Exam 1

Matthew Tiger

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Problem 1. You pay into an annuity a sum of $\$P$ dollars. This annuity pays you $\$\alpha$ per month. The annual interest is $r\%$ and is calculated as simple interest on the remaining balance at the end of each year. If $A(n)$ is the amount remaining at the end of the n-th month, with $A(0) = P$, write down $A(n+1)$ in terms of $A(n)$ and deduce a closed form solution for $A(n)$.

If $P = \$100,000$, $\alpha = \$500$, and the interest rate is 4% per year, how long will the annuity last?

Solution. Let $A(n)$ be the amount remaining in the annuity at the end of month n. If the amount initially paid into the annuity is $\$P$, then $A(0) = P$. If the annual interest rate is $r\%$, then the monthly interest rate is $r/12\%$. Assuming each month a payment of $\$\alpha$ is taken from the annuity, a difference equation representing the amount remaining in the annuity at the end of month n is given by

$$\begin{aligned} A(n+1) &= A(n) + A(n)\left[\frac{r}{12(100)}\right] - \alpha \\ &= \left[1 + \frac{r}{12(100)}\right] A(n) - \alpha \end{aligned}$$

for $n \in \mathbb{N}$.

Using the closed form solution for difference equations in the form of affine maps, the solution to the difference equation is given by

$$\begin{aligned} A(n) &= \left(A(0) + \frac{\alpha}{1 - \left(1 + \frac{r}{12(100)}\right)}\right)\left(1 + \frac{r}{12(100)}\right)^n - \frac{\alpha}{1 - \left(1 + \frac{r}{12(100)}\right)} \\ &= \left(P - \frac{1200\alpha}{r}\right)\left(1 + \frac{r}{1200}\right)^n + \frac{1200\alpha}{r}. \end{aligned}$$

The annuity will run out after $k \in \mathbb{R}$ months when $A(k) = 0$ from which we can gather that the annuity will run out after $n = \lceil k \rceil$ full months. Solving

$$A(k) = \left(100000 - \frac{1200(500)}{4}\right)\left(1 + \frac{4}{1200}\right)^k + \frac{1200(500)}{4} = 0$$

shows that $k = 330.133$. Therefore, the annuity will last for 331 months.

$\square$

Problem 2. Let $g_\mu(x) = \mu x \dfrac{(1-x)}{(1+x)}$, for $\mu > 0$.

a) Show that g_μ has a maximum at $x = \sqrt{2} - 1$ and the maximum value is $\mu(3 - 2\sqrt{2})$.

b) Deduce that g_μ is a dynamical system on $[0, 1]$ for $0 \le \mu \le 3 + 2\sqrt{2}$, i.e. $g_\mu([0, 1]) \subseteq [0, 1]$.

c) Find the fixed points of g_μ for $\mu \ge 1$.

d) Find g'_μ and determine whether the fixed points are attracting or repelling.

e) Use a graphing utility to graph g^2_μ and g^3_μ and estimate when a period 2 point is created.

Solution. a) If $g_\mu(x) = \mu x \dfrac{(1-x)}{(1+x)}$, then we see that

$$\begin{aligned} g'_\mu(x) &= \mu\left[\frac{(1-x)}{(1+x)} - \frac{2x}{(1+x)^2}\right] \\ &= \mu\left[\frac{-x^2 - 2x + 1}{(1+x)^2}\right]. \end{aligned} \tag{1}$$

Thus, $g'_\mu(x) = 0$ if $x = \pm\sqrt{2} - 1$. Since $g'_\mu(0) = \mu > 0$ with $0 < \sqrt{2} - 1$ and $g'_\mu(1) = -\mu/2 < 0$ for $\sqrt{2} - 1 < 1$, we see that $x = \sqrt{2} - 1$ is a local maximum of $g_\mu(x)$. The maximum value is thus given by

$$g_\mu(\sqrt{2} - 1) = \mu(\sqrt{2} - 1)\frac{(1 - (\sqrt{2} - 1))}{(1 + (\sqrt{2} - 1))} = \mu(3 - 2\sqrt{2}).$$

b) The function $g_\mu : [0, 1] \to [0, 1]$ will be a dynamical system for $0 \le \mu \le 3 + 2\sqrt{2}$ if $g_\mu([0, 1]) \subseteq [0, 1]$. Note that on $[0, 1]$, we have that the global minimum of g_μ is 0 and can easily see using the previous result that the global maximum of g_μ is $\mu(3-2\sqrt{2})$. Thus, since g_μ is continuous, we must have that $g_\mu([0, 1]) = [0, \mu(3-2\sqrt{2})]$. If $0 \le \mu \le 3 + 2\sqrt{2}$, we see that

$$0 \le \mu(3 - 2\sqrt{2}) \le (3 + 2\sqrt{2})(3 - 2\sqrt{2}) = 1.$$

Therefore, $g_\mu([0, 1]) = [0, \mu(3 - 2\sqrt{2})] \subseteq [0, 1]$ and g_μ is a dynamical system on $[0, 1]$.

