Homework Assignment 4

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Problem 2.3.1. For each of the following functions, $c = 0$ lies on a periodic cycle. Classify this cycle as attracting, repelling, or neutral (non-hyperbolic). State if it is super attracting.

$$\text{i. } f(x) = \frac{\pi}{2}\cos(x), \quad \text{ii. } g(x) = -\frac{1}{2}x^3 - \frac{3}{2}x^2 + 1.$$

Solution. Recall that if c is a point of period r, then c is stable, asymptotically stable, unstable, if $f^r(c)$ is stable, asymptotically stable, unstable, respectively. Thus, if c is a point of period r and $f'(x)$ is continuous at $x = c$, then c is asymptotically stable (attracting) if

$$\left|(f^r(c))'\right| = \left|f'(f^0(c)) \cdot f'(f^1(c)) \cdot \dots \cdot f'(f^{r-1}(c))\right| < 1$$

and c is unstable (repelling) if

$$\left|(f^r(c))'\right| = \left|f'(f^0(c)) \cdot f'(f^1(c)) \cdot \dots \cdot f'(f^{r-1}(c))\right| > 1.$$

i. Let $f(x) = \frac{\pi}{2}\cos(x)$. It is clear that $f^2(0) = 0$ so that $c = 0$ is a period 2 point and $\{0, f(0)\}$ forms a 2-cycle. Note that $f'(x) = -\frac{\pi}{2}\sin(x)$, which is continuous, and that

$$|f'(0) \cdot f'(f(0))| = \left|\left(-\frac{\pi}{2}\sin(0)\right)\left(-\frac{\pi}{2}\sin\left(\frac{\pi}{2}\right)\right)\right| = 0 < 1$$

so that the 2-cycle $\{0, f(0)\}$ is asymptotically stable. Since

$$(f^2(0))' = (f(f(0)))' = f'(0) \cdot f'(f(0)) = 0,$$

we have that $c = 0$ is a super-attracting point of f^2 and the 2-cycle $\{0, f(0)\}$ is a super-attracting, asymptotically stable cycle.

ii. Let $g(x) = -\frac{1}{2}x^3 - \frac{3}{2}x^2 + 1$. It is clear that $g^3(0) = 0$ so that $c = 0$ is a period 3 point and $\{0, g(0), g^2(0)\}$ forms a 3-cycle. Note that $g'(x) = -\frac{3}{2}x^2 - 3x$, which is continuous, and that

$$\left|g'(0) \cdot g'(g(0)) \cdot g'(g^2(0))\right| = \left|0\left(-\frac{9}{2}\right)\left(\frac{3}{2}\right)\right| = 0 < 1$$

so that the 2-cycle $\{0, g(0), g^2(0)\}$ is asymptotically stable. Since

$$(g^3(0))' = (g(g(g(0))))' = g'(0) \cdot g'(g(0)) \cdot g'(g^2(0)) = 0,$$

we have that $c = 0$ is a super-attracting point of g^3 and the 3-cycle $\{0, g(0), g^2(0)\}$ is a super-attracting, asymptotically stable cycle.

$\square$

Problem 2.3.2. Let $f_c(x) = x^2 + c$. Show that for $c < -3/4$, f_c has a 2-cycle, and find it explicitly. For what values of c is the 2-cycle attracting?

Solution. Note that f_c has a 2-cycle if it has a period 2 point, i.e. if $f_c^2(x) - x = 0$ has a solution $x = x_0$ with $f_c(x_0) - x_0 \neq 0$. Thus, we must have that

$$f_c^2(x) - x = (x^2 + c)^2 + c - x = x^4 + 2cx^2 - x + c^2 + c = 0 \tag{1}$$

has a solution. As was shown earlier, $x = \left(1 \pm \sqrt{-4c}\right)/2$ are fixed points of f_c and thus must satisfy $f_c^2(x) - x = 0$. This allows to easily factor (1) and we see that

$$x^4 + 2cx^2 - x + c^2 + c = \left(x - \frac{1 + \sqrt{-4c}}{2}\right)\left(x - \frac{1 - \sqrt{-4c}}{2}\right)(x^2 + x + c + 1).$$

