Homework Assignment 6

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Problem 4.3. Find the solutions of the following systems of equations with the initial data:

a. $\frac{dx}{dt} = x - 2y, \quad x(0) = 1$
$\frac{dy}{dt} = y - 2x, \quad y(0) = 0$

Solution. a. Applying the Laplace transform to the system yields

$$\mathscr{L}\left\{\frac{dx}{dt}\right\} = s\bar{x}(s) - x(0) = \bar{x}(s) - 2\bar{y}(s) = \mathscr{L}\left\{x - 2y\right\}$$
$$\mathscr{L}\left\{\frac{dy}{dt}\right\} = s\bar{y}(s) - y(0) = \bar{y}(s) - 2\bar{x}(s) = \mathscr{L}\left\{y - 2x\right\}.$$

Using the initial data, the transformed system becomes

$$(s-1)\bar{x}(s) + 2\bar{y}(s) = 1$$
$$2\bar{x}(s) + (s-1)\bar{y}(s) = 0,$$

or, equivalently,

$$\begin{bmatrix} s-1 & 2 \\ 2 & s-1 \end{bmatrix}\begin{bmatrix} \bar{x}(s) \\ \bar{y}(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This implies that the solution to the transformed system of equations is given by

$$\begin{bmatrix} \bar{x}(s) \\ \bar{y}(s) \end{bmatrix} = \begin{bmatrix} s-1 & 2 \\ 2 & s-1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{s-1}{(s-3)(s+1)} & -\frac{2}{(s-3)(s+1)} \\ -\frac{2}{(s-3)(s+1)} & \frac{s-1}{(s-3)(s+1)} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{s-1}{(s-3)(s+1)} \\ -\frac{2}{(s-3)(s+1)} \end{bmatrix}$$

i.e. the solution is given by $\bar{x}(s) = \dfrac{s-1}{(s-3)(s+1)}$ and $\bar{y}(s) = -\dfrac{2}{(s-3)(s+1)}$.

From our table of Laplace Transforms, we know that

$$\mathscr{L}\left\{e^{at} - e^{bt}\right\} = \frac{a-b}{(s-a)(s-b)}$$

and

$$\mathscr{L}\left\{\frac{ae^{at} - be^{bt}}{a-b}\right\} = \frac{s}{(s-a)(s-b)}.$$

Therefore, the solution to the original system of differential equations is given by

$$\begin{aligned}
x(t) = \mathscr{L}^{-1}\left\{\bar{x}(s)\right\} &= \mathscr{L}^{-1}\left\{\frac{s-1}{(s-3)(s+1)}\right\} \\
&= \mathscr{L}^{-1}\left\{\frac{s}{(s-3)(s+1)}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{(s-3)(s+1)}\right\} \\
&= \frac{3e^{3t}+e^{-t}}{4} - \frac{e^{3t}-e^{-t}}{r} \\
&= \frac{e^{3t}+e^{-t}}{2}
\end{aligned}$$

and

$$\begin{aligned}
y(t) = \mathscr{L}^{-1}\left\{\bar{y}(s)\right\} &= \mathscr{L}^{-1}\left\{-\frac{2}{(s-3)(s+1)}\right\} \\
&= \frac{e^{-t}-e^{3t}}{2}
\end{aligned}$$

$\square$

Problem 4.12. Solve the following initial value problems:

a. $\ddot{x} + \omega^2 x = \cos nt$, $x(0) = 1$, $\dot{x}(0) = 0$ where $\omega \neq n$.

Solution. a. We begin by applying the Laplace transform to the equation. Doing so yields

$$\mathscr{L}\left\{\ddot{x} + \omega^2 x\right\} = (s^2 + \omega^2)\bar{x}(s) - sx(0) - \dot{x}(0) = \frac{s}{s^2 + n^2} = \mathscr{L}\left\{\cos nt\right\}.$$

Using the initial data, the transformed equation becomes

$$(s^2 + \omega^2)\bar{x}(s) - s = \frac{s}{s^2 + n^2}.$$

Solve the above equation yields that the solution to the transformed equation is

$$\bar{x}(s) = \frac{s^3 + (n^2 + 1)s}{(s^2 + n^2)(s^2 + \omega^2)}.$$

From the partial fractions method we see that

$$\bar{x}(s) = \frac{s^3 + (n^2 + 1)s}{(s^2 + n^2)(s^2 + \omega^2)} = \frac{a_1 s + a_0}{s^2 + n^2} + \frac{b_1 s + b_0}{s^2 + \omega^2}.$$

