Homework Assignment 10

Matthew Tiger

May 21, 2017

Problem 12.1. Find the Z-transform of the following functions:

a. $f(n) = n^3$,

b. $f(n) = \dfrac{a^n}{n!}$

Solution. For a function $f(n)$, the Z-transform of $f(n)$ is defined as

$$Z\left\{f(n)\right\} = F(z) = \sum_{n=0}^{\infty} f(n) z^{-n}.$$

a. Let $g(n) = n^2$ and $f(n) = n^3 = ng(n)$. From our table of Z-transforms we know that

$$G(z) = Z\left\{g(n)\right\} = \sum_{n=0}^{\infty} n^2 z^{-n} = \frac{z(z+1)}{(z-1)^3}$$

given that $|z| > 1$. The multiplication theorem states that if $F(z) = Z\left\{f(n)\right\}$, then

$$Z\left\{nf(n)\right\} = -z\frac{d}{dz}\left[F(z)\right].$$

Thus, we have that

$$\begin{aligned} F(z) = Z\left\{f(n)\right\} &= Z\left\{ng(n)\right\} \\ &= -z\frac{d}{dz}\left[\frac{z(z+1)}{(z-1)^3}\right] \\ &= \frac{z(z^2+4z+1)}{(z-1)^4} \end{aligned}$$

b. Let $g(n) = \frac{1}{n!}$ and $f(n) = \frac{a^n}{n!} = a^n g(n)$. From our knowledge of infinite series, we know from the definition of the Z-transform that

$$G(z) = Z\left\{g(n)\right\} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = e^{\frac{1}{z}}.$$

From the multiplication theorem, if $F(z) = Z\left\{f(n)\right\}$, then

$$Z\left\{a^n f(n)\right\} = F\left(\frac{z}{a}\right).$$

Therefore, we have that

$$F(z) = Z\left\{f(n)\right\} = Z\left\{a^n g(n)\right\} = G\left(\frac{z}{a}\right) = e^{\frac{a}{z}}.$$

$\square$

Problem 12.3. Show that

$$Z\left\{na^n f(n)\right\} = -z\frac{d}{dz}\left[F\left(\frac{z}{a}\right)\right].$$

Solution. Suppose that $G(z) = Z\left\{g(n)\right\}$. Then the multiplication theorem states that

$$Z\left\{a^n g(n)\right\} = G\left(\frac{z}{a}\right) \tag{1}$$

and

$$Z\left\{ng(n)\right\} = -z\frac{d}{dz}\left[G(z)\right]. \tag{2}$$

Let $g(n) = a^n f(n)$ and suppose that $F(z) = Z\left\{f(n)\right\}$. By (1), we have that

$$G(z) = Z\left\{g(n)\right\} = Z\left\{a^n f(n)\right\} = F\left(\frac{z}{a}\right).$$

Therefore, by (2), we have that

$$Z\left\{na^n f(n)\right\} = Z\left\{ng(n)\right\} = -z\frac{d}{dz}\left[G(z)\right] = -z\frac{d}{dz}\left[F\left(\frac{z}{a}\right)\right].$$

$\square$

Problem 12.5. Show that

a. $Z\left\{na^{n-1}\right\} = \sum_{n=0}^{\infty} \frac{z}{(z-a)^2}$.

Solution. a. Let $f(n) = a^{n-1}$. Then from the definition of the Z-transform, we have that

$$\begin{aligned} F(z) = Z\{f(n)\} &= \sum_{n=0}^{\infty} a^{n-1} z^{-n} \\ &= \frac{1}{a} \sum_{n=0}^{\infty} \left(\frac{z}{a}\right)^{-n} \\ &= \frac{z}{a(z-a)}. \end{aligned}$$

By the multiplication theorem (2), we therefore have that

$$\begin{aligned} Z\left\{na^{n-1}\right\} = Z\{nf(n)\} &= -z\frac{d}{dz}\left[F(z)\right] \\ &= -z\frac{a(z-a) - za}{a^2(z-a)^2} \\ &= \frac{z}{(z-a)^2}. \end{aligned}$$

$\square$

Problem 12.6. Find the inverse Z-transform of the following functions:

a. $F(z) = \dfrac{z^2}{(z-2)(z-3)}$,

e. $F(z) = \dfrac{1}{(z-a)^2}$.

