Homework Assignment 4

Matthew Tiger

April 13, 2016

Problem 1. Find the first three terms in the asymptotic expansions of $x \to 0^+$ of the following integrals:

$$\int_x^1 \cos(xt)dt, \qquad \int_0^{1/x} e^{-t^2} dt.$$

Solution. If the function $f(t,x)$ possesses the asymptotic expansion

$$f(t,x) \sim \sum_{n=0}^{\infty} f_n(t)(x-x_0)^{\alpha n} \qquad \text{as } x \to x_0$$

for some $\alpha > 0$, uniformly for $a \le t \le b$, then the asymptotic expansion of the integral

$$I(x) = \int_a^b f(t,x)dt$$

as $x \to x_0$ is given by

$$I(x) \sim \sum_{n=0}^{\infty} (x-x_0)^{\alpha n} \int_a^b f_n(t)dt \qquad \text{as } x \to x_0.$$

We begin with finding the first three terms of the asymptotic expansion of the integral

$$I_1(x) = \int_x^1 \cos(xt)dt \qquad \text{as } x \to 0^+.$$

Note that $f(t,x) = \cos(xt)$ has the following asymptotic expansion as $x \to 0^+$:

$$f(t,x) = \cos(xt) \sim 1 - \frac{t^2x^2}{2} + \frac{t^4x^4}{24}.$$

This expansion converges uniformly for all $x \le t \le 1$ as $x \to 0^+$. Therefore, we have that the first three terms of the asymptotic expansion of $I_1(x)$ as $x \to 0^+$ are given by

$$I_1(x) \sim \int_x^1 dt - \frac{x^2}{2}\int_x^1 t^2 dt + \frac{x^4}{24}\int_x^1 t^4 dt = (1-x) - \frac{x^2}{2}\left[\frac{1-x^3}{3}\right] + \frac{x^4}{24}\left[\frac{1-x^5}{5}\right].$$

Similar to what was shown above, we have that if

$$f(t,x) \sim f_0(t) \qquad \text{as } x \to x_0$$

uniformly for $a \leq t \leq b$, then the asymptotic expansion of the integral is given by

$$I(x) = \int_a^b f(t,x)dt \sim \int_a^b f_0(t)dt \qquad \text{as } x \to x_0.$$

Let us continue by finding the first three terms of the asymptotic expansion of the integral

$$I_2(x) = \int_0^{1/x} e^{-t^2} dt \qquad \text{as } x \to 0^+.$$

Note that $f(t,x) = e^{-t^2}$ has the following asymptotic expansion as $x \to 0^+$:

$$f(t,x) = e^{-t^2} \sim 1 - t^2 + \frac{t^4}{2}.$$

This expansion converges uniformly for all finite points, so it converges uniformly for $0 \leq t \leq 1/x$ as $x \to 0^+$. Therefore, we may integrate the expansion term by term and we have that the first three terms of the asymptotic expansion of $I_2(x)$ as $x \to 0^+$ are given by

$$I_2(x) \sim \int_0^{1/x} dt - \int_0^{1/x} t^2 dt + \frac{1}{2}\int_0^{1/x} t^4 dt = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{10x^5}.$$

□

Problem 2. Find the full asymptotic behavior as $x \to 0^+$ of the following integral:

$$\int_0^1 \frac{e^{-t}}{1+x^2t^3}dt$$

Solution. Note that the function $f(t,x) = e^{-t}/(1+x^2t^3)$ has the asymptotic expansion

$$f(t,x) = \frac{e^{-t}}{1+x^2t^3} \sim e^{-t}\sum_{n=0}^{\infty}\left[(-1)^n t^{3n}\right]x^{2n} \qquad \text{as } x \to 0^+.$$

Note that this asymptotic expansion converges uniformly for $0 \leq x \leq t < 1-\epsilon$ for all $\epsilon > 0$. To see this, we note that for $0 < m < n$, we have that

$$\left|\sum_{k=m+1}^{n}(-1)^k(x^2t^3)^k\right| < \sum_{k=m+1}^{n}(1-\epsilon)^{5k}.$$

Since $(1-\epsilon)^5 < 1$, we have that its geometric series converges and we can make it as small as we wish. Thus, by the Cauchy criterion we have uniform convergence for $0 \leq x \leq t < 1-\epsilon$ for all $\epsilon > 0$.

