Midterm 1

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Problem 1.a. Consider the process

$$X_t + 0.4X_{t-1} - 0.32X_{t-2} = Z_t - 0.8Z_{t-1} + 0.16Z_{t-2}. \tag{1}$$

Determine whether the model is a stationary process.

Solution. The model $\{X_t\}$ is a stationary process if $\{X_t\}$ is a stationary solution of the equations (1). By the existence and uniqueness theorem of ARMA(p, q) processes, a stationary solution $\{X_t\}$ of the equations

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$$

that define the model exists if and only if

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0 \quad \text{for all } |z| = 1,$$

i.e. if and only if the roots of $\phi(z)$ do not lie on the unit circle.

For our model, we have $\phi_1 = -0.4$ and $\phi_2 = 0.32$ so that $\phi(z) = 1 + 0.4z - 0.32z^2$. Note that the roots of $\phi(z)$ are $z_1 = -1.25$ and $z_2 = 2.5$. As $|z_i| \neq 1$ for $i = 1, 2$, we conclude that the roots of $\phi(z)$ do not lie on the unit circle and that the model $\{X_t\}$ is a stationary process assuming that $\{Z_t\} \sim \text{WN}(0, \sigma^2)$. □

Problem 1.b. Considering the model in problem 1.a, what is R_3, i.e. the correlation matrix of size 3?

Solution. The covariance matrix of size 3 for our model $\{X_t\}$ is given by

$$\Gamma_3 = \begin{bmatrix} \gamma(0) & \gamma(1) & \gamma(2) \\ \gamma(1) & \gamma(0) & \gamma(1) \\ \gamma(2) & \gamma(1) & \gamma(0) \end{bmatrix}$$

where $\gamma(h)$ is the autocovariance function of the process $\{X_t\}$. For an ARMA(p, q) process $X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$, the autocovariance function $\gamma(h)$ satisfies the equations

$$\gamma(k) - \phi_1 \gamma(k-1) - \cdots - \phi_p \gamma(k-p) = \sigma^2 \sum_{j=0}^{\infty} \theta_{k+j} \psi_j \quad \text{for } 0 \leq k < \max(p, q+1)$$

where $\psi_j - \sum_{k=1}^{p} \phi_k \psi_{j-k} = \theta_j$ for $j \geq 0$ and $\psi_j = 0$ for $j < 0$. For our process, this corresponds to the system of equations

$$\begin{aligned}
\gamma(0) - \phi_1\gamma(1) - \phi_2\gamma(2) &= \sigma^2(\psi_0 + \theta_1\psi_1 + \theta_2\psi_2) \\
\gamma(1) - \phi_1\gamma(0) - \phi_2\gamma(1) &= \sigma^2(\theta_1\psi_0 + \theta_2\psi_1) \\
\gamma(2) - \phi_1\gamma(1) - \phi_2\gamma(0) &= \sigma^2\theta_2\psi_0
\end{aligned} \tag{2}$$

where $\psi_0 = 1$, $\psi_1 = \theta_1 + \phi_1$, and $\psi_2 = \theta_2 + \phi_1^2 + \phi_1\theta_1 + \phi_2$. Using the parameters ϕ_j and θ_k defining our model, the system of equations (2) becomes

$$\begin{aligned}
\gamma(0) + 0.4\gamma(1) - 0.32\gamma(2) &= 2.1136\sigma^2 \\
\gamma(1) + 0.4\gamma(0) - 0.32\gamma(1) &= -0.992\sigma^2 \\
\gamma(2) + 0.4\gamma(1) - 0.32\gamma(0) &= 0.16\sigma^2
\end{aligned}$$

the solution of which is $\gamma(0) = 5\sigma^2$, $\gamma(1) = -4.4\sigma^2$, and $\gamma(2) = 3.52\sigma^2$. Thus, the covariance matrix Γ_3 is given by

$$\Gamma_3 = \sigma^2 \begin{bmatrix} 5.00 & -4.40 & 3.52 \\ -4.40 & 5.00 & -4.40 \\ 3.52 & -4.40 & 5.00 \end{bmatrix}.$$

Note that the correlation matrix R_3 is given by $(1/\gamma(0))\Gamma_3$. Therefore,

$$R_3 = \begin{bmatrix} 1.000 & -0.880 & 0.704 \\ -0.880 & 1.000 & -0.880 \\ 0.704 & -0.880 & 1.000 \end{bmatrix}.$$

□

Problem 1.c. Express the process in problem 1.a as a pure MA process in the form of $X_t = \sum_{j=0}^{\infty} \psi_j Z_t$.

