Exam 2

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Problem 1. Find the inverse Laplace transform of the function

$$\bar{f}(s) = \frac{s}{(s-a)(s^2+b^2)}$$

for $a, b > 0$, by using the following three different approaches:

i. Using partial fraction decomposition,

ii. Applying the Convolution Theorem,

iii. Applying Heaviside's Expansion Theorem.

Solution. We will now find the inverse Laplace transform of $\bar{f}(s)$ using the respective approaches listed above:

i. From the partial fractions method, we see that

$$\bar{f}(s) = \frac{s}{(s-a)(s^2+b^2)} = \frac{c_0}{s-a} + \frac{d_1 s + d_0}{s^2+b^2}.$$

Combining the rational fractions on the right side under a common denominator and equating the coefficients in the numerator we arrive at the following system of equations

$$\begin{aligned} c_0 + d_1 &= 0 \\ d_0 - ad_1 &= 0 \\ c_0 b^2 - ad_0 &= 0. \end{aligned}$$

Solving this system, we see that $c_0 = \dfrac{a}{a^2+b^2}$, $d_1 = -\dfrac{a}{a^2+b^2}$, and $d_0 = \dfrac{b^2}{a^2+b^2}$. Thus, we have that

$$\bar{f}(s) = \frac{1}{a^2+b^2}\left[\frac{a}{s-a} - \frac{as}{s^2+b^2} + \frac{b^2}{s^2+b^2}\right].$$

From our table of Laplace transforms, we know that

$$\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$
$$\mathscr{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\} = \cos bt$$
$$\mathscr{L}^{-1}\left\{\frac{b}{s^2+b^2}\right\} = \sin bt.$$

Therefore, the inverse Laplace transform of $\bar{f}(s)$ is

$$\begin{aligned} f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} &= \frac{1}{a^2+b^2}\left[a\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} - a\mathscr{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\} + b\mathscr{L}^{-1}\left\{\frac{b}{s^2+b^2}\right\}\right] \\ &= \frac{1}{a^2+b^2}\left[ae^{at} - a\cos bt + b\sin bt\right]. \end{aligned}$$

ii. The Convolution Theorem states that if $\bar{f}(s) = \bar{g}(s)\bar{h}(s)$, then

$$f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} = \mathscr{L}^{-1}\left\{\bar{g}(s)\bar{h}(s)\right\} = (g * h)(t)$$

where

$$(g * h)(t) = \int_0^t g(t-\tau)h(\tau)d\tau.$$

Now, suppose that $\bar{f}(s) = \bar{g}(s)\bar{h}(s)$, where $\bar{g}(s) = \dfrac{1}{s-a}$ and $\bar{h}(s) = \dfrac{s}{s^2+b^2}$.

From our table of Laplace transforms we know that $g(t) = \mathscr{L}^{-1}\left\{\dfrac{1}{s-a}\right\} = e^{at}$ and $h(t) = \mathscr{L}^{-1}\left\{\dfrac{s}{s^2+b^2}\right\} = \cos bt$.

Thus, by the Convolution Theorem, we have that

$$f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} = \mathscr{L}^{-1}\left\{\bar{g}(s)\bar{h}(s)\right\} = \int_0^t g(t-\tau)h(\tau)d\tau.$$

Therefore, using a computer algebra system, we see that

$$\begin{aligned} f(t) &= \int_0^t g(t-\tau)h(\tau)d\tau \\ &= \int_0^t e^{a(t-\tau)}\cos b\tau d\tau \\ &= e^{at}\int_0^t e^{-a\tau)}\cos b\tau d\tau \\ &= \frac{1}{a^2+b^2}\left[ae^{at} - a\cos bt + b\sin bt\right]. \end{aligned}$$

iii. Heaviside's Expansion Theorem states that if $\bar{f}(s) = \dfrac{\bar{p}(s)}{\bar{q}(s)}$, where $\bar{p}(s)$ and $\bar{q}(s)$ are polynomials in s and the degree of $\bar{q}$ is higher than that of $\bar{p}$, then

$$f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} = \sum_{k=1}^{n} \frac{\bar{p}(\alpha_k)}{\bar{q}'(\alpha_k)} e^{t\alpha_k}$$

where α_k are the distinct root of $\bar{q}(s) = 0$.

