Unbinned Maximum Likelihood for LAT Data: Diffuse Emission

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Original version December 2, 2002
Revised December 7, 2002

1 Source Models

Both extragalactic and Galactic diffuse emission are modeled as power-laws:

$$S_{\rm eg}(E,\hat{p}) = S_{\rm eg0}\left(\frac{E}{E_0}\right)^{-\Gamma_{\rm eg}} \tag{1}$$

$$S_G(E,\hat{p}) = S_{G0}\tilde{S}_G(\hat{p})\left(\frac{E}{E_0}\right)^{-\Gamma_G}, \tag{2}$$

where $\tilde{S}_G(\hat{p})$ is the spatial distribution of the Galactic diffuse emission.

2 Individual Photon Contributions to $\log\mathcal{L}$

Using the notation of LikeMemos 0 and 2, the individual photon contributions to the log-likelihood from diffuse emission are

$$b_j = b_{{\rm eg}j} + b_{Gj} \tag{3}$$

$$b_{{\rm eg}j} = \int dE\, d\hat{p}\, S_{\rm eg}(E,\hat{p})R(E'_j,\hat{p}'_j,t_j;E,\hat{p}) \tag{4}$$

$$= S_{\rm eg0}\int dE\left(\frac{E}{E_0}\right)^{-\Gamma_{\rm eg}}\int d\hat{p}\, R(E'_j,\hat{p}'_j,t_j;E,\hat{p}) \tag{5}$$

$$b_{Gj} = \int dE\, d\hat{p}\, S_G(E,\hat{p})R(E'_j,\hat{p}'_j,t_j;E,\hat{p}) \tag{6}$$

$$= S_{G0}\int dE\left(\frac{E}{E_0}\right)^{-\Gamma_G}\int d\hat{p}\, \tilde{S}_G(\hat{p})R(E'_j,\hat{p}'_j,t_j;E,\hat{p}). \tag{7}$$

The $\hat{p}$-integrals should be performed over the entire sky, but in practice they will be performed over a smaller region that would ideally be centered on the individual photon positions; or alternatively, the integration region may be the "source region", SR, of LikeMemo 0. The size of the source region should be sufficiently large to encompass all sources of emission (point-like and diffuse) that contribute significantly to the extraction region, the ROI.

Expressing these integrals as Riemann sums over true energies E_k, we have

$$b_{{\rm eg}j} = S_{\rm eg0}\sum_k \Delta E_k\left(\frac{E_k}{E_0}\right)^{-\Gamma_{\rm eg}}\hat{b}_{{\rm eg}jk} \tag{8}$$

$$b_{Gj} = S_{G0}\sum_k \Delta E_k\left(\frac{E_k}{E_0}\right)^{-\Gamma_G}\hat{b}_{Gjk}, \tag{9}$$

where

$$\hat{b}_{\mathrm{eg}jk} \equiv \int d\hat{p}\, R(E'_j, \hat{p}'_j, t_j; E_k, \hat{p}) \tag{10}$$

$$\hat{b}_{Gjk} \equiv \int d\hat{p}\, \tilde{S}_G(\hat{p}) R(E'_j, \hat{p}'_j, t_j; E_k, \hat{p}) \tag{11}$$

In the case of infinite energy resolution, the integrals over true energy are replaced by their integrands evaluated at $E = E'_j$:

$$b_{\mathrm{eg}j} = S_{\mathrm{eg}0}\left(\frac{E'_j}{E_0}\right)^{-\Gamma_{\mathrm{eg}}} \hat{b}_{\mathrm{eg}j} \tag{12}$$

$$b_{Gj} = S_{G0}\left(\frac{E'_j}{E_0}\right)^{-\Gamma_G} \hat{b}_{Gj}, \tag{13}$$

where

$$\hat{b}_{\mathrm{eg}j} \equiv \int d\hat{p}\, P(\hat{p}'_j; E'_j, \hat{p}, t_j) A(E'_j, \hat{p}, t_j) \tag{14}$$

$$\hat{b}_{Gj} \equiv \int d\hat{p}\, \tilde{S}_G(\hat{p}) P(\hat{p}'_j; E'_j, \hat{p}, t_j) A(E'_j, \hat{p}, t_j). \tag{15}$$

Since the $\hat{b}$'s are independent of any source model or any specified cuts (at least for now), they may, in principle, be computed once and made part of the data associated with each photon in the event list files. For the finite energy resolution case, that would involve specifying a sufficiently broad and fine energy grid to be stored with each photon, or perhaps parameterizing the $\hat{b}_{[\mathrm{eg},G]jk}$ values using Gaussian functions or some other appropriate analytic form.

