Homework Assignment 2

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February 19, 2017

Problem 2.10. Solve the Cauchy problem for the Klein-Gordon equation

$$u_{tt} - c^2 u_{xx} + a^2 u = 0, \qquad -\infty < x < \infty, \quad t > 0,$$
$$u(x, 0) = f(x) \qquad \text{for } -\infty < x < \infty,$$
$$\left[\frac{\partial u}{\partial t}\right]_{t=0} = g(x) \qquad \text{for } -\infty < x < \infty.$$

Solution. Consider the function $u(x, y)$. The Fourier transform of u with respect to x is defined as

$$\mathscr{F}\{u(x, y)\} = U(k, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} u(x, y) dx. \tag{1}$$

From this definition and the Leibniz integral rule, we can see by induction that

$$\mathscr{F}\left\{\frac{\partial^n}{\partial y^n}[u(x, y)]\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^n}{\partial y^n}[u(x, y)] e^{-ikx} dx$$
$$= \frac{d^n}{dy^n}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, y) e^{-ikx} dx\right]$$
$$= \frac{d^n}{dy^n}[\mathscr{F}\{u(x, y)\}]. \tag{2}$$

Similarly, we see from definition (1) and previous theorems regarding the Fourier transform that

$$\mathscr{F}\left\{\frac{\partial^n}{\partial x^n}[u(x, y)]\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^n}{\partial x^n}[u(x, y)] e^{-ikx} dx$$
$$= (ik)^n\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, y) e^{-ikx} dx\right]$$
$$= (ik)^n \mathscr{F}\{u(x, y)\}. \tag{3}$$

Now, applying the Fourier transform to the first equation, we see that

$$\mathscr{F}\{u_{tt} - c^2 u_{xx} + a^2 u\} = \frac{d^2}{dt^2}[U(k, t)] - c^2(ik)^2 U(k, t) + a^2 U(k, t)$$
$$= \frac{d^2}{dt^2}[U(k, t)] + \left(c^2 k^2 + a^2\right) U(k, t).$$

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Thus, setting $\omega^2 = c^2 k^2 + a^2$, we see that

$$\frac{d^2}{dt^2}[U(k,t)] + \omega^2 U(k,t) = 0.$$

This is a second-order linear homogeneous ordinary differential equation, the solution to which we readily see is given by

$$U(k,t) = c_1 e^{-i\omega t} + c_2 e^{i\omega t}. \tag{4}$$

Applying the Fourier transform to the last two equations yields

$$\mathscr{F}\{u(x,0)\} = U(k,0) = F(k) = \mathscr{F}\{f(x)\}$$

and

$$\mathscr{F}\left\{\left[\frac{\partial u}{\partial t}\right]_{t=0}\right\} = \frac{d}{dt}[U(k,t)]_{t=0} = G(k) = \mathscr{F}\{g(x)\}.$$

Using (5), we see that the first equation reduces to

$$c_1 + c_2 = F(k).$$

Taking the derivative of $U(k,t)$ with respect to t yields

$$\frac{d}{dt}[U(k,t)] = -i\omega c_1 e^{-i\omega t} + i\omega c_2 e^{i\omega t}$$

and evaluating when $t = 0$ produces a second equation

$$i\omega(c_2 - c_1) = G(k).$$

This results in a system of two equations in two unknowns; the solution of which is given by

$$c_1 = \frac{\omega F(k) + iG(k)}{2\omega}, \qquad c_2 = \frac{\omega F(k) - iG(k)}{2\omega}.$$

Therefore, (5) becomes

$$U(k,t) = \left(\frac{\omega F(k) + iG(k)}{2\omega}\right)e^{-i\omega t} + \left(\frac{\omega F(k) - iG(k)}{2\omega}\right)e^{i\omega t}.$$