For $\bar{f}(s) = \dfrac{s}{(s-a)(s^2+b^2)}$, we identify $\bar{p}(s) = s$ and $\bar{q}(s) = (s-a)(s^2+b^2)$. Since $\bar{p}$ and $\bar{q}$ are polynomials in s with the degree of $\bar{q}$ greater than that of the degree of $\bar{p}$, the assumptions of Heaviside's Expansion Theorem are satisfied.

Note that $\bar{q}'(s) = s(3s-2a) + b^2$ and $\alpha_1 = a$, $\alpha_2 = bi$, and $\alpha_3 = -bi$ are the roots of $\bar{q}(s)$.

Therefore, by the Heaviside's Expansion Theorem, we have that

$$\begin{aligned}
f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} &= \sum_{k=1}^{n} \frac{\bar{p}(\alpha_k)}{\bar{q}'(\alpha_k)} e^{t\alpha_k} \\
&= \frac{a}{a^2+b^2} e^{at} - \frac{bi}{2bi(a-ib)} e^{bit} - \frac{bi}{2bi(a+ib)} e^{-bit} \\
&= \frac{1}{a^2+b^2}\left[ae^{at} - \frac{a+ib}{2} e^{bit} - \frac{a-ib}{2} e^{-bit}\right] \\
&= \frac{1}{a^2+b^2}\left[ae^{at} - a\cos bt + b\sin bt\right].
\end{aligned}$$

□

Problem 2. a. Evaluate the improper definite integral

$$\int_{-\infty}^{\infty} \frac{\cos tx}{x^2 + a^2} dx$$

where $a, t > 0$.

b. Show that

$$\int_{0}^{\infty} \frac{\sin \pi tx}{x(1 + x^2)} dx = \frac{\pi}{2}(1 - e^{-\pi t})$$

where $t > 0$.

Solution. a. Suppose that

$$f(t) = \int_{-\infty}^{\infty} \frac{\cos tx}{x^2 + a^2} dx.$$

In order to evaluate this integral, we take the Laplace transform of $f(t)$ with respect to t. Now, due to uniform convergence, we have that

$$\begin{aligned}
\bar{f}(s) = \mathscr{L}\{f(t)\} = \mathscr{L}\left\{\int_{-\infty}^{\infty} \frac{\cos tx}{x^2 + a^2} dx\right\} &= \int_{-\infty}^{\infty} \mathscr{L}\left\{\frac{\cos tx}{x^2 + a^2}\right\} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{x^2 + a^2} \mathscr{L}\{\cos tx\}\, dx \\
&= \int_{-\infty}^{\infty} \frac{s}{(x^2 + a^2)(x^2 + s^2)} dx.
\end{aligned}$$

Using the method of partial fraction decomposition, we see that this last integral becomes

$$\begin{aligned}
\bar{f}(s) &= \int_{-\infty}^{\infty} \frac{s dx}{(x^2 + a^2)(x^2 + s^2)} dx \\
&= \frac{s}{s^2 - a^2} \int_{-\infty}^{\infty} \frac{1}{x^2 + a^2} - \frac{1}{x^2 + s^2} dx.
\end{aligned}$$

Thus, we see that

$$\begin{aligned}
\bar{f}(s) &= \frac{s}{s^2 - a^2} \int_{-\infty}^{\infty} \frac{1}{x^2 + a^2} - \frac{1}{x^2 + s^2} dx \\
&= \frac{s}{s^2 - a^2} \left[\tan^{-1} \frac{x}{a}\Big|_{-\infty}^{\infty} - \tan^{-1} \frac{x}{s}\Big|_{-\infty}^{\infty} \right] \\
&= \frac{s}{s^2 - a^2} \left[\frac{\pi}{a} - \frac{\pi}{s} \right] \\
&= \frac{\pi}{a} \left[\frac{s}{s^2 - a^2} - \frac{a}{s^2 - a^2} \right].
\end{aligned}$$

Using the table of Laplace transforms, we know that $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2-a^2}\right\} = \cosh at$ and $\mathscr{L}^{-1}\left\{\dfrac{a}{s^2-a^2}\right\} = \sinh at$. Therefore, we have that

