

Homework Assignment 4

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Problem 2.3. Find the ACVF of the time series $X_t = Z_t + aZ_{t-1} + bZ_{t-2}$ where $Z_t \sim WN(0, \sigma^2)$ when:

a. $a = 0.3$, $b = -0.4$, and $\sigma^2 = 1$.

b. $a = -1.2$, $b = -1.6$, and $\sigma^2 = 0.25$.

Solution. The ACVF of the time series $\{X_t\}$, $\gamma_X(h)$, is by definition:

$$\begin{aligned}\gamma_X(h) &= \text{Cov}(X_{t+h}, X_t) \\ &= \text{Cov}(Z_{t+h} + aZ_{t+h-1} + bZ_{t+h-2}, Z_t + aZ_{t-1} + bZ_{t-2}) \\ &= \text{Cov}(Z_{t+h}, Z_t) + a\text{Cov}(Z_{t+h}, Z_{t-1}) + b\text{Cov}(Z_{t+h}, Z_{t-2}) \\ &\quad + a\text{Cov}(Z_{t+h-1}, Z_t) + a^2\text{Cov}(Z_{t+h-1}, Z_{t-1}) + ab\text{Cov}(Z_{t+h-1}, Z_{t-2}) \\ &\quad + b\text{Cov}(Z_{t+h-2}, Z_t) + ab\text{Cov}(Z_{t+h-2}, Z_{t-1}) + b^2\text{Cov}(Z_{t+h-2}, Z_{t-2}).\end{aligned}\tag{1}$$

Using (1), we can see that since $Z_t \sim WN(0, \sigma^2)$,

$$\gamma_X(h) = \begin{cases} (1 + a^2 + b^2)\sigma^2 & \text{if } h = 0 \\ a(1 + b)\sigma^2 & \text{if } h = \pm 1 \\ b\sigma^2 & \text{if } h = \pm 2 \\ 0 & \text{otherwise} \end{cases}.$$

Therefore, when

a. $a = 0.3$, $b = -0.4$, and $\sigma^2 = 1$, the ACVF of $\{X_t\}$ is:

$$\begin{cases} 1.25 & \text{if } h = 0 \\ 0.18 & \text{if } h = \pm 1 \\ -0.4 & \text{if } h = \pm 2 \\ 0 & \text{otherwise} \end{cases}$$

b. $a = -1.2$, $b = -1.6$, and $\sigma^2 = 0.25$, the ACVF of $\{X_t\}$ is:

$$\begin{cases} 1.25 & \text{if } h = 0 \\ 0.18 & \text{if } h = \pm 1 \\ -0.4 & \text{if } h = \pm 2 \\ 0 & \text{otherwise} \end{cases}$$

□

Problem 2.5. Suppose that $\{X_t, t = 0, \pm 1, \dots\}$ is stationary and that $|\theta| < 1$. Show that for each fixed n the sequence

$$S_m = \sum_{j=1}^m \theta^j X_{n-j}$$

is convergent absolutely and in mean square as $m \rightarrow \infty$.

Solution. Let $a_j = \theta^j X_{n-j}$. Since each X_i is a random variable, each X_i maps to a real, non-infinite value so let $X = \max\{|X_i|\}$. Then to see that S_m is convergent absolutely as $m \rightarrow \infty$, notice that

$$\begin{aligned} \sum_{j=1}^m |a_j| &= \sum_{j=1}^m |\theta^j X_{n-j}| \\ &= \sum_{j=1}^m |\theta|^j |X_{n-j}| \\ &\leq \sum_{j=1}^m X |\theta|^j = \sum_{j=1}^m b_j = T_m \end{aligned}$$

Since $|\theta| < 1$, we know that as $m \rightarrow \infty$, the partial sum $\sum_{j=1}^m X |\theta|^j \rightarrow 0$ and it must hold that $T_m \rightarrow 0$. Thus, we know that as $m \rightarrow \infty$, $\sum_{j=1}^m |a_j|$ converges to some L since $|a_j| \leq b_j$ and T_m is convergent. Therefore, S_m is convergent absolutely.

To see that S_m is convergent in the mean square, it suffices to show that $E(S_m - S_l)^2 \rightarrow 0$ as $m, l \rightarrow \infty$.

