

Homework Assignment 3

Matthew Tiger

March 22, 2016

Problem 1. a. Give an example of an asymptotic relation $f(x) \sim g(x)$ ($x \rightarrow \infty$) that cannot be exponentiated; that is $e^{f(x)} \sim e^{g(x)}$ ($x \rightarrow \infty$) is false.

b. Show that if $f(x) - g(x) \ll 1$ ($x \rightarrow \infty$), then $e^{f(x)} \sim e^{g(x)}$ ($x \rightarrow \infty$).

Solution. a. Note that for $x \rightarrow \infty$ we have that $e^{f(x)} \not\sim e^{g(x)}$ if and only if

$$\lim_{x \rightarrow \infty} \frac{e^{f(x)}}{e^{g(x)}} = \lim_{x \rightarrow \infty} e^{f(x)-g(x)} \neq 1.$$

Thus, if $\lim_{x \rightarrow \infty} f(x) - g(x) \neq 0$, then $\lim_{x \rightarrow \infty} e^{f(x)-g(x)} \neq 1$ and $e^{f(x)} \not\sim e^{g(x)}$. Therefore, take for instance the functions $f(x) = x + 1$ and $g(x) = x$. These functions are clearly asymptotic as

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{x+1}{x} = \lim_{x \rightarrow \infty} 1 + \frac{1}{x} = 1.$$

However,

$$\lim_{x \rightarrow \infty} \frac{e^{f(x)}}{e^{g(x)}} = \lim_{x \rightarrow \infty} e^{f(x)-g(x)} = \lim_{x \rightarrow \infty} e^{(x+1)-x} = e \neq 1$$

so that the asymptotic relation between $f(x)$ and $g(x)$ cannot be exponentiated.

b. Suppose that $f(x) - g(x) \ll 1$ as $x \rightarrow \infty$. Then

$$\lim_{x \rightarrow \infty} \frac{f(x) - g(x)}{1} = \lim_{x \rightarrow \infty} f(x) - g(x) = 0.$$

If this is true then we must have that

$$\lim_{x \rightarrow \infty} \frac{e^{f(x)}}{e^{g(x)}} = \lim_{x \rightarrow \infty} e^{f(x)-g(x)} = e^{\lim_{x \rightarrow \infty} f(x)-g(x)} = e^0 = 1$$

or that $e^{f(x)} \sim e^{g(x)}$ and we are done.

□

Problem 2. Find and classify all the singular points (including the point at ∞) of the equations:

$$x(1-x)y'' + [2 - (a+b)x]y' - aby = 0, \quad (x^2 + 1)y'' - xy = 0.$$

Here, $a, b \in \mathbb{R}$.

Solution. If a homogeneous differential equation is of the form

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_1(x)y^{(1)}(x) + p_0(x)y = 0 \quad (1)$$

then the point $x = x_0$ is:

- *ordinary* if the coefficient functions $p_i(x)$ for $i = 0, \dots, n-1$ are all analytic in a neighborhood of $x = x_0$.
- *singular* if some coefficient function $p_i(x)$ for $i = 0, \dots, n-1$ is not analytic in some neighborhood of $x = x_0$.

If the point $x = x_0$ is singular, then the point is:

- *regular singular* if the functions $(x - x_0)p_{n-1}(x), \dots, (x - x_0)^{n-1}p_1(x), (x - x_0)^n p_0(x)$ are all analytic in a neighborhood of $x = x_0$.
- *irregular singular* if one of the functions $(x - x_0)p_{n-1}(x), \dots, (x - x_0)^{n-1}p_1(x), (x - x_0)^n p_0(x)$ are not analytic in some neighborhood of $x = x_0$.

