

Homework Assignment 4

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Problem 1. Find the first three terms in the asymptotic expansions of $x \rightarrow 0^+$ of the following integrals:

$$\int_x^1 \cos(xt) dt, \quad \int_0^{1/x} e^{-t^2} dt.$$

Solution. If the function $f(t, x)$ possesses the asymptotic expansion

$$f(t, x) \sim \sum_{n=0}^{\infty} f_n(t)(x - x_0)^{\alpha n} \quad \text{as } x \rightarrow x_0$$

for some $\alpha > 0$, uniformly for $a \leq t \leq b$, then the asymptotic expansion of the integral

$$I(x) = \int_a^b f(t, x) dt$$

as $x \rightarrow x_0$ is given by

$$I(x) \sim \sum_{n=0}^{\infty} (x - x_0)^{\alpha n} \int_a^b f_n(t) dt \quad \text{as } x \rightarrow x_0.$$

We begin with finding the first three terms of the asymptotic expansion of the integral

$$I_1(x) = \int_x^1 \cos(xt) dt \quad \text{as } x \rightarrow 0^+.$$

Note that $f(t, x) = \cos(xt)$ has the following asymptotic expansion as $x \rightarrow 0^+$:

$$f(t, x) = \cos(xt) \sim 1 - \frac{t^2 x^2}{2} + \frac{t^4 x^4}{24}.$$

This expansion converges uniformly for all $x \leq t \leq 1$ as $x \rightarrow 0^+$. Therefore, we have that the first three terms of the asymptotic expansion of $I_1(x)$ as $x \rightarrow 0^+$ are given by

$$I_1(x) \sim \int_x^1 dt - \frac{x^2}{2} \int_x^1 t^2 dt + \frac{x^4}{24} \int_x^1 t^4 dt = (1 - x) - \frac{x^2}{2} \left[\frac{1 - x^3}{3} \right] + \frac{x^4}{24} \left[\frac{1 - x^5}{5} \right].$$

Let us continue by finding the first three terms of the asymptotic expansion of the integral

$$I_2(x) = \int_0^{1/x} e^{-t^2} dt \quad \text{as } x \rightarrow 0^+.$$

Note that $f(t, x) = e^{-t^2}$ has the following asymptotic expansion as $x \rightarrow 0^+$:

$$f(t, x) = e^{-t^2} \sim 1 - t^2 + \frac{t^4}{2}.$$

This expansion converges uniformly for all finite points, so it converges uniformly for $0 \leq t \leq 1/x$ as $x \rightarrow 0^+$. Therefore, we may integrate the expansion term by term and we have that the first three terms of the asymptotic expansion of $I_2(x)$ as $x \rightarrow 0^+$ are given by

$$I_2(x) \sim \int_0^{1/x} dt - \int_0^{1/x} t^2 dt + \frac{1}{2} \int_0^{1/x} t^4 dt = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{10x^5}.$$

□

Problem 2. Find the full asymptotic behavior as $x \rightarrow 0^+$ of the following integral:

$$\int_0^1 \frac{e^{-t}}{1+x^2 t^3} dt$$

Solution.

□

Problem 3. Find the full asymptotic expansion of $\int_0^x \text{Bi}(t)dt$ as $x \rightarrow +\infty$.

Solution.

□

Problem 4. Find the first five terms in the asymptotic expansion as $x \rightarrow +\infty$ of the integral

$$\int_0^{\pi/4} e^{-xt^2} \sqrt{\tan t} dt$$

- a. by using a suitable change of variables and then applying Watson's lemma.
- b. by applying Laplace's method directly to the given integral.

Solution.

□

Problem 5. Use Laplace's method of moving maxima to obtain the first two terms in the asymptotic expansion as $x \rightarrow +\infty$ of the integral

$$\int_0^\infty \exp \left[-t - \frac{x}{\sqrt{t}} \right] dt. \quad (1)$$

Solution.

□

Problem 6. Let $f(x, t)$ be differentiable in x and continuous in (x, t) on $I \times J$, where I and J are intervals, and suppose that there exist functions $g(t)$ and $g_1(t)$ that are integrable on J such that for all $(x, t) \in I \times J$ we have that

$$|f(x, t)| \leq g(t) \quad \text{and} \quad |\partial_x f(x, t)| \leq g_1(t).$$

Then

$$\frac{d}{dx} \int_J f(x, t) dt = \int_J \partial_x f(x, t) dt.$$

- a. Let $0 < a < b < \infty$. Use the above theorem to show that if $x \in (a, b)$, then

$$\frac{d^3}{dx^3} \int_0^\infty \exp \left[-t - \frac{x}{\sqrt{t}} \right] dt = - \int_0^\infty t^{-3/2} \exp \left[-t - \frac{x}{\sqrt{t}} \right] dt.$$

- b. Use integration by parts to show that

$$\int_0^\infty \exp \left[-t - \frac{x}{\sqrt{t}} \right] dt = \frac{x}{2} \int_0^\infty t^{-3/2} \exp \left[-t - \frac{x}{\sqrt{t}} \right] dt.$$

- c. Combine parts (a) and (b) to prove that integral (1) is a solution of the differential equation $xy''' + 2y = 0$ that also satisfies the initial condition $y(0) = 1$. Then use integration by parts to give an easy direct proof that the integral also satisfies the condition $y(+\infty) = 0$.

Solution.

□

Problem 7. a. Find the leading behavior as $x \rightarrow +\infty$ of Laplace integrals of the form

$$\int_a^b (t-a)^\alpha g(t) e^{x\phi(t)} dt$$

where $\phi(t)$ has a maximum at $t = a$, $g(a) = 1$. Suppose further that $\alpha > -1$ and $\phi'(a) < 0$.

b. Repeat the analysis of part (a) when $\alpha > -1$ and $\phi'(a) = \phi''(a) = \dots = \phi^{(p-1)}(a) = 0$ and $\phi^{(p)}(a) < 0$.

Solution.

□