Homework Assignment 6

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Problem 4.2.1. Prove that every open ball $B_\varepsilon(a)$ in a metric space (X, d) is an open set and that every finite subset of X is a closed set.

Solution. Recall that a set $A \subseteq X$ is an open set if for all $a \in A$, there exists an $\varepsilon > 0$ such that $B_\varepsilon(a) \subseteq A$. Thus, to show that $B_\varepsilon(a)$ is an open set, we will show that for each point in the open ball of radius ε centered at a there exists a neighborhood of that point that is completely contained in the open ball.

So, let $x \in B_\varepsilon(a) = \{x \in X \mid d(x, a) < \varepsilon\}$ and suppose that $d(x, a) = \delta < \varepsilon$. We wish to find some $\varepsilon_1 > 0$ such that $B_{\varepsilon_1}(x) \subseteq B_\varepsilon(a)$. Consider $B_{\varepsilon_1}(x)$, the open ball centered at x of radius $\varepsilon_1 = \varepsilon - \delta > 0$ and let $y \in B_{\varepsilon_1}(x)$. If $y \in B_{\varepsilon_1}(x)$, then $y \in X$ and $d(y, x) < \varepsilon_1 = \varepsilon - \delta$. Since $a, x, y \in X$ and X is a metric space, we must have that

$$d(y, a) \leq d(y, x) + d(x, a) < \varepsilon - \delta + \delta = \varepsilon.$$

Thus, we have that $d(y, a) < \varepsilon$ and $y \in B_\varepsilon(a)$. Therefore, for every $x \in B_\varepsilon(a)$ we have that there exists an $\varepsilon_1 > 0$ such that $B_{\varepsilon_1}(x) \subseteq B_\varepsilon(a)$ and the set $B_\varepsilon(a)$ must be open.

We now wish to show that a finite subset $A = \{a_0, a_1, \ldots, a_n\} \subseteq X$ is a closed set. Recall that a set $A \subseteq X$ is closed if and only if $X \setminus A$ is open. Let $x \in X \setminus A$ and consider $B_\varepsilon(x)$, the open ball centered at x of radius $\varepsilon = \min_i\{d(x, a_i)\}$. Since $x \in X$ and $x \neq a_0, \ldots, a_n$, we know that $\varepsilon = \min_i\{d(x, a_i)\} > 0$.

Suppose to the contrary that $y \in B_\varepsilon(x)$ and $y = a_i$ for some $i = 0, \ldots, n$. Since $y, a_i \in X$ and X is a metric space with $y = a_i$, we have that $d(y, a_i) = 0$. Thus, under the properties of the distance function of this metric space, we must have that

$$d(x, a_i) \leq d(x, y) + d(y, a_i) < \varepsilon = \min_i\{d(x, a_i)\}.$$

However, this is a contradiction since an element of a set cannot be strictly less than the minimum of that set. Thus, if $y \in B_\varepsilon(x)$, then $y \neq a_i$ for any $i = 0, \ldots, n$. Therefore, for every $x \in X \setminus A$, there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq X \setminus A$ and $X \setminus A$ is open so that A is closed. $\square$

Problem 4.2.2. Show that the closed ball $B_\varepsilon[a] = \{x \in X \mid d(a,x) \le \varepsilon\}$ in a metric space is a closed set, but it need not be equal to the closure of the open ball $B_\varepsilon(a)$. (Hint: Consider the two point space $\mathcal{A} = \{0,1\}$ with metric $d(0,1) = 1$).

