Homework Assignment 8

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Problem 7.1. Show that

a. $\mathscr{H}_0\left\{(a^2 - r^2)H(a-r)\right\} = \dfrac{4a}{\kappa^3}J_1(a\kappa) - \dfrac{2a^2}{\kappa^2}J_0(a\kappa).$

Solution. a. Let J_n be the integral representation of the Bessel function of order n, i.e.

$$J_n(\kappa r) = \frac{1}{2\pi}\int_{\pi/2-\phi}^{5\pi/2-\phi} \exp\left[i(n\alpha - \kappa r \sin\alpha)\right] d\alpha$$

Then the Hankel transformation of order n of $f(r)$ is defined to be

$$\mathscr{H}_n\left\{f(r)\right\} = \int_0^\infty r J_n(\kappa r) f(r) dr.$$

Using the table of Hankel transforms we see that

$$\mathscr{H}_0\left\{(a^2 - r^2)H(a-r)\right\} = \frac{4a}{\kappa^3}J_1(a\kappa) - \frac{2a^2}{\kappa^2}J_0(a\kappa),$$

and we are done.

$\square$

Problem 7.2. a. Show that the solution of the boundary value problem

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + u_{zz} &= 0, \qquad 0 < r < \infty, \quad 0 < z < \infty, \\ u(r,0) &= \frac{1}{\sqrt{a^2 + r^2}}, \qquad 0 < r < \infty, \end{aligned}$$

is

$$u(r,z) = \int_0^\infty e^{-\kappa(z+a)} J_0(\kappa r) d\kappa = \left[(z+a)^2 + r^2\right]^{-1/2}.$$

Solution. a. Let

$$u(r,z) = \left[(z+a)^2 + r^2\right]^{-1/2}.$$

Then it is clear that for $0 < r < \infty$ we have that

$$u(r,0) = \frac{1}{\sqrt{a^2 + r^2}}$$

and $u(r,z)$ satisfies the boundary condition.
Now, note from the definition of $u(r,z)$ that

$$\begin{aligned} u_r &= -r\left[(z+a)^2 + r^2\right]^{-3/2}, \\ u_{rr} &= -\left[(z+a)^2 + r^2\right]^{-3/2} + 3r^2\left[(z+a)^2 + r^2\right]^{-5/2}, \\ u_z &= -(z+a)\left[(z+a)^2 + r^2\right]^{-3/2}, \\ u_{zz} &= -\left[(z+a)^2 + r^2\right]^{-3/2} + 3(z+a)^2\left[(z+a)^2 + r^2\right]^{-5/2}. \end{aligned}$$

Therefore, we see that

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + u_{zz} &= \frac{3r^2 + 3(z+a)^2}{[(z+a)^2 + r^2]^{5/2}} - \frac{3}{[(z+a)^2 + r^2]^{3/2}} \\ &= \frac{3r^2 + 3(z+a)^2 - 3\left[(z+a)^2 + r^2\right]}{[(z+a)^2 + r^2]^{5/2}} \\ &= 0, \end{aligned}$$

and we see that $u(r,z)$ is a solution of the boundary value problem. $\square$

Problem 7.9. Solve the problem of the electrified unit disk in the (x, y) plane with center at the origin. The electric potential $u(r, z)$ is axisymmetric and satisfies the boundary value problem

$$\begin{aligned}
&u_{rr} + \frac{1}{r}u_r + u_{zz} = 0, \qquad 0 < r < \infty, \quad 0 < z < \infty, \\
&u(r, 0) = u_0, \qquad 0 \leq r < a \\
&\frac{\partial u}{\partial z} = 0, \qquad \text{on } z = 0 \text{ for } a < r < \infty, \\
&u(r, z) \to 0 \qquad \text{as } z \to \infty \text{ for all } r,
\end{aligned}$$

where u_0 is constant. Show that the solution is

$$u(r, z) = \left(\frac{2au_0}{\pi}\right) \int_0^\infty kJ_0(kr) \left(\frac{\sin ak}{k^2}\right) e^{-kz} dk.$$

Solution. In order to find the solution to the boundary value problem, we will apply the 0-th order Hankel transform to the system of differential equations.