Solution. For our process, the roots of the equation $\phi(z) = 1 + 0.4z - 0.32z^2 = 0$ are $z_1 = -1.25$ and $z_2 = 2.5$. As $|z_i| > 1$ for $i = 1, 2$, this process is causal and can be represented as an MA(∞) process, i.e. $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$, where the coefficients ψ_j are determined by the equations $\psi_j - \sum_{k=1}^{p} \phi_k \psi_{j-k} = \theta_j$ for $j \geq 0$ and $\psi_j = 0$ for $j < 0$.

Note that for an ARMA(p, q) process, as $\theta_j = 0$ for $j > q$, the equations determining the coefficients are difference equations determined by the boundary conditions

$$\psi_j - \sum_{k=1}^{p} \phi_k \psi_{j-k} = \theta_j \quad \text{for } 0 \leq j < \max(p, q+1)$$

and the homogeneous equation

$$\psi_j - \sum_{k=1}^{p} \phi_k \psi_{j-k} = 0 \quad \text{for } j \geq \max(p, q+1).$$

For our process, the characteristic equation of these difference equations is $\phi(z)$. The roots of this characteristic equation are, as shown above, $z_1 = -1.25$ and $z_2 = 2.5$. As these roots are distinct, the solution to the homogeneous difference equation is

$$\psi_j = \alpha_1 z_1^{-j} + \alpha_2 z_2^{-j} = \alpha_1(-1.25)^{-j} + \alpha_2(2.5)^{-j} \quad \text{for } j \geq 1$$

where the coefficients are determined by the boundary conditions $\psi_0 = 1$, $\psi_1 = \theta_1 + \phi_1 = -1.2$, and $\psi_2 = \theta_2 + \phi_1^2 + \phi_1\theta_1 + \phi_2 = 0.96$. Using the method of undetermined coefficients, we can see that $\alpha_1 = 1.5$ and $\alpha_2 = 0$. Therefore $\psi_j = 1.5(-1.25)^{-j}$ for $j \geq 1$, $\psi_0 = 1$, and

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} = Z_t + 1.5 \sum_{j=1}^{\infty} (-1.25)^{-j} Z_{t-j}.$$

□

Problem 2.a. Let X_t be the AR(2) process such that $X_t = 0.8X_{t-2} + Z_t$ where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$. Find the autocorrelation function of X_t.

Solution. This AR(2) process is defined by the parameters $\phi_1 = 0$ and $\phi_2 = 0.8$. This process has characteristic equation $\phi(z) = 1 - 0.8z^2 = 0$ of which the roots are $z_1 = 1.11803$ and $z_2 = -1.11803$. As these roots lie outside the unit circle this process is causal.

Note that $\{X_t\}$ can be represented as $(1 - \xi_1^{-1}B)(1 - \xi_2^{-1}B)X_t = Z_t$ where $0 = \phi_1 = \xi_1^{-1} + \xi_2^{-1}$ and $0.8 = \phi_2 = -\xi_1^{-1}\xi_2^{-1}$. Thus, $\xi_1^{-1} = -\frac{2}{\sqrt{5}}$ and $\xi_2^{-1} = \frac{2}{\sqrt{5}}$ so

$$X_t - 0.8X_{t-2} = \left(1 + \frac{2}{\sqrt{5}}B\right)\left(1 - \frac{2}{\sqrt{5}}B\right)X_t = Z_t.$$

The covariance function of this AR(2) process is given by

$$\gamma(h) = \frac{\sigma^2 \xi_1^2 \xi_2^2}{(\xi_1\xi_2 - 1)(\xi_2 - \xi_1)} \left[\frac{\xi_1^{1-|h|}}{\xi_1^2 - 1} - \frac{\xi_2^{1-|h|}}{\xi_2^2 - 1}\right].$$

Using $\xi_1 = -\frac{\sqrt{5}}{2}$ and $\xi_2 = \frac{\sqrt{5}}{2}$, we see that for our process,

$$\gamma(h) = \frac{5\sqrt{5}\sigma^2}{9}\left[\left(\frac{\sqrt{5}}{2}\right)^{1-|h|} - \left(\frac{-\sqrt{5}}{2}\right)^{1-|h|}\right].$$

□