Homework Assignment 5

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Problem 1. a. Explain in a specific example why, when A and $\boldsymbol{b}$ have integer components, a general integer programming problem

$$\begin{array}{lll} \text{(GILP)} & \text{Minimize (Maximize)} & f(\boldsymbol{x}) = \boldsymbol{c}^\mathsf{T}\boldsymbol{x} \\ & \text{subject to} & A\boldsymbol{x} \quad \leq (\geq, =)\boldsymbol{b} \\ & & \boldsymbol{x} \geq (\leq)\boldsymbol{0}, \boldsymbol{x} \in \mathbb{Z}^n \end{array}$$

can be reduced (or is equivalent) to a standard integer programming problem

$$\begin{array}{lll} \text{(ILP)} & \text{Minimize (Maximize)} & f(\boldsymbol{X}) = \boldsymbol{C}^\mathsf{T}\boldsymbol{X} \\ & \text{subject to} & \mathscr{A}\boldsymbol{X} \quad = \boldsymbol{B} \\ & & \boldsymbol{X} \geq \boldsymbol{0}, \boldsymbol{X} \in \mathbb{Z}^n \end{array}$$

by adding variables or any of the transformations discussed in class that change $\boldsymbol{x}$ into $\boldsymbol{X}$.

More precisely, explain why (GILP) has a solution $\boldsymbol{x} \in \mathbb{Z}^n$ if and only if (ILP) has a solution $\boldsymbol{X} \in \mathbb{Z}^n$.

b. How do we solve (GILP) when A or $\boldsymbol{b}$ do not have integer components?

Solution. a. In general, the transformations needed to change the constraints in the (GILP) to the standard form (ILP) all involve operations on integers so that the requirement that the constraints all involve integer components is satisfied. As such, if either the (GILP) or (ILP) has a solution, then the other is guaranteed to have a feasible solution since the additional slack or excess variables all involve integer operations. Thus, one of the feasible solutions will be the optimum and the other problem is solved.

To illustrate this idea, take the (GILP)

$$\begin{array}{lll} \text{(GILP)} & \text{Minimize} & f(\boldsymbol{x}) = \begin{bmatrix} -4 & -3 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\mathsf{T} \\ & \text{subject to} & \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 5 \\ 3 \end{bmatrix} \\ & & \boldsymbol{x} \geq \boldsymbol{0}, \boldsymbol{x} \in \mathbb{Z}^n \end{array}$$

The solution to (GILP) is $\boldsymbol{x}^\mathsf{T} = \begin{bmatrix} 4 & 3 \end{bmatrix}$.

In order to change the problem (GILP) to standard form (ILP) we need to add slack variables to the matrix A. Doing so results in a matrix $\mathscr{A}$ that still has all integer components and $\boldsymbol{b}$ remains unchanged so that it too still has all integer components. To see this, in standard form the problem becomes

$$\begin{aligned} \text{(ILP)} \quad & \text{Minimize} \quad f(\boldsymbol{x}) = \begin{bmatrix} -4 & -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^\mathsf{T} \\ & \text{subject to} \quad \begin{bmatrix} 2 & -1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 0 \\ 0 \end{bmatrix} . \\ & \qquad \boldsymbol{X} \geq \boldsymbol{0}, \boldsymbol{X} \in \mathbb{Z}^n \end{aligned}$$

the solution of which is $\boldsymbol{x}^\mathsf{T} = \begin{bmatrix} 4 & 3 & 0 & 0 \end{bmatrix}$.

b. When the matrix A or the constraint column $\boldsymbol{b}$ do not have integer components, we can solve (GILP) by converting the problem to standard form and performing the simplex method in conjunction with the Gomory Cutting-Plan Method to obtain an integer solution.

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Problem 2. Solve the shipping problem studied in MATH 111 with the replaced constraints over integers using the Gomory Cutting-Plane Method.