Let $\tilde{u}_0(k, z) = \mathscr{H}_0 \{u(r, z)\}$. Then from a previous theorem we have that

$$\mathscr{H}_0 \left\{ u_{rr} + \frac{1}{r}u_r \right\} = -k^2 \tilde{u}_0(k, z). \tag{1}$$

Thus, from the above result in combination with Leibniz's integral rule, we see that applying the 0-th order Hankel transform to the boundary value problem yields

$$\frac{d^2}{dz^2} [\tilde{u}_0(k, z)] - k^2 \tilde{u}_0(k, z) = 0, \qquad 0 < r < \infty, \quad 0 < z < \infty.$$

This is a homogeneous linear ordinary differential equation and we readily see that the solution to the equation is

$$\tilde{u}_0(k, z) = c_1(k)e^{-kz} + c_2(k)e^{kz}. \tag{2}$$

Note that if $u(r, z) \to 0$ as $z \to \infty$ for all r, then $\tilde{u}_0(k, z) \to 0$ as $z \to \infty$ for all k. Thus, if $\tilde{u}_0(k, z)$ is of the form (2), then $\tilde{u}_0(k, z) \to 0$ as $z \to \infty$ for all k if and only if $c_2(k) = 0$. Thus, (2) reduces to

$$\tilde{u}_0(k, z) = c_1(k)e^{-kz}. \tag{3}$$

Thus, taking the inverse 0-th order Hankel transform of (3), we see that the solution to the original differential equation is

$$u(r, z) = \mathscr{H}_0^{-1} \{\tilde{u}_0(k, z)\} = \int_0^\infty kc_1(k)J_0(kr)e^{-kz} dk.$$

From this solution the boundary conditions become

$$\begin{aligned}
\int_0^\infty kc_1(k)J_0(kr) dk &= u_0 \\
\int_0^\infty k^2 c_1(k)J_0(kr) dk &= 0.
\end{aligned}$$

Using our table of Hankel transforms, we see that

$$\mathscr{H}_0\left\{\frac{\sin ak}{k^2}\right\} = \int_0^\infty k\left(\frac{\sin ak}{k^2}\right)J_0(ak)dk = \begin{cases}\frac{\pi}{2} & \text{if } k < a\\ \sin^{-1}\left(\frac{a}{k}\right) & \text{if } k > a\end{cases}$$

$$\mathscr{H}_0\left\{\frac{\sin ak}{k}\right\} = \int_0^\infty k\left(\frac{\sin ak}{k}\right)J_0(ak)dk = \begin{cases}\left(a^2 - k^2\right)^{-1/2} & \text{if } k < a\\ 0 & \text{if } k > a\end{cases}$$

These transforms imply that the solution to the system of integral equations resulting from the boundary conditions is

$$c_1(k) = \left(\frac{2u_0}{\pi}\right)\frac{\sin ak}{k^2}.$$

Therefore, the solution to the transformed equation is

$$\tilde{u}_0(k,z) = \left(\frac{2u_0}{\pi}\right)\left(\frac{\sin ak}{k^2}\right)e^{-kz}$$

and the solution to the original equation is

$$u(r,z) = \mathscr{H}_0^{-1}\left\{\tilde{u}_0(k,z)\right\} = \int_0^\infty k\left(\frac{2u_0}{\pi}\right)\left(\frac{\sin ak}{k^2}\right)e^{-kz}J_0(rk)dk.$$

$\square$

Problem 7.12. Solve the Cauchy problem for the wave equation in a dissipating medium

$$u_{tt} + 2\kappa u_t = c^2\left(u_{rr} + \frac{1}{r}u_r\right), \qquad 0 < r < \infty, \quad 0 < t,$$
$$u(r,0) = f(r), \quad u_t(r,0) = g(r), \qquad 0 < r < \infty.$$

where κ is a constant.

