

# Homework Assignment 4

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**Problem 2.3.1.** For each of the following functions,  $c = 0$  lies on a periodic cycle. Classify this cycle as attracting, repelling, or neutral (non-hyperbolic). State if it is super attracting.

$$\text{i. } f(x) = \frac{\pi}{2} \cos(x), \quad \text{ii. } g(x) = -\frac{1}{2}x^3 - \frac{3}{2}x^2 + 1.$$

*Solution.* Recall that if  $c$  is a point of period  $r$ , then  $c$  is stable, asymptotically stable, unstable, if  $f^r(c)$  is stable, asymptotically stable, unstable, respectively. Thus, if  $c$  is a point of period  $r$  and  $f'(x)$  is continuous at  $x = c$ , then  $c$  is asymptotically stable (attracting) if

$$|(f^r(c))'| = |f'(f^0(c)) \cdot f'(f^1(c)) \cdots f'(f^{r-1}(c))| < 1$$

and  $c$  is unstable (repelling) if

$$|(f^r(c))'| = |f'(f^0(c)) \cdot f'(f^1(c)) \cdots f'(f^{r-1}(c))| > 1.$$

- i. Let  $f(x) = \frac{\pi}{2} \cos(x)$ . It is clear that  $f^2(0) = 0$  so that  $c = 0$  is a period 2 point and  $\{0, f(0)\}$  forms a 2-cycle. Note that  $f'(x) = -\frac{\pi}{2} \sin(x)$ , which is continuous, and that

$$|f'(0) \cdot f'(f(0))| = \left| \left( -\frac{\pi}{2} \sin(0) \right) \left( -\frac{\pi}{2} \sin\left(\frac{\pi}{2}\right) \right) \right| = 0 < 1$$

so that the 2-cycle  $\{0, f(0)\}$  is asymptotically stable. Since

$$(f^2(0))' = (f(f(0)))' = f'(0) \cdot f'(f(0)) = 0,$$

we have that  $c = 0$  is a super-attracting point of  $f^2$  and the 2-cycle  $\{0, f(0)\}$  is a super-attracting, asymptotically stable cycle.

- ii. Let  $g(x) = -\frac{1}{2}x^3 - \frac{3}{2}x^2 + 1$ . It is clear that  $g^3(0) = 0$  so that  $c = 0$  is a period 3 point and  $\{0, g(0), g^2(0)\}$  forms a 3-cycle. Note that  $g'(x) = -\frac{3}{2}x^2 - 3x$ , which is continuous, and that

$$|g'(0) \cdot g'(g(0)) \cdot g'(g^2(0))| = \left| 0 \left( -\frac{9}{2} \right) \left( \frac{3}{2} \right) \right| = 0 < 1$$

so that the 2-cycle  $\{0, g(0), g^2(0)\}$  is asymptotically stable. Since

$$(g^3(0))' = (g(g(g(0))))' = g'(0) \cdot g'(g(0)) \cdot g'(g^2(0)) = 0,$$

we have that  $c = 0$  is a super-attracting point of  $g^3$  and the 3-cycle  $\{0, g(0), g^2(0)\}$  is a super-attracting, asymptotically stable cycle.

□

**Problem 2.3.2.** Let  $f_c(x) = x^2 + c$ . Show that for  $c < -3/4$ ,  $f_c$  has a 2-cycle, and find it explicitly. For what values of  $c$  is the 2-cycle attracting?

*Solution.* Note that  $f_c$  has a 2-cycle if it has a period 2 point, i.e. if  $f_c^2(x) - x = 0$  has a solution  $x = x_0$  with  $f_c(x_0) - x_0 \neq 0$ . Thus, we must have that

$$f_c^2(x) - x = (x^2 + c)^2 + c - x = x^4 + 2cx^2 - x + c^2 + c = 0 \quad (1)$$

has a solution. As was shown earlier,  $x = (1 \pm \sqrt{-4c})/2$  are fixed points of  $f_c$  and thus must satisfy  $f_c^2(x) - x = 0$ . This allows to easily factor (1) and we see that

$$x^4 + 2cx^2 - x + c^2 + c = \left(x - \frac{1 + \sqrt{-4c}}{2}\right) \left(x - \frac{1 - \sqrt{-4c}}{2}\right) (x^2 + x + c + 1).$$