- a. We begin by transforming the differential equation $x(1-x)y'' + [2 - (a+b)x]y' - aby = 0$ into the form presented in (1):

$$y'' + \left[\frac{2}{x(1-x)} - \frac{a+b}{1-x} \right] y' + \left[-\frac{ab}{x(1-x)} \right] y = 0.$$

Thus, we identify the coefficient functions

$$p_1(x) = \frac{2}{x(1-x)} - \frac{a+b}{1-x}, \quad p_0(x) = -\frac{ab}{x(1-x)}.$$

In order to classify all finite points, we see that these coefficient functions are analytic everywhere on the real line except for the points $x_0 = 0$ and $x_1 = 1$. To classify the singular points x_i for $i = 0, 1$, we must determine if the functions $(x - x_i)p_1(x)$ and $(x - x_i)^2 p_0(x)$ are analytic.

For the point $x_0 = 0$, we know that

$$\begin{aligned} xp_1(x) &= \frac{2}{(1-x)} - \frac{(a+b)x}{1-x} \\ x^2 p_0(x) &= -\frac{abx}{(1-x)} \end{aligned}$$

both of which are analytic at $x = 0$ showing that $x_0 = 0$ is a regular singular point.

For the point $x_0 = 1$, we know that

$$(x-1)p_1(x) = -\frac{2}{x} + a + b$$

$$(x-1)^2p_0(x) = \frac{ab(x-1)}{x}$$

both of which are analytic at $x = 1$ showing that $x_0 = 1$ is a regular singular point.

In order to classify the point at ∞ , we make the transformation $x = \frac{1}{t}$ and classifying the point $t = 0$. Thus, the equation becomes

$$t^4y'' + \left[2t^3 - \frac{2t^4}{t-1} + \frac{(a+b)t^3}{t-1}\right]y' + \left[-\frac{abt^2}{t-1}\right]y = 0.$$

The coefficient functions are then

$$p_1(t) = \frac{2}{t} - \frac{2}{t-1} + \frac{a+b}{t(t-1)}, \quad p_0(t) = -\frac{ab}{t^2(t-1)}.$$

It is clear that $x_2 = \infty$ is a singular point since the coefficient functions are not analytic at $t = 0$. Since both $tp_1(t)$ and $t^2p_0(t)$ are analytic at $t = 0$, the point $x_2 = \infty$ is a regular singular point.

- b. We rewrite the differential equation $(x^2 + 1)y'' - xy = 0$ in the form presented in (1):

$$y'' + \left[-\frac{x}{x^2 + 1}\right]y = 0.$$

Thus, the coefficient function is

$$p_0(x) = -\frac{x}{x^2 + 1} = -\frac{x}{(x-i)(x+i)}.$$

In order to classify all finite points, we see that this coefficient function is analytic everywhere on the real line so that all finite real points are ordinary.

In order to classify the point at ∞ , we make the transformation $x = \frac{1}{t}$ and classifying the point $t = 0$. Thus, the equation becomes

$$t^4y'' + 2t^3y' - \frac{t}{t^2 + 1}y = 0.$$

The coefficient functions are then

$$p_1(t) = \frac{2}{t}, \quad p_0(t) = -\frac{1}{t^3(t^2 + 1)}.$$

The point $t = 0$ is clearly a singular point since both coefficient functions are not analytic at $t = 0$. Since $t^2p_0(t)$ is also not analytic at $t = 0$, the point $x_2 = \infty$ is an irregular singular point.

□

Problem 3. Find the Taylor series solution of the IVP

$$Ly = (1 - x^3)y''' + 2xy' = 0, \quad y(0) = 3, y'(0) = 3, y''(0) = 0.$$

Solution. Note that the point $x = 0$ is an ordinary point of this differential equation. Therefore, we may assume the solution to the differential equation is a Taylor series of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Using this form, we see that the differential equation becomes

$$\begin{aligned} Ly &= (1 - x^3) \sum_{n=0}^{\infty} n(n-1)(n-2)a_n x^{n-3} + 2x \sum_{n=0}^{\infty} n a_n x^{n-1} \\ &= \sum_{n=0}^{\infty} n(n-1)(n-2)a_n x^{n-3} - \sum_{n=0}^{\infty} n(n-1)(n-2)a_n x^n + \sum_{n=0}^{\infty} 2n a_n x^n \\ &= \sum_{n=0}^{\infty} n(n-1)(n-2)a_n x^{n-3} - \sum_{n=3}^{\infty} (n-3)(n-4)(n-5)a_{n-3} x^{n-3} + \sum_{n=3}^{\infty} 2(n-3)a_{n-3} x^{n-3} \\ &= \sum_{n=3}^{\infty} [n(n-1)(n-2)a_n - ((n-3)(n-4)(n-5) - 2(n-3))a_{n-3}] x^{n-3} = 0. \end{aligned} \quad (2)$$

