Homework Assignment 4

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Problem 3.1. Find the Laplace transforms of the following functions:

  b. $f(t) = (1 - 2t)e^{-2t}$

  c. $f(t) = t \cos at$

  d. $f(t) = t^{3/2}$

  g. $f(t) = (t-3)^2 H(t-3)$

Solution. Recall that the Laplace transform of the function $f(t)$ defined for $t > 0$ is given by

$$\mathscr{L}\{f(t)\} = \bar{f}(s) = \int_0^\infty f(t)e^{-st}dt. \tag{1}$$

  b. Let $g(t) = 1 - 2t$. Then $f(t) = (1 - 2t)e^{-2t} = g(t)e^{-2t}$. From the definition of the Laplace transform, we have that

$$\begin{aligned}
\mathscr{L}\{g(t)\} = \bar{g}(s) &= \int_0^\infty (1 - 2t)e^{-st}dt \\
&= \int_0^\infty t^0 e^{-st}dt - 2\int_0^\infty t^1 e^{-st}dt \\
&= \mathscr{L}\{t^0\} - 2\mathscr{L}\{t^1\}.
\end{aligned}$$

From a previous theorem, we know for $n \in \mathbb{N}$ that

$$\mathscr{L}\{t^n\} = \int_0^\infty t^n e^{-st}dt = \frac{n!}{s^{n+1}}.$$

Thus,

$$\bar{g}(s) = \mathscr{L}\{t^0\} - 2\mathscr{L}\{t^1\} = \frac{1}{s} - \frac{2}{s^2} = \frac{s-2}{s^2}.$$

From Heaviside's First Shifting Theorem, we know that for $\bar{g}(s) = \mathscr{L}\{g(t)\}$ that

$$\mathscr{L}\{g(t)e^{-at}\} = \bar{g}(s+a).$$

Therefore, the Laplace transform of $f(t) = (1 - 2t)e^{-2t} = g(t)e^{-2t}$ is

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{g(t)e^{-2t}\} = \bar{g}(s+2) = \frac{s}{(s+2)^2}.$$

c. From the definition of the complex exponential, we have that $f(t) = t \cos at = \frac{t}{2}\left(e^{-iat} + e^{iat}\right)$. From the definition of the Laplace transform, we have that

$$\mathscr{L}\{f(t)\} = \bar{f}(s) = \int_0^\infty \frac{t}{2}\left(e^{-iat} + e^{iat}\right)e^{-st}dt$$

$$= \frac{1}{2}\left[\int_0^\infty te^{-(s+ia)t}dt + \int_0^\infty te^{-(s-ia)t}dt\right].$$

We readily see by integrating by parts using $u = t$ and $dv = e^{-(s\pm ia)t}dt$ that

$$\int_0^\infty te^{-(s\pm ia)t}dt = -\frac{t}{s \pm ia}e^{-(s\pm ia)t}\Big|_0^\infty + \frac{1}{s \pm ia}\int_0^\infty e^{-(s\pm ia)t}dt$$

$$= -\frac{1}{(s \pm ia)^2}e^{-(s\pm ia)t}\Big|_0^\infty$$

$$= \frac{1}{(s \pm ia)^2}.$$

Therefore, the Laplace transform of $f(t)$ is given by

$$\mathscr{L}\{f(t)\} = \bar{f}(s) = \frac{1}{2}\left[\int_0^\infty te^{-(s+ia)t}dt + \int_0^\infty te^{-(s-ia)t}dt\right]$$

$$= \frac{1}{2}\left[\frac{1}{(s + ia)^2} + \frac{1}{(s - ia)^2}\right]$$

$$= \frac{s^2 - a^2}{(s + ia)^2(s - ia)^2}$$

$$= \frac{s^2 - a^2}{(s^2 + a^2)^2}.$$

d. By definition, the Laplace transform of $f(t)$ is given by

$$\mathscr{L}\{f(t)\} = \bar{f}(s) = \int_0^\infty t^{3/2}e^{-st}dt.$$

Let $u = st$, then $du/s = dt$ and

$$\mathscr{L}\{f(t)\} = \bar{f}(s) = \frac{1}{s}\int_0^\infty \left(\frac{u}{s}\right)^{3/2}e^{-u}dt$$

$$= \frac{1}{s^{5/2}}\int_0^\infty u^{3/2}e^{-u}dt.$$

Recall that the definition of the Gamma function is given by

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}du.$$

Therefore, the Laplace transform of $f(t) = t^{3/2}$ is

$$\mathscr{L}\left\{f(t)\right\} = \bar{f}(s) = \frac{1}{s^{5/2}} \int_0^\infty u^{5/2-1} e^{-u} dt$$
$$= \frac{\Gamma\left(\frac{5}{2}\right)}{s^{5/2}}.$$

g. Let $g(t) = t^2$ and suppose that $\mathscr{L}\left\{g(t)\right\} = \bar{g}(s)$. Then Heaviside's Second Shifting Theorem shows that

$$\mathscr{L}\left\{f(t)\right\} = \mathscr{L}\left\{g(t-3)H(t-3)\right\} = e^{-3s}\bar{g}(s).$$

