Homework Assignment 5

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Problem 1. Use the method of stationary phase to find the leading behavior of the following integral as $x \to +\infty$:

$$I(x) = \int_0^1 e^{ixt^2} \cosh t^2 dt.$$

Solution. We begin by noting that the integral $I(x)$ is a generalized Fourier integral which can be written as

$$I(x) = \int_0^1 f(t) e^{ix\psi(t)} dt$$

where $f(t) = \cosh t^2$ and $\psi(t) = t^2$. The leading asymptotic behavior of such integrals as $x \to +\infty$ may be found, in general, using integration by parts. However, this method may fail at *stationary points*, i.e. any point on the interval of definition such that $\psi'(t) = 0$. For the integral $I(x)$ we note that $t = 0$ is a stationary point. Thus, we proceed by writing $I(x)$ as follows:

$$I(x) = I_1(x) + I_2(x) = \int_0^\varepsilon f(t) e^{ix\psi(t)} dt + \int_\varepsilon^1 f(t) e^{ix\psi(t)} dt$$

for some $\varepsilon > 0$. Since $I_2(x)$ does not have any stationary points and the function $f(t) = \cosh t^2 \in L^1$ over the interval $[0, 1]$, i.e. we have that $\int_0^1 |f(t)|\, dt < +\infty$, integration by parts works on $I_2(x)$ and by the Riemann-Lebesgue lemma, $I_2(x) \to 0$ as $x \to +\infty$. Thus, as $x \to +\infty$,

$$I(x) \sim I_1(x) = \int_0^\varepsilon f(t) e^{ix\psi(t)} dt = \int_0^\varepsilon \cosh t^2 e^{ixt^2} dt.$$

We continue by replacing $f(t)$ with $f(0) = \cosh 0 = 1$ and ε with ∞, since these are the parts that contribute the most to the integral, introducing error terms that vanish as $x \to +\infty$ so that

$$I(x) \sim \int_0^\infty e^{ixt^2} dt$$

Making the substitution

$$t = e^{i\pi/4} \left[\frac{u}{x}\right]^{1/2}$$

yields that

$$\int_0^\infty e^{ixt^2} dt = e^{i\pi/4} \left[\frac{1}{x}\right]^{1/2} \frac{\Gamma(1/2)}{2} = \frac{e^{i\pi/4}}{2}\sqrt{\frac{\pi}{x}}.$$

Therefore, as $x \to +\infty$,

$$I(x) \sim \int_0^\infty e^{ixt^2} dt = \frac{e^{i\pi/4}}{2}\sqrt{\frac{\pi}{x}}.$$

$\square$

Problem 2. Use second-order perturbation theory to find approximations to the roots of the following equation:

$$x^3 + \varepsilon x^2 - x = 0.$$

Solution. If we assume that the roots of the above equation are functions of ε, then the roots x_i for $i = 0, 1, 2$ of the equation are of the form

$$x_i(\varepsilon) = \sum_{k=0}^{\infty} a_{i_k} \varepsilon^k.$$

Second-order perturbation theory prescribes that the roots are of the form

$$x_i(\varepsilon) = a_{i_0} + a_{i_1}\varepsilon + a_{i_2}\varepsilon^2 + O(\varepsilon^3)$$

where we disregard terms of order ε^3 or greater. Substituting $\varepsilon = 0$ into the equation yields the new equation $x^{+3} - x = 0$, the roots of which are $-1, 0$, and 1 which we will say correspond to the coefficients $a_{0_0} = -1, a_{1_0} = 0$, and $a_{2_0} = 1$.

In order to find the values of the coefficients a_{i_k} for $k \geq 1$, we substitute the expression $x_i(\varepsilon) = a_{i_0} + a_{i_1}\varepsilon + a_{i_2}\varepsilon^2 + O(\varepsilon^3)$ into the original equation yielding

$$a_{i_0}^3 - a_{i_0} + (a_{i_0}^2 - a_{i_1} + 3a_{i_0}^2 a_{i_1})\varepsilon + (2a_{i_0}a_{i_1} + 3a_{i_0}a_{i_1}^2 - a_{i_2} + 3a_{i_0}^2 a_{i_2})\varepsilon^2 = O(\varepsilon^3).$$

Since ε is variable we must have that the coefficients of ε in the above equation are 0. This yields two equations for each root:

$$\begin{aligned} a_{i_0}^2 - a_{i_1} + 3a_{i_0}^2 a_{i_1} &= 0 \\ 2a_{i_0}a_{i_1} + 3a_{i_0}a_{i_1}^2 - a_{i_2} + 3a_{i_0}^2 a_{i_2} &= 0. \end{aligned}$$