Without loss of generality, assume that $m > l > 0$. Notice that $S_m - S_l = \sum_{j=1}^m a_j - \sum_{j=1}^l a_j = \sum_{j=l+1}^m a_j$. Thus,

$$E(S_m - S_l) = E\left(\sum_{j=l+1}^m a_j\right) = \sum_{j=l+1}^m E(a_j).$$

It is clear that $E(a_j) = E(\theta^j X_{n-j}) = \theta^j E(X_{n-j})$. Since $\{X_t\}$ is a stationary time series, its expectation does not depend on t , so say $E(X_{n-j}) = \mu_X$. Then

$$\begin{aligned} E(S_m - S_l) &= \sum_{j=l+1}^m \theta^j E(X_{n-j}) \\ &= \mu_X \sum_{j=l+1}^m \theta^j \\ &= \frac{\mu_X \theta^{l+1} (1 - \theta^{m-l-1})}{1 - \theta} \end{aligned}$$

Since $|\theta| < 1$, it is clear then that $E(S_m - S_l)^2 \rightarrow 0$ as $m, l \rightarrow \infty$ showing that S_m is convergent in mean square for any n . □

Problem 2.11. Suppose that in a sample of size 100 from an AR(1) process with mean μ , $\phi = 0.6$, and $\sigma^2 = 2$ we obtain $\bar{x}_{100} = 0.271$. Construct an approximate 95% confidence interval for μ . Are the data compatible with the hypothesis that $\mu = 0$.

Solution. Note that since AR(1) is a linear model, \bar{X}_n is approximately normal with mean μ for large n and an approximate 95% confidence interval for μ is

$$\left(\bar{X}_n - \frac{1.96\nu^{1/2}}{\sqrt{n}}, \bar{X}_n + \frac{1.96\nu^{1/2}}{\sqrt{n}} \right)$$

where $\nu = \sum_{|h|<\infty} \gamma_X(h)$.

Since $\{X_t\}$ is an AR(1) process, we know that $\gamma_X(h) = \gamma_X(0)\phi^{|h|}$ where $\gamma_X(0) = \sigma^2/(1 - \phi^2)$. Thus

$$\begin{aligned} \nu &= \sum_{|h|<\infty} \gamma_X(h) = \sum_{|h|<\infty} \frac{\sigma^2 \phi^{|h|}}{1 - \phi^2} \\ &= \frac{\sigma^2}{1 - \phi^2} \left(1 + 2 \sum_{h=1}^{\infty} \phi^h \right) \\ &= \frac{\sigma^2}{1 - \phi^2} \left(1 + \frac{2\phi}{1 - \phi} \right) \\ &= \frac{\sigma^2(1 + \phi)}{(1 - \phi)(1 - \phi^2)} = \frac{\sigma^2}{(1 - \phi)^2} \end{aligned}$$

If $\phi = 0.6$ and $\sigma^2 = 2$, then $\nu = 2/(1 - 0.6)^2 = 12.5$. Since $n = 100$, $\bar{x}_n = \bar{x}_{100} = 0.271$, and an approximate 95% confidence interval for μ is

$$\left(0.271 - \frac{1.96(12.5)^{1/2}}{\sqrt{100}}, 0.271 + \frac{1.96(12.5)^{1/2}}{\sqrt{100}} \right)$$

or $(-0.42197, 0.96397)$. Given this confidence interval, it is plausible that $\mu = 0$. □

Problem 2.12. Suppose that in a sample of size 100 from an MA(1) process with mean μ , $\theta = -0.6$, and $\sigma^2 = 1$ we obtain $\bar{x}_{100} = 0.157$. Construct an approximate 95% confidence interval for μ . Are the data compatible with the hypothesis that $\mu = 0$.

Solution. Note that since MA(1) is a linear model, \bar{X}_n is approximately normal with mean μ for large n and an approximate 95% confidence interval for μ is

$$\left(\bar{X}_n - \frac{1.96\nu^{1/2}}{\sqrt{n}}, \bar{X}_n + \frac{1.96\nu^{1/2}}{\sqrt{n}} \right)$$

where $\nu = \sum_{|h|<\infty} \gamma_X(h)$.

