Homework Assignment 9

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Problem 1. Verify that the forward Euler scheme (9.29) has first order accuracy on a smooth solution $u = u(x)$ of problem (9.30).

Solution. Suppose we have the problem $Lu = f$, as defined in 9.30 i.e.

$$Lu = \begin{cases} \frac{du}{dx} - G(x,u), \ 0 < x \leq 1 \\ 0 \end{cases} \qquad \text{and } f = \begin{cases} 0 \\ a \end{cases}.$$

The forward Euler scheme $L_h u^{(h)} = f^{(h)}$ is given by

$$L_h u^{(h)} = \begin{cases} \frac{u_{n+1} - u_n}{h} - G(x_n, u_n), \ n = 0, 1, \ldots N-1 \\ u_0 \end{cases} \qquad \text{and } f^{(h)} = \begin{cases} 0, \ n = 0, 1, \ldots N-1 \\ a \end{cases}.$$

Let $[u]_h$ denote the discretized solution to $Lu = f$. This scheme has first order accuracy if $\left\| L_h[u]_h - L_h u^{(h)} \right\| \leq Ch$ where C is a constant that does not depend on h.

Note that the Taylor series expansion of $u(x+h)$ centered at x is given by

$$u(x+h) = u(x) + u'(x)h + \frac{u''(\xi)h^2}{2}$$

for $x \leq \xi \leq x + h$. This implies that

$$u'(x) = \frac{u(x+h) - u(x)}{h} - \frac{u''(\xi)h}{2}$$

or that

$$u'(x) - G(x,u) = \frac{u(x+h) - u(x)}{h} - \frac{u''(\xi)h}{2} - G(x,u).$$

As $u'(x) - G(x,u) = 0$ is the exact solution to $Lu = f$, we know that the discretized exact solution is given by

$$u'(x) - G(x,u) = \frac{u(x_{n+1}) - u(x_n)}{h} - \frac{u''(\xi(x_n))h}{2} - G(x_n, u_n) = 0$$

where $\xi(x_n)$ depends on the node x_n. But under the forward Euler scheme, $L_h[u]_h = \frac{u_{n+1} - u_n}{h} - G(x_n, u_n)$ so that

$$u'(x) - G(x,u) = L_h[u]_h - \frac{u''(\xi(x_n))h}{2} = 0$$

i.e.

$$u'(x) - G(x,u) = L_h[u]_h - L_h u^{(h)} = \frac{u''(\xi(x_n))h}{2}$$

since $L_h u^{(h)} = 0$. If $|u''(x)| \leq M$ for $x \in [0,1]$, then the above implies that

$$\left\| L_h[u]_h - L_h u^{(h)} \right\| = \left\| \frac{u''(\xi(x_n))h}{2} \right\| \leq \frac{M}{2}h.$$

As $M/2$ does not depend on h, we have shown $\left\| L_h[u]_h - L_h u^{(h)} \right\| \leq Ch$ where $C = M/2$ and that the forward Euler scheme has first order of accuracy. □

Problem 2. Verify that the Crank-Nicolson scheme (9.33) has second order accuracy on a smooth solution $u = u(x)$ of problem (9.30).

Solution. Suppose we have the problem $Lu = f$, as defined in 9.30 i.e.

$$Lu = \begin{cases} \frac{du}{dx} - G(x,u), \ 0 < x \leq 1 \\ 0 \end{cases} \quad \text{and } f = \begin{cases} 0 \\ a \end{cases} .$$

The Crank-Nicolson scheme $L_h u^{(h)} = f^{(h)}$ is given by

$$L_h u^{(h)} = \begin{cases} \frac{u_{n+1} - u_n}{h}, \ n = 0, \ldots N-1 \\ u_0 \end{cases}$$

and

$$f^{(h)} = \begin{cases} \frac{1}{2}[G(x_n, u_n) + G(x_{n+1}, u_{n+1})], \ n = 0, \ldots N-1 \\ a \end{cases} .$$

□

Problem 3. Create a difference scheme that is not consistent.

Solution. □

Problem 4. Prove that the scheme

$$4\frac{u_{n+1} - u_{n-1}}{2h} - 3\frac{u_{n+1} - u_n}{h} + u_n = 0, \quad n = 1, 2, \ldots, N-1$$

with initial conditions $u_0 = 1$ and $u_1 = e^{-h}$ is consistent for the problem

$$\frac{du}{dx} + u = 0, \quad 0 \leq x \leq 1$$

with initial condition $u(0) = 1$.

Solution. If $[u]_h$ is the discretized solution to the problem $Lu = f$ as defined above, then the scheme $L_h u^{(h)} = f^{(h)}$ is consistent if $\left\| L_h[u]_h - L_h u^{(h)} \right\| \to 0$ as $h \to 0$.

