Homework Assignment 7

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Problem 6.5.2. Let $\Sigma = \{(a_1, a_2, a_3, \dots) \mid a_i \in \{0, 1\}\}$, the sequence space of zeroes and ones with the metric defined previously. Let C be the Cantor set and define $f : \Sigma \to C$ by

$$f((a_1, a_2, a_3, \dots)) = .b_1 b_2 b_3 \dots \quad \text{where } b_i = 0 \text{ if } a_i = 0 \text{ and } b_i = 2 \text{ if } a_i = 1$$

giving the ternary expansion of a real number in $[0, 1]$. Show that f defines a homeomorphism between Σ and C, the Cantor set.

Solution. Note that $f : \Sigma \to C$ is a homemorphism if f is a continuous bijection with continuous inverse.

We begin by showing that f is a bijection. Suppose that $x_1 = (a_{11}, a_{12}, a_{13}, \dots) \in \Sigma$ and $x_2 = (a_{21}, a_{22}, a_{23}, \dots) \in \Sigma$ with $x_1 \neq x_2$. Then $a_{1k} \neq a_{2k}$ for some $k \in \mathbb{Z}^+$. Since $x_1, x_2 \in \Sigma$, this implies that if $a_{1k} = 1$ then $a_{2k} = 0$ and if $a_{1k} = 0$ then $a_{2k} = 1$. Now, we see from the definition of f that

$$f(x_1) = .b_{11} b_{12} b_{13} \dots b_{1k} \dots \neq .b_{21} b_{22} b_{23} \dots b_{2k} \dots = f(x_2)$$

since if $a_{1k} = 0$ then $b_{1k} = 0 \neq 2 = b_{2k}$ and if $a_{1k} = 1$ then $b_{1k} = 2 \neq 0 = b_{2k}$. Thus, $f(x_1) \neq f(x_2)$ and f is injective.

Now let $y = .b_1 b_2 b_3 \dots \in C$ be the ternary expansion of a real number in $[0, 1]$. Then $b_i \in \{0, 2\}$ for all $i \in \mathbb{Z}^+$. Take $x = (a_1, a_2, a_3, \dots)$ where $a_i = 0$ if $b_i = 0$ and $a_i = 1$ if $b_i = 2$. Then $x \in \Sigma$ and we see from the definition of f that

$$f(x) = f((a_1, a_2, a_3, \dots)) = .b_1 b_2 b_3 \dots = y$$

so that f is surjective, making f a bijection.

To show that f is continuous, we must show that if the distance between two points is small in the metric space Σ, then the distance between their mapped points in C is small, i.e. if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $d(x_1, x_2) < \delta$, then $d(f(x_1), f(x_2)) < \varepsilon$. So, suppose that $x_1 = (a_{11}, a_{12}, a_{13}, \dots) \in \Sigma$ and $x_2 = (a_{21}, a_{22}, a_{23}, \dots) \in \Sigma$. Then,

$$f(x_k) = .b_{k1} b_{k2} b_{k3} \dots = \sum_{n=1}^{\infty} \frac{b_{kn}}{3^n} \in C$$

for $k = 1, 2$. Let $S = \{k \in \mathbb{Z}^+ \mid a_{1k} \neq a_{2k}\}$. Then

$$d(x_1, x_2) = \sum_{n=1}^{\infty} \frac{|a_{1n} - a_{2n}|}{2^n} = \sum_{k \in S} \frac{1}{2^k}.$$

Similarly, if $a_{1k} \neq a_{2k}$, then $b_{1k} \neq b_{2k}$ so that

$$d(f(x_1), f(x_2)) = |f(x_1) - f(x_2)| = \sum_{n=1}^{\infty} \frac{|b_{1n} - b_{2n}|}{2^n} = \sum_{k \in S} \frac{2}{3^k}.$$

Choose $\delta = \varepsilon/2 > 0$. Then we have that

$$d(x_1, x_2) = \sum_{k \in S} \frac{1}{2^k} < \delta = \frac{\varepsilon}{2}$$

which implies that

$$d(f(x_1), f(x_2)) = \sum_{k \in S} \frac{2}{3^k} < \sum_{k \in S} \frac{2}{2^k} < \varepsilon.$$

Therefore, f is continuous.

