Homework Assignment 10

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Problem 10.5.1. Show that the Cantor set C is a fixed point of the map $F : \mathcal{C}(\mathbb{R}) \to \mathcal{C}(\mathbb{R})$ defined by

$$F(A) = f_1(A) \cup f_2(A)$$

where $f_1(x) = x/3$ and $f_2(x) = x/3 + 2/3$ are contractions on $\mathbb{R}$.

Solution. Recall that the Cantor set C is defined as

$$C = \left\{ x \in [0,1] \;\middle|\; x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, \quad a_n \in \{0,2\} \right\}. \tag{1}$$

We wish to show that $F(C) = C$. From the definitions of f_1 and f_2, we have that

$$\begin{aligned}
f_1(C) &= \left\{ x \in [0,1] \;\middle|\; x = \sum_{n=1}^{\infty} \frac{a_n}{3^{n+1}}, \quad a_n \in \{0,2\} \right\} \\
&= \left\{ x \in [0,1] \;\middle|\; x = \sum_{n=1}^{\infty} \frac{b_n}{3^n}, \quad b_1 = 0,\, b_n = a_{n-1} \text{ for } n > 1 \right\} \\
f_2(C) &= \left\{ x \in [0,1] \;\middle|\; x = \frac{2}{3} + \sum_{n=1}^{\infty} \frac{a_n}{3^{n+1}}, \quad a_n \in \{0,2\} \right\} \\
&= \left\{ x \in [0,1] \;\middle|\; x = \sum_{n=1}^{\infty} \frac{b_n}{3^n}, \quad b_1 = 2,\, b_n = a_{n-1} \text{ for } n > 1 \right\} \qquad (2)
\end{aligned}$$

From (2) and the definition of F, we see that

$$\begin{aligned}
F(C) &= f_1(C) \cup f_2(C) \\
&= \left\{ x \in [0,1] \;\middle|\; x = \sum_{n=1}^{\infty} \frac{b_n}{3^n}, \quad b_n \in \{0,2\} \right\}.
\end{aligned}$$

But this is precisely the definition of the Cantor set C given in (1). Therefore, $F(C) = C$ and the Cantor set C is a fixed point of F. $\square$

Problem 10.5.2.

Solution. □

Problem 10.5.4.

Solution. □

Problem 10.5.7.

Solution. □