c) Suppose that $\mu \ge 1$. The fixed points of g_μ are the roots of the function

$$f(x) = g_\mu(x) - x = -\frac{x[x(\mu+1) - (\mu-1)]}{(x+1)}.$$

Thus, the fixed points of g_μ are given by

$$x_0 = 0 \quad \text{and} \quad x_1 = \frac{\mu - 1}{\mu + 1}. \tag{2}$$

d) Recall that a fixed point c of a function f that is hyperbolic is attracting if $|f'(c)| < 1$ and repelling if $|f'(c)| > 1$. The derivative of g_μ is provided by (1). Thus, we readily see that for the fixed points provided by (2) that

$$|g'_\mu(x_0)| = |g'_\mu(0)| = |\mu|$$

and

$$\begin{aligned} |g'_\mu(x_1)| &= \left| g'_\mu\left(\frac{\mu-1}{\mu+1}\right) \right| \\ &= \frac{1}{2}\left| \left(-\mu + \frac{1}{\mu} + 2\right) \right|. \end{aligned}$$

Consider $\mu \geq 1$. We see that if $\mu > 1$ then the fixed point x_0 will be a hyperbolic fixed point and will be repelling. If, however, $\mu = 1$, we see that $g'_\mu(x_0) = 1$ and x_0 is a non-hyperbolic fixed point. We rely on a previous theorem that states that we can use the second and third derivative of g_μ in order to classify the non-hyperbolic fixed point. Note that

$$g''_\mu(x) = -\frac{4\mu}{(1+x)^3} \quad \text{and} \quad g'''_\mu(x) = \frac{12\mu}{(1+x)^4}. \tag{3}$$

Since $g''_\mu(x_0) = -4\mu = -4 < 0$ for $\mu = 1$, the fixed point $x_0 = 0$ is one-sided asymptotically stable to the right of 0.

For the fixed point x_1, we see that if $1 < \mu < 2 + \sqrt{5}$, then $|g'_\mu(x_1)| < 1$ so that x_1 is a hyperbolic, attracting fixed point. On the other hand, if $2 + \sqrt{5} < \mu$, then $|g'_\mu(x_1)| > 1$ so that x_1 is a hyperbolic, repelling fixed point. In the case that $\mu = 1$ or $\mu = 2 + \sqrt{5}$, the fixed point x_1 is non-hyperbolic.

If $\mu = 1$, we see that $x_1 = 0 = x_0$ and so it must have the same classification as x_0 when $\mu = 1$, i.e. it is a non-hyperbolic fixed point that is one-sided asymptotically stable to the right of 0. If $\mu = 2 + \sqrt{5}$, then we see that $g'_\mu(x_1) = -1$. Note that we can use the Schwarzian derivative of g_μ to classify this non-hyperbolic fixed point. The Schwarzian derivative of g_μ evaluated at x_1 is given by

$$\begin{aligned} Sg_\mu(x_1) &= -g'''_\mu(x_1) - \frac{3g''_\mu(x_1)^2}{2} \\ &= 6 - 6\sqrt{5} - \frac{3(-4)^2}{2} \\ &= -18 - 6\sqrt{5}. \end{aligned}$$

Since $Sg_\mu(x_1) < 0$, the fixed point x_1 is asymptotically stable when $\mu = 2 + \sqrt{5}$.

e) Using the Mathematica `Manipulate` command, we can plot the parametric families g_μ^2 and g_μ^3 for $0 \leq \mu \leq 3+2\sqrt{2}$. After plotting these families we see that a bifurcation point for the system occurs approximately when $\mu \approx 4.23607$. For values of $\mu > 4.23607$ a 2-cycle is born for the dynamical system.

□

Problem 3. Consider the family of functions $f_\lambda(x) = x^3 - \lambda x$ for some parameter $\lambda \in \mathbb{R}$.

a) Find all fixed points and determine their nature and where they are created as λ varies.

b) Find where a 2-cycle is created and give the graph of where this happens. Determine the stability of the hyperbolic 2-cycles.

c) Use a graphing utility to find an approximate value of λ where the 3-cycle is created. Give the graph of this situation.