The above argument extends to show that for every point $x \in \Sigma$ and every neighborhood U of x, there exists a neighborhood V of $f(x)$ such that $V \subseteq f(U)$. Explicitly, let $x = (a_1, a_2, a_3, \ldots) \in \Sigma$ and let $\varepsilon > 0$ be given. Then $B_{\varepsilon/2}(x) = \{a \in \Sigma \mid d(a, x) < \varepsilon/2\}$ is a neighborhood of x and we see that $f(B_{\varepsilon/2}(x)) = \{y \in C \mid d(y, f(x)) < \varepsilon\}$. Thus, the open ball of radius ε is a neighborhood of $f(x)$ contained in $f(B_{\varepsilon/2}(x))$.

Since f maps open sets to open sets, we have that f is an open map. Therefore, since f is a continuous bijection, we must have that its inverse is continuous or that f is a homeomorphism. $\square$

Problem 6.5.3. Let $f : I \to I$ be a transitive map with I an interval. Show that if U and V are non-empty open sets in I, then there exists $m \in \mathbb{Z}^+$ with $U \cap f^m(V) \neq \emptyset$

Solution. Note that if f is a transitive map, then there exists some $x \in I$ such that $O(x) = \{x, f(x), f^2(x), \ldots\}$, the orbit of x, is dense in I. This implies that every point in the interval I is either in $O(x)$ or is a limit point of $O(x)$, i.e. if $y \in I$ with $y \neq x$, then for some $m \in \mathbb{Z}^+$ either $y = f^m(x)$ or $|y - f^m(x)| < \varepsilon$ for every $\varepsilon > 0$. So in either case, there exists some $m \in \mathbb{Z}^+$ such that $y = f^m(x)$.

Suppose that U and V are non-empty open sets in I. Then both U and V are unions of pairwise disjoint open intervals contained in I. So say

$$U = \bigcup_{n\in\mathbb{N}} J_n, \quad V = \bigcup_{n\in\mathbb{N}} K_n$$

where J_n and K_n are open intervals contained in I with $J_p \cap J_q = \emptyset$ and $K_p \cap K_q = \emptyset$ for all $p \neq q$.

Let $y \in U$. Then $y \in J_n$ for some $n \in \mathbb{N}$. Since $y \in J_n \subseteq I$, there exists some $k \in \mathbb{Z}^+$ such that $y = f^k(x)$ due to the fact that $O(x)$ is dense in I.

Now let $y \in f^m(V)$ for some $m \in \mathbb{Z}^+$. Then $y \in f^m(K_n)$ for some $n \in \mathbb{N}$. This implies that $y = f^m(z)$ for some $z \in K_n \subseteq I$. Thus, since $O(x)$ is dense in I, there exists $p \in \mathbb{Z}^+$ such that $z = f^p(x)$. Therefore, $y \in f^m(V)$ if and only if $y = f^{m+p}(x)$ for some $p \in \mathbb{Z}^+$.

If $k > p$, then choose $m = k - p \in \mathbb{Z}^+$. It is then clear that $f^k(x) \in U \cap f^m(V)$ so that $U \cap f^m(V) \neq \emptyset$. If on the other hand, suppose that $k \leq p$. Note that if f is transitive, then there is a dense set of transitive points in I, since each member of $O(x)$ is a transitive point. This implies that for some $l_1 \in \mathbb{Z}^+$, we have that $|f^k(x) - f^{l_1}(x)| < \varepsilon$ for every $\varepsilon > 0$. Thus, we can eventually find some sequence $l_i \in \mathbb{Z}$ such that $|f^{l_i}(x) - f^{l_{i-1}}(x)| < \varepsilon$ for every $\varepsilon > 0$ with $l_j > p$ for some l_j. This implies that $|f^k(x) - f^{l_j}(x)| < \varepsilon$ for every $\varepsilon > 0$ so that $f^k(x) = f^{l_j}(x)$. We can then choose $m = l_j - p \in \mathbb{Z}^+$ which implies that $f^{l_j}(x) \in U \cap f^m(V)$ so that $U \cap f^m(V) \neq \emptyset$.

□

Problem 6.5.4. Let $F : [0,1) \to [0,1)$ be the tripling map. Show that F is transitive and that its periodic points are dense in $[0,1)$.