As shown previously, we know for $n \in \mathbb{N}$ that

$$\mathscr{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}}.$$

Therefore, the Laplace transform of $f(t)$ is

$$\mathscr{L}\left\{f(t)\right\} = \bar{f}(s) = e^{-3s}\bar{g}(s) = \frac{2e^{-3s}}{s^3}.$$

$\square$

Problem 3.3. The following is a result relating the Laplace transform of a function's derivative to the Laplace transform of that function:

$$\mathscr{L}\left\{f'(t)\right\} = s\mathscr{L}\left\{f(t)\right\} - f(0). \tag{2}$$

Use the result to find

   a. $\mathscr{L}\left\{\cos at\right\}$,

   b. $\mathscr{L}\left\{\sin at\right\}$.

Solution.   a. Let $f(t) = \cos at$. Then $f'(t) = -a\sin at$ and from (2) we have

$$-a\mathscr{L}\left\{\sin at\right\} = s\mathscr{L}\left\{\cos at\right\} - 1. \tag{3}$$

Now let $g(t) = \sin at$. Then $g'(t) = a\cos at$ and applying (2) to $g(t)$ yields

$$a\mathscr{L}\left\{\cos at\right\} = s\mathscr{L}\left\{\sin at\right\}.$$

Therefore, from (3) we have that

$$-a\left(\frac{a}{s}\mathscr{L}\left\{\cos at\right\}\right) = s\mathscr{L}\left\{\cos at\right\} - 1$$

which implies that

$$\mathscr{L}\left\{\cos at\right\} = \frac{s}{s^2 + a^2}.$$

   b. Let $f(t) = \sin at$. Then $f'(t) = a\cos at$ and from (2) we have

$$a\mathscr{L}\left\{\cos at\right\} = s\mathscr{L}\left\{\sin at\right\}. \tag{4}$$

Now let $g(t) = \cos at$. Then $g'(t) = -a\sin at$ and applying (2) to $g(t)$ yields

$$-a\mathscr{L}\left\{\sin at\right\} = s\mathscr{L}\left\{\cos at\right\} - 1$$

which implies that

$$\mathscr{L}\left\{\cos at\right\} = \frac{1}{s} - \frac{a}{s}\mathscr{L}\left\{\sin at\right\}.$$

Therefore, from (4) we have that

$$a\left(\frac{1}{s} - \frac{a}{s}\mathscr{L}\left\{\sin at\right\}\right) = s\mathscr{L}\left\{\sin at\right\}$$

which implies that

$$\mathscr{L}\left\{\sin at\right\} = \frac{a}{s^2 + a^2}.$$

$\square$

Problem 3.6. Show that

$$\mathscr{L}\left\{\int_0^t \frac{f(u)}{u}du\right\} = \frac{1}{s}\int_s^\infty \bar{f}(x)dx.$$

Solution. From the definition of the Laplace transform we see that

$$\mathscr{L}\left\{\int_0^t \frac{f(u)}{u}du\right\} = \int_0^\infty e^{-st}\left[\int_0^t \frac{f(u)}{u}du\right]dt.$$

Interchanging the order of integration from u to t where $0 \le t < \infty$, we see that $u \le t < \infty$ as $0 \le u < \infty$ and

$$\begin{aligned}
\mathscr{L}\left\{\int_0^t \frac{f(u)}{u}du\right\} &= \int_0^\infty e^{-st}\left[\int_0^t \frac{f(u)}{u}du\right]dt \\
&= \int_0^\infty \frac{f(u)}{u}\left[\int_u^\infty e^{-st}dt\right]du \\
&= \frac{1}{s}\int_0^\infty \frac{f(u)}{u}e^{-su}du.
\end{aligned}$$

We note that $\dfrac{d}{ds}\left[\dfrac{e^{-su}}{u}\right] = -e^{-su}$ so that in particular we have that

$$-\int_s^\infty e^{-su}ds = -\frac{e^{-su}}{u}\bigg|_s^\infty = -\frac{e^{-su}}{u}$$

or that

$$\int_s^\infty e^{-su}ds = \frac{e^{-su}}{u}.$$

Thus,

$$\begin{aligned}
\mathscr{L}\left\{\int_0^t \frac{f(u)}{u}du\right\} &= \frac{1}{s}\int_0^\infty \frac{f(u)}{u}e^{-su}du \\
&= \frac{1}{s}\int_0^\infty f(u)\left[\int_s^\infty e^{-su}ds\right]du.
\end{aligned}$$

Interchanging the order of integration yet again from s to u where $s \le s < \infty$ as $0 \le u < \infty$, we see that the integration limits remain unchanged and therefore that

$$\begin{aligned}
\mathscr{L}\left\{\int_0^t \frac{f(u)}{u}du\right\} &= \frac{1}{s}\int_0^\infty f(u)\left[\int_s^\infty e^{-su}ds\right]du \\
&= \frac{1}{s}\int_s^\infty \left[\int_0^\infty f(u)e^{-su}du\right]ds \\
&= \frac{1}{s}\int_s^\infty \bar{f}(s)ds \\
&= \frac{1}{s}\int_s^\infty \bar{f}(x)dx,
\end{aligned}$$

and we are done.

