

# Exam 3

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**Problem 1.** Solve the non-homogeneous diffusion problem by the Hankel transform

$$\begin{aligned} u_t &= a \left( u_{rr} + \frac{1}{r} u_r \right) + Q(r, t), \quad 0 < r < \infty, \quad 0 < t \\ u(r, 0) &= f(r), \quad 0 < r < \infty. \end{aligned}$$

*Solution.* Application of the 0-th order Hankel transform will transform the above Partial Differential Equation into an Ordinary Differential Equation. The following property of the 0-th order Hankel transform will aid in the application; if  $\mathcal{H}_0 \{u(r, t)\} = \tilde{u}_0(\kappa, t)$ , then

$$\mathcal{H}_0 \left\{ \frac{1}{r} \frac{\partial}{\partial r} [u(r, t)] + \frac{\partial^2}{\partial r^2} [u(r, t)] \right\} = -\kappa^2 \tilde{u}_0(\kappa, t). \quad (1)$$

Now, with the above property, we see that applying the 0-th order Hankel transform to the diffusion problem yields

$$\begin{aligned} \frac{d}{dt} [\tilde{u}_0(\kappa, t)] + a\kappa^2 \tilde{u}_0(\kappa, t) &= \tilde{Q}_0(\kappa, t), \quad 0 < \kappa < \infty, \quad 0 < t \\ \tilde{u}_0(\kappa, 0) &= \tilde{f}_0(\kappa), \quad 0 < \kappa < \infty. \end{aligned}$$

This is a first order linear Ordinary Differential Equation, the solution to which is

$$\tilde{u}_0(\kappa, t) = c_1(\kappa) e^{-a\kappa^2 t} + e^{-a\kappa^2 t} \int_0^t e^{a\kappa^2 x} \tilde{Q}_0(\kappa, x) dx.$$

Thus, from this solution and the transformed boundary condition, we see that  $c_1(\kappa) = \tilde{f}_0(\kappa)$  and the solution to the transformed boundary value problem is

$$\tilde{u}_0(\kappa, t) = \tilde{f}_0(\kappa) e^{-a\kappa^2 t} + e^{-a\kappa^2 t} \int_0^t e^{a\kappa^2 x} \tilde{Q}_0(\kappa, x) dx.$$

Therefore, the solution to the initial diffusion problem is

$$\begin{aligned} u(r, t) &= \mathcal{H}_0^{-1} \{ \tilde{u}_0(\kappa, t) \} = \mathcal{H}_0^{-1} \left\{ \tilde{f}_0(\kappa) e^{-a\kappa^2 t} + e^{-a\kappa^2 t} \int_0^t e^{a\kappa^2 x} \tilde{Q}_0(\kappa, x) dx \right\} \\ &= \int_0^\infty \kappa J_0(\kappa r) \left[ \tilde{f}_0(\kappa) e^{-a\kappa^2 t} + e^{-a\kappa^2 t} \int_0^t e^{a\kappa^2 x} \tilde{Q}_0(\kappa, x) dx \right] d\kappa, \end{aligned}$$

where  $J_0(\kappa r)$  is the Bessel function of order 0.

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**Problem 2.***Solution.*

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**Problem 3.** Solve the following integral equation by the Mellin transform

$$f(x) = \sin ax + \int_0^\infty \frac{f(xt)}{1+t^2} dt.$$

*Solution.* Let  $g(x) = \frac{1}{1+x^2}$  and  $h(x) = \sin ax$ . Recall that  $(f \circ g)(x)$  is defined to be

$$(f \circ g)(x) = \int_0^\infty f(xt)g(t)dt.$$

Thus, with this knowledge, the integral equation becomes

$$\begin{aligned} f(x) &= h(x) + \int_0^\infty f(xt)g(t)dt \\ &= h(x) + (f \circ g)(x). \end{aligned}$$

Let  $\mathcal{M}\{f(x)\} = \tilde{f}(p)$ ,  $\mathcal{M}\{g(x)\} = \tilde{g}(p)$ , and  $\mathcal{M}\{h(x)\} = \tilde{h}(p)$ . Then from the Convolution Type theorem regarding the Mellin transform, we see that application of the Mellin transform to the integral equation yields

$$\begin{aligned} \tilde{f}(p) &= \mathcal{M}\{h(x)\} + \mathcal{M}\{(f \circ g)(x)\} \\ &= \tilde{h}(p) + \tilde{f}(p)\tilde{g}(1-p). \end{aligned}$$

