Test 2

Matthew Tiger

May 17, 2016

Problem 1. Approximate the roots of $\varepsilon x^3 + x^2 + x - 2 = 0$, $\varepsilon \to 0^+$, with precision $O(\varepsilon^2)$.

Solution. When $\varepsilon = 0$, the unperturbed equation only has two roots while the original equation has three roots implying that this is a singular perturbation problem.

To find the other root of this equation we must employ the method of dominant balance on the original equation. There are six possible two-term balances to consider as $\varepsilon \to 0^+$:

a. Suppose that $\varepsilon x^3 \sim x^2$ is the dominant balance and that $x \ll x^2$, $1 \ll x^2$. If $\varepsilon x^3 \sim x^2$ as $\varepsilon \to 0^+$, then $x = O(\varepsilon^{-1})$, which is consistent with the assumptions that $x \ll x^2$ and $1 \ll x^2$ and the balance itself is consistent.

b. Suppose that $\varepsilon x^3 \sim x$ is the dominant balance and that $x^2 \ll x$, $1 \ll x$. If $\varepsilon x^3 \sim x$ as $\varepsilon \to 0^+$, then $x = O(\varepsilon^{-1/2})$. However, if $x = O(\varepsilon^{-1/2})$, then

$$\frac{x^2}{x} = \frac{\left(\varepsilon^{-1/2}\right)^2}{\varepsilon^{-1/2}} = \varepsilon^{-1/2}$$

which implies that as $\varepsilon \to 0^+$ the assumption that $x^2 \ll x$ is false and that this balance is inconsistent.

c. Suppose that $\varepsilon x^3 \sim 2$ is the dominant balance and that $x^2 \ll 1$, $x \ll 1$. If $\varepsilon x^3 \sim 2$ as $\varepsilon \to 0^+$, then $x = O(\varepsilon^{-1/3})$, which implies that as $\varepsilon \to 0^+$ the assumption that $x \ll 1$ is false and that this balance is inconsistent.

d. Suppose that $x^2 \sim x$ is the dominant balance and that $\varepsilon x^3 \ll x$, $1 \ll x$. If $x^2 \sim x$ as $\varepsilon \to 0^+$, then $x \sim 1$ and $x = O(1)$, which is consistent with the assumptions that $\varepsilon x^3 \ll 1$ and $x^2 \ll 1$ and the balance is consistent. Using this balance will recover the root $x = 1$ from the unperturbed equation and the root can be expanded with a perturbation series in ε in the normal way.

e. Suppose that $x^2 \sim 2$ is the dominant balance and that $\varepsilon x^3 \ll 1$, $x \ll 1$. If $x^2 \sim 2$ as $\varepsilon \to 0^+$, then $x = O(1)$, which implies that as $\varepsilon \to 0^+$ the assumption that $x \ll 1$ is false and that this balance is inconsistent.

f. Suppose that $x \sim -2$ is the dominant balance and that $\varepsilon x^3 \ll 1$, $x^2 \ll 1$. If $x \sim -2$ as $\varepsilon \to 0^+$, then $x = O(1)$, which is consistent with the assumptions that $\varepsilon x^3 \ll 1$ and $x^2 \ll 1$ and the balance is consistent. Using this balance will recover the root $x = -2$ from the unperturbed equation and the root can be expanded with a perturbation series in ε in the normal way.

Assuming the balance in case a., we see that the roots of the equation are of order ε^{-1}. Making the transformation $x = \varepsilon^{-1}y$ we see that the original equation becomes

$$y^3 + y^2 + \varepsilon y - 2\varepsilon^2 = 0. \tag{1}$$

Suppose that the roots of the equation (1) can be expressed in terms of ε, i.e.

$$y = \sum_{n=0}^{\infty} a_n \varepsilon^n.$$

Suppose $y = a_0 + a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3 + O(\varepsilon^4)$. Substituting y into (1) and equating coefficients of ε, we see that the following equations must be satisfied:

$$\begin{aligned}
&a_0^2 + a_0^3 &&= 0\\
&a_0 + 2a_0a_1 + 3a_0^2a_1 &&= 0\\
&-2 + a_1 + a_1^2 + 3a_0a_1^2 + 2a_0a_2 + 3a_0^2a_2 &&= 0\\
&a_1^3 + a_2 + 2a_1a_2 + 6a_0a_1a_2 + 2a_0a_3 + 3a_0^2a_3 &&= 0
\end{aligned}$$

When $a_0 = -1$, we see from the second equation that $a_1 = 1$, from the third equation that $a_2 = 3$, and from the fourth equation that $a_3 = 8$. Thus, the root x_1 to the original equation is given by

$$x_1 = \varepsilon^{-1}y = \varepsilon^{-1}\left(-1 + \varepsilon + 3\varepsilon^2 + 8\varepsilon^3 + O(\varepsilon^4)\right) = -\varepsilon^{-1} + 1 + 3\varepsilon + 8\varepsilon^2 + O(\varepsilon^3).$$

accurate to precision $O(\varepsilon^2)$.

