Homework Assignment 4

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Problem 2.3.1. For each of the following functions, $c = 0$ lies on a periodic cycle. Classify this cycle as attracting, repelling, or neutral (non-hyperbolic). State if it is super attracting.

$$\text{i. } f(x) = \frac{\pi}{2}\cos(x), \quad \text{ii. } g(x) = -\frac{1}{2}x^3 - \frac{3}{2}x^2 + 1.$$

Solution. Recall that if c is a point of period r, then c is stable, asymptotically stable, unstable, if $f^r(c)$ is stable, asymptotically stable, unstable, respectively. Thus, if c is a point of period r and $f'(x)$ is continuous at $x = c$, then c is asymptotically stable (attracting) if

$$\left|(f^r(c))'\right| = \left|f'(f^0(c)) \cdot f'(f^1(c)) \cdot \dots \cdot f'(f^{r-1}(c))\right| < 1$$

and c is unstable (repelling) if

$$\left|(f^r(c))'\right| = \left|f'(f^0(c)) \cdot f'(f^1(c)) \cdot \dots \cdot f'(f^{r-1}(c))\right| > 1.$$

i. Let $f(x) = \frac{\pi}{2}\cos(x)$. It is clear that $f^2(0) = 0$ so that $c = 0$ is a period 2 point and $\{0, f(0)\}$ forms a 2-cycle. Note that $f'(x) = -\frac{\pi}{2}\sin(x)$, which is continuous, and that

$$|f'(0) \cdot f'(f(0))| = \left|\left(-\frac{\pi}{2}\sin(0)\right)\left(-\frac{\pi}{2}\sin\left(\frac{\pi}{2}\right)\right)\right| = 0 < 1$$

so that the 2-cycle $\{0, f(0)\}$ is asymptotically stable. Since

$$(f^2(0))' = (f(f(0)))' = f'(0) \cdot f'(f(0)) = 0,$$

we have that $c = 0$ is a super-attracting point of f^2 and the 2-cycle $\{0, f(0)\}$ is a super-attracting, asymptotically stable cycle.

ii. Let $g(x) = -\frac{1}{2}x^3 - \frac{3}{2}x^2 + 1$. It is clear that $g^3(0) = 0$ so that $c = 0$ is a period 3 point and $\{0, g(0), g^2(0)\}$ forms a 3-cycle. Note that $g'(x) = -\frac{3}{2}x^2 - 3x$, which is continuous, and that

$$\left|g'(0) \cdot g'(g(0)) \cdot g'(g^2(0))\right| = \left|0\left(-\frac{9}{2}\right)\left(\frac{3}{2}\right)\right| = 0 < 1$$

so that the 2-cycle $\{0, g(0), g^2(0)\}$ is asymptotically stable. Since

$$(g^3(0))' = (g(g(g(0))))' = g'(0) \cdot g'(g(0)) \cdot g'(g^2(0)) = 0,$$

we have that $c = 0$ is a super-attracting point of g^3 and the 3-cycle $\{0, g(0), g^2(0)\}$ is a super-attracting, asymptotically stable cycle.

$\square$

Problem 2.3.2. Let $f_c(x) = x^2 + c$. Show that for $c < -3/4$, f_c has a 2-cycle, and find it explicitly. For what values of c is the 2-cycle attracting?

Solution. Note that f_c has a 2-cycle if it has a period 2 point, i.e. if $f_c^2(x) - x = 0$ has a solution $x = x_0$ with $f_c(x_0) - x_0 \neq 0$. Thus, we must have that

$$f_c^2(x) - x = (x^2 + c)^2 + c - x = x^4 + 2cx^2 - x + c^2 + c = 0 \tag{1}$$

has a solution. As was shown earlier, $x = \left(1 \pm \sqrt{-4c}\right)/2$ are fixed points of f_c and thus must satisfy $f_c^2(x) - x = 0$. This allows to easily factor (1) and we see that

$$x^4 + 2cx^2 - x + c^2 + c = \left(x - \frac{1+\sqrt{-4c}}{2}\right)\left(x - \frac{1-\sqrt{-4c}}{2}\right)(x^2 + x + c + 1).$$

