Homework Assignment 6

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Problem 4.3. Find the solutions of the following systems of equations with the initial data:

a. $\frac{dx}{dt} = x - 2y, \quad x(0) = 1$
$\frac{dy}{dt} = y - 2x, \quad y(0) = 0$

Solution. a. Applying the Laplace transform to the system yields

$$\mathscr{L}\left\{\frac{dx}{dt}\right\} = s\bar{x}(s) - x(0) = \bar{x}(s) - 2\bar{y}(s) = \mathscr{L}\left\{x - 2y\right\}$$
$$\mathscr{L}\left\{\frac{dy}{dt}\right\} = s\bar{y}(s) - y(0) = \bar{y}(s) - 2\bar{x}(s) = \mathscr{L}\left\{y - 2x\right\}.$$

Using the initial data, the transformed system becomes

$$(s-1)\bar{x}(s) + 2\bar{y}(s) = 1$$
$$2\bar{x}(s) + (s-1)\bar{y}(s) = 0,$$

or, equivalently,

$$\begin{bmatrix} s-1 & 2 \\ 2 & s-1 \end{bmatrix} \begin{bmatrix} \bar{x}(s) \\ \bar{y}(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This implies that the solution to the transformed system of equations is given by

$$\begin{bmatrix} \bar{x}(s) \\ \bar{y}(s) \end{bmatrix} = \begin{bmatrix} s-1 & 2 \\ 2 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{s-1}{(s-3)(s+1)} & -\frac{2}{(s-3)(s+1)} \\ -\frac{2}{(s-3)(s+1)} & \frac{s-1}{(s-3)(s+1)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{s-1}{(s-3)(s+1)} \\ -\frac{2}{(s-3)(s+1)} \end{bmatrix}$$

i.e. the solution is given by $\bar{x}(s) = \dfrac{s-1}{(s-3)(s+1)}$ and $\bar{y}(s) = -\dfrac{2}{(s-3)(s+1)}$.

From our table of Laplace Transforms, we know that

$$\mathscr{L}\left\{e^{at} - e^{bt}\right\} = \frac{a-b}{(s-a)(s-b)}$$

and

$$\mathscr{L}\left\{\frac{ae^{at} - be^{bt}}{a-b}\right\} = \frac{s}{(s-a)(s-b)}.$$

Therefore, the solution to the original system of differential equations is given by

$$\begin{aligned}
x(t) = \mathscr{L}^{-1}\left\{\bar{x}(s)\right\} &= \mathscr{L}^{-1}\left\{\frac{s-1}{(s-3)(s+1)}\right\} \\
&= \mathscr{L}^{-1}\left\{\frac{s}{(s-3)(s+1)}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{(s-3)(s+1)}\right\} \\
&= \frac{3e^{3t}+e^{-t}}{4} - \frac{e^{3t}-e^{-t}}{r} \\
&= \frac{e^{3t}+e^{-t}}{2}
\end{aligned}$$

and

$$\begin{aligned}
y(t) = \mathscr{L}^{-1}\left\{\bar{y}(s)\right\} &= \mathscr{L}^{-1}\left\{-\frac{2}{(s-3)(s+1)}\right\} \\
&= \frac{e^{-t}-e^{3t}}{2}
\end{aligned}$$

□

Problem 4.12.

Solution. $\square$

Problem 4.14.

Solution. □

Problem 4.22.

Solution. □

Problem 4.25.

Solution. $\square$