Solving the above algebraic equation shows that

$$\tilde{f}(p) = \frac{\tilde{h}(p)}{1 - \tilde{g}(1-p)}.$$

From our table of Mellin transforms we know that

$$\tilde{g}(p) = \frac{\pi}{2} \csc\left(\frac{\pi p}{2}\right)$$

and

$$\tilde{h}(p) = a^{-p}\Gamma(p) \sin\left(\frac{\pi p}{2}\right).$$

Therefore, we see that

$$\begin{aligned} \tilde{f}(p) &= \frac{a^{-p}\Gamma(p) \sin\left(\frac{\pi p}{2}\right)}{1 - \frac{\pi}{2} \csc\left(\frac{\pi(1-p)}{2}\right)} \\ &= \frac{2a^{-p}\Gamma(p) \sin\left(\frac{\pi p}{2}\right)}{2 - \pi \sec\left(\frac{\pi p}{2}\right)} \end{aligned}$$

and the solution to the integral equation is

$$\begin{aligned} f(x) &= \mathcal{M}^{-1}\{\tilde{f}(p)\} = \mathcal{M}^{-1}\left\{\frac{2a^{-p}\Gamma(p) \sin\left(\frac{\pi p}{2}\right)}{2 - \pi \sec\left(\frac{\pi p}{2}\right)}\right\} \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-p} \left[\frac{2a^{-p}\Gamma(p) \sin\left(\frac{\pi p}{2}\right)}{2 - \pi \sec\left(\frac{\pi p}{2}\right)}\right] dp. \end{aligned}$$

□

**Problem 4.** Solve the following Partial Differential Equation by the Mellin transform

$$\begin{aligned} r^2 \phi_{rr} + r \phi_r + \phi_{\theta\theta} &= 0, \quad 0 < r < \infty, \quad 0 < \theta < \pi \\ \phi(r, 0) &= \begin{cases} (1-r)^2 & 0 < r < 1 \\ 0 & 1 < r \end{cases} \\ \phi(r, \pi) &= \begin{cases} 1 & 0 < r < 1 \\ 0 & 1 < r \end{cases}, \end{aligned}$$

*Solution.* Recall that if  $\mathcal{M}\{\phi(r, \theta)\} = \tilde{\phi}(p, \theta)$ , then the following property holds

$$\mathcal{M}\left\{r^2 \frac{\partial^2}{\partial r^2} [\phi(r, \theta)] + r \frac{\partial}{\partial r} [\phi(r, \theta)]\right\} = p^2 \tilde{\phi}(p, \theta).$$

Thus, applying the Mellin transform to the Partial Differential Equation and using our table of Mellin transforms, we see that

$$\begin{aligned} \frac{d^2}{d\theta^2} [\tilde{\phi}(p, \theta)] + p^2 \tilde{\phi}(p, \theta) &= 0, \quad 0 < p < \infty, \quad 0 < \theta < \pi \\ \tilde{\phi}(p, 0) &= \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)}, \quad \tilde{\phi}(p, \pi) = \frac{1}{p}. \end{aligned}$$

The solution to the resulting homogeneous linear Ordinary Differential Equation is

$$\tilde{\phi}(p, \theta) = c_1(p) \cos p\theta + c_2(p) \sin p\theta.$$

Using the above solution and the transformed boundary conditions, we see that

$$\begin{aligned} c_1(p) &= \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)} \\ c_1(p) \cos p\pi + c_2(p) \sin p\pi &= \frac{1}{p}. \end{aligned}$$

Solving, we see that

$$\begin{aligned} c_2(p) &= \left( \frac{1}{p} - \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)} \cos p\pi \right) \csc p\pi \\ &= \frac{\csc p\pi}{p} - \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)} \cot p\pi. \end{aligned}$$

Thus, the solution to the transformed boundary value problem is

$$\tilde{\phi}(p, \theta) = \left[ \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)} \right] \cos p\theta + \left[ \frac{\csc p\pi}{p} - \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)} \cot p\pi \right] \sin p\theta.$$