Since a period 2 point x_0 is such that $f_c(x_0) - x_0 \neq 0$, we know that

$$\left(x_0 - \frac{1+\sqrt{-4c}}{2}\right) \neq 0, \quad \left(x_0 - \frac{1-\sqrt{-4c}}{2}\right) \neq 0$$

so that $x^4 + 2cx^2 - x + c^2 + c = 0$ only if $x^2 + x + c + 1 = 0$. We readily see that since $c < -3/4$, the polynomial $x^2 + x + c + 1$ has real solutions, and that

$$x^2 + x + c + 1 = \left(x - \frac{-1+\sqrt{-3-4c}}{2}\right)\left(x - \frac{-1-\sqrt{-3-4c}}{2}\right)$$

from which we identify the 2-cycle of f_c as

$$\{c_0, f_c(c_0)\} = \left\{\frac{-1+\sqrt{-3-4c}}{2}, \frac{-1-\sqrt{-3-4c}}{2}\right\}.$$

This 2-cycle will be attracting for f_c if c_0 is attracting for f_c^2, i.e. if

$$\left|\left(f_c^2(c_0)\right)'\right| = |f_c'(c_0) f_c'(f_c(c_0))| < 1.$$

Note that $f_c'(x) = 2x$ from which we see that

$$|f_c'(c_0) f_c'(f_c(c_0))| = \left|\left(-1+\sqrt{-3-4c}\right)\left(-1-\sqrt{-3-4c}\right)\right| = \left|4(1+c)\right|.$$

Therefore, the 2-cycle of f_c is attracting if $|4(1+c)| < 1$, which occurs if and only if $-5/4 < c < -3/4$.

$\square$

Problem 2.3.3. Let $a, b, c \in \mathbb{R}$. Investigate the existence of 2-cycles for the following maps:

i. $f(x) = ax + b$, $a \neq 0$.

ii. $f(x) = ax^2 - x + c$, $a, c > 0$.

iii. $f(x) = a - \frac{b}{x}$, $a \neq 0, b \neq 0$.

iv. $f(x) = \frac{ax+b}{cx-a}$, $a^2 + bc \neq 0$.

Solution. As outlined in a previous problem, a 2-cycle for a function f exists if there is a period 2 point of f, i.e. if there is a point $x = x_0$ such that $f^2(x_0) - x_0 = 0$ but $f(x_0) - x_0 \neq 0$. Thus, to identify the period 2 points, we first identify the fixed points $c_0, \dots, c_n$ of a function. The fixed points $x = c_0, \dots, c_n$ will satisfy $f(x) - x = 0$ and thus must satisfy $f^2(x) - x = 0$ so that $(x - c_i)$ is a factor of $f^2(x) - x$ for $i = 0, \dots, n$. Therefore, the remaining solutions of $f^2(x) - x$, if they exist, form the 2-cycles of f.

i. Suppose that $f(x) = ax + b$ with $a \neq 0$. We readily see that $f(x) - x = 0$ has the solution $x = -b/(a-1)$ if $a \neq 1$ and is the only fixed point of f. Note that if $a = 1$, then $f(x) - x = 0$ only if $b = 0$ giving rise to the identity map for which the solution is trivial. However, note that

$$f^2(x) - x = (a^2 - 1)x + b(a+1) = (a+1)(b + (a-1)x) = 0$$

from which the only solution is $x = -b/(a-1)$. Since this is the fixed point of f, it cannot be a period 2 point. Therefore, there are no 2-cycles for $f(x) = ax + b$ for $a \neq 0, 1$.

ii. Suppose that $f(x) = ax^2 - x + c$ with $a, c > 0$. Note that $f(x) - x = ax^2 - 2x + c = 0$ has real solutions $x = \left(1 \pm \sqrt{1 - ac}\right)/a$ if $ac \leq 1$. Since a and c are positive, this is equivalent to requiring that $a, c \in (0, 1]$. Then $\left(x - \frac{1+\sqrt{1-ac}}{a}\right)$ and $\left(x - \frac{1-\sqrt{1-ac}}{a}\right)$ are factors of $f^2(x) - x$ and we see that

$$\begin{aligned} f^2(x) - x &= a\left(ax^2 - x + c\right)^2 - x + c \\ &= \left(x - \frac{1+\sqrt{1-ac}}{a}\right)\left(x - \frac{1-\sqrt{1-ac}}{a}\right)\left(a^2x^2 + ca\right) = 0. \end{aligned}$$