For the root x_0, we have that $a_{0_0} = -1$ and the two equations become

$$\begin{aligned} (-1)^2 - a_{0_1} + 3(-1)^2 a_{0_1} &= 0 \\ -2a_{0_1} - 3a_{0_1}^2 - a_{0_2} + 3(-1)^2 a_{0_2} &= 0. \end{aligned}$$

The first equation yields that $a_{0_1} = -1/2$ and substituting into the second equation yields that $a_{0_2} = -1/8$. Thus, $x_0 = -1 + (-1/2)\varepsilon + (-1/8)\varepsilon^2 + O(\varepsilon^3)$.

For the root $x_1 = 0$, we see that $a_{1_0} = 0$ and consequently from the equations that $a_{1_1} = 0$ and $a_{1_2} = 0$. Thus, $x_1 = 0 + 0\varepsilon + 0\varepsilon^2 + O(\varepsilon^3)$.

Proceeding in the same way above we see for the root x_2, we have that $a_{2_0} = 1$ and the above two equations become

$$\begin{aligned} (1)^2 - a_{2_1} + 3(1)^2 a_{2_1} &= 0 \\ 2a_{2_1} + 3a_{2_1}^2 - a_{2_2} + 3(1)^2 a_{2_2} &= 0. \end{aligned}$$

The first equation yields that $a_{2_1} = -1/2$ and substituting into the second equation yields that $a_{2_2} = 1/8$. Therefore, $x_2 = 1 + (-1/2)\varepsilon + (1/8)\varepsilon^2 + O(\varepsilon^3)$ and we have found second-order approximations for all of the roots of the original equation.

$\square$

Problem 3. Analyze in the limit $\varepsilon \to 0$ the roots of the polynomial

$$\varepsilon x^8 - \varepsilon^2 x^6 + x - 2 = 0.$$

Solution. When $\varepsilon = 0$, the unperturbed equation only has one root while the original equation has eight roots implying that this is a singular perturbation problem.

The root of the unperturbed problem is given by $x_0 = 2$. To find the other roots, we employ the method of dominant balance to the original equation and find a consistent balance in order to understand the order of magnitude of the roots as $\varepsilon \to 0$.

As there are four terms in the original equation, there are 6 possible dominant balances to consider:

i. Suppose $\varepsilon x^8 \sim \varepsilon^2 x^6$ as $\varepsilon \to 0$ is the dominant balance. Then $x = O(\varepsilon^{1/2})$ as $\varepsilon \to 0$. However, $\varepsilon x^8 \ll 2$ as $\varepsilon \to 0$ violating the assumption that εx^8 and $\varepsilon^2 x^6$ are the dominant terms showing that the original balance is inconsistent.

ii. Suppose $\varepsilon x^8 \sim x$ as $\varepsilon \to 0$ is the dominant balance. Then $x = O(\varepsilon^{-1/7})$ as $\varepsilon \to 0$. Since $2 \ll x$ and $\varepsilon^2 x^6 = O(\varepsilon^{8/7})$ is such that $\varepsilon^2 x^6 \ll x$, this dominant balance is consistent.

iii. Suppose $\varepsilon x^8 \sim 2$ as $\varepsilon \to 0$ is the dominant balance. Then $x = O(\varepsilon^{-1/8})$ as $\varepsilon \to 0$. However, $2 \ll x$ as $\varepsilon \to 0$ violating the assumption that εx^8 and 2 are the dominant terms showing that the original balance is inconsistent.

iv. Suppose $\varepsilon^2 x^6 \sim x$ as $\varepsilon \to 0$ is the dominant balance. Then $x = O(\varepsilon^{-2/5})$ as $\varepsilon \to 0$. However, $\varepsilon x^8 = O(\varepsilon^{-11/5})$ is such that $x \ll \varepsilon x^8$ violating the assumption that $\varepsilon^2 x^6$ and x are the dominant terms showing that the original balance is inconsistent.

v. Suppose $\varepsilon^2 x^6 \sim 2$ as $\varepsilon \to 0$ is the dominant balance. Then $x = O(\varepsilon^{-1/3})$ as $\varepsilon \to 0$. However, $\varepsilon x^8 = O(\varepsilon^{-5/3})$ is such that $2 \ll \varepsilon x^8$ violating the assumption that $\varepsilon^2 x^6$ and 2 are the dominant terms showing that the original balance is inconsistent.

vi. Suppose $x \sim 2$ as $\varepsilon \to 0$ is the dominant balance. This balance is consistent since $x = O(1)$ and we recover the original root with this equation.