Solution. We begin by applying the 0-th order Hankel transform to the first equation. Letting $\tilde{u}_0(k,t) = \mathscr{H}_0\{u(r,t)\}$ and using (1), we see this results in the following transformed equation

$$\frac{d^2}{dt^2}\left[\tilde{u}_0(k,t)\right] + 2\kappa\frac{d}{dt}\left[\tilde{u}_0(k,t)\right] = -(kc)^2\tilde{u}_0(k,t),$$

or, equivalently,

$$\frac{d^2}{dt^2}\left[\tilde{u}_0(k,t)\right] + 2\kappa\frac{d}{dt}\left[\tilde{u}_0(k,t)\right] + (kc)^2\tilde{u}_0(k,t) = 0.$$

This is a homogeneous, linear ordinary differential equation, the solution to which we readily see is

$$\tilde{u}_0(k,t) = c_1 e^{\left(-\kappa-\sqrt{\kappa^2-(ck)^2}\right)t} + c_2 e^{\left(-\kappa+\sqrt{\kappa^2-(ck)^2}\right)t}. \tag{4}$$

Taking the 0-th order Hankel transform of the boundary conditions, we see that

$$\tilde{u}_0(k,0) = \tilde{f}_0(k), \qquad 0 < r < \infty$$
$$\frac{d}{dt}\left[\tilde{u}_0(k,0)\right] = \tilde{g}_0(k), \qquad 0 < r < \infty$$

Using the solution (4) and the first transformed boundary condition, we see that

$$c_1 + c_2 = \tilde{f}_0(k).$$

Similarly, using the solution (4) and the second transformed initial condition, we see that

$$\left(-\kappa - \sqrt{\kappa^2-(ck)^2}\right)c_1 + \left(-\kappa + \sqrt{\kappa^2-(ck)^2}\right)c_2 = \tilde{f}_0(k).$$

Solving the resulting system of equation for c_1 and c_2 shows that

$$c_1 = -\frac{\tilde{g}_0(k) + \tilde{f}_0(k)\kappa - \tilde{f}_0(k)\sqrt{\kappa^2-(ck)^2}}{2\sqrt{\kappa^2-(ck)^2}}$$
$$c_2 = \frac{\tilde{g}_0(k) + \tilde{f}_0(k)\kappa + \tilde{f}_0(k)\sqrt{\kappa^2-(ck)^2}}{2\sqrt{\kappa^2-(ck)^2}}.$$

Thus, letting $c_1(k) = c_1$ and $c_2(k) = c_2$, the solution to the transformed differential equation with the specified initial conditions is

$$\tilde{u}_0(k,t) = c_1(k)e^{\left(-\kappa-\sqrt{\kappa^2-(ck)^2}\right)t} + c_2(k)e^{\left(-\kappa+\sqrt{\kappa^2-(ck)^2}\right)t}.$$

Therefore, the solution to the original differential equation satisfying the specified initial conditions is

$$u(r,t) = \mathscr{H}_0^{-1}\{\tilde{u}_0(k,t)\} = \int_0^\infty kJ_0(kr)\left[c_1(k)e^{\left(-\kappa-\sqrt{\kappa^2-(ck)^2}\right)t} + c_2(k)e^{\left(-\kappa+\sqrt{\kappa^2-(ck)^2}\right)t}\right]dk.$$

$\square$

Problem 7.14. Find the steady temperature $u(r, z)$ on a beam, $0 \leq r < \infty$, $0 \leq z \leq a$ when the face $z = 0$ is kept at temperature $u(r, 0) = 0$ and the face $z = a$ is insulated except that the heat is supplied through a circular hole such that

$$u_z(r, a) = H(b - r).$$

The temperature $u(r, z)$ satisfies the axisymmetric equation

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = 0, \qquad 0 \leq r < \infty, \quad 0 \leq z \leq a.$$

