Homework Assignment 5

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Problem 3.23. Show that:

a. $\mathscr{L}\left\{t\cos(at)e^{-bt}\right\} = \dfrac{(s+b)^2 - a^2}{\left[(s+b)^2 + a^2\right]^2}$.

Solution. a. Let $f(t) = t\cos(at)$ and suppose that $\bar{f}(s) = \mathscr{L}\{f(t)\}$.

As shown previously, we know that

$$\bar{f}(s) = \mathscr{L}\{f(t)\} = \mathscr{L}\{t\cos(at)\} = \frac{s^2 - a^2}{(s^2 + a^2)^2}.$$

Therefore, by Heaviside's First Shifting Theorem,

$$\mathscr{L}\left\{t\cos(at)e^{-bt}\right\} = \mathscr{L}\left\{f(t)e^{-bt}\right\} = \bar{f}(s+b) = \frac{(s+b)^2 - a^2}{\left[(s+b)^2 + a^2\right]^2},$$

and we are done.

□

Problem 3.24. Suppose that $\mathscr{L}\left\{f(t)\right\} = \bar{f}(s)$ and $\mathscr{L}\left\{g(x,t)\right\} = \bar{h}(s)\exp(-x\bar{h}(s))$. Prove that:

a. $\mathscr{L}\left\{\int_0^\infty g(x,t)f(x)dx\right\} = \bar{h}(s)\bar{f}(\bar{h}(s))$.

Solution. a. From the definition of the Laplace transform, we have that

$$\mathscr{L}\left\{\int_0^\infty g(x,t)f(x)dx\right\} = \int_0^\infty \left[\int_0^\infty g(x,t)f(x)dx\right] e^{-st}dt.$$

Interchanging the order of integration yields that

$$\begin{aligned}\mathscr{L}\left\{\int_0^\infty g(x,t)f(x)dx\right\} &= \int_0^\infty \left[\int_0^\infty g(x,t)f(x)dx\right] e^{-st}dt \\ &= \int_0^\infty f(x)\left[\int_0^\infty g(x,t)e^{-st}dt\right] dx \\ &= \int_0^\infty f(x)\mathscr{L}\left\{g(x,t)\right\} dx.\end{aligned}$$

From the relation $\mathscr{L}\left\{g(x,t)\right\} = \bar{h}(s)\exp(-x\bar{h}(s))$, we thus see that

$$\begin{aligned}\mathscr{L}\left\{\int_0^\infty g(x,t)f(x)dx\right\} &= \int_0^\infty f(x)\mathscr{L}\left\{g(x,t)\right\} dx \\ &= \int_0^\infty f(x)\bar{h}(s)\exp(-x\bar{h}(s))dx.\end{aligned}$$

Using the definition of the Laplace transform, we see that

$$\bar{f}(\bar{h}(s)) = \int_0^\infty f(t)\exp(-\bar{h}(s)t)dt.$$

Therefore,

$$\begin{aligned}\mathscr{L}\left\{\int_0^\infty g(x,t)f(x)dx\right\} &= \int_0^\infty f(x)\bar{h}(s)\exp(-x\bar{h}(s))dx \\ &= \bar{h}(s)\int_0^\infty f(x)\exp(-x\bar{h}(s))dx \\ &= \bar{h}(s)\bar{f}(\bar{h}(s)).\end{aligned}$$

and we are done.

$\square$

Problem 3.27. Use the Initial Value Theorem to find $f(0)$ and $f'(0)$ from the following functions:

a. $\bar{f}(s) = \dfrac{s}{s^2 - 5s + 12}$,

c. $\bar{f}(s) = \dfrac{e^{-sa}}{s^2 + 3s + 5}$, $a > 0$.

Solution. The Initial Value Theorem states that if $f(t)$ and its derivatives exist as $t \to 0$, then

$$\text{i.} \quad \lim_{s\to\infty} s\bar{f}(s) = f(0) \tag{1a}$$

$$\text{ii.} \quad \lim_{s\to\infty} [s^2\bar{f}(s) - sf(0)] = f'(0). \tag{1b}$$

a. If $\bar{f}(s) = \dfrac{s}{s^2 - 5s + 12}$, then (1a) of the Initial Value Theorem shows that

$$f(0) = \lim_{s\to\infty} s\bar{f}(s) = \lim_{s\to\infty} \frac{s^2}{s^2 - 5s + 12} = 1.$$

