Homework Assignment 7

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Problem 1. State all of the KKT conditions for (N-max). More precisely state all of the following results for (N-max): KKT-FONC, KKT-FOSC, KKT-SONC, KKT-SOSC.

Solution. For the following theorems, we assume (N- max) has the following form

$$\begin{array}{lll} (N\text{- max}) & \text{maximize} & f(\boldsymbol{x}) \\ & \text{subject to} & \boldsymbol{h}(\boldsymbol{x}) = \mathbf{0} \\ & & \boldsymbol{g}(\boldsymbol{x}) \leq \mathbf{0} \end{array}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^m$, and $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^p$ with $m \leq n$. Additionally, define the following Lagrangian function to be $\boldsymbol{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) := f(\boldsymbol{x}) - \boldsymbol{\lambda}^\intercal \boldsymbol{h}(\boldsymbol{x}) - \boldsymbol{\mu}^\intercal \boldsymbol{g}(\boldsymbol{x})$.

Theorem 1 (KKT-FONC for (N- max))**.** Let f, $\boldsymbol{g}$, $\boldsymbol{h} \in C^1$ and let $\boldsymbol{x}^*$ be a regular point and local maximizer for the problem (N- max). Then, there exist $\boldsymbol{\lambda}^* \in \mathbb{R}^m$ and $\boldsymbol{\mu}^* \in \mathbb{R}^p$ such that:

i. $\boldsymbol{\mu}^* \geq \mathbf{0}$.

ii. $D_x \boldsymbol{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) = Df(\boldsymbol{x}^*) - \boldsymbol{\lambda}^{*\intercal} D\boldsymbol{h}(\boldsymbol{x}^*) - \boldsymbol{\mu}^{*\intercal} D\boldsymbol{g}(\boldsymbol{x}^*) = \mathbf{0}^\intercal$.

iii. $\mu^{*\intercal} \boldsymbol{g}(\boldsymbol{x}^*) = 0$.

Note that there are no explicit first-order conditions that are sufficient in general to show optimality.

Theorem 2 (KKT-SONC for (N- max))**.** Let f, $\boldsymbol{g}$, $\boldsymbol{h} \in C^2$ and let $\boldsymbol{x}^*$ be a regular point and local maximizer for the problem (N- max). Then, there exist $\boldsymbol{\lambda}^* \in \mathbb{R}^m$ and $\boldsymbol{\mu}^* \in \mathbb{R}^p$ such that:

i. $\boldsymbol{\mu}^* \geq \mathbf{0}, D_x \boldsymbol{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) = \mathbf{0}^\intercal, \mu^{*\intercal} \boldsymbol{g}(\boldsymbol{x}^*) = 0$.

ii. For all $\boldsymbol{y} \in T(\boldsymbol{x}^*)$, we have that $\boldsymbol{y}^\intercal D_x^2 \boldsymbol{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) \boldsymbol{y} \leq 0$.

Theorem 3 (KKT-SOSC for (N- max))**.** Let f, $\boldsymbol{g}$, $\boldsymbol{h} \in C^2$ and suppose there exists a feasible point $\boldsymbol{x}^*$ and vectors $\boldsymbol{\lambda}^* \in \mathbb{R}^m$ and $\boldsymbol{\mu}^* \in \mathbb{R}^p$ such that:

i. $\boldsymbol{\mu}^* \geq \mathbf{0}, D_x \boldsymbol{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) = \mathbf{0}^\intercal, \mu^{*\intercal} \boldsymbol{g}(\boldsymbol{x}^*) = 0$.

ii. For all $\boldsymbol{y} \in T(\boldsymbol{x}^*)$, we have that $\boldsymbol{y}^\intercal D_x^2 \boldsymbol{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) \boldsymbol{y} < 0$.

Then $\boldsymbol{x}^*$ is a strict local maximizer for the problem (N- max).

□

Problem 2. Find local minimizers for

$$
\begin{array}{lrl}
(N\text{- min}) & \text{minimize} & x_1^2 + 6x_1x_2 - 4x_1 - 2x_2 \\
& \text{subject to} & x_1^2 + 2x_2 \leq 1 \\
& & 2x_1 - 2x_2 \leq 1.
\end{array}
$$

Solution. □

Problem 3. Consider the problem of optimizing

$$
\begin{array}{lll}
(N) & \text{minimize (maximize)} & (x_1-2)^2+(x_2-1)^2 \\
 & & x_2 - x_1^2 \quad\quad \geq 0 \\
 & \text{subject to} & 2 - x_1 - x_2 \ \ \geq 0 \\
 & & x_1 \quad\quad\quad\quad\ \ \geq 0.
\end{array}
$$

Let $x^* = [0, 0]$.

a. Does x^* satisfy the KKT-FONC for minimization or maximization? What are the KKT multipliers?

b. Does x^* satisfy the KKT-SOSC? Justify your answer.

Solution. □

Problem 4. Consider the problem with equality constraint

$$\begin{array}{rl} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0}. \end{array}$$

We can convert the above into the equivalent optimization problem

$$\begin{array}{rl} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \frac{1}{2}\left\|\boldsymbol{h}(\boldsymbol{x})\right\|^2 \leq 0. \end{array}$$

Write down the KKT condition for the equivalent problem and explain why the KKT theorem cannot be applied in this case.

Solution. □