Gregory Lypny

Financial Economics

Class Notes

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About These Notes

These are draft class notes for the topics I cover in my financial economics courses. If you intend to print them, print one topic at a time because I sometimes post updates. The notes are also available in EPUB format, but I still a have a lot of formatting to do; it's tricky to get an eBook looking right when it contains a lot of math typesetting, big tables, and elaborate graphs. I'll keep at it.

The best way to learn the material is to work through all of the examples and figures, and to do the exercises at the end of each chapter. Numerical results are usually presented with at least six decimal points precision. If you don't get the same result as I do, it is probably because you have rounded intermediate results, which in turn have affected your final result. Always solve algebraically, then interpret the expression or model in plain language and with no jargon. If you can do that, you understand; if you can't, well, you're fooling yourself or faking it.

If you feel the need to refer to textbooks, these three may be helpful. Any edition is fine. The book by Fabozzi, Neave, and Zhou is pretty close to the how I present the material.

Varian, H., *Intermediate Microeconomics: A Modern Approach.* W.W. Norton and Company, New York.

Bodie, Z.; A. Kane; A.J. Marcus; S. Perrakis; and P.J. Ryan, *Investments*. McGraw-Hill Ryerson, Toronto.

Fabozzi, Frank, J.; Edwin H. Neave, and Guofu Zhou, *Financial Economics*, John Wiley & Sons, Inc. Hoboken, New Jersey. Referred to as Fabozzi et. al.

Gregory Lypny

Thursday, July 29, 2021

1 PORTFOLIO THEORY

In this chapter you'll learn how to form portfolios of risky assets and how to expand risk-return choices by borrowing or lending at a risk-free rate of interest. Make sure to work through all of the numerical examples. Some are based on real-world stock prices. You can find the data in “Stock Prices.csv” or “Stock Prices.xlsx”. The file contains monthly closing prices in US dollars on 109 S&P big cap stocks, for the 30 months, January 2009 to June 2011.

Portfolio Return

If you invest \$40 out of \$100 at five per cent and \$60 at eight per cent, the return on your portfolio will be 6.8 per cent.

$$0.068 = .40 \times 0.05 + .60 \times 0.08$$

You could invest \$125 at eight per cent but, if your capital is \$100, you'd have to borrow the \$25 at five per cent or short sell \$25 worth of the asset earning five per cent. The return on your portfolio in that case would be 8.75 per cent.

$$0.0875 = -\,0.25 \times .05 + 1.25 \times 0.08$$

What's true for the actual return on a portfolio must also be true for the expected return, so for an investment in any two risky assets, the expected return on a portfolio is

$$E(r_p) = w_1 E(r_1) + w_2 E(r_2)$$

where $w_1 + w_2 = 1$. The weights are the percentage of your wealth or capital invested in each asset. The weights obviously have to add up to one unless you are a government, in which case they can be anything. And it doesn't matter how many assets are in your portfolio—one, 50, 637, or some number n.

$$E(r_p) = \sum_{i=1}^{n} w_i E(r_i), \text{where } \sum_{i=1}^{n} w_i = 1$$

The same expression can be written in matrix notation as

$$E(r_p) = \mathbf{w}^\top \cdot \bar{\mathbf{r}}, \text{ where } \mathbf{w}^\top \cdot \mathbf{1} = 1$$

$\mathbf{w}^\top = (w_1 \quad w_2 \quad \cdots \quad w_n)$ is the vector of weights; $\bar{\mathbf{r}}^\top = \big(E(r_1) \quad E(r_2) \quad \ldots \quad E(r_n)\big)$ is a vector of expected returns; and **1** (a boldface one) is a vector of n ones.

Portfolio Risk

The riskiness of a portfolio can be measured by the variance or standard deviation of its returns, which in turn depends on the riskiness of the assets in it. Suppose that asset 1's standard deviation is 12 per cent and asset 2's is 17 per cent. If their prices always moved in the same direction—they are perfectly positively correlated—the standard deviation of a portfolio of the two would be, like expected return, just a simple weighted average of the standard deviations of the two assets.

$$\sigma_p = w_1\sigma_1 + w_2\sigma_2$$

Your 40-60 portfolio would have a standard deviation of 15 per cent.

$$0.15 = .4 \times 0.12 + .6 \times 0.17$$

But there are very few things in this world, particularly the returns on financial assets, that are *perfectly* correlated, either positively or negatively. The price of the two assets may move in the same direction most of the time, but sometimes the price of asset 1 will go up when the price of asset 2 goes down or vice versa. The correlation of their returns is less than one. If their correlation was, say, 0.8, then the standard deviation of your 40-60 portfolio would be less than 15 per cent because some of the price declines of one will be, on occasion, cancelled partly, entirely, or more than entirely, by the price increases of the other. The calculation of portfolio standard deviation captures washing away of risk or *diversification* that results from return correlations, ρ_{12}, being different from +1,

$$\begin{aligned}\sigma_p^2 &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2 \\ &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12} \\ \sigma_p &= +\sqrt{\sigma_p^2}\end{aligned}$$

where $\sigma_{12} = \rho_{12}\sigma_1\sigma_2$ is the covariance of the returns of the two assets. Your 40-60 portfolio would have a standard deviation of 14.3 per cent if the correlation between the returns of assets 1 and 2 was 0.8 or, equivalently, their covariance was 0.01632.

$$\sigma_p = +\sqrt{\sigma_p^2} = +\sqrt{0.0205416} = 0.143323$$

What if there are three assets in your portfolio? Use your intuition to write down the equation for its variance. The pattern that you should see applies to a portfolio of any number of assets. In summation notation

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}$$

where, as always, the weights add to one and, for handy notation, $\sigma_{ii} \equiv \sigma_i^2$ and $\sigma_{jj} \equiv \sigma_j^2$ (the covariance of an asset with itself is just its own variance). The same expression is elegant when written in matrix notation,

$$\sigma_p^2 = \mathbf{w}^\top \cdot \mathbf{V} \cdot \mathbf{w}$$

$\mathbf{V}$ is called the variance-covariance matrix or just the covariance matrix. For a two-asset portfolio $\mathbf{V} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix}$ or $\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$. The diagonal elements are variances, and the off-diagonal elements are covariances. $\mathbf{V}$ is symmetric because $\sigma_{ij} = \sigma_{ji}$. The covariance matrix for assets 1 and 2 is

$$\mathbf{V} = \begin{pmatrix} 0.0144 & 0.01632 \\ 0.01632 & 0.0289 \end{pmatrix}$$

and the variance of your 40-60 portfolio can be computed as

$$\sigma_p^2 = (0.4 \quad 0.6) \begin{pmatrix} 0.01440 & 0.01632 \\ 0.01632 & 0.02890 \end{pmatrix} \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix} = 0.0205416$$

Don't be lazy. Check my math by calculating the variance using matrix math.

Portfolios of Two Risky Assets

A portfolio frontier shows how a portfolio's risk and return depends on the investment in a set of assets. The frontier described in this section is formed from two real-world stocks, Haliburton (HAL) and Time Warner Cable (TWC), using their US monthly closing prices from December 2008 to June 2011. You can find their prices in the file "Stock Prices.csv" mentioned at the beginning of this chapter. HAL and TWC's estimated expected returns, variances, and covariances are what you need to form portfolios for a frontier.

	HAL (asset 1)	*TWC (asset 2)*
Mean	0.0407485	0.0234194
Variance	0.00987177	0.0142828
Standard deviation	0.0993568	0.119511

Their covariance show up in the covariance matrix (makes sense),

$$\bar{\mathbf{r}}^{\top} = (0.0407485 \quad 0.0234194)$$

$$\mathbf{V} = \begin{pmatrix} 0.00987177 & 0.00594239 \\ 0.00594239 & 0.0142828 \end{pmatrix}$$

$$\rho_{12} = 0.500446$$

The table of portfolios on the next page was made using this information for investment weights running from -20 per cent to 120 per cent in increments of 20 percentage points. For example, when the investment in asset 1 is 40 per cent, the investment in asset 2 is 60 per cent. The 40-60 weighting makes for a portfolio with a three per cent expected return and a 9.8 per cent standard deviation. The table could also have been made just as easily by choosing the expected returns first, using those to figure out the weights, and then using weights to get the standard deviations.

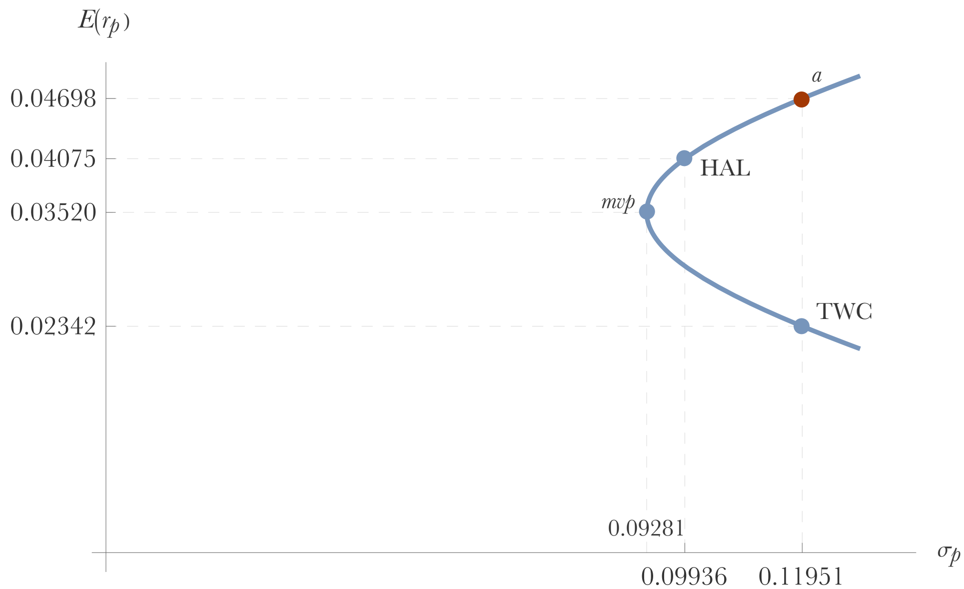
			Investment Weights	
	σ_p	$E(r_p)$	*HAL (asset 1)*	*TWC (asset 2)*
	0.134573	0.0199536	-0.2	1.2
TWC	0.119511	0.0234194	0	1
	0.106946	0.0268853	0.2	0.8
	0.097845	0.0303511	0.4	0.6
	0.0932279	0.0338169	0.6	0.4
mvp	0.0928084	0.0351989	0.679752	0.320248
	0.0937593	0.0372827	0.8	0.2
HAL	0.0993568	0.0407485	1	0
	0.109244	0.0442143	1.2	-0.2
a	0.119511	0.0469784	1.3595	-0.359504

There are two reference portfolios in the table. One is the *minimum-variance portfolio* or *mvp* for short. It is the portfolio having the smallest variance or standard deviation. You can find the minimum-variance portfolio for two-asset frontiers by solving for the investment weight in either stock that minimizes portfolio variance. The *mvp*'s weight in asset 1 is

$$w_{1(mvp)} = \frac{\sigma_{22} - \sigma_{12}}{\sigma_{11} + \sigma_{22} - 2\sigma_{12}} = 0.679752$$

The other reference portfolio in the table is *a*. Portfolio *a* has the same standard deviation as TWC, but notice that it has a higher expected return (about twice that of TWC)? Anyone willing to bear the risk of TWC but wanting the highest possible expected return would choose portfolio *a* over holding TWC by itself. Portfolio *a* is called an *efficient portfolio* because there is no other portfolio of HAL and TWC that offers a higher expected return for its level of risk.

The graph of expected portfolio return versus its standard deviation is a hyperbola. If variance is used instead of standard deviation, the frontier is a parabola. Both are pretty and are interpreted in much the same way. To see this algebraically, derive an expression for portfolio variance as a function of expected return, without the weights being there.



Portfolios on a two-asset frontier are unique. There is only one portfolio of HAL and TWC that gives an expected return of 2.5 per cent, only one that gives 3.75 per cent, and only one that gives 7.4 per cent. When you choose the weight in one asset, the weight in the other is one minus the weight in the first. That is why a two-asset frontier always passes through the points representing the two assets.

All portfolios on the upward-sloping part of the frontier, including the *mvp*, are *efficient* because they offer the highest expected return for a given level of risk. You already know that portfolio *a* is efficient but asset 2 (TWC) by itself is not. All efficient portfolios on this frontier are at least 68 per cent invested in HAL. If you wanted a higher expected return than HAL, you'd have to take a short position in TWC.

Portfolios of More Than Two Risky Assets

There is just one snag in working with portfolios of more than two risky assets: there are many ways—infinitely many ways—to allocate your capital among them and get the exact same expected return. It is a big difference, but it is the only difference between portfolios of two risky assets and those with more than two. The problem then is how to choose the investment weights. To work through this we'll add the stocks of Apple Inc. (APPL), Anadarko Petroleum Corp. (APC), and Covalon Technologies Ltd. (COV) to our set of risky assets. They will be assets 1, 2, and 3. HAL and TWC are relabelled 4 and 5. Here is the summary of risk and return for the five stocks. Their covariances and correlations will come later.

	AAPL (asset 1)	*APC (asset 2)*	*COV (asset 3)*	*HAL (asset 4)*	*TWC (asset 5)*
Mean	0.0489129	0.0315206	0.0166714	0.0407485	0.0234194
Variance	0.00469201	0.0154948	0.00460735	0.00987177	0.0142828
Std. Dev.	0.0684982	0.124478	0.0678774	0.0993568	0.119511

The only way to get an expected portfolio return of 3.75 per cent with just HAL and TWC in your portfolio is to put 81.3 per cent in HAL and 18.7 per cent in TWC. But what if you decided to make a portfolio of APPL, APC, and COV? Allocating 61.9 per cent to APPL, 5.9 per cent to APC, and 32.2 per cent to COV would get you a 3.75 per cent expected return, but so would 54.9 per cent APPL, 21.1 per cent APC, and 24 per cent COV. There are countless ways to invest in those three stocks for a 3.75 per cent return. This is true for all 16 combinations of three, four, or all of the five stocks.

{{1,2,3}, {1,2,4}, {1,2,5}, <<10>> , {1,3,4,5}, {2,3,4,5}, {1,2,3,4,5}}

Add to those the 10 combinations of two stocks, and that's a lot to choose from!

So, how should you invest? If, like most people, you don't take on risk unnecessarily, you should allocate your capital so that your portfolio has the smallest possible risk for the expected return you have chosen. Turning that into a mathematical statement, the optimal investment is one that minimizes portfolio variance with respect to the investment weights, for a given portfolio expected return and the weights adding up to one.

$$\text{Minimize } \sigma_p^2 \equiv \mathbf{w}^\top \cdot \mathbf{V} \cdot \mathbf{w} \ \text{ with respect to } \ \mathbf{w}$$
$$\text{subject to } \mathbf{w}^\top \cdot \bar{\mathbf{r}} = E(r_p) \text{ and } \mathbf{w}^\top \cdot \mathbf{1} = 1$$

I bet you recognized this as a constrained optimization problem from your high school calculus class. You're right. Turns out there is a unique solution for **w** and it is

$$\mathbf{w}^* = \mathbf{g} + \mathbf{h} \cdot E(r_p)$$

$\mathbf{w^*}$ works for portfolios of any number of risky assets, even the easy textbook case of just two. Vectors $\mathbf{g}$ and $\mathbf{h}$ are computed as

$$
\begin{aligned}
\mathbf{g} &= \frac{1}{D}\left[B\left(\mathbf{V^{-1}} \cdot \mathbf{1}\right) - A\left(\mathbf{V^{-1}} \cdot \bar{\mathbf{r}}\right)\right] \\
\mathbf{h} &= \frac{1}{D}\left[C\left(\mathbf{V^{-1}} \cdot \bar{\mathbf{r}}\right) - A\left(\mathbf{V^{-1}} \cdot \mathbf{1}\right)\right] \\
A &= \bar{\mathbf{r}}^{\top} \cdot \mathbf{V^{-1}} \cdot \mathbf{1} \\
B &= \bar{\mathbf{r}}^{\top} \cdot \mathbf{V^{-1}} \cdot \bar{\mathbf{r}} \\
C &= \mathbf{1}^{\top} \cdot \mathbf{V^{-1}} \cdot \mathbf{1} \\
D &= BC - A^2
\end{aligned}
$$

$\mathbf{w^*}$ is linear in $E(r_p)$. $\mathbf{g}$ is like an intercept, more precisely, a vector of intercepts, and $\mathbf{h}$ is a vector of slopes. You can think of the expression for $\mathbf{w^*}$ as a stack of lines.

$$
\mathbf{w}* = \begin{pmatrix} g_1 + h_1 \times E(r_p) \\ g_2 + h_2 \times E(r_p) \\ g_3 + h_3 \times E(r_p) \\ \vdots \\ g_n + h_n \times E(r_p) \end{pmatrix}
$$

To compute $\mathbf{g}$ and $\mathbf{h}$ for our sample of five stocks, start with the vector of expected returns and the covariance matrix.

$$
\bar{\mathbf{r}}^{\top} = (0.0489129 \quad 0.0315206 \quad 0.0166714 \quad 0.0407485 \quad 0.0234194)
$$

$$
\mathbf{V} = \begin{pmatrix}
0.00469201 & 0.00347929 & 0.00165498 & 0.00283357 & 0.000803448 \\
0.00347929 & 0.0154948 & 0.00333506 & 0.00772093 & 0.0035537 \\
0.00165498 & 0.00333506 & 0.00460735 & 0.00210297 & 0.000702788 \\
0.00283357 & 0.00772093 & 0.00210297 & 0.00987177 & 0.00594239 \\
0.000803448 & 0.0035537 & 0.000702788 & 0.00594239 & 0.0142828
\end{pmatrix}
$$

Next compute the inverse of the covariance matrix,

$$
\mathbf{V^{-1}} = \begin{pmatrix}
286.555 & -25.4972 & -59.5777 & -60.7008 & 18.4106 \\
-25.4972 & 117.228 & -40.1493 & -80.4612 & 7.71844 \\
-59.5777 & -40.1493 & 272.502 & -12.6842 & 5.20969 \\
-60.7008 & -80.4612 & -12.6842 & 226.632 & -70.2325 \\
18.4106 & 7.71844 & 5.20969 & -70.2325 & 96.0222
\end{pmatrix}
$$

Then A, B, C, and D,

$$
A = 11.3172, \;\; B = 0.54648, \;\; C = 363.012, \;\; D = 70.2994
$$

And finally, $\mathbf{g}$ and $\mathbf{h}$,

$$\mathbf{g} = \begin{pmatrix} -0.400875 \\ 0.0478901 \\ 1.29006 \\ -0.281744 \\ 0.344673 \end{pmatrix}, \mathbf{h} = \begin{pmatrix} 26.9247 \\ -3.40596 \\ -26.7738 \\ 9.26287 \\ -6.00783 \end{pmatrix}$$

Now you have a vector equation for computing the percentage investment in each of the five stocks that minimizes the portfolio's risk for any portfolio expected return that you chose.

$$\mathbf{w}^* = \mathbf{g} + \mathbf{h} \cdot E(r_p) = \begin{pmatrix} -0.400875 \\ 0.0478901 \\ 1.29006 \\ -0.281744 \\ 0.344673 \end{pmatrix} + \begin{pmatrix} 26.9247 \\ -3.40596 \\ -26.7738 \\ 9.26287 \\ -6.00783 \end{pmatrix} \cdot E(r_p)$$

The optimal investment weights for your 3.75 per cent portfolio are

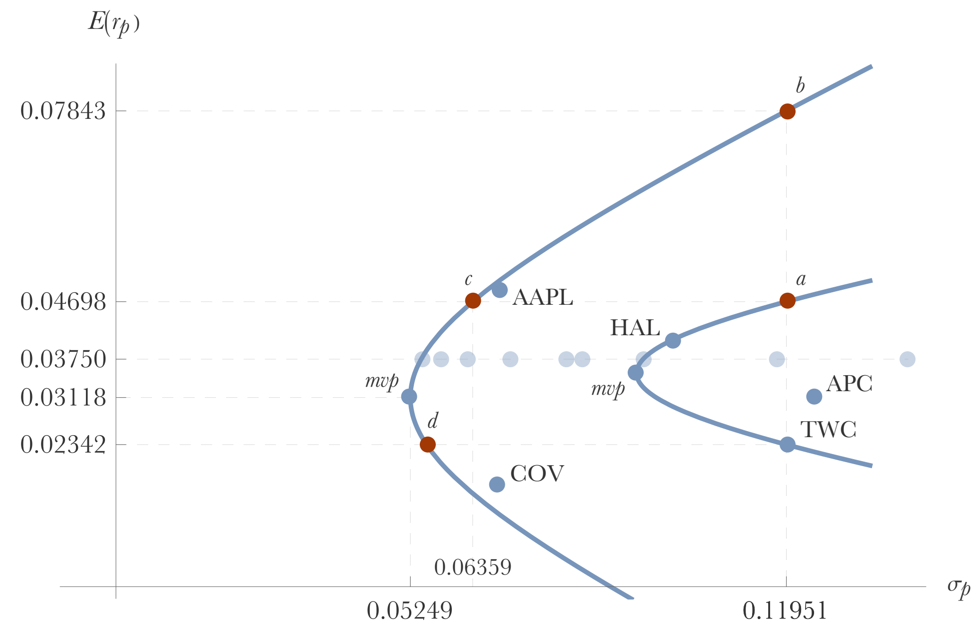
$$\mathbf{w}^* = \begin{pmatrix} -0.400875 \\ 0.0478901 \\ 1.29006 \\ -0.281744 \\ 0.344673 \end{pmatrix} + \begin{pmatrix} 26.9247 \\ -3.40596 \\ -26.7738 \\ 9.26287 \\ -6.00783 \end{pmatrix} \times 0.0375 = \begin{pmatrix} 0.608802 \\ -0.0798334 \\ 0.286039 \\ 0.0656134 \\ 0.119379 \end{pmatrix}$$

and the portfolio's standard deviation is about 5.4 per cent ($\sqrt{\mathbf{w}^\top \cdot \mathbf{V} \cdot \mathbf{w}} = 0.0544174$).

