Homework Assignment 3

Matthew Tiger

March 19, 2016

Problem 1. a. Give an example of an asymptotic relation $f(x) \sim g(x)$ $(x \to \infty)$ that cannot be exponentiated; that is $e^{f(x)} \sim e^{g(x)}$ $(x \to \infty)$ is false.

b. Show that if $f(x) - g(x) \ll 1$ $(x \to \infty)$, then $e^{f(x)} \sim e^{g(x)}$ $(x \to \infty)$.

Solution. a. Note that for $x \to \infty$ we have that $e^{f(x)} \not\sim e^{g(x)}$ if and only if

$$\lim_{x\to\infty} \frac{e^{f(x)}}{e^{g(x)}} = \lim_{x\to\infty} e^{f(x)-g(x)} \neq 1.$$

Thus, if $\lim_{x\to\infty} f(x) - g(x) \neq 0$, then $\lim_{x\to\infty} e^{f(x)-g(x)} \neq 1$ and $e^{f(x)} \not\sim e^{g(x)}$. Therefore, take for instance the functions $f(x) = x + 1$ and $g(x) = x$. These functions are clearly asymptotic as

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x+1}{x} = \lim_{x\to\infty} 1 + \frac{1}{x} = 1.$$

However,

$$\lim_{x\to\infty} \frac{e^{f(x)}}{e^{g(x)}} = \lim_{x\to\infty} e^{f(x)-g(x)} = \lim_{x\to\infty} e^{(x+1)-x} = e \neq 1$$

so that the asymptotic relation between $f(x)$ and $g(x)$ cannot be exponentiated.

b. Suppose that $f(x) - g(x) \ll 1$ as $x \to \infty$. Then

$$\lim_{x\to\infty} \frac{f(x) - g(x)}{1} = \lim_{x\to\infty} f(x) - g(x) = 0.$$

If this is true then we must have that

$$\lim_{x\to\infty} \frac{e^{f(x)}}{e^{g(x)}} = \lim_{x\to\infty} e^{f(x)-g(x)} = e^{\lim_{x\to\infty} f(x)-g(x)} = e^0 = 1$$

or that $e^{f(x)} \sim e^{g(x)}$ and we are done.

$\square$

Problem 2. Find and classify all the singular points (including the point at ∞) of the equations:

$$x(1-x)y'' + [2-(a+b)x]y' - aby = 0, \qquad (x^2+1)y'' - xy = 0.$$

Here, $a, b \in \mathbb{R}$.

Solution. $\square$

Problem 3. Find the Taylor series solution of the IVP

$$(1 - x^3)y''' + 2xy' = 0, \quad y(0) = 3, y'(0) = 3, y''(0) = 0.$$

Solution. □

Problem 4. Find two linearly independent solutions to $x(1-x)y'' - 3xy' - y = 0$.

Solution. $\square$

Problem 5. Find two linearly independent solutions to $x^2y'' + 3xy' + (1 - 2x)y = 0$.

Solution. $\square$

Problem 6. Find the leading behavior of both solutions of $x^5y'' - y = 0$ near $x = 0$.

Solution. $\square$

Problem 7. Find the first four terms in the asymptotic series for the solutions of $y'' = e^{-2/x}y$ as $x \to +\infty$.

Hint: When you are performing the asymptotic analysis to extract the leading behavior of the solution as $x \to +\infty$, you may (and probably want) to replace $e^{-2/x}$ with a reasonable simpler approximation.

Solution. □