Therefore, the solution to the original boundary value problem is

$$\begin{aligned} \phi(r, \theta) &= \mathcal{M}^{-1} \left\{ \left[ \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)} \right] \cos p\theta + \left[ \frac{\csc p\pi}{p} - \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)} \cot p\pi \right] \sin p\theta \right\} \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} r^{-p} \left\{ \left[ \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)} \right] \cos p\theta + \left[ \frac{\csc p\pi}{p} - \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)} \cot p\pi \right] \sin p\theta \right\} dp. \end{aligned}$$

□

**Problem 5.** Show that

$$\int_0^\infty e^{-ax} \left( \frac{\cos px - \cos qx}{x} \right) dx = \frac{1}{2} \log \frac{q^2 + a^2}{p^2 + a^2}$$

*Solution.* Recall that the Laplace transform of a function  $f(t)$  is defined as

$$\bar{f}(s) = \mathcal{L} \{f(t)\}_s = \int_0^\infty f(t) e^{-st} dt$$

where we explicitly note the transformation variable  $s$ . Thus, we see that the integral above becomes

$$\begin{aligned} \int_0^\infty e^{-ax} \left( \frac{\cos px - \cos qx}{x} \right) dx &= \int_0^\infty e^{-ax} \left( \frac{\cos px}{x} \right) dx - \int_0^\infty e^{-ax} \left( \frac{\cos qx}{x} \right) dx \\ &= \mathcal{L} \left\{ \frac{\cos px}{x} \right\}_a - \mathcal{L} \left\{ \frac{\cos qx}{x} \right\}_a. \end{aligned} \quad (2)$$

From a previous theorem, if  $\mathcal{L} \{f(t)\} = \bar{f}(s)$ , then we have that

$$\mathcal{L} \left\{ \frac{f(t)}{t} \right\} = \int_s^\infty \bar{f}(z) dz.$$

Thus, we see that (2) becomes

$$\begin{aligned} \int_0^\infty e^{-ax} \left( \frac{\cos px - \cos qx}{x} \right) dx &= \mathcal{L} \left\{ \frac{\cos px}{x} \right\}_a - \mathcal{L} \left\{ \frac{\cos qx}{x} \right\}_a \\ &= \int_a^\infty \mathcal{L} \{ \cos px \}_z dz - \int_a^\infty \mathcal{L} \{ \cos qx \}_z dz \end{aligned}$$

From the table of Laplace transforms, we have that

$$\mathcal{L} \{ \cos bx \}_z = \frac{z}{b^2 + z^2}.$$

Thus, we have that

$$\begin{aligned} \int_0^\infty e^{-ax} \left( \frac{\cos px - \cos qx}{x} \right) dx &= \int_a^\infty \mathcal{L} \{ \cos px \}_z dz - \int_a^\infty \mathcal{L} \{ \cos qx \}_z dz \\ &= \int_a^\infty \frac{z}{p^2 + z^2} dz - \int_a^\infty \frac{z}{q^2 + z^2} dz \end{aligned}$$

Using a  $u$ -substitution to solve the resulting integrals, we have that

$$\begin{aligned} \int_0^\infty e^{-ax} \left( \frac{\cos px - \cos qx}{x} \right) dx &= \int_a^\infty \frac{z}{p^2 + z^2} dz - \int_a^\infty \frac{z}{q^2 + z^2} dz \\ &= \frac{1}{2} \log p^2 + z^2 \Big|_a^\infty - \frac{1}{2} \log p^2 + z^2 \Big|_a^\infty. \end{aligned}$$

Using the properties of logarithms, we therefore see that

$$\begin{aligned}
 \int_0^\infty e^{-ax} \left( \frac{\cos px - \cos qx}{x} \right) dx &= \frac{1}{2} \log p^2 + z^2 \Big|_a^\infty - \frac{1}{2} \log p^2 + z^2 \Big|_a^\infty \\
 &= \frac{1}{2} \left[ \log \frac{p^2 + z^2}{q^2 + z^2} \Big|_{z=\infty} - \log \frac{p^2 + z^2}{q^2 + z^2} \Big|_{z=a} \right] \\
 &= -\frac{1}{2} \log \frac{p^2 + a^2}{q^2 + a^2} \\
 &= \frac{1}{2} \log \frac{q^2 + a^2}{p^2 + a^2}.
 \end{aligned}$$

