Homework Assignment 3

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Problem 1. Solve the following linear program using the Simplex Algorithm in conjunction with Bland's rule:

$$\begin{array}{rrl} \text{maximize} & 2x_1 + 5x_2 & \\ \text{subject to} & x_1 & \leq 4 \\ & x_2 & \leq 6 \\ & x_1 + x_2 & \leq 8 \\ & x_1, x_2 & \geq 0. \end{array}$$

Solution. To start, we must transform this LP into standard form. This is achieved by changing the objective from *maximize* to *minimize* and adding three slack variables. In standard form, the problem becomes

$$\begin{array}{rrl} \text{minimize} & -2x_1 - 5x_2 & \\ \text{subject to} & x_1 + x_3 & = 4 \\ & x_2 + x_4 & = 6 \\ & x_1 + x_2 + x_5 & = 8 \\ & x_1, x_2, x_3, x_4, x_5 & \geq 0. \end{array}$$

The initial tableau associated to this problem is then:

	$\boldsymbol{a_1}$	$\boldsymbol{a_2}$	$\boldsymbol{a_3}$	$\boldsymbol{a_4}$	$\boldsymbol{a_5}$	$\boldsymbol{b}$
	1	0	1	0	0	4
	0	1	0	1	0	6
	1	1	0	0	1	8
$\boldsymbol{c}^\intercal$	-2	-5	0	0	0	0

Note that this tableau is in canonical form with respect to the basis $[\boldsymbol{a_3}, \boldsymbol{a_4}, \boldsymbol{a_5}]$. Thus, the last row of the tableau contains the reduced cost coefficients. Bland's rule prescribes how to choose the column-index q and the row-index p to pivot around. According to Bland's rule, choose

$$\begin{aligned} q &= \min\{i \mid r_i < 0\} \\ p &= \min\{j \mid y_{j0}/y_{jq} = \min_i\{y_{i0}/y_{iq} \mid y_{iq} > 0\}\}. \end{aligned}$$

Thus, we proceed by choosing the column-index to pivot around to be the smallest index pertaining to negative reduced cost coefficients in the bottom vector of the tableau and by then choosing the row-index to pivot around to be the index pertaining to the row with the lowest ratio between the right hand side and the positive coefficients of the q-th column in matrix A of the tableau. If there are two such row-indexes, choose the smaller one.

From the initial tableau, Bland's rule prescribes that we pivot around column $q = 1$ since this is the smallest index with a negative reduced cost coefficient. The smallest ratio between the right hand side and the positive coefficients of the q-th column in matrix A is $4/1$ so we pivot around row $p = 1$. Thus, $\boldsymbol{a_1}$ enters the basis, $\boldsymbol{a_3}$ leaves the basis, and we move from the initial tableau to the updated tableau:

$$
\begin{array}{ccccccc}
 & \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} & \boldsymbol{a_4} & \boldsymbol{a_5} & \boldsymbol{b} \\
 & 1 & 0 & 1 & 0 & 0 & 4 \\
 & 0 & 1 & 0 & 1 & 0 & 6 \\
 & 1 & 1 & 0 & 0 & 1 & 8 \\
\boldsymbol{c}^\intercal & -2 & -5 & 0 & 0 & 0 & 0 \\
 & \uparrow & & & & &
\end{array}
\quad \xrightarrow[{[4]+2[1]}]{{[3]-[1]}} \quad
\begin{array}{ccccccc}
 & \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} & \boldsymbol{a_4} & \boldsymbol{a_5} & \boldsymbol{b} \\
 & 1 & 0 & 1 & 0 & 0 & 4 \\
 & 0 & 1 & 0 & 1 & 0 & 6 \\
 & 0 & 1 & -1 & 0 & 1 & 4 \\
\boldsymbol{c}^\intercal & 0 & -5 & 2 & 0 & 0 & 8
\end{array}
$$