Solution. a. Let $G(z) = \dfrac{z}{z-3}$ and $H(z) = \dfrac{z}{z-2}$. Recall that

$$Z^{-1}\left\{\frac{z}{z-a}\right\} = a^n.$$

Thus, we see that $g(n) = Z^{-1}\{G(z)\} = 3^n$ and $h(n) = Z^{-1}\{H(z)\} = 2^n$. The Convolution theorem states that if $G(z) = Z\{g(n)\}$ and $H(z) = Z\{h(n)\}$, then

$$(g * h)(n) = \sum_{m=0}^{n} g(m)h(n-m) = Z^{-1}\{G(z)H(z)\}. \tag{3}$$

Therefore, by the Convolution theorem (3), we have that

$$\begin{aligned} Z^{-1}\left\{\frac{z^2}{(z-2)(z-3)}\right\} = Z^{-1}\{F(z)\} &= Z^{-1}\{G(z)H(z)\} \\ &= \sum_{m=0}^{n} 3^m 2^{n-m} \\ &= 2^n \sum_{m=0}^{n} \left(\frac{3}{2}\right)^m \\ &= 2^n \left(3^{n+1}2^{-n} - 2\right) \\ &= 3^{n+1} - 2^{n+1}. \end{aligned}$$

e. Let $G(z) = \dfrac{1}{z-a}$. Then we know that

$$\begin{aligned} G(z) = \frac{1}{z-a} &= \frac{1}{z}\left(\frac{z}{z-a}\right) \\ &= \frac{1}{z}\sum_{n=0}^{\infty} a^n z^{-n} \\ &= \sum_{n=0}^{\infty} a^n z^{-(n+1)} \end{aligned}$$

□

Problem 12.7. Solve the following difference equations:

a. $f(n+1) + 3f(n) = n, \qquad f(0) = 1.$

e. $f(n+2) - f(n+1) - 6f(n) = 0, \qquad f(0) = 0, \quad f(1) = 3$

Solution. Recall that if $Z\{f(n)\} = F(z)$ and $m \geq 0$, then the following property holds:

$$Z\{f(n+m)\} = z^m \left[F(z) - \sum_{r=0}^{m-1} f(r) z^{-r} \right].$$

a. Applying the Z-transform to the difference equation, we have that

$$zF(z) - zf(0) + 3F(z) = \frac{z}{(z-1)^2}.$$

In light of the initial data, this reduces to

$$(z+3)F(z) - z = \frac{z}{(z-1)^2}.$$

Solving the resulting algebraic equation yields

$$F(z) = \frac{z(z^2 - 2z + 2)}{(z+3)(z-1)^2}$$

Applying the method of partial fraction decomposition to this transformed function shows that

$$\begin{aligned} F(z) &= \frac{z(z^2 - 2z + 2)}{(z+3)(z-1)^2} \\ &= z \left[\frac{a_1}{z+3} + \frac{a_2}{z-1} + \frac{a_3}{(z-1)^2} \right] \\ &= \frac{17}{16} \left(\frac{z}{z+3} \right) - \frac{1}{16} \left(\frac{z}{z-1} \right) + \frac{1}{4} \left[\frac{z}{(z-1)^2} \right]. \end{aligned}$$