□

**Problem 6.** Suppose that  $I_n f(x)$  denotes the  $n$ -th repeated integral of  $f(x)$  defined by

$$I_n f(x) = \int_x^\infty I_{n-1} f(t) dt$$

and that  $\mathcal{M}\{f(x)\} = \tilde{f}(p)$ . Show that

- a.  $\mathcal{M}\left\{\int_x^\infty f(t) dt\right\} = \frac{1}{p} \tilde{f}(p+1),$   
 b.  $\mathcal{M}\{I_n f(x)\} = \frac{\Gamma(p)}{\Gamma(p+n)} \tilde{f}(p+n)$

*Solution.* Recall that the Mellin transform of the function  $f(x)$  is defined as

$$\mathcal{M}\{f(x)\} = \int_0^\infty x^{p-1} f(x) dx. \quad (3)$$

a. From the definition of the Mellin transform (3), we see that

$$\mathcal{M}\left\{\int_x^\infty f(t) dt\right\} = \int_0^\infty x^{p-1} \left[\int_x^\infty f(t) dt\right] dx.$$

Interchanging the order of integration from  $t$  to  $x$ , we see that

$$\begin{aligned} \mathcal{M}\left\{\int_x^\infty f(t) dt\right\} &= \int_0^\infty x^{p-1} \left[\int_x^\infty f(t) dt\right] dx \\ &= \int_0^\infty f(t) \left[\int_0^t x^{p-1} dx\right] dt \\ &= \frac{1}{p} \int_0^\infty t^p f(t) dt. \end{aligned}$$

If  $\tilde{f}(p) = \mathcal{M}\{f(x)\}$ , then from the definition of the Mellin transform (3), we see that the above integral becomes

$$\begin{aligned} \mathcal{M}\left\{\int_x^\infty f(t) dt\right\} &= \frac{1}{p} \int_0^\infty t^p f(t) dt \\ &= \frac{1}{p} \tilde{f}(p+1), \end{aligned}$$

and we are done.

b. We will now prove the relation

$$\mathcal{M}\{I_n f(x)\} = \frac{\Gamma(p)}{\Gamma(p+n)} \tilde{f}(p+n)$$

by induction. The results of the previous exercise show that

$$\mathcal{M}\{I_1 f(x)\} = \mathcal{M}\left\{\int_x^\infty f(t) dt\right\} = \frac{1}{p} \tilde{f}(p+1) = \frac{\Gamma(p)}{\Gamma(p+1)} \tilde{f}(p+1)$$

and the base step is established.

Now assume the relation holds for  $n$ , i.e. assume that

$$\mathcal{M} \{I_n f(x)\} = \frac{\Gamma(p)}{\Gamma(p+n)} \tilde{f}(p+n).$$

Now, we see from the definition of the Mellin transform (3) that

$$\begin{aligned} \mathcal{M} \{I_{n+1} f(x)\} &= \mathcal{M} \left\{ \int_x^\infty I_n f(t) dt \right\} \\ &= \int_0^\infty x^{p-1} \left[ \int_x^\infty I_n f(t) dt \right] dx. \end{aligned}$$

Let  $g(t) = I_n f(t)$ . Then, proceeding as we did in establishing the result of the base step, interchanging the order of integration from  $t$  to  $x$  yields

$$\begin{aligned} \mathcal{M} \{I_{n+1} f(x)\} &= \int_0^\infty x^{p-1} \left[ \int_x^\infty g(t) dt \right] dx \\ &= \int_0^\infty g(t) \left[ \int_0^t x^{p-1} dx \right] dt \\ &= \frac{1}{p} \int_0^\infty t^p g(t) dt \\ &= \frac{1}{p} \tilde{g}(p+1) \end{aligned} \tag{4}$$

where  $\tilde{g}(p) = \mathcal{M} \{g(t)\}$ .

