$ denote the differential equation $y'' + 2y' + 2y = f(x)$ together with the initial conditions $y(0) = y'(0) = 0$. The Green function $G(x, a)$ of the IVP $Ly = f(x)$ is defined by the equations

$$\frac{\partial^2 G(x,a)}{\partial x^2} + \frac{2\partial G(x,a)}{\partial x} + 2G(x,a) = \delta(x-a), \qquad G(0,a) = 0, \qquad \frac{\partial G}{\partial x}(0,a) = 0$$

where $\delta(x-a)$ is the Dirac Delta function such that $\int_{-\infty}^{\infty} \delta(x-a)f(x)dx = f(x)$. Note that $G(x, a)$ is continuous at $x = a$ and $\partial G/\partial x$ has a jump discontinuity of magnitude 1 at $x = a$.

If y_1 and y_2 are linearly independent solutions of the homogeneous equation $Ly = 0$, then

$$G(x,a) = \begin{cases} A_1 y_1 + A_2 y_2 & \text{if } x < a \\ B_1 y_1 + B_2 y_2 & \text{if } x > a \end{cases}$$

where A_1, A_2, B_1, and B_2 are undetermined functions. The continuity of $G(x, a)$ at $x = a$ gives the equation

$$A_1 y_1(a) + A_2 y_2(a) = B_1 y_1(a) + B_2 y_2(a).$$

Further, the fact that $\partial G/\partial x$ has a jump discontinuity of magnitude 1 at $x = a$ yields the second equation

$$(B_1 y_1'(a) + B_2 y_2'(a)) - (A_1 y_1'(a) + A_2 y_2'(a)) = 1.$$

Combining these equations, we see that A_1, A_2, B_1, and B_2 are given by

$$B_1 = A_1 - \frac{y_2(a)}{W[y_1(a), y_2(a)]}$$
$$B_2 = A_2 + \frac{y_1(a)}{W[y_1(a), y_2(a)]}$$

From (1), we know that the linearly independent solutions to the homogeneous equation $Ly = 0$ are $y_1(x) = e^{-x}\sin x$ and $y_2(x) = e^{-x}\cos x$. Also, the Wronskian of these solutions is $W[y_1(a), y_2(a)] = -e^{-2a}$. Thus,

$$B_1 = A_1 - \frac{y_2(a)}{W[y_1(a), y_2(a)]} = A_1 + e^a \cos a$$
$$B_2 = A_2 + \frac{y_1(a)}{W[y_1(a), y_2(a)]} = A_2 - e^a \sin a$$

Using the two initial conditions, we can uniquely determine A_1 and A_2 since $G(x, a) = A_1 y_1(a) + A_2 y_2(a)$ satisfies $LG = f(x)$. Since $y(0) = 0$ we see that $A_2 = 0$ and since $y'(0) = 0$ we see that $A_1 - A_2 = 0$ implying that $A_1 = A_2 = 0$. Therefore, the Green function for the IVP $Ly = f(x)$ is

$$G(x,a) = \begin{cases} 0 & \text{if } x < a \\ e^{a-x}(\sin x \cos a - \cos x \sin a) = e^{a-x}\sin(x-a) & \text{if } x > a \end{cases} \tag{6}$$

□

Problem 3. Use your answer to Problem 2 to solve the IVP

$$y'' + 2y' + 2y = \sin x, \quad y(0) = y'(0) = 0.$$

Solution. If we can find the Green function $G(x, a)$ associated to the IVP we know that the particular solution to the IVP can be represented as

$$y_p(x) = \int_{-\infty}^{\infty} G(x, a)\sin(a)da.$$

Note that in problem 2, the Green function (6) is precisely the function we are after. For that Green function, we know that $G(x, a) = 0$ if $x > a$ so that

$$y_p(x) = \int_{-\infty}^{\infty} G(x, a)\sin(a)da = \int_{-\infty}^{x} G(x, a)\sin(a)da.$$

Using the expression for the Green function found in (6), we see that

$$\begin{aligned}
y_p(x) &= \int_{-\infty}^{x} G(x, a)\sin(a)da \\
&= \int_{-\infty}^{x} e^{a-x}\sin(x-a)\sin(a)da \\
&= \frac{-2\cos x + \sin x}{5}.
\end{aligned}$$

