Homework Assignment 1

Matthew Tiger

February 13, 2016

Problem 1. Solve the IVP:

$$y' = y^2 \cos(x), \quad y(0) = 2.$$

Solution. Note that this is a separable differential equation and after separating we see that

$$\begin{aligned}
\frac{dy}{y^2} &= \cos(x)dx \\
\int \frac{dy}{y^2} &= \int \cos(x)dx \\
-\frac{1}{y} &= \sin(x) + c_1
\end{aligned}$$

so that

$$y = -\frac{1}{\sin(x) + c_1}$$

is the general solution to the differential equation. Using the initial value $y(0) = 2$ and solving for c_1 we see that $c_1 = -1/2$ and the solution to the IVP is given by

$$y = -\frac{1}{\sin(x) - 1/2}.$$

$\square$

Problem 2. Review solutions of first-order linear ODEs (p. 14) and solve the IVP:

$$y' - xy = x^3, \quad y(1) = \frac{1}{2}.$$

Solution. The solution to the first-order linear ODE

$$y'(x) + p_0(x)y(x) = f(x)$$

is given by

$$y(x) = \frac{c_1}{I(x)} + \frac{1}{I(x)} \int_0^x f(t)I(t)dt, \quad I(x) = \exp\left(\int_0^x p_0(t)dt\right).$$

For this problem, we set $p_0(x) = -x$ and $f(x) = x^3$ and see that

$$I(x) = \exp\left(\int_0^x p_0(t)dt\right) = \exp\left(\int_0^x -tdt\right) = \exp\left(-\frac{x^2}{2}\right).$$

Thus the general solution to the ODE $y' - xy = x^3$ is given by

$$\begin{aligned}
y &= \frac{c_1}{\exp\left(-\frac{x^2}{2}\right)} + \frac{1}{\exp\left(-\frac{x^2}{2}\right)} \int_0^x t^3 \exp\left(-\frac{t^2}{2}\right) dt \\
&= \frac{c_1}{\exp\left(-\frac{x^2}{2}\right)} - \frac{\exp\left(-\frac{x^2}{2}\right)}{\exp\left(-\frac{x^2}{2}\right)}(2 + x^2) \\
&= \frac{c_1}{\exp\left(-\frac{x^2}{2}\right)} - (2 + x^2)
\end{aligned}$$

Using the initial value $y(1) = \frac{1}{2}$, we see that $c_1 = \frac{7}{2}\exp\left(-\frac{1}{2}\right)$ and the solution to the IVP is

$$y = \frac{7\exp\left(-\frac{1}{2}\right)}{2\exp\left(-\frac{x^2}{2}\right)} - (2 + x^2).$$

□

Problem 3. Let $Ly = y^{(4)} - 4y''' + 3y'' + 4y' - 4y$.

a. Find the general solutions of the homogeneous ODE $Ly = 0$.

b. Solve the IVP:

$$Ly = 0, \quad y(0) = 0, \quad y'(0) = -7, \quad y''(0) = 5, \quad y'''(0) = 9.$$

c. Solve the BVP:

$$Ly = 0, \quad y(0) = 1, \quad \lim_{x\to\infty} y(x) = 0.$$

Is this BVP well-posed?

d. Solve the BVP:

$$Ly = 0, \quad y(0) = 1, \quad \lim_{x\to-\infty} y(x) = 0.$$

Is this BVP well-posed?

Solution. a. The characteristic equation associated to the homogeneous ODE $Ly = 0$ is $m(x) = x^4 - 4x^3 + 3x^2 + 4x - 4$. The roots of the characteristic polynomial are $r_1 = -1$, $r_2 = 1$, $r_3 = 2$, and $r_4 = 2$.