Since a period 2 point x_0 is such that $f_c(x_0) - x_0 \neq 0$, we know that

$$\left(x_0 - \frac{1 + \sqrt{-4c}}{2}\right) \neq 0, \quad \left(x_0 - \frac{1 - \sqrt{-4c}}{2}\right) \neq 0$$

so that $x^4 + 2cx^2 - x + c^2 + c = 0$ only if $x^2 + x + c + 1 = 0$. We readily see that since $c < -3/4$, the polynomial $x^2 + x + c + 1$ has real solutions, and that

$$x^2 + x + c + 1 = \left(x - \frac{-1 + \sqrt{-3 - 4c}}{2}\right)\left(x - \frac{-1 - \sqrt{-3 - 4c}}{2}\right)$$

from which we identify the 2-cycle of f_c as

$$\{c_0, f_c(c_0)\} = \left\{\frac{-1 + \sqrt{-3 - 4c}}{2}, \frac{-1 - \sqrt{-3 - 4c}}{2}\right\}.$$

This 2-cycle will be attracting for f_c if c_0 is attracting for f_c^2, i.e. if

$$\left|\left(f_c^2(c_0)\right)'\right| = |f_c'(c_0) f_c'(f_c(c_0))| < 1.$$

Note that $f_c'(x) = 2x$ from which we see that

$$|f_c'(c_0) f_c'(f_c(c_0))| = \left|\left(-1 + \sqrt{-3 - 4c}\right)\left(-1 - \sqrt{-3 - 4c}\right)\right| = |4(1 + c)|$$

Therefore, the 2-cycle of f_c is attracting if $|4(1 + c)| < 1$, which occurs if and only if $-5/4 < c < -3/4$.

$\square$

Problem 2.3.3. Let $a, b, c \in \mathbb{R}$. Investigate the existence of 2-cycles for the following maps:

i. $f(x) = ax + b$, $a \neq 0$.

ii. $f(x) = ax^2 - x + c$, $a, c > 0$.

iii. $f(x) = a - \frac{b}{x}$, $b \neq 0$.

iv. $f(x) = \frac{ax+b}{cx-a}$, $b \neq 0$.

Solution. $\square$

Problem 2.3.4. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous.

i. If f has a 2-cycle $\{x_0, x_1\}$, show that f has a fixed point.

ii. If f has a 3-cycle $\{x_0, x_1, x_2\}$, $x_0 < x_1 < x_2$ with $f(x_0) = x_1$, $f(x_1) = x_2$, and $f(x_2) = x_0$, show that there is a fixed point y_0 with $x_1 < y_0 < x_2$ and a point y_1 with $x_0 < y_1 < x_1$ with $f^2(y_1) = y_1$.

Solution. □

Problem 2.3.7. Let $f(x) = ax^3 + bx + 1$, $a \neq 0$. If $\{0, 1\}$ is a 2-cycle for $f(x)$, find a and b so that the 2-cycle is non-hyperbolic and determine the stability.

Solution. $\square$

Problem 2.3.17. Suppose that $f(x) = ax^2 + bx + c$, $a \neq 0$ has a 2-cycle $\{x_0, x_1\}$. Show that the 2-cycle cannot be non-hyperbolic of the type $f'(x_0)f'(x_1) = 1$.

Solution. $\square$

Problem 2.3.18. Let $f(x)$ be a polynomial for which $g(x) = f^2(x) - x$ has a repeated root at x_0 (where $f(x_0) = x_1 \neq x_0$).Show that $\{x_0, x_1\}$ is a non-hyperbolic 2-cycle for f of the type where $f'(x_0)f'(x_1) = 1$. Does the converse hold?

Solution. □

Problem 2.4.1. Let $f_c(x) = x^2 + c$, $c \in \mathbb{R}$.

i. For what values of c does f_c have a super-attracting fixed point and what is the fixed point?

ii. For what values of c does f_c have a super-attracting 2-cycle and what is the 2-cycle?

iii. Show that if f_c has a super-attracting 3-cycle, then c satisfies the equation

$$c^3 + 2c^2 + c + 1 = 0$$

and the 3-cycle is given by $\{0, c, c^2 + c\}$.

Solution. $\square$