

Homework Assignment 5

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Problem 1. Find the 1, 2, ∞ norms of the matrices

$$\mathbf{T} := \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \mathbf{I} := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{J} := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then find their p -condition numbers $\text{cond}_p(\mathbf{T})$ for $p = 1, 2, \infty$.

Solution.

- **1-Norm** Note that if \mathbf{X} is a 2×2 matrix then $\|\mathbf{X}\|_1 = \max_{1 \leq j \leq 2} \sum_{i=1}^2 |x_{ij}|$ where x_{ij} is the entry in the i -th row and the j -th column of \mathbf{X} . So,

$$\|\mathbf{X}\|_1 = \max_{1 \leq j \leq 2} \{|x_{1j}| + |x_{2j}|\} = \max \{|x_{11}| + |x_{21}|, |x_{12}| + |x_{22}|\}$$

If $\mathbf{T} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, then $\mathbf{T}^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$. So,

$$\|\mathbf{T}\|_1 = \max \{|2| + |-1|, |-1| + |2|\} = \max \{3, 3\} = 3$$

and

$$\|\mathbf{T}^{-1}\|_1 = \max \{|2/3| + |1/3|, |1/3| + |2/3|\} = \max \{1, 1\} = 1.$$

Thus, $\|\mathbf{T}\|_1 = 3$ and $\text{cond}_p(\mathbf{T}) = \|\mathbf{T}\|_1 \|\mathbf{T}^{-1}\|_1 = 3 \cdot 1 = 3$.

If $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $\mathbf{I}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$. So,

$$\|\mathbf{I}\|_1 = \max \{|1| + |0|, |0| + |1|\} = \max \{1, 1\} = 1.$$

Thus, $\|\mathbf{I}\|_1 = 1$ and $\text{cond}_p(\mathbf{I}) = \|\mathbf{I}\|_1 \|\mathbf{I}^{-1}\|_1 = 1 \cdot 1 = 1$.

If $\mathbf{J} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $\mathbf{J}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. So,

$$\|\mathbf{J}\|_1 = \max \{|1| + |0|, |1| + |1|\} = \max \{1, 2\} = 2$$

and

$$\|\mathbf{J}^{-1}\|_1 = \max \{|1| + |0|, |-1| + |1|\} = \max \{1, 2\} = 2.$$

Thus, $\|\mathbf{J}\|_1 = 2$ and $\text{cond}_p(\mathbf{J}) = \|\mathbf{J}\|_1 \|\mathbf{J}^{-1}\|_1 = 2 \cdot 2 = 4$.

- **∞ -norm** Note that if \mathbf{X} is a 2×2 matrix then $\|\mathbf{X}\|_\infty = \max_{1 \leq i \leq 2} \sum_{j=1}^2 |x_{ij}|$ where x_{ij} is the entry in the i -th row and the j -th column of \mathbf{X} . So,

$$\|\mathbf{X}\|_\infty = \max_{1 \leq i \leq 2} \{|x_{i1}| + |x_{i2}|\} = \max \{|x_{11}| + |x_{12}|, |x_{21}| + |x_{22}|\}$$

If $\mathbf{T} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, then $\mathbf{T}^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$. So,

$$\|\mathbf{T}\|_\infty = \max \{|2| + |-1|, |-1| + |2|\} = \max \{3, 3\} = 3$$

and

$$\|\mathbf{T}^{-1}\|_\infty = \max \{|2/3| + |1/3|, |1/3| + |2/3|\} = \max \{1, 1\} = 1.$$

Thus, $\|\mathbf{T}\|_\infty = 3$ and $\text{cond}_p(\mathbf{T}) = \|\mathbf{T}\|_\infty \|\mathbf{T}^{-1}\|_\infty = 3 \cdot 1 = 3$.

If $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $\mathbf{I}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$. So,

$$\|\mathbf{I}\|_\infty = \max \{|1| + |0|, |0| + |1|\} = \max \{1, 1\} = 1.$$

Thus, $\|\mathbf{I}\|_\infty = 1$ and $\text{cond}_p(\mathbf{I}) = \|\mathbf{I}\|_\infty \|\mathbf{I}^{-1}\|_\infty = 1 \cdot 1 = 1$.

If $\mathbf{J} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $\mathbf{J}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. So,

$$\|\mathbf{J}\|_\infty = \max \{|1| + |1|, |0| + |1|\} = \max \{2, 1\} = 2$$

and

$$\|\mathbf{J}^{-1}\|_\infty = \max \{|1| + |-1|, |0| + |1|\} = \max \{2, 1\} = 2.$$

Thus, $\|\mathbf{J}\|_\infty = 2$ and $\text{cond}_p(\mathbf{J}) = \|\mathbf{J}\|_\infty \|\mathbf{J}^{-1}\|_\infty = 2 \cdot 2 = 4$.

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