Therefore, the general solution of the homogeneous ODE is

$$y(x) = c_1 e^{-x} + c_2 e^{x} + c_3 e^{2x} + c_4 x e^{2x}. \tag{1}$$

b. Through an abuse of notation, we note that the matrix associated to the Wronskian of this equation as function of x is given by

$$W(x) = \begin{bmatrix} e^{-x} & e^{x} & e^{2x} & xe^{2x} \\ -e^{-x} & e^{x} & 2e^{2x} & e^{2x} + 2xe^{2x} \\ e^{-x} & e^{x} & 4e^{2x} & 4e^{2x} + 4xe^{2x} \\ -e^{-x} & e^{x} & 8e^{2x} & 12e^{2x} + 8xe^{2x} \end{bmatrix}.$$

The solution to the IVP is determined by particular values of the coefficients in the general solution (1). These coefficients are found as the solution to the system of equations $W(0)\boldsymbol{c} = \boldsymbol{b}$ where

$$\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ -7 \\ 5 \\ 9 \end{bmatrix}.$$

The solution to this system is given by $\boldsymbol{c} = \langle 4, -3, -1, 2 \rangle$. Therefore, the solution to the IVP is

$$y(x) = 4e^{-x} - 3e^{x} - e^{2x} + 2xe^{2x}.$$

c. The general solution to the ODE, $y(x)$, is given by (1). The second condition that $\lim_{x\to\infty} y(x) = 0$ can not be satisfied by the general solution since $\lim_{x\to\infty} e^{ax} = \infty$ for $a > 0$. Therefore, the BVP as stated is not well-posed.

d. The general solution to the ODE, $y(x)$, is given by (1). The second condition that $\lim_{x\to-\infty} y(x) = 0$ can not be satisfied by the general solution since $\lim_{x\to-\infty} e^{ax} = \infty$ for $a < 0$. Therefore, the BVP as stated is not well-posed.

$\square$

Problem 4. Read §1.6 and then solve the ODEs:

$$xy' + 2y = x^2\sqrt{y}, \quad y' = \frac{4x^3 - 6xy^2 - 2xy}{x^2 + 6x^2y - 3y^2}, \quad y' + y^2 + (2x+1)y + 1 + x + x^2 = 0.$$

Solution. We begin with the differential equation

$$xy' + 2y = x^2\sqrt{y}.$$

Note that this equation can be rewritten as

$$y' = \left(-\frac{2}{x}\right) y + xy^{1/2}, \tag{2}$$

which is a Bernoulli equation with $P = 1/2$. Dividing (2) by $y^{1/2}$ and making the substitution $u(x) = y(x)^{1-1/2}$ yields the new linear differential equation

$$u'(x) = -\left(\frac{1}{x}\right) u(x) + \frac{x}{2}.$$

The solution to this linear equation is $u(x) = x^2/6 + c_1/x$ suggesting that

$$y(x) = u(x)^2 = \left(\frac{x^2}{6} + \frac{c_1}{x}\right)^2$$

is the solution to (2).

Let us next investigate

$$y' = \frac{4x^3 - 6xy^2 - 2xy}{x^2 + 6x^2y - 3y^2}.$$

Note that this equation can be written as

$$-(4x^3 - 6xy^2 - 2xy) + (x^2 + 6x^2y - 3y^2)y'(x) = 0.$$

Identifying $M(x,y) = -(4x^3 - 6xy^2 - 2xy)$ and $N(x,y) = (x^2 + 6x^2y - 3y^2)$, we notice that

$$\frac{\partial M(x,y)}{\partial y} = 12xy + 2x = \frac{\partial N(x,y)}{\partial x}$$

making this equation exact. The solution to the exact differential equation is then $f(x,y) = c_1$ where $f_x = M(x,y)$ and $f_y = N(x,y)$. Thus,

$$f(x,y) = \int f_x(x,y)dx = -\int (4x^3 - 6xy^2 - 2xy)dx = -x^4 + 3x^2y^2 + x^2y + h(y). \tag{3}$$

In order to find out what $h(y)$ is, we take the partial derivative of (3) and compare it with $N(x,y)$. Doing so, we see that

$$f_y(x,y) = x^2 + 6x^2y + h'(y) = x^2 + 6x^2y - 3y^2 = N(x,y)$$

implying that $h'(y) = -3y^2$ and that $h(y) = -y^3$. Therefore, the solution to the differential equation is

$$f(x, y) = -x^4 + 3x^2y^2 + x^2y - y^3 = c_1.$$

Finally let us investigate the differential equation

$$y' + y^2 + (2x + 1)y + 1 + x + x^2 = 0.$$

This equation can be rewritten as

$$y' = -y^2 - (2x + 1)y - (1 + x + x^2) \tag{4}$$

which is a Riccati equation. The procedure to find the solution of such equations is to produce a particular solution $y_p(x)$ to the equation and then find the general solution which will be in the form $y(x) = y_p(x) + u(x)$ by using this formula in the original equation. Note that $y_p(x) = -x$ is a particular solution of (4). Thus the general solution is of the form $y(x) = -x + u(x)$.

