Homework Assignment 5

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November 30, 2016

Problem 5.2. Suppose that you arrive at a single-teller bank to find five other customers in the bank, one being served and the other four waiting in line. You join the end of the line. If the service times are exponential with rate μ, what is the expected amount of time you will spend in the bank?

Solution. Let T_i denote the time that the i-th person spends at the teller in order to complete his or her transaction. Then the total amount of time I will spend in the bank is the amount of time the other five customers spend at the teller plus the time that I, the sixth customer, will spend at the teller. If S denotes the total amount of time that I spend at the bank, then

$$S = \sum_{i=1}^{6} T_i.$$

Note that even though the first customer is currently being served, the service time is exponentially distributed with rate μ, i.e. the waiting time is memory-less so that the expected service time of the first customer is still $1/\mu$. Since the other T_i are exponential random variables with mean $1/\mu$, we have that

$$E[S] = \sum_{i=1}^{6} E[T_i] = \frac{6}{\mu}.$$

□

Problem 5.8. If X and Y are independent exponential random variables with respective rates λ and μ, what is the conditional distribution of X given that $X < Y$?

Solution. $\square$

Problem 5.15. One hundred items are simultaneously put on a life test. Suppose the lifetimes of the individual items are independent exponential random variables with mean 200 hours. The test will end when there have been a total of 5 failures. If T is the time at which the test ends, find $E[T]$ and $\mathrm{Var}[T]$.

Solution. Let T_i be the time between the $(i-1)$-th and the i-th failure. Note that from a previous proposition, we have that if $X_1, \ldots, X_n$ are independent exponential random variables all with rate $\lambda = 1/200$, then $\min_i X_i$ is exponential with rate $n\lambda$. Thus, the shortest lifetime among the 100 initial items is exponential with rate 100λ, i.e. the time that it takes for the first failure to occur, T_1, is exponential with rate 100λ.

In general, after the $(i-1)$-th failure, there will be $101-i$ items left to test. Thus, the shortest lifetime among the $101-i$ items is exponential with rate $(101-i)\lambda$, i.e. the time that it takes for the i-t failure to occur, T_i, is exponential with rate $(101-i)\lambda$.

Since each T_i is exponential with rate $(101-i)\lambda$, we have that

$$E[T_i] = \frac{1}{(101-i)\lambda} = \frac{200}{(101-i)}, \quad \mathrm{Var}[T_i] = \frac{1}{(101-i)^2\lambda^2} = \frac{200^2}{(101-i)^2}.$$

If the test ends after 5 failures and if T is the total time that the test takes, then $T = T_1 + \cdots + T_5$ and

$$E[T] = \sum_{i=1}^{5} E[T_i] = \sum_{i=1}^{5} \frac{200}{(101-i)} = 10.2062.$$

Since each T_i is independent, we have that

$$\mathrm{Var}[T] = \sum_{i=1}^{5} \mathrm{Var}[T_i] = \sum_{i=1}^{5} \frac{200^2}{(101-i)^2} = 20.8377.$$

□

Problem 5.43.

Solution. □

Problem 5.44.

Solution. □

Problem 5.50.

Solution. □