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\cos tx}{x^2+a^2}dx = f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} &= \mathscr{L}^{-1}\left\{\frac{\pi}{a}\left[\frac{s}{s^2-a^2} - \frac{a}{s^2-a^2}\right]\right\} \\
&= \frac{\pi}{a}\left[\mathscr{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} - \mathscr{L}^{-1}\left\{\frac{a}{s^2-a^2}\right\}\right] \\
&= \frac{\pi}{a}\left[\cosh at - \sinh at\right] \\
&= \frac{\pi}{a}e^{-at}.
\end{aligned}$$

b. Suppose that

$$f(t) = \int_0^{\infty} \frac{\sin \pi tx}{x(1+x^2)}dx.$$

In order to evaluate this integral, we take the Laplace transform of $f(t)$ with respect to t. Now, due to uniform convergence, we have that

$$\begin{aligned}
\bar{f}(s) = \mathscr{L}\left\{f(t)\right\} = \mathscr{L}\left\{\int_0^{\infty} \frac{\sin \pi tx}{x(1+x^2)}dx\right\} &= \int_0^{\infty} \mathscr{L}\left\{\frac{\sin \pi tx}{x(1+x^2)}\right\}dx \\
&= \int_0^{\infty} \frac{1}{x(1+x^2)}\mathscr{L}\left\{\sin \pi tx\right\}dx \\
&= \int_0^{\infty} \frac{\pi}{(x^2+1)(\pi^2x^2+s^2)}dx.
\end{aligned}$$

Using a computer algebra system, we see that this last integral reduces to

$$\begin{aligned}
\bar{f}(s) &= \int_0^{\infty} \frac{\pi}{(x^2+1)(\pi^2x^2+s^2)}dx \\
&= \frac{\pi^2}{2s(\pi+s)} \\
&= \frac{\pi}{2}\left[\frac{1}{s} - \frac{1}{s+\pi}\right].
\end{aligned}$$

Therefore, from our table of Laplace transforms, we have that

$$\begin{aligned}
\int_0^{\infty} \frac{\sin \pi tx}{x(1+x^2)}dx = f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} &= \frac{\pi}{2}\left[\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{s+\pi}\right\}\right] \\
&= \frac{\pi}{2}\left(1-e^{-\pi t}\right).
\end{aligned}$$

$\square$

Problem 3. Apply the Laplace transform to solve the following Initial Value Problems:

a. $y'' + 2ay' + (a^2+4)y = f(t)$
   $y(0) = 1, \quad y'(0) = -a.$

b. $u_{tt} = c^2 u_{xx} + \sin x, \quad 0 < x < \pi, \quad t > 0$
   $u(0,t) = u(\pi,t) = 1, \quad u(x,0) = u_t(x,0) = 0.$

Solution. Recall that if $\bar{y}(s) = \mathscr{L}\{y(t)\}$, then the Laplace transform of the n-th derivative of $y(t)$ is given by

$$\mathscr{L}\left\{y^{(n)}(t)\right\} = s^n \bar{y}(s) - \sum_{k=0}^{n-1} s^{n-1-k} y^{(k)}(0). \tag{1}$$

a. Suppose that $Ly \equiv y''(t) + 2ay'(t) + (a^2+4)y(t)$. Using (1), application of the Laplace transform to $Ly = f(t)$ yields that

$$\mathscr{L}\{Ly\} = (s^2 + 2as + a^2 + 4)\bar{y}(s) - 2ay(0) - sy(0) - y'(0) = \bar{f}(s) = \mathscr{L}\{f(t)\}.$$

From the initial data, we see that this reduces to

$$(s^2 + 2as + a^2 + 4)\bar{y}(s) - (s + a) = \bar{f}(s).$$

Solving for $\bar{y}(s)$ yields

$$\bar{y}(s) = \frac{\bar{f}(s) + s + a}{s^2 + 2as + a^2 + 4} = \frac{\bar{f}(s) + s + a}{(s + a + 2i)(s + a - 2i)}.$$

Note that from our table of Laplace transforms that

$$\mathscr{L}^{-1}\left\{\frac{a-b}{(s-a)(s-b)}\right\} = e^{at} - e^{bt}$$

and

$$\mathscr{L}^{-1}\left\{\frac{s}{(s-a)(s-b)}\right\} = \frac{ae^{at} - be^{bt}}{a-b}.$$

Therefore, the solution to the original differential equation is given by

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} \\ &= \mathscr{L}^{-1}\left\{\frac{\bar{f}(s) + s + a}{(s + a + 2i)(s + a - 2i)}\right\} \\ &= \mathscr{L}^{-1}\left\{\frac{\bar{f}(s)}{(s + a + 2i)(s + a - 2i)}\right\} + \frac{e^{-(2i+a)t}}{4}\left[(2 - i + ia)e^{4it} + 2 + i - ia\right]. \end{aligned}$$

b. Let $u(x,t)$ be a function in x and t. The Laplace transform of $u(x,t)$ with respect to t is given by

$$\mathscr{L}\left\{u(x,t)\right\} = \bar{u}(x,s) = \int_0^\infty u(x,t)e^{-st}dt.$$

