

Homework Assignment 7

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November 3, 2015

Problem 9.1. For the following problems for 9.1, suppose a function $f : [a, b] \rightarrow \mathbb{R}$ is only known at distinct sites $x = [x_1, x_2, \dots, x_n]$ where $x_i \in [a, b]$, for $i = 1, 2, \dots, n$. Let $p_n(f, t)$ be the Lagrange interpolating polynomial at these sites.

Problem 9.1.1. Show that the basic quadrature $J(f) := \int_a^b p_n(f, t) dt$ satisfies $J(f) = \sum_{j=1}^n w_j f(x_j)$ where the weights w_j depend on the Lagrange basis.

Solution. Note the Lagrange interpolating polynomial of f through the nodes x_1, x_2, \dots, x_n is given by

$$p_n(f, t) = \sum_{j=1}^n f(x_j) \prod_{\substack{i=1 \\ i \neq j}}^n \frac{t - x_i}{x_j - x_i}.$$

If $J(f) := \int_a^b p_n(f, t) dt$, then, using this definition of the Lagrange interpolating polynomial, it is clear that

$$\begin{aligned} J(f) &= \int_a^b p_n(f, t) dt = \int_a^b \left[\sum_{j=1}^n f(x_j) \prod_{\substack{i=1 \\ i \neq j}}^n \frac{t - x_i}{x_j - x_i} \right] dt \\ &= \sum_{j=1}^n \left[\int_a^b \prod_{\substack{i=1 \\ i \neq j}}^n \frac{t - x_i}{x_j - x_i} dt \right] f(x_j) = \sum_{j=1}^n w_j f(x_j). \end{aligned}$$

Thus, $J(f)$ is of the form $\sum_{j=1}^n w_j f(x_j)$ where w_j depends on the Lagrange basis $l_j(t) = \prod_{\substack{i=1 \\ i \neq j}}^n \frac{t - x_i}{x_j - x_i}$. □

Problem 9.1.2. Show that $J(f)$ has degree of precision at least $n - 1$.

Solution. Let $q(t)$ be a polynomial of degree $n - 1$. Then,

$$q(t) = \sum_{j=1}^n q(x_j) \prod_{\substack{i=1 \\ i \neq j}}^n \frac{t - x_i}{x_j - x_i},$$

i.e. the Lagrange interpolating polynomial of q through the nodes x_1, x_2, \dots, x_n is q itself. Hence, the exact integral of q , $I(q) = \int_a^b q(t) dt$, satisfies

$$\begin{aligned} I(q) &= \int_a^b q(t) dt = \int_a^b \sum_{j=1}^n q(x_j) \prod_{\substack{i=1 \\ i \neq j}}^n \frac{t - x_i}{x_j - x_i} dt \\ &= \sum_{j=1}^n \left[\int_a^b \prod_{\substack{i=1 \\ i \neq j}}^n \frac{t - x_i}{x_j - x_i} dt \right] q(x_j) = J(q). \end{aligned}$$

Since q is a polynomial of degree $n - 1$ and $I(q) = J(q)$, we know that $J(f)$ has degree of precision at least $n - 1$. \square