Combining the rational fractions on the right side under a common denominator and equating the coefficients in the numerator we arrive at the following system of equations

$$\begin{aligned} a_1 + b + 1 &= 1 \\ a_0 + b_0 &= 0 \\ a_1\omega^2 + b_1 n^2 &= n^2 + 1 \\ a_0\omega^2 + b_0 n^2 &= 0 \end{aligned}$$

Solving this system, we see that $a_0 = b_0 = 0$, $a_1 = \dfrac{1}{\omega^2 - n^2}$, and $b_1 = \dfrac{\omega^2 - n^2 - 1}{\omega^2 - n^2}$. Thus, the solution to the transformed system is given by

$$\bar{x}(s) = \frac{s^3 + (n^2 + 1)s}{(s^2 + n^2)(s^2 + \omega^2)} = \left(\frac{1}{\omega^2 - n^2}\right)\frac{s}{s^2 + n^2} + \left(\frac{\omega^2 - n^2 - 1}{\omega^2 - n^2}\right)\frac{s}{s^2 + \omega^2}.$$

From our table of Laplace transforms, we know that

$$\mathscr{L}\left\{\cos at\right\} = \frac{s}{s^2 + a^2}.$$

Therefore, the solution to the original differential equation is

$$\begin{aligned} x(t) = \mathscr{L}^{-1}\left\{\bar{x}(s)\right\} &= \left(\frac{1}{\omega^2 - n^2}\right)\mathscr{L}^{-1}\left\{\frac{s}{s^2 + n^2}\right\} + \left(\frac{\omega^2 - n^2 - 1}{\omega^2 - n^2}\right)\mathscr{L}^{-1}\left\{\frac{s}{s^2 + \omega^2}\right\} \\ &= \left(\frac{1}{\omega^2 - n^2}\right)\cos nt + \left(\frac{\omega^2 - n^2 - 1}{\omega^2 - n^2}\right)\cos \omega t. \end{aligned}$$

□

Problem 4.14. With the aid of the Laplace transform, investigate the motion of a particle governed by the equations of motion

$$\ddot{x} - \omega\dot{y} = 0$$
$$\ddot{y} + \omega\dot{x} = \omega^2 a$$

with the initial conditions $x(0) = y(0) = \dot{x}(0) = \dot{y}(0) = 0$.

Solution. We begin by applying the Laplace transform to the system of differential equations. Doing so yields

$$\mathscr{L}\left\{\ddot{x} - \omega\dot{y}\right\} = s^2\bar{x}(s) - sx(0) - \dot{x}(0) - \omega\left(s\bar{y}(s) - y(0)\right) = 0 = \mathscr{L}\left\{0\right\}$$
$$\mathscr{L}\left\{\ddot{y} + \omega\dot{x}\right\} = s^2\bar{y}(s) - sy(0) - \dot{y}(0) + \omega\left(s\bar{x}(s) - x(0)\right) = \frac{\omega^2 a}{s} = \mathscr{L}\left\{\omega^2 a\right\}$$

Using the initial data, the above system becomes

$$s^2\bar{x}(s) - \omega s\bar{y}(s) = 0$$
$$s^2\bar{y}(s) + \omega s\bar{x}(s) = \frac{\omega^2 a}{s},$$

or, equivalently,

$$\begin{bmatrix} s^2 & -\omega s \\ \omega s & s^2 \end{bmatrix}\begin{bmatrix} \bar{x}(s) \\ \bar{y}(s) \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{\omega^2 a}{s} \end{bmatrix}$$

This implies that the solution to the transformed system of equations is

$$\begin{bmatrix} \bar{x}(s) \\ \bar{y}(s) \end{bmatrix} = \begin{bmatrix} s^2 & -\omega s \\ \omega s & s^2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \frac{\omega^2 a}{s} \end{bmatrix} = \begin{bmatrix} \frac{s^2}{s^2(s^2+\omega^2)} & \frac{s\omega}{s^2(s^2+\omega^2)} \\ -\frac{s\omega}{s^2(s^2+\omega^2)} & \frac{s^2}{s^2(s^2+\omega^2)} \end{bmatrix}\begin{bmatrix} 0 \\ \frac{\omega^2 a}{s} \end{bmatrix} = \begin{bmatrix} \frac{\omega^3 a}{s^2(s^2+\omega^2)} \\ \frac{\omega^2 sa}{s^2(s^2+\omega^2)} \end{bmatrix}$$