From this definition, we see from previous theorems that

$$\mathscr{L}\left\{\frac{\partial^n}{\partial t^n}\left[u(x,t)\right]\right\} = s^n\bar{u}(x,s) - \sum_{k=0}^{n-1} s^{n-1-k}\frac{\partial^k}{\partial t^k}\left[u(x,0)\right]$$

Similarly, we see from the Leibniz integral rule that

$$\mathscr{L}\left\{\frac{\partial^n}{\partial x^n}\left[u(x,t)\right]\right\} = \frac{d^n}{dx^n}\left[\bar{u}(x,s)\right].$$

Applying the Laplace transform with respect to t to the differential equation yields that

$$\mathscr{L}\left\{u_{tt} - c^2u_{xx}\right\} = s^2\bar{u}(x,s) - su(x,0) - u_t(x,0) - c^2\frac{d^2\bar{u}(x,s)}{dx^2} = \frac{\sin x}{s} = \mathscr{L}\left\{\sin x\right\}.$$

In light of the initial data, this equation reduces to

$$s^2\bar{u}(x,s) - c^2\frac{d^2\bar{u}(x,s)}{dx^2} = \frac{\sin x}{s},$$

or, equivalently,

$$\frac{d^2\bar{u}(x,s)}{dx^2} - \left(\frac{s}{c}\right)^2\bar{u}(x,s) = -\frac{\sin x}{sc^2}.$$

The homogeneous solution to the above differential equation is easily seen to be

$$\bar{u}_h(x,s) = c_1\exp\left(-\frac{xs}{c}\right) + c_2\exp\left(\frac{xs}{c}\right)$$

From the method of undetermined coefficients, assuming the particular solution of the equation is of the form $\bar{u}_p(x,s) = A\sin x$ for some unknown A, the particular solution of the transformed equation is given by

$$\bar{u}_p(x,s) = \frac{\sin x}{s(s^2+c^2)}.$$

Therefore, the general solution to the transformed equation is given by

$$\bar{u}(x,s) = \bar{u}_h(x,s) + \bar{u}_p(x,s) = c_1\exp\left(-\frac{xs}{c}\right) + c_2\exp\left(\frac{xs}{c}\right) + \frac{\sin x}{s(s^2+c^2)}.$$

Note that the transformed boundary data is given by $\bar{u}(0,s) = \bar{u}(\pi,s) = \dfrac{1}{s}$. Using the form of the solution to the transformed equation listed above, we see that in light of the transformed boundary data that

$$\begin{aligned} c_1 + c_2 &= \frac{1}{s} \\ c_1 \exp\left(-\frac{\pi s}{c}\right) + c_2 \exp\left(\frac{\pi s}{c}\right) &= \frac{1}{s} \end{aligned}$$

After solving the above system, we therefore see that the solution to the transformed equation is given by

$$\begin{aligned} \bar{u}(x,s) &= c_1 \exp\left(-\frac{xs}{c}\right) + c_2 \exp\left(\frac{xs}{c}\right) + \frac{\sin x}{s(s^2+c^2)} \\ &= \frac{\exp\left(\frac{\pi s}{c}\right)\exp\left(-\frac{xs}{c}\right)}{s\left(1+\exp\left(\frac{\pi s}{c}\right)\right)} + \frac{\exp\left(\frac{xs}{c}\right)}{s\left(1+\exp\left(\frac{\pi s}{c}\right)\right)} + \frac{\sin x}{s(s^2+c^2)} \end{aligned}$$

Therefore, the solution to the original differential equation is given by

$$u(x,t) = \mathscr{L}^{-1}\left\{\bar{u}(x,s)\right\} = \mathscr{L}^{-1}\left\{\frac{\exp\left(\frac{\pi s}{c}\right)\exp\left(-\frac{xs}{c}\right)}{s\left(1+\exp\left(\frac{\pi s}{c}\right)\right)} + \frac{\exp\left(\frac{xs}{c}\right)}{s\left(1+\exp\left(\frac{\pi s}{c}\right)\right)} + \frac{\sin x}{s(s^2+c^2)}\right\}.$$