Solution. a) The fixed points of f_λ are the roots of the function

$$\begin{aligned} g_\lambda(x) &= f_\lambda(x) - x \\ &= x(x^2 - \lambda - 1). \end{aligned}$$

Thus, the fixed points of f_λ are $x_0 = 0$, $x_1 = \sqrt{\lambda+1}$, and $x_2 = -\sqrt{\lambda+1}$. Note that the points x_1 and x_2 are real only if $\lambda \geq -1$, i.e. the points are only fixed points of the dynamical system if $\lambda \geq -1$.

Using the first derivative of f_λ, we can classify the above fixed points when they are hyperbolic. If the fixed point is non-hyperbolic, we can use the second and third derivatives when the fixed point is non-hyperbolic of the type $f'_\lambda(x) = 1$, and the Schwarzian derivative when the fixed point is non-hyperbolic of the type $f'_\lambda(x) = -1$. Note that

$$\begin{aligned} f'_\lambda(x) &= 3x^2 - \lambda \\ f''_\lambda(x) &= 6x \\ f'''_\lambda(x) &= 6. \end{aligned}$$

If $f'_\lambda(x) = -1$, we see that the Schwarzian derivative of f_λ is given by

$$\begin{aligned} Sf_\lambda(x) &= -f'''_\lambda(x) - \frac{3}{2}\left[f''_\lambda(x)\right]^2 \\ &= -6 - 54x^2. \end{aligned}$$

For the fixed point $x_0 = 0$, we see that $|f'_\lambda(x_0)| = |\lambda|$. Thus, the fixed point x_0 is a hyperbolic fixed point if $\lambda \neq -1$ or $\lambda \neq 1$. If $|\lambda| < 1$, then x_0 is asymptotically stable and if $|\lambda| > 1$, then x_0 is an unstable fixed point. If $\lambda = -1$, then $f'_\lambda(x_0) = 1$. Since $f''_\lambda(x_0) = 0$ and $f'''_\lambda(x_0) = 6 > 0$, the fixed point x_0 is unstable. If $\lambda = 1$, then $f'_\lambda(x_0) = -1$. The Schwarzian derivative of f_λ at x_0 is then $Sf_\lambda(x_0) = -6 < 0$. Therefore, the fixed point x_0 is an asymptotically stable fixed point.

Consider now the fixed point $x_1 = \sqrt{\lambda+1}$ for $\lambda \geq -1$. We readily see that $|f'_\lambda(x_1)| = |3 + 2\lambda|$. If $\lambda > -1$, then $|f'_\lambda(x_1)| > 1$ and x_1 is hyperbolic and unstable. If $\lambda = -1$, then $x_1 = 0 = x_0$ and from the previous classification of the fixed point x_0, we know that x_1 is unstable.

Lastly, consider the fixed point $x_2 = -\sqrt{\lambda+1}$ for $\lambda \geq -1$. We thus have that $|f'_\lambda(x_2)| = |3 + 2\lambda|$ and the same classification for x_1 holds for x_2, i.e. the fixed point x_2 is hyperbolic and unstable if $\lambda > -1$ and non-hyperbolic and unstable if $\lambda = -1$.

b) Recall that a point x is a period 2 point of f_λ if $f_\lambda^2(x) = x$ and $f_\lambda(x) \neq x$. The 2-cycle associated to the period 2 point is then $\{x, f_\lambda(x)\}$. We thus look for solutions to the equation

$$\begin{aligned} f_\lambda^2(x) - x &= \left(x^3 - \lambda x\right)^3 - \lambda\left(x^3 - \lambda x\right) - x \\ &= x^9 - 3\lambda x^7 + 3\lambda^2 x^5 - \lambda^3 x^3 - \lambda x^3 + \lambda^2 x - x \\ &= x(x^4 - \lambda x^2 + 1)(x^2 - \lambda - 1)(x^2 - \lambda + 1) = 0. \end{aligned} \tag{4}$$

Suppose first that $\lambda < -1$. Then the only fixed point of the function f_λ is $x_0 = 0$ so that $x = 0$ can be factored out of (4) since the solutions we seek satisfy $f_\lambda(x) \neq x$. After factoring x out from the above polynomial we have that

$$(x^4 - \lambda x^2 + 1)(x^2 - \lambda - 1)(x^2 - \lambda + 1) = 0.$$

However, if $\lambda < -1$, then $(x^4 - \lambda x^2 + 1) = 0$, $(x^2 - \lambda - 1) = 0$, and $(x^2 - \lambda + 1) = 0$, all have no real solutions. Therefore, if $\lambda < -1$, then f_λ has no period 2 points.