Solution. We wish to show that $B_\varepsilon[a]$ is closed, i.e. that $X \setminus B_\varepsilon[a]$ is open. Suppose that $x \in X \setminus B_\varepsilon[a]$. Then we have that $d(x,a) = \delta > \varepsilon$. Consider $B_{\varepsilon_1}(x)$, the open ball centered at x of radius $\varepsilon_1 = \delta - \varepsilon$. Suppose to the contrary that $y \in B_{\varepsilon_1}(x)$ and $y \in B_\varepsilon[a]$. Since $a, x, y \in X$ with X a metric space, we have that

$$d(x,a) \le d(x,y) + d(y,a) < \delta - \varepsilon + \varepsilon = \delta = d(x,a).$$

However, this is a contradiction since the distance between two points cannot be less than itself. Thus, we must have that if $y \in B_{\varepsilon_1}(x)$, then $y \notin B_\varepsilon[a]$. Therefore, for every $x \in X \setminus B_\varepsilon[a]$, there exists an $\varepsilon_1 > 0$ such that $B_{\varepsilon_1}(x) \subseteq X \setminus B_\varepsilon[a]$ and $X \setminus B_\varepsilon[a]$ is open so that $B_\varepsilon[a]$ is closed.

Recall that $\overline{A}$, the closure of a set A, is the union of the set A with its limit points, i.e. $\overline{A} = A \cup \mathrm{Lim}\{A\}$ where $a \in X$ is a limit point if every open ball centered at a contains a point in A different from a. We wish to show that the closed ball centered at a of a given radius does not necessarily coincide with the closure of the open ball centered at a of the same radius.

To see this, consider the metric space $X = \{0,1\}$ equipped with the discrete metric and consider the point $0 \in X$. The only open ball centered at 0 is given by

$$B_1(0) = \{x \in X \mid d(x,0) < 1\} = \{0\}.$$

Since the point 1 is not in any open ball centered at 0, we see that there are no limit points of 0 and that $\overline{B_1(0)} = \{0\}$. However, the only closed ball centered at 0 is given by

$$B_1[0] = \{x \in X \mid d(x,0) \le 1\} = \{0,1\}$$

and we see that $\overline{B_1(0)} \ne B_1[0]$. Therefore, we can clearly see that the closure of the open ball centered at a point $a \in X$ of any radius is not necessarily equal to the closed ball centered at a of the same radius.

$\square$

Problem 4.2.5. Show that the intersection of a finite number of open sets $A_1, A_2, \dots, A_n$ in a metric space (X, d) is an open set. Show that, by considering the intervals $(-1/n, 1/n)$ for all $n \in \mathbb{Z}^+$ in $\mathbb{R}$, the intersection of infinitely many open sets need not be open.

Solution. We wish to show that the set $A = \bigcap_{i=1}^n A_i$ is open where A_i for $i = 1, \dots, n$ is open. Suppose that $a \in A$ so that $a \in A_i$ for $i = 1, \dots, n$. Note that each A_i is open, so we must have that for every $x \in A_i$, there exists some $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A_i$. Thus, since $a \in A_i$ for $i = 1, \dots, n$, there exists some $\varepsilon_i > 0$ such that $B_{\varepsilon_i}(a) \subseteq A_i$ for each i.

Consider $B_\delta(a)$, the open ball centered at a of radius $\delta = \min_i\{\varepsilon_i\} > 0$ and let $x \in B_\delta(a)$. Then $d(x, a) < \delta \leq \varepsilon_i$ for $i = 1, \dots, n$. This implies that $x \in B_{\varepsilon_i}(a)$ and hence that $x \in A_i$ for $i = 1, \dots, n$, i.e. that $x \in \bigcap_{i=1}^n A_i = A$. Therefore, for every $a \in A$, there exists a $\delta > 0$ such that $B_\delta(a) \subseteq A$ and thus the set A is open.

Now consider the set $B = \bigcap_{i=1}^\infty (-1/n, 1/n)$. It is clear by the Archimedean property of the real numbers that $B = \{0\}$. However, as was proven previously, every finite subset of a metric space is closed. Therefore, we have that B is closed and the intersection of infinitely many open sets need not be open.

□

Problem 4.2.6. If $\mathcal{A} = \{0, 1\}$, then $\mathcal{A}^{\mathbb{N}}$ denotes the metric space of 0's and 1's:

$$\mathcal{A}^{\mathbb{N}} = \{\omega = (a_0, a_1, a_2, \dots) \mid a_i = 0 \text{ or } a_i = 1\},$$

with metric:

$$d(\omega_1, \omega_2) = \sum_{k=0}^{\infty} \frac{|s_k - t_k|}{2^k}, \tag{1}$$

where $\omega_1 = (s_0, s_1, s_2, \dots)$ and $\omega_2 = (t_0, t_1, t_2, \dots)$.