□

Problem 4. Apply the Laplace transform to solve the following wave equation

$$\begin{aligned}
&\frac{\partial^2 u(x,t)}{\partial t^2} - c^2 \frac{\partial^2 u(x,t)}{\partial x^2} = f(t), \\
&u(0,t) = 0, \qquad t > 0, \\
&u(x,0) = 0, \quad \frac{\partial}{\partial t}\left[u(x,0)\right] = 0, \qquad x > 0.
\end{aligned}$$

Solution. Suppose that $Lu \equiv \dfrac{\partial^2 u(x,t)}{\partial t^2} - c^2 \dfrac{\partial^2 u(x,t)}{\partial x^2}$. Then applying the Laplace transform to the equation $Lu = f(t)$ yields

$$\mathscr{L}\left\{Lu\right\} = s^2 \bar{u}(x,s) - su(x,0) - \frac{\partial}{\partial t}[u(x,0)] - c^2 \frac{d^2 \bar{u}(x,s)}{dx^2} = \bar{f}(s) = \mathscr{L}\left\{f(t)\right\}.$$

In light of the initial data, this equation reduces to

$$s^2 \bar{u}(x,s) - c^2 \frac{d^2 \bar{u}(x,s)}{dx^2} = \bar{f}(s),$$

or, equivalently,

$$\frac{d^2 \bar{u}(x,s)}{dx^2} - \left(\frac{s}{c}\right)^2 \bar{u}(x,s) = -\frac{\bar{f}(s)}{c^2}.$$

The homogeneous solution to the above differential equation is easily seen to be

$$\bar{u}_h(x,s) = c_1 \exp\left(-\frac{xs}{c}\right) + c_2 \exp\left(\frac{xs}{c}\right).$$

By inspection, we see that

$$\bar{u}_p(x,s) = \frac{\bar{f}(s)}{s^2}$$

is a particular solution of the transformed equation. Thus, the general solution to the transformed equation is

$$\bar{u}(x,s) = \bar{u}_h(x,s) + \bar{u}_p(x,s) = c_1 \exp\left(-\frac{xs}{c}\right) + c_2 \exp\left(\frac{xs}{c}\right) + \frac{\bar{f}(s)}{s^2}.$$

Note that we must have that $\bar{u}(x,s) \to 0$ as $s \to \infty$. For this reason, we must have that $c_2 = 0$. The transformed boundary data states that $\bar{u}(0,t) = 0$. Using the above solution, this implies that $c_1 = -\bar{f}(s)/s^2$. Thus, the solution to the transformed differential equation is

$$\bar{u}(x,s) = \bar{u}_h(x,s) + \bar{u}_p(x,s) = \frac{\bar{f}(s)}{s^2}\left[1 - \exp\left(-\frac{xs}{c}\right)\right]$$

We arrive at the solution to the original differential equation by taking the inverse Laplace transform of the above equation. Note from our table of Laplace transforms that $\mathscr{L}^{-1}\left\{1/s^2\right\} = t$ and from Heaviside's Second Shifting Theorem that

$$\mathscr{L}^{-1}\left\{\frac{\exp\left(-\frac{xs}{c}\right)}{s^2}\right\} = \left(t - \frac{x}{c}\right) H\left(t - \frac{x}{c}\right).$$

Now let $g(t) = t$ and $h(t) = \left(t - \frac{x}{c}\right) H\left(t - \frac{x}{c}\right)$. Then from our previous remarks, we have that

$$\begin{aligned}\bar{u}(x,s) &= \frac{\bar{f}(s)}{s^2}\left[1 - \exp\left(-\frac{xs}{c}\right)\right]\\ &= \frac{\bar{f}(s)}{s^2} - \frac{\bar{f}(s)\exp\left(-\frac{xs}{c}\right)}{s^2}\\ &= \bar{f}(s)\bar{g}(s) - \bar{f}(s)\bar{h}(s).\end{aligned}$$

Therefore, by the Convolution Theorem and the above results, the solution to the original differential equation is

$$\begin{aligned}u(x,t) = \mathscr{L}^{-1}\left\{\bar{u}(x,s)\right\} &= \mathscr{L}^{-1}\left\{\bar{f}(s)\bar{g}(s) - \bar{f}(s)\bar{h}(s)\right\}\\ &= (f * g)(t) - (f * h)(t)\\ &= \int_0^t \tau f(t-\tau)d\tau - \int_0^t \left(\tau - \frac{x}{c}\right) H\left(\tau - \frac{x}{c}\right) f(t-\tau)d\tau.\end{aligned}$$

$\square$

Problem 5.

Solution. □

Problem 6.

Solution. □

Problem 7.

Solution. □

Problem 8.

Solution. □