Homework Assignment 10

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Problem 10.5.1. Show that the Cantor set C is a fixed point of the map $F : \mathcal{C}(\mathbb{R}) \to \mathcal{C}(\mathbb{R})$ defined by

$$F(A) = f_1(A) \cup f_2(A)$$

where $f_1(x) = x/3$ and $f_2(x) = x/3 + 2/3$ are contractions on $\mathbb{R}$.

Solution. Recall that the Cantor set C is defined as

$$C = \left\{ x \in [0,1] \;\middle|\; x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, \quad a_n \in \{0,2\} \right\}. \tag{1}$$

We wish to show that $F(C) = C$. From the definitions of f_1 and f_2, we have that

$$\begin{aligned}
f_1(C) &= \left\{ x \in [0,1] \;\middle|\; x = \sum_{n=1}^{\infty} \frac{a_n}{3^{n+1}}, \quad a_n \in \{0,2\} \right\} \\
&= \left\{ x \in [0,1] \;\middle|\; x = \sum_{n=1}^{\infty} \frac{b_n}{3^n}, \quad b_1 = 0,\, b_n = a_{n-1} \text{ for } n > 1 \right\} \\
f_2(C) &= \left\{ x \in [0,1] \;\middle|\; x = \frac{2}{3} + \sum_{n=1}^{\infty} \frac{a_n}{3^{n+1}}, \quad a_n \in \{0,2\} \right\} \\
&= \left\{ x \in [0,1] \;\middle|\; x = \sum_{n=1}^{\infty} \frac{b_n}{3^n}, \quad b_1 = 2,\, b_n = a_{n-1} \text{ for } n > 1 \right\}
\end{aligned} \tag{2}$$

From (2) and the definition of F, we see that

$$\begin{aligned}
F(C) &= f_1(C) \cup f_2(C) \\
&= \left\{ x \in [0,1] \;\middle|\; x = \sum_{n=1}^{\infty} \frac{b_n}{3^n}, \quad b_n \in \{0,2\} \right\}.
\end{aligned}$$

But this is precisely the definition of the Cantor set C given in (1). Therefore, $F(C) = C$ and the Cantor set C is a fixed point of F. $\square$

Problem 10.5.2. Show that the box-counting dimension of the Sierpinski triangle is $\log 3/\log 2$.

Solution. The Sierpinski triangle is formed by iteratively removing smaller and smaller equilateral triangles from an equilateral triangle of side-length 1. For an equilateral triangle of side-length d, the minimum size square that completely covers the triangle is the square of side-length d.

For the first iteration of the Sierpinski triangle, we remove the open middle equilateral triangle, leaving 3 equilateral triangles of side-length $1/2$. Thus, we would require, at a minimum, 3 squares of side-length $1/2$ in order to completely cover the Sierpinski triangle. In the next iteration, we remove the open middle equilateral triangles of the remaining equilateral triangles, leaving 9 equilateral triangles of side-length $1/4$. Thus, we would, at a minimum, require 9 squares of side-length $1/4$ in order to completely cover the Sierpinski triangle.

In general, the n-th iteration will leave 3^n equilateral triangles of side length $1/2^n$ and we would require, at a minimum, 3^n squares of side-length $1/2^n$ in order to completely cover the Sierpinski triangle. Let K be the Sierpinski triangle and let $N_{\delta_n}(K)$ be the minimum number of boxes of equal length $\delta_n > 0$ needed to completely cover K at iteration n. Then from our previous discussions, $N_{\delta_n}(K) = 3^n$ with $\delta_n = 1/2^n$.

Therefore, the box-counting dimension of the Sierpinski triangle K is

$$\dim(K) = \lim_{\delta \to 0^+} \frac{\log N_\delta(K)}{\log 1/\delta} = \lim_{n\to\infty} \frac{\log N_{\delta_n}(K)}{\log 1/\delta_n} = \lim_{n\to\infty} \frac{\log 3^n}{\log 2^n} = \frac{\log 3}{\log 2}.$$

$\square$

Problem 10.5.4. Let $f(x) = x^2 - a$ with $1 < a < 3$ and let N_f be the corresponding Newton function of f. Show that N_f satisfies the hypothesis of the Contraction Mapping Theorem on $[1, \infty)$. What is the fixed point?

Solution. Recall that the Newton function N_f of a function f is defined by

$$N_f(x) = x - \frac{f(x)}{f'(x)}.$$

If $f(x) = x^2 - a$, then we see that

$$N_f(x) = x - \frac{x^2 - a}{2x} = \frac{x^2 + a}{2x}. \tag{3}$$

The function N_f will satisfy the hypothesis of the Contraction Mapping Theorem on $Y = [1, \infty)$ if (Y, d) is a complete metric space with d the usual metric on $\mathbb{R}$ and if $N_f : Y \to Y$ is a contraction mapping.