The next table shows the standard deviations of the 3.75 per cent portfolios we discussed earlier and a few more. The portfolio in the first row uses **w*** as its weights. There is no other 3.75 per cent portfolio that has a lower risk than that first one.

		Investment Weights				
σ_p	$E(r_p)$	*AAPL*	*APC*	*COV*	*HAL*	*TWC*
0.0544174	0.0375	0.608802	-0.0798334	0.286039	0.0656134	0.119379
0.0580537	0.0375	0.618732	0.059244	0.322024	0	0
0.0628525	0.0375	0.548726	0.211245	0.240029	0	0
0.0702113	0.0375	0.41208	0.32387	0.094359	0.09165	0.078041
0.0800822	0.0375	0.412	0.32387	-0.141302	0	0.405432
0.0830412	0.0375	0.31217	0.12734	0.463622	0.4743	-0.377432
0.0939667	0.0375	0	0	0	0.812539	0.187461
0.117803	0.0375	-0.274058	0.10956	0	1.1645	0
0.141205	0.0375	0	0.25497	-0.562507	0.4743	0.833237

Armed with a formula for the optimal investment weights, you can draw the five-asset frontier. Choose a series of expected portfolio returns, calculate the investment weights for each, and then use the weights to calculate each portfolio's standard deviation. Give it a shot, and make it pretty. Here's the frontier with some reference points. Interpret the points; calculate their investment weights; and their standard deviations if not shown.



Is the five-asset frontier a parabola in mean-variance space just like the two-asset frontier? Yes. Because both the expected return of a portfolio and its variance are functions of the optimal weights, the weights can be eliminated—as I encouraged you to do for the two asset case—by combining the two expressions. Portfolio variance can then be written directly in terms of portfolio expected return as

$$\sigma_p^2 = \frac{1}{D}\left(C E(r_p)^2 - 2A\,E(r_p) + B\right)$$

You can see that the frontier is a parabola because variance (think of it as y) is a quadratic function of expected return (x). Take the square root of both sides, and it is a hyperbola like the figure above. This means you don't have to calculate the weights if you don't need them, for example, when all you want to do is graph the frontier. But as a practical matter, you will want to know the weights (*How do I allocate my capital to get an expected return of eight per cent? The average monthly return on Lucy's portfolio was six per cent. How did she invest?*).

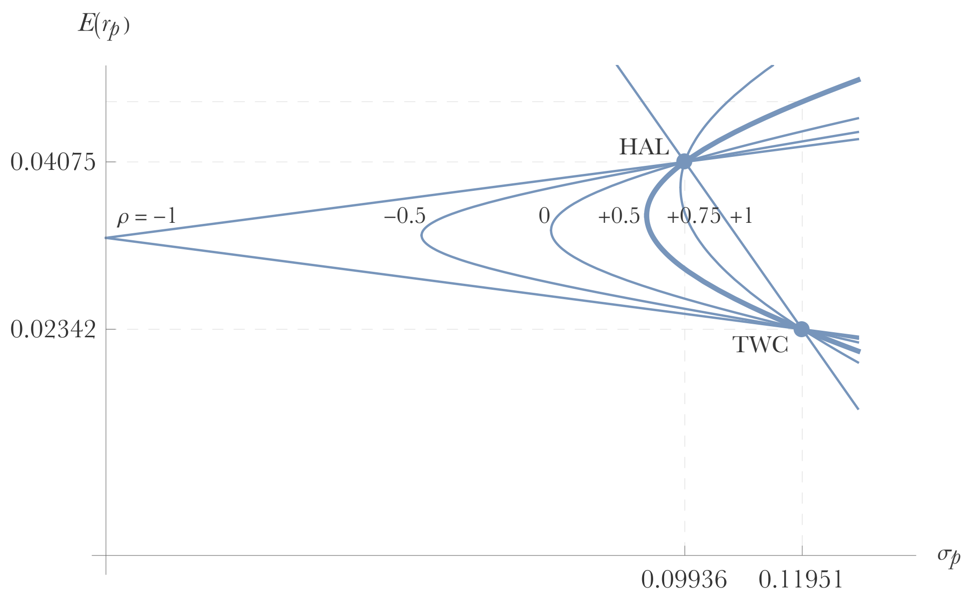
The next table shows some of the portfolios that are in the graph. Make a table of portfolios like this one both with and without calculating the investment weights. You'll be glad you did.

			Investment Weights				
	σ_p	$E(r_p)$	*APPL*	*APC*	*COV*	*HAL*	*TWC*
	0.088168	0	-0.400875	0.0478901	1.29006	-0.281744	0.344673
	0.0712058	0.01	-0.131628	0.0138305	1.02232	-0.189115	0.284595
	0.0583069	0.02	0.137619	-0.0202291	0.75458	-0.0964867	0.224516
d	0.055366	0.0234194	0.229687	-0.0318756	0.663029	-0.0648129	0.203973
	0.0525535	0.03	0.406866	-0.0542887	0.486842	-0.00385806	0.164438
mvp	0.0524855	0.0311759	0.438526	-0.0582937	0.45536	0.00703384	0.157374
	0.0561855	0.04	0.676113	-0.0883483	0.219104	0.0887706	0.10436
c	0.0635943	0.0469784	0.864005	-0.112116	0.0322662	0.153411	0.0624347
	0.067709	0.05	0.945361	-0.122408	-0.0486333	0.181399	0.0442815
b	0.119511	0.078425	1.7107	-0.219222	-0.809678	0.444696	-0.126491
	0.067709	0.05	0.945361	-0.122408	-0.0486333	0.181399	0.0442815

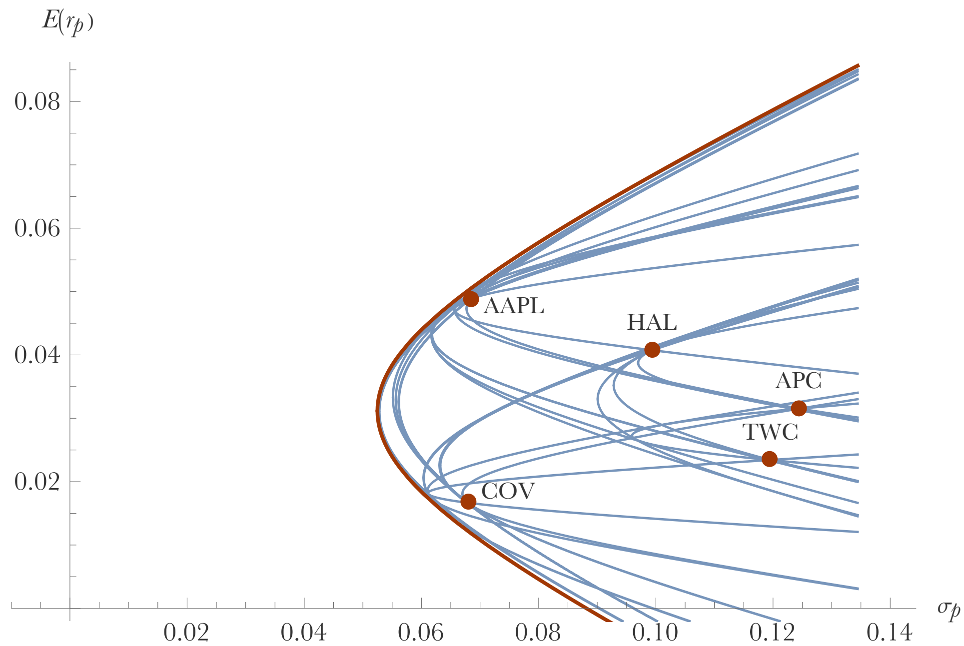
Three Ways of Looking at Diversification

Earlier in this chapter I described diversification in portfolios as a washing away of risk that happens because the prices of assets do not always move in the same direction. Here are three graphs that look more closely at that idea. I'm not going to say much about them because I want you to think about them, especially how they relate to one another.

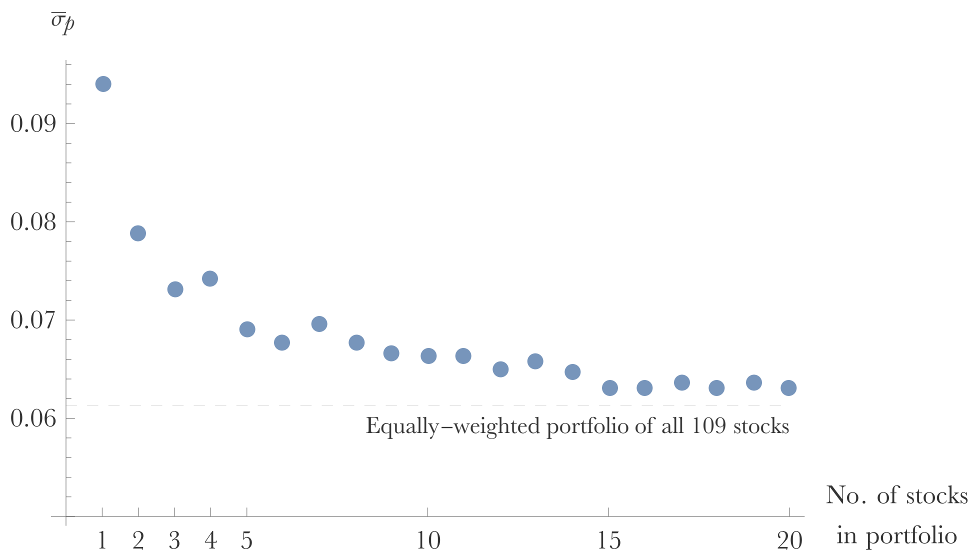
What would the portfolio frontier of HAL and TWC look like if the correlation between their returns was not 0.500446? What if their correlation was -0.5 or +0.75 or something else?



What if you graphed all 26 frontiers that are possible from our set of five stocks? There are 10 two-stock frontiers, 10 three-stock frontiers, five four-stock frontiers, and of course, one frontier of all five. Where does the five-stock frontier sit in relation to all of the others, or, for that matter, any one of the other frontiers in relation to the frontiers of portfolios with fewer stocks in them? I know, it looks like linguini.



What happens to the risk of a portfolio as the number of assets in it is increased? The final graph shows average portfolio standard deviation of samples of portfolios containing the same number of stocks. Each of the 20 samples contains 250 portfolios, randomly-selected from all 109 stocks in our data set.

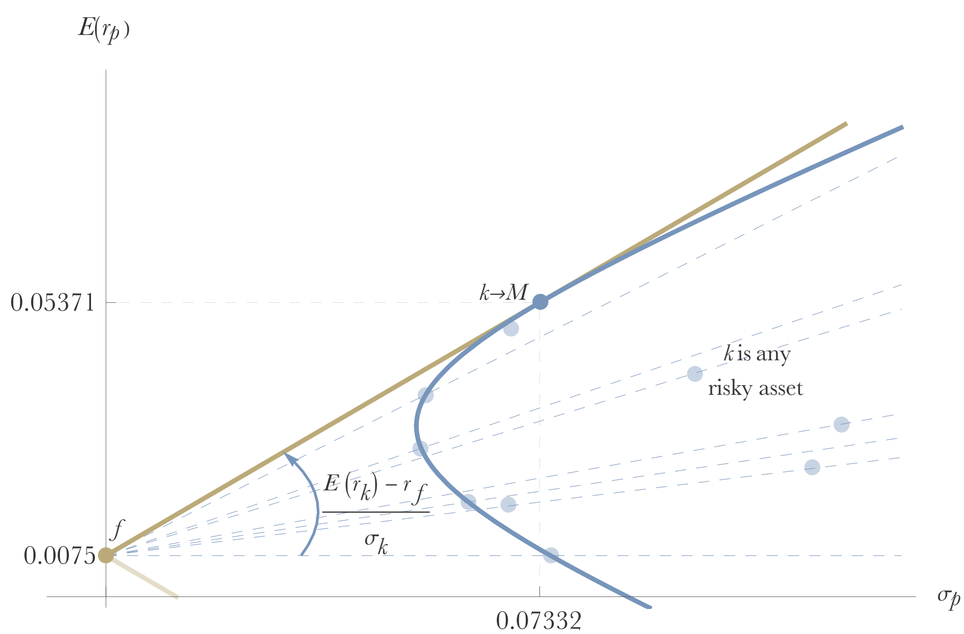


Adding A Risk-Free Asset to Portfolios

Suppose you don't want to invest all of your capital in risky assets; maybe you want \$25 out of your capital of \$100 to earn a guaranteed return, or at least as safe a return as can be. Or maybe you'd like to invest \$125 in risky assets by borrowing the extra \$25, so long as the rate on the loan is fixed. If that kind of safe borrowing and lending is possible, you are including a risk-free asset in the mix. By borrowing you are short-selling the risk-free asset (a negative investment weight) and by lending you are buying the risk-free asset (a positive investment weight). Nothing is truly risk-free, of course. The closest thing we have is government bonds, particularly short-term bonds, such as Treasury Bills. They're pretty safe because most stable governments are able to repay their debts since they have the power to tax. Inflation risk is usually low in the short run and the risk of a disaster—the country being hit by a meteor or someone in a far off place pressing the big red button is generally small too.

When risk-free borrowing and lending is possible, the elegant hyperbola or parabola that is the minimum-variance frontier of risky assets becomes an equally elegant straight line.

Call the risk-free asset f. If you invest in any risky asset, k, with expected return $E(r_k)$ and



borrow or lend at r_f, the expected return on your portfolio is a weighted average of the two, as it must be for any two-asset portfolio,

$$E(r_p) = w_f r_f + w_k E(r_k)$$

But the standard deviation of your portfolio is simpler than it would be if both assets were risky. It is now simply proportional to the standard deviation of k. That is because the standard deviation of f is zero, and f and k are uncorrelated,

$$
\begin{aligned}
\sigma_p^2 &= w_f^2\sigma_f^2 + w_k^2\sigma_k^2 + 2w_f w_k \sigma_{fk} \\
&= w_k^2\sigma_k^2 \text{ because } \sigma_f = \sigma_{fk} = 0 \\
\therefore \sigma_p &= |w_k|\sigma_k
\end{aligned}
$$

Putting expected portfolio return and standard deviation together

$$E(r_p) = r_f + \frac{E(r_k) - rf}{\sigma_k}\sigma_p$$

gives a linear frontier with intercept r_f and slope, $\dfrac{E(r_k) - rf}{\sigma_k}$, which is called the Sharpe Ratio.

In the graph of our five-asset frontier, the risk-free rate is assumed to be 0.75 per cent. Each of the light blue dashed lines is a linear frontier of portfolios of *f* and some risky asset *k*, which of course could be a portfolio itself.

Now here's where a bit of theory sneaks in again. If you like return but not risk, you'll want to choose *k* that maximizes the slope of the linear frontier because that gives portfolios of *f* and *k* that are efficient. It's another optimization problem from your calculus class. The optimal risky asset *k* is portfolio *M*, the point where the linear frontier and the minimum-variance frontier are tangent. *M* is a portfolio of all five risky assets, and has an expected return of 5.4 per cent. You can do the math to find the tangency for the case where there are only two risky assets, such as for HAL and TWC. This is sometimes presented in a footnote in finance textbooks as something that is doable. It may be doable but it is ugly, and when there are more than two risky assets, it is just plain sick. More on that in a moment.

Any portfolio on the optimal linear frontier is just an investment in *M* and some amount of borrowing or lending according to your preference. Portfolios below *M* involve lending and above *M* involve borrowing. What would be the expected return and standard deviation of your portfolio if you invested 75 per cent of your capital in *M* and lent the rest at the risk-free rate? What if you borrowed 30 per cent of your capital to lever an investment in *M*? I'll leave that to you to work out.

A nifty way to find the expected return of tangency portfolio *M* is

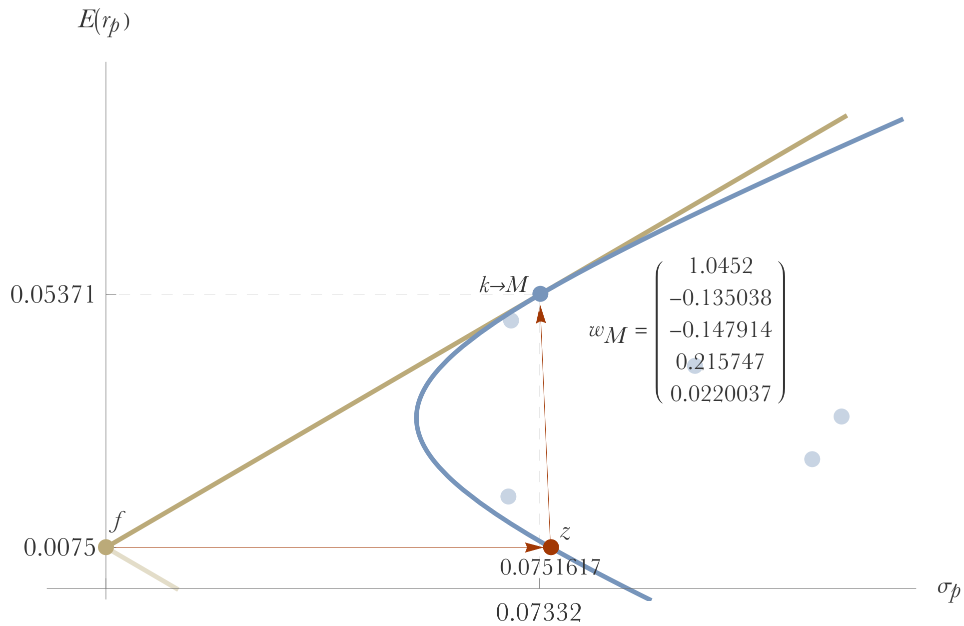
$$E(r_{k=M}) = \frac{A \cdot r_f - B}{C \cdot r_f - A}$$

A, *B*, and *C* are the efficient set math parameters for frontiers of any number risky assets. The solution exploits the fact that for any frontier portfolio *z* (next figure) there is an uncorrelated frontier portfolio *k*

$$\sigma_{kz} = \mathbf{w_k^{*\top}} \cdot \mathbf{V} \cdot \mathbf{w_z^*} = 0$$

The trick is to choose portfolio z to have an expected return equal to the risk-free rate, so that z plays the role of surrogate intercept of the line, making M the tangency. Solving $\sigma_{kw} = 0$ for $E(r_z) = r_f$ gives the solution above. I'd show you the proof, but I don't want you to drop the course.

Plugging in the values of A, B, and C for our five-asset frontier and a risk-free rate of 0.75 per cent gives $E(r_M) = 0.05371$, from which you can confirm the weights of the assets in M as shown in the figure.

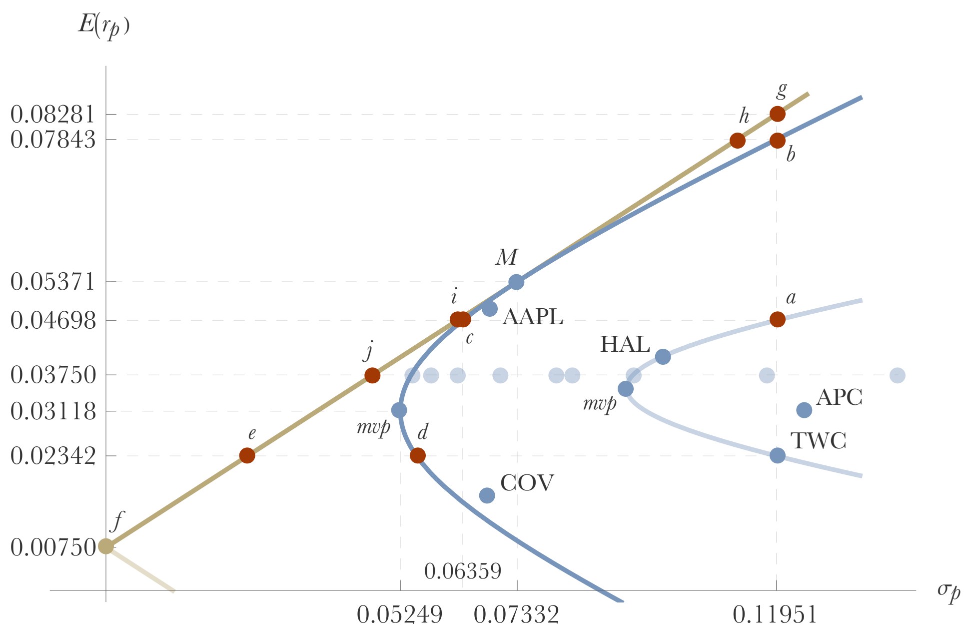


Notice that two of the asset weights in M are negative, -13.5 per cent for APC and -14.8 per cent for COV? No problem, the math is right, but the negative weights tell us that the optimal linear frontier in the example cannot represent an equilibrium in capital markets. That is because risky assets are always in positive supply, and if all the other people like you, who prefer higher return and lower risk, want to hold portfolio M, then the assets in M must be held (owned by or invested in) by everyone. In that case, the weight of every asset in M must be positive. The reason that not all of the assets weights are positive in our example is because it is just an example: the risk-free rate of 0.75 per cent is not an equilibrium interest rate—it is made up—and the expected returns and covariances of the five risky assets are not equilibrium values—they are estimated from historical data. But for our purpose of learning how to form portfolios, we will turn a blind eye to negative weights in M should they appear. We will assume that M contains all risky assets that are available, even if we are working with only five, and that the risk and return of all of the assets are equilibrium values. These assumptions, roughly speaking, let us refer to M as the *Market* portfolio, the portfolio that everyone holds when they can borrow or lend at the same risk-free rate of interest. And in that case, the name of the optimal linear frontier is elevated to the *Capital Market Line*.