Show that $\mathcal{A}^{\mathbb{N}}$ is a metric space. Find $d(\omega_1, \omega_2)$ if:

i. $\omega_1 = (0, 1, 1, 1, 1, \dots)$ and $\omega_2 = (1, 0, 1, 1, 1, \dots)$,

ii. $\omega_1 = (0, 1, 0, 1, 0, \dots)$ and $\omega_2 = (1, 0, 1, 0, 1, \dots)$.

Solution. In order for $(\mathcal{A}^{\mathbb{N}}, d)$ to be a metric space with $d : \mathcal{A}^{\mathbb{N}} \times \mathcal{A}^{\mathbb{N}} \to \mathbb{R}_{\geq 0}$ defined as in (1), we require for every $x, y, z \in \mathcal{A}^{\mathbb{N}}$ that the metric d satisfies:

i. $d(x, y) \geq 0$,

ii. $d(x, y) = 0$ if and only if $x = y$,

iii. $d(x, y) = d(y, x)$

iv. $d(x, z) \leq d(x, y) + d(y, z)$.

Suppose that $x = (x_0, x_1, \dots) \in \mathcal{A}^{\mathbb{N}}$ and $y = (y_0, y_1, \dots) \in \mathcal{A}^{\mathbb{N}}$. Note that $|x_k - y_k| \geq 0$ for every $k \in \mathbb{N}$ which implies that $|x_k - y_k|/2^k \geq 0$. Thus, $d(x, y) = \sum_{k=0}^{\infty} |x_k - y_k|/2^k \geq 0$ and property i. is satisfied.

It is clear that if $x = y$ then $x_k = y_k$ for every $k \in \mathbb{N}$ which implies that $|x_k - y_k| = 0$. Hence, $d(x, y) = 0$. On the other hand, suppose that $d(x, y) = 0$. Since $|x_k - y_k| \geq 0$ for each $k \in \mathbb{N}$, we have that $d(x, y) = \sum_{k=0}^{\infty} |x_k - y_k|/2^k = 0$ if and only if $|x_k - y_k| = 0$ for all $k \in \mathbb{N}$. This implies that $x_k = y_k$ for all $k \in \mathbb{N}$ and therefore $x = y$ and property ii. is satisfied.

Since $|x_k - y_k| = |y_k - x_k|$ it is clear that $d(x, y) = d(y, x)$ and property iii. is satisfied.

Now suppose that $z = (z_0, z_1, \dots) \in \mathcal{A}^{\mathbb{N}}$. Since $|x_k - z_k| \leq |x_k - y_k| + |y_k - z_k|$, it is clear that

$$\begin{aligned} d(x, z) = \sum_{k=0}^{\infty} \frac{|x_k - z_k|}{2^k} &\leq \sum_{k=0}^{\infty} \frac{|x_k - y_k| + |y_k - z_k|}{2^k} \\ &= \sum_{k=0}^{\infty} \frac{|x_k - y_k|}{2^k} + \sum_{k=0}^{\infty} \frac{|y_k - z_k|}{2^k} = d(x, y) + d(y, z) \end{aligned}$$

and property iv. is satisfied. Hence, the metric d satisfies all four required properties. Since it is clear that $d : \mathcal{A}^{\mathbb{N}} \times \mathcal{A}^{\mathbb{N}} \to \mathbb{R}_{\geq 0}$ is well defined and $0 \leq d(x, y) \leq 2$, the relation d is in fact a function onto the non-negative reals and therefore $(\mathcal{A}^{\mathbb{N}}, d)$ is a metric space.

