Exam 3

Matthew Tiger

May 20, 2017

Problem 1. Solve the non-homogeneous diffusion problem by the Hankel transform

$$
\begin{aligned}
&u_t = a\left(u_{rr} + \frac{1}{r}u_r\right) + Q(r,t), \qquad 0 < r < \infty, \quad 0 < t \\
&u(r,0) = f(r), \qquad 0 < r < \infty.
\end{aligned}
$$

Solution. Application of the 0-th order Hankel transform will transform the above Partial Differential Equation into an Ordinary Differential Equation. The following property of the 0-th order Hankel transform will aid in the application; if $\mathscr{H}_0\left\{u(r,t)\right\} = \tilde{u}_0(\kappa,t)$, then

$$
\mathscr{H}_0\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[u(r,t)\right] + \frac{\partial^2}{\partial r^2}\left[u(r,t)\right]\right\} = -\kappa^2\tilde{u}_0(\kappa,t). \tag{1}
$$

Now, with the above property, we see that applying the 0-th order Hankel transform to the diffusion problem yields

$$
\begin{aligned}
&\frac{d}{dt}\left[\tilde{u}_0(\kappa,t)\right] + a\kappa^2\tilde{u}_0(\kappa,t) = \tilde{Q}_0(\kappa,t), \qquad 0 < \kappa < \infty, \quad 0 < t \\
&\tilde{u}_0(\kappa,0) = \tilde{f}_0(\kappa), \qquad 0 < \kappa < \infty.
\end{aligned}
$$

This is a first order linear Ordinary Differential Equation, the solution to which is

$$
\tilde{u}_0(\kappa,t) = c_1(\kappa)e^{-a\kappa^2 t} + e^{-a\kappa^2 t}\int_0^t e^{a\kappa^2 x}\tilde{Q}_0(\kappa,x)dx.
$$

Thus, from this solution and the transformed boundary condition, we see that $c_1(\kappa) = \tilde{f}_0(\kappa)$ and the solution to the transformed boundary value problem is

$$
\tilde{u}_0(\kappa,t) = \tilde{f}_0(\kappa)e^{-a\kappa^2 t} + e^{-a\kappa^2 t}\int_0^t e^{a\kappa^2 x}\tilde{Q}_0(\kappa,x)dx.
$$

Therefore, the solution to the initial diffusion problem is

$$
\begin{aligned}
u(r,t) = \mathscr{H}_0^{-1}\left\{\tilde{u}_0(\kappa,t)\right\} &= \mathscr{H}_0^{-1}\left\{\tilde{f}_0(\kappa)e^{-a\kappa^2 t} + e^{-a\kappa^2 t}\int_0^t e^{a\kappa^2 x}\tilde{Q}_0(\kappa,x)dx\right\} \\
&= \int_0^\infty \kappa J_0(\kappa r)\left[\tilde{f}_0(\kappa)e^{-a\kappa^2 t} + e^{-a\kappa^2 t}\int_0^t e^{a\kappa^2 x}\tilde{Q}_0(\kappa,x)dx\right]d\kappa,
\end{aligned}
$$

where $J_0(\kappa r)$ is the Bessel function of order 0.

$\square$

Problem 2.

Solution. □

Problem 3.

Solution. $\square$

Problem 4.

Solution. □

Problem 5.

Solution. $\square$

Problem 6.

Solution. $\square$

Problem 7.

Solution. □

Problem 8.

Solution. □