However, if $a, c > 0$, then the only real solutions of this equation are given by $x = \left(1 \pm \sqrt{1-ac}\right)/a$ where $a, c \in (0, 1]$. But these are the fixed points of f. Therefore, there are no 2-cycles of $f(x) = ax^2 - x + c$ with $a, c > 0$.

iii. Suppose that $f(x) = a - \frac{b}{x}$ with $a \neq 0, b \neq 0$. It is easily seen that if $x \neq 0$, then $f(x) - x = x^2 - ax + b = 0$ has real solutions $x = (a \pm \sqrt{a^2 - 4b})/2$ if $a^2 \geq 4b$. Then $\left(x - \frac{a+\sqrt{a^2-4b}}{2}\right)$ and $\left(x - \frac{a-\sqrt{a^2-4b}}{2}\right)$ are factors of $f^2(x) - x$ and we see that

$$\begin{aligned} f^2(x) - x &= a - \frac{b}{\left(a - \frac{b}{x}\right)} - x \\ &= \left(x - \frac{a+\sqrt{a^2-4b}}{2}\right)\left(x - \frac{a-\sqrt{a^2-4b}}{2}\right)\left(\frac{a}{b - ax}\right) = 0 \end{aligned}$$

only when $x = (a \pm \sqrt{a^2 - 4b})/2$ which are precisely the fixed points of f. Therefore, there are no 2-cycles of $f(x) = a - \frac{b}{x}$ with $a \neq 0, b \neq 0$

iv. Suppose that $f(x) = \frac{ax+b}{cx-a}$ with $a^2 + bc \neq 0$. Note that $f(x)$ is only defined if $x \neq a/c$. We readily see that

$$f(x) - x = \frac{ax+b}{cx-a} - x = \frac{-cx^2 + 2ax + b}{cx - a} = 0$$

if $x = \left(a \pm \sqrt{a^2 + bc}\right)/c$ which is real and in the domain of f if $a^2 + bc > 0$. These are precisely the fixed points of f. Note that for any $x \neq a/c$ we have that

$$f^2(x) = \frac{b + \frac{a(b+ax)}{cx-a}}{-a + \frac{c(b+ax)}{cx-a}} = \frac{(a^2+bc)x}{a^2+bc} = x$$

if $a^2 + bc \neq 0$. Thus, every defined point satisfies $f^2(x) = x$. Therefore, every point in this function's domain generates a 2-cycle if that point is different from the fixed points

$$c_0 = \frac{a + \sqrt{a^2+bc}}{c}, \quad c_1 = \frac{a - \sqrt{a^2+bc}}{c}.$$

$\square$

Problem 2.3.4. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous.

i. If f has a 2-cycle $\{x_0, x_1\}$, show that f has a fixed point.

ii. If f has a 3-cycle $\{x_0, x_1, x_2\}$, $x_0 < x_1 < x_2$ with $f(x_0) = x_1$, $f(x_1) = x_2$, and $f(x_2) = x_0$, show that there is a fixed point y_0 with $x_1 < y_0 < x_2$ and a point y_1 with $x_0 < y_1 < x_1$ with $f^2(y_1) = y_1$.

Solution. i. Suppose that $\{x_0, x_1\}$ is a 2-cycle of the continuous function f. Then we have that $f(x_0) = x_1$ and $f(x_1) = x_0$ with $x_0 \neq x_1$. Consider the function $g(x) = f(x) - x$, which is continuous by the continuity of f. Without loss of generality, we may assume that $x_0 < x_1$.

The Intermediate Value Theorem states that for a continuous function f, for any interval $I = [a, b]$, if there is a point u such that $f(a) < u < f(b)$ or $f(a) > u > f(b)$, then there is a point $c \in (a, b)$ with $f(c) = u$.

Now, for g continuous, define $I = [x_0, x_1]$. Since $\{x_0, x_1\}$ forms a 2-cycle of f we have that

$$\begin{aligned} g(x_0) &= f(x_0) - x_0 = x_1 - x_0 > 0 \\ g(x_1) &= f(x_1) - x_1 = x_0 - x_1 = -g(x_0) < 0. \end{aligned}$$

Therefore, by the Intermediate Value Theorem, since $0 \in (g(x_1), g(x_0)) = (-g(x_0), g(x_0))$, there is some point $c \in (x_0, x_1)$ such that $g(c) = f(c) - c = 0$, i.e. c is a fixed point of f.

ii. Suppose that f is a continuous function meeting the assumptions of the problem. Consider the function $g(x) = f(x) - x$, which is continuous by the continuity of f.

