

Homework Assignment 1

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Problem 2.1. Find the Fourier transforms of each of the following functions:

c. $f(x) = \delta^{(n)}(x)$,

f. $f(x) = x \exp\left(-\frac{ax^2}{2}\right), a > 0$,

g. $f(x) = x^2 \exp\left(-\frac{x^2}{2}\right)$.

Solution. Recall that, by definition, we have that for a function $f(x) \in L^1(\mathbb{R})$, its Fourier transform is given by

$$\mathcal{F}\{f(x)\} = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx \quad (1)$$

where $k \in \mathbb{R}$.

c. The Dirac delta function $\delta(x)$ is defined such that for any good function $g(x)$ we have that

$$\int_{-\infty}^{\infty} \delta(x) g(x) dx = g(0).$$

A good function is defined as a function in C^∞ that decays sufficiently rapidly. Since it is clear that $\delta(x) \rightarrow 0$ as $|x| \rightarrow \infty$, we have by a previous theorem that

$$\mathcal{F}\{\delta'(x)\} = ik \mathcal{F}\{\delta(x)\}. \quad (2)$$

By (1) and the definition of the Dirac delta function, we see that

$$\mathcal{F}\{\delta(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \delta(x) dx = \frac{1}{\sqrt{2\pi}}.$$

Thus, using (2), we can easily see by induction for $n > 1$ that

$$\mathcal{F}\{\delta^{(n)}(x)\} = ik \mathcal{F}\{\delta^{(n-1)}(x)\} = \dots = \frac{(ik)^n}{\sqrt{2\pi}}.$$

f. From (1), we see that

$$\begin{aligned}
 \mathcal{F}\{f(x)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp(-ikx) \exp\left(-\frac{ax^2}{2}\right) dx \\
 &= \frac{\exp\left(\frac{(ik)^2}{2a}\right)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp\left(-\frac{ax^2}{2} - ikx - \frac{(ik)^2}{2a}\right) dx \\
 &= \frac{\exp\left(-\frac{k^2}{2a}\right)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp\left(-\frac{a}{2} \left(x + \frac{ik}{a}\right)^2\right) dx.
 \end{aligned}$$

Making the substitution $u = x + ik/a$, where $du = dx$, we have that

$$\begin{aligned}
 \mathcal{F}\{f(x)\} &= \frac{\exp\left(-\frac{k^2}{2a}\right)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(u - \frac{ik}{a}\right) \exp\left(-\frac{au^2}{2}\right) du \\
 &= \frac{\exp\left(-\frac{k^2}{2a}\right)}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} u \exp\left(-\frac{au^2}{2}\right) du - \frac{ik}{a} \int_{-\infty}^{\infty} \exp\left(-\frac{au^2}{2}\right) du \right]. \quad (3)
 \end{aligned}$$

Since the function $g(x) = u \exp\left(-\frac{au^2}{2}\right)$ is odd, we know that

$$\int_{-\infty}^{\infty} u \exp\left(-\frac{au^2}{2}\right) du = 0.$$

Using the formula for the general Gaussian integral we have that

$$\int_{-\infty}^{\infty} \exp\left(-\frac{au^2}{2}\right) du = \frac{\sqrt{2\pi}}{\sqrt{a}}$$

when $a > 0$.

Combining, we see from (3) that the Fourier transform of $f(x) = x \exp\left(-\frac{ax^2}{2}\right)$ for $a > 0$ is

$$\begin{aligned}
 \mathcal{F}\{f(x)\} &= \frac{\exp\left(-\frac{k^2}{2a}\right)}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} u \exp\left(-\frac{au^2}{2}\right) du - \frac{ik}{a} \int_{-\infty}^{\infty} \exp\left(-\frac{au^2}{2}\right) du \right] \\
 &= \frac{\exp\left(-\frac{k^2}{2a}\right)}{\sqrt{2\pi}} \left(-\frac{ik}{a}\right) \left(\frac{\sqrt{2\pi}}{\sqrt{a}}\right) \\
 &= -\frac{ik \exp\left(-\frac{k^2}{2a}\right)}{a\sqrt{a}}.
 \end{aligned}$$

g. From (1), we see that

$$\begin{aligned}\mathcal{F}\{f(x)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \exp(-ikx) \exp\left(-\frac{x^2}{2}\right) dx \\ &= \frac{\exp\left(\frac{(ik)^2}{2}\right)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2}{2} - ikx - \frac{(ik)^2}{2}\right) dx \\ &= \frac{\exp\left(-\frac{k^2}{2}\right)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{1}{2}(x + ik)^2\right) dx.\end{aligned}$$