Capital Market Line

$$E(r_p) = r_f + \frac{E(r_M) - rf}{\sigma_M}\sigma_p$$

The Capital Market Line is shown, with more reference points than an acupuncture chart, in the next figure. These reference points are for you to interpret. For example, what can you say about CML portfolios *g* and *h*? What are their allocations of *f* and *M*? Can you calculate *g*'s expected return only knowing that it has the same standard deviation as TWC? Can you calculate *h*'s standard deviation?



Choosing a Portfolio

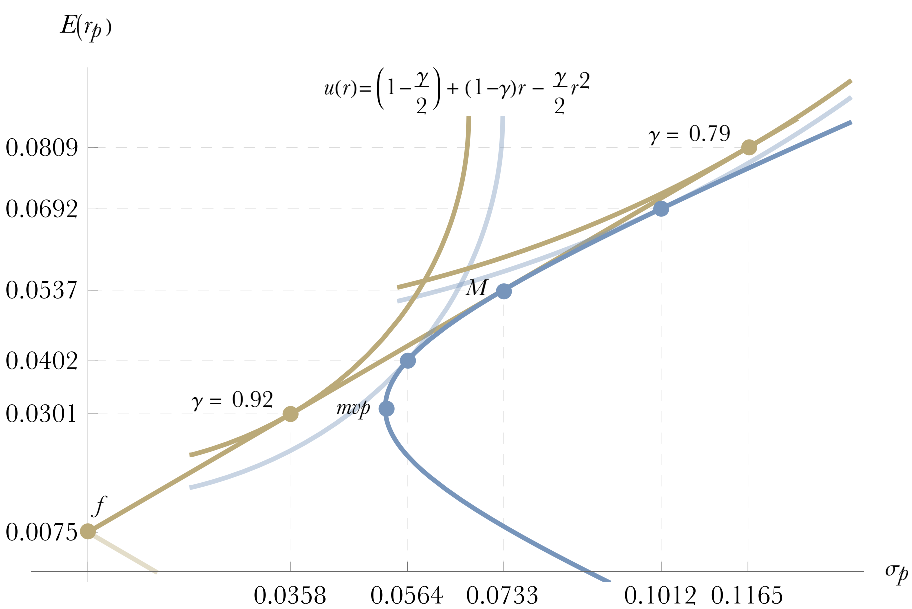
We're finally at the spot where we can talk about portfolio theory rather than just portfolio math. Utility theory says that a person will make choices that maximize their utility, but for decisions involving risk, they maximize expected utility (more on that in chapter 13). Suppose that Lucy and Ricky have utility that is quadratic in random return,

$$u(r) = \left(1 - \frac{\gamma}{2}\right) + (1 - \gamma)\, r - \frac{\gamma}{2} r^2$$

where Lucy's risk aversion parameter, γ, is 0.92 and Ricky's is 0.79. I'll let you show that, by taking the expected value of this utility function, the expected utility function is

$$E[u(r)] = \left(1 - \frac{\gamma}{2}\right) + (1 - \gamma)\, E(r) - \frac{\gamma}{2} E(r)^2 - \frac{\gamma}{2}\sigma_r^2$$

Can you see from their expected utility function why a bigger value of γ implies the person is more risk averse? Expected utility is, conveniently, a function of expected return and variance of return, which is exactly how we developed our portfolio math. Lucy and Ricky's optimal portfolios are show by the blue points and brown straddling M. The blue points are portfolios of the five assets without risk-free borrowing or lending, while the brown points are the Market with some borrowing or lending.



Some work for you: derive expressions for the optimal portfolios on the five-asset frontier and the Capital Market Line. To do that you'll need to equate the marginal rate of substitution for the indifference curve of the expected utility function,

$$MRS = \left|\frac{\delta E(r)}{\delta \sigma}\right|$$

to the slope of the five-asset frontier and the slope of the Capital Market Line. The expressions you end up with will be simple. Work out the composition of all four portfolios from the information in the figure even if you cannot work out the expressions for the optimal portfolios.

An implication of portfolio theory is that quadratic utility (mean-variance expected utility) is a sufficient condition for mean-variance choice. English translation: you have to care only about mean and variance in order choose only according to mean and variance. You like skewness? No good. How about kurtosis (no, it's not a type of facial rash)? Still no good. Quadratic utility, however, is not a necessary condition. If asset returns are jointly normally distributed, then mean-variance portfolio choice holds no matter what kind of weird utility function you may have. So, do you think asset returns are normally distributed?

Exercises

Math

1. Expand $\sigma_p^2 = E[r_p - E(r_p)]^2$ to show that the variance of return of a two-asset portfolio is

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}$$

2. Simplify the equation for the variance of a two-asset portfolio for the case where the correlation between the two assets is +1. Interpret the result.

3. Simplify the equation for the variance of a two-asset portfolio for the case where the correlation between the two assets is -1. Interpret the result.

4. Expand $\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}$ for the variance of a three-asset portfolio.

5. Expand $\sigma_p^2 = \mathbf{w}^\top \cdot \mathbf{V} \cdot \mathbf{w}$ for a three-asset portfolio to confirm your result in exercise 4. Show all steps.

6. Compute the covariance matrix of returns for stocks AGU, FTS, MFI, and VRX, using

$$\mathbf{V} = \frac{\mathbf{X}^\top \cdot \mathbf{X}}{n}$$

where $\mathbf{X}$ is a centred data matrix (the returns for each stock minus their mean) and $n = 72$ is the number of months. The data are in "Stock Returns.csv". This file contains monthly returns on a sample of Canadian stocks in the S&P/TSX index, June 2014.

Portfolios

Form two- and five-stock portfolios using the data in "Stock Returns.csv". Work with stocks AT-D.B, BA, BNS, PJC.A, and YRI. For exercises 1 to 11 your in-sample or estimation period is the first 60 months, January 2008 through December 2012. For exercise 12, use the last 12 months as your hold-out sample or actual investment period. Assume that the risk-free rate of interest is 0.18 per cent whenever your calculations require it.

7. Make a table that summarizes the returns on the five stocks. Your table should include means, medians, standard deviations, annualized holding period returns, and the percentage of months in which each stock's return was negative. Interpret your table. What is the difference between mean and median returns? What is the difference between mean return and holding period return? Does a higher mean return go hand in hand with a higher holding period return?

8. Compute the covariance and correlation matrix of returns. Which companies' returns are most strongly correlated? Which are least correlated? Negatively correlated? Can you suggest why?

9. Make a table of portfolios of ATD.B and BA. Let the investment weight in either stock go from -120 per cent to 120 per cent in increments of 10 per cent. Include the minimum-variance portfolio as one of the portfolios. Report the investment weights, expected return, and standard deviation for each portfolio. Explain your table. What are the investment weights? What does it mean for an investment weight to be negative?

10. Plot the two-stock frontier. Explain it. Why does the frontier pass through the points representing a 100 per cent investment in ATD.B or BA? Must it?

11. What would the portfolio frontier for ATD.B and BA look like if the correlation of their returns was different from what you computed in exercise 2? Plot your original frontier from exercise 4 and portfolio frontiers for assumed correlations of -1, -0.5, 0.5, and +1 all in one graph. Explain why the frontiers look the way they do. Is there practical advice that comes from this?

12. Make a table of portfolios of all five stocks like the one you made in exercise 7, but this time let the range of portfolio expected returns run from 0 per cent to 2.5 per cent in increments of 0.5 per cent. Insert the minimum-variance portfolio and portfolios that have the same expected returns as each of the five stocks. Describe your table.

13. What is the expected return on the Market portfolio if you assume that the five stocks are the entire market?

14. What is the composition of the Market portfolio?

15. Make a table like the one you made in exercise 12 but this time for portfolios that lie on the Capital Market Line. Use the same range of expected returns that you used for the table in exercise 12. Include portfolios that have the same expected returns as each of the five stocks. Explain your table, making comparisons to your table from exercise 12.

16. Plot the Capital Market Line and the five-stock frontier. Describe your plot.

17. Your friend Nigel wants an efficient portfolio of the five stocks (not with the risk-free asset and Market portfolio) that has the same expected return as Yamana Gold. What is the composition of the portfolio? What is its standard deviation?

18. How well did Nigel's portfolio perform during the 12 months following the in-sample period (estimation period)? Do not rebalance the portfolio weights during the investment period. Form the portfolio at the start of the 12 months and close your position at the end. This is called a buy-and-hold (makes sense, eh?) investment. Fill in the table and see.

Period	*Mean*	*Median*	*Std. Dev.*	*Annual Return*	*Months Neg. (%)*
Estimation					
Investment					

Answers

Math

No solutions provided for exercises 1 to 5. Compare your answers with your classmates!

1. Left to you.
2. Left to you.
3. Left to you.
4. Left to you.
5. Left to you.
6. $$V = \begin{pmatrix} & AUG & FTS & MFI & VRX \\ AUG & 0.0120616 & 0.000327956 & 0.000391855 & 0.00019593 \\ FTS & 0.000327956 & 0.00143238 & 0.000428619 & -0.000139233 \\ MFI & 0.000391855 & 0.000428619 & 0.00616395 & -0.000195186 \\ VRX & 0.00019593 & -0.000139233 & -0.000195186 & 0.00896904 \end{pmatrix}$$

Portfolios

7. Couche Tard (ATD.B) performed best, earning an average monthly return of 2.1 per cent or about 23 per cent a year. Yamana Gold (YRI) was the riskiest. All five stocks lost money 40 to 45 per cent of the time. *Annual return* is the annualized holding period return.

Monthly Return Performance, January 2008 through December 2012

Stock	*Mean*	*Median*	*Std. Dev.*	*Annual Return*	*% Months Neg.*
ATD.B	0.02133	0.00821	0.09009	0.22967	0.40
BA	0.00591	0.00535	0.03668	0.06468	0.38
BNS	0.00743	0.00521	0.05769	0.07120	0.47
PJC.A	0.00819	0.00385	0.06428	0.07663	0.45
YRI	0.01592	0.02983	0.14304	0.06975	0.45

8. Covariance matrix $\mathbf{V}$ and the correlation matrix ρ.

$$\mathbf{V} = \begin{pmatrix} 0.00811686 & 0.000266965 & 0.0000145114 & 0.00183428 & -0.00206091 \\ 0.000266965 & 0.00134522 & 0.000341095 & 0.000343396 & 0.000195863 \\ 0.0000145114 & 0.000341095 & 0.00332799 & 0.000620237 & -0.000179134 \\ 0.00183428 & 0.000343396 & 0.000620237 & 0.00413209 & 0.00042017 \\ -0.00206091 & 0.000195863 & -0.000179134 & 0.00042017 & 0.0204606 \end{pmatrix}$$

$$\rho = \begin{pmatrix} 1. & 0.0807912 & 0.00279206 & 0.316729 & -0.159922 \\ 0.0807912 & 1. & 0.161208 & 0.145651 & 0.0373333 \\ 0.00279206 & 0.161208 & 1. & 0.167256 & -0.0217084 \\ 0.316729 & 0.145651 & 0.167256 & 1. & 0.0456964 \\ -0.159922 & 0.0373333 & -0.0217084 & 0.0456964 & 1. \end{pmatrix}$$

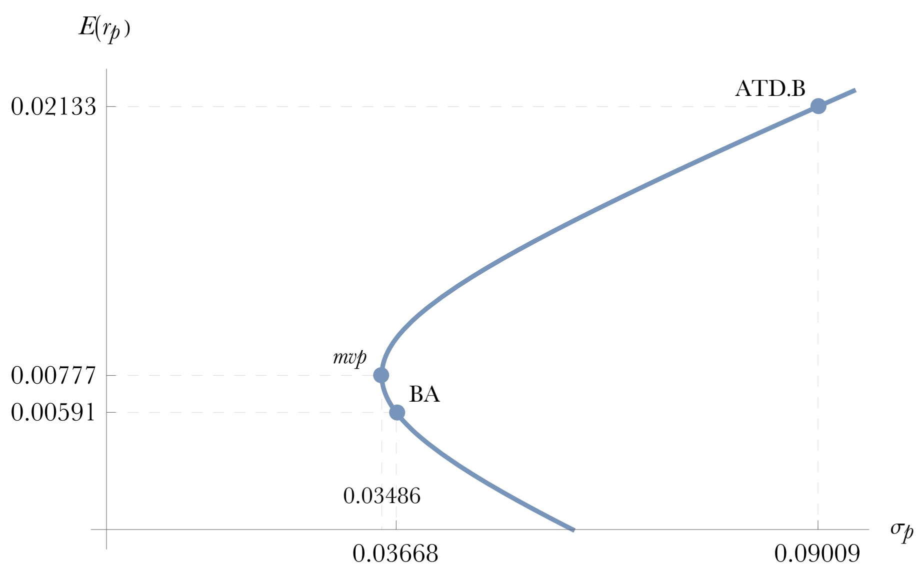
Couche Tard and Jean Coutu are the most strongly correlated. The least correlated are Couche Tard and Bank of Nova Scotia. Couche Tard and Bank of Nova Scotia are negatively correlated with Yamana Gold but Jean Coutu is not. Your thoughts?

Two-stock frontier

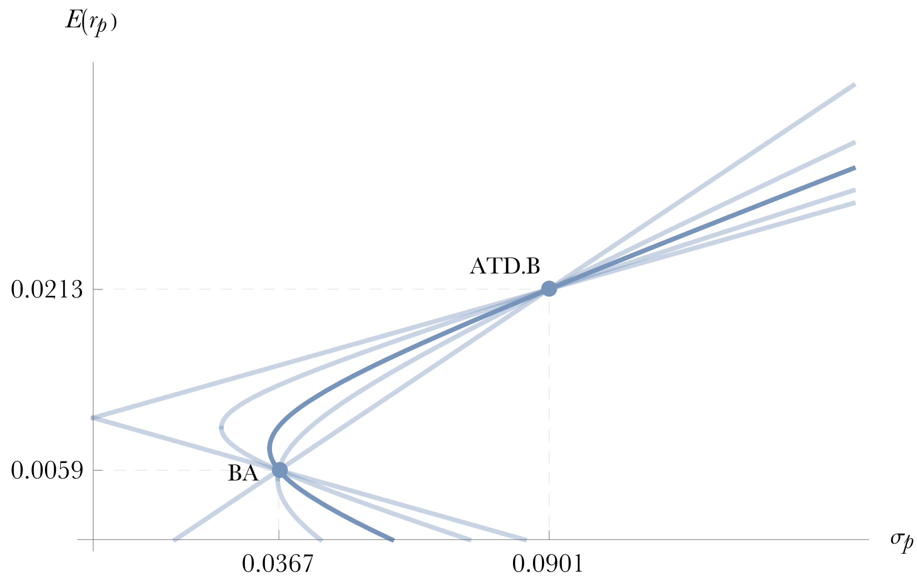
9. Portfolios of ATD.B and BA. It's pretty easy to spot the minimum-variance portfolio.

		Investment Weights	
σ_p	$E(R_p)$	*ATD.B*	*BA*
0.129575	-0.012597	-1.20000	2.20000
0.111489	-0.009513	-1.00000	2.00000
0.093725	-0.006429	-0.80000	1.80000
0.076507	-0.003345	-0.60000	1.60000
0.060302	-0.000261	-0.40000	1.40000
0.046192	0.002824	-0.20000	1.20000
0.036677	0.005908	0.00000	1.00000
0.034857	0.007770	0.12077	0.87923
0.035652	0.008992	0.20000	0.80000
0.043716	0.012076	0.40000	0.60000
0.057144	0.015160	0.60000	0.40000
0.073034	0.018245	0.80000	0.20000
0.090094	0.021329	1.00000	0.00000
0.107768	0.024413	1.20000	-0.20000

10. Two-stock frontier.



11. The effect of the correlation between two stocks on the shape of their portfolio frontier. The two linear frontiers should confirm your math in exercises 2 and 3.



Five-stock frontier

For the five-stock portfolios you needed to compute the frontier parameters

$$A = 9.0093;\ B = 0.109394;\ C = 1{,}104.41;\ D = 39.6486$$

and for the optimal portfolio weights

$$\mathbf{g} = \{-0.37991, 1.0288, 0.15962, 0.28598, -0.0944881\}$$

$$\mathbf{h} = \{56.5372, -55.684, 4.52161, -22.6687, 17.2939\}$$

12. Portfolio tables for the five stocks. The second version includes reference portfolios.

			Investment Weights				
	σ_p	$E(R_p)$	*ATD.B*	*BA*	*BNS*	*PJC.A*	*YRI*
	0.0525270	0.0000000	-0.3799100	1.0288000	0.1596200	0.2859800	-0.0944881
	0.0343973	0.0050000	-0.0972238	0.7503780	0.1822280	0.1726370	-0.0080188
	0.0316230	0.0100000	0.1854620	0.4719580	0.2048360	0.0592937	0.0784506
	0.0470064	0.0150000	0.4681480	0.1935370	0.2274440	-0.0540497	0.1649200
	0.0693682	0.0200000	0.7508340	-0.0848826	0.2500520	-0.1673930	0.2513890
	0.0938458	0.0250000	1.0335200	-0.3633030	0.2726600	-0.2807360	0.3378590

			Investment Weights				
	σ_p	$E(R_p)$	*ATD.B*	*BA*	*BNS*	*PJC.A*	*YRI*
	0.0525270	0.0000000	-0.3799100	1.0288000	0.1596200	0.2859800	-0.0944881
⦿ *F*	0.0450701	0.0018000	-0.2781430	0.9285660	0.1677590	0.2451770	-0.0633591
	0.0343973	0.0050000	-0.0972238	0.7503780	0.1822280	0.1726370	-0.0080188
⦿ *BA*	0.0323489	0.0059077	-0.0459031	0.6998310	0.1863320	0.1520600	0.0076795
⦿ *BNS*	0.0303353	0.0074294	0.0401267	0.6151000	0.1932130	0.1175660	0.0339947
⦿ *mvp*	0.0300908	0.0081576	0.0812953	0.5745520	0.1965050	0.1010600	0.0465875
⦿ *PJC.A*	0.0300914	0.0081929	0.0832957	0.5725820	0.1966650	0.1002570	0.0471994
	0.0316230	0.0100000	0.1854620	0.4719580	0.2048360	0.0592937	0.0784506
⦿ *M*	0.0404186	0.0132705	0.3703700	0.2898400	0.2196240	-0.0148453	0.1350110
	0.0470064	0.0150000	0.4681480	0.1935370	0.2274440	-0.0540497	0.1649200
⦿ *YRI*	0.0508105	0.0159150	0.5198780	0.1425880	0.2315810	-0.0747908	0.1807430
	0.0693682	0.0200000	0.7508340	-0.0848826	0.2500520	-0.1673930	0.2513890
⦿ *ATD.B*	0.0757477	0.0213287	0.8259550	-0.1588700	0.2560600	-0.1975130	0.2743680
	0.0938458	0.0250000	1.0335200	-0.3633030	0.2726600	-0.2807360	0.3378590

⦿ is a frontier portfolio having the same expected return as the corresponding stock in the portfolio. *mvp* is the minimum-variance portfolio. The risk-free rate asset *F* and the market portfolio *M* are included for reference.

13. The expected return on the Market portfolio is 1.3 per cent (0.0132705).
14. The composition of the Market portfolio is

$$\{0.37037, 0.28984, 0.219624, -0.0148453, 0.135011\}$$

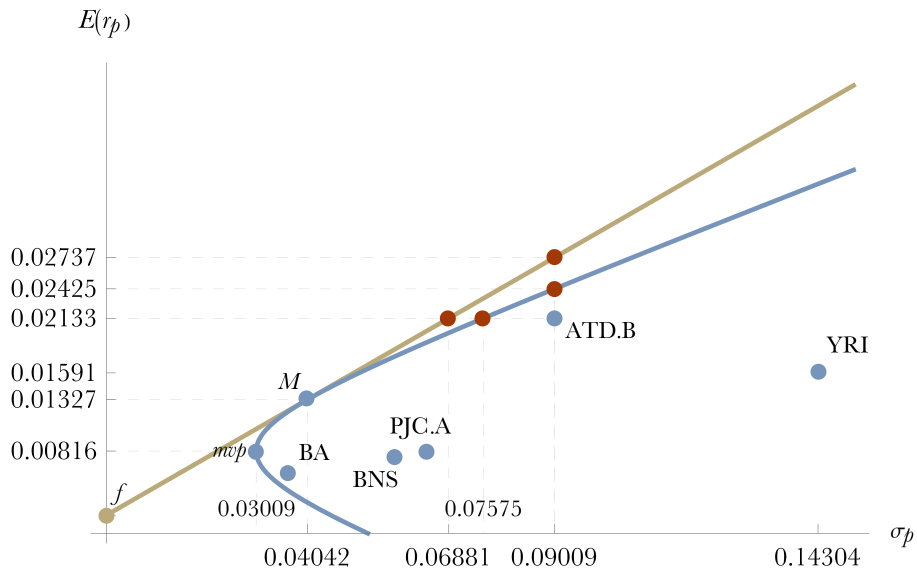
Knowing the weights, the standard deviation of the Market portfolio is 0.0404186.

15. Portfolios on the Capital Market Line.

	Frontier	*CML*		*Investment Weights*	
	σ_p	σ_p	$E(R_p)$	F	M
	0.052527	0.006343	0.000000	1.15692	-0.15692
⦿ *F*	0.045070	0.000000	0.001800	1.00000	0.00000
	0.034397	0.011276	0.005000	0.72103	0.27898
⦿ *BA*	0.032349	0.014474	0.005908	0.64189	0.35811
⦿ *BNS*	0.030335	0.019836	0.007429	0.50923	0.49077
⦿ *mvp*	0.030091	0.022402	0.008158	0.44575	0.55425
⦿ *PJC.A*	0.030091	0.022527	0.008193	0.44267	0.55734
	0.031623	0.028894	0.010000	0.28513	0.71487
⦿ *M*	0.040419	0.040419	0.013271	0.00000	1.00000
	0.047006	0.046513	0.015000	-0.15077	1.15077
⦿ *YRI*	0.050811	0.049737	0.015915	-0.23054	1.23054
	0.069368	0.064131	0.020000	-0.58667	1.58667
⦿ *ATD.B*	0.075748	0.068813	0.021329	-0.70251	1.70251
	0.093846	0.081750	0.025000	-1.02257	2.02257

⦿ is a frontier portfolio having the same expected return as the corresponding stock in the portfolio. *F* is the risk-free asset, *mvp* the minimum-variance portfolio, and *M* the market portfolio. I've included the five-stock frontier standard deviations for comparison to the CML standard deviation.