Solution. Let $x_0 \in [0,1)$ be defined using its ternary expansion as all possible combinations of n-blocks written in lexicographical order where an n-block is all possible permutations of $\{0,1,2\}$ of length n. Thus, the 1-blocks are $0,1,2$, the 2-blocks are 00, 01, 02, 10, 11, 12, 20, 21, 22, the 3-blocks are 000, 001, 002, 010, 011, 012, 020, 021, 022, 100, 101, 102, 110, 111, 112, 120, 121, 122, 200, 201, 202, 210, 211, 212, 220, 221, 222, etc. so that

$$x_0 = .012000102101112202122\ldots \in [0,1).$$

Let $y \in [0,1)$ with ternary expansion

$$y = .y_1y_2y_3\ldots = \sum_{n=1}^{\infty} \frac{y_1}{3^n}$$

where $y_i \in \{0,1,2\}$ and let $\delta > 0$.

Choose N so large that $1/3^N < \delta$. All possible finite strings of 0's, 1's, and 2's appear in the ternary expansion of x_0, so the string $y_1y_2y_3\ldots y_N$ must also appear in the ternary expansion of x_0.

It follows that $F^r(x_0) = .y_1y_2y_3\ldots y_N a_{N+1}a_{N+2}\ldots$ for some $r \in \mathbb{Z}^+$ since F^r removes the first r places of a ternary expansion. Thus,

$$\begin{aligned} |F^r(x_0) - y| &= |.y_1y_2y_3\ldots y_N a_{N+1}a_{N+2}\ldots - .y_1y_2y_3\ldots y_N y_{N+1}y_{N+2}\ldots| \\ &\le \sum_{n=N+1}^{\infty} \frac{2}{3^n} = \frac{1}{3^N} < \delta \end{aligned}$$

Therefore, the orbit of x_0 is arbitrarily close to all points in $[0,1)$ so that F is transitive and its periodic points are dense in $[0,1)$.

The points of period n are the sequences of $\{0,1,2\}$ that have an infinite repeating pattern, i.e. the points that have repeating n-blocks. For instance, the point $y_1 = .202020\ldots$ consisting of an infinite repeating sequence of the 2-block 20 is a period 2 point. To demonstrate, $F(y_1) = .02020\ldots$ and $F^2(y_1) = .20202\ldots = y_1$. $\square$

Problem 7.1.2. i. Define $f_a : \mathbb{R} \to \mathbb{R}$ by $f_a(x) = ax$ for $a \in \mathbb{R}$. Show that $f_{1/2}$ and $f_{1/4}$ are conjugate via the map

$$h(x) = \begin{cases} \sqrt{x} & x \geq 0 \\ -\sqrt{-x} & x < 0 \end{cases}. \tag{1}$$

ii. More generally, show that $f_a, f_b : [0, \infty) \to [0, \infty)$ for $0 < a, b < 1$, the f_a and f_b are conjugate via the map $h(x) = x^p$ for $p > 0$ and similarly if $a, b > 1$.

iii. Discuss the cases where $a > 1$ and $0 < b < 1$. What happens when $a = 1/2$ and $b = 2$?

Solution. i. We begin by showing that $h : \mathbb{R} \to \mathbb{R}$ where h is defined as in (1) is a homeomorphism, i.e. it is a continuous bijection with continuous inverse.

It is clear from the definition of h that if $x_1 \neq x_2$ then $h(x_1) \neq h(x_2)$ due to the uniqueness of the square root operator. Thus, h is injective.

To show that h is surjective, suppose that $y \in \mathbb{R}$ and that $y_1 = |y|$. If $y \geq 0$, then $y = y_1$, otherwise $y = -y_1$. Now, if $y \geq 0$, then set $x = y_1^2 \geq 0$, otherwise set $x = -y_1^2 < 0$. Then we have from the definition of h that if $y \geq 0$, then

$$h(x) = \sqrt{y_1^2} = y_1 = y.$$

Similarly, we have that if $y < 0$, then

$$h(x) = -\sqrt{-\left(-y_1^2\right)} = -y_1 = y.$$

Therefore, h is surjective.

It is clear that h and its inverse are continuous so that h is a homemorphism.