From this newly derived tableau, we notice that the only negative reduced cost coefficient occurs in column $q = 2$. Further, the smallest ratio between the right hand side and the positive coefficients of the q-th column in matrix A is $4/1$ so we pivot around row $p = 3$. Thus, $\boldsymbol{a_2}$ enters the basis, $\boldsymbol{a_5}$ leaves the basis, and we move from this tableau to the updated tableau:

$$
\begin{array}{ccccccc}
 & \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} & \boldsymbol{a_4} & \boldsymbol{a_5} & \boldsymbol{b} \\
 & 1 & 0 & 1 & 0 & 0 & 4 \\
 & 0 & 1 & 0 & 1 & 0 & 6 \\
 & 0 & 1 & -1 & 0 & 1 & 4 \\
\boldsymbol{c}^\intercal & 0 & -5 & 2 & 0 & 0 & 8 \\
 & & \uparrow & & & &
\end{array}
\quad \xrightarrow[{[4]+5[3]}]{{[2]-[3]}} \quad
\begin{array}{ccccccc}
 & \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} & \boldsymbol{a_4} & \boldsymbol{a_5} & \boldsymbol{b} \\
 & 1 & 0 & 1 & 0 & 0 & 4 \\
 & 0 & 0 & 1 & 1 & -1 & 2 \\
 & 0 & 1 & -1 & 0 & 1 & 4 \\
\boldsymbol{c}^\intercal & 0 & 0 & -3 & 0 & 5 & 28
\end{array}
$$

From this newly derived tableau, we notice that the only negative reduced cost coefficient occurs in column $q = 3$. Further, the smallest ratio between the right hand side and the positive coefficients of the q-th column in matrix A is $2/1$ so we pivot around row $p = 2$. Thus, $\boldsymbol{a_3}$ enters the basis, $\boldsymbol{a_4}$ leaves the basis, and we move from this tableau to the updated tableau:

$$
\begin{array}{ccccccc}
 & \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} & \boldsymbol{a_4} & \boldsymbol{a_5} & \boldsymbol{b} \\
 & 1 & 0 & 1 & 0 & 0 & 4 \\
 & 0 & 0 & 1 & 1 & -1 & 2 \\
 & 0 & 1 & -1 & 0 & 1 & 4 \\
\boldsymbol{c}^\intercal & 0 & 0 & -3 & 0 & 5 & 28 \\
 & & & \uparrow & & &
\end{array}
\quad
\begin{array}{c}
\xrightarrow[{[3]+[2]}]{{[1]-[2]}} \\
\xrightarrow[{[4]+3[2]}]{}
\end{array}
\quad
\begin{array}{ccccccc}
 & \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} & \boldsymbol{a_4} & \boldsymbol{a_5} & \boldsymbol{b} \\
 & 1 & 0 & 0 & -1 & 1 & 2 \\
 & 0 & 0 & 1 & 1 & -1 & 2 \\
 & 0 & 1 & 0 & 1 & 0 & 6 \\
\boldsymbol{c}^\intercal & 0 & 0 & 0 & 3 & 2 & 34
\end{array}
$$

In the final tableau we have no negative reduced cost coefficients. Therefore, the current basic feasible solution $\boldsymbol{x} = [2, 6, 2, 0, 0]^\intercal$ of the LP in standard form is optimal with corresponding objective function value -34. The solution to the original problem is then $x_1 = 2$, $x_2 = 6$ with corresponding objective value 34. $\square$

Problem 2. a. Prove that if (ALP) has a feasible solution $[x_1, \ldots, x_n, y_1, \ldots, y_m]^\intercal$ with objective function value zero then $y_1 = 0, \ldots, y_m = 0$.

b. What do you do if after Phase I (ALP) does not have any optimal feasible solution with objective function value zero?

Solution. Suppose we have the following LP in standard form:

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^\intercal \boldsymbol{x} \\ \text{subject to} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0. \end{aligned}$$

The *artificial problem* (ALP) associated to this problem is stated as

$$\begin{aligned} \text{minimize} \quad & y_1 + y_2 + \cdots + y_m \\ \text{subject to} \quad & [A, I_m] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \boldsymbol{b} \\ & \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} \geq \boldsymbol{0}, \end{aligned} \tag{1}$$

where $\boldsymbol{y} = [y_1, \ldots, y_m]^\intercal$.

