Homework Assignment 5

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Problem 5.2. Suppose that you arrive at a single-teller bank to find five other customers in the bank, one being served and the other four waiting in line. You join the end of the line. If the service times are exponential with rate μ, what is the expected amount of time you will spend in the bank?

Solution. Let T_i denote the time that the i-th person spends at the teller in order to complete his or her transaction. Then the total amount of time I will spend in the bank is the amount of time the other five customers spend at the teller plus the time that I, the sixth customer, will spend at the teller. If S denotes the total amount of time that I spend at the bank, then

$$S = \sum_{i=1}^{6} T_i.$$

Note that even though the first customer is currently being served, the service time is exponentially distributed with rate μ, i.e. the waiting time is memory-less so that the expected service time of the first customer is still $1/\mu$. Since the other T_i are exponential random variables with mean $1/\mu$, we have that

$$E[S] = \sum_{i=1}^{6} E[T_i] = \frac{6}{\mu}.$$

□

Problem 5.8. If X and Y are independent exponential random variables with respective rates λ and μ, what is the conditional distribution of X given that $X < Y$?

Solution. Let E be the event such that $X < Y$. Recall from Bayes' Theorem that the conditional density function of X given E is given by

$$f_{X|E}(x \mid x < y) = \frac{f_X(x)P(E \mid X = x)}{\int_0^\infty f_X(s)P(E \mid X = s)ds}.$$

Note that since X and Y are independent we have that $P(X < Y|X = x) = P(x < Y) = e^{-\mu x}$. Thus,

$$\begin{aligned} f_{X|E}(x|E) &= \frac{\lambda e^{-\lambda x}e^{-\mu x}}{\int_0^\infty \lambda e^{-\lambda s}e^{-\mu s}ds} \\ &= \frac{\lambda e^{-\lambda x}e^{-\mu x}}{\frac{\lambda}{\lambda+\mu}} \\ &= (\lambda + \mu)e^{-(\lambda+\mu)x}, \end{aligned}$$

if $x \geq 0$, and we see that $f_{X|E}(x \mid x < y)$ has the density function that of an exponential random variable with rate $\lambda + \mu$.

Therefore, the conditional distribution of X given E is

$$F_{X|E}(x \mid x < y) = \begin{cases} 1 - e^{-(\lambda+\mu)x} & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

□

Problem 5.15. One hundred items are simultaneously put on a life test. Suppose the lifetimes of the individual items are independent exponential random variables with mean 200 hours. The test will end when there have been a total of 5 failures. If T is the time at which the test ends, find $E[T]$ and $\mathrm{Var}[T]$.

Solution. Let T_i be the time between the $(i-1)$-th and the i-th failure. Note that from a previous proposition, we have that if $X_1, \ldots, X_n$ are independent exponential random variables all with rate $\lambda = 1/200$, then $\min_i X_i$ is exponential with rate $n\lambda$. Thus, the shortest lifetime among the 100 initial items is exponential with rate 100λ, i.e. the time that it takes for the first failure to occur, T_1, is exponential with rate 100λ.

In general, after the $(i-1)$-th failure, there will be $101 - i$ items left to test. Thus, the shortest lifetime among the $101 - i$ items is exponential with rate $(101 - i)\lambda$, i.e. the time that it takes for the i-t failure to occur, T_i, is exponential with rate $(101 - i)\lambda$.

Since each T_i is exponential with rate $(101 - i)\lambda$, we have that

$$E[T_i] = \frac{1}{(101-i)\lambda} = \frac{200}{(101-i)}, \quad \mathrm{Var}[T_i] = \frac{1}{(101-i)^2\lambda^2} = \frac{200^2}{(101-i)^2}.$$

If the test ends after 5 failures and if T is the total time that the test takes, then $T = T_1 + \cdots + T_5$ and

$$E[T] = \sum_{i=1}^{5} E[T_i] = \sum_{i=1}^{5} \frac{200}{(101-i)} = 10.2062.$$

Since each T_i is independent, we have that

$$\mathrm{Var}[T] = \sum_{i=1}^{5} \mathrm{Var}[T_i] = \sum_{i=1}^{5} \frac{200^2}{(101-i)^2} = 20.8377.$$

□

Problem 5.43. Customers arrive at a two-server service station according to a Poisson process with rate λ. Whenever a new customer arrives, any customer that is in the system immediately departs. A new arrival enters first with server 1 and then with server 2. If the service times are independent exponentials with respective rates μ_1 and μ_2, what proportion of entering customers complete their service with server 2?

Solution. Let T_1 denote the amount of time spent with server 1 and T_2 denote the amount of time spent with server 2. Note that T_1 and T_2 are independent and exponentially distributed with rates μ_1 and μ_2, respectively.

Let T be the amount of time before the next customer arrives. Since T is the arrival time of some event of a Poisson process with rate λ, we have that T is exponentially distributed with rate λ.

