

Homework Assignment 7

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Problem 4.28. Using the Laplace transform, evaluate the following integrals:

a. $f(t) = \int_0^\infty \frac{\sin tx}{\sqrt{x}} dx,$

e. $f(t) = \int_0^\infty e^{-tx^2} dx, 0 < t.$

Solution. a. We begin by taking the Laplace transform of $f(t)$. Doing so yields

$$\begin{aligned}\bar{f}(s) &= \mathcal{L}\{f(t)\} = \mathcal{L}\left\{\int_0^\infty \frac{\sin tx}{\sqrt{x}} dx\right\} \\ &= \int_0^\infty \mathcal{L}\left\{\frac{\sin tx}{\sqrt{x}}\right\} dx \\ &= \int_0^\infty \frac{\sqrt{x}}{s^2 + x^2} dx.\end{aligned}$$

Using a computer algebra system, we see that this integral evaluates to

$$\begin{aligned}\bar{f}(s) &= \int_0^\infty \frac{\sqrt{x}}{s^2 + x^2} dx \\ &= \frac{\pi}{\sqrt{2s}}.\end{aligned}$$

From our table of Laplace transforms, we see that

$$\mathcal{L}^{-1}\left\{\frac{\Gamma(a+1)}{s^{a+1}}\right\} = t^a.$$

In particular, for $a = -1/2$, we see that

$$\mathcal{L}^{-1}\left\{\frac{\Gamma(1/2)}{s^{-1/2}}\right\} = \mathcal{L}^{-1}\left\{\frac{\sqrt{\pi}}{s^{-1/2}}\right\} = t^{-1/2}.$$

Therefore, the evaluation of the original integral is

$$\begin{aligned} f(t) &= \mathcal{L}^{-1} \{ \bar{f}(s) \} = \mathcal{L}^{-1} \left\{ \frac{\pi}{\sqrt{2s}} \right\} \\ &= \sqrt{\frac{\pi}{2}} \mathcal{L}^{-1} \left\{ \frac{\sqrt{\pi}}{s^{-1/2}} \right\} \\ &= \sqrt{\frac{\pi}{2t}}. \end{aligned}$$

e. Applying the Laplace transform to $f(t)$ yields

$$\begin{aligned} \bar{f}(s) &= \mathcal{L} \{ f(t) \} = \mathcal{L} \left\{ \int_0^\infty e^{-tx^2} dx \right\} \\ &= \int_0^\infty \mathcal{L} \{ e^{-tx^2} \} dx \\ &= \int_0^\infty \frac{1}{s+x^2} dx \end{aligned}$$

Using a computer algebra system, we see that

$$\begin{aligned} \bar{f}(s) &= \int_0^\infty \frac{1}{s+x^2} dx \\ &= \frac{\pi}{2\sqrt{s}}. \end{aligned}$$

Therefore, using previous arguments, we see that the evaluation of the original integral is

$$\begin{aligned} f(t) &= \mathcal{L}^{-1} \{ \bar{f}(s) \} = \mathcal{L}^{-1} \left\{ \frac{\pi}{2\sqrt{s}} \right\} \\ &= \sqrt{\frac{\pi}{4}} \mathcal{L}^{-1} \left\{ \frac{\sqrt{\pi}}{s^{-1/2}} \right\} \\ &= \sqrt{\frac{\pi}{4t}}. \end{aligned}$$

□

Problem 4.29. Show that

$$\text{b. } I(a) = \int_0^\infty e^{-ax} \left(\frac{\sin qx - \sin px}{x} \right) dx = \tan^{-1} \left(\frac{q}{a} \right) - \tan^{-1} \left(\frac{p}{a} \right)$$

Solution. b. Let $f(x) = \sin qx - \sin px$ and $g(x) = \frac{f(x)}{x}$.

From the definition of the Laplace transform, we see that this integral is the Laplace transform of $\frac{f(x)}{x}$ with respect to x in the variable a , i.e.

$$I(a) = \int_0^\infty e^{-ax} \left(\frac{\sin qx - \sin px}{x} \right) dx = \mathcal{L} \left\{ \frac{f(x)}{x} \right\} = \bar{g}(a).$$

From a previous result, we know that

$$I(a) = \mathcal{L} \left\{ \frac{f(x)}{x} \right\} = \int_a^\infty \bar{f}(a) da$$

where $\bar{f}(a) = \mathcal{L} \{f(x)\}$. Our table of Laplace transforms shows that

$$\begin{aligned} \bar{f}(a) &= \mathcal{L} \{f(x)\} = \mathcal{L} \{\sin qx - \sin px\} \\ &= \frac{q}{a^2 + q^2} - \frac{p}{a^2 + p^2}. \end{aligned}$$

