Homework Assignment 6

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Problem 3.1. Determine which of the following ARMA processes are causal and which of them are invertible. (In each case $\{Z_t\}$ denotes white noise.)

a. $X_t + 0.2X_{t-1} - 0.48X_{t-2} = Z_t$

b. $X_t + 1.9X_{t-1} + 0.88X_{t-2} = Z_t + 0.2Z_{t-1} + 0.7Z_{t-2}$

c. $X_t + 0.6X_{t-1} = Z_t + 1.2Z_{t-1}$

d. $X_t + 1.8X_{t-1} + 0.81X_{t-2} = Z_t$

e. $X_t + 1.6X_{t-1} = Z_t - 0.4Z_{t-1} + 0.04Z_{t-2}$

Solution. A general ARMA(p,q) process $X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$ can be rewritten as $\phi(B)X_t = \theta(B)Z_t$, where B is the backshift operator and $\phi(x) = 1 - \phi_1 x - \cdots - \phi_p x^p$ and $\theta(x) = 1 + \theta_1 x + \cdots + \theta_q x^q$. A stationary solution of the above process exists if and only if $\phi(x) = 0$ for $|x| \neq 1$. Similarly, if a stationary solution to the process exists, then the process is causal if and only the solutions to $\phi(x) = 0$ have norm greater than 1 and the process is invertible if the solutions to $\theta(x) = 0$ have norm greater than 1.

a. For this process, $\phi(x) = 1 - (-0.2)x - (0.48)x^2$ and $\theta(x) = 1$. The solutions to $\phi(x) = 0$ are $x_1 = 1.\overline{66}$ and $x_2 = -1.25$ both of which have norm greater than 1. Therefore a stationary solution exists and the process is causal. Similarly, $\theta(x) \neq 0$ for any x so it is invertible as well.

b. For this process, $\phi(x) = 1 - (-1.9)x - (-0.88)x^2$ and $\theta(x) = 1 + 0.2x + 0.7x^2$. The solutions to $\phi(x) = 0$ are $x_1 = -0.\overline{90}$ and $x_2 = -1.25$. Therefore a stationary solution exists however since $|x_1| \leq 1$ the process is not causal. Similarly, the solutions to $\theta(x) = 0$ are $x_1 = -0.1429 + 1.1867i$ and $x_2 = -0.1429 - 1.1867i$ both of which have norm greater than 1. Therefore the process is invertible.

c. For this process, $\phi(x) = 1 - (-0.6)x$ and $\theta(x) = 1 + 1.2x$. The solution to $\phi(x) = 0$ is $x_1 = -1.\overline{66}$ which has norm greater than 1. Therefore a stationary solution exists and the process is causal. Similarly, the solution to $\theta(x) = 0$ is $x_1 = -0.8\overline{33}$ which has norm less than 1. Therefore the process is not invertible.

d. For this process, $\phi(x) = 1 - (-1.8)x - (-0.81)x^2$ and $\theta(x) = 1$. The solution to $\phi(x) = 0$ is $x_1 = -1.\overline{11}$ with multiplicity 2 which has norm greater than 1. Therefore a stationary solution exists and the process is causal. Similarly, $\theta(x) \neq 0$ for any x so it is invertible as well.

e. For this process, $\phi(x) = 1 - (-1.6)x$ and $\theta(x) = 1 + (-0.4)x + (0.04)x^2$. The solution to $\phi(x) = 0$ is $x_1 = -0.625$ which has norm less than 1. Therefore a stationary solution exists however the process is not causal. Similarly, the solution to $\theta(x) = 0$ is $x = 5$ with multiplicity 2 which has norm greater than 1. Therefore the process is invertible.

□

Problem 3.3. For those processes in Problem 3.1 that are causal, compute the first six coefficients $\psi_0, \psi_1, \ldots, \psi_5$ in the causal representation $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ of $\{X_t\}$.

