# Homework Assignment 8

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**Problem 7.2.2.** If $D : [0,1) \to [0,1)$ is the doubling map $D(x) = 2x \mod 1$ and $f : S^1 \to S^1$ is the angle doubling map, $f(z) = z^2$, show that $f$ is a factor of $D$.

*Solution.* Recall that a dynamical system $f : S^1 \to S^1$ is a factor of the dynamical system $D : [0,1) \to [0,1)$ if there exists a continuous, onto function $h : [0,1) \to S^1$ such that $h \circ D = f \circ h$.

Define $h : [0,1) \to S^1$ by $h(x) = e^{2\pi i x}$. Then it is easy to see that $h$ is continuous. To show that it is onto, let $z \in S^1$ be given. Then $z = e^{it}$ for some $t \in [0, 2\pi)$. Choose $x \in [0,1)$ such that $t = 2\pi x$. Then it is clear that $h(x) = e^{2\pi i x} = e^{it} = z$ and $h$ is onto.

Now, we see that

$$f \circ h(x) = f(e^{2\pi i x}) = e^{2\pi i x}$$

and

$$\begin{aligned} h \circ D(x) &= \begin{cases} h(2x) & \text{if } x \in [0, 1/2) \\ h(2x-1) & \text{if } x \in [1/2, 1) \end{cases} \\ &= \begin{cases} e^{4\pi i x} & \text{if } x \in [0, 1/2) \\ e^{4\pi i x - 2\pi i} & \text{if } x \in [1/2, 1) \end{cases}. \end{aligned}$$

However, $e^{4\pi i x - 2\pi i} = e^{-2\pi i} e^{4\pi i x} = e^{4\pi i x}$ so in either case $h \circ D(x) = e^{4\pi i x} = f \circ h(x)$ and $f$ is a factor of $D$.

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**Problem 7.2.3.** i. If $g : S^1 \to S^1$ is defined by $g(z) = z^3$, show that $g$ is the angle-tripling map

ii. Find the periodic points of $g$ and show they are dense in $S^1$.

iii. Let $F : [0, 1) \to [0, 1)$ be defined by $F(x) = 3x \mod 1$. Show that $g$ is a factor of $F$.

*Solution.* i. If $z \in S^1$, then $z = e^{i\theta}$ for some $\theta \in (-\pi, \pi]$. Note that if $z = x + iy$ for $x, y \in \mathbb{R}$, then $\theta$ is the angle between the vector $< x, y >$ and the real line measured counter-clockwise.

So, if $z = e^{i\theta}$, then

$$g(z) = \left(e^{i\theta}\right)^3 = e^{i3\theta}$$

and the angle between the vector $< x, y >$ and the real line measured counter-clockwise has now tripled. Therefore, $g$ is the angle-tripling map.

ii. For the map $g$, note that 0 is a fixed point and so it cannot be periodic. It is easy to see that if $g(z) = z^3$, then $g^n(z) = z^{3^n}$. Thus, for $z \neq 0$, we have that $g^n(z) = z$ if and only if $z^{3^n} = z$ or $z^{3^n - 1} = 1$. Therefore, the period $n$ points are the $(3^n - 1)$-th roots of unity.

Having identified the periodic points, we see that the periodic points of $g$ are dense in $S^1$ if for every $z \in S^1$ either $z$ is a $(3^n - 1)$-th root of unity for some $n$ or $z$ is arbitrarily close to some $(3^n - 1)$-th root of unity, i.e. if for every $z \in S^1$ and every $\varepsilon > 0$, there exists some period $n$ point $x$ such that $|z - x| < \varepsilon$.

If $x \in S^1$ then $x = e^{i\theta}$ for some $-\pi < \theta \leq \pi$. If $x$ is a period $n$ point, then $\left(e^{i\theta}\right)^{3n-1} = e^{2\pi i}$ implies that $x = e^{2k\pi i/3^n - 1}$ for some $0 \leq k < 3^n - 1$. Note that the $(3n - 1)$-th roots of unity are evenly spaced on the unity circle a distance $2\pi/(3^n - 1)$ apart. Taking $n$ arbitrarily large shows that this distance is arbitrarily small and the distance between any point on the unit circle will be arbitrarily close to a $(3^n - 1)$-th root of unity.

iii. Recall that a dynamical system $g : S^1 \to S^1$ is a factor of the dynamical system $F : [0, 1) \to [0, 1)$ if there exists a continuous, onto function $h : [0, 1) \to S^1$ such that $h \circ F = g \circ h$.

Define $h : [0, 1) \to S^1$ by $h(x) = e^{2\pi ix}$. As was shown earlier, this function is continuous and onto.