This implies from (1b) of the Initial Value Theorem that

$$\begin{aligned} f'(0) = \lim_{s\to\infty} [s^2\bar{f}(s) - sf(0)] &= \lim_{s\to\infty} \frac{s^3}{s^2 - 5s + 12} - s \\ &= \lim_{s\to\infty} \frac{s^3 - (s^3 - 5s^2 + 12s)}{s^2 - 5s + 12} \\ &= \lim_{s\to\infty} \frac{5s^2 - 12s}{s^2 - 5s + 12} \\ &= 5. \end{aligned}$$

c. Suppose that $p(s)$ and $q(s)$ are both polynomials in s and that $a > 0$. Then from L'Hospital's rule we have that

$$\lim_{s\to\infty} \frac{p(s)e^{-sa}}{q(s)} = \lim_{s\to\infty} \frac{p(s)}{e^{sa}q(s)} = 0. \tag{2}$$

If $\bar{f}(s) = \dfrac{e^{-sa}}{s^2 + 3s + 5}$ where $a > 0$, then (1a) of the Initial Value Theorem in combination with (2) shows that

$$f(0) = \lim_{s\to\infty} s\bar{f}(s) = \lim_{s\to\infty} \frac{se^{-sa}}{s^2 + 3s + 5} = 0.$$

Using this result, we have from (1b) of the Initial Value Theorem in combination with (2) that

$$f'(0) = \lim_{s\to\infty} [s^2\bar{f}(s) - sf(0)] = \lim_{s\to\infty} \frac{s^2e^{-sa}}{s^2 + 3s + 5} = 0.$$

□

Problem 3.28. Use the Final Value Theorem to find $\lim_{t\to\infty} f(t)$ if it exists from the following functions:

a. $\bar{f}(s) = \dfrac{1}{s(s^2+as+b)}$,

d. $\bar{f}(s) = \dfrac{3}{(s^2+4)^2}$.

Solution. The Final Value Theorem states that if $\bar{f}(s) = \dfrac{\bar{p}(s)}{\bar{q}(s)}$ where $\bar{p}(s)$ and $\bar{q}(s)$ are polynomials in s and the degree of $\bar{p}(s)$ is less than that of $\bar{q}(s)$, and if all roots of $\bar{q}(s)$ have negative real parts with the possible exception of the root $s = 0$, then

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t), \tag{3}$$

if the limit exists.

a. Suppose that $\bar{f}(s) = \dfrac{1}{s(s^2+as+b)} = \dfrac{\bar{p}(s)}{\bar{q}(s)}$. Note that the roots of $\bar{q}(s)$ are at $s = 0$ and $s = \frac{1}{2}(-a \pm \sqrt{a^2-4b})$.

If $a \leq 0$, then the assumptions of the Final Value Theorem are not satisfied and thus cannot be applied. However, if $a > 0$, then the assumptions are satisfied and from (3) we see that

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\bar{f}(s) = \frac{s}{s(s^2+as+b)} = \frac{1}{b}.$$

d. Suppose that $\bar{f}(s) = \dfrac{3}{(s^2+4)^2} = \dfrac{\bar{p}(s)}{\bar{q}(s)}$. Note that the roots of $\bar{q}(s)$ are $s = \pm 2i$ each with multiplicity 2. Since the real parts of these roots are not negative, the Final Value Theorem cannot be applied.

$\square$

Problem 3.29. Suppose that $\mathscr{L}\{f(t)\} = \bar{f}(s)$ and $\mathscr{L}\{g(t)\} = \bar{g}(s)$. Show that

$$\mathscr{L}^{-1}\left\{s\bar{f}(s)\bar{g}(s)\right\} = f(0)g(t) + \int_0^t g(t-\tau)f'(\tau)d\tau$$
$$\mathscr{L}^{-1}\left\{s\bar{f}(s)\bar{g}(s)\right\} = g(0)f(t) + \int_0^t f(t-\tau)g'(\tau)d\tau.$$

Solution. We wish to show that

$$\mathscr{L}^{-1}\left\{s\bar{f}(s)\bar{g}(s)\right\} = f(0)g(t) + \int_0^t g(t-\tau)f'(\tau)d\tau.$$

This is equivalent to showing that

$$\mathscr{L}\left\{f(0)g(t) + \int_0^t g(t-\tau)f'(\tau)d\tau\right\} = s\bar{f}(s)\bar{g}(s).$$

Note that we have by the definition of the convolution that

$$\int_0^t g(t-\tau)f'(\tau)d\tau = (g * f')(t).$$

Thus,

$$\mathscr{L}\left\{f(0)g(t) + \int_0^t g(t-\tau)f'(\tau)d\tau\right\} = \mathscr{L}\left\{g(t)f(0) + (g*f')(t)\right\}.$$

Using the linearity of the Laplace transform in combination with the Convolution Theorem, we have that

$$\begin{aligned}\mathscr{L}\left\{f(0)g(t) + \int_0^t g(t-\tau)f'(\tau)d\tau\right\} &= \mathscr{L}\left\{g(t)f(0) + (g*f')(t)\right\} \\ &= f(0)\mathscr{L}\{g(t)\} + \mathscr{L}\{g(t)\}\mathscr{L}\{f'(t)\}.\end{aligned}$$