Note that the Taylor series expansions of $u(x+h)$ and $u(x-h)$ centered at x are given by

$$u(x+h) = u(x) + u'(x)h + \frac{u''(\xi_1)h^2}{2}$$
$$u(x-h) = u(x) - u'(x)h + \frac{u''(\xi_2)h^2}{2}$$

for $x \leq \xi_1 \leq x+h$ and $x-h \leq \xi_2 \leq x$. From these expansions we can see that

$$u'(x) = \frac{u(x+h) - u(x-h)}{2h} - \frac{1}{4}h(u''(\xi_1) - u''(\xi_2))$$

and

$$u'(x) = \frac{u(x+h) - u(x)}{h} - \frac{1}{2}hu''(\xi_3).$$

This shows that

$$u'(x) + u(x) = 4\frac{u(x+h) - u(x-h)}{2h} - 3\frac{u(x+h) - u(x)}{h} + u(x) + h\left(\frac{3}{2}u''(\xi_3) - (u''(\xi_1) - u''(\xi_2))\right)$$

so that if $[u]_h$ is the discretized solution to the problem defined above,

$$\begin{aligned} u'(x) + u(x) &= 4\frac{u(x_{n+1}) - u(x_{n-1})}{2h} - 3\frac{u(x_{n+1}) - u(x_n)}{h} + u(x_n) + h\left(\frac{3}{2}u''(\xi_3) - (u''(\xi_1) - u''(\xi_2))\right) \\ &= L_h[u]_h + h\left(\frac{3}{2}u''(\xi_3) - (u''(\xi_1) - u''(\xi_2))\right) = 0. \end{aligned}$$

Combining the above and the fact that $L_h u^{(h)} = 0$, we see that

$$\left\| L_h[u]_h - L_h u^{(h)} \right\| = h\left\| (u''(\xi_1) - u''(\xi_2)) - \frac{3}{2}u''(\xi_3) \right\|.$$

If $|u''(x)| \leq M$, then $0 \leq \left\| L_h[u]_h - L_h u^{(h)} \right\| \leq h\left(\frac{7}{2}M\right)$ and it is then clear that $\left\| L_h[u]_h - L_h u^{(h)} \right\| \to 0$ as $h \to 0$ showing the consistency of the scheme. □

Problem 5. Prove that the scheme

$$4\frac{u_{n+1} - u_{n-1}}{2h} - 3\frac{u_{n+1} - u_n}{h} + u_n = 0, \quad n = 1, 2, \ldots, N-1$$

with initial conditions $u_0 = 1$ and $u_1 = e^{-h}$ is divergent for the problem

$$\frac{du}{dx} + u = 0, \quad 0 \leq x \leq 1$$

with initial condition $u(0) = 1$.

Solution. If $[u]_h$ is the discretized solution to the problem $Lu = f$ as defined above, then the scheme $L_h u^{(h)} = f^{(h)}$ is divergent if $\left\|[u]_h - u^{(h)}\right\|$ does not approach 0 as $h \to 0$.

The exact solution to the problem $Lu = f$ with the initial condition $u(0) = 1$ is $u(x) = e^{-x}$. Hence, $[u]_h = [1, e^{-x_1}, \ldots, e^{-x_n}]$. The solution to the difference scheme $L_h u^{(h)} = f^{(h)}$ given by $u^{(h)}$ and can be found by finding the explicit solution to the difference equation defined in the scheme.

Note that this is a second order difference equation that can be rewritten as

$$-u_{n+1} + (3+h)u_n - 2u_{n-1} = 0.$$

The characteristic equation of this difference equation is given by $-m^2 + (3+h)m - 2 = 0$. As this characteristic equation has distinct real roots, the general solution to the difference equation is $u_n = c_1 m_1^n + c_2 m_2^n$ where $m_1 = \frac{1}{2}(-\sqrt{h^2+6h+1} + h + 3)$ and $m_2 = \frac{1}{2}(\sqrt{h^2+6h+1} + h + 3)$ are the roots of the characteristic equation. Choosing the constants so that the initial conditions are satisfied gives us the general solution as

$$\begin{aligned} u_n^{(h)} = u_0 &\left[\frac{m_2(h)}{m_2(h) - m_1(h)} m_1(h)^n - \frac{m_1(h)}{m_2(h) - m_1(h)} m_2(h)^n\right] \\ + u_1 &\left[-\frac{1}{m_2(h) - m_1(h)} m_1(h)^n + \frac{1}{m_2(h) - m_1(h)} m_1(h)^n\right]. \end{aligned}$$

Combining this general solution to the scheme and the exact solution to the problem we can clearly see that $\left\|[u]_h - u^{(h)}\right\|$ does not approach 0 as $h \to 0$ and that the scheme is divergent. $\square$