Thus, we see that the series in (2) satisfies the differential equation if and only if the coefficients of the series are identically zero. From this we derive the recurrence relation

$$\begin{aligned} n(n-1)(n-2)a_n &= 0 & n &= 0, 1, 2 \\ a_n &= \frac{[(n-3)(n-4)(n-5) - 2(n-3)]a_{n-3}}{n(n-1)(n-2)} & n &= 3, 4, 5, \dots \end{aligned}$$

where the coefficients a_0 , a_1 , and a_2 are arbitrary constants. The solutions of the second equation are

$$\begin{aligned} a_{3n} &= 0, \\ a_{3n+1} &= a_1 \left[\left(\frac{(1 \cdot 0 \cdot -1) - (2 \cdot 1)}{4 \cdot 3 \cdot 2} \right) \left(\frac{(4 \cdot 3 \cdot 2) - (2 \cdot 4)}{7 \cdot 6 \cdot 5} \right) \cdots \right. \\ &\quad \cdots \left(\frac{(3n-5) \cdot (3n-6) \cdot (3n-7) - 2 \cdot (3n-5)}{(3n-2)(3n-3)(3n-4)} \right) \cdots \\ &\quad \left. \cdots \left(\frac{(3n-2) \cdot (3n-3) \cdot (3n-4) - 2 \cdot (3n-2)}{(3n+1)(3n)(3n-1)} \right) \right] \\ &= a_1 \left[\frac{\prod_{k=1}^n (3k-2)(3k-3)(3k-4) - 2(3k-2)}{(3n+1)!} \right], \end{aligned}$$

$$\begin{aligned}
a_{3n+2} &= a_2 \left[\left(\frac{(2 \cdot 1 \cdot 0) - (2 \cdot 2)}{5 \cdot 4 \cdot 3} \right) \left(\frac{(5 \cdot 4 \cdot 3) - (2 \cdot 5)}{8 \cdot 7 \cdot 6} \right) \cdots \right. \\
&\quad \cdots \left(\frac{(3n-4) \cdot (3n-5) \cdot (3n-6) - 2 \cdot (3n-4)}{(3n-1)(3n-2)(3n-3)} \right) \cdots \\
&\quad \left. \cdots \left(\frac{(3n-1) \cdot (3n-2) \cdot (3n-3) - 2 \cdot (3n-1)}{(3n+2)(3n+1)(3n)} \right) \right] \\
&= a_2 \left[\frac{2 \prod_{k=1}^n (3k-1)(3k-2)(3k-3) - 2(3k-1)}{(3n+2)!} \right].
\end{aligned}$$

Therefore, the general solution to the differential equation $Ly = 0$ is

$$\begin{aligned}
y(x) &= a_0 + a_1x + a_2x^2 + \sum_{n=3}^{\infty} a_nx^n \\
&= a_0 + a_1x + a_2x^2 + \sum_{n=1}^{\infty} a_{3n+1}x^{3n+1} + \sum_{n=1}^{\infty} a_{3n+2}x^{3n+2} \\
&= a_0 + a_1 \left[x + \sum_{n=1}^{\infty} \frac{a_{3n+1}}{a_1} x^{3n+1} \right] + a_2 \left[x^2 + \sum_{n=1}^{\infty} \frac{a_{3n+2}}{a_2} x^{3n+2} \right]. \tag{3}
\end{aligned}$$