For completeness, the relevant partial derivatives of the b_j components are

$$\frac{\partial b_{[\mathrm{eg},G]j}}{\partial S_{[\mathrm{eg},G]0}} = \frac{b_{[\mathrm{eg},G]j}}{S_{[\mathrm{eg},G]0}} \tag{16}$$

$$\frac{\partial b_{[\mathrm{eg},G]j}}{\partial \Gamma_{[\mathrm{eg},G]0}} = -S_{[\mathrm{eg},G]0} \sum_k \Delta E_k \left(\frac{E_k}{E_0}\right)^{-\Gamma_{[\mathrm{eg},G]}} \log(E_k/E_0)\, \hat{b}_{[\mathrm{eg},G]jk}, \quad \text{for } E/\Delta E < \infty \tag{17}$$

$$= -S_{[\mathrm{eg},G]0}\left(\frac{E'_j}{E_0}\right)^{-\Gamma_{[\mathrm{eg},G]}} \log(E'_j/E_0)\, \hat{b}_{[\mathrm{eg},G]j}, \quad \text{for } E/\Delta E = \infty. \tag{18}$$

3 N_{pred} Contributions

The predicted number of photons from diffuse emission is

$$d = d_{\mathrm{eg}} + d_G. \tag{19}$$

For the finite energy resolution case, the Riemann sums over E_k are

$$d_{[\mathrm{eg},G]} = S_{[\mathrm{eg},G]0} \sum_k \Delta E_k \left(\frac{E_k}{E_0}\right)^{-\Gamma_{[\mathrm{eg},G]}} \hat{d}_{[\mathrm{eg},G]k}, \tag{20}$$

where

$$
\begin{aligned}
\hat{d}_{\mathrm{eg}k} &= \int d\hat{p} \int dE'\, d\hat{p}'\, dt\, R(E', \hat{p}', t; E_k, \hat{p}) & (21)\\
\hat{d}_{Gk} &= \int d\hat{p}\, \tilde{S}_G(\hat{p}) \int dE'\, d\hat{p}'\, dt\, R(E', \hat{p}', t; E_k, \hat{p}). & (22)
\end{aligned}
$$

For infinite energy resolution,

$$
\begin{aligned}
\hat{d}_{\mathrm{eg}k} &= \int d\hat{p} \int d\hat{p}'\, dt\, P(\hat{p}'; E_k, \hat{p}, t) A(E_k, \hat{p}, t), & E'_{\mathrm{min}} \le E_k \le E'_{\mathrm{max}} \quad (23)\\
&= 0, & E_k < E'_{\mathrm{min}} \text{ or } E_k > E'_{\mathrm{max}} \quad (24)\\
\hat{d}_{Gk} &= \int d\hat{p}\, \tilde{S}_G(\hat{p}) \int d\hat{p}'\, dt\, P(\hat{p}'; E_k, \hat{p}, t) A(E_k, \hat{p}, t), & E'_{\mathrm{min}} \le E_k \le E'_{\mathrm{max}} \quad (25)\\
&= 0, & E_k < E'_{\mathrm{min}} \text{ or } E_k > E'_{\mathrm{max}}. \quad (26)
\end{aligned}
$$

These decompositions of the $d_{[\mathrm{eg},G]}$'s motivate the definition of the energy-dependent exposure:

$$
\begin{aligned}
\varepsilon(E, \hat{p}) &\equiv \int d\hat{p}'\, dt\, R(E', \hat{p}', t; E, \hat{p}) \qquad \text{for } E/\Delta E < \infty & (27)\\
&\equiv \int d\hat{p}'\, dt\, P(\hat{p}'; E, \hat{p}, t) A(E, \hat{p}, t) \qquad \text{for } E/\Delta E = \infty, & (28)
\end{aligned}
$$

so that the (non-zero) $\hat{d}$'s are

$$
\begin{aligned}
\hat{d}_{\mathrm{eg}k} &= \int d\hat{p}\, \varepsilon(E_k, \hat{p}) & (29)\\
\hat{d}_{Gk} &= \int d\hat{p}\, \tilde{S}_G(\hat{p}) \varepsilon(E_k, \hat{p}). & (30)
\end{aligned}
$$

