Exam 1

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Problem 1. Find the Fourier Transforms of the following functions:

a. $f(x) = x^2 e^{-a|x|}$, $a > 0$,

b. $f(x) = \left(1 - \frac{|x|}{2}\right) H\left(1 - \frac{|x|}{2}\right)$.

Solution. Recall that if $f(x) \in L^1(\mathbb{R})$, then the Fourier Transform of f is defined to be

$$\mathscr{F}\left\{f(x)\right\} = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx. \tag{1}$$

a. If $f(x) = x^2 e^{-a|x|}$, $a > 0$, we see from (1) that, by definition, the Fourier Transform of f is given by

$$\begin{aligned}
\mathscr{F}\left\{f(x)\right\} = F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-a|x|} e^{-ikx} dx \\
&= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} x^2 e^{-(-a+ik)x} dx + \int_{0}^{\infty} x^2 e^{-(a+ik)x} dx\right].
\end{aligned}$$

Now, we see by integration by parts that

$$\begin{aligned}
\int_c^d x^2 e^{-(b+ik)x} dx &= -\frac{x^2}{b+ik} e^{-(b+ik)x}\Big|_c^d + \frac{2}{b+ik} \int_c^d x e^{-(b+ik)x} dx \\
&= -\frac{x^2}{b+ik} e^{-(b+ik)x}\Big|_c^d + \frac{2}{b+ik}\left[-\frac{x}{b+ik} e^{-(b+ik)x}\Big|_c^d + \frac{1}{b+ik}\int_c^d e^{-(b+ik)x} dx\right] \\
&= -\frac{x^2}{b+ik} e^{-(b+ik)x}\Big|_c^d + \frac{2}{b+ik}\left[-\frac{x}{b+ik} e^{-(b+ik)x}\Big|_c^d - \frac{1}{(b+ik)^2} e^{-(b+ik)x}\Big|_c^d\right].
\end{aligned}$$

Thus,

$$\int_{-\infty}^{0} x^2 e^{-(-a+ik)x} dx = -\frac{2}{(-a+ik)^3}$$

and

$$\int_0^{\infty} x^2 e^{-(a+ik)x} dx = \frac{2}{(a+ik)^3}.$$

Therefore,

$$\begin{aligned}
\mathscr{F}\left\{f(x)\right\} = F(k) &= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} x^2 e^{-(-a+ik)x} dx + \int_0^{\infty} x^2 e^{-(a+ik)x} dx\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[-\frac{2}{(-a+ik)^3} + \frac{2}{(a+ik)^3}\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{4a(a^2-3k)}{(a^2+k^2)^3}\right] \\
&= \sqrt{\frac{2}{\pi}}\left[\frac{2a(a^2-3k)}{(a^2+k^2)^3}\right].
\end{aligned}$$

b. Recall that the Heaviside function H is defined as

$$H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Thus,

$$H\left(1 - \frac{|x|}{2}\right) = \begin{cases} 1 & \text{if } |x| < 2 \\ 0 & \text{if } |x| > 2. \end{cases}$$

If $f(x) = \left(1 - \frac{|x|}{2}\right) H\left(1 - \frac{|x|}{2}\right)$, we see from (1) that, by definition, the Fourier Transform of f is given by

$$\begin{aligned}
\mathscr{F}\left\{f(x)\right\} = F(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(1 - \frac{|x|}{2}\right) H\left(1 - \frac{|x|}{2}\right) e^{-ikx} dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-2}^{2}\left(1 - \frac{|x|}{2}\right) e^{-ikx} dx \\
&= \frac{1}{\sqrt{2\pi}}\left[\int_{-2}^{0}\left(1 + \frac{x}{2}\right) e^{-ikx} dx + \int_0^2 \left(1 - \frac{x}{2}\right) e^{-ikx} dx\right].
\end{aligned}$$

Now, we see by integration by parts that

$$\begin{aligned}
\int_c^d \left(1 \pm \frac{x}{2}\right) e^{-ikx} dx &= \frac{i}{k} e^{-ikx}\Big|_c^d \pm \frac{1}{2}\int_c^d x e^{-ikx} dx \\
&= \frac{i}{k} e^{-ikx}\Big|_c^d \pm \frac{1}{2}\left[\frac{ix}{k} e^{-ikx}\Big|_c^d - \frac{i}{k}\int_c^d e^{-ikx} dx\right] \\
&= \frac{i}{k} e^{-ikx}\Big|_c^d \pm \frac{1}{2}\left[\frac{ix}{k} e^{-ikx}\Big|_c^d + \frac{e^{-ikx}}{k^2}\Big|_c^d\right].
\end{aligned}$$

