Homework Assignment 9

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Problem 1. Verify that the forward Euler scheme (9.29) has first order accuracy on a smooth solution $u = u(x)$ of problem (9.30).

Solution. Suppose we have the problem $Lu = f$, as defined in 9.30 i.e.

$$Lu = \begin{cases} \frac{du}{dx} - G(x, u), \ 0 < x \leq 1, \\ 0 \end{cases} \quad \text{and } f = \begin{cases} 0 \\ a \end{cases}.$$

The forward Euler scheme $L_h u^{(h)} = f^{(h)}$ is given by

$$L_h u^{(h)} = \begin{cases} \frac{u_{n+1} - u_n}{h} - G(x_n, u_n), \ n = 0, 1, \ldots N-1 \\ u_0 \end{cases} \quad \text{and } f^{(h)} = \begin{cases} 0 \\ a \end{cases}.$$

Let $[u]_h$ denote the discretized solution to $Lu = f$. This scheme has first order accuracy if $\left\| L_h[u]_h - L_h u^{(h)} \right\| \leq Ch$ where C is a constant that does not depend on h.

Note that the Taylor series expansion of $u(x+h)$ centered at x is given by

$$u(x+h) = u(x) + u'(x)h + \frac{u''(\xi)h^2}{2}$$

for $x \leq \xi \leq x + h$. This implies that

$$u'(x) = \frac{u(x+h) - u(x)}{h} - \frac{u''(\xi)h}{2}$$

or that

$$u'(x) - G(x, u) = \frac{u(x+h) - u(x)}{h} - \frac{u''(\xi)h}{2} - G(x, u).$$

As $u'(x) - G(x, u) = 0$ is the exact solution to $Lu = f$, we know that the discretized exact solution is given by

$$u'(x) - G(x, u) = \frac{u(x_{n+1}) - u(x_n)}{h} - \frac{u''(\xi(x_n))h}{2} - G(x_n, u_n) = 0$$

where $\xi(x_n)$ depends on the node x_n. But under the forward Euler scheme, $L_h[u]_h = \frac{u_{n+1} - u_n}{h} - G(x_n, u_n)$ so that

$$u'(x) - G(x, u) = L_h[u]_h - \frac{u''(\xi(x_n))h}{2} = 0$$

i.e.

$$u'(x) - G(x, u) = L_h[u]_h - L_h u^{(h)} = \frac{u''(\xi(x_n))h}{2}$$

since $L_h u^{(h)} = 0$. If $|u''(x)| \leq M$ for $x \in [0, 1]$, then the above implies that

$$\left\| L_h[u]_h - L_h u^{(h)} \right\| = \left\| \frac{u''(\xi(x_n))h}{2} \right\| \leq \frac{M}{2} h.$$

As $M/2$ does not depend on h, we have shown $\left\| L_h[u]_h - L_h u^{(h)} \right\| \leq Ch$ where $C = M/2$ and that the forward Euler scheme has first order of accuracy. $\square$