

Homework Assignment 8

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Problem 7.1. Show that

$$\text{a. } \mathcal{H}_0 \{ (a^2 - r^2)H(a - r) \} = \frac{4a}{\kappa^3} J_1(a\kappa) - \frac{2a^2}{\kappa^2} J_0(a\kappa).$$

Solution. a. Let J_n be the integral representation of the Bessel function of order n , i.e.

$$J_n(\kappa r) = \frac{1}{2\pi} \int_{\pi/2-\phi}^{5\pi/2-\phi} \exp[i(n\alpha - \kappa r \sin \alpha)] d\alpha$$

Then the Hankel transformation of order n of $f(r)$ is defined to be

$$\mathcal{H}_n \{ f(r) \} = \int_0^\infty r J_n(\kappa r) f(r) dr.$$

Using the table of Hankel transforms we see that

$$\mathcal{H}_0 \{ (a^2 - r^2)H(a - r) \} = \frac{4a}{\kappa^3} J_1(a\kappa) - \frac{2a^2}{\kappa^2} J_0(a\kappa),$$

and we are done. □

Problem 7.2. a. Show that the solution of the boundary value problem

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + u_{zz} &= 0, & 0 < r < \infty, \quad 0 < z < \infty, \\ u(r, 0) &= \frac{1}{\sqrt{a^2 + r^2}}, & 0 < r < \infty, \end{aligned}$$

is

$$u(r, z) = \int_0^\infty e^{-\kappa(z+a)} J_0(\kappa r) d\kappa = [(z+a)^2 + r^2]^{-1/2}.$$

Solution. a. Let

$$u(r, z) = [(z+a)^2 + r^2]^{-1/2}.$$

Then it is clear that for $0 < r < \infty$ we have that

$$u(r, 0) = \frac{1}{\sqrt{a^2 + r^2}}$$

and $u(r, z)$ satisfies the boundary condition.

Now, note from the definition of $u(r, z)$ that

$$\begin{aligned} u_r &= -r [(z+a)^2 + r^2]^{-3/2}, \\ u_{rr} &= -[(z+a)^2 + r^2]^{-3/2} + 3r^2 [(z+a)^2 + r^2]^{-5/2}, \\ u_z &= -(z+a) [(z+a)^2 + r^2]^{-3/2}, \\ u_{zz} &= -[(z+a)^2 + r^2]^{-3/2} + 3(z+a)^2 [(z+a)^2 + r^2]^{-5/2}. \end{aligned}$$

Therefore, we see that

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + u_{zz} &= \frac{3r^2 + 3(z+a)^2}{[(z+a)^2 + r^2]^{5/2}} - \frac{3}{[(z+a)^2 + r^2]^{3/2}} \\ &= \frac{3r^2 + 3(z+a)^2 - 3[(z+a)^2 + r^2]}{[(z+a)^2 + r^2]^{5/2}} \\ &= 0, \end{aligned}$$

and we see that $u(r, z)$ is a solution of the boundary value problem.

□

Problem 7.9. Solve the problem of the electrified unit disk in the (x, y) plane with center at the origin. The electric potential $u(r, z)$ is axisymmetric and satisfies the boundary value problem

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + u_{zz} &= 0, & 0 < r < \infty, & \quad 0 < z < \infty, \\ u(r, 0) &= u_0, & 0 \leq r < a \\ \frac{\partial u}{\partial z} &= 0, & \text{on } z = 0 \text{ for } a < r < \infty, \\ u(r, z) &\rightarrow 0 & \text{as } z \rightarrow \infty \text{ for all } r, \end{aligned}$$

where u_0 is constant. Show that the solution is

$$u(r, z) = \left(\frac{2au_0}{\pi} \right) \int_0^\infty k J_0(kr) \left(\frac{\sin ak}{k^2} \right) e^{-kz} dk.$$

Solution. In order to find the solution to the boundary value problem, we will apply the 0-th order Hankel transform to the system of differential equations.

Let $\tilde{u}_0(k, z) = \mathcal{H}_0 \{u(r, z)\}$. Then from a previous theorem we have that

$$\mathcal{H}_0 \left\{ u_{rr} + \frac{1}{r}u_r \right\} = -k^2 \tilde{u}_0(k, z). \quad (1)$$

Thus, from the above result in combination with Leibniz's integral rule, we see that applying the 0-th order Hankel transform to the boundary value problem yields

$$\frac{d^2}{dz^2} [\tilde{u}_0(k, z)] - k^2 \tilde{u}_0(k, z) = 0, \quad 0 < r < \infty, \quad 0 < z < \infty.$$