Now consider $\lambda \geq -1$. Then for similar reasons we can factor $(x - x_0)(x - x_1)(x - x_2)$, where x_i for $i = 0, 1, 2$ are fixed points, out of (4) and thus see that

$$(x^4 - \lambda x^2 + 1)(x^2 - \lambda + 1) = 0$$

To continue, we note that the first polynomial, say $g(x) = x^4 - \lambda x^2 + 1$, only has real solutions if $\lambda \geq 2$ and the second polynomial, say $h(x) = (x^2 - \lambda + 1)$, only has real solutions if $\lambda \geq 1$. Thus, for $-1 \leq \lambda < 1$ there are no period 2 points.

If $1 \leq \lambda < 2$, then $h(x) = 0$ if $x = \pm\sqrt{\lambda - 1}$. Thus, $\{\sqrt{\lambda - 1}, -\sqrt{\lambda - 1}\}$ is a 2-cycle of f_λ.

If on the other hand $\lambda \geq 2$, then $h(x) = 0$ has real solutions and the previous 2-cycle is still a 2-cycle of f_λ. However, $g(x) = 0$ also real solutions. These are given by

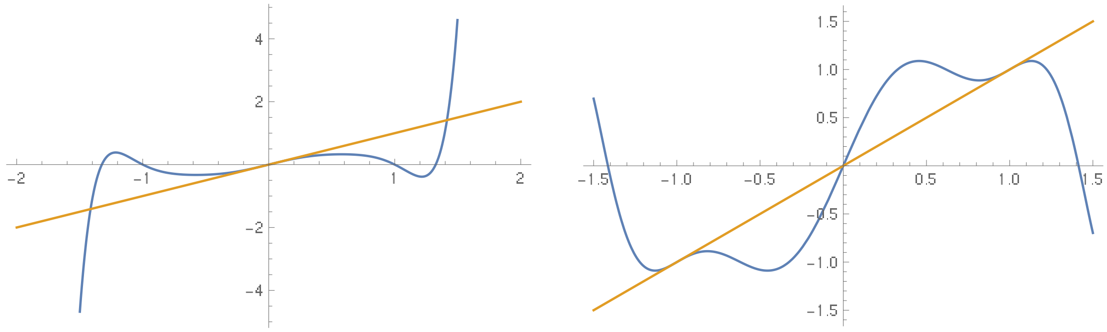
$$\begin{aligned} y_0 &= -\frac{\sqrt{\lambda - \sqrt{\lambda^2 - 4}}}{\sqrt{2}}, \quad y_1 = \frac{\sqrt{\lambda - \sqrt{\lambda^2 - 4}}}{\sqrt{2}} \\ y_2 &= -\frac{\sqrt{\lambda + \sqrt{\lambda^2 - 4}}}{\sqrt{2}}, \quad y_3 = \frac{\sqrt{\lambda + \sqrt{\lambda^2 - 4}}}{\sqrt{2}}. \end{aligned}$$

Since $f_\lambda^2(y_0) = y_0$ and $f_\lambda(y_0) = y_3 \neq y_0$, we have that $\{y_0, y_3\}$ is an additional 2-cycle. Similarly, since $f_\lambda^2(y_1) = y_1$ and $f_\lambda(y_1) = y_2 \neq y_1$, we have that $\{y_1, y_2\}$ is the last 2-cycle.

We now present the graphs of the bifurcation points $\lambda = 1$ and $\lambda = 2$ that indicate the birth of new 2-cycles in figure 1.

In figure 2, we can see where the two cycles actually arise for values of λ that occur between the bifurcations points $\lambda = 1$ and $\lambda = 2$.