From our assumption, we have that

$$\tilde{g}(p) = \mathcal{M} \{g(t)\} = \mathcal{M} \{I_n f(x)\} = \frac{\Gamma(p)}{\Gamma(p+n)} \tilde{f}(p+n).$$

Thus, the integral in (4) becomes

$$\begin{aligned} \mathcal{M} \{I_{n+1} f(x)\} &= \frac{1}{p} \tilde{g}(p+1) \\ &= \frac{\Gamma(p+1)}{p\Gamma(p+n+1)} \tilde{f}(p+n+1) \\ &= \frac{\Gamma(p)}{\Gamma(p+n+1)} \tilde{f}(p+n+1). \end{aligned}$$

Therefore, the result holds for  $n+1$  and the relation holds in general for all  $n > 0$ .

□



**Problem 7.** Solve the following Initial Value Problem by the Z-transform

$$\begin{aligned} f(n+2) - f(n+1) - 6f(n) &= \sin\left(\frac{n\pi}{2}\right), \quad n \geq 2 \\ f(0) &= 0, \quad f(1) = 1. \end{aligned}$$

*Solution.* Recall that if  $Z\{f(n)\} = F(z)$  and  $m \geq 0$ , then the following property holds:

$$Z\{f(n+m)\} = z^m \left[ F(z) - \sum_{r=0}^{m-1} f(r)z^{-r} \right].$$

Thus, applying the Z-transform to the Initial Value Problem, we have that

$$z^2 F(z) - z^2 f(0) - z f(1) - z F(z) + z f(0) - 6 F(z) = Z\left\{\sin\left(\frac{n\pi}{2}\right)\right\}.$$

In light of the initial data, this reduces to

$$(z-3)(z+2)F(z) - z = Z\left\{\sin\left(\frac{n\pi}{2}\right)\right\}.$$

From our table of Z-transforms, we know that

$$Z\left\{\sin\left(\frac{n\pi}{2}\right)\right\} = \frac{z \sin \frac{\pi}{2}}{z^2 - 2z \cos \frac{\pi}{2} + 1} = \frac{z}{z^2 + 1}$$

Thus, the solution to the transformed equation is

$$\begin{aligned} F(z) &= \frac{Z\left\{\sin\left(\frac{n\pi}{2}\right)\right\}}{(z-3)(z+2)} + \frac{z}{(z-3)(z+2)} \\ &= \frac{z}{(z^2+1)(z-3)(z+2)} + \frac{z}{(z-3)(z+2)}. \end{aligned}$$

Applying the method of partial fraction decomposition to this transformed function shows that

$$\begin{aligned} F(z) &= \frac{z}{(z^2+1)(z-3)(z+2)} + \frac{z}{(z-3)(z+2)} \\ &= z \left[ \frac{a_1 z + a_2}{z^2+1} + \frac{a_3}{z-3} + \frac{a_4}{z+2} \right] + z \left[ \frac{b_1}{z-3} + \frac{b_2}{z+2} \right] \\ &= \frac{1}{50} \left[ \frac{z(z-7)}{z^2+1} \right] + \frac{1}{50} \left[ \frac{z}{z-3} \right] - \frac{1}{25} \left[ \frac{z}{z+2} \right] + \frac{1}{5} \left[ \frac{z}{z-3} \right] - \frac{1}{5} \left[ \frac{z}{z+2} \right] \\ &= \frac{1}{50} \left[ \frac{z(z-7)}{z^2+1} \right] + \frac{11}{50} \left[ \frac{z}{z-3} \right] - \frac{6}{25} \left[ \frac{z}{z+2} \right]. \end{aligned}$$

Therefore, using the fact that

$$Z\{a^n\} = \frac{z}{z-a}$$

and a computer algebra system we see that the solution to the original finite difference equation is

$$\begin{aligned} f(n) = Z^{-1} \{F(z)\} &= \frac{1}{50} Z^{-1} \left\{ \frac{z(z-7)}{z^2+1} \right\} + \frac{11}{50} Z^{-1} \left\{ \frac{z}{z-3} \right\} - \frac{6}{25} Z^{-1} \left\{ \frac{z}{z+2} \right\} \\ &= \frac{1}{50} \left[ i^n \left( \frac{1}{2} + \frac{7i}{2} \right) + (-i)^n \left( \frac{1}{2} - \frac{7i}{2} \right) \right] + \frac{11}{50} 3^n - \frac{6}{25} (-2)^n. \end{aligned}$$

□

**Problem 8.***Solution.*

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