Let $\bar{f}(s) = \frac{1}{s^2}$, $\bar{g}(s) = \frac{\omega}{s^2+\omega^2}$, and $\bar{h}(s) = \frac{\omega}{s^2+\omega^2}$. From our table of Laplace transforms, we know that

$$f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$
$$g(t) = \mathscr{L}^{-1}\left\{\bar{g}(s)\right\} = \mathscr{L}^{-1}\left\{\frac{b}{s^2+\omega^2}\right\} = \sin\omega t$$
$$h(t) = \mathscr{L}^{-1}\left\{\bar{h}(s)\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2+\omega^2}\right\} = \cos\omega t.$$

Therefore, by the Convolution Theorem, the solution to the original system of equations is given by

$$\begin{aligned} x(t) = \mathscr{L}^{-1}\left\{\bar{x}(s)\right\} = \omega^2 a\mathscr{L}^{-1}\left\{\bar{f}(s)\bar{g}(s)\right\} &= \omega^2 a(f * g)(t) \\ &= \omega^2 a\int_0^t f(t-\tau)g(\tau)d\tau \\ &= \omega^2 a\int_0^t (t-\tau)\sin\omega\tau d\tau \\ &= a\omega t - a\sin\omega t. \end{aligned}$$

and

$$
\begin{aligned}
y(t) = \mathscr{L}^{-1}\left\{\bar{y}(s)\right\} = \omega^2 a \mathscr{L}^{-1}\left\{\bar{f}(s)\bar{h}(s)\right\} &= \omega^2 a (f * h)(t) \\
&= \omega^2 a \int_0^t f(t-\tau)h(\tau)d\tau \\
&= \omega^2 a \int_0^t (t-\tau)\cos\omega\tau d\tau \\
&= a - a\cos\omega t.
\end{aligned}
$$

□

Problem 4.22. Solve the Blasius problem of an unsteady boundary layer flow in a semi-infinite body of viscous fluid enclosed by an infinite horizontal disk at $z = 0$. The governing equation and the boundary and initial conditions are

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nu \frac{\partial^2 u}{\partial z^2} \\ u(z,t) &= Ut \quad \text{on } z = 0,\ t > 0 \\ u(z,t) &\to 0 \quad \text{as } z \to \infty,\ t > 0 \\ u(z,t) &= 0 \quad \text{at } t \le 0,\ z > 0. \end{aligned}$$

Explain the significance of the solution.

Solution. Let $u(z,t)$ be a function in z, t. The Laplace transform of $u(z,t)$ with respect to t is given by

$$\mathscr{L}\{u(z,t)\} = \bar{u}(z,s) = \int_0^\infty u(z,t)e^{-st}dt.$$

From this definition, we see from previous theorems that

$$\mathscr{L}\left\{\frac{\partial^n}{\partial t^n}[u(z,t)]\right\} = s^n\bar{u}(z,s) - \sum_{k=0}^{n-1} s^{n-1-k}\frac{\partial^k}{\partial t^k}[u(z,0)]$$

Similarly, we see from the Leibniz integral rule that

$$\mathscr{L}\left\{\frac{\partial^n}{\partial z^n}[u(z,t)]\right\} = \frac{d^n}{dz^n}[\bar{u}(z,s)].$$

We begin by applying the Laplace transform to the governing equation and the boundary conditions on the infinity horizontal disk. Doing so yields

$$\begin{aligned} s\bar{u}(z,s) - u(z,0) &= \nu \frac{d^2}{dz^2}[\bar{u}(z,s)] \\ \bar{u}(z,s) &= \frac{U}{s^2}, \quad \text{on } z = 0,\ s > 0 \\ \bar{u}(z,s) &\to 0, \quad \text{as } z \to \infty,\ s > 0. \end{aligned}$$

The initial condition $u(z,t) = 0$ when $t = 0$ shows that the first equation reduces to

$$\frac{d^2}{dz^2}[\bar{u}(z,s)] - \frac{s}{\nu}\bar{u}(z,s) = 0.$$

This is a second-order, linear homogeneous differential equation, the solution of which we readily see if

$$\bar{u}(z,s) = c_1 e^{-\sqrt{\frac{s}{\nu}}z} + c_2 e^{\sqrt{\frac{s}{\nu}}z}.$$