Let p be the proportion of customers that complete their service with server 2. Note that p will be the proportion of time between inter-arrivals that is greater than $T_1 + T_2$. This will be the probability that $T > T_1 + T_2$, i.e.

$$p = P(T > T_1 + T_2)$$

Since T is exponentially distributed and T_1 and T_2 are independent, we have that

$$\begin{aligned}
p &= P(T > T_1 + T_2) \\
&= P(T > T_1)P(T > T_2) \\
&= \int_0^\infty P(T > T_1 \mid T = t) f_T(t) dt \int_0^\infty P(T > T_2 \mid T = t) f_T(t) dt \\
&= \int_0^\infty \lambda(1 - e^{-\mu_1 t}) e^{-\lambda t} dt \int_0^\infty \lambda(1 - e^{-\mu_2 t}) e^{-\lambda t} dt \\
&= \frac{\mu_1}{\mu_1 + \lambda} \frac{\mu_2}{\mu_2 + \lambda}. \qquad (1)
\end{aligned}$$

Therefore, (1) is the proportion of customers that complete their services with server 2. □

Problem 5.44. Cars pass a certain street location according to a Poisson process with rate λ. A woman who wants to cross the street at that location waits until she can see that no cars will come by in the next T time units.

i. Find the probability that her waiting time is 0.

ii. Find her expected waiting time.

Solution. If t_0 is the time that the woman arrives at the location, then the event E that her waiting time is 0 occurs if no car passes the street during the time $(t_0, t_0 + T]$. Thus, the probability that her waiting time is 0 is given by

$$P(E) = P(0 \text{ events in } (t_0, t_0 + T]) = e^{-\lambda T}$$

by the memory-less property of the exponential distribution.

In order to find the expected waiting time of the woman, let X be the woman's waiting time. If Y is the time until the first car arrives, then we can find the expected value of X by conditioning over all possible values of Y. Note that Y is exponentially distributed with rate λ. Thus,

$$E[X] = \int_0^\infty E[X|Y = y]\lambda e^{-\lambda y} dy$$

Note that if $Y > T$, then the first car arrives some time after T and the expected wait time given that $Y = y$ with $y > T$is T, i.e. $E[X \mid Y = y] = T$. If on the other hand $Y < T$, then the expected wait time given that $Y = y$ with $y < T$ is however long it takes for the first car to arrive plus the expected wait time, i.e. $E[X \mid Y = y] = y + E[X]$. This implies that

$$\begin{aligned}
E[X] &= \int_0^\infty E[X|Y = y]\lambda e^{-\lambda y} dy \\
&= \int_0^T E[X|Y = y]\lambda e^{-\lambda y} dy + \int_T^\infty E[X|Y = y]\lambda e^{-\lambda y} dy \\
&= \int_0^T (y + E[X])\lambda e^{-\lambda y} dy + T\int_T^\infty \lambda e^{-\lambda y} dy \\
&= \frac{(1 - e^{-\lambda T})(1 + \lambda T)}{\lambda} + E[X](1 - e^{-\lambda T}) + Te^{-\lambda T}.
\end{aligned}$$

Solving for $E[X]$ shows that

$$E[X] = \frac{(e^{\lambda T} - 1)(1 + \lambda T)}{\lambda} + T$$

$\square$

Problem 5.50. The number of hours between successive train arrivals at the station is uniformly distributed on $(0, 1)$. Passengers arrive according to a Poison process with rate $\lambda = 7$ per hour. Suppose a train has just left the station. Let X denote the number of people who get on the next train. Find $E[X]$ and $\text{Var}[X]$.

Solution. Let T denote the time that elapses from the moment the train leaves the station until the next train arrives. Then $T \sim \mathcal{U}(0, 1)$. Since the number of passengers that arrive is a Poisson process, we have that X, the number of people who get on the next train during any length of time t is a Poisson random variable with parameter λt. We can obtain $E[X]$ by conditioning over all of the possible values of T. Thus, noting that X and T are independent, we have that

$$E[X] = \int_0^1 E[X|T = t]f_T(t)dt = \int_0^1 \lambda t dt = \frac{\lambda}{2}.$$

Since $\lambda = 7$, we have that $E[X] = 7/2$.

Using similar reasoning, since X is Poisson during any length of time t with parameter $t\lambda$, we have that $\text{Var}[X|T = t] = \lambda t$. Thus, by the formula for conditional variance and using our previous results, we have that

$$\begin{aligned}
\text{Var}[X] &= E\left[\text{Var}[X \mid T]\right] + \text{Var}\left[E[X \mid T]\right] \\
&= E\left[\lambda T\right] + \text{Var}\left[\lambda T\right] \\
&= \lambda E[T] + \lambda^2 \text{Var}[T] \\
&= \frac{\lambda}{2} + \lambda^2 \left[\int_0^1 t^2 dt - \left(\int_0^1 t dt\right)^2\right] \\
&= \frac{\lambda}{2} + \frac{\lambda^2}{12}.
\end{aligned}$$

Again, since $\lambda = 7$, we have that $\text{Var}[X] = 7/2 + 49/12$. □