Homework Assignment 10

Matthew Tiger

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Problem 10.5.1. Show that the Cantor set C is a fixed point of the map $F : \mathcal{C}(\mathbb{R}) \to \mathcal{C}(\mathbb{R})$ defined by

$$F(A) = f_1(A) \cup f_2(A)$$

where $f_1(x) = x/3$ and $f_2(x) = x/3 + 2/3$ are contractions on $\mathbb{R}$.

Solution. Recall that the Cantor set C is defined as

$$C = \left\{ x \in [0,1] \;\middle|\; x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, \quad a_n \in \{0,2\} \right\}. \tag{1}$$

We wish to show that $F(C) = C$. From the definitions of f_1 and f_2, we have that

$$\begin{aligned} f_1(C) &= \left\{ x \in [0,1] \;\middle|\; x = \sum_{n=1}^{\infty} \frac{a_n}{3^{n+1}}, \quad a_n \in \{0,2\} \right\} \\ &= \left\{ x \in [0,1] \;\middle|\; x = \sum_{n=1}^{\infty} \frac{b_n}{3^n}, \quad b_1 = 0,\, b_n = a_{n-1} \text{ for } n > 1 \right\} \\ f_2(C) &= \left\{ x \in [0,1] \;\middle|\; x = \frac{2}{3} + \sum_{n=1}^{\infty} \frac{a_n}{3^{n+1}}, \quad a_n \in \{0,2\} \right\} \\ &= \left\{ x \in [0,1] \;\middle|\; x = \sum_{n=1}^{\infty} \frac{b_n}{3^n}, \quad b_1 = 2,\, b_n = a_{n-1} \text{ for } n > 1 \right\} \end{aligned} \tag{2}$$

From (2) and the definition of F, we see that

$$\begin{aligned} F(C) &= f_1(C) \cup f_2(C) \\ &= \left\{ x \in [0,1] \;\middle|\; x = \sum_{n=1}^{\infty} \frac{b_n}{3^n}, \quad b_n \in \{0,2\} \right\}. \end{aligned}$$

But this is precisely the definition of the Cantor set C given in (1). Therefore, $F(C) = C$ and the Cantor set C is a fixed point of F. $\square$

Problem 10.5.2. Show that the box-counting dimension of the Sierpinski triangle is $\log 3/\log 2$.

Solution. The Sierpinski triangle is formed by iteratively removing smaller and smaller equilateral triangles from an equilateral triangle of side-length 1. For an equilateral triangle of side-length d, the minimum size square that completely covers the triangle is the square of side-length d.

For the first iteration of the Sierpinski triangle, we remove the open middle equilateral triangle, leaving 3 equilateral triangles of side-length $1/2$. Thus, we would require, at a minimum, 3 squares of side-length $1/2$ in order to completely cover the Sierpinski triangle. In the next iteration, we remove the open middle equilateral triangles of the remaining equilateral triangles, leaving 9 equilateral triangles of side-length $1/4$. Thus, we would, at a minimum, require 9 squares of side-length $1/4$ in order to completely cover the Sierpinski triangle.

In general, the n-th iteration will leave 3^n equilateral triangles of side length $1/2^n$ and we would require, at a minimum, 3^n squares of side-length $1/2^n$ in order to completely cover the Sierpinski triangle. Let K be the Sierpinski triangle and let $N_{\delta_n}(K)$ be the minimum number of boxes of equal length $\delta_n > 0$ needed to completely cover K at iteration n. Then from our previous discussions, $N_{\delta_n}(K) = 3^n$ with $\delta_n = 1/2^n$.

Therefore, the box-counting dimension of the Sierpinski triangle K is

$$\dim(K) = \lim_{\delta\to 0^+} \frac{\log N_\delta(K)}{\log 1/\delta} = \lim_{n\to\infty} \frac{\log N_{\delta_n}(K)}{\log 1/\delta_n} = \lim_{n\to\infty} \frac{\log 3^n}{\log 2^n} = \frac{\log 3}{\log 2}.$$

□

Problem 10.5.4.

Solution. □

Problem 10.5.7.

Solution. □