a. Suppose that the (ALP) has the following feasible solution $\begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{y}_0 \end{bmatrix} = [x_1, \ldots, x_n, y_1, \ldots, y_m]^\intercal$ with corresponding objective function value zero. Then $\begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{y}_0 \end{bmatrix}$ is a solution of (1), i.e. $A\boldsymbol{x}_0 + I_m\boldsymbol{y}_0 = \boldsymbol{b}$ where $\boldsymbol{x}_0 \geq \boldsymbol{0}$ and $\boldsymbol{y}_0 \geq \boldsymbol{0}$. As $\boldsymbol{y}_0 = [y_1, y_2, \ldots, y_m]^\intercal \geq \boldsymbol{0}$, every component of $\boldsymbol{y}_0$ is non-negative, i.e. $y_i \geq 0$ for $i = 1, \ldots, m$. Since the corresponding objective function value of this solution is zero, we know that

$$y_1 + y_2 + \cdots + y_m = 0.$$

Note that if $y_i \geq 0$ for $i = 1, \ldots, m$ the sum $y_1 + y_2 + \cdots + y_m = 0$ if and only each $y_i = 0$. Therefore, if $\begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{y}_0 \end{bmatrix} = [x_1, \ldots, x_n, y_1, \ldots, y_m]^\intercal$ is a feasible solution of the (ALP) with objective function value zero then $y_1 = 0, \ldots, y_m = 0$.

b. Proposition 16.1 found on p. 362 of *An Introduction to Optimization* states that the (ALP) has an optimal feasible solution with objective function value zero if and only if the original LP problem has a basic feasible solution. Thus, if the Phase I (ALP) does not have any optimal feasible solution with objective value zero, then the original LP does not have a basic feasible solution.

The Fundamental Theorem of LP states that if there exists a feasible solution of the LP, then there exists a basic feasible solution. Thus, if no basic feasible solution of the LP exists, then no feasible solution of the LP exists.

Therefore, if the Phase I (ALP) does not have any optimal feasible solution with objective value zero, then the original LP does not have a basic feasible solution and consequently the original LP has no feasible solution, i.e. the original LP is infeasible.

□

Problem 3. Consider the linear program

$$\begin{array}{rrl} \text{maximize} & 2x_1 + x_2 & \\ \text{subject to} & 0 \leq x_1 & \leq 5 \\ & 0 \leq x_2 & \leq 7 \\ & x_1 + x_2 & \leq 9. \end{array}$$

Convert the problem to standard form and solve it using the simplex method.

Solution. □

Problem 4. Solve the following linear programs using the revised simplex method:

a.

$$\begin{array}{rrl} \text{maximize} & -4x_1 - 3x_2 & \\ \text{subject to} & 5x_1 + x_2 & \geq 11 \\ & -2x_1 - x_2 & \leq -8 \\ & x_1 + 2x_2 & \geq 7 \\ & x_1, x_2 & \geq 0. \end{array}$$

b.

$$\begin{array}{rrl} \text{maximize} & 6x_1 + 4x_2 + 7x_3 + 5x_4 & \\ \text{subject to} & x_1 + 2x_2 + x_3 + 2x_4 & \leq 20 \\ & 6x_1 + 5x_2 + 3x_3 + 2x_4 & \leq 100 \\ & 3x_1 + 4x_2 + 9x_3 + 12x_4 & \leq 75 \\ & x_1, x_2, x_3, x_4 & \geq 0. \end{array}$$

Solution. □

Problem 5. Suppose that we apply the simplex method to a given linear programming problem and obtain the following canonical tableau:

$$\begin{array}{ccccc} 0 & \beta & 0 & 1 & 4 \\ 1 & \gamma & 0 & 0 & 5 \\ 0 & -3 & 1 & 0 & 6 \\ 0 & 2-\alpha & 0 & 0 & \delta \end{array}$$

For each of the following conditions, find the set of all parameter values $\alpha, \beta, \gamma, \delta$ that satisfy the condition.

a. The problem has no solution because the objective function values are unbounded.

b. The current basic feasible solution is optimal, and the corresponding objective function value is 7.

c. The current basic feasible solution is not optimal, and the objective function value strictly decreases if we remove the first column of A from the basis.

Solution. □