In a manner similar to the one used above, we may use the Intermediate Value Theorem to show that f has a fixed point on the interval $I = [x_1, x_2]$. By assumption we have that $f(x_1) = x_2$ and $f(x_2) = x_0$ with $x_0 < x_1 < x_2$. Thus, we have that

$$\begin{aligned} g(x_1) &= f(x_1) - x_1 = x_2 - x_1 > 0 \\ g(x_2) &= f(x_2) - x_2 = x_0 - x_2 < 0. \end{aligned}$$

Therefore, by the Intermediate Value Theorem, since $0 \in (g(x_2), g(x_1))$, there is some point $y_0 \in (x_1, x_2)$ such that $g(y_0) = f(y_0) - y_0 = 0$, i.e. y_0 is a fixed point of f.

Now, define the function $h(x) = f^2(x) - x$. This function is continuous since f is continuous and the composition of continuous functions is continuous. Consider the interval $I = [x_0, x_1]$. By assumption we have that $f(x_0) = x_1$, $f(x_1) = x_2$, and $f(x_2) = x_0$ with $x_0 < x_1 < x_2$. Thus, we have that

$$\begin{aligned} h(x_0) &= f(f(x_0)) - x_0 = f(x_1) - x_0 = x_2 - x_0 > 0 \\ h(x_1) &= f(f(x_1)) - x_1 = f(x_2) - x_1 = x_0 - x_1 < 0 \end{aligned}$$

Therefore, by the Intermediate Value Theorem, since $0 \in (h(x_1), h(x_0))$, there is some point $y_1 \in (x_0, x_1)$ such that $h(y_1) = f^2(y_1) - y_1 = 0$, i.e. there is a point $x_0 < y_1 < x_1$ such that $f^2(y_1) = y_1$.

$\square$

Problem 2.3.7. Let $f(x) = ax^3 + bx + 1$, $a \neq 0$. If $\{0, 1\}$ is a 2-cycle for $f(x)$, find a and b so that the 2-cycle is non-hyperbolic and determine the stability.

Solution. Note that $\{0, 1\}$ is a 2-cycle of f if $f(0) = 1$ and $f(1) = 0$, i.e. if

$$f(1) = a + b + 1 = 0.$$

Thus, $a = -b - 1$. The 2-cycle is non-hyperbolic if $|f'(0)f'(1)| = 1$. We see that $f'(x) = 3ax^2 + b$ so that

$$|f'(0)f'(1)| = |b(3a + b)| = | - (2b^2 + 3b)| = 1.$$

Thus, either $2b^2 + 3b = 1$ which implies that $b = (-3 \pm \sqrt{17})/4$ or we have $2b^2 + 3b = -1$ which implies that $b = -1$ or $b = -1/2$. Note that if $b = -1$, then $a = 0$ which violates our assumptions so we eliminate this choice. The other three possible functions are listed below:

$$\begin{aligned}
f_1(x) &= \left(\frac{-1-\sqrt{17}}{4}\right)x^3 + \left(\frac{-3+\sqrt{17}}{4}\right)x + 1 \\
f_2(x) &= \left(\frac{-1+\sqrt{17}}{4}\right)x^3 + \left(\frac{-3-\sqrt{17}}{4}\right)x + 1 \\
f_3(x) &= -\frac{1}{2}x^3 - \frac{1}{2}x + 1.
\end{aligned}$$

Recall that a period 2 point c is stable if $f^2(c)$ is stable. Note that by construction this 2-cycle is non-hyperbolic for f and in particular $(f_1^2(0))' = (f_2^2(0))' = -1$ and $(f_3^2(0))' = 1$.

