Homework Assignment 10

Matthew Tiger

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Problem 12.1. Find the Z-transform of the following functions:

a. $f(n) = n^3$,

b. $f(n) = \dfrac{a^n}{n!}$

Solution. For a function $f(n)$, the Z-transform of $f(n)$ is defined as

$$Z\{f(n)\} = F(z) = \sum_{n=0}^{\infty} f(n) z^{-n}.$$

a. Let $g(n) = n^2$ and $f(n) = n^3 = ng(n)$. From our table of Z-transforms we know that

$$G(z) = Z\{g(n)\} = \sum_{n=0}^{\infty} n^2 z^{-n} = \frac{z(z+1)}{(z-1)^3}$$

given that $|z| > 1$. The multiplication theorem states that if $F(z) = Z\{f(n)\}$, then

$$Z\{nf(n)\} = -z\frac{d}{dz}\left[F(z)\right].$$

Thus, we have that

$$\begin{aligned} F(z) = Z\{f(n)\} &= Z\{ng(n)\} \\ &= -z\frac{d}{dz}\left[\frac{z(z+1)}{(z-1)^3}\right] \\ &= \frac{z(z^2+4z+1)}{(z-1)^4} \end{aligned}$$

b. Let $g(n) = \frac{1}{n!}$ and $f(n) = \frac{a^n}{n!} = a^n g(n)$. From our knowledge of infinite series, we know from the definition of the Z-transform that

$$G(z) = Z\{g(n)\} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = e^{\frac{1}{z}}.$$

From the multiplication theorem, if $F(z) = Z\left\{f(n)\right\}$, then

$$Z\left\{a^n f(n)\right\} = F\left(\frac{z}{a}\right).$$

Therefore, we have that

$$F(z) = Z\left\{f(n)\right\} = Z\left\{a^n g(n)\right\} = G\left(\frac{z}{a}\right) = e^{\frac{a}{z}}.$$

$\square$

Problem 12.3. Show that

$$Z\left\{na^n f(n)\right\} = -z\frac{d}{dz}\left[F\left(\frac{z}{a}\right)\right].$$

Solution. Suppose that $G(z) = Z\left\{g(n)\right\}$. Then the multiplication theorem states that

$$Z\left\{a^n g(n)\right\} = G\left(\frac{z}{a}\right) \tag{1}$$

and

$$Z\left\{ng(n)\right\} = -z\frac{d}{dz}\left[G(z)\right]. \tag{2}$$

Let $g(n) = a^n f(n)$ and suppose that $F(z) = Z\left\{f(n)\right\}$. By (1), we have that

$$G(z) = Z\left\{g(n)\right\} = Z\left\{a^n f(n)\right\} = F\left(\frac{z}{a}\right).$$

Therefore, by (2), we have that

$$Z\left\{na^n f(n)\right\} = Z\left\{ng(n)\right\} = -z\frac{d}{dz}\left[G(z)\right] = -z\frac{d}{dz}\left[F\left(\frac{z}{a}\right)\right].$$

$\square$

Problem 12.5. Show that

a. $Z\left\{na^{n-1}\right\} = \sum_{n=0}^{\infty} \frac{z}{(z-a)^2}$.

Solution. a. Let $f(n) = a^{n-1}$. Then from the definition of the Z-transform, we have that

$$\begin{aligned} F(z) = Z\left\{f(n)\right\} &= \sum_{n=0}^{\infty} a^{n-1} z^{-n} \\ &= \frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{z}{a}\right)^{-n} \\ &= \frac{z}{a(z-a)}. \end{aligned}$$

By the multiplication theorem (2), we therefore have that

$$\begin{aligned} Z\left\{na^{n-1}\right\} = Z\left\{nf(n)\right\} &= -z\frac{d}{dz}\left[F(z)\right] \\ &= -z\frac{a(z-a)-za}{a^2(z-a)^2} \\ &= \frac{z}{(z-a)^2}. \end{aligned}$$

□

Problem 12.6.

Solution. □

Problem 12.7.

Solution. $\square$

Problem 12.11.

Solution. □