Thus,

$$\int_{-2}^{0} \left(1+\frac{x}{2}\right) e^{-ikx} dx = \frac{1-e^{2ik}+2ik}{2k^2}$$

and

$$\int_{0}^{2} \left(1-\frac{x}{2}\right) e^{-ikx} dx = \frac{1-e^{-2ik}-2ik}{2k^2}$$

Therefore, using the definition of the complex exponential and various trigonometric identities, we have that

$$\begin{aligned}
\mathscr{F}\left\{f(x)\right\} = F(k) &= \frac{1}{\sqrt{2\pi}}\left[\int_{-2}^{0}\left(1+\frac{x}{2}\right)e^{-ikx}dx + \int_{0}^{2}\left(1-\frac{x}{2}\right)e^{-ikx}dx\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{1-e^{2ik}+2ik}{2k^2} + \frac{1-e^{-2ik}-2ik}{2k^2}\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{k^2} - \frac{e^{-2ik}+e^{2ik}}{2k^2}\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{1-\cos 2k}{k^2}\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{1-(\cos^2 k - \sin^2 k)}{k^2}\right] \\
&= \frac{2\sin^2 k}{\sqrt{2\pi}k^2}.
\end{aligned}$$

□

Problem 2. Find the Laplace Transforms of the following functions:

a. $f(t) = \displaystyle\int_0^t \frac{\sin ax}{x} dx$,

b. $f(t) = tH(t-a)$.

Solution. Recall that if $f(t) \in L^1(\mathbb{R})$, then the Laplace Transform of f is defined to be

$$\mathscr{L}\{f(t)\} = \bar{f}(s) = \int_0^\infty f(t)e^{-st}dt, \quad \text{Re } s > 0. \tag{2}$$

Note that it can be shown that the Laplace transform satisfies the important property

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\left[\mathscr{L}\{f(t)\}\right]. \tag{3}$$

a. Let $f(t) = \displaystyle\int_0^t \frac{\sin ax}{x} dx$. Then $f(0) = 0$ and $f'(t) = \dfrac{\sin at}{t}$ so that $tf'(t) = \sin at$. This implies that

$$\mathscr{L}\{tf'(t)\} = \mathscr{L}\{\sin at\} = \frac{a}{s^2 + a^2}.$$

The Laplace transform satisfies the following property that relates a function's derivative to its Laplace Transform:

$$\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - f(0).$$

This combined with (3) shows that

$$\mathscr{L}\{tf'(t)\} = -\frac{d}{ds}\left[\mathscr{L}\{f'(t)\}\right] = -\frac{d}{ds}\left[s\mathscr{L}\{f(t)\} - f(0)\right] = \frac{a}{s^2 + a^2},$$

or that

$$\frac{d}{ds}\left[s\mathscr{L}\{f(t)\}\right] = -\frac{a}{s^2 + a^2}.$$

Integrating both sides of the above equation yields that

$$s\mathscr{L}\{f(t)\} = -\int \frac{a}{s^2 + a^2} ds = -\tan^{-1}(s/a) + C.$$

In order to determine the constant of integration, we use the Initial Value Theorem which states that

$$\lim_{s\to\infty} s\mathscr{L}\{f(t)\} = \lim_{t\to 0} f(t) = f(0).$$

Since $f(0) = 0$, this implies that

$$\lim_{s\to\infty} s\mathscr{L}\{f(t)\} = \lim_{s\to\infty}\left[-\tan^{-1}(s/a) + C\right] = 0$$

or that $C = \frac{\pi}{2}$. Therefore, we have that

$$s\mathscr{L}\left\{f(t)\right\} = -\tan^{-1}(s/a) + \frac{\pi}{2} = \tan^{-1}(a/s)$$

so that

$$\mathscr{L}\left\{f(t)\right\} = \frac{\tan^{-1}(a/s)}{s}.$$

b. Let $f(t) = tH(t-a)$ and assume that $a > 0$. By property (3) we see that

$$\mathscr{L}\left\{tH(t-a)\right\} = -\frac{d}{ds}\left[\mathscr{L}\left\{H(t-a)\right\}\right].$$