Taking the Inverse Fourier transform yields that the solution to original differential equation is given by

$$\begin{aligned}
u(x,t) &= \mathscr{F}^{-1}\left\{\left(\frac{\omega F(k) + iG(k)}{2\omega}\right)e^{-i\omega t} + \left(\frac{\omega F(k) - iG(k)}{2\omega}\right)e^{i\omega t}\right\} \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\left(\frac{\omega F(k) + iG(k)}{2\omega}\right)e^{-i\omega t} + \left(\frac{\omega F(k) - iG(k)}{2\omega}\right)e^{i\omega t}\right]e^{ikx}dk \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{\omega F(k) + iG(k)}{2\omega}\right)e^{i(kx-\omega t)} + \left(\frac{\omega F(k) - iG(k)}{2\omega}\right)e^{i(kx+\omega t)}dk.
\end{aligned}$$

$\square$

Problem 2.12. Solve the equation

$$u_{tt} + u_{xxxx} = 0, \qquad -\infty < x < \infty, \quad t > 0$$
$$u(x,0) = f(x), \quad u_t(x,0) = 0 \qquad \text{for } -\infty < x < \infty.$$

Solution. Using (2) and (3), we begin by applying the Fourier transform to the first differential equation yielding

$$\mathscr{F}\{u_{tt} + u_{xxxx}\} = \frac{d^2}{dt^2}[U(k,t)] + k^4 U(k,t) = 0 = \mathscr{F}\{0\}.$$

This is a second-order linear homogeneous differential equation, the solution to which we readily see is given by

$$U(k,t) = c_1 \cos(k^2 t) + c_2 \sin(k^2 t). \tag{5}$$

Applying the Fourier transform to the last two equations yields

$$\mathscr{F}\{u(x,0)\} = U(k,0) = F(k) = \mathscr{F}\{f(x)\}$$

and

$$\mathscr{F}\{u_t(x,0)\} = \mathscr{F}\left\{\left[\frac{\partial u}{\partial t}\right]_{t=0}\right\} = \frac{d}{dt}[U(k,t)]\Big|_{t=0} = 0 = \mathscr{F}\{0\}.$$

Using the form (5) of the solution we see from the second equation that

$$U(k,0) = c_1 = F(k).$$

Similarly, we also see from the last equation that

$$\frac{d}{dt}[U(k,t)] = -c_1 k^2 \sin(k^2 t) + c_2 k^2 \cos(k^2 t)$$

which implies that

$$\frac{d}{dt}[U(k,t)]\Big|_{t=0} = c_2 k^2 = 0.$$

Since this equation must hold for all k, this implies that $c_2 = 0$. Thus, the solution (5) becomes

$$U(k,t) = F(k)\cos(k^2 t).$$

From the Convolution Theorem, we see that

$$u(x,t) = \mathscr{F}^{-1}\{F(k)\cos(k^2 t)\} = (f * g)(x)$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi$$

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where $g(x) = \mathscr{F}^{-1}\{\cos(k^2 t)\}$.

Note that it can be shown that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik(x\pm kt)} dk = \frac{1}{2}\left(\frac{(1\pm i)e^{\mp ix^2/4t}}{\sqrt{t}}\right).$$

Thus, using the definition of the complex exponential, we have that

$$\begin{aligned}
g(x) &= \mathscr{F}^{-1}\left\{\cos(k^2 t)\right\} \\
&= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik(x+kt)} + e^{ik(x-kt)} dk \\
&= \frac{1}{4\sqrt{t}} \left[(1+i)e^{-ix^2/4t} + (1-i)e^{ix^2/4t}\right] \\
&= \frac{1}{2\sqrt{t}} \left[\frac{e^{ix^2/4t} + e^{-ix^2/4t}}{2} + \frac{e^{ix^2/4t} - e^{-ix^2/4t}}{2i}\right] \\
&= \frac{\cos\left(\frac{x^2}{4t}\right) + \sin\left(\frac{x^2}{4t}\right)}{2\sqrt{t}}.
\end{aligned}$$

Therefore, the solution to the original differential equation is given by

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi \\
&= \frac{1}{2\sqrt{2\pi t}} \int_{-\infty}^{\infty} f(x-\xi)\left(\cos\left(\frac{\xi^2}{4t}\right) + \sin\left(\frac{\xi^2}{4t}\right)\right) d\xi.
\end{aligned}$$

$\square$

Problem 2.14. Obtain the Fourier cosine transforms of the following functions:

a. xe^{-ax}, $a > 0$.