Using the initial conditions $y(0) = y'(0) = 3, y''(0) = 0$ and the general solution (3), we see that

$$\begin{aligned}
y'(x) &= a_1 + 2a_2x + \sum_{n=1}^{\infty} (3n+1)a_{3n+1}x^{3n} + \sum_{n=1}^{\infty} (3n+2)a_{3n+2}x^{3n+1} \\
y''(x) &= 2a_2 + \sum_{n=1}^{\infty} (3n+1)(3n)a_{3n+1}x^{3n-1} + \sum_{n=1}^{\infty} (3n+2)(3n+1)a_{3n+2}x^{3n}
\end{aligned}$$

which implies that

$$\begin{aligned}
y(0) &= a_0 = 3 \\
y'(0) &= a_1 = 3 \\
y''(0) &= 2a_2 = 0.
\end{aligned}$$

Therefore, the solution to the IVP is

$$y(x) = 3 + 3x + 3 \sum_{n=1}^{\infty} \left[\frac{\prod_{k=1}^n (3k-2)(3k-3)(3k-4) - 2(3k-2)}{(3n+1)!} \right] x^{3n+1}.$$

□

Problem 4. Find two linearly independent solutions to $x(1-x)y'' - 3xy' - y = 0$ about $x = 0$.

Solution. Suppose that we have a second-order differential equation in the following form

$$Ly = y'' + \left[\frac{p(x)}{x - x_0} \right] y' + \left[\frac{q(x)}{(x - x_0)^2} \right] y = 0 \quad (4)$$

where $p(x)$ and $q(x)$ are analytic at the point $x = x_0$, i.e. $p(x)$ and $q(x)$ have power series representations centered at $x = x_0$, say

$$p(x) = \sum_{n=0}^{\infty} p_n x^n, \quad q(x) = \sum_{n=0}^{\infty} q_n x^n.$$

Clearly $x = x_0$ is a regular singular point of the differential equation $Ly = 0$. As such, the results of Fuchs show that there is at least one solution to the differential equation $Ly = 0$ of the form

$$y(x) = (x - x_0)^\alpha \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+\alpha} \quad (5)$$

where $\alpha \in \mathbb{R}$. Substituting this expression for $y(x)$ into the differential equation $Ly = 0$ yields

$$Ly = (x - x_0)^{\alpha-2} \left[\sum_{n=0}^{\infty} \left[(n + \alpha)(n + \alpha - 1)a_n + \sum_{k=0}^n (k + \alpha)p_{n-k}a_k + \sum_{k=0}^n q_{n-k}a_k \right] (x - x_0)^n \right]$$

which implies that if $Ly = 0$ then the coefficients of the above series must be zero yielding the following recurrence relation:

$$\begin{aligned} [\alpha(\alpha - 1) + \alpha p_0 + q_0] a_0 &= 0 & n = 0 \\ [(n + \alpha)(n + \alpha - 1) + (n + \alpha)p_0 + q_0] a_n &= - \sum_{k=0}^{n-1} [(k + \alpha)p_{n-k} + q_{n-k}] a_k & n > 1. \end{aligned} \quad (6)$$

Thus, in order to generate a non-trivial solution that satisfies this relation, we require that α must be a root of the *indicial polynomial*

$$P(z) = z(z - 1) + zp_0 + q_0$$

since $a_0 \neq 0$ by convention. This choice of α then ensures that the solution (5) will then be a non-trivial solution that satisfies the differential equation $Ly = 0$. Depending on the roots of the indicial polynomial, a second linearly independent solution of the same form of the differential equation can be found.

Note that for the differential equation

$$Ly = x(1-x)y'' - 3xy' - y = 0, \quad (7)$$

the point $x = 0$ is a regular singular point. Thus, we begin by rewriting the equation $Ly = 0$ in the form presented in (4):

$$\begin{aligned} Ly &= y'' + \left[\frac{-\frac{3x}{1-x}}{x} \right] y' + \left[\frac{-\frac{x}{1-x}}{x^2} \right] y \\ &= y'' + \left[\frac{p(x)}{x} \right] y' + \left[\frac{q(x)}{x^2} \right] y = 0. \end{aligned}$$

