Homework Assignment 3

Matthew Tiger

September 20, 2016

Problem 1.5.1. Find the fixed points of the following maps and use the appropriate theorems to determine whether they are asymptotically stable, semi-stable, or unstable:

i. $f(x) = \frac{x^3}{2} + \frac{x}{2}$,

ii. $f(x) = \arctan(x)$,

iii. $f(x) = x^3 + x^2 + x$,

iv. $f(x) = x^3 - x^2 + x$,

v. $f(x) = \begin{cases} 3x/4 & x \leq 1/2 \\ 3(1-x)/4 & x > 1/2 \end{cases}$.

Solution. $\square$

Problem 1.5.2. Consider the family of quadratic maps $f_c(x) = x^2 + c$ where $x \in \mathbb{R}$.

i. Use the theorems of section 1.5 to determine the stability of the hyperbolic fixed points of the the family of maps for all possible values of c.

ii. Find any values of c such that f_c has a non-hyperbolic fixed point and determine the stability of these fixed points.

Solution. $\square$

Problem 1.5.3.

i. Show that $f(x) = -2x^3 + 2x^2 + x$ has two non-hyperbolic fixed points and determine their stability.

ii. If $x = 0$ and $x = 1$ are non-hyperbolic fixed points for $f : \mathbb{R} \to \mathbb{R}$ for $f(x) = ax^3 + bx^2 + cx + d$, find all possible values of a, b, c, and d.

iii. Write down the function $f(x)$ in each case of (ii) above and determine the stability of the fixed points.

Solution. □

Problem 1.5.6. Find the Schwarzian derivative of both $f(x) = e^x$ and $g(x) = \sin(x)$ and show that they are always negative.

Solution. □

Problem 1.5.9. Let $f(x)$ be a polynomial such that $f(c) = c$. (Recall that a polynomial $p(x)$ has $(x-c)^2$ as a factor if and only if both $p(c) = 0$ and $p'(c) = 0$.)

i. If $f'(c) = 1$, show that $(x-c)^2$ is a factor of $g(x) = f(x) - x$.

ii. If $|f'(c)| = 1$, show that $(x-c)^2$ is a factor of $h(x) = f^2(x) - x$.

iii. Show in the case that $f'(c) = -1$, we actually have that $(x-c)^3$ is a factor of $h(x) = f^2(x) - x$.

iv. Check that (iii) holds for the non-hyperbolic fixed point $x = 2/3$ of the logistic map $L_3(x) = 3x(1-x)$.

v. Check that (i), (ii), (iii) hold for the non-hyperbolic fixed points of the polynomial $f(x) = -2x^3 + 2x^2 + x$.

Solution. □