Since $\{X_t\}$ is an MA(1) process, we know that

$$\gamma_X(h) = \begin{cases} \sigma^2(1 + \theta^2) & \text{if } h = 0 \\ \sigma^2\theta & \text{if } h = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$\nu = \sum_{|h|<\infty} \gamma_X(h) = \sigma^2(1 + \theta^2) + 2\sigma^2\theta = \sigma^2(1 + \theta)^2$$

If $\theta = -0.6$ and $\sigma^2 = 1$, then $\nu = (1 - 0.6)^2 = 0.16$. Since $n = 100$, $\bar{x}_n = \bar{x}_{100} = 0.157$, and an approximate 95% confidence interval for μ is

$$\left(0.157 - \frac{1.96(0.16)^{1/2}}{\sqrt{100}}, 0.157 + \frac{1.96(0.16)^{1/2}}{\sqrt{100}} \right)$$

or $(0.15198, 0.16202)$. Given this confidence interval, it is not plausible that $\mu = 0$. \square

Problem 2.13. Suppose that in a sample of size 100, we obtain $\hat{\rho}(1) = 0.438$ and $\hat{\rho}(2) = 0.145$.

- Assuming that the data were generated from an AR(1) model, construct approximate 95 % confidence intervals for both $\rho(1)$ and $\rho(2)$. Based on these two confidence intervals, are the data consistent with an AR(1) model with $\phi = 0.8$
- Assuming that the data were generated from an MA(1) model, construct approximate 95 % confidence intervals for both $\rho(1)$ and $\rho(2)$. Based on these two confidence intervals, are the data consistent with an MA(1) model with $\theta = 0.6$

Solution. Note that since AR(1) and MA(1) are linear models, we know that for large n , the vector $\hat{\boldsymbol{\rho}} = (\hat{\rho}(1), \dots, \hat{\rho}(k))^T$ is a multivariate normal random variable with mean $\boldsymbol{\rho} = (\rho(1), \dots, \rho(k))^T$ and covariance matrix $n^{-1}W$ where w_{ij} is defined by Bartlett's formula.

Hence, an approximate 95% confidence interval for $\rho(i)$ is

$$(\hat{\rho}(i) - 1.96\text{Var}(\hat{\rho}(i))^{1/2}, \hat{\rho}(i) + 1.96\text{Var}(\hat{\rho}(i))^{1/2}).$$

Since $\hat{\rho}(i) \sim N(\rho(i), n^{-1}w_{ii})$, we know $\text{Var}(\hat{\rho}(i)) = n^{-1}w_{ii}$ and the approximate 95% confidence interval for $\rho(i)$ is

$$\left(\hat{\rho}(i) - \frac{1.96w_{ii}^{1/2}}{\sqrt{n}}, \hat{\rho}(i) + \frac{1.96w_{ii}^{1/2}}{\sqrt{n}} \right). \quad (2)$$

- If the data were generated from an AR(1) model, then

$$w_{ii} = (1 - \phi^{2i})(1 + \phi^2)(1 - \phi^2)^{-1} - 2i\phi^{2i}$$

If $\phi = 0.8$, then $w_{11} = 0.36$ and $w_{22} = 1.0512$. Using (2), we know that the 95% confidence interval for $\rho(1)$ is $(0.3204, 0.556)$ and the 95% confidence interval for $\rho(2)$ is $(-0.0560, 0.3640)$. If $\phi = 0.8$, then the actual values of $\rho(1) = 0.8$ and $\rho(2) = 0.64$ suggest that the data were not generated from an AR(1) model with $\phi = 0.8$.

- If the data were generated from an MA(1) model, then $w_{11} = 1 - 3\rho(1)^2 + 4\rho(1)^4$ and $w_{22} = 1 + 2\rho(1)^2$ where $\rho(1) = \theta/(1 + \theta^2)$. If $\theta = 0.6$, then $w_{11} = 0.5676$ and $w_{22} = 1.3893$. Using (2), we know that the 95% confidence interval for $\rho(1)$ is $(0.2903, 0.5857)$ and the 95% confidence interval for $\rho(2)$ is $(-0.0860, 0.3760)$. If $\theta = 0.6$, then the actual values of $\rho(1) = 0.4412$ and $\rho(2) = 0$ suggest that the data were generated from an MA(1) model with $\theta = 0.6$. \square