From this we identify the functions $p(x) = -\frac{3x}{1-x}$ and $q(x) = -\frac{x}{1-x}$ which are both analytic at the singular point $x = 0$. Note that $p(x)$ and $q(x)$ have the following power series representations centered at $x = 0$

$$\begin{aligned} p(x) &= -3x \sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} -3x^n \\ q(x) &= -x \sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} -x^n. \end{aligned}$$

Using these power series representations, we see that $p_0 = q_0 = 0$, and that the indicial polynomial of the differential equation $Ly = 0$ is

$$P(z) = z(z - 1)$$

the roots of which are $\alpha_1 = 1$ and $\alpha_2 = 0$. Using $\alpha_1 = 1$, we know that one solution of the differential equation is of the form (5), i.e. there is a solution $y_1(x)$ of the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$

where the coefficients a_n satisfy the recurrence relation (6) with $a_0 \neq 0$. Using the recurrence relation with $\alpha = 1$ we see that for $n > 0$,

$$\begin{aligned} a_n &= -\frac{\sum_{k=0}^{n-1} [(k+1)p_{n-k} + q_{n-k}]a_k}{n(n+1)} \\ &= \frac{\sum_{k=0}^{n-1} (3k+4)a_k}{n(n+1)} \end{aligned}$$

since $p_{n-k} = -3$ and $q_{n-k} = -1$ for $0 \leq k \leq n-1$. It can easily be proven through induction that the solution to this newly obtained recurrence relation is $a_n = (n+1)a_0$ for $n \geq 0$. Therefore, the first solution to the differential equation (7) is

$$y_1(x) = a_0 \sum_{n=0}^{\infty} (n+1)x^{n+1}$$

In order to find the second solution to the differential equation, we note that the roots of the indicial polynomial differ by a positive integer. Since the roots of the indicial polynomial

differ by a positive integer, there are two linearly independent solutions to the differential equation; however, without further analysis, we are unable to tell if the other solution is in the form (5). Since $P(\alpha_1) = P(N + \alpha_2) = 0$, we have from the recurrence relations (6) that

$$P(N + \alpha_2)a_N = 0a_N = - \sum_{k=0}^{N-1} [(k + \alpha_2)p_{N-k} + q_{N-k}] a_k$$

which is consistent only if the right hand side is equivalently zero. In our case, $N = 1$, $\alpha_2 = 0$ and we see that the right hand side evaluates to $-q_1a_0 \neq 0$ since $q_1 = -1 \neq 0$ and $a_0 \neq 0$ by assumption. Thus, the second solution to the differential equation is not of the form (5). To find the second solution we generalize the Frobenius solution and substitute the solution into the original differential equation to obtain

$$Ly(x, \alpha) = a_0x^{\alpha-2}P(\alpha).$$

Note that if $\alpha = \alpha_1 = 1$, then the solution is the solution already obtained to the homogeneous equation $Ly(x, \alpha) = 0$. Differentiating both sides of the equation with respect to α yields that

$$L \left[\frac{\partial}{\partial \alpha} y(x, \alpha) \Big|_{\alpha=\alpha_1} \right] = a_0 P'(\alpha_1) x^{\alpha_2+N-2} \quad (8)$$

so that $\frac{\partial}{\partial \alpha} y(x, \alpha) \Big|_{\alpha=\alpha_1}$ is a particular solution of the above differential equation. We can construct a second particular solution and subtract it from this particular solution to obtain the solution to the homogeneous equation. Note that the second particular solution has a Frobenius expansion $\sum_{n=0}^{\infty} c_n x^{n+\alpha_2}$. Using this form of the solution and substituting into (8) and equating coefficients with x^{α_2+N-2} yields the following relations

$$\begin{aligned} c_n &= - \frac{\sum_{k=0}^{n-1} [(k + \alpha_2)p_{n-k} + q_{n-k}] c_k}{P(\alpha_2 + n)} & n \neq 0, N \\ a_0 &= \frac{1}{P'(\alpha_1)} \sum_{k=0}^{N-1} [(k + \alpha_2)p_{N-k} + q_{N-k}] c_k & n = N \end{aligned} \quad (9)$$

where $c_0 \neq 0$ and $c_N \neq 0$ are arbitrary. The second solution to the homogeneous differential equation $Ly = 0$ is then

