Homework Assignment 4

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Problem 2.3. Find the ACVF of the time series $X_t = Z_t + aZ_{t-1} + bZ_{t-2}$ where $Z_t \sim WN(0, \sigma^2)$ when:

a. $a = 0.3$, $b = -0.4$, and $\sigma^2 = 1$.

b. $a = -1.2$, $b = -1.6$, and $\sigma^2 = 0.25$.

Solution. The ACVF of the time series $\{X_t\}$, $\gamma_X(h)$, is by definition:

$$\begin{aligned}
\gamma_X(h) &= \mathrm{Cov}(X_{t+h}, X_t) \\
&= \mathrm{Cov}(Z_{t+h} + aZ_{t+h-1} + bZ_{t+h-2}, Z_t + aZ_{t-1} + bZ_{t-2}) \\
&= \mathrm{Cov}(Z_{t+h}, Z_t) + a\mathrm{Cov}(Z_{t+h}, Z_{t-1}) + b\mathrm{Cov}(Z_{t+h}, Z_{t-2}) \\
&\quad + a\mathrm{Cov}(Z_{t+h-1}, Z_t) + a^2\mathrm{Cov}(Z_{t+h-1}, Z_{t-1}) + ab\mathrm{Cov}(Z_{t+h-1}, Z_{t-2}) \\
&\quad + b\mathrm{Cov}(Z_{t+h-2}, Z_t) + ab\mathrm{Cov}(Z_{t+h-2}, Z_{t-1}) + b^2\mathrm{Cov}(Z_{t+h-2}, Z_{t-2}). \qquad (1)
\end{aligned}$$

Using (1), we can see that since $Z_t \sim WN(0, \sigma^2)$,

$$\gamma_X(h) = \begin{cases} (1 + a^2 + b^2)\sigma^2 & \text{if } h = 0 \\ a(1+b)\sigma^2 & \text{if } h = \pm 1 \\ b\sigma^2 & \text{if } h = \pm 2 \\ 0 & \text{otherwise} \end{cases}.$$

Therefore, when

a. $a = 0.3$, $b = -0.4$, and $\sigma^2 = 1$, the ACVF of $\{X_t\}$ is:

$$\begin{cases} 1.25 & \text{if } h = 0 \\ 0.18 & \text{if } h = \pm 1 \\ -0.4 & \text{if } h = \pm 2 \\ 0 & \text{otherwise} \end{cases}$$

b. $a = -1.2$, $b = -1.6$, and $\sigma^2 = 0.25$, the ACVF of $\{X_t\}$ is:

$$\begin{cases} 1.25 & \text{if } h = 0 \\ 0.18 & \text{if } h = \pm 1 \\ -0.4 & \text{if } h = \pm 2 \\ 0 & \text{otherwise} \end{cases}$$

$\square$

Problem 2.5. Suppose that $\{X_t, t = 0, \pm 1, \dots\}$