Since a period 2 point  $x_0$  is such that  $f_c(x_0) - x_0 \neq 0$ , we know that

$$\left(x_0 - \frac{1 + \sqrt{-4c}}{2}\right) \neq 0, \quad \left(x_0 - \frac{1 - \sqrt{-4c}}{2}\right) \neq 0$$

so that  $x^4 + 2cx^2 - x + c^2 + c = 0$  only if  $x^2 + x + c + 1 = 0$ . We readily see that since  $c < -3/4$ , the polynomial  $x^2 + x + c + 1$  has real solutions, and that

$$x^2 + x + c + 1 = \left(x - \frac{-1 + \sqrt{-3 - 4c}}{2}\right) \left(x - \frac{-1 - \sqrt{-3 - 4c}}{2}\right)$$

from which we identify the 2-cycle of  $f_c$  as

$$\{c_0, f_c(c_0)\} = \left\{ \frac{-1 + \sqrt{-3 - 4c}}{2}, \frac{-1 - \sqrt{-3 - 4c}}{2} \right\}.$$

This 2-cycle will be attracting for  $f_c$  if  $c_0$  is attracting for  $f_c^2$ , i.e. if

$$\left| (f_c^2(c_0))' \right| = |f_c'(c_0)f_c'(f_c(c_0))| < 1.$$

Note that  $f_c'(x) = 2x$  from which we see that

$$|f_c'(c_0)f_c'(f_c(c_0))| = |(-1 + \sqrt{-3 - 4c})(-1 - \sqrt{-3 - 4c})| = |4(1 + c)|.$$

Therefore, the 2-cycle of  $f_c$  is attracting if  $|4(1 + c)| < 1$ , which occurs if and only if  $-5/4 < c < -3/4$ .

□

**Problem 2.3.3.** Let  $a, b, c \in \mathbb{R}$ . Investigate the existence of 2-cycles for the following maps:

- i.  $f(x) = ax + b$ ,  $a \neq 0$ .
- ii.  $f(x) = ax^2 - x + c$ ,  $a, c > 0$ .
- iii.  $f(x) = a - \frac{b}{x}$ ,  $a \neq 0, b \neq 0$ .
- iv.  $f(x) = \frac{ax+b}{cx-a}$ ,  $a^2 + bc \neq 0$ .

*Solution.* As outlined in a previous problem, a 2-cycle for a function  $f$  exists if there is a period 2 point of  $f$ , i.e. if there is a point  $x = x_0$  such that  $f^2(x_0) - x_0 = 0$  but  $f(x_0) - x_0 \neq 0$ . Thus, to identify the period 2 points, we first identify the fixed points  $c_0, \dots, c_n$  of a function. The fixed points  $x = c_0, \dots, c_n$  will satisfy  $f(x) - x = 0$  and thus must satisfy  $f^2(x) - x = 0$  so that  $(x - c_i)$  is a factor of  $f^2(x) - x$  for  $i = 0, \dots, n$ . Therefore, the remaining solutions of  $f^2(x) - x$ , if they exist, form the 2-cycles of  $f$ .

- i. Suppose that  $f(x) = ax + b$  with  $a \neq 0$ . We readily see that  $f(x) - x = 0$  has the solution  $x = -b/(a - 1)$  if  $a \neq 1$  and is the only fixed point of  $f$ . Note that if  $a = 1$ , then  $f(x) - x = 0$  only if  $b = 0$  giving rise to the identity map for which the solution is trivial. However, note that

$$f^2(x) - x = (a^2 - 1)x + b(a + 1) = (a + 1)(b + (a - 1)x) = 0$$

from which the only solution is  $x = -b/(a - 1)$ . Since this is the fixed point of  $f$ , it cannot be a period 2 point. Therefore, there are no 2-cycles for  $f(x) = ax + b$  for  $a \neq 0, 1$ .