When $a_0 = 0$, we have two possibilities from the above system of non-linear equations, either $a_1 = -2$, $a_2 = -8/3$, $a_3 = 0$ or $a_1 = 1$, $a_2 = -1/3$, $a_3 = 0$. Thus, the second and third roots x_2, x_3 to the original equation are given by

$$x_2 = \varepsilon^{-1}y = \varepsilon^{-1}\left(-2\varepsilon - \frac{8}{3}\varepsilon^2 + O(\varepsilon^4)\right) = -2 - \frac{8}{3}\varepsilon + O(\varepsilon^3).$$

and

$$x_3 = \varepsilon^{-1}y = \varepsilon^{-1}\left(\varepsilon - \frac{1}{3}\varepsilon^2 + O(\varepsilon^4)\right) = 1 - \frac{1}{3}\varepsilon + O(\varepsilon^3).$$

both accurate to precision $O(\varepsilon^2)$. Note that when $a_0 = 0$ we have recovered the roots to the unperturbed equation and their actual order is ε.

Therefore, the three roots to the equation $\varepsilon x^3 + x^2 + x - 2 = 0$, $\varepsilon \to 0^+$ are given by

$$\begin{aligned}
x_1 &= -\varepsilon^{-1} + 1 + 3\varepsilon + 8\varepsilon^2 + O(\varepsilon^3)\\
x_2 &= -2 - \frac{8}{3}\varepsilon + O(\varepsilon^3)\\
x_3 &= 1 - \frac{1}{3}\varepsilon + O(\varepsilon^3)
\end{aligned}$$

all accurate to precision $O(\varepsilon^2)$. □

Problem 2. Let

$$I(x) = \int_0^\infty \frac{t^{x-1}e^{-t}}{t^2+x^2}dt.$$

a. Use the method of movable maxima to transform $I(x)$ into an integral of the form

$$J(x) = g(x)\int_0^\infty f(s)e^{-x\phi(s)}ds$$

which is amenable to analysis by Laplace's method.

b. Use the result of part a. to obtain a three term approximation to $I(x)$ for $x \to +\infty$ by Laplace's method.

Solution. Omitted. $\square$

Problem 3. Use the method of stationary phase to obtain the leading behavior for $x \to +\infty$ of the integral

$$I(x) = \int_0^1 e^{ix(t-\sin t)}dt.$$

Solution. We begin by noting that the integral $I(x)$ is a generalized Fourier integral which can be written as

$$I(x) = \int_0^1 f(t)e^{ix\psi(t)}dt$$

where $f(t) = 1$ and $\psi(t) = t - \sin t$. The leading asymptotic behavior of such integrals as $x \to +\infty$ may be found, in general, using integration by parts. However, this method may fail at *stationary points*, i.e. any point on the interval of definition such that $\psi'(t) = 0$. Note that $\psi'(t) = 0$ when $t = 2\pi k$ for $k \in \mathbb{Z}$. Thus, for the integral $I(x)$, we note that $t = 0$ is the only stationary point. Thus, we proceed by writing $I(x)$ as follows:

$$I(x) = I_1(x) + I_2(x) = \int_0^\varepsilon f(t)e^{ix\psi(t)}dt + \int_\varepsilon^1 f(t)e^{ix\psi(t)}dt$$

for some $\varepsilon > 0$.

Since $I_2(x)$ does not have any stationary points and the function $f(t) = 1 \in L^1$ over the interval $[0, 1]$, we have by the Riemann-Lebesgue lemma that $I_2(x) \to 0$ as $x \to +\infty$. Thus, as $x \to +\infty$,

$$I(x) \sim I_1(x) = \int_0^\varepsilon f(t)e^{ix\psi(t)}dt = \int_0^\varepsilon e^{ix(t-\sin t)}dt.$$

The leading behavior of $I(x)$ can be obtained by replacing $f(t)$ with $f(0) = 1$ and $\psi(t) = t - \sin t$ with

$$\psi(0) + \frac{\psi'''(0)t^3}{6} = (0 - \sin 0) + \frac{(\cos 0)t^3}{6} = \frac{t^3}{6},$$

since these are the parts that contribute the most to the integral. Thus,

$$I(x) \sim \int_0^\varepsilon e^{ixt^3/6}dt.$$

Replacing ε with ∞ introduces error terms that vanish as $x \to +\infty$ so that

$$I(x) \sim \int_0^\infty e^{ixt^3/6}dt.$$

Making the substitution $u = x^{1/3}t/6^{1/3}$, we see that $du = (x^{1/3}/6^{1/3})dt$, $u^3 = xt^3/6$ and that

$$I(x) \sim \int_0^\infty e^{ixt^3/6}dt = \left(\frac{6}{x}\right)^{1/3} \int_0^\infty e^{iu^3}du.$$

