Homework Assignment 1

Matthew Tiger

September 4, 2016

Problem 1.1.2. Use Example 1.1.3 for affine maps to find the solutions to the difference equations:

i. $x_{n+1} - \frac{x_n}{3} = 2,\ x_0 = 2.$

ii. $x_{n+1} + 3x_n = 4,\ x_0 = -1.$

Solution. Consider the affine map $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = ax + b$. Define the sequence $x_{n+1} = f(x_n) = ax_n + b$ where $x_0 \in \mathbb{R}$ is given. As was shown in the reading, the closed form solution to the above recurrence relation is given by

$$x_n = \left(x_0 - \frac{b}{1-a}\right) a^n + \frac{b}{1-a}. \tag{1}$$

Thus, the solutions to the provided difference equations can be solved by rewriting the equation in the form of an affine map, identifying a, b, and x_0, and using the closed solution (1).

i. For the difference equation $x_{n+1} - \frac{x_n}{3} = 2,\ x_0 = 2$, we readily see by rewriting the equation that $a = 1/3$ and $b = 2$ with $x_0 = 2$ given. Therefore, using (1), the solution to the difference equation is

$$\begin{aligned} x_n &= \left(x_0 - \frac{b}{1-a}\right) a^n + \frac{b}{1-a} \\ &= \left(2 - \frac{2}{1-1/3}\right)\left(\frac{1}{3}\right)^n + \frac{2}{1-1/3} \\ &= 3 - 3^{-n} \end{aligned}$$

ii. For the difference equation $x_{n+1} + 3x_n = 4,\ x_0 = -1$, we readily see by rewriting the equation that $a = -3$ and $b = 4$ with $x_0 = -1$ given. Therefore, using (1), the solution to the difference equation is

$$\begin{aligned} x_n &= \left(x_0 - \frac{b}{1-a}\right) a^n + \frac{b}{1-a} \\ &= \left(-1 - \frac{4}{1-(-3)}\right)(-3)^n + \frac{4}{1-(-3)} \\ &= 1 - 2(-3)^n. \end{aligned}$$

$\square$

Problem 1.1.3. A *logistic difference equation* is one of the form $x_{n+1} = \mu x_n(1 - x_n)$ for some fixed $\mu \in \mathbb{R}$. Find exact (closed form) solutions to the following logistic difference equations:

i. $x_{n+1} = 2x_n(1 - x_n)$. Hint: Use the substitution $x_n = (1 - y_n)/2$ to transform the equation into a simpler equation that is easily solved.

ii. $x_{n+1} = 4x_n(1 - x_n)$. Hint: Set $x_n = \sin^2(\theta_n)$ and simplify to get an equation that is easily solved.

Solution. i. Let $x_n = (1 - y_n)/2$ for $n \in \mathbb{N}$ with x_0 given. Substituting this expression into the original difference equation yields the new difference equation

$$\begin{aligned}\frac{1 - y_{n+1}}{2} &= 2\left(\frac{1 - y_n}{2}\right)\left[1 - \left(\frac{1 - y_n}{2}\right)\right] \\ &= (1 - y_n)\left(\frac{1 + y_n}{2}\right) \\ &= \frac{1 - y_n^2}{2}.\end{aligned}$$

This new difference equation reduces to $y_{n+1} = y_n^2$ for $n \in \mathbb{N}$, the solution of which is readily seen to be $y_{n+1} = y_0^{2^{n+1}}$. Making the substitution $y_n = 1 - 2x_n$ shows that, for $n \in \mathbb{N}$, the solution to the original difference equation is given by

$$x_{n+1} = \frac{1 - (1 - 2x_0)^{2^{n+1}}}{2}.$$

ii. Let $x_n = \sin^2(\theta_n)$ for $n \in \mathbb{N}$ with x_0 given. We may assume without loss of generality that $\theta_n \in [0, \pi)$ for if the angle θ_n isn't in the stated range, we can find an integer k such that $\theta_n + k\pi \in [0, \pi)$ and $\sin^2(\theta_n) = \sin^2(\theta_n + k\pi)$. We then declare the sum $\theta_n + k\pi$ to be the new angle θ_n. Substituting the above expression for x_n into the original difference equation yields the new difference equation

$$\begin{aligned}\sin^2(\theta_{n+1}) &= 4\sin^2(\theta_n)\left(1 - \sin^2(\theta_n)\right) \\ &= (2\sin(\theta_n)\cos(\theta_n))^2 \\ &= \sin^2(2\theta_n).\end{aligned}$$

Knowing that for $x, y \in [0, \pi)$ we have that $\sin^2(x) = \sin^2(y)$ if and only if $x = y$, the new difference equation reduces to $\theta_{n+1} = 2\theta_n$ for $n \in \mathbb{N}$ where it is implicitly understood that θ_{n+1} will be mapped to the corresponding angle between 0 and π if $2\theta_n \geq \pi$. Using the closed form solution for difference equations in the form of linear maps, the solution to the reduced difference equation is given by $\theta_{n+1} = 2^{n+1}\theta_0$ for $n \in \mathbb{N}$. Making the substitution $\theta_n = \sin^{-1}(\sqrt{x_n})$ shows that, for $n \in \mathbb{N}$, the solution to the original difference equation is given by

$$x_{n+1} = \sin^2(2^{n+1}\sin^{-1}(\sqrt{x_0}))$$

$\square$

Problem 1.1.4. You borrow $\$P$ at $r\%$ per annum and pay off $\$M$ at the end of each subsequent month. Write down a difference equation for the amount owing $A(n)$ at the end of each month (so $A(0) = P$). Solve the equation to find a closed form for $A(n)$. If $P = 100,000$, $M = 1,000$, and $r = 4$, after how long will the loan be paid off?

Solution. □

Problem 1.1.7. Let $f(x) = x^2 + bx + c$. Give conditions on b and c for $f : [0, 1] \to [0, 1]$ to be a dynamical system. Hint: Recall that the maximum and minimum values of a continuous function defined on a closed interval $[a, b]$ occur either at the end points or at the critical points of the function.

Solution. $\square$

Problem 1.2.1. Give conditions on b and c for the map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + bx + c$ to have a fixed point. Use these conditions to show that $f_c(x) = x^2 + c$ has a fixed point provided $c \leq 1/4$.

Solution. □

Problem 1.2.6. Consider the eventual fixed points of the logistic map $L_\mu : [0,1] \to [0,1]$, $L_\mu(x) = \mu x(1-x)$ for $0 < \mu < 4$.

i. Show that there are no eventual fixed points associated with the fixed point $x = 0$, other than $x = 1$.

ii. Show that for $1 < \mu \leq 2$, the only eventual fixed point associated with the fixed point $x = 1 - 1/\mu$ is $x = 1/\mu$.

iii. Show that there are additional eventual fixed points associated with $x = 1 - 1/\mu$ when $2 < \mu < 3$.

iv. Investigate the eventual fixed points of the logistic map when $\mu = 5/2$.

Solution. □