Recall that we have shown previously that

$$\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - f(0).$$

Therefore,

$$\begin{aligned}\mathscr{L}\left\{f(0)g(t) + \int_0^t g(t-\tau)f'(\tau)d\tau\right\} &= f(0)\mathscr{L}\{g(t)\} + \mathscr{L}\{g(t)\}\mathscr{L}\{f'(t)\} \\ &= \mathscr{L}\{g(t)\}\left(f(0) + s\mathscr{L}\{f(t)\} - f(0)\right) \\ &= s\mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\} \\ &= s\bar{f}(s)\bar{g}(s).\end{aligned}$$

Note the same argument can be repeated by interchanging f and g to show that

$$\mathscr{L}\left\{g(0)f(t) + \int_0^t f(t-\tau)g'(\tau)d\tau\right\} = s\bar{f}(s)\bar{g}(s),$$

and we are done. $\square$

Problem 3.32. Use Heaviside's Second Shifting Theorem to obtain the Laplace transforms of the following functions:

a. $f(t) = (t-a)^n H(t-a)$,

e. $f(t) = \cos 2t H(t-\pi)$.

Solution. Heaviside's Second Shifting Theorem states that if $\mathscr{L}\{f(t)\} = \bar{f}(s)$, then

$$\mathscr{L}\{f(t-a)H(t-a)\} = e^{-as}\bar{f}(s) \tag{4}$$

or, equivalently,

$$\mathscr{L}\{f(t)H(t-a)\} = e^{-as}\mathscr{L}\{f(t+a)\}. \tag{5}$$

a. Let $g(t) = t^n$. Then $f(t) = g(t-a)H(t-a)$ and from our table of Laplace transforms,

$$\bar{g}(s) = \mathscr{L}\{g(t)\} = \mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}.$$

Therefore, from (4), we see that

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{g(t-a)H(t-a)\} = e^{-as}\bar{g}(s) = \frac{n!e^{-as}}{s^{n+1}}.$$

e. Let $g(t) = \cos 2t$. Then $f(t) = g(t)H(t-\pi)$ and from (5), we see that

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{g(t)H(t-\pi)\} = e^{-\pi s}\mathscr{L}\{g(t+\pi)\}.$$

Note that

$$\mathscr{L}\{g(t+\pi)\} = \mathscr{L}\{\cos 2(t+\pi)\} = \mathscr{L}\{\cos 2t\} = \frac{s}{s^2+4}.$$

Therefore,

$$\mathscr{L}\{f(t)\} = e^{-\pi s}\mathscr{L}\{g(t+\pi)\} = \frac{se^{-\pi s}}{s^2+4}.$$

$\square$

Problem 3.34. Suppose that $f(t) = aH(t) - 2aH(t-1) + aH(t-2)$. Show that

$$\bar{f}(s) = \mathscr{L}\{f(t)\} = \frac{a}{s}\left(1 - e^{-s}\right)^2$$

Solution. Suppose that $g(t) = 1$. Then from Heaviside's Second Shifting Theorem (5), we see that for $b \geq 0$

$$\begin{aligned} \mathscr{L}\{H(t-b)\} = \mathscr{L}\{g(t)H(t-b)\} &= e^{-bs}\mathscr{L}\{g(t+b)\} \\ &= e^{-bs}\mathscr{L}\{1\} \\ &= \frac{e^{-bs}}{s}. \end{aligned} \tag{6}$$

Therefore, using (6) and the linearity of the Laplace transform, we see that

$$\begin{aligned} \mathscr{L}\{f(t)\} &= \mathscr{L}\{aH(t) - 2aH(t-1) + aH(t-2)\} \\ &= a\left(\mathscr{L}\{H(t)\} - 2\mathscr{L}\{H(t-1)\} + \mathscr{L}\{H(t-2)\}\right) \\ &= \frac{a}{s}\left(1 - 2e^{-s} + e^{-2s}\right) \\ &= \frac{a}{s}\left(1 - e^{-s}\right)^2, \end{aligned}$$

and we are done. □

Problem 4.1. Using the Laplace transform, solve the following initial value problems:

a. $\dfrac{dx}{dt} + ax = e^{-bt}$, $t > 0$, $a \neq b$, with $x(0) = 0$,

c. $\dfrac{dx}{dt} + 2x = \cos t$, $t > 0$, $x(0) = 1$,

h. $\dfrac{\partial u}{\partial t} + x\dfrac{\partial u}{\partial x} = xf(t)$, $u(x, 0) = 0$, $u(0, t) = 0$.