16. The five-stock frontier and the Capital Market Line.



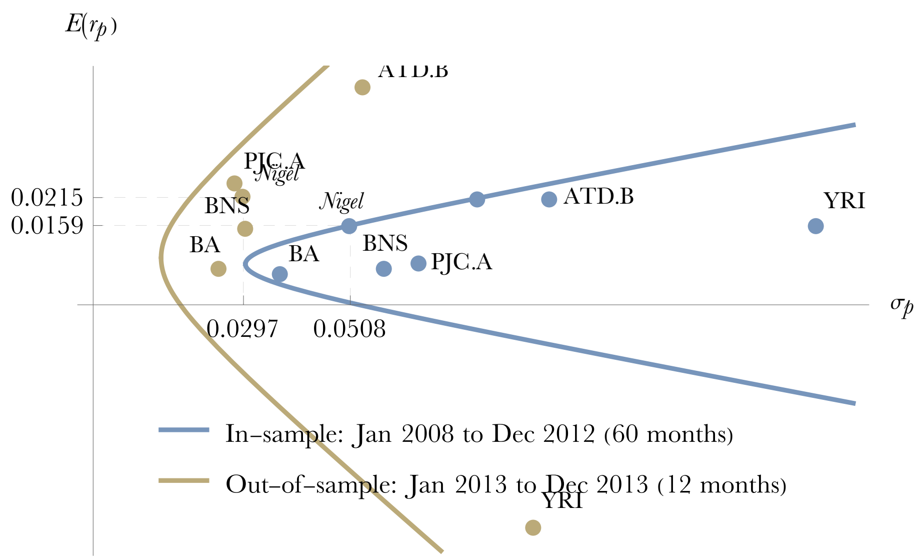
17. The composition of Nigel's portfolio is

$$\{0.519878, 0.142588, 0.231581, -0.0747908, 0.180743\}$$

The standard deviation is 5.1 per cent (0.0508105), which is considerably less than YRI's standard deviation of 14.3 per cent (0.14304).

18. Performance of Nigel's portfolio. The estimation period is Jan 2008 to Dec 2012 (60 months); the investment period is Jan 2013 to Dec 2013 (12 months). Sorry about the cluttered plot.

Period	*Mean*	*Median*	*Std. Dev.*	*Annual Return*	*Months Neg. (%)*
Estimation	0.01592	0.00577	0.05081	0.16308	0.43
Investment	0.02152	0.02100	0.02968	0.28464	0.25

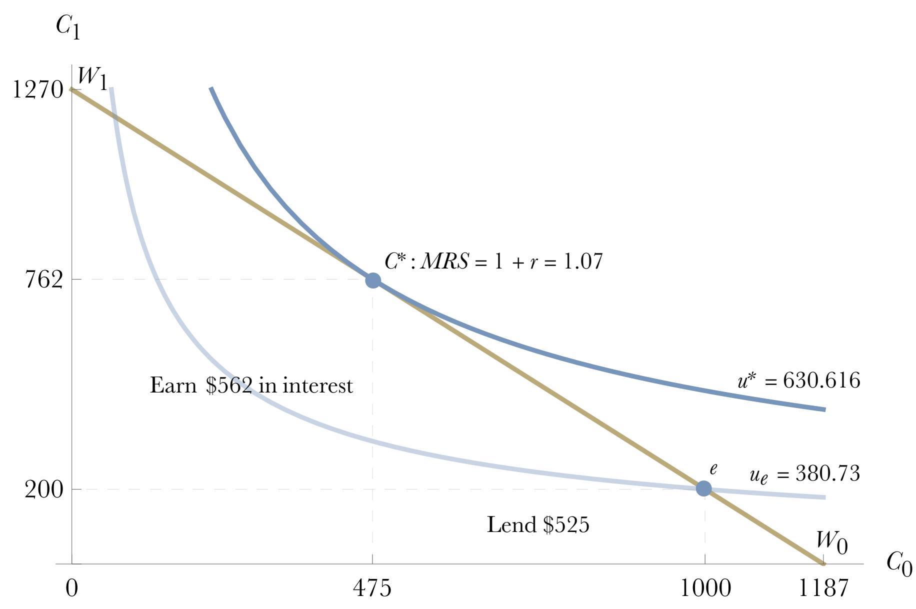


2
CHOICE UNDER CERTAINTY

A capital market may be as simple as a market for loans. Borrow to consume more now at the cost of having less later; lend to consume more later by giving up some now. Having the choice of shifting consumption across time leaves us better off.[1] The price of consuming \$1 worth of anything today is the rate of interest. That is why the rate of interest is called the *time value of money*. The opportunity to borrow can also leave us better off by making us wealthier when the money raised is used to finance real production that earns a higher return than the rate of interest.

Optimal Consumption

Like most economic theories, utility theory tries to be one-size-fits-all. It can be used to describe someone's choice of how much to spend now and in the future just as easily as we used it to describe their choice of goods, at any given time, as we did in chapter 2. Good x becomes C_0, for current consumption or the dollars that you have today, and good y becomes C_1, for future consumption or the dollars you will have next year. You can't consume dollars directly, of course, but the more you have, the more real stuff you can consume, and it really doesn't matter what the stuff is. The graph shows Lucy's intertemporal consumption optimum. Her tastes are represented



by a Cobb-Douglas utility function, $u = C_0^{\alpha} C_0^{1-\alpha}$ with $\alpha = 0.4$. Dollars plotted against dollars may seem strange at first, but they are different things because one is dollars today and the other is dollars next year. (And speaking of one-size-fits all, there is nothing stopping us from extending the

[1] Having a choice can't make you worse off, right? After all, it's your choice! Hmmm, unless of course having choices overwhelms you.

dates to year 2, 3, 4, right up to when Lucy kicks the proverbial bucket, and even beyond if she wants to leave an inheritance to her pet chihuahua, Ms Tinkles.) We'll assume that the future is certain despite the fact that it is not. This allows us to consider Lucy's choice as involving only her preference for consumption over time independent of her preferences for risk.

Lucy has an endowment of \$1,000 today and \$200 next year (point *e*). At a seven per cent rate of interest, her wealth is \$1,186.92, which is where her budget line crosses the horizontal axis. The budget line or wealth constraint is exactly the same as it is for real good and services x and y except that it is now represents choices for consumption dollars, so

$$W_0 = p_x x + p_y y$$

becomes

$$W_0 = p_0 C_0 + p_1 C_1$$

and the only thing that needs to be done is to interpret p_0 and p_1. If C_0 is current consumption measured in dollars, then p_0 must be 1 because the value of one dollar today is, well, one dollar. And if C_1 is consumption next year but wealth is being measured in dollars today, then p_1 must be the present value of one dollar, $\frac{1}{1+r}$, where r is the risk-free rate of interest. Substituting the interpreted prices into the wealth constraint gives

$$W_0 = C_0 + \frac{C_1}{1+r} = \$1{,}000 + \frac{\$200}{1.07} = \$1{,}186.92$$

which says, very neatly, that your wealth is the present value of the cash flow stream to which you are entitled over the course of your life. Now rearrange the equation so that it is in the standard format for an equation for a line

$$C_1 = -(1+r)C_0 + (1+r)W_0 = -(1+r)C_0 + W_1$$

The price ratio is $1 + r$, and it is the absolute value of the slope of the budget line as before. Consuming one more dollar today means foregoing a dollar plus interest later and vice versa. Another way to say this is that $1 + r$ is the price of current consumption. $(1 + r)W_0$ is the intercept on the C_1 or vertical axis, which is simply wealth measured in one year's time or W_1.

Lucy maximizes her utility by choosing to consume \$474.77 today and \$762 next year. She gave up \$525.23 this year, which at seven per cent, gives her an extra \$562 next year for a total of \$762. You can see why Lucy is a lender and not a borrower. Her marginal rate of substitution at his endowment point is less than the price ratio

$$MRS_e < 1 + r$$

With her endowment Lucy values consuming one more dollar today less than the market does. So she lends, moving up her budget line, and with each dollar lent, the dollars she has to consume now are becoming relatively scarcer, pushing up her marginal rate of substitution. She stops lending, having maximized her utility, when her marginal rate of substitution is exactly equal to because both she and everyone else values the next current dollar at exactly seven per cent. That is why the optimum is defined by

$$C^*: \; MRS = 1 + r$$

A few things to note about the intertemporal consumption optimum. One is that Lucy's utility-maximizing choice is less extreme than her endowment in that \$475 and \$762 are closer to one another than are \$1,000 and \$200. This is called *consumption smoothing*, and it makes sense: most people prefer to have an income stream that is smoother (or, better still, smooth and increasing over time) than one that is bumpy (feast one year, starve the next). You can see that consumption smoothing is a direct result of modelling with convex indifference curves. If Lucy's endowment was *e'* instead of *e*, she'd borrow to reach *C** and smooth her consumption. No matter what specific mathematical function is used to represent a person's tastes, if it is convex, the consumption optimum will always lie between *e* and *e'*; the optimum is a weighted average of the two and therefore "smoother". Another thing to note is that Lucy is not wealthier for having chosen *C**, and she is not wealthier for having lent. *C** is worth \$1,186.92 just as *e* and every other point on the budget line is. But she is better off because she prefers *C** to *e*. Despite Scotiabank's slogan, *you're richer than you think*, a capital market by itself does not create wealth. It gives us choice, and that is a good thing. For wealth creation, investment in real production is necessary. The final thing to note is more subtle. Everyone else has presumably borrowed or lent to reach their own consumption optimum, so everyone must be evaluating their next dollar to spend (borrow to take that vacation?) at the market rate of interest. Utility theory implies that all of us implicitly agree what the time value of money is. If that is true in real life—an empirical question—then governments can justify using market interest rates as discount factors in cost-benefit analysis. If not, those analyses don't mean all that much.

The Math of Optimal Consumption

Lucy choses to consume \$475 now and \$762 next year by lending \$525 at seven per cent to earn \$562 next year. How do we come up with those numbers?

Lucy's tastes for consumption are represented by a Cobb-Douglas utility function, $u(C_0, C_1) = C_0^{\alpha} C_1^{1-\alpha}$, with $\alpha = 0.4$. She maximizes her utility with respect to her current and future consumption, subject to the constraint that she cannot consume more than her endowed wealth. For any utility function, $u(C_0, C_1)$, this can be written

$$\underset{<C_0, C_1>}{Max} \; u(C_0, C_1) \text{ s.t. } C_0 + \frac{C_1}{1+r} = e_0 + \frac{e_1}{1+r} = W_0$$

The solution to the optimization problem is

$$C^*: \; MRS = \frac{u'(C_0)}{u'(C_1)} = 1 + r$$

which says that utility is maximized when a person consumes so that their marginal rate of substitution is equal to one plus the interest rate.

How does this relate to the graph? *MRS* is the absolute value of the slope of an indifference curve

$$MRS = \left|\frac{dC_1}{dC_0}\right|$$

which you can find by differentiating u separately by C_0 and C_1, and then taking the ratio of the two derivatives

$$MRS = \left|\frac{dC_1}{dC_0}\right| = \frac{\frac{\delta u}{\delta C_0}}{\frac{\delta u}{\delta C_1}} \equiv \frac{u'(C_0)}{u'(C_1)}$$

You can see that *MRS* is the ratio of marginal utilities for current and future consumption. It is a "personal" or "psychological" price ratio, one at which a person is willing trade future dollars for dollars now. The price ratio that the market trades future for current dollars is one plus the rate of interest; it is the slope of the wealth constraint

$$\frac{dC_1}{dC_0} = 1 + r$$

When the two price ratios are equal a person is consuming at the point where they place the same value as the market does on having one more dollar now. There are no transactions that can make the person better off.

For Lucy,

$$MRS = \frac{\alpha}{1-\alpha}\frac{C_1}{C_0}$$

At her optimum,

$$\frac{\alpha}{1-\alpha}\frac{C_1}{C_0} = 1 + r$$

which, substituted into the wealth constraint, gives

$$C_0^* = \alpha W_0 = \alpha\left(e_0 + \frac{e_1}{1+r}\right) = \$474.77$$
$$C_1^* = (1-\alpha)(1+r)W_0 = (1-\alpha)\big((1+r)e_0 + e_1\big) = \$762$$

for Lucy's smoother consumption of \$474.77 now and \$762 next year.

Demand for Consumption

When we say demand for consumption, we usually mean current consumption. The demand for consumption is a function of the interest rate—the price— for a given endowment and taste parameters.

$$C = f(r, e, \text{taste parameters})$$

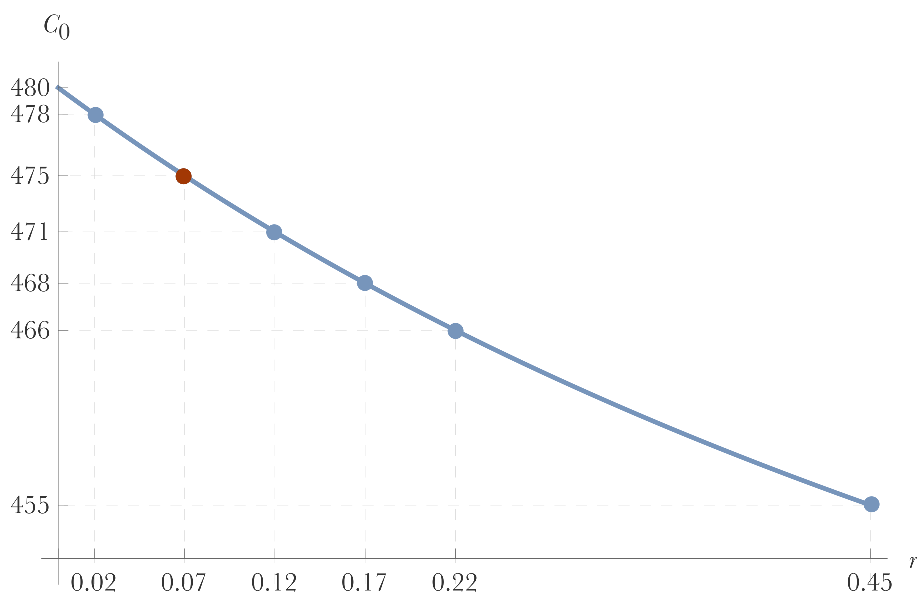
Lucy's demand function for consumption is the expression for her optimal current consumption but with the interest rate allowed to vary.

$$C_0 = \alpha\left(e_0 + \frac{e_1}{1+r}\right)$$

It can be written for any year looking one year ahead

$$C_t = \alpha\left(e_t + \frac{e_{t+1}}{1+r}\right)$$

for Lucy's Cobb-Douglas utility function. Downward-sloping demand just like we see for apples and oranges.[2]



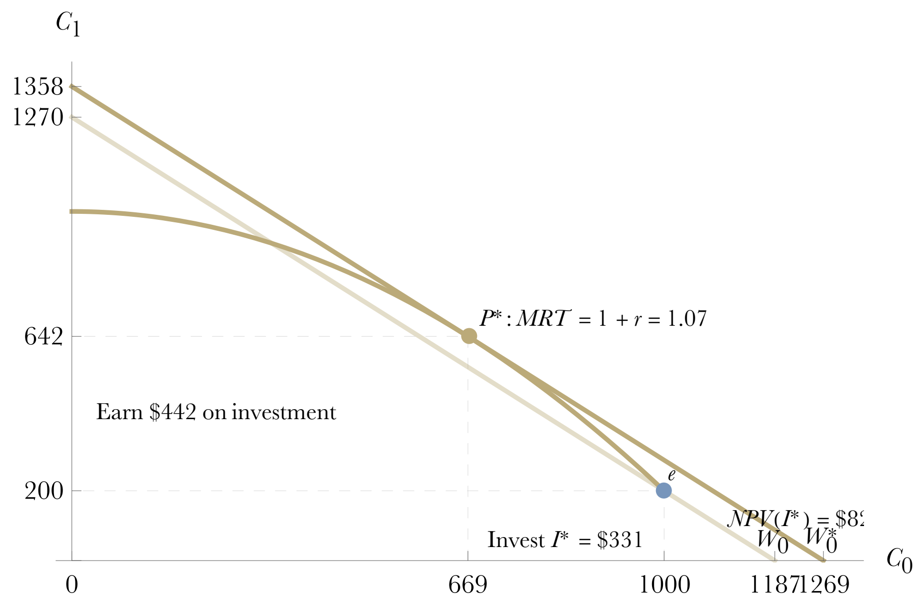
[2] In this plot quantity is on the vertical axis and price on the horizontal.

REAL INVESTMENT

If Lucy has opportunities to invest in real production, she will do so to make herself as wealthy as possible and then consume according to her tastes in line with that greater wealth. Investment followed by consumption: it is usually described as happening in that order but really the idea is that it just happens—the decision, that is—all at once.

Her investment opportunities are represented by the concave investment opportunity schedule or production function in the figure. It is a parabola in this case:

$$C_1 = g - hC_0^2,\ g = \$1{,}000,\ h = \frac{1}{\$1{,}250}$$



The concavity means that, going leftwards from e, the first dollar she invests earns the most, the next a little less, the third still less, and so on. In other words, the absolute value of the slope of her investment opportunity schedule (marginal rate of transformation or *MRT*) is one plus the marginal return on investment.

$$MRT = 1 + ROI = \left|\frac{\delta C_1}{\delta C_0}\right| = 2hC_0$$

The *MRT* on the first dollar Lucy invests is 1.6 or a return of 60 per cent. The second dollar earns 59.92 per cent and the third 59.84 per cent. You can find the *MRT* for any investment opportunity schedule written as an implicit function, $T(C_0, C_1)$, the same way we found the marginal rate of substitution for an indifference curve.

$$MRT = 1 + ROI = \left|\frac{\delta C_1}{\delta C_0}\right| = \frac{\frac{\delta T}{\delta C_0}}{\frac{\delta T}{\delta C_1}}$$

Step 1. Lucy maximizes her wealth by making sure that every dollar she invests either increases or maintains her wealth; in other words, every dollar invested earns a non-negative net present value ($NPV \geq \$0$).

$$\underset{<C_0,C_1>}{Max}\; W_0 = \left(C_0 + \frac{C_1}{1+r}\right) \text{ s.t. } T(C_0, C_1) = 0$$

The solution is point P^*, where the wealth constraint is tangent to the investment opportunity schedule as

$$P^*:\; MRT = 1 + r$$

Solving for Lucy's *IOS*,

$$\begin{aligned} MRT &= 2hC_0 = 1 + r \\ \therefore C_0 &= \frac{1+r}{2h} = 668.75 \\ \therefore P^* &= (668.75, 642.219) \end{aligned}$$

She invests about \$331 = \$1,000 - \$669 and increases her wealth by \$82 to \$1,269, the net present value of the investment.

$$\begin{aligned} NPV(I^*) &= -I^* + \frac{\$ROI}{1+r} = -\$331.25 + \frac{\$442.219}{1.07} = \$82.0386 \\ W_0^* &= W_0 + NPV(I^*) = \$1268.95 \end{aligned}$$

Step 2. Lucy chooses the consumption stream that maximizes her utility. You already know how to do this. It is the solution to

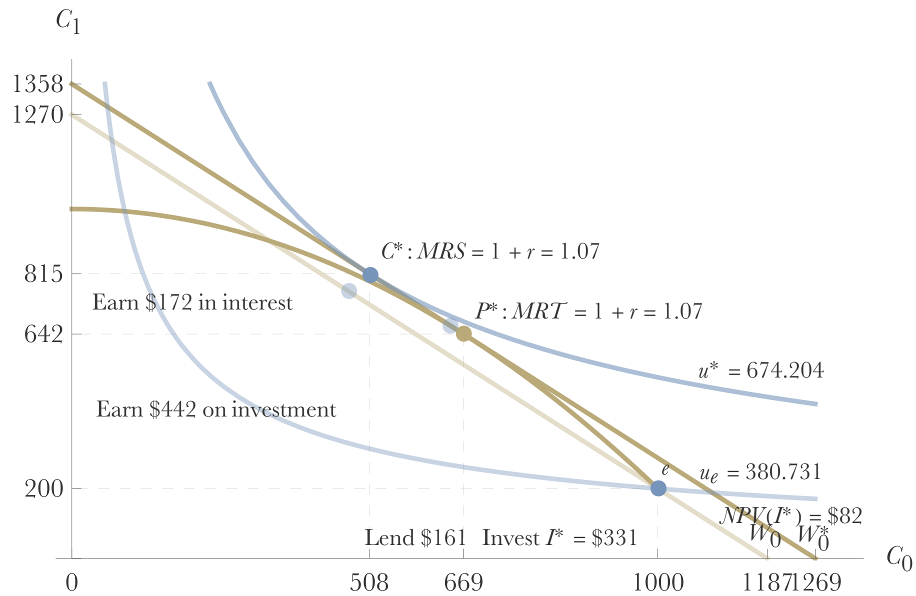
$$C^*:\; MRS = \frac{\alpha}{1-\alpha}\frac{C_1}{C_0} = 1 + r$$

but applied to Lucy's new (maximized) wealth. The details are in the figure on the next page.

The result for an exchange (borrowing and lending) and production (real investment) is called Fisher Separation because optimal consumption is independent of optimal investment. How you want to spend (the second condition below) does not interfere with how you should invest (the first condition).

