

Homework Assignment 6

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Problem 4.3. Find the solutions of the following systems of equations with the initial data:

a. $\frac{dx}{dt} = x - 2y, \quad x(0) = 1$
 $\frac{dy}{dt} = y - 2x, \quad y(0) = 0$

Solution. a. Applying the Laplace transform to the system yields

$$\mathcal{L} \left\{ \frac{dx}{dt} \right\} = s\bar{x}(s) - x(0) = \bar{x}(s) - 2\bar{y}(s) = \mathcal{L} \{x - 2y\}$$
$$\mathcal{L} \left\{ \frac{dy}{dt} \right\} = s\bar{y}(s) - y(0) = \bar{y}(s) - 2\bar{x}(s) = \mathcal{L} \{y - 2x\}.$$

Using the initial data, the transformed system becomes

$$(s-1)\bar{x}(s) + 2\bar{y}(s) = 1$$
$$2\bar{x}(s) + (s-1)\bar{y}(s) = 0,$$

or, equivalently,

$$\begin{bmatrix} s-1 & 2 \\ 2 & s-1 \end{bmatrix} \begin{bmatrix} \bar{x}(s) \\ \bar{y}(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This implies that the solution to the transformed system of equations is given by

$$\begin{bmatrix} \bar{x}(s) \\ \bar{y}(s) \end{bmatrix} = \begin{bmatrix} s-1 & 2 \\ 2 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{s-1}{(s-3)(s+1)} & -\frac{2}{(s-3)(s+1)} \\ -\frac{2}{(s-3)(s+1)} & \frac{s-1}{(s-3)(s+1)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{s-1}{(s-3)(s+1)} \\ -\frac{2}{(s-3)(s+1)} \end{bmatrix}$$

i.e. the solution is given by $\bar{x}(s) = \frac{s-1}{(s-3)(s+1)}$ and $\bar{y}(s) = -\frac{2}{(s-3)(s+1)}$.

From our table of Laplace Transforms, we know that

$$\mathcal{L} \{e^{at} - e^{bt}\} = \frac{a-b}{(s-a)(s-b)}$$

and

$$\mathcal{L} \left\{ \frac{ae^{at} - be^{bt}}{a-b} \right\} = \frac{s}{(s-a)(s-b)}.$$

Therefore, the solution to the original system of differential equations is given by

$$\begin{aligned}x(t) &= \mathcal{L}^{-1} \{ \bar{x}(s) \} = \mathcal{L}^{-1} \left\{ \frac{s-1}{(s-3)(s+1)} \right\} \\&= \mathcal{L}^{-1} \left\{ \frac{s}{(s-3)(s+1)} \right\} - \mathcal{L}^{-1} \left\{ \frac{1}{(s-3)(s+1)} \right\} \\&= \frac{3e^{3t} + e^{-t}}{4} - \frac{e^{3t} - e^{-t}}{4} \\&= \frac{e^{3t} + e^{-t}}{2}\end{aligned}$$

and

$$\begin{aligned}y(t) &= \mathcal{L}^{-1} \{ \bar{y}(s) \} = \mathcal{L}^{-1} \left\{ -\frac{2}{(s-3)(s+1)} \right\} \\&= -\frac{e^{-t} - e^{3t}}{2}\end{aligned}$$

□

Problem 4.12. Solve the following initial value problems:

- a. $\ddot{x} + \omega^2 x = \cos nt$, $x(0) = 1$, $\dot{x}(0) = 0$ where $\omega \neq n$.

Solution. a. We begin by applying the Laplace transform to the equation. Doing so yields

$$\mathcal{L}\{\ddot{x} + \omega^2 x\} = (s^2 + \omega^2)\bar{x}(s) - sx(0) - \dot{x}(0) = \frac{s}{s^2 + n^2} = \mathcal{L}\{\cos nt\}.$$

Using the initial data, the transformed equation becomes

$$(s^2 + \omega^2)\bar{x}(s) - s = \frac{s}{s^2 + n^2}.$$

Solve the above equation yields that the solution to the transformed equation is

$$\bar{x}(s) = \frac{s^3 + (n^2 + 1)s}{(s^2 + n^2)(s^2 + \omega^2)}.$$

From the partial fractions method we see that

$$\bar{x}(s) = \frac{s^3 + (n^2 + 1)s}{(s^2 + n^2)(s^2 + \omega^2)} = \frac{a_1 s + a_0}{s^2 + n^2} + \frac{b_1 s + b_0}{s^2 + \omega^2}.$$

Combining the rational fractions on the right side under a common denominator and equating the coefficients in the numerator we arrive at the following system of equations

$$\begin{aligned} a_1 + b_1 + 1 &= 1 \\ a_0 + b_0 &= 0 \\ a_1 \omega^2 + b_1 n^2 &= n^2 + 1 \\ a_0 \omega^2 + b_0 n^2 &= 0 \end{aligned}$$

Solving this system, we see that $a_0 = b_0 = 0$, $a_1 = \frac{1}{\omega^2 - n^2}$, and $b_1 = \frac{\omega^2 - n^2 - 1}{\omega^2 - n^2}$.