More precisely, solve:

$$\begin{array}{lrl} \text{Maximize} & 9x_1 + 13x_2 & \\ \text{subject to} & 4x_1 + 3x_2 & \leq 300 \\ & x_1 + 2x_2 & \leq 625/6 \\ & -2x_1 + x_2 & \leq 0 \\ & x_1, x_2 \geq 0 & \\ & x_1, x_2 \in \mathbb{Z} & \end{array}.$$

Solution. We begin by transforming the shipping problem into the equivalent problem:

$$\begin{array}{llrl} \text{ILP} & \text{Minimize} & -9x_1 - 13x_2 & \\ & \text{subject to} & 4x_1 + 3x_2 + x_3 & = 300 \\ & & x_1 + 2x_2 + x_4 & = 625/6 \\ & & -2x_1 + x_2 + x_5 & = 0 \\ & & x_1, x_2, x_3, x_4, x_5 \geq 0 & \\ & & x_1, x_2, x_3, x_4, x_5 \in \mathbb{Z} & \end{array} \tag{1}$$

The Gomory Cutting-Plane Method involves solving the LP (1) without the integer constraints using the Simplex Method.Once an optimal basic feasible solution is obtained, if the solution is a noninteger solution, additional constraints are added that remove thee current optimal noninteger solution from the feasible set without eliminating any integer feasible solutions. The process then begins anew on this system until an optimal integer solution is found.

Thus, we perform the simplex method on the initial tableau

$$\begin{array}{c|cccccc} & \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} & \boldsymbol{a_4} & \boldsymbol{a_5} & \boldsymbol{b} \\ & 4 & 3 & 1 & 0 & 0 & 300 \\ & 1 & 2 & 0 & 1 & 0 & 625/6 \\ & -2 & 1 & 0 & 0 & 1 & 0 \\ \boldsymbol{c}^\top & -9 & -13 & 0 & 0 & 0 & 0 \end{array}$$

which is in canonical form. Moving from this tableau to the next we see that $\boldsymbol{a_2}$ enters the basis and $\boldsymbol{a_5}$ leaves the basis:

$$\begin{array}{ccccccc} & \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} & \boldsymbol{a_4} & \boldsymbol{a_5} & \boldsymbol{b} \\ & 4 & 3 & 1 & 0 & 0 & 300 \\ & 1 & 2 & 0 & 1 & 0 & 625/6 \\ & -2 & 1 & 0 & 0 & 1 & 0 \\ \boldsymbol{c}^\top & -9 & -13 & 0 & 0 & 0 & 0 \\ & & \uparrow & & & & \end{array} \quad \rightarrow \quad \begin{array}{ccccccc} & \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} & \boldsymbol{a_4} & \boldsymbol{a_5} & \boldsymbol{b} \\ & 10 & 0 & 1 & 0 & -3 & 300 \\ & 5 & 0 & 0 & 1 & -2 & 625/6 \\ & -2 & 1 & 0 & 0 & 1 & 0 \\ \boldsymbol{c}^\top & -35 & 0 & 0 & 0 & 13 & 0 \end{array}$$

Moving from this tableau to the next, $\boldsymbol{a_1}$ enters the basis and $\boldsymbol{a_4}$ leaves the basis:

$$\begin{array}{cccccccc} & \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} & \boldsymbol{a_4} & \boldsymbol{a_5} & \boldsymbol{b} \\ & 10 & 0 & 1 & 0 & -3 & 300 \\ & 5 & 0 & 0 & 1 & -2 & 625/6 \\ & -2 & 1 & 0 & 0 & 1 & 0 \\ \boldsymbol{c}^\mathsf{T} & -35 & 0 & 0 & 0 & 13 & 0 \\ & \uparrow & & & & & \end{array} \rightarrow \begin{array}{ccccccc} & \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} & \boldsymbol{a_4} & \boldsymbol{a_5} & \boldsymbol{b} \\ & 0 & 0 & 1 & -2 & 1 & 275/3 \\ & 1 & 0 & 0 & 1/5 & -2/5 & 125/6 \\ & 0 & 1 & 0 & 2/5 & 1/5 & 125/3 \\ \boldsymbol{c}^\mathsf{T} & 0 & 0 & 0 & 7 & -1 & 4375/6 \end{array}$$