- ii. Suppose that  $f(x) = ax^2 - x + c$  with  $a, c > 0$ . Note that  $f(x) - x = ax^2 - 2x + c = 0$  has real solutions  $x = (1 \pm \sqrt{1 - ac})/a$  if  $ac \leq 1$ . Since  $a$  and  $c$  are positive, this is equivalent to requiring that  $a, c \in (0, 1]$ . Then  $\left(x - \frac{1 + \sqrt{1 - ac}}{a}\right)$  and  $\left(x - \frac{1 - \sqrt{1 - ac}}{a}\right)$  are factors of  $f^2(x) - x$  and we see that

$$\begin{aligned} f^2(x) - x &= a(ax^2 - x + c)^2 - x + c \\ &= \left(x - \frac{1 + \sqrt{1 - ac}}{a}\right) \left(x - \frac{1 - \sqrt{1 - ac}}{a}\right) (a^2x^2 + ca) = 0. \end{aligned}$$

However, if  $a, c > 0$ , then the only real solutions of this equation are given by  $x = (1 \pm \sqrt{1 - ac})/a$  where  $a, c \in (0, 1]$ . But these are the fixed points of  $f$ . Therefore, there are no 2-cycles of  $f(x) = ax^2 - x + c$  with  $a, c > 0$ .

- iii. Suppose that  $f(x) = a - \frac{b}{x}$  with  $a \neq 0, b \neq 0$ . It is easily seen that if  $x \neq 0$ , then  $f(x) - x = x^2 - ax + b = 0$  has real solutions  $x = (a \pm \sqrt{a^2 - 4b})/2$  if  $a^2 \geq 4b$ . Then  $\left(x - \frac{a + \sqrt{a^2 - 4b}}{2}\right)$  and  $\left(x - \frac{a - \sqrt{a^2 - 4b}}{2}\right)$  are factors of  $f^2(x) - x$  and we see that

$$\begin{aligned} f^2(x) - x &= a - \frac{b}{\left(a - \frac{b}{x}\right)} - x \\ &= \left(x - \frac{a + \sqrt{a^2 - 4b}}{2}\right) \left(x - \frac{a - \sqrt{a^2 - 4b}}{2}\right) \left(\frac{a}{b - ax}\right) = 0 \end{aligned}$$

only when  $x = (a \pm \sqrt{a^2 - 4b})/2$  which are precisely the fixed points of  $f$ . Therefore, there are no 2-cycles of  $f(x) = a - \frac{b}{x}$  with  $a \neq 0, b \neq 0$

- iv. Suppose that  $f(x) = \frac{ax+b}{cx-a}$  with  $a^2 + bc \neq 0$ . Note that  $f(x)$  is only defined if  $x \neq a/c$ . We readily see that

$$f(x) - x = \frac{ax+b}{cx-a} - x = \frac{-cx^2 + 2ax + b}{cx-a} = 0$$

if  $x = (a \pm \sqrt{a^2 + bc})/c$  which is real and in the domain of  $f$  if  $a^2 + bc > 0$ . These are precisely the fixed points of  $f$ . Note that for any  $x \neq a/c$  we have that

$$f^2(x) = \frac{b + \frac{a(b+ax)}{cx-a}}{-a + \frac{c(b+ax)}{cx-a}} = \frac{(a^2 + bc)x}{a^2 + bc} = x$$

if  $a^2 + bc \neq 0$ . Thus, every defined point satisfies  $f^2(x) = x$ . Therefore, every point in this function's domain generates a 2-cycle if that point is different from the fixed points

$$c_0 = \frac{a + \sqrt{a^2 + bc}}{c}, \quad c_1 = \frac{a - \sqrt{a^2 + bc}}{c}.$$

□

**Problem 2.3.4.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous.

- i. If  $f$  has a 2-cycle  $\{x_0, x_1\}$ , show that  $f$  has a fixed point.
- ii. If  $f$  has a 3-cycle  $\{x_0, x_1, x_2\}$ ,  $x_0 < x_1 < x_2$  with  $f(x_0) = x_1$ ,  $f(x_1) = x_2$ , and  $f(x_2) = x_0$ , show that there is a fixed point  $y_0$  with  $x_1 < y_0 < x_2$  and a point  $y_1$  with  $x_0 < y_1 < x_1$  with  $f^2(y_1) = y_1$ .

*Solution.* i. Suppose that  $\{x_0, x_1\}$  is a 2-cycle of the continuous function  $f$ . Then we have that  $f(x_0) = x_1$  and  $f(x_1) = x_0$  with  $x_0 \neq x_1$ . Consider the function  $g(x) = f(x) - x$ , which is continuous by the continuity of  $f$ . Without loss of generality, we may assume that  $x_0 < x_1$ .

The Intermediate Value Theorem states that for a continuous function  $f$ , for any interval  $I = [a, b]$ , if there is a point  $u$  such that  $f(a) < u < f(b)$  or  $f(a) > u > f(b)$ , then there is a point  $c \in (a, b)$  with  $f(c) = u$ .

