Homework Assignment 7

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Problem 9.1. For the following problems for 9.1, suppose a function $f : [a, b] \to \mathbb{R}$ is only known at distinct sites $x = [x_1, x_2, \dots, x_n]$ where $x_i \in [a, b]$, for $i = 1, 2, \dots n$. Let $p_n(f, t)$ be the Lagrange interpolating polynomial at these sites.

Problem 9.1.1. Show that the basic quadrature $J(f) := \int_a^b p_n(f, t)\,\mathrm{d}t$ satisfies $J(f) = \sum_{j=1}^n w_j f(x_j)$ where the weights w_j depend on the Lagrange basis.

Solution. Note the Lagrange interpolating polynomial of f through the nodes $x_1, x_2, \dots, x_n$ is given by

$$p_n(f, t) = \sum_{j=1}^{n} f(x_j) \prod_{\substack{i=1 \\ i \neq j}} \frac{t - x_i}{x_j - x_i}.$$

If $J(f) := \int_a^b p_n(f, t)$, then, using this definition of the Lagrange interpolating polynomial, it is clear that

$$\begin{aligned} J(f) = \int_a^b p_n(f, t)\,\mathrm{d}t &= \int_a^b \left[\sum_{j=1}^{n} f(x_j) \prod_{\substack{i=1 \\ i \neq j}} \frac{t - x_i}{x_j - x_i} \right] \mathrm{d}t \\ &= \sum_{j=1}^{n} \left[\int_a^b \prod_{\substack{i=1 \\ i \neq j}} \frac{t - x_i}{x_j - x_i}\,\mathrm{d}t \right] f(x_j) = \sum_{j=1}^{n} w_j f(x_j). \end{aligned}$$

Thus, $J(f)$ is of the form $\sum_{j=1}^n w_j f(x_j)$ where w_j depends on the Lagrange basis $l_j(t) = \prod_{\substack{i=1 \\ i \neq j}} \frac{t - x_i}{x_j - x_i}$. □

Problem 9.1.2. Show that $J(f)$ has degree of precision at least $n - 1$.

Solution. Let $q(t)$ be a polynomial of degree $n - 1$. Then,

$$q(t) = \sum_{j=1}^{n} q(x_i) \prod_{\substack{i=1 \\ i \neq j}} \frac{t - x_i}{x_j - x_i},$$

i.e. the Lagrange interpolating polynomial of q through the nodes $x_1, x_2, \dots x_n$ is q itself. Hence, the exact integral of q, $I(q) = \int_a^b q(t)\,\mathrm{d}t$, satisfies

$$\begin{aligned} I(q) = \int_a^b q(t)\,\mathrm{d}t &= \int_a^b \sum_{j=1}^{n} q(x_j) \prod_{\substack{i=1\\ i\neq j}} \frac{t-x_i}{x_j-x_i}\,\mathrm{d}t \\ &= \sum_{j=1}^{n} \left[\int_a^b \prod_{\substack{i=1\\ i\neq j}} \frac{t-x_i}{x_j-x_i}\,\mathrm{d}t \right] q(x_j) = J(q). \end{aligned}$$

Since q is a polynomial of degree $n-1$ and $I(q) = J(q)$, we know that $J(f)$ has degree of precision at least $n-1$. □

Problem 9.1.3. Show that if $f \in C^n[a,b]$, then the truncation error can be bounded in terms of the nodal polynomial as follows:

$$|R(f)| \leq \frac{1}{n!} \max_{t\in[a,b]} |f^{(n)}(t)| \int_a^b |\Pi_n(t)| \mathrm{d}t$$

Solution. Let $f \in C^n([a,b])$. Note the truncation error is given by $R(f) = I(f) - J(f)$. Since $f \in C^n([a,b])$ and the Lagrange interpolating polynomial p_n satisfies $p_n(f, x_i) = f(x_i)$ for $i = 1, 2, \dots, n$, there is a point ξ_x in the smallest interval containing $[a,b]$ and every x_i such that

$$R(f) = I(f) - J(f) = \int_a^b f(t)\,\mathrm{d}t - \int_a^b p_n(f,t)\,\mathrm{d}t = \frac{1}{n!} \int_a^b f^{(n)}(\xi_x) \Pi_n(t)\,\mathrm{d}t$$

where $\Pi_n(t)$ is the nodal polynomial $\Pi_n(t) = \prod_{j=1}^n (t - x_j)$.

From this identity, it is clear that

$$\begin{aligned} |R(f)| &= \left| \frac{1}{n!} \int_a^b f^{(n)}(\xi_x)\, \Pi_n(t)\,\mathrm{d}t \right| \\ &\leq \frac{1}{n!} \left| f^{(n)}(\xi_x) \right| \int_a^b \Pi_n(t)\,\mathrm{d}t \\ &\leq \frac{1}{n!} \max_{t\in[a,b]} \left| f^{(n)}(t) \right| \int_a^b \Pi_n(t)\,\mathrm{d}t \end{aligned}$$

since $\left|f^{(n)}(\xi_x)\right| \leq \max_{t\in[a,b]} \left|f^{(n)}(t)\right|$ as $\xi_x \in [a,b]$ and we are done. □