Homework Assignment 4

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Problem 1. Find the first three terms in the asymptotic expansions of $x \to 0^+$ of the following integrals:

$$\int_x^1 \cos(xt)dt, \qquad \int_0^{1/x} e^{-t^2} dt.$$

Solution. $\square$

Problem 2. Find the full asymptotic behavior as $x \to 0^+$ of the following integral:

$$\int_0^1 \frac{e^{-t}}{1 + x^2 t^3} dt$$

Solution. $\square$

Problem 3. Find the full asymptotic expansion of $\int_0^x \mathrm{Bi}(t)dt$ as $x \to +\infty$.

Solution. □

Problem 4. Find the first five terms in the asymptotic expansion as $x \to +\infty$ of the integral

$$\int_0^{\pi/4} e^{-xt^2} \sqrt{\tan t}\, dt$$

a. by using a suitable change of variables and then applying Watson's lemma.

b. by applying Laplace's method directly to the given integral.

Solution. $\square$

Problem 5. Use Laplace's method of moving maxima to obtain the first two terms in the asymptotic expansion as $x \to +\infty$ of the integral

$$\int_0^\infty \exp\left[-t - \frac{x}{\sqrt{t}}\right] dt. \tag{1}$$

Solution. □

Problem 6. Let $f(x,t)$ be differentiable in x and continuous in (x,t) on $I \times J$, where I and J are intervals, and suppose that there exist functions $g(t)$ and $g_1(t)$ that are integrable on J such that for all $(x,t) \in I \times J$ we have that

$$|f(x,t)| \leq g(t) \quad \text{and} \quad |\partial_x f(x,t)| \leq g_1(t).$$

Then

$$\frac{d}{dx}\int_J f(x,t)dt = \int_J \partial_x f(x,t)dt.$$

a. Let $0 < a < b < \infty$. Use the above theorem to show that if $x \in (a,b)$, then

$$\frac{d^3}{dx^3}\int_0^\infty \exp\left[-t - \frac{x}{\sqrt{t}}\right] dt = -\int_0^\infty t^{-3/2} \exp\left[-t - \frac{x}{\sqrt{t}}\right] dt.$$

b. Use integration by parts to show that

$$\int_0^\infty \exp\left[-t - \frac{x}{\sqrt{t}}\right] dt = \frac{x}{2}\int_0^\infty t^{-3/2} \exp\left[-t - \frac{x}{\sqrt{t}}\right] dt.$$

c. Combine parts (a) and (b) to prove that integral (1) is a solution of the differential equation $xy''' + 2y = 0$ that also satisfies the initial condition $y(0) = 1$. Then use integration by parts to give an easy direct proof that the integral also satisfies the condition $y(+\infty) = 0$.

Solution. □

Problem 7. a. Find the leading behavior as $x \to +\infty$ of Laplace integrals of the form

$$\int_a^b (t-a)^\alpha g(t) e^{x\phi(t)} dt$$

where $\phi(t)$ has a maximum at $t = a$, $g(a) = 1$. Suppose further that $\alpha > -1$ and $\phi'(a) < 0$.

b. Repeat the analysis of part (a) when $\alpha > -1$ and $\phi'(a) = \phi''(a) = \cdots = \phi^{(p-1)}(a) = 0$ and $\phi^{(p)}(a) < 0$.

Solution. □