Note that the homogeneous solution associated to this differential equation is given by $y_h(x) = c_1 e^{-x}\sin x + c_2 e^{-x}\cos x$ so that the general solution to the differential equation is

$$\begin{aligned}
y(x) &= y_h(x) + y_p(x) \\
&= c_1 e^{-x}\sin x + c_2 e^{-x}\cos x + \frac{-2\cos x + \sin x}{5}.
\end{aligned}$$

Using the initial conditions, we see that $c_1 = 1/5$ and $c_2 = 2/5$. Thus, the solution to the IVP is

$$y(x) = (1/5)e^{-x}\sin x + (2/5)e^{-x}\cos x + \frac{-2\cos x + \sin x}{5} \tag{7}$$

□

Problem 4. Show that if y_1, y_2, and y_3 are three linearly independent solutions of the linear ODE

$$y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = 0$$

and u_1, u_2, u_3 are solutions of the system

$$\left\{\begin{array}{ccccccc} u_1'y_1 & + & u_2'y_2 & + & u_3'y_3 & = & 0, \\ u_1'y_1' & + & u_2'y_2' & + & u_3'y_3' & = & 0, \\ u_1'y_1'' & + & u_2'y_2'' & + & u_3'y_3'' & = & f(x), \end{array}\right. \tag{8}$$

then the function $u = u_1y_1 + u_2y_2 + u_3y_3$ is a solution of

$$Ly = y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = f(x)$$

Solution. We wish to show that $y = \sum_{i=1}^3 u_iy_i$ is a solution of the equation $Ly = f(x)$ given that y_i are linearly independent solutions of the homogeneous equation $Ly = 0$ and u_i are solutions of the system (8). Using the form $y = \sum_{i=1}^3 u_iy_i$, we see that

$$\begin{aligned} y' &= \sum_{i=1}^3 u_iy_i' + u_i'y_i \\ y'' &= \sum_{i=1}^3 u_iy_i'' + 2u_i'y_i' + u_i''y_i \\ y''' &= \sum_{i=1}^3 u_iy_i''' + 3u_i'y_i'' + 3u_i''y_i' + u_i'''y_i. \end{aligned}$$

Thus, we find that for $y = \sum_{i=1}^3 u_iy_i$,

$$\begin{aligned} Ly &= \sum_{i=1}^3 u_iy_i''' + 3u_i'y_i'' + 3u_i''y_i' + u_i'''y_i + p_2(x)\sum_{i=1}^3 u_iy_i'' + 2u_i'y_i' + u_i''y_i \\ &\quad + p_1(x)\sum_{i=1}^3 u_iy_i' + u_i'y_i + p_0(x)\sum_{i=1}^3 u_iy_i \\ &= \sum_{i=1}^3 u_i[y_i''' + p_2(x)u_i'' + p_1(x)y_i' + p_0(x)y_i] \\ &\quad + \sum_{i=1}^3 3u_i'y_i'' + 3u_i''y_i' + u_i'''y_i + 2p_2(x)u_i'y_i' + p_2(x)u_i''y_i + p_1(x)u_i'y_i. \end{aligned}$$

Since y_i are solutions of the homogeneous equation $Ly = 0$, we see that the first sum is 0 and

$$Ly = \sum_{i=1}^3 3u_i'y_i'' + 3u_i''y_i' + u_i'''y_i + 2p_2(x)u_i'y_i' + p_2(x)u_i''y_i + p_1(x)u_i'y_i. \tag{9}$$

We also know that since u_1, u_2, and u_3 are solutions of the system (8) the following implications are true

$$\sum_{i=1}^{3} u_i' y_i = 0 \implies \left[\sum_{i=1}^{3} u_i' y_i\right]' = \sum_{i=1}^{3} u_i'' y_i + u_i' y_i' = 0$$
$$\sum_{i=1}^{3} u_i'' y_i + u_i' y_i' = 0 \implies \left[\sum_{i=1}^{3} u_i'' y_i + u_i' y_i'\right]' = \sum_{i=1}^{3} u_i''' y_i + 2u_i'' y_i' + u_i' y_i'' = 0$$
$$\sum_{i=1}^{3} u_i' y_i' = 0 \implies \left[\sum_{i=1}^{3} u_i' y_i'\right]' = \sum_{i=1}^{3} u_i'' y_i' + u_i' y_i'' = 0$$