Thus, the solution to the transformed system is given by

$$\bar{x}(s) = \frac{s^3 + (n^2 + 1)s}{(s^2 + n^2)(s^2 + \omega^2)} = \left(\frac{1}{\omega^2 - n^2}\right) \frac{s}{s^2 + n^2} + \left(\frac{\omega^2 - n^2 - 1}{\omega^2 - n^2}\right) \frac{s}{s^2 + \omega^2}.$$

From our table of Laplace transforms, we know that

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}.$$

Therefore, the solution to the original differential equation is

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\{\bar{x}(s)\} = \left(\frac{1}{\omega^2 - n^2}\right) \mathcal{L}^{-1}\left\{\frac{s}{s^2 + n^2}\right\} + \left(\frac{\omega^2 - n^2 - 1}{\omega^2 - n^2}\right) \mathcal{L}^{-1}\left\{\frac{s}{s^2 + \omega^2}\right\} \\ &= \left(\frac{1}{\omega^2 - n^2}\right) \cos nt + \left(\frac{\omega^2 - n^2 - 1}{\omega^2 - n^2}\right) \cos \omega t. \end{aligned}$$

□

Problem 4.14. With the aid of the Laplace transform, investigate the motion of a particle governed by the equations of motion

$$\begin{aligned}\ddot{x} - \omega\dot{y} &= 0 \\ \ddot{y} + \omega\dot{x} &= \omega^2 a\end{aligned}$$

with the initial conditions $x(0) = y(0) = \dot{x}(0) = \dot{y}(0) = 0$.

Solution. We begin by applying the Laplace transform to the system of differential equations. Doing so yields

$$\begin{aligned}\mathcal{L}\{\ddot{x} - \omega\dot{y}\} &= s^2\bar{x}(s) - sx(0) - \dot{x}(0) - \omega(s\bar{y}(s) - y(0)) = 0 = \mathcal{L}\{0\} \\ \mathcal{L}\{\ddot{y} + \omega\dot{x}\} &= s^2\bar{y}(s) - sy(0) - \dot{y}(0) + \omega(s\bar{x}(s) - x(0)) = \frac{\omega^2 a}{s} = \mathcal{L}\left\{\frac{\omega^2 a}{s}\right\}\end{aligned}$$

Using the initial data, the above system becomes

$$\begin{aligned}s^2\bar{x}(s) - \omega s\bar{y}(s) &= 0 \\ s^2\bar{y}(s) + \omega s\bar{x}(s) &= \frac{\omega^2 a}{s},\end{aligned}$$

or, equivalently,

$$\begin{bmatrix} s^2 & -\omega s \\ \omega s & s^2 \end{bmatrix} \begin{bmatrix} \bar{x}(s) \\ \bar{y}(s) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\omega^2 a}{s} \end{bmatrix}$$

This implies that the solution to the transformed system of equations is

$$\begin{bmatrix} \bar{x}(s) \\ \bar{y}(s) \end{bmatrix} = \begin{bmatrix} s^2 & -\omega s \\ \omega s & s^2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{\omega^2 a}{s} \end{bmatrix} = \begin{bmatrix} \frac{s^2}{s^2(s^2+\omega^2)} & \frac{s\omega}{s^2(s^2+\omega^2)} \\ -\frac{s\omega}{s^2(s^2+\omega^2)} & \frac{s^2}{s^2(s^2+\omega^2)} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{\omega^2 a}{s} \end{bmatrix} = \begin{bmatrix} \frac{\omega^3 a}{s^2(s^2+\omega^2)} \\ \frac{\omega^2 a}{s^2(s^2+\omega^2)} \end{bmatrix}$$

Let $\bar{f}(s) = \frac{1}{s^2}$, $\bar{g}(s) = \frac{\omega}{s^2+\omega^2}$, and $\bar{h}(s) = \frac{\omega}{s^2+\omega^2}$. From our table of Laplace transforms, we know that

$$\begin{aligned}f(t) &= \mathcal{L}^{-1}\{\bar{f}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t \\ g(t) &= \mathcal{L}^{-1}\{\bar{g}(s)\} = \mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\} = \sin \omega t \\ h(t) &= \mathcal{L}^{-1}\{\bar{h}(s)\} = \mathcal{L}^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\} = \cos \omega t.\end{aligned}$$