Now, for  $g$  continuous, define  $I = [x_0, x_1]$ . Since  $\{x_0, x_1\}$  forms a 2-cycle of  $f$  we have that

$$\begin{aligned} g(x_0) &= f(x_0) - x_0 = x_1 - x_0 > 0 \\ g(x_1) &= f(x_1) - x_1 = x_0 - x_1 = -g(x_0) < 0. \end{aligned}$$

Therefore, by the Intermediate Value Theorem, since  $0 \in (g(x_1), g(x_0)) = (-g(x_0), g(x_0))$ , there is some point  $c \in (x_0, x_1)$  such that  $g(c) = f(c) - c = 0$ , i.e.  $c$  is a fixed point of  $f$ .

- ii. Suppose that  $f$  is a continuous function meeting the assumptions of the problem. Consider the function  $g(x) = f(x) - x$ , which is continuous by the continuity of  $f$ .

In a manner similar to the one used above, we may use the Intermediate Value Theorem to show that  $f$  has a fixed point on the interval  $I = [x_1, x_2]$ . By assumption we have that  $f(x_1) = x_2$  and  $f(x_2) = x_0$  with  $x_0 < x_1 < x_2$ . Thus, we have that

$$\begin{aligned} g(x_1) &= f(x_1) - x_1 = x_2 - x_1 > 0 \\ g(x_2) &= f(x_2) - x_2 = x_0 - x_2 < 0. \end{aligned}$$

Therefore, by the Intermediate Value Theorem, since  $0 \in (g(x_2), g(x_1))$ , there is some point  $y_0 \in (x_1, x_2)$  such that  $g(y_0) = f(y_0) - y_0 = 0$ , i.e.  $y_0$  is a fixed point of  $f$ .

Now, define the function  $h(x) = f^2(x) - x$ . This function is continuous since  $f$  is continuous and the composition of continuous functions is continuous. Consider the interval  $I = [x_0, x_1]$ . By assumption we have that  $f(x_0) = x_1$ ,  $f(x_1) = x_2$ , and  $f(x_2) = x_0$  with  $x_0 < x_1 < x_2$ . Thus, we have that

$$\begin{aligned} h(x_0) &= f(f(x_0)) - x_0 = f(x_1) - x_0 = x_2 - x_0 > 0 \\ h(x_1) &= f(f(x_1)) - x_1 = f(x_2) - x_1 = x_0 - x_1 < 0 \end{aligned}$$

Therefore, by the Intermediate Value Theorem, since  $0 \in (h(x_1), h(x_0))$ , there is some point  $y_1 \in (x_0, x_1)$  such that  $h(y_1) = f^2(y_1) - y_1 = 0$ , i.e. there is a point  $x_0 < y_1 < x_1$  such that  $f^2(y_1) = y_1$ .

□

**Problem 2.3.7.** Let  $f(x) = ax^3 + bx + 1$ ,  $a \neq 0$ . If  $\{0, 1\}$  is a 2-cycle for  $f(x)$ , find  $a$  and  $b$  so that the 2-cycle is non-hyperbolic and determine the stability.

*Solution.* Note that  $\{0, 1\}$  is a 2-cycle of  $f$  if  $f(0) = 1$  and  $f(1) = 0$ , i.e. if

$$f(1) = a + b + 1 = 0.$$

Thus,  $a = -b - 1$ . The 2-cycle is non-hyperbolic if  $|f'(0)f'(1)| = 1$ . We see that  $f'(x) = 3ax^2 + b$  so that

$$|f'(0)f'(1)| = |b(3a + b)| = |- (2b^2 + 3b)| = 1.$$

Thus, either  $2b^2 + 3b = 1$  which implies that  $b = (-3 \pm \sqrt{17})/4$  or we have  $2b^2 + 3b = -1$  which implies that  $b = -1$  or  $b = -1/2$ . Note that if  $b = -1$ , then  $a = 0$  which violates our assumptions so we eliminate this choice. The other three possible functions are listed below:

$$\begin{aligned} f_1(x) &= \left( \frac{-1 - \sqrt{17}}{4} \right) x^3 + \left( \frac{-3 + \sqrt{17}}{4} \right) x + 1 \\ f_2(x) &= \left( \frac{-1 + \sqrt{17}}{4} \right) x^3 + \left( \frac{-3 - \sqrt{17}}{4} \right) x + 1 \\ f_3(x) &= -\frac{1}{2}x^3 - \frac{1}{2}x + 1. \end{aligned}$$

Recall that a period 2 point  $c$  is stable if  $f^2(c)$  is stable. Note that by construction this 2-cycle is non-hyperbolic for  $f$  and in particular  $(f_1^2(0))' = (f_2^2(0))' = -1$  and  $(f_3^2(0))' = 1$ .