The above analysis suggests that, from the only consistent dominant balance that provides additional information, the magnitudes of the missing roots are $O(\varepsilon^{-1/7})$ as $\varepsilon \to 0$. Making the scale transformation $x = \varepsilon^{-1/7} y$ and substituting into the original equation yields

$$y^8 - \varepsilon^{9/7} y^6 + y - 2\varepsilon^{1/7} = 0 \tag{1}$$

The unperturbed problem now does not vanish and the problem is now a regular perturbation problem. Thus, a perturbation expansion exists for the roots y in terms of powers of $\varepsilon^{1//7}$, i.e. the roots to (1) are of the form

$$y = \sum_{n=0}^{\infty} y_n \left(\varepsilon^{1/7}\right)^n .$$

□

Problem 4. Solve perturbatively

$$\begin{cases} y'' = (\sin x)y \\ y(0) = 1 \\ y'(0) = 1 \end{cases}.$$

Is the resulting perturbation series uniformly valid for $0 \leq x \leq \infty$? Why?

Solution. We begin by introducing the small perturbation factor ε into the problem as follows:

$$y'' = \varepsilon(\sin x)y, \quad y(0) = 1,\ y'(0) = 1.$$

Next, we assume a solution $y(x)$ exists in the form

$$y(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n(x) \tag{2}$$

where $y_0(0) = 1$, $y_0'(0) = 1$, and $y_n(0) = y_n'(0) = 0$ for $n > 0$. Using the perturbation expansion (2) and substituting into the perturbed problem yields after comparing the coefficients of ε that

$$\begin{cases} y_0''(x) = 0 & y_0(0) = 1,\ y_0'(0) = 1 \\ y_n''(x) = (\sin x)y_{n-1}(x) & y_n(0) = 0,\ y_n'(0) = 0 \quad \text{for } n > 0. \end{cases}$$

Thus, the solution to the zeroth-order problem is simply $y_0(x) = c_1 x + c_2$ where the values of c_1 and c_2 are determined by the initial conditions $y_0(0) = 1,\ y_0'(0) = 1$. Therefore, the solution to the zeroth order problem is $y_0(x) = x + 1$.

The solution to the n-th order problem can be found simply by integration:

$$y_n(x) = \int_0^x dx_2 \int_0^{x_2} (\sin x_1) y_{n-1}(x_1) dx_1.$$

Thus, from this equation we can see that for $n > 0$

$$y_n(x) = \int_0^x dx_{2n} \int_0^{x_{x_{2n}}} dx_{2n-1}(\sin x_{2n-1}) \int_0^{x_{2n-1}} d_{x_{2n-2}} \int_0^{x_{x_{2n-2}}} d_{x_{2n-3}}(\sin x_{2n-3}) \cdots$$
$$\cdots \int_0^{x_3} dx_2 \int_0^{x_2} (\sin x_1)(1 + x_1) dx_1$$

and therefore that our perturbation series is

$$y(x) = 1 + x + \left(\int_0^x dx_2 \int_0^{x_2} (\sin x_1)(1 + x_1) dx_1 \right) \varepsilon + y_2(x)\varepsilon^2 + \dots .$$

Since $|\sin x| \leq 1$, using the above formula, we see that the N-th term of the perturbation series is bounded above, i.e.

$$\left|\varepsilon^N y_N(x)\right| \leq \frac{\varepsilon^N x^{2N}(1 + |x|)}{(2N)!}.$$

Thus, the series is convergent for all finite x. However, this series is not uniformly valid over the interval $0 \leq x \leq \infty$. To see why, note that the leading behavior of the perturbation series is that of $1 + x = O(x)$. However, for $\varepsilon > 0$, the perturbation series is a linear combination of sin and cos functions and therefore a linear combination of exponentially increasing and decreasing functions. For large x these exponentially increasing and decreasing functions are not negligible and contribute to the perturbation series. This is a change in character of the perturbation series from the unperturbed problem and thus we may conclude that the perturbation series is not uniformly valid over the interval $0 \leq x \leq \infty$. $\square$

Problem 5. Find leading-order uniform asymptotic approximations to the solution of the following equation in the limit $\varepsilon \to 0^+$:

$$\varepsilon y'' + (x^2+1)y' - x^3 y = 0$$
$$y(0) = 1,\ y(1) = 1.$$

Solution. In obtaining a leading-order uniform asymptotic approximation to the solution of the above differential equation, we make the assumption that $y(x)$ develops an isolated boundary layer in the neighborhood of $x = 0$.