Now, we see that

$$g \circ h(x) = g(e^{2\pi ix}) = e^{6\pi ix}$$

and

$$h \circ F(x) = \begin{cases} h(3x) & \text{if } x \in [0, 1/3) \\ h(3x - 1) & \text{if } x \in [1/3, 2/3) \\ h(3x - 2) & \text{if } x \in [2/3, 1) \end{cases}$$

$$= \begin{cases} e^{6\pi ix} & \text{if } x \in [0, 1/3) \\ e^{6\pi ix - 2\pi i} & \text{if } x \in [1/3, 2/3) \\ e^{6\pi ix - 4\pi i} & \text{if } x \in [2/3, 1) \end{cases} .$$

Note that $e^{2k\pi i} = 1$ for all $k \in \mathbb{Z}$, so in either case $h \circ F(x) = e^{6\pi i x} = g \circ h(x)$ and $g$ is a factor of $F$.

$\square$

**Problem 7.3.2.** Check that for $0 < \mu \leq 4$, if $f_c(x) = x^2 + c$ with $c = (2\mu - \mu^2)/4$, then $f_c$ is a dynamical system on $[-\mu/2, \mu/2]$.

*Solution.* Recall that $f_c$ is a dynamical system on $[-\mu/2, \mu/2]$ if $f_c([-\mu/2, \mu/2]) \subseteq [-\mu/2, \mu/2]$. Note that $f_c'(x) = 2x = 0$ if $x = 0$ so it is at this point that a relative extremum exists for $f_c$. It is easy to see that $f_c(0) = c$ is the absolute minimum of $f_c$ on $[-\mu/2, \mu/2]$.

The maximum on the bounded interval $[-\mu/2, \mu/2]$ must therefore occur at one of the end points. In either case, $f_c(\mu/2) = f_c(-\mu/2) = \mu/2$. Since $f_c$ is continuous, we have by the Intermediate Value Theorem that $f_c([-\mu/2, \mu/2]) = [(2\mu - \mu^2)/4, \mu/2]$.

If $0 < \mu \leq 4$, then we have that $\mu^2 \leq 4\mu$ which implies that $0 \leq \mu - \mu^2/4$. Thus, $-\mu/2 \leq (2\mu - \mu^2)/4$ and we have that $[(2\mu - \mu^2)/4, \mu/2] \subseteq [-\mu/2, \mu/2]$.

Therefore, $f_c([-\mu/2, \mu/2]) \subseteq [-\mu/2, \mu/2]$ and $f_c$ is a dynamical system.

$\square$

**Problem 7.3.4.** i. Let $f_a(x) = ax$ and $f_b(x) = bx$ with $a, b \in \mathbb{R}$ be defined on $\mathbb{R}$. Under which conditions are $f_a$ and $f_b$ linearly conjugate?

ii. Show that any conjugation $h$ between $f_a$ and $f_b$ cannot be a diffeomorphism unless $a = b$.

iii. Let $0 < a, b < 1$ and $f_a, f_b : [0,1] \to [0,1]$. Show that any conjugacy $h$ between $f_a$ and $f_b$ must satisfy $h(0) = 0$, $h(1) = 1$, and $h(a^n) = b^n$ for all $n \in \mathbb{Z}^+$

*Solution.* i. Recall that $f_a$ and $f_b$ are linearly conjugate if there exists a function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = c_1 x + c_0$ with $c_1 \neq 0$ such that $f_a \circ h = h \circ f_b$. Thus, $f_a$ and $f_b$ are linearly conjugate if

$$f_a \circ h(x) = ac_1 x + ac_0 = bc_1 x + c_0 = h \circ f_b(x).$$

Equating the coefficients of these polynomials, we see that we must have that $ac_1 = bc_1$ and $ac_0 = c_0$. Since $c_1 \neq 0$, we must have that $a = b$. If $c_0 \neq 0$, then we must have that $a = 1 = b$, otherwise no additional restrictions are necessary for $f_a$ and $f_b$ to be linearly conjugate. Thus, $f_a$ and $f_b$ are linearly conjugate if $a = b$ and if the conjugate map is such that $c_0 \neq 0$, then we must have that $a = b = 1$.

ii. Suppose that $h$ is a continuous bijection such that $f_a \circ h = h \circ f_b$. Suppose to the contrary that $h$ is a diffeomorphism but $a \neq b$. Then we have that $h$ and its inverse are differentiable so that

$$(f_a \circ h)'(x) = (ah(x))' = ah'(x)$$

and that

$$(h \circ f_b)'(x) = (h(bx))' = bh'(bx).$$

Since $h$ is the conjugate map, we have that $ah'(x) = bh'(bx)$. If $a \neq b$, then we must have that $h'(0) = 0$. However, this contradicts the assumption that $h$ is a diffeomorphism since

$$\left(h^{-1}(y)\right)' = \frac{1}{h'(x)}$$

for any $h(x) = y$, i.e. the derivative of $h^{-1}$ is defined only if $h'(x) \neq 0$. Therefore, we must have that $a = b$ if $h$ is a diffeomorphism.

iii. Suppose that $h : [0,1] \to [0,1]$ is a conjugate map between $f_a$ and $f_b$, i.e. $f_b \circ h = h \circ f_a$. Then we have that $f_b \circ h(0) = bh(0) = h(0) = h \circ f_a(0)$. Since $0 < b < 1$, this implies that $h(0) = 0$.