From our table of Laplace Transforms, we have that

$$\mathscr{L}\left\{H(t-a)\right\} = \frac{e^{-as}}{s},$$

assuming that $a > 0$. Therefore,

$$\begin{aligned}\mathscr{L}\left\{f(t)\right\} = -\frac{d}{ds}\left[\mathscr{L}\left\{H(t-a)\right\}\right] &= -\frac{d}{ds}\left[s^{-1}e^{-as}\right] \\ &= -\left[-s^{-2}e^{-as} - as^{-1}e^{-as}\right] \\ &= \frac{(1+as)e^{-as}}{s^2}.\end{aligned}$$

$\square$

Problem 3. Solve the following integral equations:

a. $\displaystyle\int_0^\infty f(x)\sin kx dx = \begin{cases} 1-k & k<1 \\ 0 & k>1 \end{cases},$

b. $\displaystyle\int_{-\infty}^\infty \frac{f(t)}{(x-t)^2+4}dt = \frac{1}{x^2+9}.$

Solution. a. Recall that the definition of the Fourier Sine Transform is given by

$$\mathscr{F}_s\{f(x)\} = F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin kx dx.$$

Thus, we see that

$$\int_0^\infty f(x)\sin kx dx = \sqrt{\frac{\pi}{2}}\mathscr{F}_s\{f(x)\} = \sqrt{\frac{\pi}{2}}F_s(k).$$

Let $G_s(k) = \begin{cases} 1-k & k<1 \\ 0 & k>1 \end{cases}$. Then the above integral equation becomes

$$F_s(k) = \sqrt{\frac{2}{\pi}}G_s(k).$$

Thus, applying the inverse Fourier Sine Transform, we have that

$$f(x) = \mathscr{F}_s^{-1}\{F_s(k)\} = \sqrt{\frac{2}{\pi}}\mathscr{F}_s^{-1}\{G_s(k)\}$$

where the inverse Fourier Sine Transform is defined as

$$g(x) = \mathscr{F}_s^{-1}\{G_s(k)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty G_s(k)\sin kx dk. \tag{4}$$

Therefore, the solution to the integral equation is

$$\begin{aligned} f(x) = \sqrt{\frac{2}{\pi}}\mathscr{F}_s^{-1}\{G_s(k)\} &= \frac{2}{\pi}\int_0^\infty G_s(k)\sin kx dk \\ &= \frac{2}{\pi}\int_0^1 (1-k)\sin kx dk \\ &= \frac{2}{\pi}\left[\frac{1-\cos x}{x} - \frac{\sin x - x\cos x}{x^2}\right] \\ &= \frac{2}{\pi}\left[\frac{x-\sin x}{x^2}\right]. \end{aligned}$$

b. Recall that the convolution of two functions f and g is defined such that

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi.$$

Let $g(x) = \dfrac{1}{x^2+2^2}$. Then we see that

$$\int_{-\infty}^{\infty} \frac{f(t)}{(x-t)^2+4}dt = \int_{-\infty}^{\infty} f(t)g(x-t)dt = \sqrt{2\pi}(g * f)(x) = \sqrt{2\pi}(f * g)(x).$$

Now, let $h(x) = \dfrac{1}{x^2+3^2}$. Then in light of the above remarks, the integral equation becomes

$$\int_{-\infty}^{\infty} \frac{f(t)}{(x-t)^2+4}dt = \sqrt{2\pi}(f * g)(x) = h(x) = \frac{1}{x^2+9}.$$

Applying the Fourier transform to the integral equation, we see by the Convolution Theorem that

$$\mathscr{F}\left\{\sqrt{2\pi}(f * g)(x)\right\} = \sqrt{2\pi}F(k)G(k) = H(k) = \mathscr{F}\left\{h(x)\right\},$$

where $\mathscr{F}\{f(x)\} = F(k), \mathscr{F}\{g(x)\} = G(k)$, and $\mathscr{F}\{h(x)\} = H(k)$, respectively. From our table of Fourier transforms, we see that for $a > 0$ we have that

$$\mathscr{F}\left\{\frac{1}{x^2+a^2}\right\} = \sqrt{\frac{\pi}{2}}\frac{e^{-a|k|}}{a}. \tag{5}$$

Thus, from (5) we have that

$$F(k) = \frac{1}{\sqrt{2\pi}}\frac{H(k)}{G(k)} = \frac{2}{3\sqrt{2\pi}}\frac{e^{-3|k|}}{e^{-2|k|}} = \frac{2e^{-|k|}}{3\sqrt{2\pi}}.$$

Applying the inverse Fourier Transform to this equation yields that

$$f(x) = \mathscr{F}^{-1}\left\{F(k)\right\} = \frac{2}{3\sqrt{2\pi}}\mathscr{F}^{-1}\left\{e^{-|k|}\right\}.$$

