

Exam 3

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Problem 1. Solve the non-homogeneous diffusion problem by the Hankel transform

$$\begin{aligned} u_t &= a \left(u_{rr} + \frac{1}{r} u_r \right) + Q(r, t), \quad 0 < r < \infty, \quad 0 < t \\ u(r, 0) &= f(r), \quad 0 < r < \infty. \end{aligned}$$

Solution. Application of the 0-th order Hankel transform will transform the above Partial Differential Equation into an Ordinary Differential Equation. The following property of the 0-th order Hankel transform will aid in the application; if $\mathcal{H}_0 \{u(r, t)\} = \tilde{u}_0(\kappa, t)$, then

$$\mathcal{H}_0 \left\{ \frac{1}{r} \frac{\partial}{\partial r} [u(r, t)] + \frac{\partial^2}{\partial r^2} [u(r, t)] \right\} = -\kappa^2 \tilde{u}_0(\kappa, t). \quad (1)$$

Now, with the above property, we see that applying the 0-th order Hankel transform to the diffusion problem yields

$$\begin{aligned} \frac{d}{dt} [\tilde{u}_0(\kappa, t)] + a\kappa^2 \tilde{u}_0(\kappa, t) &= \tilde{Q}_0(\kappa, t), \quad 0 < \kappa < \infty, \quad 0 < t \\ \tilde{u}_0(\kappa, 0) &= \tilde{f}_0(\kappa), \quad 0 < \kappa < \infty. \end{aligned}$$

This is a first order linear Ordinary Differential Equation, the solution to which is

$$\tilde{u}_0(\kappa, t) = c_1(\kappa) e^{-a\kappa^2 t} + e^{-a\kappa^2 t} \int_0^t e^{a\kappa^2 x} \tilde{Q}_0(\kappa, x) dx.$$

Thus, from this solution and the transformed boundary condition, we see that $c_1(\kappa) = \tilde{f}_0(\kappa)$ and the solution to the transformed boundary value problem is

$$\tilde{u}_0(\kappa, t) = \tilde{f}_0(\kappa) e^{-a\kappa^2 t} + e^{-a\kappa^2 t} \int_0^t e^{a\kappa^2 x} \tilde{Q}_0(\kappa, x) dx.$$

Therefore, the solution to the initial diffusion problem is

$$\begin{aligned} u(r, t) &= \mathcal{H}_0^{-1} \{ \tilde{u}_0(\kappa, t) \} = \mathcal{H}_0^{-1} \left\{ \tilde{f}_0(\kappa) e^{-a\kappa^2 t} + e^{-a\kappa^2 t} \int_0^t e^{a\kappa^2 x} \tilde{Q}_0(\kappa, x) dx \right\} \\ &= \int_0^\infty \kappa J_0(\kappa r) \left[\tilde{f}_0(\kappa) e^{-a\kappa^2 t} + e^{-a\kappa^2 t} \int_0^t e^{a\kappa^2 x} \tilde{Q}_0(\kappa, x) dx \right] d\kappa, \end{aligned}$$

where $J_0(\kappa r)$ is the Bessel function of order 0.

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Problem 2.*Solution.*

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Problem 3. Solve the following integral equation by the Mellin transform

$$f(x) = \sin ax + \int_0^\infty \frac{f(xt)}{1+t^2} dt.$$

Solution. Let $g(x) = \frac{1}{1+x^2}$ and $h(x) = \sin ax$. Recall that $(f \circ g)(x)$ is defined to be

$$(f \circ g)(x) = \int_0^\infty f(xt)g(t)dt.$$

Thus, with this knowledge, the integral equation becomes

$$\begin{aligned} f(x) &= h(x) + \int_0^\infty f(xt)g(t)dt \\ &= h(x) + (f \circ g)(x). \end{aligned}$$

Let $\mathcal{M}\{f(x)\} = \tilde{f}(p)$, $\mathcal{M}\{g(x)\} = \tilde{g}(p)$, and $\mathcal{M}\{h(x)\} = \tilde{h}(p)$. Then from the Convolution Type theorem regarding the Mellin transform, we see that application of the Mellin transform to the integral equation yields

$$\begin{aligned} \tilde{f}(p) &= \mathcal{M}\{h(x)\} + \mathcal{M}\{(f \circ g)(x)\} \\ &= \tilde{h}(p) + \tilde{f}(p)\tilde{g}(1-p). \end{aligned}$$