Rearranging the terms of (9) and using the above relations we see that

$$\begin{aligned} Ly &= \sum_{i=1}^{3} u_i' y_i'' + \left[\sum_{i=1}^{3} u_i''' y_i + 2u_i'' y_i' + u_i' y_i''\right] + \left[\sum_{i=1}^{3} u_i' y_i'' + u_i'' y_i'\right] \\ &\quad + p_2(x)\left[\sum_{i=1}^{3} u_i' y_i' + u_i'' y_i\right] + p_2(x)\left[\sum_{i=1}^{3} u_i' y_i'\right] + p_1(x)\left[\sum_{i=1}^{3} u_i' y_i\right] \\ &= \sum_{i=1}^{3} u_i' y_i'' \end{aligned}$$

where every term in brackets is 0 as a consequence of the above derived relations or the fact that u_1, u_2, and u_3 are solutions of the system (8). From the third equation of the system (8) we know that $\sum_{i=1}^{3} u_i' y_i'' = f(x)$. Therefore, for $y = \sum_{i=1}^{3} u_i y_i$ satisfying the assumptions of the problem,

$$Ly = \sum_{i=1}^{3} u_i' y_i'' = f(x)$$

showing that y is a solution of the equation $Ly = f(x)$. $\square$

Problem 5. Find the eigenvalues and the respective eigenfunctions for the BVP

$$x^2y'' + xy' + \lambda y = 0, \quad y'(1) = 0, \quad y'(b) = 0$$

where $b > 1$.

Solution. The differential equation stated in this problem is an Euler differential equation. The equation can be transformed into a constant coefficient second order linear differential equation by making the substitution $x(t) = e^t$ and rewriting the differential equation in terms of the independent variable t.

To see this, we note that

$$\begin{aligned} \frac{d}{dt}\left[y(x(t))\right] &= \frac{dy(x(t))}{dx}\frac{dx(t)}{dt} \\ &= \frac{dy(x(t))}{dx}\frac{dx(t)}{dt} \\ &= y'(x(t))x(t) \end{aligned}$$

since $x'(t) = [e^t]' = e^t = x(t)$. Similarly, using the above relation,

$$\begin{aligned} \frac{d^2}{dt^2}[y(x(t))] &= \frac{d}{dt}\left[\frac{dy(x(t))}{dt}\right] \\ &= \frac{d}{dt}\left[\frac{dy(x(t))}{dx}\right]x(t) + \frac{dy(x(t))}{dx}\frac{d}{dt}\left[x(t)\right] \\ &= \left[\frac{dy(x(t))}{dx}\frac{dx(t)}{dt}\right]x(t) + \left[\frac{dy(x(t))}{dx}\right]x(t) \\ &= x(t)^2\frac{d^2y(x(t))}{dx^2} + x(t)\frac{dy(x(t))}{dx} \\ &= x(t)^2y''(x(t)) + x(t)y'(x(t)). \end{aligned}$$

Thus, the original differential equation in the independent variable x can be written as the following differential equation in the independent variable t after making the change of variables $x(t) = e^t$:

$$\left[x^2y''(x) + xy'(x)\right] + \lambda y(x) = [y''(x(t))] + \lambda y(x(t)) = 0. \tag{10}$$

The characteristic equation of the homogeneous second order linear differential equation in the variable t is given by

$$m(z) = z^2 + \lambda. \tag{11}$$

The roots of $m(z)$ are $z_1 = \sqrt{-\lambda}$ and $z_2 = -\sqrt{-\lambda}$. The solution to (10) is thus dependent on the value of λ and as such there are three cases to consider, when $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$.

Case 1: $\lambda < 0$
If $\lambda < 0$, then $\sqrt{-\lambda}$ is a positive real number and the roots of the characteristic equation (11) are real and distinct. Thus, the solution to (10) is

$$y(t) = c_1 e^{\sqrt{-\lambda}t} + c_2 e^{-\sqrt{-\lambda}t}.$$

Using the substitution $t = \log x$, the solution to the differential equation with respect to x becomes

$$\begin{aligned} y(t(x)) = y(x) &= c_1 e^{\sqrt{-\lambda}\log x} + c_2 e^{-\sqrt{-\lambda}\log x} \\ &= c_1 x^{\sqrt{-\lambda}} + c_2 x^{-\sqrt{-\lambda}} \end{aligned}$$

For this solution, we see that

$$y'(x) = c_1\sqrt{-\lambda}x^{\sqrt{-\lambda}-1} - c_2\sqrt{-\lambda}x^{-\sqrt{-\lambda}-1}$$