Now, we see that

$$h \circ f_{1/4}(x) = h\left(\frac{x}{4}\right) = \begin{cases} \frac{\sqrt{x}}{2} & x \geq 0 \\ -\frac{\sqrt{-x}}{2} & x < 0 \end{cases}$$

and that

$$\begin{aligned} f_{1/2} \circ h(x) &= \begin{cases} f_{1/2}\left(\sqrt{x}\right) & x \geq 0 \\ f_{1/2}\left(-\sqrt{-x}\right) & x < 0 \end{cases} \\ &= \begin{cases} \frac{\sqrt{x}}{2} & x \geq 0 \\ -\frac{\sqrt{-x}}{2} & x < 0 \end{cases} \end{aligned}$$

so that h is a conjugate map of $f_{1/4}$ and $f_{1/2}$.

ii. From the previous remarks, we see that if $p > 0$, then $h : [0, \infty) \to [0, \infty)$ with $h(x) = x^p$ is a homeomorphism. Let $f_c : [0, \infty) \to [0, \infty)$ be a function defined by $f_c(x) = cx$. Consider the maps f_a and f_b. Then we see that

$$h \circ f_a(x) = h(ax) = (ax)^p = a^p x^p$$

and that

$$f_b \circ h(x) = f_b(x^p) = bx^p.$$

Thus, if $a^p = b$, then $h \circ f_a = f_b \circ h$ so that f_a and f_b are conjugate via h. Note that for $a, b > 0$ we have that $a^p = b$ if and only if $0 < a, b < 1$ or $a, b > 1$.

iii. Suppose that $a > 1$ and $0 < b < 1$. Then for any $p > 0$, $a^p > 1$, so that $a^p > b$. Thus, f_a and f_b will not be conjugate via h.

Suppose that $a = 1/2$ and $b = 2$. Then $a^p = 1/2^p < 2 = b$ for any positive p and $f_{1/2}$ and f_2 are not conjugate via h.

$\square$

Problem 7.1.3. Prove that if $f : X \to X$ and $g : Y \to Y$ are conjugate maps of metric spaces, then f is one-to-one if and only if g is one-to-one and f is onto if and only if g is onto.

Solution. If f and g are conjugate maps of metric spaces, then there exists a map $h : X \to Y$, with h a bijection, such that $g \circ h = h \circ f$.

Suppose that f is one-to-one and that $g(y_1) = g(y_2)$. Since h is onto, there exist $x_1 \in X$ and $x_2 \in X$ such that $h(x_1) = y_1$ and $h(x_2) = y_2$. Thus, if $g(y_1) = g(y_2)$, then $g \circ h(x_1) = g \circ h(x_2)$. By the conjugacy of h, we then have that $h \circ f(x_1) = h \circ f(x_2)$ and since h and f are one-to-one, we have that $x_1 = x_2$. Due to the fact that h is a well-defined function, if $x_1 = x_2$, then $y_1 = h(x_1) = h(x_2) = y_2$ and we therefore have that g is one-to-one.

Now suppose that g is one-to-one and that $f(x_1) = f(x_2)$. Since h is well-defined, we have that $h \circ f(x_1) = h \circ f(x_2)$. By the conjugacy of h, we then have that $g \circ h(x_1) = g \circ h(x_2)$. Since g and h are one-to-one, it follows that $x_1 = x_2$ and f is therefore one-to-one.

Suppose that f is onto and let $y_2 \in Y$ be given. Since h is onto, there exists $x_2 \in X$ such that $h(x_2) = y_2$. Thus, since f is onto, there exists $x_1 \in X$ such that $f(x_1) = x_2$ which implies that $h \circ f(x_1) = y_2$. By the conjugacy of h we have that

$$g \circ h(x_1) = h \circ f(x_1) = y_2.$$

Hence, there exists $y_1 = h(x_1) \in Y$ such that $g(y_1) = y_2$. Therefore, since $y_2 \in Y$ was arbitary, we have that h is onto.

Now suppose that g is onto. Since g and h are onto, for every $y \in Y$, there exists $x_1 \in X$ such that $g \circ h(x_1) = y$. By the conjugacy of h, we then have that $h \circ f(x_1) = y$. So, for every $y \in Y$, there exists $f(x_1) \in X$ such that $h \circ f(x_1) = y$. However, since h is onto, we also have that for every $y \in Y$, there exists $x_2 \in X$ such that $h(x_2) = y$. Thus, for every $x_2 \in X$ we have that $h(x_2) = y = h(f(x_1))$ for some $x_1 \in X$. The fact that h is one-to-one then shows that $f(x_1) = x_2$ or that f is onto. □