We now wish to find the distance between two points $\omega_1 = (s_0, s_1, s_2, \dots)$ and $\omega_2 = (t_0, t_1, t_2, \dots)$ in $\mathcal{A}^{\mathbb{N}}$ for various representations of ω_1 and ω_2.

i. Suppose that $\omega_1 = (0, 1, 1, 1, 1, \dots)$ and $\omega_2 = (1, 0, 1, 1, 1, \dots)$. Then we have that

$$|s_k - t_k| = \begin{cases} 1 & \text{if } 0 \leq k \leq 1 \\ 0 & \text{if } 1 < k \end{cases}.$$

Therefore, we have that

$$d(\omega_1, \omega_2) = \sum_{k=0}^{\infty} \frac{|s_k - t_k|}{2^k} = \frac{1}{2^0} + \frac{1}{2^1} = \frac{3}{2}.$$

ii. Suppose that $\omega_1 = (0, 1, 0, 1, 0, \dots)$ and $\omega_2 = (1, 0, 1, 0, 1, \dots)$. Then we have that $|s_k - t_k| = 1$ for all $k \in \mathbb{N}$. Therefore, we have that

$$d(\omega_1, \omega_2) = \sum_{k=0}^{\infty} \frac{|s_k - t_k|}{2^k} = \sum_{k=0}^{\infty} \frac{1}{2^k} = 2.$$

$\square$

Problem 4.2.7. Let $f : I \to I$ be a continuous function defined on an interval I.

i. What can you say about the graph of f, if f has a dense set of points with $f^2(x) = x$?
ii. Show that the inverse of f must exist and that f must have at least one fixed point.
iii. Deduce that if there exists an $x \in I$ with $f(x) \neq x$, then f must be strictly decreasing.
iv. If $f'(x)$ exists for all $x \in I$, show that the 2-cycles are non-hyperbolic, and any fixed point x_0 is non-hyperbolic of the type $f'(x_0) = -1$, when f is not the identity map.
v. Give an example of a function of the type appearing in iv.

Solution. Throughout, we assume that $f : I \to I$ is a continuous function defined on an interval I.

i. Let $A = \{x \in I \mid f^2(x) = x\}$ and suppose that A is dense in I. Then for every $y \in I$ and for all $\varepsilon > 0$, there is some $x \in (y - \varepsilon, y + \varepsilon)$ such that $x \in A$, i.e. $x = f^2(x)$. This also implies that for every $x \in I$, there is a sequence (x_n) in A such that $\lim_n x_n = x$. So, suppose that (x_n) is a sequence in A. By the continuity of f, we have that

$$\lim_n f(x_n) = f(\lim_n x_n).$$

which implies that

$$\lim_n f^2(x_n) = \lim_n f(f(x_n)) = f(\lim_n f(x_n)) = f^2(\lim_n x_n).$$

Since A is dense in I, we have for every $x \in I$ that $\lim_n x_n = x$ and thus

$$x = \lim_n x_n = \lim_n f^2(x_n) = f^2(\lim_n x_n) = f^2(x).$$

This implies that every point in the domain is in A.

The graph of f is given by $G(f) = \{(x, f(x)) \in I \times I \mid x = f(x)\}$. If every point $x \in I$ satisfies $f^2(x) = x$, then for all $(x, f(x)) \in G(f)$, we must have that $(f(x), x) \in G(f)$ since $f(x) \in I$ if $x \in I$ and $f^2(x) = x$.

ii. We will show that if every point in the domain of f is a period-2 point, then f must be injective and surjective and hence must be invertible.

Suppose that $f(x) = f(y)$ with $f(x), f(y) \in I$. Since f is a function, we must have that $f^2(x) = f^2(y)$, but this implies that $x = y$ since every point in the domain of f is a period-2 point. Thus, f is injective.

It is clear that f is surjective since it is injective and the codomain of the function f is the same as its domain. Therefore, f must have an inverse.

As was shown previously, a continuous function from an interval onto itself must have a fixed point.

iii. It follows from the Intermediate Value Theorem that if f is a continuous function that is not strictly monotonic, then f is not injective. Therefore, since f is a continuous, injective function we must have that it is a strictly monotonic function.

