Homework Assignment 8

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Problem 7.2.2. If $D : [0, 1) \to [0, 1)$ is the doubling map $D(x) = 2x \mod 1$ and $f : S^1 \to S^1$ is the angle doubling map, $f(z) = z^2$, show that f is a factor of D.

Solution. Recall that a dynamical system $f : S^1 \to S^1$ is a factor of the dynamical system $D : [0, 1) \to [0, 1)$ if there exists a continuous, onto function $h : [0, 1) \to S^1$ such that $h \circ D = f \circ h$.

Define $h : [0, 1) \to S^1$ by $h(x) = e^{2\pi ix}$. Then it is easy to see that h is continuous. To show that it is onto, let $z \in S^1$ be given. Then $z = e^{it}$ for some $t \in [0, 2\pi)$. Choose $x \in [0, 1)$ such that $t = 2\pi x$. Then it is clear that $h(x) = e^{2\pi ix} = e^{it} = z$ and h is onto.

Now, we see that

$$f \circ h(x) = f(e^{2\pi ix}) = e^{2\pi ix}$$

and

$$\begin{aligned} h \circ D(x) &= \begin{cases} h(2x) & \text{if } x \in [0, 1/2) \\ h(2x - 1) & \text{if } x \in [1/2, 1) \end{cases} \\ &= \begin{cases} e^{4\pi ix} & \text{if } x \in [0, 1/2) \\ e^{4\pi ix - 2\pi i} & \text{if } x \in [1/2, 1) \end{cases}. \end{aligned}$$

However, $e^{4\pi ix-2\pi i} = e^{-2\pi i}e^{4\pi ix} = e^{4\pi ix}$ so in either case $h \circ D(x) = e^{4\pi ix} = f \circ h(x)$ and f is a factor of D.

$\square$

Problem 7.2.3. i. If $g : S^1 \to S^1$ is defined by $g(z) = z^3$, show that g is the angle-tripling map

ii. Find the periodic points of g and show they are dense in S^1.

iii. Let $F : [0,1) \to [0,1)$ be defined by $F(x) = 3x \mod 1$. Show that g is a factor of F.

Solution. i. If $z \in S^1$, then $z = e^{i\theta}$ for some $\theta \in (-\pi, \pi]$. Note that if $z = x + iy$ for $x, y \in \mathbb{R}$, then θ is the angle between the vector $< x, y >$ and the real line measured counter-clockwise.

So, if $z = e^{i\theta}$, then

$$g(z) = \left(e^{i\theta}\right)^3 = e^{i3\theta}$$

and the angle between the vector $< x, y >$ and the real line measured counter-clockwise has now tripled. Therefore, g is the angle-tripling map.

ii. For the map g, note that 0 is a fixed point and so it cannot be periodic. It is easy to see that if $g(z) = z^3$, then $g^n(z) = z^{3^n}$. Thus, for $z \neq 0$, we have that $g^n(z) = z$ if and only if $z^{3^n} = z$ or $z^{3^n - 1} = 1$. Therefore, the period n points are the $(3^n - 1)$-th roots of unity.

Having identified the periodic points, we see that the periodic points of g are dense in S^1 if for every $z \in S^1$ either z is a $(3^n - 1)$-th root of unity for some n or z is arbitrarily close to some $(3^n - 1)$-th root of unity, i.e. if for every $z \in S^1$ and every $\varepsilon > 0$, there exists some period n point x such that $|z - x| < \varepsilon$.

If $x \in S^1$ then $x = e^{i\theta}$ for some $-\pi < \theta \leq \pi$. If x is a period n point, then $\left(e^{i\theta}\right)^{3n-1} = e^{2\pi i}$ implies that $x = e^{2k\pi i/3^n - 1}$ for some $0 \leq k < 3^n - 1$. Note that the $(3n-1)$-th roots of unity are evenly spaced on the unity circle a distance $2\pi/(3^n - 1)$ apart. Taking n arbitrarily large shows that this distance is arbitrarily small and the distance between any point on the unit circle will be arbitrarily close to a $(3^n - 1)$-th root of unity.

iii. Recall that a dynamical system $g : S^1 \to S^1$ is a factor of the dynamical system $F : [0,1) \to [0,1)$ if there exists a continuous, onto function $h : [0,1) \to S^1$ such that $h \circ F = g \circ h$.

Define $h : [0,1) \to S^1$ by $h(x) = e^{2\pi i x}$. As was shown earlier, this function is continuous and onto.

Now, we see that

$$g \circ h(x) = g(e^{2\pi i x}) = e^{6\pi i x}$$

and

$$h \circ F(x) = \begin{cases} h(3x) & \text{if } x \in [0, 1/3) \\ h(3x - 1) & \text{if } x \in [1/3, 2/3) \\ h(3x - 2) & \text{if } x \in [2/3, 1) \end{cases}$$

$$= \begin{cases} e^{6\pi i x} & \text{if } x \in [0, 1/3) \\ e^{6\pi i x - 2\pi i} & \text{if } x \in [1/3, 2/3) \\ e^{6\pi i x - 4\pi i} & \text{if } x \in [2/3, 1) \end{cases}.$$

Note that $e^{2k\pi i} = 1$ for all $k \in \mathbb{Z}$, so in either case $h \circ F(x) = e^{6\pi i x} = g \circ h(x)$ and g is a factor of F.

$\square$

Problem 7.2.4.

Solution. □

Problem 7.3.2.

Solution. □

Problem 7.3.4.

Solution. □

Problem 7.3.5.

Solution. $\square$