Exam 2

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Problem 1. Find the inverse Laplace transform of the function

$$\bar{f}(s) = \frac{s}{(s-a)(s^2+b^2)}$$

for $a, b > 0$, by using the following three different approaches:

i. Using partial fraction decomposition,

ii. Applying the Convolution Theorem,

iii. Applying Heaviside's Expansion Theorem.

Solution. We will now find the inverse Laplace transform of $\bar{f}(s)$ using the respective approaches listed above:

i. From the partial fractions method, we see that

$$\bar{f}(s) = \frac{s}{(s-a)(s^2+b^2)} = \frac{c_0}{s-a} + \frac{d_1 s + d_0}{s^2+b^2}.$$

Combining the rational fractions on the right side under a common denominator and equating the coefficients in the numerator we arrive at the following system of equations

$$\begin{aligned} c_0 + d_1 &= 0 \\ d_0 - ad_1 &= 0 \\ c_0 b^2 - ad_0 &= 0. \end{aligned}$$

Solving this system, we see that $c_0 = \dfrac{a}{a^2+b^2}$, $d_1 = -\dfrac{a}{a^2+b^2}$, and $d_0 = \dfrac{b^2}{a^2+b^2}$. Thus, we have that

$$\bar{f}(s) = \frac{1}{a^2+b^2}\left[\frac{a}{s-a} - \frac{as}{s^2+b^2} + \frac{b^2}{s^2+b^2}\right].$$

From our table of Laplace transforms, we know that

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} &= e^{at}\\ \mathscr{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\} &= \cos bt\\ \mathscr{L}^{-1}\left\{\frac{b}{s^2+b^2}\right\} &= \sin bt.\end{aligned}$$

Therefore, the inverse Laplace transform of $\bar{f}(s)$ is

$$\begin{aligned}f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} &= \frac{1}{a^2+b^2}\left[a\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} - a\mathscr{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\} + b\mathscr{L}^{-1}\left\{\frac{b}{s^2+b^2}\right\}\right]\\ &= \frac{1}{a^2+b^2}\left[ae^{at} - a\cos bt + b\sin bt\right].\end{aligned}$$

ii. The Convolution Theorem states that if $\bar{f}(s) = \bar{g}(s)\bar{h}(s)$, then

$$f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} = \mathscr{L}^{-1}\left\{\bar{g}(s)\bar{h}(s)\right\} = (g * h)(t)$$

where

$$(g*h)(t) = \int_0^t g(t-\tau)h(\tau)d\tau.$$

Now, suppose that $\bar{f}(s) = \bar{g}(s)\bar{h}(s)$, where $\bar{g}(s) = \dfrac{1}{s-a}$ and $\bar{h}(s) = \dfrac{s}{s^2+b^2}$.

From our table of Laplace transforms we know that $g(t) = \mathscr{L}^{-1}\left\{\dfrac{1}{s-a}\right\} = e^{at}$ and $h(t) = \mathscr{L}^{-1}\left\{\dfrac{s}{s^2+b^2}\right\} = \cos bt$.

Thus, by the Convolution Theorem, we have that

$$f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} = \mathscr{L}^{-1}\left\{\bar{g}(s)\bar{h}(s)\right\} = \int_0^t g(t-\tau)h(\tau)d\tau.$$

Therefore, using a computer algebra system, we see that

$$\begin{aligned}f(t) &= \int_0^t g(t-\tau)h(\tau)d\tau\\ &= \int_0^t e^{a(t-\tau)}\cos b\tau d\tau\\ &= e^{at}\int_0^t e^{-a\tau)}\cos b\tau d\tau\\ &= \frac{1}{a^2+b^2}\left[ae^{at} - a\cos bt + b\sin bt\right].\end{aligned}$$

iii. Heaviside's Expansion Theorem states that if $\bar{f}(s) = \dfrac{\bar{p}(s)}{\bar{q}(s)}$, where $\bar{p}(s)$ and $\bar{q}(s)$ are polynomials in s and the degree of $\bar{q}$ is higher than that of $\bar{p}$, then

$$f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} = \sum_{k=1}^{n} \frac{\bar{p}(\alpha_k)}{\bar{q}'(\alpha_k)} e^{t\alpha_k}$$

where α_k are the distinct root of $\bar{q}(s) = 0$.

For $\bar{f}(s) = \dfrac{s}{(s-a)(s^2+b^2)}$, we identify $\bar{p}(s) = s$ and $\bar{q}(s) = (s-a)(s^2+b^2)$. Since $\bar{p}$ and $\bar{q}$ are polynomials in s with the degree of $\bar{q}$ greater than that of the degree of $\bar{p}$, the assumptions of Heaviside's Expansion Theorem are satisfied.

Note that $\bar{q}'(s) = s(3s-2a) + b^2$ and $\alpha_1 = a$, $\alpha_2 = bi$, and $\alpha_3 = -bi$ are the roots of $\bar{q}(s)$.