Per the discussion in Problem 1, we using this uniformly convergent asymptotic expansion, we have that as $x \to 0^+$

$$\int_0^1 \frac{e^{-t}}{1+x^2t^3}dt \sim \sum_{n=0}^{\infty}(-1)^n x^{2n}\int_0^1 e^{-t}t^{3n}dt = \sum_{n=0}^{\infty}(-1)^n x^{2n}\left[\Gamma(3n+1) - \Gamma(3n+1,1)\right]$$

where $\Gamma(a,k) = \int_k^\infty t^{a-1}e^{-t}dt$. $\square$

Problem 3. Find the full asymptotic expansion of $\int_0^x \mathrm{Bi}(t)dt$ as $x \to +\infty$.

Solution. $\square$

Problem 4. Find the first five terms in the asymptotic expansion as $x \to +\infty$ of the integral

$$\int_0^{\pi/4} e^{-xt^2} \sqrt{\tan t} dt$$

a. by using a suitable change of variables and then applying Watson's lemma.

b. by applying Laplace's method directly to the given integral.

Solution. □

Problem 5. Use Laplace's method of moving maxima to obtain the first two terms in the asymptotic expansion as $x \to +\infty$ of the integral

$$\int_0^\infty \exp\left[-t - \frac{x}{\sqrt{t}}\right] dt. \tag{1}$$

Solution. □

Problem 6. Let $f(x,t)$ be differentiable in x and continuous in (x,t) on $I \times J$, where I and J are intervals, and suppose that there exist functions $g(t)$ and $g_1(t)$ that are integrable on J such that for all $(x,t) \in I \times J$ we have that

$$|f(x,t)| \leq g(t) \quad \text{and} \quad |\partial_x f(x,t)| \leq g_1(t).$$

Then

$$\frac{d}{dx}\int_J f(x,t)dt = \int_J \partial_x f(x,t)dt.$$

a. Let $0 < a < b < \infty$. Use the above theorem to show that if $x \in (a,b)$, then

$$\frac{d^3}{dx^3}\int_0^\infty \exp\left[-t - \frac{x}{\sqrt{t}}\right] dt = -\int_0^\infty t^{-3/2} \exp\left[-t - \frac{x}{\sqrt{t}}\right] dt.$$

b. Use integration by parts to show that

$$\int_0^\infty \exp\left[-t - \frac{x}{\sqrt{t}}\right] dt = \frac{x}{2}\int_0^\infty t^{-3/2} \exp\left[-t - \frac{x}{\sqrt{t}}\right] dt.$$

c. Combine parts (a) and (b) to prove that integral (1) is a solution of the differential equation $xy''' + 2y = 0$ that also satisfies the initial condition $y(0) = 1$. Then use integration by parts to give an easy direct proof that the integral also satisfies the condition $y(+\infty) = 0$.

Solution. □

Problem 7. a. Find the leading behavior as $x \to +\infty$ of Laplace integrals of the form

$$\int_a^b (t-a)^\alpha g(t) e^{x\phi(t)} dt$$

where $\phi(t)$ has a maximum at $t = a$, $g(a) = 1$. Suppose further that $\alpha > -1$ and $\phi'(a) < 0$.

b. Repeat the analysis of part (a) when $\alpha > -1$ and $\phi'(a) = \phi''(a) = \cdots = \phi^{(p-1)}(a) = 0$ and $\phi^{(p)}(a) < 0$.

Solution. □