Making this substitution reveals the following Bernoulli equation in $u(x)$:

$$u'(x) = -u(x) - u(x)^2$$

The solution to this differential equation is $u(x) = -(e^{c_1}/(-e^x + e^{c_1}))$. Therefore, the general solution to (4) is

$$y(x) = -x - \frac{e^{c_1}}{-e^x + e^{c_1}}.$$

$\square$

Problem 5. a. Use mathematical induction to prove Leibnitz's differentiation rule:

$$D^k(fg) = \sum_{j=0}^{k} \binom{k}{j} (D^j f)(D^{k-j} g).$$

Here $f = f(x)$ and $g = g(x)$ are k-times differentiable functions and $D^k = \frac{d^k}{dx^k}$.

b. Consider the constant-coefficient ODE

$$Ly = D^n y + p_{n-1} D^{n-1} y + \cdots + p_1 Dy + p_0 y = 0, \tag{5}$$

where $p_0, p_1, \ldots, p_{n-1}$ are real numbers. Let r be a double root of the characteristic polynomial $P(z) = z^n + p_{n-1} z^{n-1} + \cdots + p_1 z + p_0$. Use Leibnitz's rule to show that the function xe^{rx} is a solution of (5).

c. Let r be a triple root of the characteristic polynomial $P(z)$ from part (b). Use Leibnitz's rule to show that the function $x^2 e^{rx}$ is then also a solution of (5).

d. Let r be a real number. Show that the functions e^{rx}, xe^{rx}, and $x^2 e^{rx}$ are linearly independent on $\mathbb{R}$.

Solution. a. Suppose that $k = 1$. Then our formula yields

$$D(fg) = \sum_{j=0}^{1} \binom{1}{j} (D^j f)(D^{1-j} g) = fD(g) + D(f)g,$$

which is the product rule for derivatives and the base case is established.

Now suppose the formula holds for $k = n$. Then, using the linear properties of the derivative, we see that

$$\begin{aligned} D^{n+1}(fg) = D(D^n(fg)) &= D\left(\sum_{j=0}^{n} \binom{n}{j} (D^j f)(D^{n-j} g)\right) \\ &= \sum_{j=0}^{n} \binom{n}{j} D((D^j f)(D^{n-j} g)) \end{aligned} \tag{6}$$

Using the product rule, we note that

$$D((D^j f)(D^{n-j} g)) = (D^j f)(D^{(n+1)-j} g) + (D^{j+1} f)(D^{n-j} g)$$

and replacing in (6) we have that

$$\begin{aligned} \sum_{j=0}^{n} \binom{n}{j} D((D^j f)(D^{n-j} g)) &= \sum_{j=0}^{n} \binom{n}{j} \left[(D^j f)(D^{(n+1)-j} g) + (D^{j+1} f)(D^{n-j} g)\right] \\ &= \sum_{j=0}^{n} \binom{n}{j} (D^j f)(D^{(n+1)-j} g) + \sum_{j=0}^{n} \binom{n}{j} (D^{j+1} f)(D^{n-j} g) \\ &= \sum_{j=0}^{n} \binom{n}{j} (D^j f)(D^{(n+1)-j} g) + \sum_{j=1}^{n+1} \binom{n}{j-1} (D^j f)(D^{(n+1)-j} g). \end{aligned} \tag{7}$$

Combining terms along with Pascal's rule allows us to combine the binomial coefficients in (7) and thus

$$\begin{aligned} D^{n+1}(fg) &= (D^0 f)(D^{n+1}g) + \sum_{j=1}^{n} \binom{n+1}{j} (D^j f)(D^{(n+1)-j}g) + (D^{n+1}f)(D^0 g) \\ &= \sum_{j=0}^{n+1} \binom{n+1}{j} (D^j f)(D^{k-j}g). \end{aligned}$$

