

Homework Assignment 6

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Problem 4.3. Find the solutions of the following systems of equations with the initial data:

a. $\frac{dx}{dt} = x - 2y, \quad x(0) = 1$
 $\frac{dy}{dt} = y - 2x, \quad y(0) = 0$

Solution. a. Applying the Laplace transform to the system yields

$$\mathcal{L} \left\{ \frac{dx}{dt} \right\} = s\bar{x}(s) - x(0) = \bar{x}(s) - 2\bar{y}(s) = \mathcal{L} \{x - 2y\}$$
$$\mathcal{L} \left\{ \frac{dy}{dt} \right\} = s\bar{y}(s) - y(0) = \bar{y}(s) - 2\bar{x}(s) = \mathcal{L} \{y - 2x\}.$$

Using the initial data, the transformed system becomes

$$(s - 1)\bar{x}(s) + 2\bar{y}(s) = 1$$
$$2\bar{x}(s) + (s - 1)\bar{y}(s) = 0,$$

or, equivalently,

$$\begin{bmatrix} s - 1 & 2 \\ 2 & s - 1 \end{bmatrix} \begin{bmatrix} \bar{x}(s) \\ \bar{y}(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This implies that the solution to the transformed system of equations is given by

$$\begin{bmatrix} \bar{x}(s) \\ \bar{y}(s) \end{bmatrix} = \begin{bmatrix} s - 1 & 2 \\ 2 & s - 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{s-1}{(s-3)(s+1)} & -\frac{2}{(s-3)(s+1)} \\ -\frac{2}{(s-3)(s+1)} & \frac{s-1}{(s-3)(s+1)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{s-1}{(s-3)(s+1)} \\ -\frac{2}{(s-3)(s+1)} \end{bmatrix}$$

i.e. the solution is given by $\bar{x}(s) = \frac{s-1}{(s-3)(s+1)}$ and $\bar{y}(s) = -\frac{2}{(s-3)(s+1)}$.

From our table of Laplace Transforms, we know that

$$\mathcal{L} \{e^{at} - e^{bt}\} = \frac{a - b}{(s - a)(s - b)}$$

and

$$\mathcal{L} \left\{ \frac{ae^{at} - be^{bt}}{a - b} \right\} = \frac{s}{(s - a)(s - b)}.$$

Therefore, the solution to the original system of differential equations is given by

$$\begin{aligned}x(t) &= \mathcal{L}^{-1} \{ \bar{x}(s) \} = \mathcal{L}^{-1} \left\{ \frac{s-1}{(s-3)(s+1)} \right\} \\&= \mathcal{L}^{-1} \left\{ \frac{s}{(s-3)(s+1)} \right\} - \mathcal{L}^{-1} \left\{ \frac{1}{(s-3)(s+1)} \right\} \\&= \frac{3e^{3t} + e^{-t}}{4} - \frac{e^{3t} - e^{-t}}{4} \\&= \frac{e^{3t} + e^{-t}}{2}\end{aligned}$$

and

$$\begin{aligned}y(t) &= \mathcal{L}^{-1} \{ \bar{y}(s) \} = \mathcal{L}^{-1} \left\{ -\frac{2}{(s-3)(s+1)} \right\} \\&= -\frac{e^{-t} - e^{3t}}{2}\end{aligned}$$

□

Problem 4.12. Solve the following initial value problems:

- a. $\ddot{x} + \omega^2 x = \cos nt$, $x(0) = 1$, $\dot{x}(0) = 0$ where $\omega \neq n$.

Solution. a. We begin by applying the Laplace transform to the equation. Doing so yields

$$\mathcal{L}\{\ddot{x} + \omega^2 x\} = (s^2 + \omega^2)\bar{x}(s) - sx(0) - \dot{x}(0) = \frac{s}{s^2 + n^2} = \mathcal{L}\{\cos nt\}.$$

Using the initial data, the transformed equation becomes

$$(s^2 + \omega^2)\bar{x}(s) - s = \frac{s}{s^2 + n^2}.$$

Solve the above equation yields that the solution to the transformed equation is

$$\bar{x}(s) = \frac{s^3 + (n^2 + 1)s}{(s^2 + n^2)(s^2 + \omega^2)}.$$

From the partial fractions method we see that

$$\bar{x}(s) = \frac{s^3 + (n^2 + 1)s}{(s^2 + n^2)(s^2 + \omega^2)} = \frac{a_1 s + a_0}{s^2 + n^2} + \frac{b_1 s + b_0}{s^2 + \omega^2}.$$

Combining the rational fractions on the right side under a common denominator and equating the coefficients in the numerator we arrive at the following system of equations

$$\begin{aligned} a_1 + b_1 + 1 &= 1 \\ a_0 + b_0 &= 0 \\ a_1 \omega^2 + b_1 n^2 &= n^2 + 1 \\ a_0 \omega^2 + b_0 n^2 &= 0 \end{aligned}$$

Solving this system, we see that $a_0 = b_0 = 0$, $a_1 = \frac{1}{\omega^2 - n^2}$, and $b_1 = \frac{\omega^2 - n^2 - 1}{\omega^2 - n^2}$.

Thus, the solution to the transformed system is given by

$$\bar{x}(s) = \frac{s^3 + (n^2 + 1)s}{(s^2 + n^2)(s^2 + \omega^2)} = \left(\frac{1}{\omega^2 - n^2}\right) \frac{s}{s^2 + n^2} + \left(\frac{\omega^2 - n^2 - 1}{\omega^2 - n^2}\right) \frac{s}{s^2 + \omega^2}.$$

From our table of Laplace transforms, we know that

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}.$$

Therefore, the solution to the original differential equation is

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\{\bar{x}(s)\} = \left(\frac{1}{\omega^2 - n^2}\right) \mathcal{L}^{-1}\left\{\frac{s}{s^2 + n^2}\right\} + \left(\frac{\omega^2 - n^2 - 1}{\omega^2 - n^2}\right) \mathcal{L}^{-1}\left\{\frac{s}{s^2 + \omega^2}\right\} \\ &= \left(\frac{1}{\omega^2 - n^2}\right) \cos nt + \left(\frac{\omega^2 - n^2 - 1}{\omega^2 - n^2}\right) \cos \omega t. \end{aligned}$$

□

Problem 4.14.*Solution.*

Problem 4.22.*Solution.*

Problem 4.25.*Solution.*