Since Y is a closed subset of $\mathbb{R}$, a complete metric space, we know that Y must be a complete metric space. This is since any Cauchy sequence in Y will converge to some point $x \in \mathbb{R}$, but since Y is closed, i.e. it contains all of its limit points, the point x must be in Y.

All that is left is to show that N_f is a contraction mapping, i.e. for all $x, y \in Y$, there is some $\alpha \in (0, 1)$ such that

$$|N_f(x) - N_f(y)| \leq \alpha |x - y|.$$

Note that

$$\begin{aligned}
|N_f(x) - N_f(y)| &= \left| \frac{x^2 + a}{2x} - \frac{y^2 + a}{2y} \right| \\
&= \left| \frac{x^2 y + ay - xy^2 - ax}{2xy} \right| \\
&= \left| \frac{(x - y)(xy - a)}{2xy} \right| \\
&= \frac{|x - y|}{2} \left| \frac{xy - a}{xy} \right|.
\end{aligned}$$

Since $a > 1$ we have that $|xy - a| < |xy|$ which implies that $|(xy - a)/xy| \leq 1$. Thus,

$$|N_f(x) - N_f(y)| = \frac{|x - y|}{2} \left| \frac{xy - a}{xy} \right| \leq \frac{1}{2} |x - y|$$

and N_f is a contraction mapping with contraction constant $\alpha = 1/2$. Therefore, N_f satisfies the hypothesis of the Contraction Mapping Theorem on $Y = [1, \infty)$.

By the Contraction Mapping Theorem, the fixed point of N_f is the limit of its iterates. Since N_f is the Newton function, we know that its iterates converge to a root of the function $f(x) = x^2 - a$. Note that the iterates of N_f are positive so they will converge to the positive root of f, i.e. the fixed point of N_f is $p = \sqrt{a}$. $\square$

Problem 10.5.7. Find the distance between the sets $A = \{0, 1/n, 2/n, \ldots, (n-1)/n, 1\}$ and $B = [0, 1]$ in the Hausdorff metric. Deduce that the distance between an infinite set and a finite set can be arbitrarily small.

Solution. For a compact subset A of $\mathbb{R}$, let $U_\delta(A)$ be the closed set containing A whose boundary lies within $\delta > 0$ of A, i.e.

$$U_\delta(A) = \{x \in X \mid d(x, y) \leq \delta \text{ for some } x \in A\}.$$

Then the distance between two compact subsets A and B in the Hausdorff metric is given by the smallest $\delta > 0$ such that A is contained in the closed set containing B whose boundary lies within δ of B and vice versa. More precisely,

$$D(A, B) = \inf\{\delta > 0 \mid A \subseteq U_\delta(B) \text{ and } B \subseteq U_\delta(A)\}. \tag{4}$$

It is clear that for any $\delta > 0$, $A \subseteq U_\delta(B)$ since $A \subseteq B$. However, if A is the set of $n+1$ equally spaced points a distance of $1/n$ apart on $[0, 1]$, then the smallest $\delta > 0$ such that $B \subseteq U_\delta(A)$ is $\delta = 1/2n$. To demonstrate this, if $\delta = 1/2n$, then we see that

$$U_\delta(A) = \bigcup_{k=0}^{n} \left[\frac{k}{n} - \frac{1}{2n}, \frac{k}{n} + \frac{1}{2n}\right] = \bigcup_{k=0}^{n} \left[\frac{2k-1}{2n}, \frac{2k+1}{2n}\right] = \left[-\frac{1}{2n}, 1 + \frac{1}{2n}\right]$$

so that $B = [0, 1] \subseteq U_\delta(A)$. If on the other hand, $\delta < 1/2n$, then $\delta = 1/2n - \varepsilon$ for some $\varepsilon > 0$ and we see that

$$U_\delta(A) = \bigcup_{k=0}^{n} \left[\frac{k}{n} - \left(\frac{1}{2n} - \varepsilon\right), \frac{k}{n} + \left(\frac{1}{2n} - \varepsilon\right)\right]$$

Note that $x = 1/2n \in B$, but $x \notin U_\delta(A)$. This shows that $\delta = 1/2n$ is the smallest $\delta > 0$ such that $B \subseteq U_\delta(A)$.

Therefore, we see that $D(A, B) = 1/2n$. We then see that as the size of the finite set A tends towards infinity, the distance between A and B tends to 0 and can be made small.

$\square$