Therefore, the formula holds for $k = n+1$ and the rule holds.

b. We wish to see if $y(x) = xe^{rx}$ is a solution of (5) given that r is a double root of the characteristic polynomial. Using Leibnitz's formula, note that for $k > 0$

$$\begin{aligned} D^k y(x) = D^k(xe^{rx}) &= \sum_{j=0}^{k} \binom{k}{j} (D^j x)(D^{k-j}e^{rx}) \\ &= xD^k e^{rx} + kD^{k-1}e^{rx} \\ &= xr^k e^{rx} + kr^{k-1}e^{rx} = e^{rx}(xr^k + kr^{k-1}) \end{aligned} \tag{8}$$

since $D^j x = 0$ if $j > 1$. Using the formula in (8) and replacing into the ODE, we see that for $y(x) = xe^{rx}$

$$\begin{aligned} Ly(x) &= e^{rx}(xr^n + nr^{n-1}) + p_{n-1}e^{rx}(xr^{n-1} + (n-1)r^{n-2}) + \cdots + p_1 e^{rx}(xr+1) + p_0 xe^{rx} \\ &= e^{rx}\left[(xr^n + nr^{n-1}) + p_{n-1}(xr^{n-1} + (n-1)r^{n-2}) + \cdots + p_1(xr+1) + p_0 x\right] \\ &= e^{rx}\left[x(r^n + p_{n-1}r^{n-1} + \cdots + p_1 r + p_0) + (nr^{n-1} + p_{n-1}(n-1)r^{n-2} + \cdots + p_1)\right] \\ &= e^{rx}\left[P(r) + P'(r)\right]. \end{aligned}$$

Since r is a root with multiplicity 2 of the polynomial $P(x)$, we know that $P(x) = (x-r)^2 q(x)$ and $P'(x) = 2(x-r)q(x) + (x-r)^2 q'(x)$ where the degree of $q(x)$ is $n-2$. This shows that $P(r) = P'(r) = 0$ and that $Ly(x) = 0$ for $y(x) = xe^{rx}$, i.e. $y(x)$ is a solution of the differential equation.

c. We wish to similarly see if $y(x) = x^2 e^{rx}$ is a solution of (5) given that r is a triple root of the characteristic polynomial. Using Leibnitz's formula, note that for $k > 0$

$$\begin{aligned} D^k y(x) = D^k(x^2 e^{rx}) &= \sum_{j=0}^{k} \binom{k}{j} (D^j x^2)(D^{k-j}e^{rx}) \\ &= x^2 D^k e^{rx} + 2kxD^{k-1}e^{rx} + k(k-1)D^{k-2}e^{rx} \\ &= x^2 r^k e^{rx} + 2kxr^{k-1}e^{rx} + k(k-1)r^{k-2}e^{rx} \\ &= e^{rx}\left(x^2 r^k + 2kxr^{k-1} + k(k-1)r^{k-2}\right) \end{aligned} \tag{9}$$

since $D^j x = 0$ if $j > 2$. Using the formula in (9) and replacing into the ODE, we see that for $y(x) = x^2 e^{rx}$

$$\begin{aligned}
Ly(x) &= e^{rx}\left(x^2 r^n + 2nxr^{n-1} + n(n-1)r^{n-2}\right) + \\
&\quad + p_{n-1}e^{rx}\left(x^2 r^{n-1} + 2(n-1)xr^{n-2} + (n-1)(n-2)r^{n-3}\right) + \cdots + \\
&\quad + p_1 e^{rx}(x^2 r + 2x) + p_0 x^2 e^{rx} \\
&= e^{rx}x^2(r^n + p_{n-1}r^{n-1} + \cdots + p_0) + \\
&\quad + e^{rx}2x(nr^{n-1} + p_{n-1}(n-1)r^{n-2} + \cdots + p_1) + \\
&\quad + e^{rx}(n(n-1)r^{n-2} + (n-1)(n-2)r^{n-3} + \cdots + 2p_2) \\
&= e^{rx}\left[P(r) + 2P'(r) + P''(r)\right].
\end{aligned}$$