Therefore, by the Heaviside's Expansion Theorem, we have that

$$\begin{aligned} f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} &= \sum_{k=1}^{n} \frac{\bar{p}(\alpha_k)}{\bar{q}'(\alpha_k)} e^{t\alpha_k} \\ &= \frac{a}{a^2+b^2}e^{at} - \frac{bi}{2bi(a-ib)}e^{bit} - \frac{bi}{2bi(a+ib)}e^{-bit} \\ &= \frac{1}{a^2+b^2}\left[ae^{at} - \frac{a+ib}{2}e^{bit} - \frac{a-ib}{2}e^{-bit}\right] \\ &= \frac{1}{a^2+b^2}\left[ae^{at} - a\cos bt + b\sin bt\right]. \end{aligned}$$

□

Problem 2. a. Evaluate the improper definite integral

$$\int_{-\infty}^{\infty} \frac{\cos tx}{x^2+a^2}dx$$

where $a, t > 0$.

b. Show that

$$\int_{0}^{\infty} \frac{\sin \pi tx}{x(1+x^2)}dx = \frac{\pi}{2}(1-e^{-\pi t})$$

where $t > 0$.

Solution. a. Suppose that

$$f(t) = \int_{-\infty}^{\infty} \frac{\cos tx}{x^2+a^2}dx.$$

In order to evaluate this integral, we take the Laplace transform of $f(t)$ with respect to t. Now, due to uniform convergence, we have that

$$\begin{aligned}
\bar{f}(s) = \mathscr{L}\{f(t)\} = \mathscr{L}\left\{\int_{-\infty}^{\infty} \frac{\cos tx}{x^2+a^2}dx\right\} &= \int_{-\infty}^{\infty} \mathscr{L}\left\{\frac{\cos tx}{x^2+a^2}\right\}dx \\
&= \int_{-\infty}^{\infty} \frac{1}{x^2+a^2}\mathscr{L}\{\cos tx\}\,dx \\
&= \int_{-\infty}^{\infty} \frac{s}{(x^2+a^2)(x^2+s^2)}dx.
\end{aligned}$$

Using the method of partial fraction decomposition, we see that this last integral becomes

$$\begin{aligned}
\bar{f}(s) &= \int_{-\infty}^{\infty} \frac{sdx}{(x^2+a^2)(x^2+s^2)}dx \\
&= \frac{s}{s^2-a^2}\int_{-\infty}^{\infty} \frac{1}{x^2+a^2} - \frac{1}{x^2+s^2}dx.
\end{aligned}$$

Thus, we see that

$$\begin{aligned}
\bar{f}(s) &= \frac{s}{s^2-a^2}\int_{-\infty}^{\infty} \frac{1}{x^2+a^2} - \frac{1}{x^2+s^2}dx \\
&= \frac{s}{s^2-a^2}\left[\tan^{-1}\frac{x}{a}\Big|_{-\infty}^{\infty} - \tan^{-1}\frac{x}{s}\Big|_{-\infty}^{\infty}\right] \\
&= \frac{s}{s^2-a^2}\left[\frac{\pi}{a} - \frac{\pi}{s}\right] \\
&= \frac{\pi}{a}\left[\frac{s}{s^2-a^2} - \frac{a}{s^2-a^2}\right].
\end{aligned}$$

Using the table of Laplace transforms, we know that $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2-a^2}\right\} = \cosh at$ and $\mathscr{L}^{-1}\left\{\dfrac{a}{s^2-a^2}\right\} = \sinh at$. Therefore, we have that

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\cos tx}{x^2+a^2}dx = f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} &= \mathscr{L}^{-1}\left\{\frac{\pi}{a}\left[\frac{s}{s^2-a^2} - \frac{a}{s^2-a^2}\right]\right\} \\
&= \frac{\pi}{a}\left[\mathscr{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} - \mathscr{L}^{-1}\left\{\frac{a}{s^2-a^2}\right\}\right] \\
&= \frac{\pi}{a}\left[\cosh at - \sinh at\right] \\
&= \frac{\pi}{a}e^{-at}.
\end{aligned}$$

b. Suppose that

$$f(t) = \int_0^{\infty} \frac{\sin \pi tx}{x(1+x^2)}dx.$$

In order to evaluate this integral, we take the Laplace transform of $f(t)$ with respect to t. Now, due to uniform convergence, we have that

$$\begin{aligned}
\bar{f}(s) = \mathscr{L}\left\{f(t)\right\} = \mathscr{L}\left\{\int_0^{\infty} \frac{\sin \pi tx}{x(1+x^2)}dx\right\} &= \int_0^{\infty} \mathscr{L}\left\{\frac{\sin \pi tx}{x(1+x^2)}\right\}dx \\
&= \int_0^{\infty} \frac{1}{x(1+x^2)}\mathscr{L}\left\{\sin \pi tx\right\}dx \\
&= \int_0^{\infty} \frac{\pi}{(x^2+1)(\pi^2x^2+s^2)}dx.
\end{aligned}$$

Using a computer algebra system, we see that this last integral reduces to

$$\begin{aligned}
\bar{f}(s) &= \int_0^{\infty} \frac{\pi}{(x^2+1)(\pi^2x^2+s^2)}dx \\
&= \frac{\pi^2}{2s(\pi+s)} \\
&= \frac{\pi}{2}\left[\frac{1}{s} - \frac{1}{s+\pi}\right].
\end{aligned}$$

Therefore, from our table of Laplace transforms, we have that

$$\begin{aligned}
\int_0^{\infty} \frac{\sin \pi tx}{x(1+x^2)}dx = f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} &= \frac{\pi}{2}\left[\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{s+\pi}\right\}\right] \\
&= \frac{\pi}{2}\left(1 - e^{-\pi t}\right).
\end{aligned}$$

$\square$

Problem 3.

Solution. $\square$

Problem 4.

Solution. □

Problem 5.

Solution. □

Problem 6.

Solution. □

Problem 7.

Solution. □

Problem 8.

Solution. □