Homework Assignment 5

Matthew Tiger

October 7, 2016

Problem 2.6.9. i. Use the results of section 2.6 to show that the logistic map $L_4(x) = 4x(1-x)$ cannot have a super-attracting cycle.

ii. Find a point $x_0 \in (0, 1)$ which is not a periodic point for L_4.

Solution. i. Suppose that $k > 1$ and x_k is a period k point so that $\{x_1, x_2, \ldots, x_k\}$ is a k-cycle with $L_4^i(x_1) = x_i$ for $0 < i < k$ and $L_4^k(x_1) = x_1$. This cycle will be super-attracting if

$$\prod_{i=1}^{k} L_4'(x_i) = 0.$$

Note that $L_4'(x) = 4 - 8x = 0$ only if $x = 1/2$. Thus, the cycle will be super attracting if and only if $x_i = 1/2$ for some $i = 1, \ldots k$. Note that the point $x = 1/2$ does not generate a cycle since $L_4(1/2) = 1$ and $L_4^n(1/2) = 0$ for $n > 1$ so $x_1 \neq 1/2$.

We will now demonstrate that there is no point $x \in [0, 1]$ such that $L_4^n(x) = 1/2$ for $n > 0$. It has been shown previously that

$$L_4^n(x) = \sin^2\left(2^n \sin^{-1}\left(\sqrt{x}\right)\right) = \sin^2(\theta)$$

for some $\theta \in (0, \pi/2]$. Note that for $\theta_1, \theta_2 \in (0, \pi/2]$, we have that $\sin^2(\theta_1) = 1/2$ if and only if $\theta_1 = \pi/4$ and $\sin^2(\theta_2) = \pi/4$ if and only if $\theta_2 = \sin^{-1}(\sqrt{\pi}/2) > 1$. However, since $\theta_2 > 1$, there is no $\theta \in (0, \pi/2]$ such that $\sin^2(\theta) = \theta_2$.

So there is no $x \in [0, 1]$ such that $L_4^n(x) = \theta_2$ for any $n > 0$ and hence no $n > 0$ such that $L_4^n(x) = 1/2$. Thus, $x_i = L_4^i(x_1) \neq 1/2$ for any $i > 0$ so that $L_4'(x_i) \neq 0$. Therefore, L_4 has no super-attracting cycle.

ii. As was shown previously, $x = 1/2$ is such that $L_4(x) = 1$ and $L_4^n(x) = 0 \neq 1/2$ for $n > 1$. Therefore $x = 1/2$ is not a periodic point of L_4.

□

Problem 2.8.3. Show that

$$\left\{\frac{\mu}{1+\mu^3}, \frac{\mu^2}{1+\mu^3}, \frac{\mu^3}{1+\mu^3}\right\}$$

is a 3-cycle for T_μ when $\mu \geq (1+\sqrt{5})/2$.

Solution. The tent map is defined as

$$T_\mu(x) := \begin{cases} \mu x & 0 \leq x \leq 1/2 \\ \mu(1-x) & 1/2 < x \leq 1 \end{cases}.$$

Now suppose that $\mu \geq (1+\sqrt{5})/2 > 1$ and let $x_1 = \frac{\mu}{1+\mu^3}$. We will now show that

$$T_\mu(x_1) = \frac{\mu^2}{1+\mu^3} = x_2, \quad T_\mu^2(x_1) = T_\mu(x_2) = \frac{\mu^3}{1+\mu^3} = x_3, \quad T_\mu^3(x_1) = T_\mu(x_3) = \frac{\mu}{1+\mu^3} = x_1$$

demonstrating that $\{x_1, x_2, x_3\}$ is a 3-cycle.

Note that $\mu/(1+\mu^3)$ is monotonically decreasing if $\mu \geq (1+\sqrt{5})/2$. Thus,

$$0 \leq \frac{\mu}{1+\mu^3} \leq \frac{\frac{1+\sqrt{5}}{2}}{1+\left(\frac{1+\sqrt{5}}{2}\right)^3} = \frac{-1+\sqrt{5}}{4} \leq \frac{1}{2}.$$

Hence,

$$T_\mu(x_1) = \mu\left(\frac{\mu}{1+\mu^3}\right) = \frac{\mu^2}{1+\mu^3} = x_2.$$

Similarly, we see that $\mu^2/(1+\mu^3)$ is monotonically decreasing if $\mu \geq (1+\sqrt{5})/2$ so

$$0 \leq \frac{\mu^2}{1+\mu^3} \leq \frac{\left(\frac{1+\sqrt{5}}{2}\right)^2}{1+\left(\frac{1+\sqrt{5}}{2}\right)^3} = \frac{1}{2}.$$

Thus,

$$T_\mu(x_2) = \mu\left(\frac{\mu^2}{1+\mu^3}\right) = \frac{\mu^3}{1+\mu^3} = x_3.$$

Lastly, if $\mu \geq (1+\sqrt{5})/2$ then $\mu^3/(1+\mu^3)$ is monotonically increasing so that

$$\frac{1}{2} \leq \frac{1+\sqrt{5}}{4} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^3}{1+\left(\frac{1+\sqrt{5}}{2}\right)^3} \leq \frac{\mu^3}{1+\mu^3} \leq 1$$

Therefore,

$$T_\mu(x_3) = \mu\left(1 - \frac{\mu^3}{1+\mu^3}\right) = \frac{\mu}{1+\mu^3} = x_1$$

and $\{x_1, x_2, x_3\}$ is a 3-cycle.

