Exam 1

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Problem 1. Find the Fourier Transforms of the following functions:

a. $f(x) = x^2 e^{-a|x|}$, $a > 0$,

b. $f(x) = \left(1 - \frac{|x|}{2}\right) H \left(1 - \frac{|x|}{2}\right)$.

Solution. $\square$

Problem 2. Find the Laplace Transforms of the following functions:

a. $f(t) = \displaystyle\int_0^t \frac{\sin ax}{x} dx$,

b. $f(t) = tH(t-a)$.

Solution. $\square$

Problem 3. Solve the following integral equations:

a. $$\int_0^\infty f(x)\sin kx dx = \begin{cases} 1-k & k<1 \\ 0 & k>1 \end{cases},$$

b. $$\int_{-\infty}^\infty \frac{f(t)}{(x-t)^2+4}dt = \frac{1}{x^2+9}.$$

Solution. □

Problem 4. Show that

$$\int_0^\infty F_s(k)G_c(k)\sin kx dx = \frac{1}{2}\int_0^\infty g(\xi)\left[f(\xi+x) - f(\xi-x)\right]d\xi$$

where

$$F_s(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin kx dx$$

and

$$G_c(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\cos kx dx.$$

Solution. □

Problem 5. Apply the Fourier Transform to solve the following initial value problem for the heat equation:

$$\begin{aligned}&\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2} + f(x,t), \quad -\infty < x < \infty,\\ &u(x,0) = \phi(x), \quad t > 0.\end{aligned}$$

Solution. □

Problem 6. Evaluate the following definite integrals:

a. $\displaystyle\int_0^\infty \frac{\sin ax \sin bx}{x^2} dx,$

b. $\displaystyle\int_0^\infty \frac{(a^2 - x^2)^2}{(x^2 + a^2)^4} dx, \quad a > 0.$

Solution. $\square$

Problem 7. Use the Fourier Sine Transform to solve the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \infty$$

with the boundary data $0 < y < L$

$$\begin{aligned} &u(x, L) = 0, \quad u(x, 0) = f(x), \\ &u(0, y) = 0, \quad u(x, y) \to 0 \text{as } x \to \infty \text{ uniformly in } y. \end{aligned}$$

Solution. $\square$

Problem 8. Apply the Fourier Transform to solve the 3-dimensional wave problem

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \quad -\infty < x, y, z < \infty,$$

subject to the initial conditions

$$\begin{gathered} u(x,y,z,t)|_{t=0} = 0 \\ \left.\frac{\partial u(x,y,z,t)}{\partial t}\right|_{t=0} = \delta(x,y,z). \end{gathered}$$

Solution. □

Problem 9. Show that if E is a solution of an m-th order partial differential equation

$$P(\partial)u = \sum_{k=0}^{m} a_k \partial^k u = \delta,$$

where δ is the Dirac delta function, then $E * f$ is the solution of the partial differential equation $P(\partial)u = f$, where $*$ is the convolution.

Solution. □