Solving the above algebraic equation shows that

$$\tilde{f}(p) = \frac{\tilde{h}(p)}{1 - \tilde{g}(1-p)}.$$

From our table of Mellin transforms we know that

$$\tilde{g}(p) = \frac{\pi}{2} \csc\left(\frac{\pi p}{2}\right)$$

and

$$\tilde{h}(p) = a^{-p}\Gamma(p) \sin\left(\frac{\pi p}{2}\right).$$

Therefore, we see that

$$\begin{aligned} \tilde{f}(p) &= \frac{a^{-p}\Gamma(p) \sin\left(\frac{\pi p}{2}\right)}{1 - \frac{\pi}{2} \csc\left(\frac{\pi(1-p)}{2}\right)} \\ &= \frac{2a^{-p}\Gamma(p) \sin\left(\frac{\pi p}{2}\right)}{2 - \pi \sec\left(\frac{\pi p}{2}\right)} \end{aligned}$$

and the solution to the integral equation is

$$\begin{aligned} f(x) &= \mathcal{M}^{-1}\{\tilde{f}(p)\} = \mathcal{M}^{-1}\left\{\frac{2a^{-p}\Gamma(p) \sin\left(\frac{\pi p}{2}\right)}{2 - \pi \sec\left(\frac{\pi p}{2}\right)}\right\} \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-p} \left[\frac{2a^{-p}\Gamma(p) \sin\left(\frac{\pi p}{2}\right)}{2 - \pi \sec\left(\frac{\pi p}{2}\right)}\right] dp. \end{aligned}$$

□

Problem 4. Solve the following Partial Differential Equation by the Mellin transform

$$r^2 \phi_{rr} + r \phi_r + \phi_{\theta\theta} = 0, \quad 0 < r < \infty, \quad 0 < \theta < \pi$$

$$\phi(r, 0) = \begin{cases} (1-r)^2 & 0 < r < 1 \\ 0 & 1 < r \end{cases}$$

$$\phi(r, \pi) = \begin{cases} 1 & 0 < r < 1 \\ 0 & 1 < r \end{cases},$$

Solution. Recall that if $\mathcal{M}\{\phi(r, \theta)\} = \tilde{\phi}(p, \theta)$, then the following property holds

$$\mathcal{M}\left\{r^2 \frac{\partial^2}{\partial r^2} [\phi(r, \theta)] + r \frac{\partial}{\partial r} [\phi(r, \theta)]\right\} = p^2 \tilde{\phi}(p, \theta).$$

Thus, applying the Mellin transform to the Partial Differential Equation and using our table of Mellin transforms, we see that

$$\frac{d^2}{d\theta^2} [\tilde{\phi}(p, \theta)] + p^2 \tilde{\phi}(p, \theta) = 0, \quad 0 < p < \infty, \quad 0 < \theta < \pi$$

$$\tilde{\phi}(p, 0) = \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)}, \quad \tilde{\phi}(p, \pi) = \frac{1}{p}.$$

The solution to the resulting homogeneous linear Ordinary Differential Equation is

$$\tilde{\phi}(p, \theta) = c_1(p) \cos p\theta + c_2(p) \sin p\theta.$$

Using the above solution and the transformed boundary conditions, we see that

$$c_1(p) = \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)}$$

$$c_1(p) \cos p\pi + c_2(p) \sin p\pi = \frac{1}{p}.$$

Solving, we see that

$$c_2(p) = \left(\frac{1}{p} - \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)} \cos p\pi\right) \csc p\pi$$

$$= \frac{\csc p\pi}{p} - \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)} \cot p\pi.$$

Thus, the solution to the transformed boundary value problem is

$$\tilde{\phi}(p, \theta) = \left[\frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)}\right] \cos p\theta + \left[\frac{\csc p\pi}{p} - \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)} \cot p\pi\right] \sin p\theta.$$

Therefore, the solution to the original boundary value problem is

$$\phi(r, \theta) = \mathcal{M}^{-1} \left\{ \left[\frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)}\right] \cos p\theta + \left[\frac{\csc p\pi}{p} - \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)} \cot p\pi\right] \sin p\theta \right\}$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} r^{-p} \left\{ \left[\frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)}\right] \cos p\theta + \left[\frac{\csc p\pi}{p} - \frac{\Gamma(3)\Gamma(p)}{\Gamma(p+3)} \cot p\pi\right] \sin p\theta \right\} dp.$$