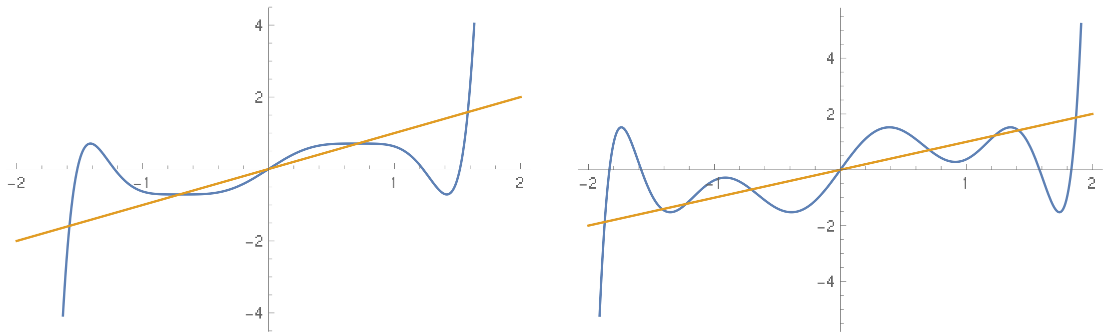
We will now determine the stability of the hyperbolic two cycle $\{z_0, z_1\} = \{\sqrt{\lambda - 1}, -\sqrt{\lambda - 1}\}$ when $1 \leq \lambda < 2$ and the stability of the hyperbolic two cycles $\{z_0, z_1\}$, $\{y_0, y_3\}$, and $\{y_1, y_2\}$ when $\lambda \geq 2$.



(a) The graphs of $f_\lambda^2(x)$ (blue) and $y = x$ (orange) for $\lambda = 1$.

(b) The graphs of $f_\lambda^2(x)$ (blue) and $y = x$ (orange) for $\lambda = 2$.

Figure 1: The graphs of f_λ^2 at the bifurcation points $\lambda = 1$ and $\lambda = 2$ for the birth of 2-cycles.



(a) The graphs of $f_\lambda^2(x)$ (blue) and $y = x$ (orange) for $\lambda = 3/2$.

(b) The graphs of $f_\lambda^2(x)$ (blue) and $y = x$ (orange) for $\lambda = 5/2$.

Figure 2: The graphs of f_λ^2 for values of λ different from the bifurcation points $\lambda = 1$ and $\lambda = 2$.

Recall that for a function g that a 2-cycle $\{z_0, z_1\}$ is hyperbolic and stable if z_0 is a stable fixed point of g^2, i.e. if

$$|(g^2(z_0))'| = |g'(g(z_0))g'(z_0)| = |g'(z_0)g'(z_1)| < 1.$$

Note that $f_\lambda'(x) = 3x^2 - \lambda.$, Thus, we see for the period 2 point z_0 that

$$\begin{aligned}
\left|(g^2(z_0))'\right| &= \left|g'\left(\sqrt{\lambda - 1}\right) g'\left(-\sqrt{\lambda - 1}\right)\right| \\
&= \left|\left(3\left(\sqrt{\lambda - 1}\right)^2 - \lambda\right)\left(3\left(-\sqrt{\lambda - 1}\right)^2 - \lambda\right)\right| \\
&= \left|(2\lambda - 3)^2\right|.
\end{aligned}$$

Similarly for the period 2 point y_0 we have that

$$\begin{aligned}
\left|(g^2(y_0))'\right| &= \left|g'\left(-\frac{\sqrt{\lambda-\sqrt{\lambda^2-4}}}{\sqrt{2}}\right) g'\left(\frac{\sqrt{\lambda+\sqrt{\lambda^2-4}}}{\sqrt{2}}\right)\right| \\
&= \left|\left(\frac{3\left(-\sqrt{\lambda^2-4}+\lambda\right)}{2}-\lambda\right)\left(\frac{3\left(\sqrt{\lambda^2-4}+\lambda\right)}{2}-\lambda\right)\right| \\
&= \left|-2\lambda^2+9\right|
\end{aligned}$$

and for the period 2 point y_1 we have that

$$\begin{aligned}
\left|(g^2(y_1))'\right| &= \left|g'\left(\frac{\sqrt{\lambda-\sqrt{\lambda^2-4}}}{\sqrt{2}}\right) g'\left(-\frac{\sqrt{\lambda+\sqrt{\lambda^2-4}}}{\sqrt{2}}\right)\right| \\
&= \left|\left(\frac{3\left(-\sqrt{\lambda^2-4}+\lambda\right)}{2}-\lambda\right)\left(\frac{3\left(\sqrt{\lambda^2-4}+\lambda\right)}{2}-\lambda\right)\right| \\
&= \left|-2\lambda^2+9\right|.
\end{aligned}$$

For the 2-cycle $\{z_0, z_1\}$ of f_λ, we see that $|(g^2(z_0))'| = |(2\lambda-3)^2| < 1$ only if $1 < \lambda < 2$. Therefore, $\{z_0, z_1\}$ is a hyperbolic, stable 2-cycle if $1 < \lambda < 2$.