A previous result allows us to determine the stability of  $f_i^2$  by evaluating the derivatives of  $f_i^2$  at  $c = 0$ . In general, for  $f(x) = ax^3 + bx + 1$  we have that  $f'(x) = 3ax^2 + b$ ,  $f''(x) = 6ax$ , and  $f'''(x) = 6a$  so that

$$\begin{aligned} (f^2(0))' &= f'(f(0))f'(0) &= b^2 + 3ab \\ (f^2(0))'' &= f''(f(0))f'(0)^2 + f''(0)f'(f(0)) &= 6ab^2 \\ (f^2(0))''' &= f'''(f(0))f'(0)^3 + f'''(0)f'(f(0)) + 3f''(f(0))f''(0)f'(0) &= 6ab^3 + 6a(3a + b). \end{aligned}$$

Thus, we see that

$$\begin{aligned} (f_1^2(0))' &= -1 & (f_2^2(0))' &= -1 & (f_3^2(0))' &= 1 \\ (f_1^2(0))'' &= \frac{3(19-5\sqrt{17})}{8} & (f_2^2(0))'' &= \frac{3(19+5\sqrt{17})}{8} & (f_3^2(0))'' &= -3/4 \\ (f_1^2(0))''' &= \frac{9(3+11\sqrt{17})}{16} & (f_2^2(0))''' &= \frac{-9(-3+11\sqrt{17})}{16} & (f_3^2(0))''' &= 51/8. \end{aligned}$$

Since  $f_3^2(0)' = 1$  and  $f_3^2(0)'' < 0$ , we have by a previous theorem that the point 0 is one-sided stable to the left of 0 for  $f_3^2(x)$  and hence the 2-cycle  $\{0, 1\}$  is one-sided asymptotically stable to the left of 0.

Note that when  $g'(x) = -1$ , the Schwarzian derivative of a function is given by  $Sg(x) = -g'''(x) - (3/2)g''(x)^2$ . Thus, we see that

$$\begin{aligned} Sf_1^2(0) &= -(f_1^2(0))''' - \frac{3}{2}((f_1^2(0))'')^2 = \frac{9(-1191 + 241\sqrt{17})}{64} < 0 \\ Sf_2^2(0) &= -(f_2^2(0))''' - \frac{3}{2}((f_2^2(0))'')^2 = \frac{9(-1191 - 241\sqrt{17})}{64} < 0 \end{aligned}$$

so that by a previous theorem, the point 0 is asymptotically stable for  $f_1^2(x)$  and  $f_2^2(x)$ . Thus, the 2-cycle  $\{0, 1\}$  is asymptotically stable for  $f_1(x)$  and  $f_2(x)$ .  $\square$

**Problem 2.3.17.** Suppose that  $f(x) = ax^2 + bx + c$ ,  $a \neq 0$  has a 2-cycle  $\{x_0, x_1\}$ . Show that the 2-cycle cannot be non-hyperbolic of the type  $f'(x_0)f'(x_1) = 1$ .

*Solution.*

□



**Problem 2.3.18.** Let  $f(x)$  be a polynomial for which  $g(x) = f^2(x) - x$  has a repeated root at  $x_0$  (where  $f(x_0) = x_1 \neq x_0$ ). Show that  $\{x_0, x_1\}$  is a non-hyperbolic 2-cycle for  $f$  of the type where  $f'(x_0)f'(x_1) = 1$ . Does the converse hold?

*Solution.*

□

**Problem 2.4.1.** Let  $f_c(x) = x^2 + c$ ,  $c \in \mathbb{R}$ .

- i. For what values of  $c$  does  $f_c$  have a super-attracting fixed point and what is the fixed point?
- ii. For what values of  $c$  does  $f_c$  have a super-attracting 2-cycle and what is the 2-cycle?
- iii. Show that if  $f_c$  has a super-attracting 3-cycle, then  $c$  satisfies the equation

$$c^3 + 2c^2 + c + 1 = 0$$

and the 3-cycle is given by  $\{0, c, c^2 + c\}$ .