Making the substitution $u = x + ik$, where $du = dx$, we have that

$$\begin{aligned}\mathcal{F}\{f(x)\} &= \frac{\exp\left(-\frac{k^2}{2}\right)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (u - ik)^2 \exp\left(-\frac{u^2}{2}\right) du \\ &= \frac{\exp\left(-\frac{k^2}{2}\right)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (u^2 - 2iku - k^2) \exp\left(-\frac{u^2}{2}\right) du.\end{aligned}\quad (4)$$

After distributing the exponential term to the polynomial in (4) and splitting the integral using the operator's linearity, the first integral may be computed by parts by setting $w = u$ and $dv = -u \exp\left(-\frac{u^2}{2}\right) du$ as so:

$$\int_{-\infty}^{\infty} u^2 \exp\left(-\frac{u^2}{2}\right) du = -\left[u \exp\left(-\frac{u^2}{2}\right)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \exp\left(-\frac{u^2}{2}\right) du = \sqrt{2\pi}$$

where the last equality follows from the formula for the general Gaussian integral.

The other two resulting integrals in (4) are calculated very similarly to their corresponding integrals in 2.1.f, i.e.

$$\begin{aligned}2ik \int_{-\infty}^{\infty} u \exp\left(-\frac{u^2}{2}\right) du &= 0, \\ k^2 \int_{-\infty}^{\infty} \exp\left(-\frac{u^2}{2}\right) du &= k^2 \sqrt{2\pi}.\end{aligned}$$

Therefore, we have that the Fourier transform of $f(x) = x^2 \exp\left(-\frac{x^2}{2}\right)$ is given by

$$\begin{aligned}\mathcal{F}\{f(x)\} &= \frac{\exp\left(-\frac{k^2}{2}\right)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (u^2 - 2iku - k^2) \exp\left(-\frac{u^2}{2}\right) du \\ &= \frac{\exp\left(-\frac{k^2}{2}\right)}{\sqrt{2\pi}} \left[\sqrt{2\pi} - k^2 \sqrt{2\pi}\right] \\ &= (1 - k^2) \exp\left(-\frac{k^2}{2}\right).\end{aligned}$$

□

Problem 2.2. Show that

a. $\mathcal{F} \{ \delta(x - ct) + \delta(x + ct) \} = \sqrt{\frac{2}{\pi}} \cos(kct),$

b. $\mathcal{F} \{ H(ct - |x|) \} = \mathcal{F} \{ \chi_{[-ct, ct]}(x) \} = \sqrt{\frac{2}{\pi}} \frac{\sin(kct)}{k}.$

Solution.

□

Problem 2.3. Show that

a. $i \frac{d}{dk} F(k) = \mathcal{F} \{ x f(x) \}$

b. $i^n \frac{d^n}{dk^n} F(k) = \mathcal{F} \{ x^n f(x) \}$

Solution.

□

Problem 2.5. Prove the following:

c. If $f(x)$ has a finite discontinuity at a point $x = a$, then

$$\mathcal{F} \{f'(x)\} = (ik)F(k) - \frac{1}{\sqrt{2\pi}} \exp(-ika)[f]_a,$$

where $[f]_a = f(a+0) - f(a-0)$.

Generalize this result for $\mathcal{F} \{f^{(n)}(x)\}$.

Solution.

□

Problem 2.7. Prove the following results for the convolution:

c. $\frac{d}{dx} [f(x) * g(x)] = f'(x) * g(x) = f(x) * g'(x),$

d. $\int_{-\infty}^{\infty} (f * g)(x) dx = \int_{-\infty}^{\infty} f(u) du \int_{-\infty}^{\infty} g(v) dv.$

Solution.

□

Problem 2.8. Use the Fourier transform to solve the following ordinary differential equations for $-\infty < x < \infty$:

a. $y''(x) - y(x) + 2f(x) = 0$, where $f(x) = 0$ when $x < -a$ and when $x > a$ and its derivatives vanish at $x = \pm\infty$,

b. $2y''(x) + xy'(x) + y(x) = 0$.

Solution.

□

Problem 2.9. Solve the following integral equations for an unknown function $f(x)$:

a. $\int_{-\infty}^{\infty} \phi(x-t)f(t)dt = g(x),$

b. $\int_{-\infty}^{\infty} \exp(-at^2)f(x-t)dt = \exp(-at^2), a > b > 0,$

d. $\int_{-\infty}^{\infty} f(x-t)f(t)dt = \frac{b}{x^2 + b^2}.$

Solution.

□