LINEAR AND NONLINEAR PROGRAMMING

Spring 2016

Abstract

This is an introductory optimization course containing linear up to convex programming. The problems are studied in conjunction with applications and algorithms.

1 Introduction

1.1 Optimization problems

The general form of an *optimization problem* is:

$$\textbf{(G)} \text{ (Locally) Minimize (Maximize) } F(x_1, x_2, \dots, x_n) \longleftarrow$$

$$\text{subject to: } \bar{x} = (x_1, x_2, \dots, x_n) \in \underset{\uparrow}{S} \subset \mathbb{R}^n \longleftarrow$$

Depending on the cost functional and constraint structure we get different names for optimization problems as follows:

• If $S = \mathbb{R}^n$ the problem is called . If $S \subsetneq \mathbb{R}^n$ the problem is called

• If $F(\bar{x})$ is a linear (affine) function and S is defined by a system of linear equations and/or inequalities, the problem is called ; otherwise, the problem is called

• If $F(\bar{x})$ is a convex function and S is a convex set the problem is called

• If $F(\bar{x})$ is quadratic and S is defined by a system of linear equations and/or inequalities, the problem is called

Since every maximization problem can be seen as a minimization problem via: $\max F = -\min(-F)$ in the sequel we study only minimization problems:

$$\textbf{(G-min)} \text{ (Locally) Minimize } F(x_1, x_2, \dots, x_n), \text{ subject to: } \bar{x} = (x_1, x_2, \dots, x_n) \in S \subset \mathbb{R}^n$$

Here $F : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ (can take the $+\infty$ value). Reason:
(see section 2.1 - extended value functions for more details)

A (local) *solution* of (G-min) is called a (local) minimizer. By definition $\bar{x}_0 \in \mathbb{R}^n$ is a

• *global minimizer* if $\bar{x}_0 \in S$ and, for every $\bar{x} \in S$, $F(\bar{x}_0) \le F(\bar{x})$

• *strict global minimizer* if $\bar{x}_0 \in S$ and, for every $\bar{x} \in S$ such that $\bar{x} \neq \bar{x}_0$, $F(\bar{x}_0) < F(\bar{x})$

• *local minimizer* if $\bar{x}_0 \in S$ and

• *strict local minimizer* if $\bar{x}_0 \in S$ and

What is the meaning of $\bar{x}$ is near $\bar{x}_0$?

What is the meaning of $\bar{x}_0 \in S$?

1.2 Basic examples

$\boxed{\text{Calculus I}}$: Min $f(x)$, $x \in \mathbb{R}$ or $x \in [a, b]$. Facts:

- Fermat's (Candidate) Theorem. Every relative (local) extrema is a

  What is a critical number for a constrained problem ($x \in [a, b]$)?

- *Necessary Condition*: A condition that must be satisfied by any solution of an optimization problem

- *Sufficient Condition*: A condition that ensures that a critical number is a (local) extrema

The ideas that solved this basic problem are followed throughout optimization and form the main body of knowledge on the subject.

$\boxed{\text{Calculus III-Unconstrained Problems}}$: Min (Max) $f(x,y)$, $x \in \mathbb{R}^2$. Facts:

Def. A two-variable function $f = f(x,y)$ has a *local maximum at* (a,b) if $f(x,y) \leq f(a,b)$, when (x,y) is near (a,b). The number $f(a,b)$ is called a *local maximum value.*

near?

Criterion-NC: If f_x, f_y exist and f has a local extrema at (a,b) then $f_x(a,b) = f_y(a,b) = 0$ (or the tangent plane to $z = f(x,y)$ at (a,b) is) or the gradient $\nabla f(a,b) =$

Def. (a,b) is called a **critical point of** f if $f_x(a,b) = f_y(a,b) = 0$ or one of the f_x, f_y does not exist.

The criterion says:

Example. Study the local extrema for $f(x,y) = x^2 + y^2 - 2x - 6y + 14$

Is every critical point a local extrema?

Example. $f(x,y) = y^2 - x^2$

Is there a general way to decide whether a critical point is a local extrema?