Using the identity

$$\int_0^\infty e^{iu^3}du = e^{i\pi/6}\Gamma(4/3),$$

we see that

$$I(x) \sim \left(\frac{6}{x}\right)^{1/3} \int_0^\infty e^{iu^3}du = \left(\frac{6}{x}\right)^{1/3} e^{i\pi/6}\Gamma(4/3).$$

Therefore, as $x \to +\infty$,

$$I(x) \sim \left(\frac{6}{x}\right)^{1/3} e^{i\pi/6}\Gamma(4/3).$$

$\square$

Problem 4. a. Show that the BVP

$$\varepsilon y'' + x^{1/3}y' - y = 0, \qquad y(0) = 0,\ y(1) = e^2$$

has a boundary layer of thickness $\varepsilon^{3/4}$ at $x = 0$.

b. Find a leading order uniform asymptotic approximation to the solution of the BVP.

Solution. a. Note that for this BVP $x^{1/3} > 0$ for all $0 < x \leq 1$. From section 9.1 we conclude then that no boundary layer appears for $0 < x \leq 1$ and that the only other possible appearance of a boundary layer must occur at $x = 0$.

Assume that the boundary layer is of thickness $\delta(\varepsilon)$. In the inner region, i.e. when $x \ll \delta(\varepsilon)$, let $y(x) = Y_{\text{in}}(Z)$ where $Z = x/\delta(\varepsilon)$. Thus, the original differential equation becomes

$$\frac{\varepsilon}{\delta(\varepsilon)^2}\frac{d^2Y_{\text{in}}(Z)}{dZ^2} + \frac{Z^{1/3}}{\delta(\varepsilon)^{2/3}}\frac{dY_{\text{in}}(Z)}{dZ} - Y_{\text{in}}(Z) = 0$$

There are three distinguished limits to consider as $\varepsilon \to 0^+$:

i. $\varepsilon/\delta(\varepsilon)^2 \sim 1$ with $\delta(\varepsilon)^{-2/3} \ll 1$.

ii. $\delta(\varepsilon)^{-2/3} \sim 1$ with $\varepsilon/\delta(\varepsilon)^2 \ll 1$.

iii. $\varepsilon/\delta(\varepsilon)^2 \sim \delta(\varepsilon)^{-2/3}$ with $1 \ll \delta(\varepsilon)^{-2/3}$.

In case i. we see that if $\varepsilon/\delta(\varepsilon)^2 \sim 1$ as $\varepsilon \to 0^+$, then $\delta(\varepsilon) \sim \varepsilon^{1/2}$. However, if $\delta(\varepsilon) \sim \varepsilon^{1/2}$ as $\varepsilon \to 0^+$, then $\delta(\varepsilon)^{-2/3} \sim \varepsilon^{-1/3}$ which is inconsistent with the assumption that $\delta(\varepsilon)^{-2/3} \ll 1$ as $\varepsilon \to 0^+$.

In case ii. we see that if $\delta(\varepsilon)^{-2/3} \sim 1$ as $\varepsilon \to 0^+$, then $\delta(\varepsilon)^2 \sim 1$ which is consistent with the assumption that $\varepsilon/\delta(\varepsilon)^2 \ll 1$ as $\varepsilon \to 0^+$. This distinguished limit will reproduce the assumptions used to create the outer solution.

In case iii. we see that if $\varepsilon/\delta(\varepsilon)^2 \sim \delta(\varepsilon)^{-2/3}$ as $\varepsilon \to 0^+$, then $\delta(\varepsilon) \sim \varepsilon^{3/4}$. If $\delta(\varepsilon) \sim \varepsilon^{3/4}$ as $\varepsilon \to 0^+$, then $\delta(\varepsilon)^{-2/3} \sim \varepsilon^{-1/2}$. This is consistent with the assumption that $1 \ll \delta(\varepsilon)^{-2/3}$ as $\varepsilon \to 0^+$. This distinguished limit is the only consistent choice of boundary layer scale.

Therefore, the boundary layer must be of thickness $\varepsilon^{3/4}$ at $x = 0$.

b. In order to find a uniform asymptotic approximation to the solution of the BVP, we must first find the leading-order outer solution outside the boundary and the leading-order inner solution in the boundary layer and then asymptotically match the two solutions.