Note that $h$ is continuous and one-to-one on $[0,1]$ and so it is either strictly increasing or strictly decreasing. Since $h(0) = 0$, it must be strictly increasing. Thus, since $h$ maps $[0,1]$ onto $[0,1]$, we must have that $h(1) = 1$.

Since $h(1) = 1$, we have by the conjugacy of $h$ that

$$f_b \circ h(1) = bh(1) = h(a) = h \circ f_a(1)$$

or that $h(a) = b$ . So now suppose that $h(a^n) = b^n$ for $n \in \mathbb{Z}^+$. By the conjugacy of $h$, we then see that

$$h(f_a(a^n)) = h(a^{n+1}) = b^{n+1} = f_b(b^n) = f_b(h(a^n))$$

and the formula holds for $n + 1$. Therefore, we have that $h(a^n) = b^n$ for any $n \in \mathbb{Z}^+$. $\square$

**Problem 7.3.5.** Show that every quadratic polynomial $p(x) = a_2x^2 + a_1x + a_0$ is linearly conjugate to a unique polynomial of the form $f_c(x) = x^2 + c$.

*Solution.* In order for $p$ and $f_c$ to be linearly conjugate, we wish to find a function $h : \mathbb{R} \to \mathbb{R}$ of the form $h(x) = b_1x + b_0$ such that $h \circ p = f \circ h$ with $b_1 \neq 0$. Note that any such $h$ is a continuous bijection so we need only check $h \circ p = f \circ h$.

Checking, we have that

$$\begin{aligned} h \circ p(x) &= b_1p(x) + b - 0 \\ &= b_1(a_2x^2 + a_1x + a_0) + b_0 \\ &= a_2b_1x^2 + a_1b_1x + a_0b_1 + b_0 \end{aligned}$$

and

$$\begin{aligned} f \circ h(x) &= (b_1x + b_0)^2 + c \\ &= b_1^2x^2 + 2b_0b_1x + b_0^2 + c. \end{aligned}$$

Thus, $h \circ p = f \circ h$ if and only if the coefficients of the resulting polynomials are the same if and only if

$$\begin{aligned} b_1^2 - a_2b_1 &= 0 \\ 2b_0b_1 - a_1b_1 &= 0 \\ c + b_0^2 - a_0b_1 - b_0 &= 0. \end{aligned}$$

Since $b_1 \neq 0$, we can solve this system so that

$$\begin{aligned} b_1 &= a_2 \\ b_0 &= \frac{a_1}{2} \\ c &= a_0b_1 + b_0 - b_0^2 \\ &= a_0a_2 + \frac{a_1}{2} - \frac{a_1^2}{4}. \end{aligned}$$

Therefore, $p(x) = a_2x^2 + a_1x + a_0$ is linearly conjugate to $f_c(x) = x^2 + c$ via $h(x) = a_2x + a_1/2$ if $c = a_0a_2 + a_1/2 - a_1^2/4$.

To show that $f_c$ is unique, suppose that $p(x) = a_2x^2 + a_1x + a_0$ is linearly conjugate to some other quadratic polynomial $g(x) = d_2x^2 + d_1x + d_0$ via $h(x) = b_1x + b_0 = a_2x + a_1/2$. Then we have that $h \circ p = g \circ h$ and equating coefficients we see that

$$\begin{aligned} d_0 &= \frac{a_2b_0^2 + b_0b_1 - a1b_0b_1 + a_0b_1^2}{b_1} \\ d_1 &= \frac{-2a_2b_0 + a_1b_1}{b_1} \\ d_2 &= \frac{a_2}{b_1}. \end{aligned}$$

Using the fact that $b_1 = a_2$ and $b_0 = a_1/2$, we have that $d_0 = a_0a_2 + a_1/2 - a_1^2/4$, $d_1 = 0$, and $d_2 = 1$. Thus, $g(x) = x^2 + a_0a_2 + a_1/2 - a_1^2/4 = f_c(x)$ and $f_c(x)$ is unique.

$\square$