$\square$

Problem 3.7. Obtain the inverse Laplace transforms of the following functions:

b. $\bar{f}(s) = \dfrac{1}{s^2(s^2+c^2)}$.

Solution.   b. Let $\bar{f}(s) = \bar{g}(s)\bar{h}(s)$ where $\bar{g}(s) = \dfrac{1}{s^2}$ and $\bar{h}(s) = \frac{1}{s^2+c^2}$.

Using previous results, we know that

$$g(t) = \mathscr{L}^{-1}\left\{\bar{g}(s)\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

and

$$h(t) = \mathscr{L}^{-1}\left\{\bar{h}(s)\right\} = \mathscr{L}^{-1}\left\{\frac{1}{c}\left(\frac{c}{s^2+c^2}\right)\right\} = \frac{\sin ct}{c}.$$

Now, by the Convolution Theorem for the Laplace transform, we have that

$$f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} = \mathscr{L}^{-1}\left\{\bar{g}(s)\bar{h}(s)\right\} = (g * h)(t)$$

where

$$(g * h)(s) = \int_0^t g(t-\tau)h(\tau)d\tau.$$

Therefore,

$$\begin{aligned}
f(t) = (g * h)(t) &= \frac{1}{c}\int_0^t (t-\tau)\sin c\tau\, d\tau \\
&= \frac{t}{c}\int_0^t \sin c\tau\, d\tau - \frac{1}{c}\int_0^t \tau \sin c\tau\, d\tau \\
&= \frac{t}{c}\left[\frac{1}{c} - \frac{\cos ct}{c}\right] - \frac{1}{c}\left[-\frac{t\cos ct}{c} + \frac{\sin ct}{c^2}\right] \\
&= \frac{t}{c^2} - \frac{\sin ct}{c^3}.
\end{aligned}$$

□

Problem 3.8. Use the Convolution Theorem to fin the inverse Laplace transforms of the following functions:

a. $\bar{f}(s) = \dfrac{s^2}{(s^2 + a^2)^2}$.

Solution. Let $\bar{f}(s) = \bar{g}(s)^2$ where $\bar{g}(s) = \dfrac{s}{s^2 + a^2}$. Then we know that

$$g(t) = \mathscr{L}^{-1}\left\{\bar{g}(s)\right\} = \cos at.$$

From the Convolution Theorem, we then have that

$$f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} = \mathscr{L}^{-1}\left\{\bar{g}(s)\bar{g}(s)\right\} = (g * g)(t).$$

Therefore,

$$
\begin{aligned}
f(t) = (g * g)(t) &= \int_0^t \cos a(t - \tau) \cos a\tau d\tau \\
&= \cos at \int_0^t \cos^2 a\tau d\tau + \sin at \int_0^t \sin a\tau \cos a\tau d\tau \\
&= \cos at \left[\frac{2at + \sin 2at}{4a}\right] + \sin at \left[\frac{\sin^2 at}{2a}\right] \\
&= \frac{at \cos at + \sin at}{2a}.
\end{aligned}
$$

$\square$

Problem 3.10. Show that

a. $\mathscr{L}\left\{\frac{1}{t}(\sin at - at\cos at)\right\} = \tan^{-1}\left(\frac{a}{s}\right) - \frac{as}{s^2+a^2}$,

b. $\mathscr{L}\left\{\int_0^t \frac{1}{\tau}(\sin a\tau - a\tau\cos a\tau)d\tau\right\} = \frac{1}{s}\left[\tan^{-1}\left(\frac{a}{s}\right) - \frac{as}{s^2+a^2}\right]$.