$\boxed{\text{THE SECOND DERIVATIVE TEST.}}$ Suppose that second partial derivatives of f exist and are continuous on a disk centered at a critical point (a, b). Let

$$D = D(a,b) := \det \underbrace{\begin{pmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{pmatrix}}_{Hessian\ Matrix} =$$

(a) If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$ then (a, b) is a local minimum;

(b) If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$ then (a, b) is a local maximum;

(c) If $D(a, b) < 0$ (a, b) is a *saddle point*, that is, it is neither a local minimum nor a local maximum point.

Remark. The same conclusions hold if $f_{xx}(a, b)$ is replaced by $f_{yy}(a, b)$ (in case $f_{xx}(a, b) = 0$)

PROCEDURE. Step 1. Find critical points. Example $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$

Step 2. Compute in general the Hessian Matrix

Step 3. Compute D for all critical points and draw conclusion.

Calculus III-Constrained Problems Min(Max) $f(x, y)$ subject to $g(x, y) = k$.

We introduce a new variable λ which will be called:
and define a new function (Lagrangian)

$$L(x, y, \lambda) := f(x, y) +$$

Fact: To find relative extrema of $f(x, y)$ subject to $g(x, y) = k$ is equivalent to finding (free) relative extrema of

Procedure. Step 1. Find critical points of $L(x, y, \lambda)$. Example $f(x, y) = x^2 + 2y^2$ subject to $x^2 + y^2 = 1$.

Step 2. Evaluate f at all (x, y) that result from step 1. The largest [smallest] of these values is the (global) maximum [minimum] of f.

Explain under what (special) assumptions is Step 2 valid?

Math 111-LINEAR PROGRAMMING A truck traveling from New York to Baltimore is to be loaded with two types of cargo. Each crate of cargo A is 4 cubic feet in volume, weighs 100 pounds, and earns \$13 for the driver. Each crate of cargo B is 3 cubic feet in volume, weighs 200 pounds, and earns \$9 for the driver. The truck can carry no more than 300 cubic feet of crates and no more than $10,000$ pounds. Also, the number of crates of cargo B must be less than or equal to twice the number of crates of cargo A.

(a) Fill in the following chart:

	A	**B**	**Truck capacity**
Volume			
Weight			
Earnings			

(b) Let x be the number of crates of cargo A and y the number of crates of cargo B. Referring to the chart, give the two inequalities that x and y must satisfy because of the truck's capacity for volume and weight.

(c) Give the inequalities that x and y must satisfy because of the last sentence of the problem and also because x and y cannot be negative.

(d) Express the earnings from carrying x crates of cargo A and y crates of cargo B.

(e) Graph the feasible set for the shipping problem.

(f) Formulate it as an optimization problem.

(g) How many crates of each cargo should be shipped in order to satisfy the shipping requirements and yield the greatest earning?

Recall the FUNDAMENTAL THEOREM OF LINEAR PROGRAMMING: The maximum (or minimum) value of the objective function is achieved at one of the vertices of the feasible set.

2 Linear Programming

2.1 Convex Sets. Convex Functions

A set $C \subset \mathbb{R}^n$ is *convex* if

Extended values function: Every function $f : S \subset \mathbb{R}^n \to \mathbb{R}$ is extended outside S with the value $+\infty$. In this way its extension becomes

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if} \quad x \in S \\ +\infty & \end{cases}$$

Conversely, if $\tilde{f} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ then $D(f) = \{x \in \mathbb{R}^n \mid f(x) < +\infty\}$ is called the of f. A function is called *proper* if

If $f : S \subset \mathbb{R}^n \to \mathbb{R}$ what is the domain of its extension?

From the point of view of minimization problems f and $\tilde{f}$ are equivalent in the sense that

$$\min_S f = \min_{\mathbb{R}^n} \tilde{f} \text{ (and have the same optimal solutions)}$$

Convex function: $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is *convex* if

$$\forall x, y \in \mathbb{R}^n \ \forall \lambda \in [0, 1], \ f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y)$$

Fact: If f is convex then $D(f)$ is convex:

2.2 Examples of Convex Functions

Linear Function: $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = c^T x = \sum_{k=1}^{n} c_k x_k$,

where $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$, $x^T = (x_1, x_2, \dots, x_n)$, $c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n$.

Quadratic form: $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = x^T Q x$

where Q is a $n \times n-$matrix.

Without loss of generality Q can be taken to be symmetric.