From this form, the transformed boundary condition, $\bar{u}(z,s) \to 0$ as $z \to \infty$ for $s > 0$, indicates that $c_2 = 0$, since $\sqrt{\frac{s}{\nu}} > 0$. Similarly the transformed boundary condition, $\bar{u}(z,s) =$

$\frac{U}{s^2}$ on $z = 0$ for $s > 0$, indicates that $c_1 = \frac{U}{s^2}$. Thus, the solution to the transformed equation obeying the initial and boundary conditions is

$$\bar{u}(z, s) = \frac{U}{s^2} e^{-\sqrt{\frac{s}{\nu}} z}.$$

From our table of Laplace transforms, we know that

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2} e^{-\sqrt{\frac{s}{\nu}} z}\right\} = t\left[1 + 2\zeta^2 \operatorname{erfc}(\zeta) - \frac{2\zeta}{\sqrt{\pi}} e^{-\zeta^2}\right]$$

where $\zeta = \frac{z}{2\sqrt{\nu t}}$ and $\operatorname{erfc}(z) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_z^\infty e^{-t^2} dt$. Therefore, the solution to the original differential equation is

$$u(z, t) = \mathscr{L}^{-1}\left\{\bar{u}(z, s)\right\} = U\mathscr{L}^{-1}\left\{\frac{1}{s^2} e^{-\sqrt{\frac{s}{\nu}} z}\right\} = Ut\left[1 + 2\zeta^2 \operatorname{erfc}(\zeta) - \frac{2\zeta}{\sqrt{\pi}} e^{-\zeta^2}\right].$$

$\square$

Problem 4.25. Solve the following integral equations:

a. $f(t) = \sin 2t + \int_0^t f(t-\tau)\sin\tau d\tau.$

b. $f(t) = \frac{t}{2}\sin t + \int_0^t f(\tau)\sin(t-\tau)d\tau.$

d. $f(t) = \sin t + \int_0^t f(\tau)\sin 2(t-\tau)d\tau.$

Solution. a. Let $g(t) = \sin t$. Then

$$\begin{aligned} f(t) &= g(2t) + \int_0^t f(t-\tau)g(\tau)d\tau \\ &= g(2t) + (f * g)(t). \end{aligned}$$

Applying the Laplace transform to this equation and using the Convolution Theorem, we have that

$$\mathscr{L}\{f(t)\} = \bar{f}(s) = \mathscr{L}\{g(2t)\} + \bar{f}(s)\bar{g}(s) = \mathscr{L}\{g(2t) + (f * g)(t)\}.$$

From our table of Laplace transforms, we know that

$$\mathscr{L}\{\sin nt\} = \frac{n}{s^2+n^2}.$$

Thus, the transformed equation becomes

$$\bar{f}(s) = \frac{2}{s^2+4} + \bar{f}(s)\frac{1}{s^2+1},$$

or, equivalently,

$$\bar{f}(s) = \frac{2(s^2+1)}{s^2(s^2+4)}.$$

From the partial fractions method we see that

$$\bar{f}(s) = \frac{2(s^2+1)}{s^2(s^2+4)} = \frac{a_1 s + a_0}{s^2} + \frac{b_1 s + b_0}{s^2+4}.$$

Combining the rational fractions on the right side under a common denominator and equating the coefficients in the numerator of the left side we arrive at the following system of equations:

$$\begin{aligned} a_1 + b_1 &= 0 \\ a_0 + b_0 &= 2 \\ 4a_1 &= 0 \\ 4a_0 &= 2. \end{aligned}$$

By inspection, we see that $a_1 = b_1 = 0$, $a_0 = \frac{1}{2}$, and $b_0 = \frac{3}{2}$.