Solution. The temperature described above satisfies the following partial differential equation with associated boundary conditions

$$\begin{aligned}
&u_{rr} + \frac{1}{r}u_r + u_{zz} = 0, \qquad 0 \leq r < \infty, \quad 0 \leq z \leq a,\\
&u(r, 0) = 0, \qquad 0 \leq r < \infty,\\
&u_z(r, a) = H(b - r), \qquad 0 \leq r < \infty.
\end{aligned}$$

Let $\tilde{u}_0(k, z) = \mathscr{H}_0 \{u(r, z)\}$ be the 0-th order Hankel transform of $u(r, z)$. Now, applying the 0-th order Hankel transform to the differential equation yields that

$$\frac{d^2}{dz^2}\left[\tilde{u}_0(k, z)\right] - k^2\tilde{u}_0(k, z) = 0.$$

The resulting equation is a homogeneous linear ordinary differential equation, the solution to which is

$$\tilde{u}_0(k, z) = c_1(k)e^{-kz} + c_2(k)e^{kz}. \tag{5}$$

Applying the 0-th order Hankel transform to the boundary conditions yields

$$\begin{aligned}
\tilde{u}_0(k, 0) &= 0, \qquad 0 \leq k < \infty,\\
\frac{d}{dz}\left[\tilde{u}_0(k, z)\right]\bigg|_{z=a} &= \frac{b}{k}J_1(bk), \qquad 0 \leq k < \infty.
\end{aligned}$$

Using the solution (5) and the transformed boundary conditions, we see that

$$\begin{aligned}
c_1(k) + c_2(k) &= 0\\
-c_1(k)ke^{-ka} + c_2(k)ke^{ka} &= \frac{b}{k}J_1(bk).
\end{aligned}$$

Solving this system of equation we see that

$$\begin{aligned}
c_1(k) &= -\frac{be^{ak}J_1(bk)}{k\left(1 + e^{2ak}\right)}\\
c_2(k) &= \frac{be^{ak}J_1(bk)}{k\left(1 + e^{2ak}\right)}.
\end{aligned}$$

Thus, the solution to the transformed differential equation is

$$\begin{aligned}
\tilde{u}_0(k,z) &= \left[-\frac{be^{ak}J_1(bk)}{k\left(1+e^{2ak}\right)}\right]e^{-kz} + \left[\frac{be^{ak}J_1(bk)}{k\left(1+e^{2ak}\right)}\right]e^{kz} \\
&= \frac{bJ_1(bk)e^{k(a-z)}\left(-1+e^{2kz}\right)}{k\left(1+e^{2ak}\right)} \\
&= \left(\frac{bJ_1(bk)}{k}\right)\frac{\sinh kz}{\cosh ka}.
\end{aligned}$$

Therefore, the solution to the original differential equation is

$$\begin{aligned}
u(r,z) = \mathscr{H}_0^{-1}\left\{\tilde{u}_0(k,z)\right\} &= \int_0^\infty k\left(\frac{bJ_1(bk)}{k}\right)\frac{\sinh kz}{\cosh ka}J_0(kr)dk \\
&= b\int_0^\infty \frac{\sinh kz}{\cosh ka}J_0(kr)J_1(bk)dk.
\end{aligned}$$

$\square$

Problem 7.19. Use the joint Hankel and Laplace transform method to solve the initial boundary value problem

$$\begin{aligned}
&u_{rr} + \frac{1}{r}u_r - u_{tt} - 2\varepsilon u_t = a\frac{\delta(r)}{r}\delta(t), \qquad 0 < r < \infty, \quad 0 < t,\\
&u(r,t) \to 0 \qquad \text{as } r \to \infty,\\
&u(0,t) \qquad \text{is finite for } 0 < t,\\
&u(r,0) = u_t(r,0) = 0 \qquad \text{for } 0 < r < \infty.
\end{aligned}$$