In this case, the initial condition $y'(1) = 0$ shows that

$$y'(1) = c_1\sqrt{-\lambda} - c_2\sqrt{-\lambda} = \sqrt{-\lambda}(c_1 - c_2) = 0.$$

Since $\lambda < 0$, we know that $\sqrt{-\lambda} \neq 0$ and so $c_1 - c_2 = 0$ or that $c_1 = c_2$.
The initial condition $y'(b) = 0$ for $b > 1$ together with the fact that $c_1 = c_2$ shows that

$$\begin{aligned} y'(b) &= c_1\sqrt{-\lambda}b^{\sqrt{-\lambda}-1} - c_2\sqrt{-\lambda}b^{-\sqrt{-\lambda}-1} \\ &= c_1\sqrt{-\lambda}\left(b^{\sqrt{-\lambda}-1} - b^{-\sqrt{-\lambda}-1}\right) = 0 \end{aligned}$$

showing that since $\lambda < 0$ we must have that $c_1 = 0$ since $\sqrt{-\lambda} \neq 0$ and $b^{\sqrt{-\lambda}-1} \neq b^{-\sqrt{-\lambda}-1}$. Therefore, for $\lambda < 0$, the only solution to the differential equation is the trivial solution and in this case there are no eigenvalues of this equation.

Case 2: $\lambda = 0$
If $\lambda = 0$, then the root of the characteristic equation (11) is $z = 0$ with multiplicity 2. As this is a repeated root, the solution to (10) is

$$y(t) = c_1 + c_2 t.$$

Making the substitution $t = \log x$, we see that

$$y(t(x)) = y(x) = c_1 + c_2 \log x.$$

In this case, we see that $y'(x) = c_2 x^{-1}$. Using the initial condition that $y'(1) = 0$, we see that $c_2 = 0$. Similarly, the condition $y'(b) = 0$ for $b > 1$ yields the same result. Thus, c_1 is free and we see that $y(x) = c_1$ is a non-trivial solution to this problem. Therefore, $\lambda_0 = 0$ is an eigenvalue of this differential equation with associated eigenfunction $y_{\lambda_0}(x) = 1$.

Case 3: $\lambda > 0$
If $\lambda > 0$, then the roots to the characteristic equation (11) are $z_1 = i\sqrt{\lambda}$ and $z_2 = -i\sqrt{\lambda}$ which are complex roots. Thus, the solution to (10) is

$$y(t) = c_1 \cos\left(t\sqrt{\lambda}\right) + c_2 \sin\left(t\sqrt{\lambda}\right).$$

Making the substitution $t = \log x$, we see that

$$y(t(x)) = y(x) = c_1 \cos\left(\sqrt{\lambda}\log x\right) + c_2 \sin\left(\sqrt{\lambda}\log x\right).$$

In this case, we see that

$$y'(x) = -c_1 x^{-1}\sqrt{\lambda}\sin\left(\sqrt{\lambda}\log x\right) + c_2 x^{-1}\sqrt{\lambda}\cos\left(\sqrt{\lambda}\log x\right).$$

The initial condition $y'(1) = 0$ shows that

$$y'(x) = -c_1\sqrt{\lambda}\sin(0) + c_2\sqrt{\lambda}\cos(0) = c_2\sqrt{\lambda} = 0.$$

Since $\sqrt{\lambda} > 0$, we must have that $c_2 = 0$. This fact, combined with $y'(b) = 0$ for $b > 1$, shows that

$$y'(b) = -c_1\sqrt{\lambda}\sin\left(\sqrt{\lambda}\log b\right) = 0 \implies c_1 \sin\left(\sqrt{\lambda}\log b\right) = 0.$$

So either $c_1 = 0$, which leads to the trivial solution, or $\sqrt{\lambda}\log b = n\pi$ for $n = 1, 2, \ldots$. Since $\lambda > 0$ no other values of n will yield $\sin\left(\sqrt{\lambda}\log b\right) = 0$. Thus,

$$\lambda_n = \left(\frac{n\pi}{\log b}\right)^2 \qquad \text{for } n = 1, 2, \ldots$$

are eigenvalues associated to this problem with associated eigenfunctions

$$y_{\lambda_n}(x) = \cos\left(\frac{n\pi \log x}{\log b}\right) \qquad \text{for } b > 1 \text{ and } n = 1, 2, \ldots.$$

We have therefore exhausted all cases and found all eigenvalues associated to the differential equation $x^2 y''(x) + xy'(x) + \lambda y(x) = 0$ along with their eigenfunctions. □