Fisher Separation

$$P^* : MRT = 1 + r$$
$$C^* : MRS = 1 + r$$



The Equilibrium Interest Rate

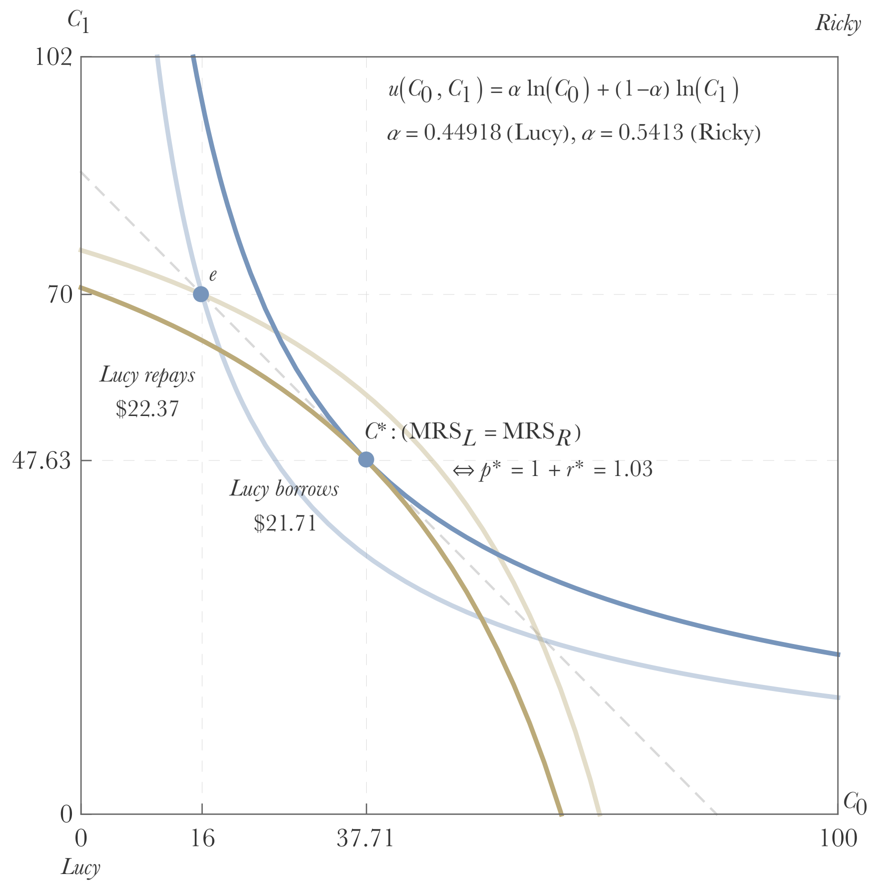
Up to this point we have been looking at intertemporal consumption in *partial equilibrium*. Lucy achieved her preferred consumption stream by investing part of her current endowment and borrowing *given* that the interest rate was seven per cent.

Where does the seven per cent interest rate come from? Equilibrium in the market for money. The equilibrium interest rate is the one that equates the number of dollars potential lenders are willing to lend with the number of dollars that potential borrowers are willing to borrow. You can guess that the supply and demand for dollars today depends upon everyones' endowments (think of this as the distribution of income both now and in the future), their tastes for consumption ('high' or 'low' marginal rates of substitution at the endowments), and individual real investment opportunities. All three of these variables interact.

People who have little today compared to what they will have in the future will have biggish marginal rates of substitution, relatively speaking. If these same people comprise a big share of the aggregate current endowment of the economy, the borrowing that they do from those whose current endowments are also comparatively small will occur at a higher interest rate than it otherwise would. But we can't talk about endowments without talking about tastes. You know that two people with the same endowment will have different marginal rates of substitution if their tastes differ. That is, after all, what it means to have different tastes. It is also possible that someone with a small current endowment relative to their future endowment has a smaller marginal rate of substitution than someone with a bigger current endowment relative to their future consumption. (Confused? Draw some indifference curves.) The person with the smaller relative current endowment values more current consumption less than the person with the bigger relative current endowment because of differences in their preferences (I don't have much now, but it doesn't matter because I'm not interested in consuming more now). So, endowments and tastes necessarily interact.

The figure below is an example of the determination of a three per cent rate of interest in *general equilibrium*. Including real investment opportunities in the Edgeworth box is a little advanced for this course, so the equilibrium is presented for a pure exchange economy. The aggregate current endowment is \$100, of which \$16 belongs to Lucy and \$84 to Ricky. The aggregate future endowment is \$102 with \$70 going to Lucy and \$32 to Ricky. Both Lucy and Ricky have the same natural log utility functions noted in the figure, but value of their taste parameter, α, differs. The economy is growing at a rate of two per cent.

At the endowment point, *e*, Lucy's marginal rate of substitution is bigger than Ricky's, so she borrows \$21.71 from Ricky and they agree that the repayment should be \$22.37. At the optimum, C^*, Lucy's marginal rate of substitution is equal to Ricky's is equal to 1.03. That implies an interest rate of three per cent (work the slope of the dashed line that is tangent to both Lucy and Ricky's indifference curves at C^*).



Change any one of the parameters—individual endowments, the growth rate of the economy (aggregate endowment), or tastes—and the equilibrium changes. For example, had the value of Lucy's taste parameter, , been a little smaller, 0.415964 rather than 0.44918, the equilibrium rate of interest would be minus three per cent!

THE MATH OF INTEREST RATE DETERMINATION

Let's work out the three per cent equilibrium interest rate. The equilibrium is shown as a Pareto Optimum in an Edgeworth box. The solution is a general equilibrium because both the interest rate and the allocation of optimal consumption are determined jointly. We'll show that the interest rate and optimal consumption depend on tastes (culture, demography) and endowments (the distribution of income) but do not depend directly on each other.

Step 1. Lucy and Ricky, acting as price takers, maximize their utilities. This means that each seeks to borrow or lend an amount so that their marginal rate of substitution is equal to one plus the interest rate. But the interest rate doesn't exist, except in equilibrium, so Lucy and Ricky are not "comparing," for lack of a better term, their own marginal rate of substitution to an actual interest rate but are comparing it implicitly to the other's marginal rate of substitution because they are, after all, transacting with one another. There are mutually-beneficial loans to be made as long as their marginal rates of substitution differ; equilibrium exists when they are the same.

Since Lucy and Ricky have the same utility function, $u(C_0, C_1) = \alpha \ln(C_0) + (1-\alpha)\ln(C_1)$, the condition for utility maximization is the same for both

$$MRS = \frac{\alpha}{1-\alpha} \cdot \frac{C_1}{C_0} = 1 + r$$

implying each will consume

$$C_0^* = \alpha W_0 = \alpha \left(e_0 + \frac{e_1}{1+r} \right)$$

$$C_1^* = (1-\alpha)(1+r)W_0 = (1-\alpha)\big((1+r)e_0 + e_1\big)$$

according to the value of their individual taste parameter α and their endowments.

Step 2. Solve for the equilibrium interest rate by imposing the market-clearing condition. It's just Supply = Demand, the bookkeeping that ensures everything adds up. The total of Lucy and Ricky's optimal current consumption from step 1, for example, must equal the total of their current endowments.

$$\begin{aligned} e_{0_L} + e_{0_R} &= C_{0_L}^* + C_{0_R}^* \\ &= \alpha_L \left(e_{0_L} + \frac{e_{1_L}}{1+r^*} \right) + \alpha_R \left(e_{0_R} + \frac{e_{1_R}}{1+r^*} \right) \end{aligned}$$

You could also apply the analogous market-clearing condition for future consumption, but you don't need to because one is all it takes to balance the books. Now solve for the equilibrium interest rate.

$$1 + r^* = \frac{\alpha_L e_{1_L} + \alpha_R e_{1_R}}{(1-\alpha_L)e_{0_L} + (1-\alpha_R)e_{0_R}}$$

Notice that interest rate depends on tastes (the α's) and endowments (the e's) but not on final optimal consumption (the C's). Evaluating r^* with the parameter values in our example gives an equilibrium interest of three per cent.

Step 3. Solve for the optimal allocation of consumption. Here is Lucy's optimal current consumption from step 1

$$C^*_{0_L} = \alpha_L \left(e_{0_L} + \frac{e_{1_L}}{1 + r^*} \right)$$

Eliminate r^* by substituting in the expression for r^* from step 2 to get (it's ugly)

$$C^*_{0_L} = \alpha_L \left(e_{0_L} + e_{1_L} \frac{(1 - \alpha_L) e_{0_L} + (1 - \alpha_R) e_{0_R}}{\alpha_L e_{1_L} + \alpha_R e_{1_R}} \right)$$

which, like the interest rate, depends only on tastes and endowments. It evaluates to 37.7137 in the example. Work out the other three consumption values (see table). You don't need more fancy equations to do it.

A Positive Interest Rate in a Pure Exchange Economy

	Lucy	*Ricky*
Taste parameter α	0.44918	0.5413
Endowment e	{16, 70}	{84, 32}
Optimum C^*	{37.7137, 47.6349}	{62.2863, 54.3651}
MRS(e)	3.56771	0.449552
MRS(C^*) = price ratio	1.03	1.03
Utility(e)	3.58555	3.98813
Utility(C^*)	3.75866	4.06935

EXERCISES

1. Lucy lives in a nice neighbourhood in a riskless, two-period, Fisher-type economy. This is what we know about her situation.

   The interest rate: r = 3.9 per cent

   Endowment: e = (\$240,000, \$106,400)

   Utility function: $u(C_0, C_1) = -e^{-\alpha C_0} - e^{-\alpha C_1}$, $\alpha = \dfrac{31}{10{,}000{,}000}$

   Investment opportunity schedule: $1 = \dfrac{C_0^2}{a^2} + \dfrac{C_1^2}{b^2}$, $a = 250{,}000$, $b = 380{,}000$

   Fill in the table and draw a graph, as well-labelled as practical, showing the Fisher Separation results. Explain it using plain language. Also draw a second graph showing Lucy's demand for current consumption as a function of the interest rate.

	Without Real Investment	*With Real Investment*
MRS at *C**		
MRT at *P** (1 + Marginal ROI)	-	
Optimal Production (*P**)	-	
Optimal Investment (*I**)	-	
Dollar ROI	-	
ROI	-	
NPV(*I**)	-	
Wealth		
Optimal Consumption (*C**)		
Utility at *C**		
Amount Lent (Borrowed)		
Principal + Interest Earned (Repayed)		

2. Read Loewenstein and Prelec (1992), and write a short description of the anomalies of intertemporal choice discussed in the article. Concentrate on the ideas; ignore the math.

3. Compute the equilibrium interest rate and consumption optimum for a two-person, two-period, riskless economy. Both Lucy and Ricky have Cobb-Douglas utility functions of the form

$$u(C_0, C_1) = \alpha \ln(C_0) + (1-\alpha)\ln(C_1)$$

Lucy's α is 0.46 and Ricky's 0.54. Her endowment is (16, 70) and his is (84, 33).

a. What is the growth rate of the economy?

b. What is the equilibrium rate of interest?

c. Is Ricky a borrower or lender in equilibrium? How much?

d. What is the consumption (Pareto) optimum? Illustrate the Pareto optimum in a drop-dead gorgeous, well-labeled Edgeworth box diagram.

e. All things being equal, what would be the effect on the interest rate if the growth rate of economy was higher?

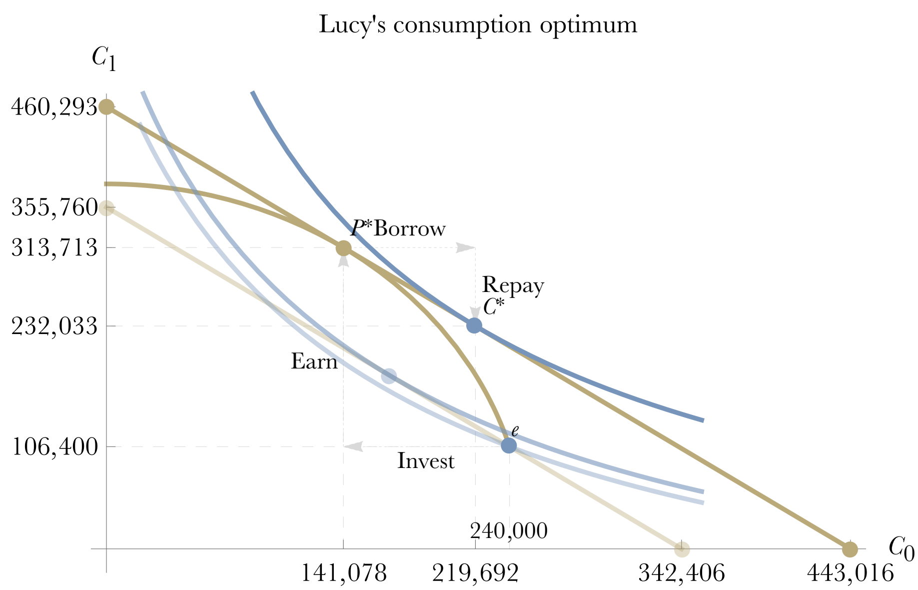
f. All things being equal, what would be the effect on the interest rate if Ricky's α was smaller?

Answers

1. Fisher Separation. Make sure you can interpret all of the figures. Math follows.

	Without Real Investment	*With Real Investment*
MRS at *C**	1.039	1.039
MRT at *P** (1 + Marginal ROI)	-	1.039
Optimal Production(*P**)	-	{$141,078, $313,713.}
Optimal Investment (*I**)	-	$98,921.50
Dollar ROI	-	$207,313.00
ROI	-	109.6%
NPV(*I**)	-	$100,610.00
Wealth	$342,406.00	$443,016.00
Optimal Consumption (*C**)	{$168,425, $180,766}	{$219,692, $232,033}
Utility at *C**	-1.16426	-0.993179
Amount Lent (borrowed)	$71,575.00	-$78,613.50
Principal + Interest Earned (repayed)	$74,366.50	-$81,679.40

The results graphed.



Math for the Fisher-Separation result. Solve for the optimal production point P^* and the optimal investment I^* by equating the marginal rate of transformation to one plus the rate of interest

$$P^*: \; MRT = \frac{b^2}{a^2} \cdot \frac{C_0}{C_1} = 1 + r$$

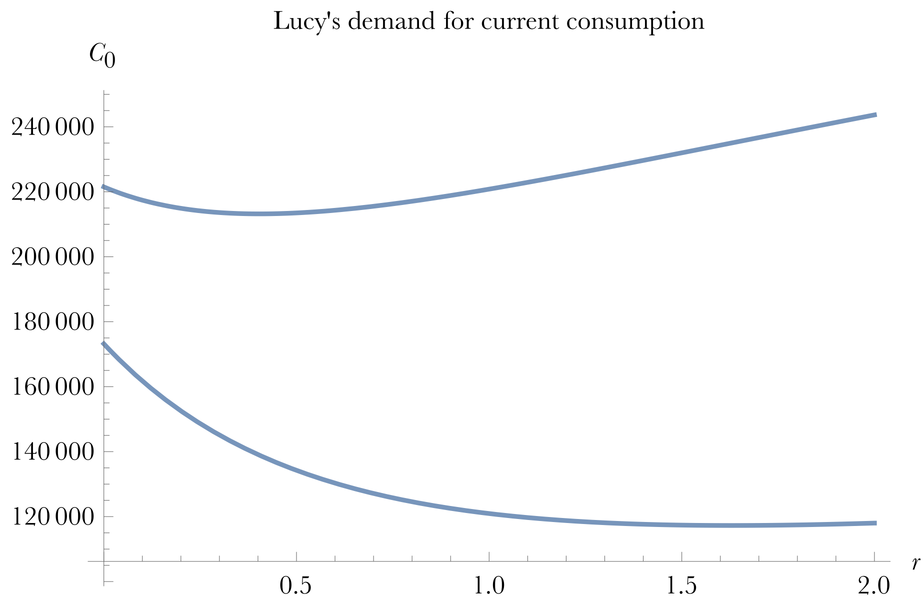
Plug that back into the equation for the IOS to get P^*. Then solve for Lucy's maximized wealth and find her consumption optimum.

$$C^*: \; MRS = e^{\alpha(C_1 - C_0)} = 1 + r$$
$$\therefore C_1^* = C_0^* + \frac{1}{\alpha} \ln(1 + r)$$

Throw that into the wealth constraint to get

$$C_0^* = \frac{1+r}{2+r}\left(W_0 - \frac{1}{\alpha} \cdot \frac{\ln(1+r)}{1+r}\right)$$
$$C_1^* = \frac{1+r}{2+r}\left(W_0 + \frac{1}{\alpha} \ln(1+r)\right)$$

Lucy's demand for consumption as a function of her wealth with and without real investment. Can you tell which is which? Demand is downward sloping for "normal" interest rates but turns upwards when interest rates are at stratospheric level.



Words with little economic mumbo jumbo. Lucy is worth $342,406 if she doesn't invest in her business. This amount is the $240,000 she has today plus $102,406, which is the value today at 3.9 per cent of the $106,400 she has coming to her next year. Given her tastes and the fact that next year will be leaner than this year for her, Lucy has a strong desire to have more cash available next year, in fact, so much so that she values her next current dollar at -33.9 per cent. This negative "personal rate of interest" means she'd pay someone to help her build a bigger nest egg! No need for that though. She can make herself better off by investing in her business—to increase her wealth obviously—and then borrowing or lending to spread her cash more evenly between this year and next. Her real business opportunities are such that she'll get back about 4.21 dollars on the first dollar she invests, and earn more than the rate of interest on every dollar after that up to $98,922 invested for a total income of $207,313—an average return of almost 109.6 per cent! This makes her richer by $100,610. Lucy's tastes at this higher level of wealth are such that she'd like to spend more next year without having to cut back by much this year, so she'll borrow $78,613, yet still end up with a smoother spending stream of $219,692 and $232,033.

2. There are three inconsistencies observed in intertemporal decisions. (1) Discount rates decline with the time to be waited. An amount of money has a bigger implied present value if received later (dynamic inconsistency & self control). (2) Discount rates decline with the total amount of money. Small amounts are discounted at implausibly high rates (magnitude effects, & absolute differences & mental accounts). (3) Discount rates are higher for gains than for losses. People need to be paid a lot to wait for a reward but are unwilling to pay to delay a fine (sign effects, reference points & debt aversion). The authors attribute these to the behavioural tendencies indicated in parentheses above, all of which are associated with an overriding tendency to make decisions relative to context-dependent reference points that are not accounted for in economic models. Loss aversion is used to explain our preference for increasing consumption profiles, and costly self-control is used to explain our preference for increasing income profiles, with our various decisions being further tempered by savouring and dread. What policy implications does this have for the use of social discount rates? The answer is that social discount rates become all but useless.

3. The equilibrium interest rate.

   a. The growth rate of the economy is three per cent.

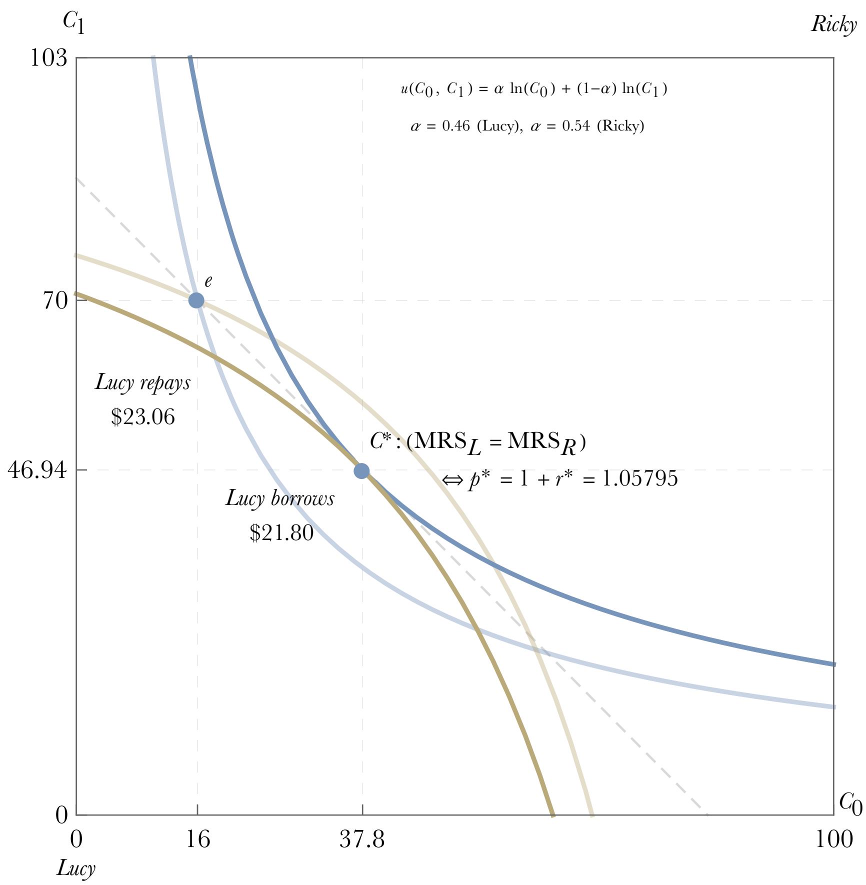
   b. The equilibrium rate of interest is 5.7953 per cent. Why is it different from the growth rate?

$$1 + r^* = \frac{\alpha_L e_{1,L} + \alpha_R e_{1,R}}{(1-\alpha_L)e_{0,L} + (1-\alpha_R)e_{0,R}} = 1.0579526$$

   c. Ricky lends \$21.80 to Lucy.

   d. The Pareto optimum or consumption optimum read from Lucy's origin is {37.7961, 46.9407}. Here's the expression for current consumption.

$$\begin{aligned} C^*_{0_L} &= \alpha_L W_L = \alpha_L \left(e_{0,L} + \frac{e_{1,L}}{1+r^*} \right) \\ &= \alpha_L \left(e_{0,L} + e_{1,L} \frac{(1-\alpha_L)e_{0,L} + (1-\alpha_R)e_{0,R}}{\alpha_L e_{1,L} + \alpha_R e_{1,R}} \right) = 37.7961 \end{aligned}$$



e. The interest rate would be higher if the growth rate was higher. See the expression for r^* above. A bigger e_1 or smaller e_0 for either Lucy or Ricky or both of them implies a higher growth rate growth rate and higher r^*. Does that make sense?

f. The interest rate would be lower if Ricky's α was smaller. Once again, see the expression for r^*. It doesn't matter whose α we're talking about because, for both Lucy and Ricky, α gauges the preference for current consumption. A smaller α means a relatively weaker preference for current consumption, which implies a lower interest rate.