Therefore, by the Convolution Theorem, the solution to the original system of equations is given by

$$\begin{aligned}x(t) &= \mathcal{L}^{-1}\{\bar{x}(s)\} = \omega^2 a \mathcal{L}^{-1}\{\bar{f}(s)\bar{g}(s)\} = \omega^2 a (f * g)(t) \\ &= \omega^2 a \int_0^t f(t-\tau)g(\tau)d\tau \\ &= \omega^2 a \int_0^t (t-\tau) \sin \omega \tau d\tau \\ &= a\omega t - a \sin \omega t.\end{aligned}$$

and

$$\begin{aligned}y(t) &= \mathcal{L}^{-1}\{\bar{y}(s)\} = \omega^2 a \mathcal{L}^{-1}\{\bar{f}(s)\bar{h}(s)\} = \omega^2 a (f * h)(t) \\&= \omega^2 a \int_0^t f(t-\tau)h(\tau)d\tau \\&= \omega^2 a \int_0^t (t-\tau)\cos\omega\tau d\tau \\&= a - a\cos\omega t.\end{aligned}$$

□

Problem 4.22. Solve the Blasius problem of an unsteady boundary layer flow in a semi-infinite body of viscous fluid enclosed by an infinite horizontal disk at $z = 0$. The governing equation and the boundary and initial conditions are

$$\begin{aligned}\frac{\partial u}{\partial t} &= \nu \frac{\partial^2 u}{\partial z^2} \\ u(z, t) &= Ut \quad \text{on } z = 0, t > 0 \\ u(z, t) &\rightarrow 0 \quad \text{as } z \rightarrow \infty, t > 0 \\ u(z, t) &= 0 \quad \text{at } t \leq 0, z > 0.\end{aligned}$$

Explain the significance of the solution.

Solution. Let $u(z, t)$ be a function in z, t . The Laplace transform of $u(z, t)$ with respect to t is given by

$$\mathcal{L}\{u(z, t)\} = \bar{u}(z, s) = \int_0^\infty u(z, t)e^{-st} dt.$$

From this definition, we see from previous theorems that

$$\mathcal{L}\left\{\frac{\partial^n}{\partial t^n}[u(z, t)]\right\} = s^n \bar{u}(z, s) - \sum_{k=0}^{n-1} s^{n-1-k} \frac{\partial^k}{\partial t^k}[u(z, 0)]$$

Similarly, we see from the Leibniz integral rule that

$$\mathcal{L}\left\{\frac{\partial^n}{\partial z^n}[u(z, t)]\right\} = \frac{d^n}{dz^n}[\bar{u}(z, s)].$$

We begin by applying the Laplace transform to the governing equation and the boundary conditions on the infinity horizontal disk. Doing so yields

$$\begin{aligned}s\bar{u}(z, s) - u(z, 0) &= \nu \frac{d^2}{dz^2}[\bar{u}(z, s)] \\ \bar{u}(z, s) &= \frac{U}{s^2}, \quad \text{on } z = 0, s > 0 \\ \bar{u}(z, s) &\rightarrow 0, \quad \text{as } z \rightarrow \infty, s > 0.\end{aligned}$$

The initial condition $u(z, t) = 0$ when $t = 0$ shows that the first equation reduces to

$$\frac{d^2}{dz^2}[\bar{u}(z, s)] - \frac{s}{\nu}\bar{u}(z, s) = 0.$$

This is a second-order, linear homogeneous differential equation, the solution of which we readily see if

$$\bar{u}(z, s) = c_1 e^{-\sqrt{\frac{s}{\nu}}z} + c_2 e^{\sqrt{\frac{s}{\nu}}z}.$$

From this form, the transformed boundary condition, $\bar{u}(z, s) \rightarrow 0$ as $z \rightarrow \infty$ for $s > 0$, indicates that $c_2 = 0$, since $\sqrt{\frac{s}{\nu}} > 0$. Similarly the transformed boundary condition, $\bar{u}(z, s) =$

$\frac{U}{s^2}$ on $z = 0$ for $s > 0$, indicates that $c_1 = \frac{U}{s^2}$. Thus, the solution to the transformed equation obeying the initial and boundary conditions is

$$\bar{u}(z, s) = \frac{U}{s^2} e^{-\sqrt{\frac{s}{\nu}} z}.$$

From our table of Laplace transforms, we know that

$$\mathcal{L}^{-1} \left\{ \frac{1}{s^2} e^{-\sqrt{\frac{s}{\nu}} z} \right\} = t \left[1 + 2\zeta^2 \operatorname{erfc}(\zeta) - \frac{2\zeta}{\sqrt{\pi}} e^{-\zeta^2} \right]$$

where $\zeta = \frac{z}{2\sqrt{\nu t}}$ and $\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt$. Therefore, the solution to the original differential equation is

$$u(z, t) = \mathcal{L}^{-1} \{ \bar{u}(z, s) \} = U \mathcal{L}^{-1} \left\{ \frac{1}{s^2} e^{-\sqrt{\frac{s}{\nu}} z} \right\} = Ut \left[1 + 2\zeta^2 \operatorname{erfc}(\zeta) - \frac{2\zeta}{\sqrt{\pi}} e^{-\zeta^2} \right].$$

□

Problem 4.25.*Solution.*