Outside of this boundary layer there are no rapid variations of $y(x)$ and we may assume that $\varepsilon y''(x)$ is negligible as $\varepsilon \to 0^+$. Thus, we obtain the approximation

$$(x^2+1)y'_{\text{out}}(x) = x^3 y_{\text{out}}(x),$$

the solution of which is

$$y_{\text{out}}(x) = c_1 \exp\left[\int_x^1 \frac{t^3}{t^2+1}dt\right] = \frac{c_1\sqrt{x^2+1}e^{\frac{1-x^2}{2}}}{\sqrt{2}}.$$

Since $x = 1$ is outside of the boundary layer we have that $y_{\text{out}}(x)$ must satisfy the boundary condition $y_{\text{out}}(1) = 1$. This implies that $c_1 = 1$. This solution is a uniform aproximation to the solution $y(x)$ as $\varepsilon \to 0^+$ on the interval $\delta \ll x \leq 1$ for some $\delta > 0$.

In order to determine the behavior of the solution near the boundary layer, we make the following approximations $(x^2+1)y' \sim y'$ and $-x^3 y \sim 0$ so that the approximation of the inner solution becomes

$$\varepsilon y''_{\text{in}}(x) + y'_{\text{in}}(x) = 0.$$

The solution to this second-order differential equation with constant cofficients is simply

$$y_{\text{in}}(x) = c_1 + c_2 e^{-x/\varepsilon}.$$

Since we must have that the boundary condition is met, we see that $y_{\text{in}}(0) = 1$. This implies that $c_1 + c_2 = 1$ or that

$$y_{\text{in}}(x) = 1 + c_2\left(e^{-x/\varepsilon} - 1\right).$$

The boundary layer occurs when $x = O(\varepsilon)$. In order to asymptotically match the inner and outer solutions, let's assume that $x = O(\varepsilon^{1/2})$. Thus, as $\varepsilon \to 0^+$, we see that

$$y_{\text{in}}(x) \sim 1 - c_2$$
$$y_{\text{out}}(x) \sim y_{\text{out}}(0) = \frac{e^{1/2}}{\sqrt{2}}.$$

In order for the approximation to be valid, we require that $c_2 = 1 - \frac{e^{1/2}}{\sqrt{2}}$. Thus, as $\varepsilon \to 0^+$, the boundary layer approximation is given by

$$y(x) \sim \frac{\sqrt{x^2+1}e^{\frac{1-x^2}{2}}}{\sqrt{2}} \quad \text{for } 0 < x \leq 1$$
$$y(x) \sim e^{-x/\varepsilon} + \frac{e^{1/2}}{\sqrt{2}}\left(1 - e^{-x/\varepsilon}\right)$$

A uniform approximation valid for all $0 \le x \le 1$ is then given by

$$\begin{aligned}
y(x) &= y_{\text{out}}(x) + y_{\text{in}}(x) - y_{\text{match}}(x) \\
&= \frac{\sqrt{x^2+1}e^{\frac{1-x^2}{2}}}{\sqrt{2}} + e^{-x/\varepsilon} + \frac{e^{1/2}}{\sqrt{2}}\left(1 - e^{-x/\varepsilon}\right) - \frac{e^{1/2}}{\sqrt{2}} \\
&= \frac{\sqrt{x^2+1}e^{\frac{1-x^2}{2}}}{\sqrt{2}} + \left(1 - \frac{e^{1/2}}{\sqrt{2}}\right) e^{-x/\varepsilon},
\end{aligned}$$

where $y_{\text{match}}(x) = 1 - (1 - y_{\text{out}}(0))$. □

Problem 6. Obtain a uniform approximation accurate to order ε^2 as $\varepsilon \to 0^+$ for the problem

$$\begin{aligned} &\varepsilon y'' + (1+x)^2 y' + y = 0 \\ &y(0) = 1,\ y(1) = 1. \end{aligned}$$

Solution. In order to ensure that the approximation is accurate to order ε^2 as $\varepsilon \to 0^+$, we seek an outer solution in the form of the following perturbation series

$$y_{\text{out}} \sim y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x).$$