A previous result allows us to determine the stability of f_i^2 by evaluating the derivatives of f_i^2 at $c = 0$. In general, for $f(x) = ax^3 + bx + 1$ we have that $f'(x) = 3ax^2 + b$, $f''(x) = 6ax$, and $f'''(x) = 6a$ so that

$$\begin{aligned}
(f^2(0))' &= f'(f(0))f'(0) && = b^2 + 3ab \\
(f^2(0))'' &= f''(f(0))f'(0)^2 + f''(0)f'(f(0)) && = 6ab^2 \\
(f^2(0))''' &= f'''(f(0))f'(0)^3 + f'''(0)f'(f(0)) + 3f''(f(0))f''(0)f'(0) && = 6ab^3 + 6a(3a + b).
\end{aligned}$$

Thus, we see that

$$\begin{array}{lll}
(f_1^2(0))' = -1 & (f_2^2(0))' = -1 & (f_3^2(0))' = 1 \\
(f_1^2(0))'' = \frac{3(19-5\sqrt{17})}{8} & (f_2^2(0))'' = \frac{3(19+5\sqrt{17})}{8} & (f_3^2(0))'' = -3/4 \\
(f_1^2(0))'' = \frac{9(3+11\sqrt{17})}{16} & (f_2^2(0))'' = \frac{-9(-3+11\sqrt{17})}{16} & (f_3^2(0))'' = 51/8.
\end{array}$$

Since $f_3^2(0)' = 1$ and $f_3^2(0)'' < 0$, we have by a previous theorem that the point 0 is one-sided stable to the left of 0 for $f_3^2(x)$ and hence the 2-cycle $\{0, 1\}$ is one-sided asymptotically stable to the left of 0.

Note that when $g'(x) = -1$, the Schwarzian derivative of a function is given by $Sg(x) = -g'''(x) - (3/2)g''(x)^2$. Thus, we see that

$$\begin{aligned}
Sf_1^2(0) &= -(f_1^2(0))''' - \frac{3}{2}((f_1^2(0))'')^2 = \frac{9(-1191 + 241\sqrt{17})}{64} < 0 \\
Sf_2^2(0) &= -(f_2^2(0))''' - \frac{3}{2}((f_2^2(0))'')^2 = \frac{9(-1191 - 241\sqrt{17})}{64} < 0
\end{aligned}$$

so that by a previous theorem, the point 0 is asymptotically stable for $f_1^2(x)$ and $f_2^2(x)$. Thus, the 2-cycle $\{0, 1\}$ is asymptotically stable for $f_1(x)$ and $f_2(x)$. $\square$

Problem 2.3.17. Suppose that $f(x) = ax^2 + bx + c$, $a \neq 0$ has a 2-cycle $\{x_0, x_1\}$. Show that the 2-cycle cannot be non-hyperbolic of the type $f'(x_0)f'(x_1) = 1$.

Solution. □

Problem 2.3.18. Let $f(x)$ be a polynomial for which $g(x) = f^2(x) - x$ has a repeated root at x_0 (where $f(x_0) = x_1 \neq x_0$).Show that $\{x_0, x_1\}$ is a non-hyperbolic 2-cycle for f of the type where $f'(x_0)f'(x_1) = 1$. Does the converse hold?

Solution. Suppose that $f(x)$ is a polynomial with $g(x) = f^2(x) - x$. Let x_0 be a repeated root of $g(x)$ such that $f(x_0) = x_1 \neq x_0$. Since x_0 is a repeated root of $g(x)$ with $g(x)$ a polynomial, we have that $g(x_0) = 0$ and $g'(x_0) = 0$. If $g(x_0) = 0$, then $f^2(x_0) = x_0$ with $f(x_0) \neq x_0$ implying that x_0 is a period 2 point and $\{x_0, x_1\}$ is a 2-cycle. Note that

$$g'(x) = (f^2(x) - x)' = f'(f(x))f'(x) - 1. \tag{2}$$

Since $g'(x_0) = 0$ and $f(x_0) = x_1$, we have that

$$g'(x_0) = f'(f(x_0))f'(x_0) - 1 = f'(x_1)f'(x_0) - 1 = 0$$

which implies that $f'(x_1)f'(x_0) = 1$ and the 2-cycle is non-hyperbolic.

If on the other hand $\{x_0, x_1\}$ is a 2-cycle such that $f(x_0) = x_1 \neq x_0$ with $f'(x_1)f'(x_0) = 1$, then by (2) we have that $g'(x_0) = 0$. Since x_0 is a period 2 point, $f^2(x_0) = x_0$ and $g(x_0) = 0$ so that x_0 is a repeated root of $g(x)$. $\square$

Problem 2.4.1. Let $f_c(x) = x^2 + c$, $c \in \mathbb{R}$.

i. For what values of c does f_c have a super-attracting fixed point and what is the fixed point?

ii. For what values of c does f_c have a super-attracting 2-cycle and what is the 2-cycle?

iii. Show that if f_c has a super-attracting 3-cycle, then c satisfies the equation

$$c^3 + 2c^2 + c + 1 = 0$$

and the 3-cycle is given by $\{0, c, c^2 + c\}$.