This is a homogeneous linear ordinary differential equation and we readily see that the solution to the equation is

$$\tilde{u}_0(k, z) = c_1(k)e^{-kz} + c_2(k)e^{kz}. \quad (2)$$

Note that if $u(r, z) \rightarrow 0$ as $z \rightarrow \infty$ for all r , then $\tilde{u}_0(k, z) \rightarrow 0$ as $z \rightarrow \infty$ for all k . Thus, if $\tilde{u}_0(k, z)$ is of the form (2), then $\tilde{u}_0(k, z) \rightarrow 0$ as $z \rightarrow \infty$ for all k if and only if $c_2(k) = 0$. Thus, (2) reduces to

$$\tilde{u}_0(k, z) = c_1(k)e^{-kz}. \quad (3)$$

Thus, taking the inverse 0-th order Hankel transform of (3), we see that the solution to the original differential equation is

$$u(r, z) = \mathcal{H}_0^{-1} \{ \tilde{u}_0(k, z) \} = \int_0^\infty k c_1(k) J_0(kr) e^{-kz} dk.$$

From this solution the boundary conditions become

$$\begin{aligned} \int_0^\infty k c_1(k) J_0(kr) dk &= u_0 \\ \int_0^\infty k^2 c_1(k) J_0(kr) dk &= 0. \end{aligned}$$

Using our table of Hankel transforms, we see that

$$\begin{aligned}\mathcal{H}_0 \left\{ \frac{\sin ak}{k^2} \right\} &= \int_0^\infty k \left(\frac{\sin ak}{k^2} \right) J_0(ak) dk = \begin{cases} \frac{\pi}{2} & \text{if } k < a \\ \sin^{-1} \left(\frac{a}{k} \right) & \text{if } k > a \end{cases} \\ \mathcal{H}_0 \left\{ \frac{\sin ak}{k} \right\} &= \int_0^\infty k \left(\frac{\sin ak}{k} \right) J_0(ak) dk = \begin{cases} (a^2 - k^2)^{-1/2} & \text{if } k < a \\ 0 & \text{if } k > a \end{cases}\end{aligned}$$

These transforms imply that the solution to the system of integral equations resulting from the boundary conditions is

$$c_1(k) = \left(\frac{2u_0}{\pi} \right) \frac{\sin ak}{k^2}.$$

Therefore, the solution to the transformed equation is

$$\tilde{u}_0(k, z) = \left(\frac{2u_0}{\pi} \right) \left(\frac{\sin ak}{k^2} \right) e^{-kz}$$

and the solution to the original equation is

$$u(r, z) = \mathcal{H}_0^{-1} \{ \tilde{u}_0(k, z) \} = \int_0^\infty k \left(\frac{2u_0}{\pi} \right) \left(\frac{\sin ak}{k^2} \right) e^{-kz} J_0(rk) dk.$$

□

Problem 7.12. Solve the Cauchy problem for the wave equation in a dissipating medium

$$u_{tt} + 2\kappa u_t = c^2 \left(u_{rr} + \frac{1}{r} u_r \right), \quad 0 < r < \infty, \quad 0 < t,$$

$$u(r, 0) = f(r), \quad u_t(r, 0) = g(r), \quad 0 < r < \infty.$$

where κ is a constant.

Solution. We begin by applying the 0-th order Hankel transform to the first equation. Letting $\tilde{u}_0(k, t) = \mathcal{H}_0 \{u(r, t)\}$ and using (1), we see this results in the following transformed equation

$$\frac{d^2}{dt^2} [\tilde{u}_0(k, t)] + 2\kappa \frac{d}{dt} [\tilde{u}_0(k, t)] = -(kc)^2 \tilde{u}_0(k, t),$$

or, equivalently,

$$\frac{d^2}{dt^2} [\tilde{u}_0(k, t)] + 2\kappa \frac{d}{dt} [\tilde{u}_0(k, t)] + (kc)^2 \tilde{u}_0(k, t) = 0.$$

This is a homogeneous, linear ordinary differential equation, the solution to which we readily see is

$$\tilde{u}_0(k, t) = c_1 e^{(-\kappa - \sqrt{\kappa^2 - (ck)^2})t} + c_2 e^{(-\kappa + \sqrt{\kappa^2 - (ck)^2})t}. \quad (4)$$

Taking the 0-th order Hankel transform of the boundary conditions, we see that

$$\tilde{u}_0(k, 0) = \tilde{f}_0(k), \quad 0 < r < \infty$$

$$\frac{d}{dt} [\tilde{u}_0(k, 0)] = \tilde{g}_0(k), \quad 0 < r < \infty$$

Using the solution (4) and the first transformed boundary condition, we see that

$$c_1 + c_2 = \tilde{f}_0(k).$$

Similarly, using the solution (4) and the second transformed initial condition, we see that

$$\left(-\kappa - \sqrt{\kappa^2 - (ck)^2} \right) c_1 + \left(-\kappa + \sqrt{\kappa^2 - (ck)^2} \right) c_2 = \tilde{g}_0(k).$$