Using the same argument as in (b), we know that since r is a root with multiplicity 3 of the polynomial $P(x)$, we see $P(r) = P'(r) = P''(r) = 0$ and that $Ly(x) = 0$ for $y(x) = x^2 e^{rx}$, i.e. $y(x)$ is a solution of the differential equation.

d. Note that e^{rx}, xe^{rx}, and $x^2 e^{rx}$ are linearly independent on $\mathbb{R}$ if the Wronskian of these functions is nonzero. It is clear that

$$W(x) = \begin{vmatrix} 1 & x & x^2 \\ r & (xr+1) & (rx^2+2x) \\ r^2 & (r^2 x + 2r) & (r^2x^2 + 4rx + 2) \end{vmatrix} = 2e^{3rx} \neq 0$$

if $x \in \mathbb{R}$. Therefore, the functions are linearly independent.

$\square$

Problem 6. Use the formula for the derivative of a determinant from the lectures, other properties of determinants, and the linear ODE (1.3.1) to verify identity (1.3.4) in the textbook.

Solution. We wish to show that

$$W'(x) = -p_{n-1}(x)W(x). \tag{10}$$

We know for a set of functions $\{y_1, y_2, \ldots, y_n\}$ that

$$\begin{aligned}
W'(x) &= \begin{vmatrix} y_1' & y_2' & y_3' & \cdots & y_n' \\ y_1' & y_2' & y_3' & \cdots & y_n' \\ y_1'' & y_2'' & y_3'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots & \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1'' & y_2'' & y_3'' & \cdots & y_n'' \\ y_1'' & y_2'' & y_3'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots & \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} + \\
&\quad + \cdots + \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1' & y_2' & y_3' & \cdots & y_n' \\ y_1'' & y_2'' & y_3'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots & \\ y_1^{(n)} & y_2^{(n)} & y_3^{(n)} & \cdots & y_n^{(n)} \end{vmatrix} \\
&= \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1' & y_2' & y_3' & \cdots & y_n' \\ y_1'' & y_2'' & y_3'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots & \\ y_1^{(n)} & y_2^{(n)} & y_3^{(n)} & \cdots & y_n^{(n)} \end{vmatrix} \qquad (11)
\end{aligned}$$

since the determinant of a matrix with dependent rows is zero. Note that the elementary row operation of adding a multiple of one row to another row does not change the determinant. Thus, we perform the following $n-1$ row operations on row $[n]$: add $p_{i-1}[i]$ to row $[n]$ where $1 \leq i \leq n-2$. Using the identity found in (11), we see that after the row operations the determinant becomes

$$\begin{aligned}
W'(x) &= \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1' & y_2' & y_3' & \cdots & y_n' \\ y_1'' & y_2'' & y_3'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots & \\ y_1^{(n)} + \sum_{j=0}^{n-2} p_j y_1^{(j)} & y_2^{(n)} + \sum_{j=0}^{n-2} p_j y_2^{(j)} & y_3^{(n)} + \sum_{j=0}^{n-2} p_j y_3^{(j)} & \cdots & y_n^{(n)} + \sum_{j=0}^{n-2} p_j y_n^{(j)} \end{vmatrix} \\
&= \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1' & y_2' & y_3' & \cdots & y_n' \\ y_1'' & y_2'' & y_3'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots & \\ -p_{n-1}y_1^{(n-1)} & -p_{n-1}y_2^{(n-1)} & -p_{n-1}y_3^{(n-1)} & \cdots & -p_{n-1}y_n^{(n-1)} \end{vmatrix}
\end{aligned}$$

where $y_i^{(n)} + \sum_{j=0}^{n-2} p_j y_i^{(j)} = -p_{n-1} y_i^{n-1}$ since each y_i satisfies the original differential equation. Thus, after removing the p_{n-1} term from the determinant we have that

$$W'(x) = -p_{n-1} \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1' & y_2' & y_3' & \cdots & y_n' \\ y_1'' & y_2'' & y_3'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots & \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = -p_{n-1} W(x)$$

and the original identity (10) is satisfied. $\square$