Therefore, using the fact that

$$Z\{a^n\} = \frac{z}{z-a}$$

and

$$Z\{n\} = \frac{z}{(z-1)^2},$$

we see that the solution to the original difference equation is

$$\begin{aligned} f(n) = Z^{-1}\{F(z)\} &= \frac{17}{16} Z^{-1} \left\{ \frac{z}{z+3} \right\} - \frac{1}{16} Z^{-1} \left\{ \frac{z}{z-1} \right\} + \frac{1}{4} Z^{-1} \left\{ \frac{z}{(z-1)^2} \right\} \\ &= \frac{17}{16}(-3)^n - \frac{1}{16} + \frac{1}{4} n \end{aligned}$$

e. Applying the Z-transform to the Initial Value Problem, we have that

$$z^2F(z) - z^2f(0) - zf(1) - zF(z) + zf(0) - 6F(z) = 0.$$

In light of the initial data, this reduces to

$$(z-3)(z+2)F(z) - 3z = 0$$

Thus, the solution to the transformed equation is

$$F(z) = \frac{3z}{(z-3)(z+2)}.$$

Applying the method of partial fraction decomposition to this transformed function shows that

$$\begin{aligned} F(z) &= \frac{3z}{(z-3)(z+2)} \\ &= 3z\left[\frac{a_1}{z-3} + \frac{a_2}{z+2}\right] \\ &= \frac{3}{5}\left[\frac{z}{z-3} - \frac{z}{z+2}\right]. \end{aligned}$$

Therefore, using the fact that

$$Z\{a^n\} = \frac{z}{z-a},$$

we see that the solution to the original difference equation is

$$\begin{aligned} f(n) = Z^{-1}\{F(z)\} &= \frac{3}{5}\left[Z^{-1}\left\{\frac{z}{z-3}\right\} - Z^{-1}\left\{\frac{z}{z+2}\right\}\right] \\ &= \frac{3}{5}\left[3^n - (-2)^n\right]. \end{aligned}$$

$\square$

Problem 12.11. Find the sum of the following series using the Z-transform:

a. $\displaystyle\sum_{n=0}^{\infty} a^n e^{inx}$,

c. $\displaystyle\sum_{n=0}^{\infty} e^{-x(2n+1)}$,

Solution. By a previous theorem, if $Z\{f(n)\} = F(z)$, then

$$\sum_{n=0}^{\infty} f(n) = \lim_{z\to 1} F(z). \tag{4}$$

Thus, in order to compute the above series, we need merely find the Z-transforms of the sequences and then evaluate the above limit.

a. Let $g(n) = e^{inx}$ and $f(n) = a^n g(n)$. From the table of Z-transforms, we know that

$$G(z) = Z\left\{e^{inx}\right\} = \frac{z}{z - e^{ix}}.$$

Thus, from the multiplication theorem, we have that

$$\begin{aligned} F(z) = Z\{a^n g(n)\} &= G\left(\frac{z}{a}\right) \\ &= \frac{z}{z - ae^{ix}}. \end{aligned}$$

Therefore, by (4), we have that

$$\begin{aligned} \sum_{n=0}^{\infty} a^n e^{inx} = \sum_{n=0}^{\infty} f(n) &= \lim_{z\to 1} F(z) \\ &= \lim_{z\to 1} \frac{z}{z - ae^{ix}} \\ &= \frac{1}{1 - ae^{ix}}. \end{aligned}$$

c. Let $f(n) = e^{-x(2n+1)} = e^{-x}e^{-2xn}$. From the table of Z-transforms, we know that

$$F(z) = Z\{f(n)\} = e^{-x} Z\left\{e^{-2xn}\right\} = e^{-x}\left[\frac{z}{z - e^{-2x}}\right].$$

Therefore, we have that

$$\begin{aligned} \sum_{n=0}^{\infty} e^{-x(2n+1)} = \sum_{n=0}^{\infty} f(n) &= \lim_{z\to 1} F(z) \\ &= \lim_{z\to 1} e^{-x}\left[\frac{z}{z - e^{-2x}}\right] \\ &= \frac{e^{-x}}{1 - e^{-2x}}. \end{aligned}$$

$\square$