MATH 635 Final Assessment

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Problem 1. Provide a rigorous proof of the case $x_0 = a$ in the Fundamental Lemma of the Calculus of Variations:

Theorem 1 (Fundamental Lemma of the Calculus of Variations)**.** *Suppose $M(x)$ is a continuous function defined on the interval $a \leq x \leq b$. Suppose further that for every continuous function $\zeta(x)$,*

$$\int_a^b M(x)\zeta(x)dx = 0.$$

Then

$$M(x) = 0 \text{ for all } x \in [a, b].$$

Solution. Suppose to the contrary that $M(x) \neq 0$ at the point $x_0 = a$. In that case either $M(a) > 0$ or $M(a) < 0$. Let us first assume that $M(a) > 0$. Due to the continuity of $M(x)$ there is some neighborhood of a where the function is positive, i.e. there is some $\delta > 0$ such that if $|x - a| < \delta$ then

$$|M(x) - M(a)| < \frac{M(a)}{2} \quad \text{for } x \in [a, b].$$

Thus, $0 < M(a)/2 < M(x)$ for $x \in [a, a + \delta)$. Choose the function $\zeta(x)$ to be the linear spline interpolating the points $(a, 3M(a)/2)$ and $(a + \delta, 0)$ with support on $[a, a + \delta)$, i.e.

$$\zeta(x) := \begin{cases} \frac{-3M(a)}{2\delta}(x - (a + \delta)) & \text{if } a \leq x < a + \delta \\ 0 & \text{if } a + \delta \leq x \leq b. \end{cases}$$

Clearly $\zeta(x)$ is continuous and positive on the interval $[a, a + \delta)$. Thus,

$$\int_a^b M(x)\zeta(x)dx = \int_a^{a+\delta} M(x)\zeta(x)dx > \frac{M(a)}{2}\int_a^{a+\delta} \zeta(x)dx > 0.$$

However, by our supposition

$$\int_a^b M(x)\zeta(x)dx = 0,$$

a contradiction. Therefore, if $M(a) > 0$, the function $M(x) \equiv 0$ on the interval $[a, b]$.

If $M(a) < 0$, then we can repeat the argument above replacing $M(x)$ with $-M(x)$. To demonstrate, let us investigate the case when $M(a) < 0$. Due to the continuity of $M(x)$ there is some neighborhood of a where $-M(x)$ is positive, i.e. there is some $\delta > 0$ such that if $|x - a| < \delta$ then

$$|-M(x) + M(a)| < \frac{-M(a)}{2} \quad \text{for } x \in [a, b].$$

Thus, $0 < -M(a)/2 < -M(x)$ for $x \in [a, a + \delta)$. Choose the function $\zeta(x)$ to be the linear spline interpolating the points $(a, -3M(a)/2)$ and $(a + \delta, 0)$ with support on $[a, a + \delta)$, i.e.

$$\zeta(x) := \begin{cases} \frac{3M(a)}{2\delta}(x - (a + \delta)) & \text{if } a \leq x < a + \delta \\ 0 & \text{if } a + \delta \leq x \leq b. \end{cases}$$

Clearly $\zeta(x)$ is continuous and positive on the interval $[a, a+\delta)$. Thus,

$$\int_a^b -M(x)\zeta(x)dx = \int_a^{a+\delta} -M(x)\zeta(x)dx > \frac{-M(a)}{2}\int_a^{a+\delta} \zeta(x)dx > 0.$$

However, by our supposition

$$\int_a^b M(x)\zeta(x)dx = 0,$$

a contradiction. Therefore, if $M(a) < 0$, the function $M(x) \equiv 0$ on the interval $[a, b]$ and we have proven both cases. $\square$

Problem 2. Consider the differential equation

$$y'' - y = -x, \quad 0 < x < 1 \quad y(0) = y(1) = 0 \tag{1}$$

as in Example 15.12 on page 502. Use the basis $\{\phi_j(x)\} = \{x^j(1-x)^j\}$, as in section 15.5.1, to compute approximations to the exact solution using the finite-element method.

Provide relative errors at the points 0.25, 0.50, and 0.75 of the approximations using the first $n = 2, 3, 4$ basis functions. Plot the corresponding approximations y_2, y_3, y_4, and the exact solution y. Then find the first value of j for which the relative error at all three points is less than 0.5%.

Solution. We wish to approximate the solution to the above differential equation, $y(x)$, with a linear combination of the basis functions, i.e. find an approximation $y_n(x)$ where

$$y_n(x) = \sum_{j=1}^{n} a_j \phi_j(x). \tag{2}$$

Note that the basis functions $\phi_j(x) = x^j(1-x)^j$ satisfy the boundary conditions $\phi_j(0) = \phi_j(1) = 0$ so that $y_n(x)$ also satisfies the boundary conditions.

Corollary 15.2 suggests that if

$$\int_0^1 (y_n'' - y_n + x)\phi_i(x)dx = 0 \quad \text{for } i = 1, \dots, n$$

then $y_n'' - y_n + x = 0$, i.e $y_n(x)$ satisfies the differential equation (1). If $y_n(x)$ satisfies the differential equation and the boundary conditions, then we know that $y_n(x)$ approximates the exact solution $y(x)$.

Therefore, we choose the coefficients a_k such that they satisfy the system of equations

$$\sum_{j=1}^{n} a_j \int_0^1 \phi_j''(x)\phi_i(x) - \phi_j(x)\phi_i(x)dx = -\int_0^1 x\phi_i(x)dx \quad \text{for } i = 1, \dots, n.$$

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