Solution.   a. If $\bar{f}(s) = \mathscr{L}\{f(t)\}$, then from a previous theorem we have that

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)ds.$$

Thus, we have that

$$\mathscr{L}\left\{\frac{1}{t}(\sin at - at\cos at)\right\} = \int_s^\infty \mathscr{L}\{\sin at - at\cos at\}\,ds$$
$$= \int_s^\infty \mathscr{L}\{\sin at\} - a\mathscr{L}\{t\cos at\}\,ds.$$

From our table of Laplace transforms, we know that

$$\mathscr{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

and

$$\mathscr{L}\{t\cos at\} = \frac{s^2-a^2}{(s^2+a^2)^2}.$$

This implies that

$$\mathscr{L}\left\{\frac{1}{t}(\sin at - at\cos at)\right\} = \int_s^\infty \mathscr{L}\{\sin at\} - a\mathscr{L}\{t\cos at\}\,ds$$
$$= \int_s^\infty \frac{a}{s^2+a^2} - a\left(\frac{s^2-a^2}{(s^2+a^2)^2}\right)ds$$
$$= \tan^{-1}\left(\frac{s}{a}\right)\Big|_s^\infty + \frac{as}{s^2+a^2}\Big|_s^\infty$$
$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) - \frac{as}{s^2+a^2}$$
$$= \tan^{-1}\left(\frac{a}{s}\right) - \frac{as}{s^2+a^2}.$$

b. From the theorem regarding the Laplace transform of an integral, if $\bar{f}(s) = \mathscr{L}\{f(t)\}$, then

$$\mathscr{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{\bar{f}(s)}{s}. \tag{5}$$

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Suppose that $f(t) = \dfrac{1}{t}(\sin at - at \cos at)$. Then we have shown previously that

$$\bar{f}(s) = \mathscr{L}\{f(t)\} = \tan^{-1}\left(\frac{a}{s}\right) - \frac{as}{s^2 + a^2}.$$

Therefore, we have by (5) that

$$\mathscr{L}\left\{\int_0^t \frac{1}{\tau}(\sin a\tau - a\tau \cos a\tau)d\tau\right\} = \mathscr{L}\left\{\int_0^t f(\tau)d\tau\right\}$$

$$= \frac{\bar{f}(s)}{s}$$

$$= \frac{1}{s}\left[\tan^{-1}\left(\frac{a}{s}\right) - \frac{as}{s^2 + a^2}\right].$$

$\square$

Problem 3.12. If $\mathscr{L}\{f(t)\} = \bar{f}(s)$, show that

ii. $\mathscr{L}^{-1}\left\{\dfrac{\bar{f}(s)}{s^2}\right\} = \displaystyle\int_0^t \left[\int_0^{t_1} f(\tau)d\tau\right] dt_1 = \int_0^t (t-\tau)f(\tau)d\tau.$

Solution.   ii. Let $\bar{g}(s) = \dfrac{1}{s}$ and $\bar{h}(s) = \dfrac{\bar{f}(s)}{s}$. From (5) we know that

$$h(t) = \mathscr{L}^{-1}\left\{\bar{h}(s)\right\} = \mathscr{L}^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(\tau)d\tau.$$

It is easy to see that if $\bar{g}(s) = \dfrac{1}{s}$ then $g(t) = 1$. Now, by the Convolution Theorem, we have that

$$\mathscr{L}^{-1}\left\{\frac{\bar{f}(s)}{s^2}\right\} = \mathscr{L}^{-1}\left\{\bar{g}(s)\bar{h}(s)\right\} = (g*h)(t) = \int_0^t g(t-t_1)h(t_1)dt_1$$
$$= \int_0^t \int_0^{t_1} f(\tau)d\tau dt_1.$$

Thus,

$$\mathscr{L}^{-1}\left\{\frac{\bar{f}(s)}{s^2}\right\} = \int_0^t \int_0^{t_1} f(\tau)d\tau dt_1. \tag{6}$$

By interchanging the order of integration from τ to t_1, where $0 \leq \tau \leq t_1$ as $0 \leq t_1 \leq t$, we see that $\tau \leq t_1 \leq t$ and $0 \leq \tau \leq t$ so that

$$\mathscr{L}^{-1}\left\{\frac{\bar{f}(s)}{s^2}\right\} = \int_0^t \int_0^{t_1} f(\tau)d\tau dt_1 = \int_0^t f(\tau)\left[\int_\tau^t dt_1\right] d\tau = \int_0^t (t-\tau)f(\tau)d\tau,$$

and we are done.

$\square$

Problem 3.15. Show that

b. $\mathscr{L}\left\{t^n e^{at}\right\} = \dfrac{n!}{(s-a)^{n+1}}.$

Solution.   b. Let $f(t) = t^n$. By Heaviside's First Shifting Theorem, we have that

$$\mathscr{L}\left\{t^n e^{at}\right\} = \mathscr{L}\left\{f(t)e^{at}\right\} = \bar{f}(s-a).$$

As shown previously,

$$\bar{f}(s) = \mathscr{L}\left\{f(t)\right\} = \frac{n!}{s^{n+1}}.$$

Therefore,

$$\mathscr{L}\left\{t^n e^{at}\right\} = \bar{f}(s-a) = \frac{n!}{(s-a)^{n+1}}.$$

$\square$

Problem 3.18.

Solution.   □