For the other 2-cycles $\{y_0, y_3\}$ and $\{y_1, y_2\}$, we see that $|(g^2(y_0))'| = |(g^2(y_1))'| = |-2\lambda^2+9| < 1$ only if $2 < \lambda < \sqrt{5}$. Therefore, it is for these values of λ that the 2-cycles $\{y_0, y_3\}$ and $\{y_1, y_2\}$ are hyperbolic and stable.

c) The plot in figure 3 shows that when $\lambda \approx 2.6995$, the graph of f_λ^3 touches the line $y = x$ at 6 points that differ from the fixed points of f_λ. Therefore, it is around this value of λ that two 3-cycles occur for f_λ.

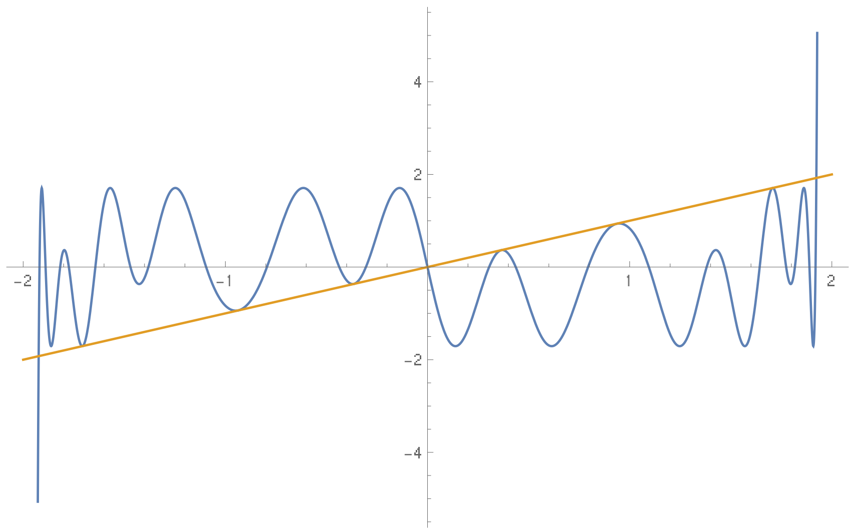


Figure 3: The graphs of f_λ^3 and $y = x$ for $\lambda = 2.6995$.

□

Problem 4. Let f be a 4-times continuously differentiable function. Its Newton function is $N_f(x) = x - f(x)/f'(x)$. Suppose that c is a zero of f. If $Sf(x)$ is the Schwarzian derivative of f, show that

$$N_f'''(c) = 2Sf(c)$$

Solution. □

Problem 5. Let $f : [0,1] \to [0,1]$ be continuous on $[0,1]$ and differentiable on (0, 1) with $|f'(x)| < 1$ for all $x \in (0,1)$.

a) Prove that f has a unique fixed point p in $[0,1]$.

b) Prove that f cannot have a point of period 2 in $[a,b]$.

c) Prove that $f^n(x) \to p$ as $n \to \infty$ for all $x \in (0,1)$.

Solution. □

Problem 6. Let $f(x) = ax^3 + bx + c$ where a and b satisfy $a/b > 0$. Denote by N_f the corresponding Newton function.

a) Show that N_f has a unique fixed point.

b) Show that N_f cannot have any period 2 points.

c) Why does it follow that N_f has no points of period n for $n > 2$?

Solution. □

Problem 7. a) Show that the function $f(x) = -1/(x+1)$ has the property that $f^3(x) = x$ for all $x \neq -1, 0$.

b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined on a set I, with $f^3(x) = x$ for all $x \in I$. Set $g(x) = f^2(x)$. Show that $g^3(x) = x$ for all $x \in I$. Deduce a function different from that in a) that has this property.

c) In general, show that such a function cannot have a 2-cycle.

d) Deduce that a function $f : \mathbb{R} \to \mathbb{R}$ with the property $f^3(x) = x$ cannot be continuous.

e) Show that the inverse of f must exist.

f) If $f'(x)$ exists for all $x \in I$, show that the 3-cycles are non-hyperbolic where f is not the identity map.

g) Suppose that $f(x) = \dfrac{ax+b}{cx+d}$ satisfies $f^3(x) = x$. Show that if f is not the identity map and $a \neq d$, then $a^2 + bc + ad + d^2 = 0$.

  i) Use this to find other functions with the property $f^3(x) = x$.

  ii) Deduce that if $ad - bc > 0$, then such a function cannot have any fixed points.

Solution. □