Since $(1+x)^2 > 0$ for $0 \le x \le 1$, we expect a boundary layer only at $x = 0$. The perturbation series above must satisfy the boundary condition $y_{\text{out}}(1) = 1$. Thus,

$$y_0(1) = 1, \quad y_1(1) = 0, \quad y_2(1) = 0.$$

Replacing the perturbation series into the original equation yields the following three equations

$$\begin{aligned} (1+x)^2 y_0' + y_0 &= 0 \\ (1+x)^2 y_1' + y_1 &= -y_0'' \\ (1+x)^2 y_2' + y_2 &= -y_1''. \end{aligned}$$

From these equations and the boundary conditions, we see that

$$\begin{aligned} y_0(x) &= e^{1/(1+x)-1/2} \\ y_1(x) &= -\frac{e^{1/(1+x)-1/2}\left(-53 - 25x + 30x^2 + 30x^3 + 15x^4 + 3x^5\right)}{80(1+x)^5} \\ y_2(x) &= -\frac{e^{1/(1+x)-1/2}\left(-5005759 - 11370550x - 7410195x^2 + 1474680x^3 + 5856690x^4\right)}{1612800(1+x)^{10}} + \\ &\quad - \frac{e^{1/(1+x)-1/2}\left(6604668x^5 + 5372850x^6 + 3052920x^7 + 1144845x^8 + 254410x^9 + 25441x^{10}\right)}{1612800(1+x)^{10}} \end{aligned}$$

Thus, as $\varepsilon \to 0^+$,

$$y_{\text{out}}(x) \sim e^{1/(1+x)-1/2} + \varepsilon \left[\frac{e^{1/(1+x)-1/2}\left(-53 - 25x + 30x^2 + 30x^3 + 15x^4 + 3x^5\right)}{80(1+x)^5}\right] + \varepsilon^2 y_2(x).$$

In order to determine the inner solution, we expect the boundary layer to be of thickness ε and introduce the variables $X = \frac{x}{\varepsilon}$ and $Y_{\text{in}}(X) \equiv y_{\text{in}}(x)$. The original equation is then

$$\frac{\varepsilon}{\varepsilon^2}\frac{d^2 Y_{\text{in}}}{dX^2} + \frac{(1+X\varepsilon)^2}{\varepsilon}\frac{dY_{\text{in}}}{dX} + Y_{\text{in}} = 0 \implies \frac{d^2 Y_{\text{in}}}{dX^2} + (1+X\varepsilon)^2 \frac{dY_{\text{in}}}{dX} + \varepsilon Y_{\text{in}} = 0.$$

If we express $Y_{\text{in}}(x)$ as a perturbation series as $\varepsilon \to 0^+$, then

$$Y_{\text{in}}(x) \sim Y_0(x) + \varepsilon Y_1(x) + \varepsilon^2 Y_2(X)$$

and the boundary condition $y(0) = 1$ is then

$$Y_0(0) = 1, \quad Y_1(0) = 0, \quad Y_2(0) = 0.$$

Substituting the above perturbation series into the derived equation yields the following three equations:

$$\begin{aligned}
&\frac{d^2Y_0}{dX^2} + \frac{dY_0}{dX} = 0 \\
&\frac{d^2Y_1}{dX^2} + \frac{dY_1}{dX} = -2X\frac{dY_0}{dX} - Y_0 \\
&\frac{d^2Y_2}{dX^2} + \frac{dY_2}{dX} = -X^2\frac{dY_0}{dX} - 2X\frac{dY_1}{dX} - Y_1.
\end{aligned}$$

From these equations and the boundary conditions we see that

$$\begin{aligned}
Y_0(X) &= 1 + c_0(e^{-X} - 1) \\
Y_1(X) &= -X + c_0e^{-X}\left(-1 + e^X - X + Xe^X - X^2\right) + c_1e^{-X}(e^X - 1) \\
Y_2(X) &= -\frac{1}{6}c_0e^{-X}(-12 + 12e^X - 12X - 12e^XX - 9X^2 + 9e^XX^2 - 4X^3 - 3X^4) + \\
&\quad - e^{-X}(3e^XX - \frac{3}{2}e^XX^2) - c_1e^{-X}(-1 + e^X - X + e^XX - X^2) + c_2e^{-X}(e^X - 1).
\end{aligned}$$

We must now determine the constants c_0, c_1, c_2 through asymptotically matching the inner and outer solutions. □

Problem 7. For what real values of the constant α does the singular perturbation problem

$$\begin{aligned} &\varepsilon y''(x) + y'(x) - x^{\alpha} y(x) = 0 \\ &y(0) = 1,\ y(1) = 1. \end{aligned}$$

have a solution with a boundary layer near $x = 0$ as $\varepsilon \to 0^{+}$?

Solution. □