Solution. Recall that the definition of the Fourier cosine transform of a function $f(x)$ is given by

$$\mathscr{F}_c\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos kx f(x) dx.$$

a. From the definition of the Fourier cosine transform we have that

$$\mathscr{F}_c\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}} \int_0^\infty xe^{-ax} \cos kx dx.$$

Using the definition of the complex exponential, we see that

$$\mathscr{F}_c\{xe^{-ax}\} = \sqrt{\frac{2}{\pi}} \int_0^\infty xe^{-ax} \left[\frac{e^{-ikx} + e^{ikx}}{2}\right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty x\left[e^{-(a+ik)x} + e^{-(a-ik)x}\right] dx.$$

Now, for $w = a \pm ik$ with $a > 0$, we see using integration by parts with $u = x$ and $dv = e^{-wx}dx$ that

$$\int_0^\infty xe^{-wx} dx = -\frac{xe^{-wx}}{w}\bigg|_0^\infty + \frac{1}{w}\int_0^\infty e^{-wx} dx.$$

Note that

$$\lim_{x\to\infty} \left|e^{-wx}\right| = \lim_{x\to\infty} \left|e^{-(a\pm ik)x}\right| = \lim_{x\to\infty} \left|e^{-ax}\right|\left|e^{\mp ikx}\right| \le \lim_{x\to\infty} \left|e^{-ax}\right| = 0.$$

This implies that $\lim_{x\to\infty} e^{-wx} = 0$. Thus,

$$\int_0^\infty xe^{-wx} dx = -\frac{xe^{-wx}}{w}\bigg|_0^\infty + \frac{1}{w}\int_0^\infty e^{-wx} dx$$

$$= -\frac{1}{w^2}\left[e^{-wx}\big|_0^\infty\right]$$

$$= \frac{1}{w^2}.$$

Therefore,

$$\mathscr{F}_c\{xe^{-ax}\} = \frac{1}{\sqrt{2\pi}} \left[\int_0^\infty xe^{-(a+ik)x} dx + \int_0^\infty xe^{-(a-ik)x} dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{(a+ik)^2} + \frac{1}{(a-ik)^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{a^2 - k^2}{(a^2 + k^2)^2}.$$

$\square$

Problem 2.15. Find the Fourier sine transform of the following functions:

a. xe^{-ax}, $a > 0$.

b. $\dfrac{e^{-ax}}{x}$, $a > 0$.

Solution. Recall that the definition of the Fourier sine transform of a function $f(x)$ is given by

$$\mathscr{F}_s \{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin kx f(x) dx.$$

a. From the definition of the Fourier sine transform we have that

$$\mathscr{F}_s \{xe^{-ax}\} = \sqrt{\frac{2}{\pi}} \int_0^\infty xe^{-ax} \sin kx dx.$$

Using the definition of the complex exponential, we see that

$$\mathscr{F}_s \{xe^{-ax}\} = \sqrt{\frac{2}{\pi}} \int_0^\infty xe^{-ax} \left[\frac{e^{ikx} - e^{-ikx}}{2i} \right] dx$$

$$= -\frac{i}{\sqrt{2\pi}} \int_0^\infty x \left[e^{-(a-ik)x} - e^{-(a+ik)x} \right] dx.$$

Now, for $w = a \pm ik$ with $a > 0$, we see using integration by parts with $u = x$ and $dv = e^{-wx} dx$ that

$$\int_0^\infty xe^{-wx} dx = -\frac{xe^{-wx}}{w} \Big|_0^\infty + \frac{1}{w} \int_0^\infty e^{-wx} dx.$$