Moving from this tableau to the next, $\boldsymbol{a_5}$ enters the basis and $\boldsymbol{a_3}$ leaves the basis:

$$\begin{array}{ccccccc} & \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} & \boldsymbol{a_4} & \boldsymbol{a_5} & \boldsymbol{b} \\ & 0 & 0 & 1 & -2 & 1 & 275/3 \\ & 1 & 0 & 0 & 1/5 & -2/5 & 125/6 \\ & 0 & 1 & 0 & 2/5 & 1/5 & 125/3 \\ \boldsymbol{c}^\mathsf{T} & 0 & 0 & 0 & 7 & -1 & 4375/6 \\ & & & & & \uparrow & \end{array} \rightarrow \begin{array}{ccccccc} & \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} & \boldsymbol{a_4} & \boldsymbol{a_5} & \boldsymbol{b} \\ & 0 & 0 & 1 & -2 & 1 & 275/3 \\ & 1 & 0 & 2/5 & -3/5 & 0 & 115/2 \\ & 0 & 1 & -1/5 & 4/5 & 0 & 70/3 \\ \boldsymbol{c}^\mathsf{T} & 0 & 0 & 1 & 5 & 0 & 4925/6 \end{array}$$

so that the current tableau is optimal. This optimal basic feasible solution does not satisfy the integer constraint in terms of the decision variables x_1 and x_2. Thus, we introduce the Gomory cut

$$\begin{aligned} [2/5 - \lfloor 2/5 \rfloor]x_3 + [-3/5 - \lfloor -3/5 \rfloor]x_4 - x_6 &= 115/2 - \lfloor 115/2 \rfloor \\ 2/5x_3 + 2/5x_4 - x_6 &= 1/2. \end{aligned}$$

The tableau then becomes

$$\begin{array}{cccccccc} & \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} & \boldsymbol{a_4} & \boldsymbol{a_5} & \boldsymbol{a_6} & \boldsymbol{b} \\ & 0 & 0 & 1 & -2 & 1 & 0 & 275/3 \\ & 1 & 0 & 2/5 & -3/5 & 0 & 0 & 115/2 \\ & 0 & 1 & -1/5 & 4/5 & 0 & 0 & 70/3 \\ & 0 & 0 & 2/5 & 2/5 & 0 & -1 & 1/2 \\ \boldsymbol{c}^\mathsf{T} & 0 & 0 & 1 & 5 & 0 & 0 & 4925/6 \end{array}$$

Pivoting around (4,3) yields

$$\begin{array}{cccccccc} & \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} & \boldsymbol{a_4} & \boldsymbol{a_5} & \boldsymbol{a_6} & \boldsymbol{b} \\ & 0 & 0 & 0 & -3 & 1 & 5/2 & 1085/12 \\ & 1 & 0 & 0 & -1 & 0 & 1 & 57 \\ & 0 & 1 & 0 & 1 & 0 & -1/2 & 283/12 \\ & 0 & 0 & 1 & 1 & 0 & -5/2 & 5/4 \\ \boldsymbol{c}^\mathsf{T} & 0 & 0 & 0 & 4 & 0 & 5/2 & 9835/12 \end{array}$$

which is optimal, but the optimal basic feasible solution still does not satisfy the integer constraints in terms of the decision variables x_1 and x_2. Thus, we introduce the Gomory cut

$$\begin{aligned} [-1/2 - \lfloor -1/2 \rfloor]x_6 - x_7 &= 283/12 - \lfloor 283/12 \rfloor \\ 1/2x_6 - x_7 &= 7/12. \end{aligned}$$

The tableau then becomes

	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{a_4}$	$\boldsymbol{a_5}$	$\boldsymbol{a_6}$	$\boldsymbol{a_7}$	$\boldsymbol{b}$
	0	0	0	-3	1	$5/2$	0	$1085/12$
	1	0	0	-1	0	1	0	57
	0	1	0	1	0	$-1/2$	0	$283/12$
	0	0	1	1	0	$-5/2$	0	$5/4$
	0	0	0	0	0	$1/2$	-1	$7/12$
$\boldsymbol{c}^\top$	0	0	0	4	0	$5/2$	0	$9835/12$