*Solution.* i. As was shown in problem 1.2.1, we know that  $f_c : \mathbb{R} \rightarrow \mathbb{R}$  with  $f_c(x) = x^2 + c$  has two fixed points given by

$$x_1 = \frac{1 - \sqrt{1 - 4c}}{2}, \quad x_2 = \frac{1 + \sqrt{1 - 4c}}{2} \quad (2)$$

provided that  $c \leq 1/4$ .

The fixed point  $x$  will be a super-attracting fixed point if  $f'_c(x) = 0$ . We note that  $f'_c(x) = 2x$  so that  $f'_c(x) = 0$  only if  $x = 0$ . There is no real value of  $c$  that will allow  $x_2 = 0$  so  $x_2$  is never a super-attracting fixed point. On the other hand, if  $c = 0$ , then  $x_1 = 0$  is a super-attracting fixed point.

- ii. Note that  $f_c$  will have a super-attracting 2-cycle if  $f_c^2$  has a super-attracting period 2 point. A point  $x$  will be a super-attracting period 2 point if  $f_c^2(x) = x$  with  $f_c(x) \neq x$  and if  $(f_c^2(x))' = 0$ .

Since (2) are fixed points, we know that  $(x - x_1)$  and  $(x - x_2)$  must factor  $f_c^2(x) - x$  so that

$$\begin{aligned} f_c^2(x) - x &= (x^2 + c)^2 + c - x \\ &= (x - x_1)(x - x_2) \left( x - \frac{-1 + \sqrt{-3 - 4c}}{2} \right) \left( x - \frac{-1 - \sqrt{-3 - 4c}}{2} \right). \end{aligned}$$

Thus,

$$\{x_3, x_4\} = \left\{ \frac{-1 + \sqrt{-3 - 4c}}{2}, \frac{-1 - \sqrt{-3 - 4c}}{2} \right\}$$

forms a 2-cycle of  $f_c$ . To analyze when this 2-cycle is super attracting, we analyze when

$$\begin{aligned} (f_c^2(x_3))' &= f'_c(x_3)f'_c(x_4) \\ &= (-1 - \sqrt{-3 - 4c})(-1 + \sqrt{-3 - 4c}) \\ &= 4(1 + c) = 0. \end{aligned}$$

We readily see that  $(f_c^2(x_3))' = 0$  only if  $c = -1$  so that only  $f_c(x) = x^2 - 1$  has a super-attracting 2-cycle given by  $\{0, -1\}$ .

- iii. If  $f_c$  has a 3-cycle then  $f_c$  has a period 3 point  $x_0$  with  $f_c^3(x_0) = x_0$  such that  $f_c(x_0) = x_1 \neq x_0$  and  $f_c^2(x_0) = x_2 \neq x_0$ . Note that  $f'_c(x) = 2x$ . Thus, this 3-cycle is super attracting if

$$(f_c^3(x_0))' = f'_c(x_0)f'_c(x_1)f'_c(x_2) = 2^3x_0x_1x_2 = 0$$

which implies that  $x_0 = 0$ ,  $x_1 = 0$ , or  $x_2 = 0$ . Without loss of generality, we may assume that  $x_0 = 0$ . Using the fact that  $f_c(x_0) = x_1 \neq x_0$  and  $f_c^2(x_0) = x_2 \neq x_0$ , we see that

$$\begin{aligned} x_1 &= f_c(x_0) = x_0^2 + c = c \\ x_2 &= f_c^2(x_0) = (x_0^2 + c)^2 + c = c^2 + c \end{aligned}$$

In order for this to be a 3-cycle, we require that  $f_c^3(x_0) = x_0 = 0$ , i.e. we require that

$$\begin{aligned} f_c^3(x_0) &= f_c(f_c^2(x_0)) = (c^2 + c)^2 + c \\ &= c^4 + 2c^3 + c^2 + c \\ &= c(c^3 + 2c^2 + c + 1) = 0. \end{aligned}$$

However we must have that  $c \neq 0$  or  $x_0 = 0$  would not generate a 3-cycle. Thus, we require that  $(c^3 + 2c^2 + c + 1) = 0$ . If this condition is met and  $f_c$  has a super-attracting 3-cycle, then that 3-cycle is given by  $\{0, c, c^2 + c\}$ .

□