□

Problem 5. Show that

$$\int_0^\infty e^{-ax} \left(\frac{\cos px - \cos qx}{x} \right) dx = \frac{1}{2} \log \frac{q^2 + a^2}{p^2 + a^2}$$

Solution. Recall that the Laplace transform of a function $f(t)$ is defined as

$$\bar{f}(s) = \mathcal{L} \{f(t)\}_s = \int_0^\infty f(t) e^{-st} dt$$

where we explicitly note the transformation variable s . Thus, we see that the integral above becomes

$$\begin{aligned} \int_0^\infty e^{-ax} \left(\frac{\cos px - \cos qx}{x} \right) dx &= \int_0^\infty e^{-ax} \left(\frac{\cos px}{x} \right) dx - \int_0^\infty e^{-ax} \left(\frac{\cos qx}{x} \right) dx \\ &= \mathcal{L} \left\{ \frac{\cos px}{x} \right\}_a - \mathcal{L} \left\{ \frac{\cos qx}{x} \right\}_a. \end{aligned} \quad (2)$$

From a previous theorem, if $\mathcal{L} \{f(t)\} = \bar{f}(s)$, then we have that

$$\mathcal{L} \left\{ \frac{f(t)}{t} \right\} = \int_s^\infty \bar{f}(z) dz.$$

Thus, we see that (2) becomes

$$\begin{aligned} \int_0^\infty e^{-ax} \left(\frac{\cos px - \cos qx}{x} \right) dx &= \mathcal{L} \left\{ \frac{\cos px}{x} \right\}_a - \mathcal{L} \left\{ \frac{\cos qx}{x} \right\}_a \\ &= \int_a^\infty \mathcal{L} \{ \cos px \}_z dz - \int_a^\infty \mathcal{L} \{ \cos qx \}_z dz \end{aligned}$$

From the table of Laplace transforms, we have that

$$\mathcal{L} \{ \cos bx \}_z = \frac{z}{b^2 + z^2}.$$

Thus, we have that

$$\begin{aligned} \int_0^\infty e^{-ax} \left(\frac{\cos px - \cos qx}{x} \right) dx &= \int_a^\infty \mathcal{L} \{ \cos px \}_z dz - \int_a^\infty \mathcal{L} \{ \cos qx \}_z dz \\ &= \int_a^\infty \frac{z}{p^2 + z^2} dz - \int_a^\infty \frac{z}{q^2 + z^2} dz \end{aligned}$$

Using a u -substitution to solve the resulting integrals, we have that

$$\begin{aligned} \int_0^\infty e^{-ax} \left(\frac{\cos px - \cos qx}{x} \right) dx &= \int_a^\infty \frac{z}{p^2 + z^2} dz - \int_a^\infty \frac{z}{q^2 + z^2} dz \\ &= \frac{1}{2} \log p^2 + z^2 \Big|_a^\infty - \frac{1}{2} \log p^2 + z^2 \Big|_a^\infty. \end{aligned}$$

Using the properties of logarithms, we therefore see that

$$\begin{aligned}
 \int_0^\infty e^{-ax} \left(\frac{\cos px - \cos qx}{x} \right) dx &= \frac{1}{2} \log p^2 + z^2 \Big|_a^\infty - \frac{1}{2} \log p^2 + z^2 \Big|_a^\infty \\
 &= \frac{1}{2} \left[\log \frac{p^2 + z^2}{q^2 + z^2} \Big|_{z=\infty} - \log \frac{p^2 + z^2}{q^2 + z^2} \Big|_{z=a} \right] \\
 &= -\frac{1}{2} \log \frac{p^2 + a^2}{q^2 + a^2} \\
 &= \frac{1}{2} \log \frac{q^2 + a^2}{p^2 + a^2}.
 \end{aligned}$$