Solution. For causal ARMA(p,q) processes, the causal representation of $\{X_t\}$ is given by $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ where for $j \geq 1$,

$$\psi_j - \sum_{k=1}^{p} \phi_k \psi_{j-k} = \theta_j. \tag{1}$$

In the above formula, $\psi_j = 0$ for $j < 0$ and ϕ_k and θ_j are the coefficients found in the polynomials $\phi(x)$ and $\theta(x)$. We already know $\psi_0 = 1$ for any causal ARMA(p,q) process, so we need only compute ψ_j for $1 \leq j \leq 5$.

a. Note that for this ARMA$(2,0)$ process $\phi(x) = 1 - (-0.2)x - (0.48)x^2$ and $\theta(x) = 1$. So $\phi_1 = -0.2$, $\phi_2 = 0.48$, and $\phi_k = 0$ for $k \geq 3$. Similarly, $\theta_k = 0$ for $k \geq 1$. Using (1), we can see that since $\theta_j = 0$, $\psi_j = \sum_{k=1}^{2} \phi_k \psi_{j-k}$ for $j \geq 1$.

   For $j = 1$:

   $\psi_1 = \phi_1 \psi_0 = (-0.2)(1) = -0.2$

   For $j = 2$:

   $\psi_2 = \phi_1 \psi_1 + \phi_2 \psi_0 = (-0.2)(-0.2) + (0.48)(1) = 0.52$

   For $j = 3$:

   $\psi_3 = \phi_1 \psi_2 + \phi_2 \psi_1 = (-0.2)(0.52) + (0.48)(-0.2) = -0.2$

   For $j = 4$:

   $\psi_4 = \phi_1 \psi_3 + \phi_2 \psi_2 = (-0.2)(-0.2) + (0.48)(0.52) = 0.2896$

   For $j = 5$:

   $\psi_5 = \phi_1 \psi_4 + \phi_2 \psi_3 = (-0.2)(0.2896) + (0.48)(-0.2) = -0.1539$

b. This process is not causal.

c. Note that for this ARMA(1,1) process, $\phi(x) = 1 - (-0.6)x$ and $\theta(x) = 1 + 1.2x$. So $\phi_1 = -0.6$ and $\phi_k = 0$ for $k \geq 2$. Similarly, $\theta_1 = 1.2$ and $\theta_k = 0$ for $k \geq 2$. Using (1), we can see that $\psi_j = \theta_j + \phi_1\psi_{j-1}$.

For $j = 1$:

$\psi_1 = \theta_1 + \phi_1\psi_0 = 1.2 + (-0.6)(1) = 0.6$

For $j = 2$:

$\psi_2 = \theta_2 + \phi_1\psi_1 = (-0.6)(0.6) = -0.36$

For $j = 3$:

$\psi_3 = \theta_3 + \phi_1\psi_2 = (-0.6)(-0.36) = 0.2160$

For $j = 4$:

$\psi_4 = \theta_4 + \phi_1\psi_3 = (-0.6)(0.2160) = -0.1296$

For $j = 5$:

$\psi_5 = \theta_5 + \phi_1\psi_4 = (-0.6)(-0.1296) = 0.0778$

d. Note that for this ARMA(2,0) process $\phi(x) = 1 - (-1.8)x - (-0.81)x^2$ and $\theta(x) = 1$. So $\phi_1 = -1.8$, $\phi_2 = -0.81$, and $\phi_k = 0$ for $k \geq 3$. Similarly, $\theta_k = 0$ for $k \geq 1$. Using (1), we can see that since $\theta_j = 0$, $\psi_j = \sum_{k=1}^{2} \phi_k\psi_{j-k}$ for $j \geq 1$.

For $j = 1$:

$\psi_1 = \phi_1\psi_0 = (-1.8)(1) = -1.8$

For $j = 2$:

$\psi_2 = \phi_1\psi_1 + \phi_2\psi_0 = (-1.8)(-1.8) + (-0.81)(1) = 2.43$

For $j = 3$:

$\psi_3 = \phi_1\psi_2 + \phi_2\psi_1 = (-1.8)(2.43) + (-0.81)(-1.8) = -2.916$

For $j = 4$:

$\psi_4 = \phi_1\psi_3 + \phi_2\psi_2 = (-1.8)(-2.916) + (-0.81)(2.43) = 3.2805$

For $j = 5$:

$\psi_5 = \phi_1\psi_4 + \phi_2\psi_3 = (-1.8)(3.2805) + (-0.81)(-2.916) = -3.5429$

e. This process is not causal.

$\square$