3
CHOICE UNDER UNCERTAINTY

In the St. Petersburg gamble a coin is tossed until it comes up tails, and for every toss that it comes up heads, a starting amount of money, say \$1, is doubled. The prize is the amount to which \$1 has grown, $\$2^n$, where n is the number of consecutive heads. Daniel Bernoulli (1700-1782), mathematician and physicist, considered the gamble a paradox because the expected prize is infinite yet most people are not willing to pay more than a modest amount to play the game.

$$E(prize) = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{2^n}\$2^n = \$1 + \$1 + \$1 + \ldots = \infty$$

This led Bernoulli to suggest that perhaps people place an expected psychological value, v, on the gamble that is a concave function of the prize

$$v(prize) = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{2^n} f\left(\$2^n\right) < \infty,$$

where f is a strictly concave function. If f is square root, for example, the expected psychological value is only ⅓.

$$v(prize) = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{2^n}\sqrt{\$2^n} = \frac{1}{3}$$

Expected Utility Theory

Bernoulli's mathematical solution to the St. Petersburg paradox parallels economics' paradigm of treating people as displaying diminishing marginal utility of consumption. Every slice of pizza provides satisfaction but less satisfaction than the slice before. And so with money. Suppose your wealth is \$100 right now but there's a 52 per cent chance that in the next instant you'll lose \$72 and a 48 per cent chance that you'll gain \$78. The change in your wealth in the immediate future is the gamble

$$\Delta W \sim \left(\Delta W_1 = -\$72, \Delta W_2 = \$78;\ \pi_1 = 0.52\right)$$

and, in this case, a *fair gamble* because it has an expected value of \$0, or saying the same thing, your expected wealth under the gamble is equal to your current wealth.

$$\begin{aligned} E(\Delta W) &= \pi_1 \cdot \Delta W_1 + \pi_2 \cdot \Delta W_2 \\ &= 0.52(-\$72) + 0.48(\$78) = \$0 \\ \therefore E(W) &= W_0 = \$100 \end{aligned}$$

Your wealth is the distribution

$$W \sim \left(W_1,\ W_2;\ \pi_1\right) = \left(\$28,\ \$178;\ 0.52\right)$$

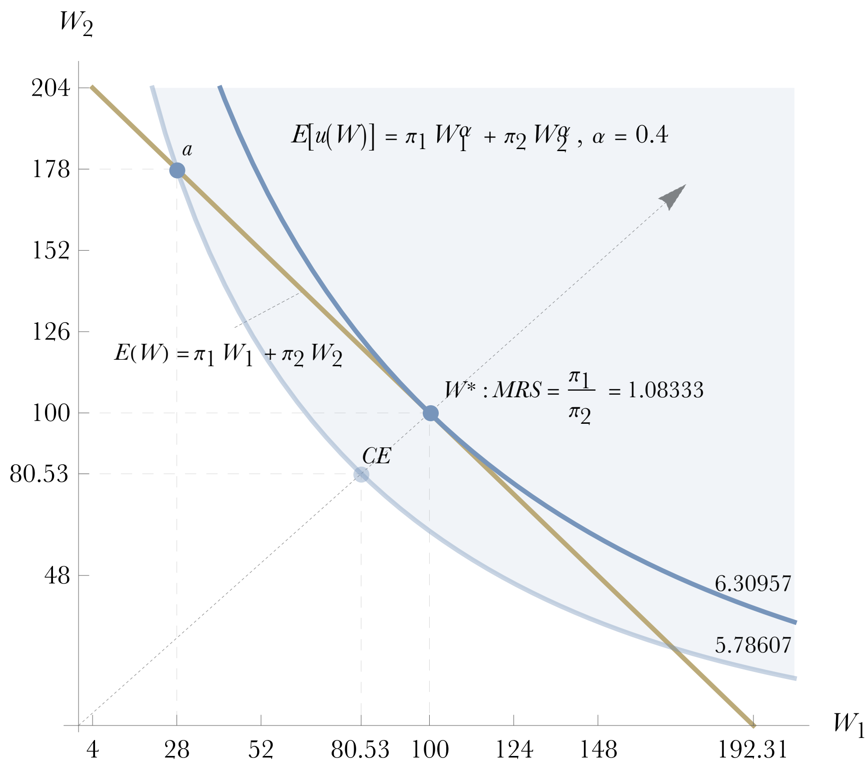
In economics, Bernouli's expected psychological value becomes expected utility

$$E[u(W)] = \pi_1 u(W_1) + \pi_2 u(W_2)$$

with the only change in the marginal rate of substitution compared to non-risky choices of apple and oranges being the appearance of the odds ratio, $\frac{\pi_1}{\pi_2}$,

$$MRS = \frac{\pi_1}{\pi_2}\frac{u'(W_1)}{u'(W_2)}$$

If wealth yields diminishing marginal utility, then risky outcomes like ($28, $178) must lie on a convex indifference curve (point *a* in the figure), where here the person is assumed to have a



utility function of the form

$$u(W) = W^\alpha,\ \alpha = 0.4$$

and therefore expected utility

$$E[u(W)] = \pi_1 W_1^\alpha + \pi_2 W_2^\alpha$$

All of the distributions of wealth on the indifference curve passing through *a* yield the same expected utility: 5.79. There is one distribution in particular on that initial indifference curve that tells us the person is *risk averse*. That distribution is completely free of risk and is denoted by *CE* for *certainty equivalent*. You can find the certainty equivalent at the intersection of an indifference curve and a 45-degree line passing through the origin. The certainty equivalent in the example is \$80.53. How does the certainty equivalent tell us that the person is risk averse? It does so because it is less than the person's expected wealth of \$100 under distribution *a*. Think about it. The person is indifferent between risky distribution *a* and riskless amount *CE*

$$u(CE) = E[u(W)]$$

In everyday language, *CE* is the answer to the question: what is the smallest sure amount you'd be willing to accept if all risk was taken away from you? A risk averse person is always willing to accept a sure amount (\$80.53) that is less than their expected wealth (\$100) when faced with risk. Do the math.

$$\begin{aligned} u(CE) &= E[u(W)] \\ \therefore CE^\alpha &= \pi_1 W_1^\alpha + \pi_2 W_2^\alpha \\ \therefore CE &= \sqrt[\alpha]{\pi_1 W_1^\alpha + \pi_2 W_2^\alpha} \\ \therefore CE &= \sqrt[0.4]{5.78607} \\ \therefore CE &\approx \$80.53 \end{aligned}$$

What about the line passing through *a*?

$$E(W) = \pi_1 W_1 + \pi_2 W_2 = \$100$$

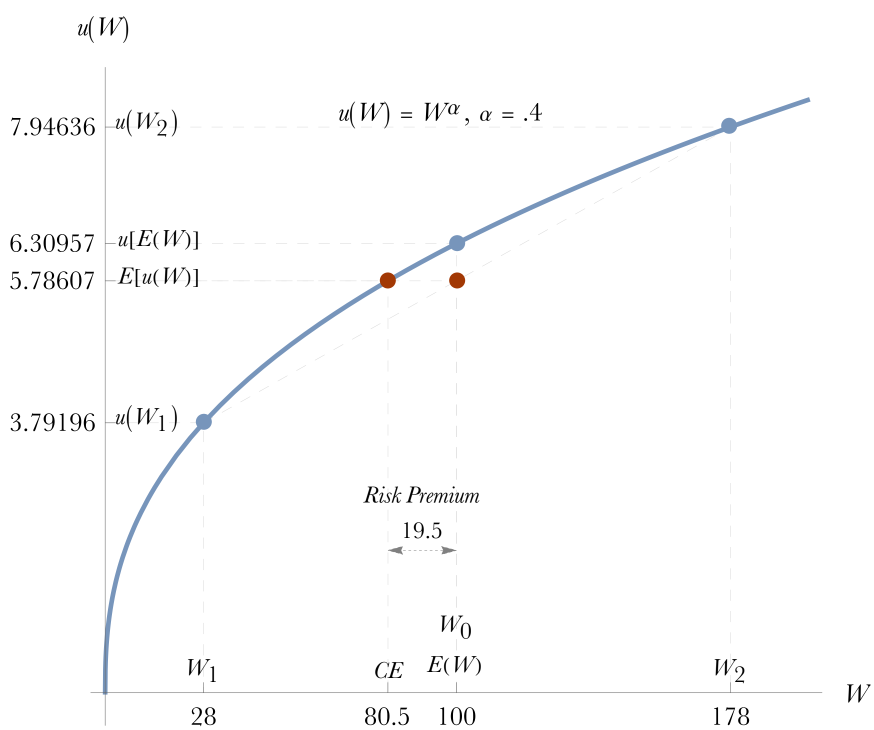
It is not a wealth constraint or a budget line; it is line of distributions of equal expected wealth, and its slope is equal to the odds ratio, $\frac{\pi_1}{\pi_2}$. Since W_0 is \$100, all of the distributions on this particular line are fair gambles. Points lying above the line are distributions with expected wealth greater than \$100, and those below, less than \$100. The distributions become less risky moving down the line from the top left because the spread between W_1 and W_2 is smaller for the fixed probabilities. Risk falls until there is no risk at all at $W_1 = W_2 = \$100$, where a ray from the origin at 45 degrees intersects the line, and then risk increases all the way down to the horizontal intercept.

The line of equal expected wealth is not a constraint; the person represented in the figure wouldn't ordinarily be restricted to choosing only from among the distributions on the line. All that can be said is that the outcomes the person currently faces are represented by *a*, and paired with the fixed probabilities, that is the distribution of their wealth. Their preferred distributions are the ones that lie above the indifference curve passing through *a*—expected utility is higher. Can you pick out areas in the graph matching the nine distribution classes in the table? Which are preferred?

		Risk Compared to Point a		
		Lower	*Same*	*Higher*
Expected Wealth Compared to Point a	*Higher*			
	Same			
	Lower			

Working with Utility Functions Instead of Indifference Curves

From this point on we'll work with utility functions. They provide the same information as indifference curves but are somewhat handier when dealing with risky prospects because all of the outcomes (W_1 and W_2) can be shown on one axis and utility on the other. Here is our original example for risky wealth, $W \sim (W_1, W_2;\ \pi_1) = (\$28, \$178;\ 0.52)$, shown on the utility function for $u(W) = W^{\alpha}$, $\alpha = 0.4$, rather than point *a* on one of the utility function's indifference curves. Once



again, we know this is a fair gamble because current wealth is equal to expected wealth. The expected utility of the gamble can be found at the intersection of the line connecting $u(W_1)$ and $u(W_2)$ and the vertical line through $E(W)$. It is the red point on the right. You already know how to calculate the certainty equivalent. The red point on the left picks off *CE* below ($80.53) because that sure amount yields the same utility as the expected utility of the gamble (the red point on the right).

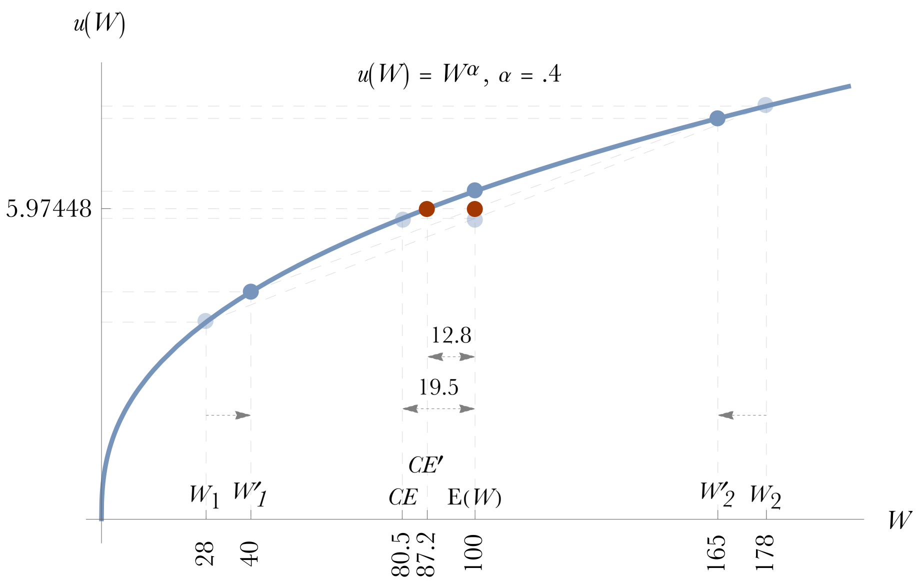
There are three ways that a concave utility tells us a person is risk averse:

- $u[E(W)] > E[u(W)]$:—Your utility of expected wealth is greater than the expected utility of wealth. You'd rather have $100 for sure than a gamble paying $28 or $178 and expected payoff of $100.

- $CE < E(W)$:—Your certainty equivalent is less than your expected wealth. You are indifferent between *CE* ($80.53) and the gamble. You would trade the risk for any sure amount greater than or equal to *CE*.

- Risk Premium $\equiv E(W) - CE > 0$:—Your risk premium for the gamble is positive. This is just another way of looking at the certainty equivalent. You'd be willing to pay an amount up to the risk premium in order to have the risk removed. Think of it as a personal valuation of an insurance premium.

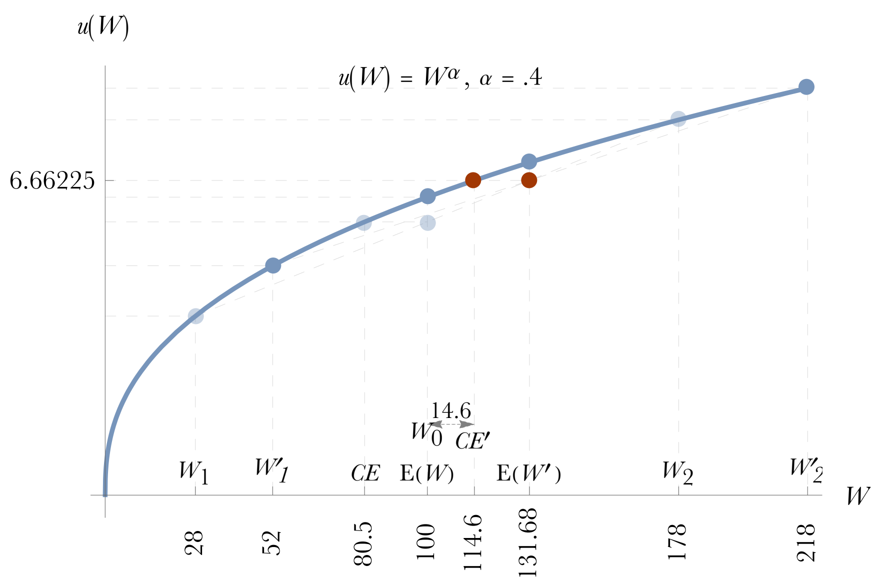
The next graph illustrates the definition of risk aversion. Expected utility is higher (5.974) under a safer gamble, $(W_1', W_2';\ \pi_1) = (\$40, \$165;\ 0.52)$ with the same expected wealth as the original gamble. The changes with the certainty equivalent and the risk premium follow suit. A change in risk without a change in the expected outcome is called a *mean-preserving change in spread*. For this safer gamble, it is a mean-preserving decrease in spread. Can you visualize in the graph



what would happen to expected utility with repeated applications of a mean-preserving decrease in risk?

> **Risk aversion**:—If faced with two or more gambles having the same expected payoff, a risk averse person prefers the least risky.

Most risks we face are not fair gambles; expected wealth is bigger or smaller than its current level. Consider taking on a gamble with payoffs $(\Delta W_1 = \$24, \Delta W_2 = 40;\ 0.52)$ and a positive expected payoff of \$31.68, which would increase your expected wealth to \$131.68 from \$100. Taking on the gamble would increase your expected utility to 6.662, so you clearly prefer the distribu-



tion of your wealth with the gamble than without it. But would you be willing to pay to take on this new risk? The answer is yes; you'd be willing to pay up to \$14.60, the difference between *CE'* and W_0.

Coefficient of Risk Aversion

It's hard to say that one person is more risk averse than another unless they have similar wealth and you get to see them respond to a similar risk, for example, the amount each is willing to pay for a risky investment. Economics uses a standardized measure of sorts in modelling to represent differences in risk aversion. ARA is the coefficient of absolute risk aversion, and RRA is the coefficient of relative risk aversion.

$$ARA(W) = -\frac{u''(W)}{u'(W)}$$

$$RRA(W) = -W\frac{u''(W)}{u'(W)} = -W \cdot ARA(W)$$

ARA is simply the ratio of second derivative of the utility function to the first derivative. Because the second derivative of a concave function is negative, the negative sign in front of the fraction ensures that ARA is positive for anyone who is risk averse. The intuition, although perhaps not a straightforward intuition, behind the coefficient is this: it is the rate of decline in marginal utility (the numerator) per unit of marginal utility (the denominator). A rougher way to say the same thing is that it tells us how fast marginal utility is declining. The faster the decline, the more risk averse the person is given the risk that they face. Here's how to interpret the coefficients for someone with log utility, $u(W) = \ln(W)$. The coefficient of absolute risk aversion is $\frac{1}{W}$, which is decreasing in W; the person becomes less risk averse as they get wealthier, and they will put more of their wealth in dollar terms at risk, for example, by investing in risky assets. The coefficient of relative risk aversion is 1, a constant. The person's relative risk aversion is unchanged as their wealth changes; they keep a constant percentage of their wealth at risk. So a log utility maximizer display declining absolute risk aversion (puts more *dollars* = absolute at risk as they become wealthier) and constant relative risk aversion (they keep a constant *percentage* of their wealth put at risk)

Pricing a Risky Asset

Here's how expected utility theory is used to developed a rudimentary pricing model, that is, one that tells us the return that is required by an investor depending on their level of risk aversion and the riskiness of the asset.

Let an investor's wealth be comprised of the market values of just two assets: risk-free debt M and a risky stock S.

$$W_0 = M + S$$

If the person looks just one period ahead, say a year, their uncertain end-of-period income, y, is

$$\begin{aligned} y &= r_f M + rS \\ &= r_f W_0 + S(r - r_f) \end{aligned}$$

where r_f is the return on the safe bond and r is the risky return on the stock. The investor maximizes their expected utility with respect to their dollar investment S in the stock

$$\text{Max } E[u(y)] = \text{Max } E[u(r_f W_0 + S(r - r_f))]$$

The condition for the maximum (called a *first-order condition*) is

$$\frac{dE[u(y)]}{dS} = E[u'(y)(r - r_f)] = 0$$

The argument of the expectation $E[u'(y)(r - r_f)]$ is the product of two terms, and can be written using the definition of covariance as

$$E[u'(y)]E[r - r_f] + \text{cov}[u'(y), (r - r_f)] = 0$$

Solving for $E(r)$ gives an expression for the return that the investor "requires"

$$E(r) = r_f - \frac{\text{cov}[u'(y), (r - r_f)]}{E[u'(y)]}$$

If the fraction in the second term is negative, the expression has the nice interpretation that return an investor requires is equal to the risk-free rate plus a positive risk premium, which depends on the investor's taste for risk and return. The denominator of the second term is positive because marginal utility, $u'(y)$, is always positive, so must be its expected value. That leaves the question of whether the covariance in the numerator is negative. It is. When the return on the stock is higher than the risk-free rate, that is, $r - r_f > 0$, income y will rise, and the marginal utility of income, $u'(y)$, will

For the specific case of quadratic utility, $u(y) = y - \alpha y^2$, $\alpha > 0$, the risk premium can be broken down into a risk aversion part and the riskiness of the stock. To get there, note the first and second derivative of the utility function and its coefficient of absolute risk aversion

$$u'(y) = 1 - 2\alpha y \text{ and } u''(y) = -2\alpha$$
$$\therefore ARA(y) = \frac{2\alpha}{1 - 2\alpha y}$$

Substitute the first derivative into the expression for required return

$$E(r) = r_f - \frac{\text{cov}[1 - 2\alpha y, (r - r_f)]}{E[1 - 2\alpha y]}$$

Now substitute in the definition of income y

$$E(r) = r_f - \frac{\text{cov}[1 - 2\alpha(r_f W_0 + S(r - r_f)), (r - r_f)]}{E[1 - 2\alpha(r_f W_0 + S(r - r_f))]}$$

Additive constants like 1 and $r_f W_0$ do not affect covariance, so they can be dropped, and multiplicative constants like 2α, and S can be factored out, leaving

$$\begin{aligned}
E(r) &= r_f - \frac{2\alpha}{1 - 2\alpha E(y)} \text{cov}[y, r - r_f] \\
&= r_f - \frac{2\alpha}{1 - 2\alpha E(y)} \text{cov}[S(r - r_f), r - r_f] \\
&= r_f - \frac{2\alpha}{1 - 2\alpha E(y)} \cdot S \cdot \text{cov}[r - r_f, r - r_f]
\end{aligned}$$

In the last line above, you can see $ARA(y) = \dfrac{2\alpha}{1 - 2\alpha E(y)}$, solved at $E(y)$, and $\text{cov}(r - r_f, r - r_f)$ is the variance of return of the stock (the risk-free rate in there does nothing because it is a constant). The last line can then be written

$$E(r) = r_f + ARA(y) \cdot S \cdot \sigma_r^2$$

The risk premium on stock, demanded by a quadratic utility maximizer is proportional to their coefficient of absolute risk aversion and the variance of the stock's return. Tastes for risk and the "quantity" of risk have been separated.