Pivoting around (5,6) yields

	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{a_4}$	$\boldsymbol{a_5}$	$\boldsymbol{a_6}$	$\boldsymbol{a_7}$	$\boldsymbol{b}$
	0	0	0	-3	1	0	5	$175/2$
	1	0	0	-1	0	0	2	$335/6$
	0	1	0	1	0	0	-1	$145/6$
	0	0	1	1	0	0	-5	$25/6$
	0	0	0	0	0	1	-2	$7/6$
$\boldsymbol{c}^\top$	0	0	0	4	0	0	5	$2450/3$

which is optimal, but the optimal basic feasible solution still does not satisfy the integer constraints in terms of the decision variables x_1 and x_2. Thus, we introduce the Gomory cut

$$-x_8 = 145/6 - \lfloor 145/6 \rfloor = 1/6$$

$\square$

Problem 3. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $\Omega \subset \mathbb{R}^n$ be an open subset. Explain the meaning of $f \in C^1(\Omega)$. More precisely, give all the definitions needed and present some examples and results concerning $C^1(\Omega)$ functions.

Solution. The set Ω is open in $\mathbb{R}^n$ if Ω is a union of open balls where an open ball of radius r centered at $\boldsymbol{x_0}$ is defined to be the set $B_r(\boldsymbol{x_0}) = \{\boldsymbol{x} \in \mathbb{R}^n \mid ||\boldsymbol{x} - \boldsymbol{x_0}|| < r\}$ where $||\boldsymbol{x} - \boldsymbol{x_0}||$ can be taken to be the Euclidean distance.

The space $C^1(\Omega)$ is the set of all continuous functions that have continuous partial derivatives over the open set Ω. Thus, if $f \in C^1(\Omega)$, then the function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is necessarily continuous and all partial derivatives of f exist and are continuous.

Let us make this more precise. If f is a function from the open set Ω to $\mathbb{R}^m$, then f has m component functions $f_1, \ldots, f_m$,i.e. $f(\boldsymbol{x}) = [f_1(\boldsymbol{x}), \ldots, f_m(\boldsymbol{x})]$. We require first that f be continuous. Thus, for every $\boldsymbol{x}, \boldsymbol{a} \in \Omega$, we require that

$$\lim_{\boldsymbol{x} \to \boldsymbol{a}} f(\boldsymbol{x}) = \lim_{\boldsymbol{x} \to \boldsymbol{a}} [f_1(\boldsymbol{x}), \ldots, f_m(\boldsymbol{x})] = [f_1(\boldsymbol{a}), \ldots, f_m(\boldsymbol{a})] = f(\boldsymbol{a}),$$

i.e. for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $||\boldsymbol{x} - \boldsymbol{a}|| < \delta$, then $||f(\boldsymbol{x}) - f(\boldsymbol{a})|| < \epsilon$. We also require that all of the partial derivatives of f to exist and be continuous. That is, for all $1 \leq i \leq n$and $1 \leq k \leq m$ the partial derivatives $\partial f_k(\boldsymbol{x})/\partial x_i$ exist and

$$\lim_{\boldsymbol{x} \to \boldsymbol{a}} \frac{\partial f(\boldsymbol{x})}{\partial x_i} = \lim_{\boldsymbol{x} \to \boldsymbol{a}} \left[\frac{\partial f_1(\boldsymbol{x})}{\partial x_i}, \ldots, \frac{\partial f_m(\boldsymbol{x})}{\partial x_i}\right] = \left[\left.\frac{\partial f_1(\boldsymbol{x})}{\partial x_i}\right|_{\boldsymbol{x}=\boldsymbol{a}}, \ldots, \left.\frac{\partial f_m(\boldsymbol{x})}{\partial x_i}\right|_{\boldsymbol{x}=\boldsymbol{a}}\right] = \left.\frac{\partial f(\boldsymbol{x})}{\partial x_i}\right|_{\boldsymbol{x}=\boldsymbol{a}}.$$

An example of a function in this space, take for instance $f : \mathbb{R}^2 \to \mathbb{R}$ where $f(x_1, x_2) = x_1 + x_2$. This function is clearly C^1 smooth over the open set $\Omega = \{\boldsymbol{x} \in \mathbb{R}^2 \mid ||\boldsymbol{x}|| < 1\}$ since it is continuous over this set and its partial derivatives are both the constant 1, guaranteeing that its partial derivatives are continuous as well.

One result concerning $C^1(\Omega)$ functions is that the set of all such functions forms a vector space.

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