□

Problem 6. Suppose that $I_n f(x)$ denotes the n -th repeated integral of $f(x)$ defined by

$$I_n f(x) = \int_x^\infty I_{n-1} f(t) dt$$

and that $\mathcal{M}\{f(x)\} = \tilde{f}(p)$. Show that

- a. $\mathcal{M}\left\{\int_x^\infty f(t) dt\right\} = \frac{1}{p} \tilde{f}(p+1),$
 b. $\mathcal{M}\{I_n f(x)\} = \frac{\Gamma(p)}{\Gamma(p+n)} \tilde{f}(p+n)$

Solution. Recall that the Mellin transform of the function $f(x)$ is defined as

$$\mathcal{M}\{f(x)\} = \int_0^\infty x^{p-1} f(x) dx. \quad (3)$$

a. From the definition of the Mellin transform (3), we see that

$$\mathcal{M}\left\{\int_x^\infty f(t) dt\right\} = \int_0^\infty x^{p-1} \left[\int_x^\infty f(t) dt\right] dx.$$

Interchanging the order of integration from t to x , we see that

$$\begin{aligned} \mathcal{M}\left\{\int_x^\infty f(t) dt\right\} &= \int_0^\infty x^{p-1} \left[\int_x^\infty f(t) dt\right] dx \\ &= \int_0^\infty f(t) \left[\int_0^t x^{p-1} dx\right] dt \\ &= \frac{1}{p} \int_0^\infty t^p f(t) dt. \end{aligned}$$

If $\tilde{f}(p) = \mathcal{M}\{f(x)\}$, then from the definition of the Mellin transform (3), we see that the above integral becomes

$$\begin{aligned} \mathcal{M}\left\{\int_x^\infty f(t) dt\right\} &= \frac{1}{p} \int_0^\infty t^p f(t) dt \\ &= \frac{1}{p} \tilde{f}(p+1), \end{aligned}$$

and we are done.

b. We will now prove the relation

$$\mathcal{M}\{I_n f(x)\} = \frac{\Gamma(p)}{\Gamma(p+n)} \tilde{f}(p+n)$$

by induction. The results of the previous exercise show that

$$\mathcal{M}\{I_1 f(x)\} = \mathcal{M}\left\{\int_x^\infty f(t) dt\right\} = \frac{1}{p} \tilde{f}(p+1) = \frac{\Gamma(p)}{\Gamma(p+1)} \tilde{f}(p+1)$$

and the base step is established.

Now assume the relation holds for n , i.e. assume that

$$\mathcal{M} \{I_n f(x)\} = \frac{\Gamma(p)}{\Gamma(p+n)} \tilde{f}(p+n).$$

Now, we see from the definition of the Mellin transform (3) that

$$\begin{aligned} \mathcal{M} \{I_{n+1} f(x)\} &= \mathcal{M} \left\{ \int_x^\infty I_n f(t) dt \right\} \\ &= \int_0^\infty x^{p-1} \left[\int_x^\infty I_n f(t) dt \right] dx. \end{aligned}$$

Let $g(t) = I_n f(t)$. Then, proceeding as we did in establishing the result of the base step, interchanging the order of integration from t to x yields

$$\begin{aligned} \mathcal{M} \{I_{n+1} f(x)\} &= \int_0^\infty x^{p-1} \left[\int_x^\infty g(t) dt \right] dx \\ &= \int_0^\infty g(t) \left[\int_0^t x^{p-1} dx \right] dt \\ &= \frac{1}{p} \int_0^\infty t^p g(t) dt \\ &= \frac{1}{p} \tilde{g}(p+1) \end{aligned} \tag{4}$$

where $\tilde{g}(p) = \mathcal{M} \{g(t)\}$.

From our assumption, we have that

$$\tilde{g}(p) = \mathcal{M} \{g(t)\} = \mathcal{M} \{I_n f(x)\} = \frac{\Gamma(p)}{\Gamma(p+n)} \tilde{f}(p+n).$$

Thus, the integral in (4) becomes

$$\begin{aligned} \mathcal{M} \{I_{n+1} f(x)\} &= \frac{1}{p} \tilde{g}(p+1) \\ &= \frac{\Gamma(p+1)}{p\Gamma(p+n+1)} \tilde{f}(p+n+1) \\ &= \frac{\Gamma(p)}{\Gamma(p+n+1)} \tilde{f}(p+n+1). \end{aligned}$$

Therefore, the result holds for $n+1$ and the relation holds in general for all $n > 0$.

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Problem 7.*Solution.*

Problem 8.*Solution.*

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