Note that

$$\lim_{x \to \infty} \left| e^{-wx} \right| = \lim_{x \to \infty} \left| e^{-(a \pm ik)x} \right| = \lim_{x \to \infty} \left| e^{-ax} \right| \left| e^{\mp ikx} \right| \le \lim_{x \to \infty} \left| e^{-ax} \right| = 0.$$

This implies that $\lim_{x \to \infty} e^{-wx} = 0$. Thus,

$$\int_0^\infty xe^{-wx} dx = -\frac{xe^{-wx}}{w} \Big|_0^\infty + \frac{1}{w} \int_0^\infty e^{-wx} dx$$

$$= -\frac{1}{w^2} \left[e^{-wx} \Big|_0^\infty \right]$$

$$= \frac{1}{w^2}.$$

Therefore,

$$\mathscr{F}_s \{xe^{-ax}\} = -\frac{i}{\sqrt{2\pi}} \left[\int_0^\infty xe^{-(a-ik)x} dx - \int_0^\infty xe^{-(a+ik)x} dx \right]$$

$$= -\frac{i}{\sqrt{2\pi}} \left[\frac{1}{(a-ik)^2} - \frac{1}{(a+ik)^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \frac{2ak}{(a^2 + k^2)^2}.$$

b. From our table of Fourier sine transforms, we see for $a > 0$ that

$$\mathscr{F}_s\left\{e^{-ax}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin(kx)dx = \sqrt{\frac{2}{\pi}}\left(\frac{k}{k^2+a^2}\right).$$

Thus, we must have that both sides are equal after integrating with respect to a from a to ∞, i.e.

$$\sqrt{\frac{2}{\pi}}\int_a^\infty\left[\int_0^\infty e^{-ax}\sin(kx)dx\right]da = \sqrt{\frac{2}{\pi}}\int_a^\infty\left(\frac{k}{k^2+a^2}\right)da.$$

For the integral on the left, since the integrand is continuous on the domain of integration, we can interchange the order of integration and we see that

$$\sqrt{\frac{2}{\pi}}\int_a^\infty\left[\int_0^\infty e^{-ax}\sin(kx)dx\right]da = \sqrt{\frac{2}{\pi}}\int_0^\infty\sin(kx)\left[\int_a^\infty e^{-ax}da\right]dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty\frac{e^{-ax}}{x}\sin(kx)dx$$

$$= \mathscr{F}_s\left\{\frac{e^{-ax}}{x}\right\}.$$

Therefore,

$$\mathscr{F}_s\left\{\frac{e^{-ax}}{x}\right\} = \sqrt{\frac{2}{\pi}}\int_a^\infty\left(\frac{k}{k^2+a^2}\right)da.$$

Note, the indefinite integral evaluates to $\tan^{-1}\left(\frac{a}{k}\right)$ and we therefore see that

$$\mathscr{F}_s\left\{\frac{e^{-ax}}{x}\right\} = \sqrt{\frac{2}{\pi}}\int_a^\infty\left(\frac{k}{k^2+a^2}\right)da$$

$$= \sqrt{\frac{2}{\pi}}\left(\tan^{-1}\left(\frac{a}{k}\right)\Big|_a^\infty\right)$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{a}{k}\right)\right)$$

$\square$

Problem 2.20. Apply the Fourier cosine transform to find the solution $u(x, y)$ of the problem

$$
\begin{aligned}
u_{xx} + u_{yy} &= 0, & 0 < x < \infty, \quad 0 < y < \infty \\
u(x, 0) &= H(a - x), & x < a \\
u_x(0, y) &= 0, & 0 < x, \quad y < \infty.
\end{aligned}
$$

Solution.									□

Problem 2.22. Solve the diffusion equation in the semi-infinite line

$$u_t = \kappa u_x x, \qquad 0 \le x < \infty, \quad t > 0,$$

with the boundary and initial data

$$
\begin{aligned}
u(0, t) &= 0 && \text{for } t > 0, \\
u(x, t) &\to 0 && \text{as } x \to \infty \text{ for } t > 0, \\
u(x, 0) &= f(x) && \text{for } 0 < x < \infty.
\end{aligned}
$$

Solution. $\square$