Exercises

1. There's no two ways about it. Mostafa, an investment advisor, is going to be out some money. He has been cited by the provincial securities regulator for dubious bookkeeping (we don't have a federal securities regulator in Canada, how lame). Mostafa is required by law to present himself at the regulator's office today. He can either plead guilty and pay a fine of \$55,000 on the spot, in which case it is a sure loss, or take a chance by pleading not guilty to argue his case. If he argues his case, there is a 63.2 per cent chance that the judge will rule against him, and administrative and legal fees will raise his fine to \$82,500, also payable right away. As big fan of Wintersleep and a risk averse expected utility maximizer with utility function

$$u(W) = \frac{1}{\beta} W^{\alpha}, \ \alpha = 0.4, \ \beta = 44$$

what will he do? Use at least four-decimal place precision in your calculations. Drawing a generic utility diagram (you don't need to draw the actual function) with W on the horizontal axis and $u(W)$ on the vertical will help a lot.

	Without Real Investment
Current wealth	\$700,000
Fine if he pleads guilty	\$55,000
Fine if he pleads guilty and loses his case	\$82,500
Probability of losing his case	0.632

   a. What is Mostafa's expected utility if he pleads guilty? If he pleads not guilty?

   b. What will Mostafa do? Would all risk averse people do the same?

2. You are a property insurer and one of your clients, Renata, whose current wealth is \$1.2 million, wants to insure her \$590,000 house (which would be a dump in Toronto in today's market). The chances of the house burning down in any given year are 1.3 in a thousand, and Renata's utility function is $u(W) = \sqrt{W}$. She doesn't face any other risks.

   a. Will Renata buy insurance for \$805? What is the most that she is willing to pay?

   b. Is the amount that Renata is willing to pay for insurance relatively elastic or inelastic with respect to her current wealth? What about with respect to the probability of her house burning down? Recall from your introductory economics class that price elasticity of demand or supply is the percentage change in quantity for a given percentage change in price, $\frac{\% \, \Delta Q}{\% \, \Delta P}$. All you have to do is rename the variables. Q becomes the most Renata is willing to pay for insurance, and depending on which elasticity you are computing, P is either her current wealth, W_0, or the probability of her house burning down, π. Compute both elasticities, the first for a one per cent increase in her wealth and the other for a one per cent increase in the probability of her house burning down.

3. Your current wealth is $1,400,000 and you have logarithmic preferences, $u(W) = \ln(W)$. At the moment, you don't face any risks, but an old acquaintance, Zelda, asks you to be her partner in an investment where there is a 10 per cent chance that you will lose $100,000 and a 90 per cent chance that you will earn $250,000.

   Are you interested in Zelda's offer? If so, how much are you willing to invest?

ANSWERS

1. Oh, Mostafa. When will you learn?
    a. Mostafa's expected utility is 4.79039 if he pleads guilty, which is his utility of $645,000 (his current wealth $700,000 less the sure loss of $55,000). If he pleads not guilty, his expected utility is 4.79674, which is the expected value of his utility if he loses, 4.70763, and if he wins, 4.94979.
    b. Mostafa will plead not guilty and take his chances before the judge because his expected utility under the risky loss, 4.79674, is higher than it is under the sure loss, 4.79039. You come to the same conclusion by noting that his certainty equivalent under the risky loss, $647,139, is greater than his wealth if he pleads guilty and pays the fine, $645,000. Whether another risk-averse person would take the same decision depends on their tastes because wealth under the sure loss is less than expected wealth under the risky loss. Can you show this with a utility diagram? Can you also show with the same diagram that if wealth under the sure loss was bigger than expected wealth under the risky loss, any risk averse person would choose the sure loss? Turns out that people do not behave that way. See Kahneman and Tversky (1979) for experiments that illustrate the failure of expected utility theory in this regard.
2. Renata has smoke detectors installed on every floor of her home.
    a. Yes she will buy insurance from you at that premium because her expected utility insured, 1095.08, is higher than it is uninsured, 1095.04. Renata is willing to pay up to $895.35 for insurance, which is her wealth of $1.2 million less her certainty equivalent of $1,199,104.65.
    b. If Renata was one per cent wealthier, the most she would be willing to pay for insurance would drop by 0.2 per cent (-0.198204). This is considered to be *inelastic* because the percentage change is less than one. That utility theory implies a negative relationship between wealth and willingness to pay for insurance seems counter intuitive. You could also compute *point elasticity*, that is, using calculus, and the answer is almost the same, -0.201063, but the derivative is ugly[3]. Renata is more sensitive to the probability of loss but not by much. If the probability of her house burning down was one per cent higher, she be willing to pay about one per cent more for insurance (0.999812). A one-for-one change such as this, in either direction, is referred to as *unitary elasticity*. The answer is almost identical when calculated as a point elasticity (0.999813).
3. Zelda is always able to sniff out the good opportunities. You are interested in her proposal because your expected utility with the investment, 14.2924, is greater than your utility for your current wealth, 14.152. The most you are willing to invest is $211,127. This is the difference between your certainty equivalent under the investment, $1,611,127, and your current wealth of $1,400,000.

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[3] Ask me if you want to see the math.

4
EQUILIBRIUM IN CAPITAL MARKETS

A capital market not only allows us to shift consumption over time, it allows us to shift risk. Both are done through trade. Those who are willing to take on more risk buy it from those who want to get rid of some. Risk-averse buyers will demand a higher rate of return for taking bigger risks, and risk-averse sellers decide how much return they are willing to give up to shed risk. Markets for risk, such as those for bonds, stocks, and insurance, are no different from any other in that they benefit society by providing choice.

States and Dates

Suppose that there is no risk now. Lucy and Ricky know everything about their present circumstances: bank balances, pairs of clean socks in the dresser drawer, whether the salami in the refrigerators is still fresh, and the condition of the shingles on the roof. You know that you can't know everything even if there is no uncertainty, but there is an awful lot that is knowable. That is period 0 in the intertemporal choice model in chapter 9. The future, however, is uncertain, so consumption in period 1 must now be treated differently (likewise, periods 2, 3, 4 and so on, if there are more than two periods in the model). Risk and time are intertwined. In a world without risk, Lucy's period 0 endowment might be \$180 and her period 1 endowment \$214. If she chooses to consume her endowment, she knows right now that \$214 will be waiting for her next year for sure.

Not so in a risky world. Lucy's current endowment is still a sure \$180, but her future endowment might be any amount. Probably not \$1 billion and hopefully more than \$3.57. How is risk to be brought into the model? One way is to assume that future, risky consumption is a random variable that behaves according to a known probability distribution. Probability distributions are made up of two things: outcomes (consumption or income in our case) and the probabilities of those outcomes occurring. Lucy, for example, might face a 0.4 chance of ending up with \$160 next year and a 0.6 chance of \$250. That there are only two outcomes is arbitrary. There could be any number, say, 11 or 546. In real life I don't think we can truly know what the outcomes are or how many are possible, and we would struggle to attach probabilities to them. Economists refer to the circumstances in which particular outcomes occur as *states* or *states of nature*. If there are two, then they might be called state *a* and state *b* (like Thing 1 and Thing 2 from Dr. Seuss).

States are mutually exclusive: rain or shine, war or peace, boom or bust. State *a* has a 0.4 chance of occurring, and if it does, Lucy will have \$160; state *b* has a 0.6 chance of occurring, in which case Lucy will have \$250. In a riskless model, Lucy's endowment is written as

$$e = (e_0, e_1) = (\$180, \$214)$$

but in a *time-state preference model*, her endowment or any consumption bundle is written as

$$C = (C_0, C_1) = (C_0, (C_a, C_b;\ \pi_a)) = (\$180, (\$160, \$250;\ 0.4))$$

where C_1 is now a random variable; π_a is the probability of state a happening and having consumption C_a; $\pi_b = 1 - \pi_a$ is the probability of state b and having C_b. You can also refer to C_a and C_b as possible future income. It is important to remember that Lucy will never have \$160 *and* \$250 next year. Today she knows that later she will have \$160 *or* \$250 depending on what happens. *Que sera sera*.

Lucy's wealth in a riskless world is the present value of her endowment computed at whatever the risk-free rate of interest happens to be

$$W_0 = C_0 + \frac{C_1}{1 + r_F} = \$180 + \frac{\$214}{1 + r_F}$$

In a risky world, her wealth is still the present value of her endowment but it is expected future consumption, $E(C_1)$, that must be discounted at a higher rate, k, to take account of the risk

$$W_0 = C_0 + \frac{E(C_1)}{1+k} = \$180 + \frac{0.4 \times \$160 + 0.6 \times \$250}{1+k} = \$180 + \frac{\$214}{1+k},\ k > r_F$$

In the *time-state preference model*, present value, including wealth, is expressed in terms of theoretical prices rather than interest rates or discount factors. We can figure out the rates later.

$$W_0 = p_0 C_0 + p_1 E(C_1) = p_0 C_0 + p_a C_a + p_b C_b$$

Instead of having one price for future consumption, $p_1 = \dfrac{1}{1+k}$, there is a price today, p_a and p_b, for consumption in each future state of nature. These are called *pure state prices*, *primitive security prices*, or *Arrow-Debreu prices*, after the two economists who, independently, developed the model. Take a moment to think about what a pure state price is. It is the price or value today of having exactly one dollar in a particular future state and nothing in all of the other states. That means that you can think of a pure state price as a present value factor that is attached to a particular future event or, even easier, you can think of it as the market price of a gamble that pays one dollar if a particular event occurs and nothing if anything else occurs.

- p_a is the price today of a gamble that pays \$1 in state a and \$0 in state b
- p_b is the price today of a gamble that pays \$1 in state b and \$0 in state a

While pure state prices are theoretical—you can't look them up anywhere—they do exist in real markets under various guises, some of which we will discuss in class. For now, here's an example: the price of a lottery ticket that pays \$1 if you earn an A or better in my course and \$0 if you don't. Trade that online.

In the example economy we'll be discussing, $p_a = 0.328$ and $p_b = 0.615$. You'll learn how to calculate those prices in the last section of this chapter. But for now, their interpretation is more important than their computation: A dollar next year in state *a* is worth 33 cents now and a dollar in state *b* is 62 cents. You'll be able to use them to calculate Lucy and Ricky's wealth, as well as the equilibrium risk-free rate of interest and the expected return on the stock market index.

Description of the Economy

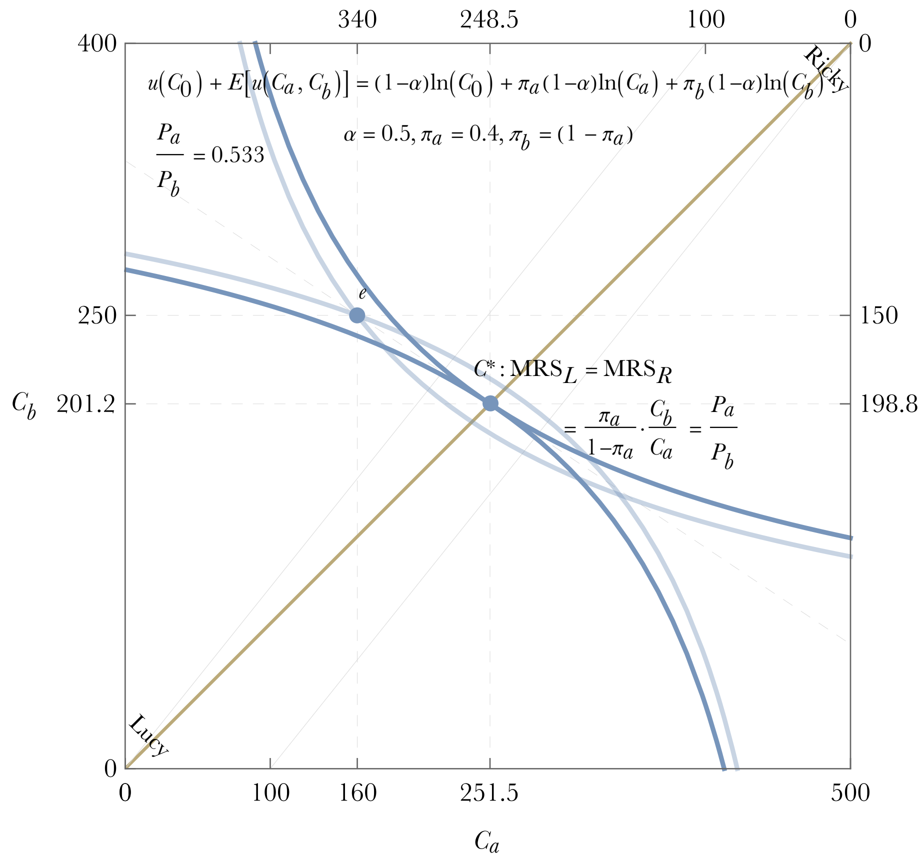
The table shows information about a two-period economy. Pretend it is Canada.

Contingencies				*Endowments*			*Securities*	
Dates	*States*	*Probs.*	p_s	*Lucy*	*Ricky*	*Canada*	*M*	*F*
0	-	1	1	180	230	410	4.10	0.943
1	*a*	0.4	0.328	160	340	500	5.00	1.00
1	*b*	0.6	0.615	250	150	400	4.00	1.00

The states and dates are in the first panel. The state probabilities and pure state prices are carried over from the previous section. The state prices are in the rows labeled time period 1 to associate them with states *a* and *b*, but they are time 0 or *now* prices. The *Endowments* columns show endowments for Lucy, Ricky, and Canada as the two of them combined. Since Lucy and Ricky are Canada's only inhabitants, the country is endowed with $410 today and will grow to $500 next year if state *a* occurs or decline to $400 if it's *b*. You can think of the aggregate endowment as Canada's gross domestic product. The last panel introduces two financial securities. *M* is a risky stock that pays a dividend of $5 in state *a* and $4 in state *b*. *M*'s price now is P_M. *M* is like a stock market index or super stock that includes all firms in the economy, so it pays out the aggregate endowment of the economy. *M*'s payoff is one one-hundredth the aggregate endowment, so the supply of shares of *M* must be 100. If *M*'s payoffs were $2.50 and $2, there would be 200 shares outstanding because the payoffs have to be proportional to the aggregate endowment. Anyone holding nothing but shares in *M* is bearing the same risk as the economy as a whole. *F* is a riskless security because it pays $1 no matter what happens. It is like a pure discount bond with price P_F now because it does not pay interest, and its payoff is not related to the aggregate income of the economy. You can think of *F* as a Treasury bill with a face value of $1. Since *F* is a debt security, it's net supply must be zero—for every dollar borrowed there is a dollar lent. By figuring out the equilibrium price of *M* and *F*, we will know the expected return on risky stock and the risk-free rate of interest.

The figure on the next page shows the economy represented in an Edgeworth box. It includes the Pareto optimum, $C^* = (C_0^*, (C_a^*, C_b^*)) = (\$180, (\$251.50, \$201.20))$. Notice there is no C_0 axis. If it was drawn, the C_0 axis would come straight out into your face from Lucy's origin, because the Edgeworth box would be three dimensional. $e_0 = \$180$ could be shown on a third axis but it isn't necessary because the time-state preference model describes how Lucy and Ricky trade their future endowment now, *e*, to end up at the Pareto optimum, *C**, and that is also why optimal current consumption is fixed at $180. What you are looking at in two dimensions is next year or period 1, and the axes are the two outcomes. Even though the probabilities are not shown anywhere (you'll have to remember them or look back at the table), every point in the Edgeworth box is a distribution of future consumption or income. Endowment point *e* is the distribution ($160, $250; 0.4) for Lucy, and by subtracting from the aggregate endowment of the economy, ($340, $150; 0.4) for Ricky. Canada is expected to grow by 7.3 per cent

$$E(g) = \frac{E(E_1)}{E_0} - 1 = \frac{\$440}{\$410} - 1 = 0.0731707$$



Pareto Optimum

The Pareto optimum is a general equilibrium and is determined in the usual way: Lucy and Ricky's marginal rates of substitution over uncertain state consumption are equal, and this happens at a value of 0.533, which is the implied equilibrium price ratio. One dollar of consumption in state *a* is worth 0.533 dollars of consumption in state *b*.

$$C^*: \; MRS_L = MRS_R = \frac{p_a^*}{p_b^*} = \frac{0.328}{0.615} = 0.53\dot{3}$$

Notice that you cannot compute the pure state prices individually from just the implied price ratio. But knowing $p_a = 0.328$ and $p_b = 0.615$, you can compute Lucy's wealth to be $386.23, Ricky's as $433.77, making $820 the wealth of the nation. Why is a dollar of consumption worth so much less in state *a* than in *b*? You can find two reasons in the table.

If wealth is the present value of a person's income, then the price or market value of a security, in the same way, is the present value of the income it provides. The equilibrium price of risky stock M is about \$4.10 for an expected return of 7.3 per cent.[4] The price of safe bond F is \$0.943, making the risk-free rate of interest 6.04 per cent. Letting X_s be the state income or payoff of any security, then

$$
\begin{aligned}
P_M &= p_a X_{aM} + p_b X_{bM} = 0.328 \times \$5 + 0.615 \times \$4 = \$4.10 \\
\therefore E(r_M) &= \frac{E(X_M)}{P_M} - 1 = \frac{\$4.40}{\$4.10} - 1 = 0.07317 \\
P_F &= p_a X_F + p_b X_F = X_F(p_a + p_b) = \$1(0.328 + 0.615) = \$0.943 \\
\therefore r_F &= \frac{X_F}{P_F} - 1 = \frac{\$1}{P_F} - 1 = \frac{\$1}{\$0.943} - 1 = 0.0604454
\end{aligned}
$$

The difference between the expected return on the stock market index and risk-free bonds is called the *market risk premium*. It is the extra return that the market compensates us for bearing risk. The risk premium is only 1.3 per cent in this economy. That's tiny. A market risk premium in the range of three to six per cent was typical during the 20th century. Tweak the parameters of the time-state preference model all you want and you won't be able to crank out realistic risk premiums unless you make Lucy and Ricky pathologically risk averse. This anomaly, that real life risk premiums are so different from what theory predicts, is known as *the equity premium puzzle*.

Portfolios and Trade

C^* is an equilibrium because, mutually, Lucy and Ricky prefer it to any other distribution of future consumption. At any other point in the Edgeworth box, their marginal rates of substitution would differ from one another, and they would then do something to get to C^*. How do they get to C^* if, for whatever reason, they are somewhere else? The government could try to reallocate their endowments. So could a benevolent monarch or dictator. But we haven't included a government in this model. Moving around the Edgeworth box means that Lucy and Ricky must be trading future distributions of consumption. Since the only thing to trade in this economy is the two securities, and there is no production, every distribution (point) must be a portfolio of shares of risky stock M and safe bond F. You can see this by considering two references. One is the main diagonal, and the other is the two dashed grey lines, one leaving Lucy's origin at 45 degrees and the other leaving Ricky's at the same angle. The main diagonal or Market line is the set of all portfolios that are comprised of nothing but shares of M (no bonds) because every distribution along the line pays proportionally to the economy as a whole, \$4 in state b for every \$5 in a. For example, the point (220, 160) on the main diagonal (the point is not shown in the figure) represents the distribution of future income that Lucy would earn if she holds 44 shares of M because \$220 = \$5 x 44 and \$160 = \$4 x 44. Ricky holds the other 56 shares at that point. Now consider the 45-degree line starting from Lucy's origin. It is her *certainty line* because for every point on it her income is the same in both states. At (193, 193), for example, Lucy must own 193 bonds and no shares since the bond is guaranteed to pay \$1. She has lent \$193 and Ricky must be the bor-

[4] Notice that the expected growth rate of the economy is the same as the expected return on the market. It does not have to be. It is true in our example because we are assuming Cobb-Douglas or log utility.

rower. Ricky would likewise be in a riskless position on his certainty line. Lucy and Ricky cannot both be in riskless positions; the economy is, after all, risky, and they make up the economy.

Any point not on the Market line or one of the two certainty lines must be an income distribution for a portfolio of shares of *M* and safe bonds. Lucy and Ricky's shares add up to 100, and their bonds add up to 0. Points *e* and *C** are the two distributions that are important to the time-state preference model. Working out the composition of these portfolios is as simple as solving two equations in two unknowns. At *e*, Lucy receives \$160 in state *a* or \$250 in state *b*.

$$e_{Lucy}\begin{cases} e_a & = \$160 = \$5q_M + \$1q_F \\ e_b & = \$250 = \$4q_M + \$1q_F \end{cases} \rightarrow q_M = -90,\ q_F = 610$$

Those amounts, \$160 and \$250, have to come from the payoffs of *M* and *F*. Lucy is short 90 shares. She borrowed those from Ricky. This is a risky loan because the amount she'll have to repay Ricky depends on which state prevails. If *a* then she has to pay him \$450 but if *b*, only \$360. She used the loan to buy 610 bonds that will pay her \$610 no matter what. Kind of strange, isn't it? Taking out a risky loan to buy something safe?[5] Strange but not impossible; this is a theoretical model, and *e* was arbitrary. Ricky is long 90 shares and short 610 bonds (he borrowed \$610). No need to solve equations; it's just bookkeeping. I'll let you confirm that at the optimum Lucy holds about 50.3 shares, Ricky holds 49.7, and there is no debt outstanding (*C** is on the Market line in the example). Could you have gotten a sense of the composition of Lucy's *e* portfolio just by looking at its location in the Edgeworth box? Move *e* around. What happens to its composition if it is moved closed to Lucy's certainty line? Beyond that, as it is moved closer to the Market line? Beyond the Market line to the lower-right of the lower-right?

Here's some homework for you. Pick any point in the Edgeworth box and see if you can figure out the composition of Lucy and Ricky's portfolios at that point using the payoffs of securities *M* and *F*. You'll find that you can because the following conditions are met: there are at least as many securities as there are states of nature (*M* and *F*, *a* and *b*); the securities are not perfect substitutes (you cannot create the payoffs of *M* by multiplying the payoffs of *F* by some constant); and negative quantities are allowed (short selling). These three conditions are necessary for a *complete market*, which is one where any distribution of payoffs can be created by combining existing securities into portfolios. The upshot of this is that Lucy and Ricky can trade to any point in the Edgeworth box that they prefer, and that includes *C**. Choice is good.