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} c_n x^{n+\alpha_2} - \frac{\partial}{\partial \alpha} y(x, \alpha) \Big|_{\alpha=\alpha_1} \\ &= \sum_{n=0}^{\infty} c_n x^{n+\alpha_2} - \log x \sum_{n=0}^{\infty} a_n x^{n+\alpha_1} - \sum_{n=0}^{\infty} \frac{\partial a_n(\alpha)}{\partial \alpha} \Big|_{\alpha=\alpha_1} x^{n+\alpha_1} \end{aligned}$$

where a_n is the sequence of coefficients defining the first solution to the homogeneous differential equation.

For this problem $N = 1$, $\alpha_1 = 1$, and $\alpha_2 = 0$. Using the recurrence relations (9) we readily see that $c_0 = -a_0$, c_1 is arbitrary, and that for $n > 1$:

$$\begin{aligned} c_n &= -\frac{\sum_{k=0}^{n-1} [kp_{n-k} + q_{n-k}]c_k}{n(n-1)} \\ &= \frac{\sum_{k=0}^{n-1} (3k+1)c_k}{n(n-1)} \end{aligned}$$

It can be proven through induction that $c_n = n(c_0 + 4c_1)/4 = n(4c_1 - a_0)/4$ completely determining the coefficients of the power series in the second solution.

We now investigate the partial derivative of $a_n(\alpha)$ with respect to α . Recall that for this problem the indicial polynomial is given by $P(z) = z(z-1)$. Looking, for a closed form representation of the solution to $a_n(\alpha)$ we see that for $a_0(\alpha) = a_0 \neq 0$,

$$\begin{aligned} a_1(\alpha) &= \frac{(3\alpha+1)a_0}{P(1+\alpha)} && \text{for } n = 1 \\ a_n(\alpha) &= \frac{(3\alpha+1)a_0 \prod_{k=1}^{n-1} [P(k+\alpha) + 3(k+\alpha) + 1]}{\prod_{k=1}^n P(k+\alpha)} && \text{for } n > 1 \end{aligned}$$

Using this recurrence relation in conjunction with Mathematica we see that the sequence

$$\left. \frac{\partial a_n(\alpha)}{\partial \alpha} \right|_{\alpha=\alpha_1} = b_n = \begin{cases} a_0 & \text{for } n = 0 \\ -\frac{3a_0}{2} & \text{for } n = 1 \\ \frac{3a_0(2n^4+11n^3+18n^2-71n+40)}{2n(n+1)^2(n+2)} & \text{for } n > 1 \end{cases}$$

Therefore, the solution to the differential equation (7) is

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} \frac{n(4c_1 - a_0)}{4} x^n - \log x \sum_{n=0}^{\infty} a_0(n+1)x^{n+1} - \sum_{n=0}^{\infty} \left. \frac{\partial a_n(\alpha)}{\partial \alpha} \right|_{\alpha=\alpha_1} x^{n+\alpha_1} \\ &= c_1 \sum_{n=0}^{\infty} nx^n - a_0 \left[\sum_{n=0}^{\infty} \frac{n}{4} x^n + \log x \sum_{n=0}^{\infty} (n+1)x^{n+1} + \sum_{n=0}^{\infty} \frac{b_n}{a_0} x^{n+1} \right]. \end{aligned}$$

□

Problem 5. Find two linearly independent solutions to $x^2y'' + 3xy' + (1 - 2x)y = 0$ about $x = 0$.

Solution.

□

Problem 6. Find the leading behavior of both solutions of $x^5 y'' - y = 0$ near $x = 0$.

Solution.

□

Problem 7. Find the first four terms in the asymptotic series for the solutions of $y'' = e^{-2/x}y$ as $x \rightarrow +\infty$.

Hint: When you are performing the asymptotic analysis to extract the leading behavior of the solution as $x \rightarrow +\infty$, you may (and probably want) to replace $e^{-2/x}$ with a reasonable simpler approximation.

Solution.

□