Solving the resulting system of equation for c_1 and c_2 shows that

$$c_1 = -\frac{\tilde{g}_0(k) + \tilde{f}_0(k)\kappa - \tilde{f}_0(k)\sqrt{\kappa^2 - (ck)^2}}{2\sqrt{\kappa^2 - (ck)^2}}$$

$$c_2 = \frac{\tilde{g}_0(k) + \tilde{f}_0(k)\kappa + \tilde{f}_0(k)\sqrt{\kappa^2 - (ck)^2}}{2\sqrt{\kappa^2 - (ck)^2}}.$$

Thus, letting $c_1(k) = c_1$ and $c_2(k) = c_2$, the solution to the transformed differential equation with the specified initial conditions is

$$\tilde{u}_0(k, t) = c_1(k) e^{(-\kappa - \sqrt{\kappa^2 - (ck)^2})t} + c_2(k) e^{(-\kappa + \sqrt{\kappa^2 - (ck)^2})t}.$$

Therefore, the solution to the original differential equation satisfying the specified initial conditions is

$$u(r, t) = \mathcal{H}_0^{-1} \{ \tilde{u}_0(k, t) \} = \int_0^\infty k J_0(kr) \left[c_1(k) e^{(-\kappa - \sqrt{\kappa^2 - (ck)^2})t} + c_2(k) e^{(-\kappa + \sqrt{\kappa^2 - (ck)^2})t} \right] dk.$$

□

Problem 7.14. Find the steady temperature $u(r, z)$ on a beam, $0 \leq r < \infty$, $0 \leq z \leq a$ when the face $z = 0$ is kept at temperature $u(r, 0) = 0$ and the face $z = a$ is insulated except that the heat is supplied through a circular hole such that

$$u_z(r, a) = H(b - r).$$

The temperature $u(r, z)$ satisfies the axisymmetric equation

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = 0, \quad 0 \leq r < \infty, \quad 0 \leq z \leq a.$$

Solution. The temperature described above satisfies the following partial differential equation with associated boundary conditions

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + u_{zz} &= 0, & 0 \leq r < \infty, & \quad 0 \leq z \leq a, \\ u(r, 0) &= 0, & 0 \leq r < \infty, \\ u_z(r, a) &= H(b - r), & 0 \leq r < \infty. \end{aligned}$$

Let $\tilde{u}_0(k, z) = \mathcal{H}_0 \{u(r, z)\}$ be the 0-th order Hankel transform of $u(r, z)$. Now, applying the 0-th order Hankel transform to the differential equation yields that

$$\frac{d^2}{dz^2} [\tilde{u}_0(k, z)] - k^2 \tilde{u}_0(k, z) = 0.$$

The resulting equation is a homogeneous linear ordinary differential equation, the solution to which is

$$\tilde{u}_0(k, z) = c_1(k)e^{-kz} + c_2(k)e^{kz}. \quad (5)$$

Applying the 0-th order Hankel transform to the boundary conditions yields

$$\begin{aligned} \tilde{u}_0(k, 0) &= 0, & 0 \leq k < \infty, \\ \frac{d}{dz} [\tilde{u}_0(k, z)] \Big|_{z=a} &= \frac{b}{k} J_1(bk), & 0 \leq k < \infty. \end{aligned}$$

Using the solution (5) and the transformed boundary conditions, we see that

$$\begin{aligned} c_1(k) + c_2(k) &= 0 \\ -c_1(k)ke^{-ka} + c_2(k)ke^{ka} &= \frac{b}{k} J_1(bk). \end{aligned}$$

Solving this system of equation we see that

$$\begin{aligned} c_1(k) &= -\frac{be^{ak} J_1(bk)}{k(1 + e^{2ak})} \\ c_2(k) &= \frac{be^{ak} J_1(bk)}{k(1 + e^{2ak})}. \end{aligned}$$

Thus, the solution to the transformed differential equation is

$$\begin{aligned}
 \tilde{u}_0(k, z) &= \left[-\frac{be^{ak} J_1(bk)}{k(1+e^{2ak})} \right] e^{-kz} + \left[\frac{be^{ak} J_1(bk)}{k(1+e^{2ak})} \right] e^{kz} \\
 &= \frac{bJ_1(bk)e^{k(a-z)}(-1+e^{2kz})}{k(1+e^{2ak})} \\
 &= \left(\frac{bJ_1(bk)}{k} \right) \frac{\sinh kz}{\cosh ka}.
 \end{aligned}$$

Therefore, the solution to the original differential equation is

$$\begin{aligned}
 u(r, z) &= \mathcal{H}_0^{-1} \{ \tilde{u}_0(k, z) \} = \int_0^\infty k \left(\frac{bJ_1(bk)}{k} \right) \frac{\sinh kz}{\cosh ka} J_0(kr) dk \\
 &= \int_0^\infty (bJ_1(bk)) \frac{\sinh kz}{\cosh ka} J_0(kr) dk.
 \end{aligned}$$

□

Problem 7.19.*Solution.*