Solution. We begin by applying the 0-th order Hankel transform to the differential equation and its initial and boundary conditions. Doing so yields

$$\begin{aligned}
&-k^2\tilde{u}_0(k,t) - \frac{d^2}{dt^2}\left[\tilde{u}_0(k,t)\right] - 2\varepsilon\frac{d}{dt}\left[\tilde{u}_0(k,t)\right] = a\delta(t), \qquad 0 < r < \infty, \quad 0 < t,\\
&\tilde{u}_0(k,t) \to 0 \qquad \text{as } k \to \infty,\\
&\tilde{u}_0(0,t) \qquad \text{is finite for } 0 < t,\\
&\tilde{u}_0(k,0) = \frac{d}{dt}\left[\tilde{u}_0(k,t)\right]\bigg|_{t=0} = 0 \qquad \text{for } 0 < k < \infty.
\end{aligned}$$

Now, let $\bar{\tilde{u}}_0(k,s) = \mathscr{L}\left\{\tilde{u}_0(k,t)\right\}$. Applying the Laplace transform to the resulting equation shows that

$$-k^2\bar{\tilde{u}}_0(k,s) - s^2\bar{\tilde{u}}_0(k,s) + s\tilde{u}_0(k,0) + \frac{d}{dt}\left[\tilde{u}_0(k,t)\right]\bigg|_{t=0} - 2\varepsilon\left(s\bar{\tilde{u}}_0(k,s) - \tilde{u}_0(k,0)\right) = a.$$

In light of the initial data, this equation reduces to

$$-k^2\bar{\tilde{u}}_0(k,s) - s^2\bar{\tilde{u}}_0(k,s) - 2\varepsilon s\bar{\tilde{u}}_0(k,s) = a,$$

or, equivalently,

$$\left[s^2 + 2\varepsilon s + k^2\right]\bar{\tilde{u}}_0(k,s) = -a.$$

Thus, the solution to the Laplace-transformed equation is

$$\bar{\tilde{u}}_0(k,s) = -\frac{a}{s^2 + 2\varepsilon s + k^2} = -\frac{a}{\left[s - \left(-\varepsilon - \sqrt{\varepsilon^2 - k^2}\right)\right]\left[s - \left(-\varepsilon + \sqrt{\varepsilon^2 - k^2}\right)\right]}.$$

Recall that

$$\frac{1}{a-b}\mathscr{L}\left\{e^{at} - e^{bt}\right\} = \frac{1}{(s-a)(s-b)}.$$

This implies that

$$\begin{aligned}
\tilde{u}_0(k,t) = \mathscr{L}^{-1}\left\{\bar{\tilde{u}}_0(k,s)\right\} &= -a\mathscr{L}^{-1}\left\{\frac{1}{\left[s - \left(-\varepsilon - \sqrt{\varepsilon^2 - k^2}\right)\right]\left[s - \left(-\varepsilon + \sqrt{\varepsilon^2 - k^2}\right)\right]}\right\}\\
&= \frac{a\left[e^{\left(-\varepsilon - \sqrt{\varepsilon^2 - k^2}\right)t} - e^{\left(-\varepsilon + \sqrt{\varepsilon^2 - k^2}\right)t}\right]}{2\sqrt{\varepsilon^2 - k^2}}.
\end{aligned}$$

Therefore, the solution to the original differential equation is

$$u(r,t) = \mathscr{H}_0^{-1}\left\{\tilde{u}_0(k,t)\right\} = \int_0^\infty k\left\{\frac{a\left[e^{\left(-\varepsilon-\sqrt{\varepsilon^2-k^2}\right)t} - e^{\left(-\varepsilon+\sqrt{\varepsilon^2-k^2}\right)t}\right]}{2\sqrt{\varepsilon^2-k^2}}\right\}J_0(kr)dk.$$

$\square$