Homework Assignment 6

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Problem 4.2.1. Prove that every open ball $B_\varepsilon(a)$ in a metric space (X, d) is an open set and that every finite subset of X is a closed set.

Solution. $\square$

Problem 4.2.2. Show that the closed ball $B_\varepsilon[a] = \{x \in X \mid d(a,x) \le \varepsilon\}$ in a metric space is a closed set, but it need not be equal to the closure of the open ball $B_\varepsilon(a)$. (Hint: Consider the two point space $\mathcal{A} = \{0,1\}$ with metric $d(0,1) = 1$).

Solution. □

Problem 4.2.5. Show that the intersection of a finite number of open sets $A_1, A_2, \ldots, A_n$ in a metric space (X, d) is an open set. Show that, by considering the intervals $(-1/n, 1/n)$ for all $n \in \mathbb{Z}^+$ in $\mathbb{R}$, the intersection of infinitely many open sets need not be open.

Solution. $\square$

Problem 4.2.6. If $\mathcal{A} = \{0, 1\}$, then $\mathcal{A}^{\mathbb{N}}$ denotes the metric space of 0's and 1's:

$$\mathcal{A}^{\mathbb{N}} = \{\omega = (a_0, a_1, a_2, \dots) \mid a_i = 0 \text{ or } a_i = 1\},$$

with metric:

$$d(\omega_1, \omega_2) = \sum_{k=0}^{\infty} \frac{|s_k - t_k|}{2^k},$$

where $\omega_1 = (s_0, s_1, s_2, \dots)$ and $\omega_2 = (t_0, t_1, t_2, \dots)$.

Show that $\mathcal{A}^{\mathbb{N}}$ is a metric space. Find $d(\omega_1, \omega_2)$ if:

i. $\omega_1 = (0, 1, 1, 1, 1, \dots)$ and $\omega_2 = (1, 0, 1, 1, 1, \dots)$,

ii. $\omega_1 = (0, 1, 0, 1, 0, \dots)$ and $\omega_2 = (1, 0, 1, 0, 1, \dots)$.

Solution. □

Problem 4.2.7. Let $f : I \to I$ be a continuous function defined on an interval I.

i. What can you say about the graph of f, if f has a dense set of points with $f^2(x) = x$?

ii. Show that the inverse of f must exist and that f must have at least one fixed point.

iii. Deduce that if there exists an $x \in I$ with $f(x) \neq x$, then f must be strictly decreasing.

iv. If $f'(x)$ exists for all $x \in I$, show that the 2-cycles are non-hyperbolic, and any fixed point x_0 is non-hyperbolic of the type $f'(x_0) = -1$, when f is not the identity map.

v. Give an example of a function of the type appearing in iv.

Solution. □

Problem 4.3.4. Show that if $f : [a, b] \to [a, b]$ is a homeomorphism, then either a and b are fixed points or $\{a, b\}$ is a 2-cycle.

Solution. $\square$

Problem 4.3.8. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map with fixed point c and basin of attraction $B_f(c) = (a, b)$, an interval. Show that one of the following must hold:

i. a and b are fixed points.

ii. a or b is fixed and the other is eventually fixed.

iii. $\{a, b\}$ is a 2-cycle.

Solution. $\square$