Homework Assignment 0

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September 1, 2016

Problem 1. Suppose that the number of claims each policy holder will have in a given year is Poisson distributed with mean randomly distributed with density function $g(\lambda) = e^{-\lambda}$ for $\lambda \geq 0$. What is the probability that a policy holder will have n claims in that year?

Solution. Let X be the discrete random variable representing the number of claims a policy holder will have in a given year. Let Y be the continuous random variable representing the Poisson parameter λ with density function $g(\lambda) = e^{-\lambda}$ with $\lambda \geq 0$.

Conditioning the probability that $X = n$ on the random variable Y shows that

$$\begin{aligned} P(X = n) &= \int_{-\infty}^{\infty} P(X = n | Y = \lambda) g(\lambda) d\lambda \\ &= \int_{0}^{\infty} P(X = n | Y = \lambda) e^{-\lambda} d\lambda \end{aligned}$$

where the limits of integration have changed since $P(X = n|Y = \lambda) = 0$ for $\lambda < 0$ given our initial assumptions. Using the probability mass function for the Poisson random variable X in conjunction with the fact that for $Y = \lambda$ the random variable X will have parameter λ, we see that

$$P(X = n|Y = \lambda) = \frac{e^{-\lambda}\lambda^n}{n!}.$$

Thus, from the above computation, we have that

$$\begin{aligned} P(X = n) &= \int_0^{\infty} e^{-\lambda} \left[\frac{e^{-\lambda}\lambda^n}{n!} \right] d\lambda \\ &= \frac{1}{n!} \int_0^{\infty} \lambda^n e^{-2\lambda} d\lambda. \end{aligned}$$

Making the u-substitution $u = 2\lambda$, the previous integral transforms into

$$\begin{aligned} P(X = n) &= \frac{1}{n!} \int_0^{\infty} 2^{-(n+1)} u^n e^{-u} du \\ &= \frac{2^{-(n+1)}}{n!} \int_0^{\infty} u^n e^{-u} du \\ &= \frac{2^{-(n+1)} \Gamma(n+1)}{n!} \\ &= 2^{-(n+1)}, \end{aligned}$$

where we have used the fact that, for an integer k,

$$\Gamma(k) = \int_0^\infty x^{k-1}e^{-x}dx = (k-1)!.$$

Therefore, the probability a policy holder will have n claims in that year is $2^{-(n+1)}$. $\square$