Solution. Throughout, we use the following relation

$$\mathscr{L}\left\{\frac{dx}{dt}\right\} = s\bar{x}(s) - x(0). \tag{7}$$

a. Applying the Laplace transform to the differential equation shows that

$$\mathscr{L}\left\{\frac{dx}{dt} + ax\right\} = (s + a)\bar{x}(s) - x(0) = \frac{1}{s + b} = \mathscr{L}\left\{e^{-bt}\right\}$$

where we use the relation (7).

Thus,

$$\bar{x}(s) = \frac{1}{(s + a)(s + b)}.$$

Let $\bar{f}(s) = 1/(s + a)$ and $\bar{g}(s) = 1/(s + b)$. Then $f(t) = e^{-at}$ and $g(t) = e^{-bt}$ and by the Convolution Theorem,

$$\begin{aligned} x(t) = \mathscr{L}^{-1}\left\{\bar{x}(s)\right\} = \mathscr{L}^{-1}\left\{\bar{f}(s)\bar{g}(s)\right\} &= (f * g)(t) \\ &= \int_0^t f(t - \tau)g(\tau)d\tau. \end{aligned}$$

Therefore, the solution to the original initial value problem is

$$\begin{aligned} x(t) = \int_0^t f(t - \tau)g(\tau)d\tau &= \int_0^t e^{-a(t-\tau)}e^{-b\tau}d\tau \\ &= e^{-at}\int_0^t e^{-(b-a)\tau}d\tau \\ &= -\frac{e^{-at}}{b - a}\left(e^{-(b-a)t} - 1\right) \\ &= \frac{e^{-at}}{a - b}\left(e^{-bt} - e^{-at}\right). \end{aligned}$$

c. Applying the Laplace transform to the differential equation shows that

$$\mathscr{L}\left\{\frac{dx}{dt} + 2x\right\} = (s+2)\bar{x}(s) - x(0) = \frac{s}{s^2+1} = \mathscr{L}\left\{\cos t\right\}$$

where we use the relation (7).

Thus, using the initial value $x(0) = 1$

$$\bar{x}(s) = \frac{s}{(s+2)(s^2+1)} + \frac{1}{s+2}$$

Let $\bar{f}(s) = 1/(s+2)$ and $\bar{g}(s) = s/(s^2+1)$. Then $f(t) = e^{-2t}$ and $g(t) = \cos t$ and by the Convolution Theorem,

$$\begin{aligned} x(t) = \mathscr{L}^{-1}\left\{\bar{x}(s)\right\} = \mathscr{L}^{-1}\left\{\bar{f}(s)\bar{g}(s) + \bar{f}(s)\right\} &= (f * g)(t) - f(t) \\ &= \int_0^t f(t-\tau)g(\tau)d\tau + f(t). \end{aligned}$$

Therefore, using a computer algebra system, we see that the solution to the original initial value problem is

$$\begin{aligned} x(t) = \int_0^t f(t-\tau)g(\tau)d\tau + f(t) &= \int_0^t e^{-2(t-\tau)}\cos\tau d\tau + e^{-2t} \\ &= \frac{1}{5}\left(-2e^{-2t} + 2\cos t + \sin t\right) + e^{-2t} \\ &= \frac{1}{5}\left(3e^{-2t} + 2\cos t + \sin t\right). \end{aligned}$$

h. Applying the Laplace transform to the partial differential equation yields that

$$\begin{aligned} &s\bar{u}(x,s) + x\frac{d\bar{u}(x,s)}{dx} = x\bar{f}(s) \\ &\bar{u}(0,s) = 0 \end{aligned}$$

where we used the initial value $u(x,0) = 0$.

Thus,

$$\frac{d\bar{u}(x,s)}{dx} + \frac{s}{x}\bar{u}(x,s) = \bar{f}(s).$$

Using a computer algebra system, we see that the solution to this differential equation is

$$\bar{u}(x,s) = \frac{c_1}{x^s} + \frac{x\bar{f}(s)}{s+1}.$$

Note that $\bar{u}(0, s) = 0$ implies that $c_1 = 0$ and that

$$\bar{u}(x, s) = \frac{x\bar{f}(s)}{s + 1}.$$

Let $\bar{g}(s) = 1/(s + 1)$. Therefore, $g(t) = e^{-t}$ and using the Convolution Theoreom, the solution to the original differential equation is

$$\begin{aligned} u(x, t) = \mathscr{L}^{-1}\left\{\bar{u}(x, s)\right\} &= x\mathscr{L}^{-1}\left\{\bar{f}(s)\bar{g}(s)\right\} \\ &= x(f * g)(t) \\ &= x\int_0^t f(t - \tau)e^{-\tau}d\tau. \end{aligned}$$

$\square$