The $\hat{p}'$-integrals should be performed over the extraction region, i.e., the ROI. Just as with the $\hat{b}$'s of equations 14 and 15, it will be desirable to precompute maps of the exposure $\varepsilon(E, \hat{p})$. It should be noted that the values in the ε maps will depend on the ROI, and the angular extent of these maps should cover the SR.

The partial derivatives of the $d_{[\mathrm{eg},G]}$'s are

$$
\begin{aligned}
\frac{\partial d_{[\mathrm{eg},G]}}{\partial S_{[\mathrm{eg},G]0}} &= \frac{d_{[\mathrm{eg},G]}}{S_{[\mathrm{eg},G]0}} & (31)\\
\frac{\partial d_{[\mathrm{eg},G]}}{\partial \Gamma_{[\mathrm{eg},G]0}} &= -S_{[\mathrm{eg},G]0} \sum_k \Delta E_k \left(\frac{E_k}{E_0}\right)^{-\Gamma_{[\mathrm{eg},G]}} \log(E_k/E_0)\, \hat{d}_{[\mathrm{eg},G]k}. & (32)
\end{aligned}
$$

4 Coordinate Systems

For the exposure maps, it is convenient to use a coordinate system such that the center of the ROI (which should correspond to the center of the SR), is at latitude and longitude $(\alpha'_0, \delta'_0) = (0, 0)$. If the center of the ROI is (α_0, δ_0) in celestial coordinates, then the following transformation maps locations (α, δ) to (α', δ') in the new coordinate system:

$$
\begin{pmatrix} \cos\delta' \cos\alpha' \\ \cos\delta' \sin\alpha' \\ \sin\delta' \end{pmatrix} = \begin{pmatrix} \cos\delta_0 \cos\alpha_0 & \cos\delta_0 \sin\alpha_0 & \sin\delta_0 \\ -\sin\alpha_0 & \cos\alpha_0 & 0 \\ -\sin\delta_0 \cos\alpha_0 & -\sin\delta_0 \sin\alpha_0 & \cos\delta_0 \end{pmatrix} \begin{pmatrix} \cos\delta \cos\alpha \\ \cos\delta \sin\alpha \\ \sin\delta \end{pmatrix} \qquad (33)
$$

If the SR angular radius is $\theta_{\rm SR}$, then since coordinates $(\alpha', 0)$ and $(0, \delta')$ lie along great circles, computing the exposure for ranges of values $\alpha' \in (-\theta_{\rm SR}, \theta_{\rm SR})$ and $\delta' \in (-\theta_{\rm SR}, \theta_{\rm SR})$ ensures that the SR will be covered by the exposure maps. In addition, as noted in LikeMemo 0, the poles of the celestial sphere are avoided.

Yet another coordinate system is useful for computing the $\hat{p}$-integrals. Identifying the z-axis of this coordinate system with the center of the SR, we define $(\tilde{\mu}, \tilde{\phi})$ such that

$$\int_{\rm SR} d\hat{p} = \int_{\cos\theta_{\rm SR}}^{1} d\tilde{\mu} \int_0^{2\pi} d\tilde{\phi}. \tag{34}$$

In terms of $(\tilde{\mu}, \tilde{\phi})$, we have

$$\begin{aligned}
\alpha' &= \cos^{-1}\left[\frac{\tilde{\mu}}{(1-(1-\tilde{\mu}^2)\sin^2\tilde{\phi})^{1/2}}\right], \qquad \text{for } \cos\tilde{\phi} \geq 0 && (35)\\
&= 2\pi - \cos^{-1}\left[\frac{\tilde{\mu}}{(1-(1-\tilde{\mu}^2)\sin^2\tilde{\phi})^{1/2}}\right], \qquad \text{for } \cos\tilde{\phi} < 0 && (36)\\
\delta' &= \sin^{-1}\left[(1-\tilde{\mu}^2)^{1/2}\sin\tilde{\phi}\right]. && (37)
\end{aligned}$$