Homework Assignment 7

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Problem 6.5.2. Let $\Sigma = \{(a_1, a_2, a_3, \ldots) \mid a_i \in \{0, 1\}\}$, the sequence space of zeroes and ones with the metric defined previously. Let C be the Cantor set and define $f : \Sigma \to C$ by

$$f((a_1, a_2, a_3, \ldots)) = .b_1 b_2 b_3 \ldots \quad \text{where } b_i = 0 \text{ if } a_i = 0 \text{ and } b_i = 2 \text{ if } a_i = 1$$

giving the ternary expansion of a real number in $[0, 1]$. Show that f defines a homeomorphism between $Sigma$ and C, the Cantor set.

Solution. $\square$

Problem 6.5.3. Let $f : I \to I$ be a transitive map with I an interval. Show that if U and V are non-empty open sets in I, then there exists $m \in \mathbb{Z}^+$ with $U \cap f^m(V) \neq \emptyset$

Solution. □

Problem 6.5.4. Let $F : [0, 1) \to [0, 1)$ be the tripling map. Show that F is transitive and that its periodic points are dense in $[0, 1)$.

Solution. Note that

$$F(x) := \begin{cases} 3x & \text{if } x \in [0, 1/3) \\ 3x - 1 & \text{if } x \in [1/3, 2/3) \\ 3x - 2 & \text{if } x \in [2/3, 1) \end{cases}$$

□

Problem 7.1.2. i. Define $f_a : \mathbb{R} \to \mathbb{R}$ by $f_a(x) = ax$ for $a \in \mathbb{R}$. Show that $f_{1/2}$ and $f_{1/4}$ are conjugate via the map

$$h(x) = \begin{cases} \sqrt{x} & x \geq 0 \\ -\sqrt{-x} & x < 0 \end{cases}.$$

ii. More generally, show that $f_a, f_b : [0, \infty) \to [0, \infty)$ for $0 < a, b < 1$, the f_a and f_b are conjugate via the map $h(x) = x^p$ for $p > 0$ and similarly if $a, b > 1$.

iii. Discuss the cases where $a > 1$ and $0 < b < 1$. What happens when $a = 1/2$ and $b = 2$?

Solution. □

Problem 7.1.3. Prove that if $f : X \to X$ and $g : Y \to Y$ are conjugate maps of metric spaces, then f is one-to-one if and only if g is one-to-one and f is onto if and only if g is onto.

Solution. If f and g are conjugate maps of metric spaces, then there exists a map $h : X \to Y$, with h a bijection, such that $g \circ h = h \circ f$.

Suppose that f is one-to-one and that $g(y_1) = g(y_2)$. Since h is onto, there exist $x_1 \in X$ and $x_2 \in X$ such that $h(x_1) = y_1$ and $h(x_2) = y_2$. Thus, if $g(y_1) = g(y_2)$, then $g \circ h(x_1) = g \circ h(x_2)$. By the conjugacy of h, we then have that $h \circ f(x_1) = h \circ f(x_2)$ and since h and f are one-to-one, we have that $x_1 = x_2$. Due to the fact that h is a well-defined function, if $x_1 = x_2$, then $y_1 = h(x_1) = h(x_2) = y_2$ and we therefore have that g is one-to-one.

Now suppose that g is one-to-one and that $f(x_1) = f(x_2)$. Since h is well-defined, we have that $h \circ f(x_1) = h \circ f(x_2)$. By the conjugacy of h, we then have that $g \circ h(x_1) = g \circ h(x_2)$. Since g and h are one-to-one, it follows that $x_1 = x_2$ and f is therefore one-to-one.

Suppose that f is onto and let $y_2 \in Y$ be given. Since h is onto, there exists $x_2 \in X$ such that $h(x_2) = y_2$. Thus, since f is onto, there exists $x_1 \in X$ such that $f(x_1) = x_2$ which implies that $h \circ f(x_1) = y_2$. By the conjugacy of h we have that

$$g \circ h(x_1) = h \circ f(x_1) = y_2.$$

Hence, there exists $y_1 = h(x_1) \in Y$ such that $g(y_1) = y_2$. Therefore, since $y_2 \in Y$ was arbitary, we have that h is onto.

Now suppose that g is onto. Since g and h are onto, for every $y \in Y$, there exists $x_1 \in X$ such that $g \circ h(x_1) = y$. By the conjugacy of h, we then have that $h \circ f(x_1) = y$. So, for every $y \in Y$, there exists $f(x_1) \in X$ such that $h \circ f(x_1) = y$. However, since h is onto, we also have that for every $y \in Y$, there exists $x_2 \in X$ such that $h(x_2) = y$. Thus, for every $x_2 \in X$ we have that $h(x_2) = y = h(f(x_1))$ for some $x_1 \in X$. The fact that h is one-to-one then shows that $f(x_1) = x_2$ or that f is onto. $\square$