Solution. i. As was shown in problem 1.2.1, we know that $f_c : \mathbb{R} \to \mathbb{R}$ with $f_c(x) = x^2 + c$ has two fixed points given by

$$x_1 = \frac{1 - \sqrt{1-4c}}{2}, \qquad x_2 = \frac{1 + \sqrt{1-4c}}{2} \tag{3}$$

provided that $c \leq 1/4$.

The fixed point x will be a super-attracting fixed point if $f_c'(x) = 0$. We note that $f_c'(x) = 2x$ so that $f_c'(x) = 0$ only if $x = 0$. There is no real value of c that will allow $x_2 = 0$ so x_2 is never a super-attracting fixed point. On the other hand, if $c = 0$, then $x_1 = 0$ is a super-attracting fixed point.

ii. Note that f_c will have a super-attracting 2-cycle if f_c^2 has a super-attracting period 2 point. A point x will be a super-attracting period 2 point if $f_c^2(x) = x$ with $f_c(x) \neq x$ and if $(f_c^2(x))' = 0$.

Since (3) are fixed points, we know that $(x - x_1)$ and $(x - x_2)$ must factor $f_c^2(x) - x$ so that

$$\begin{aligned} f_c^2(x) - x &= \left(x^2 + c\right)^2 + c - x \\ &= (x - x_1)(x - x_2)\left(x - \frac{-1 + \sqrt{-3-4c}}{2}\right)\left(x - \frac{-1 - \sqrt{-3-4c}}{2}\right). \end{aligned}$$

Thus,

$$\{x_3, x_4\} = \left\{\frac{-1 + \sqrt{-3-4c}}{2}, \frac{-1 - \sqrt{-3-4c}}{2}\right\}$$

forms a 2-cycle of f_c. To analyze when this 2-cycle is super attracting, we analyze when

$$\begin{aligned} (f_c^2(x_3))' &= f_c'(x_3) f_c'(x_4) \\ &= \left(-1 - \sqrt{-3-4c}\right)\left(-1 + \sqrt{-3-4c}\right) \\ &= 4(1 + c) = 0. \end{aligned}$$

We readily see that $(f_c^2(x_3))' = 0$ only if $c = -1$ so that only $f_c(x) = x^2 - 1$ has a super-attracting 2-cycle given by $\{0, -1\}$.

iii. If f_c has a 3-cycle then f_c has a period 3 point x_0 with $f_c^3(x_0) = x_0$ such that $f_c(x_0) = x_1 \neq x_0$ and $f_c^2(x_0) = x_2 \neq x_0$. Note that $f_c'(x) = 2x$. Thus, this 3-cycle is super attracting if

$$(f_c^3(x_0))' = f_c'(x_0)f_c'(x_1)f_c'(x_2) = 2^3 x_0 x_1 x_2 = 0$$

which implies that $x_0 = 0$, $x_1 = 0$, or $x_2 = 0$. Without loss of generality, we may assume that $x_0 = 0$. Using the fact that $f_c(x_0) = x_1 \neq x_0$ and $f_c^2(x_0) = x_2 \neq x_0$, we see that

$$\begin{aligned} x_1 &= f_c(x_0) = x_0^2 + c = c \\ x_2 &= f_c^2(x_0) = (x_0^2 + c)^2 + c = c^2 + c \end{aligned}$$

In order for this to be a 3-cycle, we require that $f^3(x_0) = x_0 = 0$, i.e. we require that

$$\begin{aligned} f_c^3(x_0) = f_c(f_c^2(x_0)) &= (c^2 + c)^2 + c \\ &= c^4 + 2c^3 + c^2 + c \\ &= c(c^3 + 2c^2 + c + 1) = 0. \end{aligned}$$

However we must have that $c \neq 0$ or $x_0 = 0$ would not generate a 3-cycle. Thus, we require that $(c^3 + 2c^2 + c + 1) = 0$. If this condition is met and f_c has a super-attracting 3-cycle, then that 3-cycle is given by $\{0, c, c^2 + c\}$.

□