Therefore, the solution to the original integral equation is

$$\begin{aligned} f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} = \mathscr{L}^{-1}\left\{\frac{2(s^2+1)}{s^2(s^2+4)}\right\} \\ &= \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{3}{2}\mathscr{L}^{-1}\left\{\frac{1}{s^2+4}\right\} \\ &= \frac{t}{2} + \frac{3}{4}\sin 2t. \end{aligned}$$

b. Let $g(t) = \sin t$. Then

$$\begin{aligned} f(t) &= \frac{t}{2}g(t) + \int_0^t f(\tau)g(t-\tau)d\tau \\ &= \frac{t}{2}g(t) + (g * f)(t) \\ &= \frac{t}{2}g(t) + (f * g)(t). \end{aligned}$$

Applying the Laplace transform to this equation and using the Convolution Theorem, we have that

$$\mathscr{L}\left\{f(t)\right\} = \bar{f}(s) = \frac{1}{2}\mathscr{L}\left\{tg(t)\right\} + \bar{f}(s)\bar{g}(s) = \mathscr{L}\left\{\frac{t}{2}g(t) + (f * g)(t)\right\}.$$

From our table of Laplace transforms, we know that

$$\mathscr{L}\left\{\sin nt\right\} = \frac{n}{s^2+n^2}$$

and

$$\mathscr{L}\left\{t\sin nt\right\} = \frac{2ns}{(s^2+n^2)^2}.$$

Thus, the transformed equation becomes

$$\bar{f}(s) = \frac{s}{(s^2+1)^2} + \bar{f}(s)\frac{1}{s^2+1},$$

or, equivalently,

$$\bar{f}(s) = \frac{1}{s(s^2+1)} = \bar{h}(s)\bar{g}(s),$$

where $\bar{h}(s) = \frac{1}{s}$. From our table of Laplace transforms, we know that $g(t) = \sin t$ and $h(t) = 1$.

Therefore, by the Convolution theorem, the solution to the original equation is

$$\begin{aligned} f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} &= \mathscr{L}^{-1}\left\{\bar{h}(s)\bar{g}(s)\right\} \\ &= (h * g)(t) \\ &= \int_0^t h(t-\tau)g(\tau)d\tau \\ &= \int_0^t \sin\tau d\tau \\ &= 1 - \cos t. \end{aligned}$$

d. Let $g(t) = \sin t$ and $h(t) = \sin 2t$. Then

$$\begin{aligned} f(t) &= g(t) + \int_0^t f(\tau)h(t-\tau)d\tau \\ &= g(t) + (h * f)(t) \\ &= g(t) + (f * h)(t). \end{aligned}$$

Applying the Laplace transform to this equation and using the Convolution Theorem, we have that

$$\mathscr{L}\{f(t)\} = \bar{f}(s) = \bar{g}(s) + \bar{f}(s)\bar{h}(s) = \mathscr{L}\{g(t) + (f * h)(t)\}.$$

From our table of Laplace transforms, we know that

$$\mathscr{L}\{\sin nt\} = \frac{n}{s^2+n^2}.$$

Thus, the transformed equation becomes

$$\bar{f}(s) = \frac{1}{s^2+1} + \bar{f}(s)\frac{2}{s^2+4},$$

or, equivalently,

$$\bar{f}(s) = \frac{s^2+4}{(s^2+2)(s^2+1)}.$$

From the partial fractions method we see that

$$\bar{f}(s) = \frac{s^2+4}{(s^2+2)(s^2+1)} = \frac{a_1 s + a_0}{s^2+2} + \frac{b_1 s + b_0}{s^2+1}.$$

Combining the rational fractions on the right side under a common denominator and equating the coefficients in the numerator we arrive at the following system of equations

$$\begin{aligned} a_1 + b + 1 &= 0 \\ a_0 + b_0 &= 1 \\ a_1 + 2b_1 &= 0 \\ a_0 + 2b_0 &= 4 \end{aligned}$$

Solving this system, we see that $a_1 = b_1 = 0$, $a_0 = -2$, and $b_0 = 3$. Thus, we see that

$$\bar{f}(s) = \frac{s^2+4}{(s^2+2)(s^2+1)} = -\frac{2}{s^2+2} + \frac{3}{s^2+1}.$$

Therefore, the solution to the original equation is

$$\begin{aligned} f(t) = \mathscr{L}^{-1}\left\{\bar{f}(s)\right\} &= -2\mathscr{L}^{-1}\left\{\frac{1}{s^2+2}\right\} + 3\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\ &= -\frac{2}{\sqrt{2}}\sin\sqrt{2}t + 3\sin t \\ &= -\sqrt{2}\sin\sqrt{2}t + 3\sin t. \end{aligned}$$

$\square$