Homework Assignment 6

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Problem 4.2.1. Prove that every open ball $B_\varepsilon(a)$ in a metric space (X, d) is an open set and that every finite subset of X is a closed set.

Solution. Recall that a set $A \subseteq X$ is an open set if for all $a \in A$, there exists an $\varepsilon > 0$ such that $B_\varepsilon(a) \subseteq A$. Thus, to show that $B_\varepsilon(a)$ is an open set, we will show that for each point in the open ball of radius ε centered at a there exists a neighborhood of that point that is completely contained in the open ball.

So, let $x \in B_\varepsilon(a) = \{x \in X \mid d(x, a) < \varepsilon\}$ and suppose that $d(x, a) = \delta < \varepsilon$. We wish to find some $\varepsilon_1 > 0$ such that $B_{\varepsilon_1}(x) \subseteq B_\varepsilon(a)$. Consider $B_{\varepsilon_1}(x)$, the open ball centered at x of radius $\varepsilon_1 = \varepsilon - \delta > 0$ and let $y \in B_{\varepsilon_1}(x)$. If $y \in B_{\varepsilon_1}(x)$, then $y \in X$ and $d(y, x) < \varepsilon_1 = \varepsilon - \delta$. Since $a, x, y \in X$ and X is a metric space, we must have that

$$d(y, a) \leq d(y, x) + d(x, a) < \varepsilon - \delta + \delta = \varepsilon.$$

Thus, we have that $d(y, a) < \varepsilon$ and $y \in B_\varepsilon(a)$. Therefore, for every $x \in B_\varepsilon(a)$ we have that there exists an $\varepsilon_1 > 0$ such that $B_{\varepsilon_1}(x) \subseteq B_\varepsilon(a)$ and the set $B_\varepsilon(a)$ must be open.

We now wish to show that a finite subset $A = \{a_0, a_1, \ldots, a_n\} \subseteq X$ is a closed set. Recall that a set $A \subseteq X$ is closed if and only if $X \setminus A$ is open. Let $x \in X \setminus A$ and consider $B_\varepsilon(x)$, the open ball centered at x of radius $\varepsilon = \min_i\{d(x, a_i)\}$. Since $x \in X$ and $x \neq a_0, \ldots, a_n$, we know that $\varepsilon = \min_i\{d(x, a_i)\} > 0$.

Suppose to the contrary that $y \in B_\varepsilon(x)$ and $y = a_i$ for some $i = 0, \ldots, n$. Since $y, a_i \in X$ and X is a metric space with $y = a_i$, we have that $d(y, a_i) = 0$. Thus, under the properties of the distance function of this metric space, we must have that

$$d(x, a_i) \leq d(x, y) + d(y, a_i) < \varepsilon = \min_i\{d(x, a_i)\}.$$

However, this is a contradiction since an element of a set cannot be strictly less than the minimum of that set. Thus, if $y \in B_\varepsilon(x)$, then $y \neq a_i$ for any $i = 0, \ldots, n$. Therefore, for every $x \in X \setminus A$, there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq X \setminus A$ and $X \setminus A$ is open so that A is closed. $\square$

Problem 4.2.2. Show that the closed ball $B_\varepsilon[a] = \{x \in X \mid d(a,x) \leq \varepsilon\}$ in a metric space is a closed set, but it need not be equal to the closure of the open ball $B_\varepsilon(a)$. (Hint: Consider the two point space $\mathcal{A} = \{0,1\}$ with metric $d(0,1) = 1$).

Solution. We wish to show that $B_\varepsilon[a]$ is closed, i.e. that $X \setminus B_\varepsilon[a]$ is open. Suppose that $x \in X \setminus B_\varepsilon[a]$. Then we have that $d(x,a) = \delta > \varepsilon$. Consider $B_{\varepsilon_1}(x)$, the open ball centered at x of radius $\varepsilon_1 = \delta - \varepsilon$. Suppose to the contrary that $y \in B_{\varepsilon_1}(x)$ and $y \in B_\varepsilon[a]$. Since $a, x, y \in X$ with X a metric space, we have that

$$d(x,a) \leq d(x,y) + d(y,a) < \delta - \varepsilon + \varepsilon = \delta = d(x,a).$$

However, this is a contradiction since the distance between two points cannot be less than itself. Thus, we must have that if $y \in B_{\varepsilon_1}(x)$, then $y \notin B_\varepsilon[a]$. Therefore, for every $x \in X \setminus B_\varepsilon[a]$, there exists an $\varepsilon_1 > 0$ such that $B_{\varepsilon_1}(x) \subseteq X \setminus B_\varepsilon[a]$ and $X \setminus B_\varepsilon[a]$ is open so that $B_\varepsilon[a]$ is closed.