Thus, we see that

$$I(a) = \int_a^\infty \bar{f}(a) da = \int_a^\infty \frac{q}{a^2 + q^2} - \frac{p}{a^2 + p^2} da.$$

Recall that

$$\int \frac{t}{a^2 + t^2} da = \tan^{-1} \left(\frac{a}{t} \right) + C.$$

Therefore, we have that

$$\begin{aligned} I(a) &= \int_a^\infty \frac{q}{a^2 + q^2} - \frac{p}{a^2 + p^2} da \\ &= \left[\frac{\pi}{2} - \tan^{-1} \left(\frac{a}{q} \right) \right] - \left[\frac{\pi}{2} - \tan^{-1} \left(\frac{a}{p} \right) \right] \\ &= \tan^{-1} \left(\frac{q}{a} \right) - \tan^{-1} \left(\frac{p}{a} \right). \end{aligned}$$

□

Problem 4.32.*Solution.*

Problem 4.35. Using the Laplace transform, solve the following difference equations:

a. $\Delta u_n - 2u_n = 0, u_0 = 1$

b. $\Delta^2 u_n - 2u_{n+1} + 3u_n = 0, u_0 = 0, u_1 = 1.$

Solution. Define $S_n(t) = H(t - n) - H(t - n - 1)$ for $n \leq t < n + 1$ and define

$$u(t) = \sum_{n=0}^{\infty} u_n S_n(t).$$

It follows that for $n \leq t < n + 1$ we have that $u(t) = u_n$.

By a previous theorem, if $\bar{u}(s) = \mathcal{L}\{u(t)\}$, then

$$\mathcal{L}\{u(t+1)\} = e^s [\bar{u}(s) - u_0 \bar{S}_0(s)]$$

where $\bar{S}_0 = \frac{1}{s}(1 - e^{-s})$. It then follows that

$$\mathcal{L}\{u(t+2)\} = e^{2s} [\bar{u}(s) - (u_0 + u_1 e^{-s}) \bar{S}_0(s)].$$

a. Note that this difference equation is equivalent to

$$\Delta u_n - 2u_n = u_{n+1} - 3u_n = 0.$$

Applying the Laplace transform to the difference equation yields that

$$\mathcal{L}\{u_{n+1} - 3u_n\} = e^s [\bar{u}(s) - u_0 \bar{S}_0(s)] - 3\bar{u}(s) = 0 = \mathcal{L}\{0\}.$$

In light of the initial data, this equation becomes

$$e^s [\bar{u}(s) - \bar{S}_0(s)] - 3\bar{u}(s) = 0.$$

Thus, we see that

$$\bar{u}(s) = \frac{e^s \bar{S}_0(s)}{e^s - 3}.$$

Therefore, from a previous result, we see that the solution to the original difference equation is

$$u(t) = \mathcal{L}^{-1}\{\bar{u}(s)\} = \mathcal{L}^{-1}\left\{\frac{e^s \bar{S}_0(s)}{e^s - 3}\right\} = 3^n$$

b. Note that this difference equation is equivalent to

$$\Delta^2 u_n - 2u_{n+1} + 3u_n = u_{n+2} - 4u_{n+1} + 4u_n = 0.$$

Applying the Laplace transform to the difference equation yields that

$$\mathcal{L}\{u_{n+2} - 4u_{n+1} + 4u_n\} = e^{2s} [\bar{u}(s) - (u_0 + u_1 e^{-s}) \bar{S}_0(s)] - 4e^s [\bar{u}(s) - u_0 \bar{S}_0(s)] + 4\bar{u}(s) = 0.$$

In light of the initial data, this equation becomes

$$e^{2s} [\bar{u}(s) - e^{-s} \bar{S}_0(s)] - 4e^s \bar{u}(s) + 4\bar{u}(s) = 0.$$

Thus, we see that

$$\bar{u}(s) = \frac{e^s \bar{S}_0(s)}{(e^s - 2)^2}.$$

From a previous result, we know that

$$\mathcal{L} \{na^n\} = \frac{ae^s \bar{S}_0(s)}{(e^s - a)^2}$$

Therefore, we see that the solution to the original difference equation is

$$u(t) = \mathcal{L}^{-1} \{\bar{u}(s)\} = \mathcal{L}^{-1} \left\{ \frac{e^s \bar{S}_0(s)}{(e^s - 2)^2} \right\} = n2^{n-1}.$$

□

Problem 4.36.*Solution.*

Problem 4.37.*Solution.*

Problem 4.40.*Solution.*

Problem 4.43.*Solution.*

Problem 4.50.*Solution.*