Matrices and Linear Systems

Unit 2

Linear Systems

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Overdetermined, underdetermined, square, homogeneous, number of solutions

Matrices

- Column space $span(A)$

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- Rank $rank(A)$
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- Nonsingular square matrix $A \in \mathbf{R}^{n \times n}$

Properties

$$A \in \mathbf{R}^{m\times p}, \quad B \in \mathbf{R}^{p\times n}, \quad C = AB \in \mathbf{R}^{n\times m}$$

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

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Noncommutative ring

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Multiplication count

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Coppersmith, Winograd, down to $n^{2.376}$

Potential use of group theory, Robinson

Block Multiplication

- Exercise 2.5

- If each product in the blocks can be formed, then block multiplication works

Diagonally Dominant

$A \in \mathbf{R}^{n \times n}$ is strictly row diagonally dominant if $\sigma > 0$, where

$$\sigma := \min_{i=1,\ldots,n} \sigma_i, \quad \text{and} \quad \sigma_i = |a_{ii}| - \sum_{j \neq i} |a_{ij}| > 0$$

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- Show the bound

LU Factorization of Tridiagonal Matrices

$$\begin{pmatrix} d_1 & c_1 & 0 \\ a_2 & d_2 & c_2 \\ 0 & a_3 & d_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l_2 & 1 & 0 \\ 0 & l_3 & 1 \end{pmatrix} \begin{pmatrix} u_1 & c_1 & 0 \\ 0 & u_2 & c_2 \\ 0 & 0 & u_3 \end{pmatrix}$$

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Show that $|u_1| > |c_1|, \quad |u_2| > |c_2|, \quad |u_3| > 0$

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Exercise 2.10