Suppose that there exists an $x \in I$ such that $f(x) \neq x$. Since $f^2(x) = x$ with $f(x) \neq x$, we have that x is a period 2 point and $\{x, f(x)\}$ is a 2-cycle. Without loss of generality, assume that $x < f(x)$. Since f is injective, we either have that $f(x) < f^2(x)$ or $f(x) > f^2(x)$. However, $f^2(x) = x$, so we cannot have that $f(x) < f^2(x) = x$ if $x < f(x)$. Thus, we must have that $f(x) > f^2(x)$ or that f is strictly decreasing on the interval $[x, f(x)]$. Therefore, since f is monotonic, we must have that f is strictly monotonically decreasing.

iv. Suppose that $f'(x)$ exists for every $x \in I$. We know that $f^2(x) = x$ for all $x \in I$. Then we have that

$$|f^2(x)'| = |f'(f(x))f'(x)| = |x'| = 1 \tag{2}$$

and the 2-cycle $\{x, f(x)\}$ is non-hyperbolic. Suppose on the other hand that $f(x_0) = x_0$ for some $x_0 \in I$. Note that since f is strictly decreasing we have that $f'(x) < 0$ for all $x \in I$. Since $f^2(x_0) = x_0$, we see from equation (2) that

$$|f^2(x_0)'| = |f'(x_0)^2| = 1.$$

This implies that $f'(x_0)^2 = 1$. Since $f'(x_0) < 0$ we must have that $f'(x_0) = -1$ and the fixed point x_0 is non-hyperbolic.

v. The function $f : (0, \infty) \to (0, \infty)$ with $f(x) = 1/x$ is an example of the type appearing in iv. It is clear that f is strictly decreasing on this interval. Note that if $x \in (0, \infty)$ with $x \neq 1$, then $f^2(x) = x$ with $f(x) \neq x$ so that $\{x, f(x)\}$ is a 2-cycle. Since $f'(x) = -x^{-2}$, we see that

$$|f'(f(x))f(x)| = \left| \left(-\frac{1}{(1/x)^2} \right) \left(-\frac{1}{x^2} \right) \right| = \left| \frac{x^2}{x^2} \right| = 1$$

and the 2-cycle is non-hyperbolic. The only fixed point of this function is $x_0 = 1$, we clearly see that $f'(1) = -1$ and it is non-hyperbolic.

$\square$

Problem 4.3.4. Show that if $f : [a, b] \to [a, b]$ is a homeomorphism, then either a and b are fixed points or $\{a, b\}$ is a 2-cycle.

Solution. Suppose that $f : [a, b] \to [a, b]$ is a homeomorphism, i.e. f is a continuous bijection with a continuous inverse. It was shown previously that a continuous, injective function must either be strictly monotonically increasing or decreasing.

Suppose first that $f(a) \neq a$. Note in this case that f cannot be strictly monotonically increasing for otherwise we would contradict the fact that f is a bijection. To demonstrate, suppose to the contrary that f is a homeomorphism with $f(a) \neq a$ but f is strictly monotonically increasing. Since $a < x$ for all $x \in (a, b]$ we have that $f(a) < f(x)$. Since f is a bijection that is monotonically increasing, we must have that each $x \in (a, b]$ gets mapped to some $f(x) \in (a, b]$. However, if $f(a) < f(x)$ for all $f(x) \in (a, b]$, then $f(a) = a$ if f is a bijection, a contradiction. Therefore, we have that if $f(a) \neq a$, then f is strictly monotonically decreasing. If f is a strictly monotonically decreasing bijection, then we must have that $f(a) = b$ and $\{a, b\}$ is a 2-cycle.

On the other hand, if $f(a) = a$, then following an argument similar to the one posited above, we have that f must be strictly monotonically increasing. Thus, if f is a bijection, we must have that $f(b) = b$ so that a and b are both fixed points.

□

Problem 4.3.8. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map with fixed point c and basin of attraction $B_f(c) = (a, b)$, an interval. Show that one of the following must hold:

i. a and b are fixed points.

ii. a or b is fixed and the other is eventually fixed.

iii. $\{a, b\}$ is a 2-cycle.

Solution. $\square$