Recall that $\overline{A}$, the closure of a set A, is the union of the set A with its limit points, i.e. $\overline{A} = A \cup \mathrm{Lim}\{A\}$ where $a \in X$ is a limit point if every open ball centered at a contains a point in A different from a. We wish to show that the closed ball centered at a of a given radius does not necessarily coincide with the closure of the open ball centered at a of the same radius.

To see this, consider the metric space $X = \{0,1\}$ equipped with the discrete metric and consider the point $0 \in X$. The only open ball centered at 0 is given by

$$B_1(0) = \{x \in X \mid d(x,0) < 1\} = \{0\}.$$

Since the point 1 is not in any open ball centered at 0, we see that there are no limit points of 0 and that $\overline{B_1(0)} = \{0\}$. However, the only closed ball centered at 0 is given by

$$B_1[0] = \{x \in X \mid d(x,0) \leq 1\} = \{0,1\}$$

and we see that $\overline{B_1(0)} \neq B_1[0]$. Therefore, we can clearly see that the closure of the open ball centered at a point $a \in X$ of any radius is not necessarily equal to the closed ball centered at a of the same radius.

$\square$

Problem 4.2.5. Show that the intersection of a finite number of open sets $A_1, A_2, \dots, A_n$ in a metric space (X, d) is an open set. Show that, by considering the intervals $(-1/n, 1/n)$ for all $n \in \mathbb{Z}^+$ in $\mathbb{R}$, the intersection of infinitely many open sets need not be open.

Solution. □

Problem 4.2.6. If $\mathcal{A} = \{0, 1\}$, then $\mathcal{A}^{\mathbb{N}}$ denotes the metric space of 0's and 1's:

$$\mathcal{A}^{\mathbb{N}} = \{\omega = (a_0, a_1, a_2, \dots) \mid a_i = 0 \text{ or } a_i = 1\},$$

with metric:

$$d(\omega_1, \omega_2) = \sum_{k=0}^{\infty} \frac{|s_k - t_k|}{2^k},$$

where $\omega_1 = (s_0, s_1, s_2, \dots)$ and $\omega_2 = (t_0, t_1, t_2, \dots)$.

Show that $\mathcal{A}^{\mathbb{N}}$ is a metric space. Find $d(\omega_1, \omega_2)$ if:

i. $\omega_1 = (0, 1, 1, 1, 1, \dots)$ and $\omega_2 = (1, 0, 1, 1, 1, \dots)$,

ii. $\omega_1 = (0, 1, 0, 1, 0, \dots)$ and $\omega_2 = (1, 0, 1, 0, 1, \dots)$.

Solution. $\square$

Problem 4.2.7. Let $f : I \to I$ be a continuous function defined on an interval I.

i. What can you say about the graph of f, if f has a dense set of points with $f^2(x) = x$?

ii. Show that the inverse of f must exist and that f must have at least one fixed point.

iii. Deduce that if there exists an $x \in I$ with $f(x) \neq x$, then f must be strictly decreasing.

iv. If $f'(x)$ exists for all $x \in I$, show that the 2-cycles are non-hyperbolic, and any fixed point x_0 is non-hyperbolic of the type $f'(x_0) = -1$, when f is not the identity map.

v. Give an example of a function of the type appearing in iv.

Solution. □

Problem 4.3.4. Show that if $f : [a, b] \to [a, b]$ is a homeomorphism, then either a and b are fixed points or $\{a, b\}$ is a 2-cycle.

Solution. $\square$

Problem 4.3.8. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map with fixed point c and basin of attraction $B_f(c) = (a, b)$, an interval. Show that one of the following must hold:

i. a and b are fixed points.

ii. a or b is fixed and the other is eventually fixed.

iii. $\{a, b\}$ is a 2-cycle.

Solution. $\square$