But from (5), we know that

$$\mathscr{F}^{-1}\left\{e^{-|k|}\right\} = \sqrt{\frac{2}{\pi}}\frac{1}{x^2+1}.$$

Therefore, the solution to the integral equation is given by

$$\begin{aligned} f(x) = \frac{2}{3\sqrt{2\pi}}\mathscr{F}^{-1}\left\{e^{-|k|}\right\} &= \frac{2}{3\sqrt{2\pi}}\left[\sqrt{\frac{2}{\pi}}\frac{1}{x^2+1}\right] \\ &= \frac{2}{3\pi}\left[\frac{1}{x^2+1}\right]. \end{aligned}$$

$\square$

Problem 4. Show that

$$\int_0^\infty F_s(k)G_c(k)\sin kx dk = \frac{1}{2}\int_0^\infty g(\xi)\left[f(\xi+x)-f(\xi-x)\right]d\xi$$

where

$$F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin kx dx$$

and

$$G_c(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\cos kx dx.$$

Solution. Using the definition of $G_c(k)$, we see that

$$\int_0^\infty F_s(k)G_c(k)\sin kx dk = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(k)\sin kx\left[\int_0^\infty g(\xi)\cos k\xi d\xi\right]dk$$

Interchanging the order of integration from ξ to k shows that

$$\begin{aligned}\int_0^\infty F_s(k)G_c(k)\sin kx dk &= \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(k)\sin kx\left[\int_0^\infty g(\xi)\cos k\xi d\xi\right]dk \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty g(\xi)\left[\int_0^\infty F_s(k)\cos k\xi\sin kx dk\right]d\xi\end{aligned}$$

Using the following trigonometric identity

$$\cos k\xi\sin kx = \frac{\sin k(\xi+x)-\sin k(\xi-x)}{2},$$

we then see that

$$\begin{aligned}\int_0^\infty F_s(k)G_c(k)\sin kx dk &= \sqrt{\frac{2}{\pi}}\int_0^\infty g(\xi)\left[\int_0^\infty F_s(k)\cos k\xi\sin kx dk\right]d\xi \\ &= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty g(\xi)\left[\int_0^\infty F_s(k)\sin k(\xi+x)dk - \int_0^\infty F_s(k)\sin k(\xi-x)dk\right]d\xi \\ &= \frac{1}{2}\int_0^\infty g(\xi)\left[f(\xi+x)-f(\xi-x)\right]d\xi,\end{aligned}$$

where the last line follows using (4), the definition of the inverse Fourier Sine Transform.

Therefore,

$$\int_0^\infty F_s(k)G_c(k)\sin kx dk = \frac{1}{2}\int_0^\infty g(\xi)\left[f(\xi+x)-f(\xi-x)\right]d\xi,$$

and we are done.

□

Problem 5. Apply the Fourier Transform to solve the following initial value problem for the heat equation:

$$\begin{aligned}
&\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad -\infty < x < \infty, \\
&u(x,0) = \phi(x), \quad t > 0.
\end{aligned}$$

Solution. □

Problem 6. Evaluate the following definite integrals:

a. $\displaystyle\int_0^\infty \frac{\sin ax \sin bx}{x^2} dx,$

b. $\displaystyle\int_0^\infty \frac{(a^2 - x^2)^2}{(x^2 + a^2)^4} dx, \quad a > 0.$

Solution. □

Problem 7. Use the Fourier Sine Transform to solve the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \infty$$

with the boundary data $0 < y < L$

$$\begin{aligned} &u(x, L) = 0, \quad u(x, 0) = f(x), \\ &u(0, y) = 0, \quad u(x, y) \to 0 \text{as } x \to \infty \text{ uniformly in } y. \end{aligned}$$

Solution. ☐

Problem 8. Apply the Fourier Transform to solve the 3-dimensional wave problem

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \quad -\infty < x, y, z < \infty,$$

subject to the initial conditions

$$u(x,y,z,t)|_{t=0} = 0$$
$$\left.\frac{\partial u(x,y,z,t)}{\partial t}\right|_{t=0} = \delta(x,y,z).$$

Solution. □

Problem 9. Show that if E is a solution of an m-th order partial differential equation

$$P(\partial)u = \sum_{k=0}^{m} a_k \partial^k u = \delta,$$

where δ is the Dirac delta function, then $E * f$ is the solution of the partial differential equation $P(\partial)u = f$, where $*$ is the convolution.

Solution. $\square$