Homework Assignment 3

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Problem 2.20. Apply the Fourier cosine transform to find the solution $u(x, y)$ of the problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0, && 0 < x < \infty, \quad 0 < y < \infty \\ u(x, 0) &= H(a - x), && x < a \\ u_x(0, y) &= 0, && 0 < x, y < \infty. \end{aligned}$$

Solution. Consider the function $u(x, y)$. The Fourier cosine transform of u with respect to x is defined as

$$\mathscr{F}_c \{u(x, y)\} = U_c(k, y) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x, y) \cos(kx) dx.$$

From this definition we see using the Leibniz integral rule that

$$\begin{aligned} \mathscr{F}_c \left\{ \frac{\partial^n u(x, y)}{\partial y^n} \right\} &= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^n u(x, y)}{\partial y^n} \cos(kx) dx \\ &= \frac{d^n}{dy^n} \left[\sqrt{\frac{2}{\pi}} \int_0^\infty u(x, y) \cos(kx) dx \right] \\ &= \frac{d^n}{dy^n} \left[\mathscr{F}_c \{u(x, y)\} \right]. \end{aligned}$$

The transforms of the partials of u with respect to x are not as easy to characterize. Nevertheless, we see from the properties of the Fourier cosine transform that

$$\mathscr{F}_c \left\{ \frac{\partial u(x, y)}{\partial x} \right\} = k \mathscr{F}_s \{u(x, y)\} - \sqrt{\frac{2}{\pi}} u(0, y)$$

and

$$\mathscr{F}_c \left\{ \frac{\partial^2 u(x, y)}{\partial x^2} \right\} = -k^2 \mathscr{F}_c \{u(x, y)\} - \sqrt{\frac{2}{\pi}} u_x(0, y)$$

Let $U_c(x, y) = \mathscr{F}_c \{u(x, y)\}$. Then, applying the Fourier cosine transform to the first differential equation shows that

$$\mathscr{F}_c \{u_{xx} + u_{yy}\} = -k^2 U_c(k, y) - \sqrt{\frac{2}{\pi}} u_x(0, y) + \frac{d^2}{dy^2} [U_c(k, y)] = 0 = \mathscr{F}_c \{0\}.$$

From the third equation we see that $u_x(0, y) = 0$ for all $0 < x, y < \infty$ which implies that the above equation reduces to

$$\frac{d^2}{dy^2}\left[U_c(k, y)\right] - k^2 U_c(k, y) = 0.$$

This is a second-order linear homogeneous differential equation, the solution to which is readily seen to be

$$U_c(k, y) = c_1 e^{-ky} + c_2 e^{ky}.$$

However, since $U_c(k, y) \to 0$ as $k \to \infty$, we must have that $c_2 = 0$. Thus, the solution to the previous differential equation is given by

$$U_c(k, y) = c_1 e^{-ky}. \tag{1}$$

We now apply the Fourier cosine transform to the second differential equation yielding

$$\mathscr{F}_c\left\{u(x, 0)\right\} = U_c(k, 0) = \mathscr{F}_c\left\{H(a - x)\right\}.$$

Using the form (1) of the solution to the transformed differential equation and a table of Fourier cosine transforms we see that

$$U_c(k, 0) = c_1 = \mathscr{F}_c\left\{H(a - x)\right\} = \sqrt{\frac{2}{\pi}}\left(\frac{\sin ak}{k}\right).$$

Thus, the solution to the transformed differential equation with the boundary conditions listed above is given by

$$U_c(k, y) = \mathscr{F}_c\left\{H(a - x)\right\} e^{-ky} = \sqrt{\frac{2}{\pi}}\left(\frac{\sin ak}{k}\right) e^{-ky}.$$

Therefore, taking the inverse Fourier cosine transform to both sides shows that the solution to the original differential equation is given by

$$\begin{aligned} u(x, y) = \mathscr{F}_c^{-1}\left\{U_c(k, y)\right\} &= \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{\sin ak}{k}\right) e^{-ky}\cos kx dk \\ &= \frac{2}{\pi}\int_0^\infty \left(\frac{\sin ak}{k}\right) e^{-ky}\cos kx dk. \end{aligned}$$

$\square$

Problem 2.23.

Solution. □

Problem 2.47.

Solution. □

Problem 2.48.

Solution. □

Problem 2.54.

Solution. □