$\square$

Problem 3.2.5. Show that the map $f(x) = (x - 1/x)/2$, $x \neq 0$, has no fixed points but it has period 2-points. Find the 2-cycle, and by looking at the graph of $f^3(x)$, check to see whether or not it has a 3-cycle. Why does this not contradict Sharkovskys Theorem?

Solution. The function f will have a fixed point if and only if the function $g(x) = f(x) - x$ has real roots. We see that

$$g(x) = f(x) - x = \frac{x^2 - 1}{2x} - x = -\frac{x^2 + 1}{2x} = 0$$

if and only if $x^2 + 1 = 0$. Thus, g has no real roots and f has no fixed points.

If $h(x) = f^2(x) - x$ has real solutions, then these solutions give rise to a 2-cycle of f. Thus,

$$h(x) = f(f(x)) - x = \frac{\left(\frac{x^2-1}{2x}\right)^2 - 1}{2\left(\frac{x^2-1}{2x}\right)} - x = -\frac{-3x^4 - 2x^2 + 1}{-4x^3 + 4x} = 0$$

if and only if $-3x^4 - 2x^2 + 1 = 0$, the real solutions of which are given by $3^{-1/2}$ and $-3^{-1/2}$. Hence, $\{3^{-1/2}, -3^{-1/2}\}$ is a 2-cycle of f.

The graphs of $f^3(x)$ and $y = x$ are shown in Figure 1. From these graphs we see that the graph of $f^3(x)$ crosses the line $y = x$ at 6 points. Since f has a 2-cycle, 2 of these points arise from the fact that the solutions of $f^2(x) = x$ also satisfy $f^3(x) = x$. However, of the four remaining points, the graph of $f^3(x)$ does not cross the graph $y = x$ at exactly 3 points so a 3-cycle does not arise for $f(x)$.

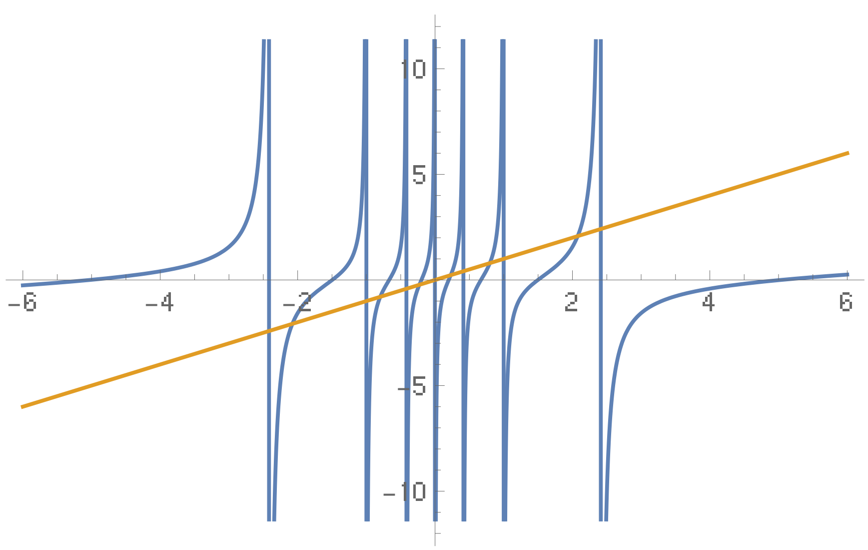

Figure 1: The graphs of $f^3(x)$ (blue) and $y = x$ (orange).

Note that the domain of f is given by $(-\infty, 0) \cup (0, \infty)$. Since the domain of f is not an interval, it does not satisfy the assumptions of Sharkovsky's Theorem and thus the theorem does not apply.

□

Problem 3.2.6. A map $f : [1, 7] \to [1, 7]$ is defined so that $f(1) = 4$, $f(2) = 7$, $f(3) = 6$, $f(4) = 5$, $f(5) = 3$, $f(6) = 2$, $f(7) = 1$, and the corresponding points are joined so the map is continuous and piece-wise linear. Show that f has a 7-cycle but no 5-cycle.

Solution. □

Problem 3.2.10. Let $f : \mathbb{R} \to \mathbb{R}$. Write down all the possibilities for a 4-cycle $\{a, b, c, d\}$ with $a < b < c < d$ for f (e.g. $f(a) = c$, $f(c) = d$, $f(d) = b$, and $f(b) = a$). Indicate which are mirror images, and which give rise to a 3-cycle.

Solution. $\square$

Problem 3.2.11. Use Sharkovskys Theorem to prove that if $f : [a, b] \to [a, b]$ is a continuous function and $\lim_n f^n(x)$ exists for every $x \in [a, b]$, then f can have no points of period $n > 1$.

Solution. □