Outside of the boundary layer, i.e. for $\varepsilon^{3/4} \ll x \leq 1$, we have that the leading-order outer solution satisfies $x^{1/3}y_0' - y_0 = 0$. Thus, when $\varepsilon^{3/4} \ll x \leq 1$, we have that

$$y_0(x) = c_0 e^{\frac{3x^{2/3}}{2}}.$$

Note that $y_0(x)$ must satisfy the condition that $y(1) = e^2$. Thus,

$$y_0(1) = c_0 e^{\frac{3}{2}} = e^2$$

so that $c_0 = e^{1/2}$ and for $\varepsilon^{3/4} \ll x \leq 1$ the outer solution is given by

$$y_0(x) = e^{\frac{1}{2}} e^{\frac{3x^{2/3}}{2}} = e^{\frac{3x^{2/3}+1}{2}}.$$

When x is inside the boundary layer, i.e. for $x \ll \delta(\varepsilon)$, let $y(x) = Y(Z)$ where $Z = x/\delta(\varepsilon)$. As shown above the original differential equation satisfies

$$\frac{\varepsilon}{\delta(\varepsilon)^2}\frac{d^2Y(Z)}{dZ^2} + \frac{Z^{1/3}}{\delta(\varepsilon)^{2/3}}\frac{dY(Z)}{dZ} - Y(Z) = 0.$$

Using the distinguished limit $\delta(\varepsilon) \sim \varepsilon^{3/4}$ as $\varepsilon \to 0^+$, the above differential equation becomes

$$\frac{d^2Y(Z)}{dZ^2} + Z^{1/3}\frac{dY(Z)}{dZ} = \varepsilon^{1/2}Y(Z).$$

Thus, for $0 \leq x \ll \varepsilon^{3/4}$ as $\varepsilon \to 0^+$, the leading-order inner solution $Y_0(Z)$ satisfies

$$\frac{d^2Y_0(Z)}{dZ^2} + Z^{1/3}\frac{dY_0(Z)}{dZ} = 0,$$

the solution of which is

$$Y_0(Z) = d_0 - d_1\left[\frac{3^{1/4}\Gamma(3/4, 3Z^{4/3}/4)}{\sqrt{2}}\right]$$

where $\Gamma(3/4, 3Z^{4/3}/4)$ is the incomplete Gamma function

$$\Gamma(3/4, 3Z^{4/3}/4) = \Gamma(3/4) - \int_0^{3Z^{4/3}/4} t^{-1/4}e^{-t}dt.$$

The boundary condition $y(0) = 0$ implies that

$$Y_0(0) = d_0 - d_1\left[\frac{3^{1/4}\Gamma(3/4, 0)}{\sqrt{2}}\right] = 0$$

or that

$$d_0 = d_1\left[\frac{3^{1/4}\Gamma(3/4)}{\sqrt{2}}\right]$$

so that for $0 \leq x \ll \varepsilon^{3/4}$ as $\varepsilon \to 0^+$ the leading-order inner solution is

$$Y_0(Z) = d_1\left[\frac{3^{1/4}\left(\Gamma(3/4) - \Gamma(3/4, 3Z^{4/3}/4)\right)}{\sqrt{2}}\right].$$

In order to determine the constant d_1, we must asymptotically match the left edge of the outer solution $y_0(x)$ to the right edge of the inner solution $Y_0(Z)$ as $\varepsilon \to 0^+$. In this area, say that $x = O(\varepsilon^{5/8})$. For such x, we know since $Z = x/\varepsilon^{3/4}$ that $Z \to \infty$ as $\varepsilon \to 0^+$ and the inner solution satisfies

$$Y_0(Z) \sim d_1 \left[\frac{3^{1/4}\Gamma(3/4)}{\sqrt{2}}\right].$$

Similarly for $x = O(\varepsilon^{5/8})$, we have that as $\varepsilon \to 0^+$ the outer solution is asymptotic to $y_0(0)$, i.e.

$$y_0(x) \sim y_0(0) = e^{1/2}.$$

Thus, for the two solutions to agree at these edges, we require that

$$d_1 = \frac{e^{1/2}\sqrt{2}}{3^{1/4}\Gamma(3/4)}.$$

Therefore, a leading order uniform asymptotic approximation to the solution of the BVP is given by

$$\begin{aligned}
y_{\text{unif}}(x) &= y_0(x) + Y_0\left(\frac{x}{\varepsilon^{3/4}}\right) - y_{\text{match}}(x) \\
&= e^{\frac{3x^{2/3}+1}{2}} + \frac{e^{1/2}\sqrt{2}}{3^{1/4}\Gamma(3/4)}\left[\frac{3^{1/4}\left(\Gamma\left(\frac{3}{4}\right) - \Gamma\left(\frac{3}{4}, \frac{3x^{4/3}}{4\varepsilon}\right)\right)}{\sqrt{2}}\right] - e^{1/2} \\
&= e^{\frac{3x^{2/3}+1}{2}} - e^{1/2}\left[\frac{\Gamma\left(\frac{3}{4}, \frac{3x^{4/3}}{4\varepsilon}\right)}{\Gamma\left(\frac{3}{4}\right)}\right].
\end{aligned}$$

$\square$