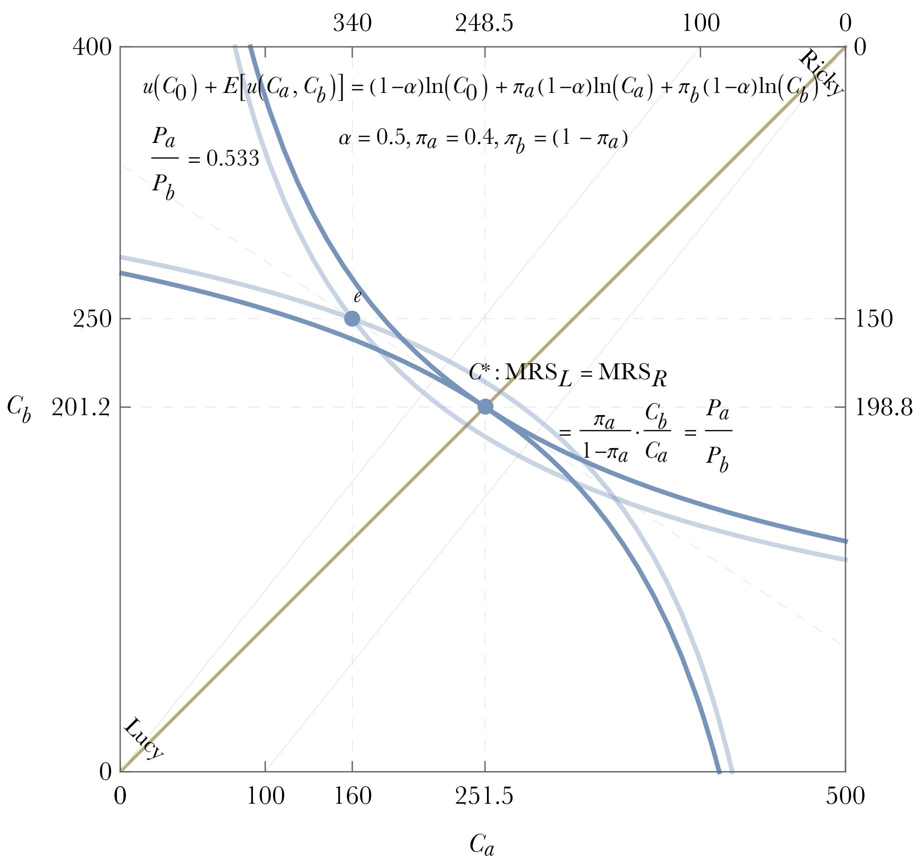
[5] Owning a positive quantity of a security is called a *long position*. Owning a negative number is called a *short position* or *short selling*, and it is the same as having borrowed.

The Mathematics of Time-State Preference Models

Let's work out the details for the equilibrium model that appears in chapter 10. I've repeated a lot of the exposition here but in more technical language. Lucy and Ricky have the same utility function:

$$u\left(C_0, C_a, C_b; \pi_a, \pi_b\right) = (1-\alpha)\ln\left(C_0\right) + \pi_a(1-\alpha)\ln\left(C_a\right) + \pi_b(1-\alpha)\ln\left(C_b\right),\ \alpha = 0.5,\ \pi_a = 0.4$$

Most of the information about the economy is in the Edgeworth diagram; all of it is in the table on the next page. We want to figure out optimal future consumption, the equilibrium inter-



est rate, and the equilibrium expected return on a risky stock. The future is next period ($t = 1$), and because the future is uncertain, optimal consumption is not a single amount; it is a distribution with C_a to be consumed if state a occurs and C_b if state b occurs.

Contingencies				*Endowments*			*Securities*	
Dates	*States*	*Probs.*	p_s	*Lucy*	*Ricky*	*Canada*	*M*	*F*
0	-	1	1	180	230	410	4.10	0.943
1	*a*	0.4	0.328	160	340	500	5.00	1.00
1	*b*	0.6	0.615	250	150	400	4.00	1.00

Lucy and Ricky are assumed to have the same tastes, the log utility function shown in the diagram. Making them twins is a big simplification, but that won't detract from the insights that the results will give. Lucy and Ricky are also assumed to hold the same beliefs about probability of each state occurring.

Recall that a pure state price, p_s, is a present value discount factor, but more specific than a regular present value factor. Instead of telling you the value today of some amount to be received in the future, it tells you the value today of an amount received in the future *if* a particular state of nature prevails. A dollar received in state *a* next period is worth 32.8 cents today, and a dollar received in state *b* is worth 61.5 cents today. All market values in a time-state preference model depend on those two equilibrium prices (or more than two if there are more than two states); they're to financial prices as carbon is to life. Lucy and Ricky's wealth, the price of a riskless bond, the price of a risky stock, and the value of any financial contract is derived from the state prices. That's why state prices are called *pure* or *primitive*.

The price ratio and the individual state prices. In equilibrium, Lucy and Ricky's marginal rates of substitution are equal, and that common marginal rate of substitution is ratio of equilibrium pure state prices.

$$C^*: \ (MRS_L = MRS_R) \equiv \frac{p_a^*}{p_b^*}$$

For *any* two utility function, equivalent MRSs means setting the slopes the indifference curves equal to one another

$$\frac{\pi_a}{\pi_b} \cdot \frac{u'\left(C_{aL}^*\right)}{u'\left(C_{bL}^*\right)} = \frac{\pi_a}{\pi_b} \cdot \frac{u'\left(C_{aR}^*\right)}{u'\left(C_{bR}^*\right)}$$

For Lucy and Ricky's particular log utility function, MRS is

$$MRS = \frac{\pi_a}{\pi_b} \cdot \frac{1-\alpha}{1-\alpha} \cdot \frac{C_b}{C_a} = \frac{\pi_a}{\pi_b} \cdot \frac{C_b}{C_a}$$

You can see that the marginal rate of substitution between consumption in states *a* and *b* does not depend on α, which, in turn, means that the equilibrium ratio of pure state prices does not depend on α. This is because utility in states *a* and *b* depends on α in the same way: $(1-\alpha)\ln(C_a)$ and $(1-\alpha)\ln(C_b)$. One dollar received in state *a* contributes the same satisfaction as one dollar received in state b. We say that, in this case, utility is *state-independent*. If the taste parameter for state a consumption was different from that for state b, $\alpha_a \neq \alpha_b$, or if a single taste parameter acted upon the utility of consumption differently in each state, such as $(1-\alpha)$ for state a and $\left(1-\frac{1}{2}\alpha\right)$ for state b, then utility would be state-dependent, and the price ratio would depend on the taste parameter or parameters.

Equating Lucy and Ricky's MRSs

$$MRS_L = MRS_R$$
$$\frac{\pi_a}{\pi_b} \cdot \frac{C^*_{bL}}{C^*_{aL}} = \frac{\pi_a}{\pi_b} \cdot \frac{C^*_{bR}}{C^*_{aR}}$$

The odds ratios drop out because we have assumed that Lucy and Ricky have the same beliefs[6]

$$\frac{C^*_{bL}}{C^*_{aL}} = \frac{C^*_{bR}}{C^*_{aR}}$$

This last expression says that in equilibrium Lucy and Ricky choose state b consumption to state a consumption to be in the same ratio. The consumption optimum C^*, therefore, must be on the main diagonal (Market line) of the Edgeworth box

$$\frac{C^*_{bL}}{C^*_{aL}} = \frac{C^*_{bR}}{C^*_{aR}} = \frac{E_b}{E_a} = \frac{400}{500} = \frac{4}{5}$$

and the ratio of aggregate endowments, $\frac{E_b}{E_a}$, can be used in the expression for MRS to find the equilibrium price ratio

$$\frac{p^*_a}{p^*_b} = \frac{\pi_a}{\pi_b} \cdot \frac{E_b}{E_a} = \frac{0.4}{0.6} \cdot \frac{400}{500} = \frac{8}{15} = 0.5\dot{3}$$

The price ratio is less than one: a state b dollar is worth more than a state a dollar. That makes sense because state b is less prosperous (\$400 < \$500)[7] and more likely to occur (a probability of 0.6). Both conditions support p_b being greater than p_a in equilibrium.

The individual equilibrium state prices can be extracted from the price ratio of each state at $t = 1$ to $t = 0$ as $\frac{p_s}{p_0}$ because $P_0 \equiv 1$ and $\pi_0 \equiv 1$. Neat trick.

$$p^*_a = \frac{p^*_a}{p_0} = MRS_{0,a} = \frac{\pi_a}{\pi_0} \cdot \frac{1-\alpha}{1-\alpha} \cdot \frac{E_0}{E_a} = \frac{\pi_a}{\pi_0} \cdot \frac{E_0}{E_a} = \frac{0.4}{1} \cdot \frac{410}{500} = 0.328$$
$$p^*_b = \frac{p^*_b}{p_0} = MRS_{0,b} = \frac{\pi_b}{\pi_0} \cdot \frac{1-\alpha}{1-\alpha} \cdot \frac{E_0}{E_b} = \frac{\pi_b}{\pi_0} \cdot \frac{E_0}{E_b} = \frac{0.6}{1} \cdot \frac{410}{400} = 0.615$$

---

[6] *Homogeneous expectations*.

[7] State b consumption is relatively scarcer.

Notice that the individual state prices *would* depend on α if it was not the same for C_0 as is for C_a or C_b. Think about what that implies and why. Notice also that the state prices do not add up to one; there's no reasons that they should. And don't confuse them with the state probabilities. The prices are not probabilities but they do depend on the probabilities.[8]

Consumption optimum. Solving for C^* means solving for the two unknowns, C_a^* and C_b^*. This requires two pieces of information or conditions. The first piece of information, which we've already come across, is that C^* sits on the Market line

$$\frac{C_b^*}{C_a^*} = \frac{E_b}{E_a} = \frac{4}{5}$$

The second piece of information is the condition that market value of Lucy and Ricky's consumption, that is, their wealth, must be equal to the market value of their endowments

$$W_0 = C_0 + p_a C_a + p_b C_b = e_0 + p_a e_a + p_b e_b$$

but current consumption is taken as given, $C_0 = e_0$, which more compactly,

$$p_a C_a + p_b C_b = p_a e_a + p_b e_b$$

Substituting the sits-on-the-Market-line condition into the wealth condition gives a simple and elegant expression for Lucy or Ricky's consumption optimum.

$$C_a^* = \frac{p_a e_a + p_b e_b}{p_a E_a + p_b E_b} Ea = \theta E_a$$
$$C_b^* = \frac{p_a e_a + p_b e_b}{p_a E_a + p_b E_b} Eb = \theta E_b$$

Optimal state a and b consumption are proportional to the aggregate endowments. The proportion θ is a person's share of the wealth of the country. Lucy consumes 50.3 per cent ($\theta_L = 0.503$) of Canada's aggregate state a endowment and 50.3 per cent of the aggregate state b endowment. Ricky consumes 49.7 per cent ($\theta_R = 0.497$). Plugging in Lucy's values gives

$$C_a^* = 0.503 \times 500 = \$251.50$$
$$C_b^* = 0.503 \times 400 = \$201.20$$

[8] From this point on, we'll drop the * from the state prices, and it will be understood that they are equilibrium prices.

Wealth. Wealth in a riskless, two-period world (chapter 11)

$$W_0 = C_0 + \frac{C_1}{1 + r_F}$$

becomes

$$W_0 = C_0 + \frac{E(C_1)}{1 + k}$$

in a risky, two-period world, where $k > r_f$ is some risk-adjusted discount rate; and, in a time-state preference model, that present value is written as

$$W_0 = C_0 + p_a C_a + p_b C_b$$

Lucy's wealth, based on her endowment, is \$386.23.

$$W_0 = 180 + 0.328 \times 160 + 0.615 \times 250 = \$386.23$$

Check that the answer is the same if Lucy's C^* is used to compute her wealth; it was, after all, the wealth condition we used to find C^*! The two equations for wealth in a risky world suggest that the connection between pure state prices and a "normal" discount rate or required k is

$$p_a C_a + p_b C_b = \frac{E(C_1)}{1 + k}$$

Work out k for Lucy and Ricky at e and C^*.

Prices and returns on a risky stock and a safe bond. It's easy to work out the prices of the risky stock and the safe bond: multiply their state payoffs by the state prices and add them up. If X_s is the payoff of asset x in state s, then the price of x is

$$P_x = p_a X_a + p_b X_b$$

and its expected return is

$$E(r_x) = \frac{E(X)}{P_x} - 1 = \frac{\pi_a X_a + \pi_b X_b}{P_x} - 1$$

Plugging in the values for Lucy and Ricky's economy,

$$\begin{aligned}
P_M &= p_a X_{aM} + p_b X_{bM} = 0.328 \times \$5 + 0.615 \times \$4 = \$4.10 \\
\therefore E(r_M) &= \frac{E(X_M)}{P_M} - 1 = \frac{\$4.40}{\$4.10} - 1 = 0.07317 \\
P_F &= p_a X_F + p_b X_F = X_F(p_a + p_b) = \$1(0.328 + 0.615) = \$0.943 \\
\therefore r_F &= \frac{X_F}{P_F} - 1 = \frac{\$1}{P_F} - 1 = \frac{\$1}{\$0.943} - 1 = 0.0604454
\end{aligned}$$

The expected return on the market, say, the TSX, is 7.3 per cent and the risk-free rate of interest is 6 per cent. The risk premium on the market is only 1.3 per cent. Historically, market risk premiums have averaged from three to six per cent. No matter how much you play with the variables in the time-state preference model—risk aversion parameters, endowments, risk—it is difficult to generate risk premiums that are close to those observed in real life. This anomaly is known as the *equity premium puzzle*.

Portfolio composition. I guess you could also call this security holdings. How many shares and bonds is Lucy endowed with? How many does she hold at the consumption optimum? Let q_x be the number of units of asset x she holds. Her consumption (read income) in either state comes is the number of shares of M she owns times their payoff and the number of bonds she owns times their payoff. Her security holdings, q_M and q_f, are the solution to this system of equations.

$$\begin{aligned}
C_a &= q_M X_{aM} + q_f X_F \\
C_b &= q_M X_{bM} + q_f X_F
\end{aligned}$$

For Lucy's endowment,

$$\begin{aligned}
e_a &= \$160 = \$5 q_M + \$1 q_F \\
e_b &= \$250 = \$4 q_M + \$1 q_F
\end{aligned}$$

She is short 90 shares ($q_M = -90$) and is long 610 bonds ($q_f = 610$).[9] Ricky must be long 190 shares because there are 100 shares of M outstanding, and he must be short 610 bonds (he's a borrower) because debt is in zero net supply. Confirm that at C^* Lucy holds 50.3 shares (where have you seen that number before?) and no bonds.

Complete markets. The system of equations representing state consumption as a function of security holdings can be written in matrix form as

$$\mathbf{C} = \mathbf{X} \,.\, \mathbf{q}$$

where

$$\mathbf{C} = \begin{pmatrix} C_a \\ C_b \end{pmatrix};\ \mathbf{X} = \begin{pmatrix} X_{aM} & X_{aF} \\ X_{bM} & X_{bF} \end{pmatrix};\ \text{and } \mathbf{q} = \begin{pmatrix} q_M \\ q_F \end{pmatrix}$$

The solution for **q**

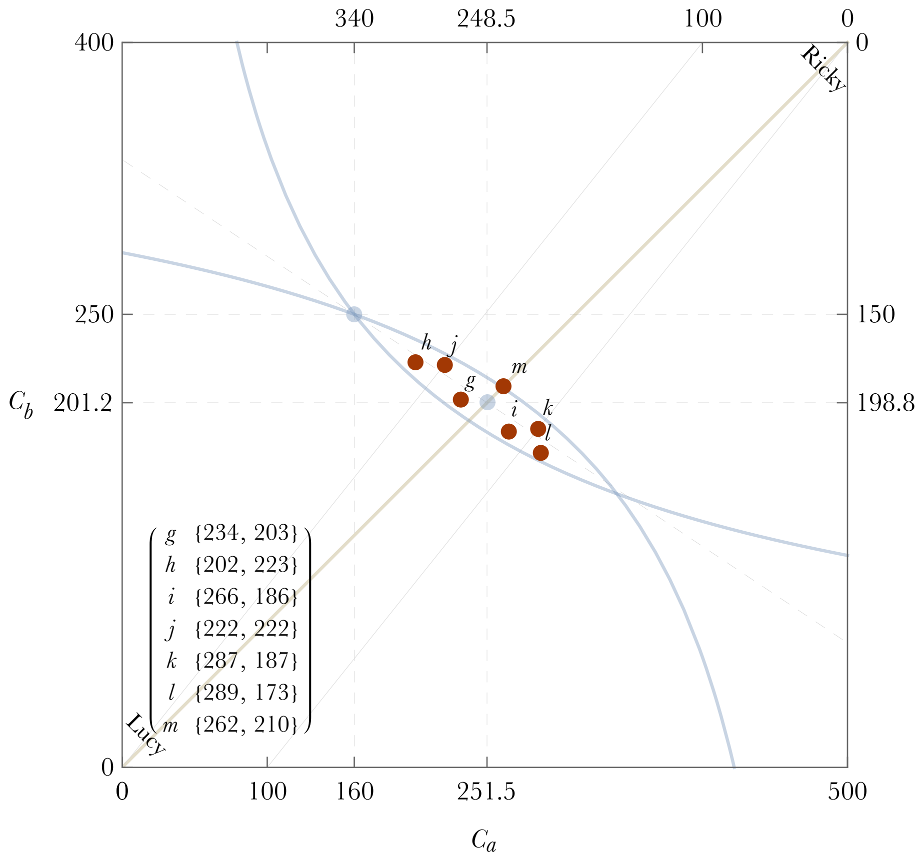
$$\mathbf{q} = \mathbf{X}^{-1} \cdot \mathbf{C}$$

exists if the payoff matrix **X** is non-singular (has an inverse). This has a neat economic interpretation. If **X** has an inverse, it means that the payoffs of the securities are linearly independent: the payoff of any one security cannot be replicated with some combination of the payoffs of the others (if there are more than two). That is really saying that all of the securities are unique—that there are no perfect substitutes. When is true, Lucy and Ricky then have maximum choice and any point in the Edgeworth box can reached through trade. It is called a *complete market*. The necessary conditions for a complete market are the following:

- There must be at least as many securities as there are states of nature (M and F for a and b)
- The payoffs of the securities must be linearly independent (**X** has an inverse)
- Short positions must be allowed (negative q's)

[9] Being long in bonds means lending, and short means borrowing.

Consider the seven alternative consumption points shown in red in the next figure. Are Lucy and Ricky able to trade M and F to reach these points?



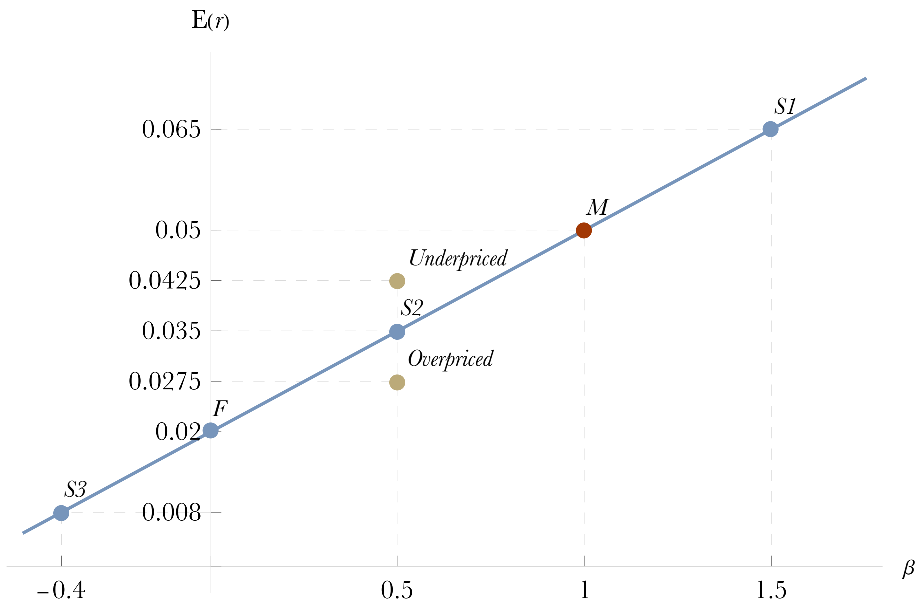
The answer is yes. Could they trade to those points if F was replaced by risky security H whose payoff is \$15 in state a and \$12 in state b. The answer is no.

The Capital Asset Pricing Model

Straight to the model because I have to get out and buy some groceries before the store closes. The CAPM is represented by the Security Market Line (SML), which says that the required

$$E(r_j) = r_f + \beta_j(E(r_m) - r_f)$$

(expected) return on any risk security *j* is equal to the risk-free rate plus a risk premium that is proportional to the risk premium on the market, $E(r_m) - r_f$.[10] In the stylized graph of the security market line below, the risk-free rate is two per cent, the expected return on the market is five per cent, making the slope of the SML or the risk premium on the market three per cent. The risk



premium is $\beta_j(E(r_m) - r_f)$, where beta is a measure of a security's systematic or market risk.

$$\beta_j = \frac{\sigma_{jm}}{\sigma_m^2} = \rho_{jm}\frac{\sigma_j}{\sigma_m}$$

Beta is the only thing that matters in the CAPM. You can think of it as a security's contribution to the risk of a well-diversified portfolio. The portfolio is the market portfolio (you can't get more diversified than that since the market includes all risky assets), and the security's contribution to the risk of the market portfolio is its covariance with the market, σ_{jm}. Dividing the covariance by the variance of the market, σ_m^2, normalizes beta to risk contribution per unit of market risk. Stock *S1* has a beta of 1.5. Its return is positively correlated with the market but fluctuates 50 per cent

[10] Unlike the time-state preference model, the risk-free rate is taken as given, not determined, in the CAPM.

more widely then the market, so it earns a premium that is 50 per cent bigger, 4.5 per cent, than the market's three per cent.

$$E(r_{S1}) = 0.02 + 1.5 \times 0.03 = 0.02 + 0.045 = 0.065$$

Stock *S2*, with a beta of 0.5, is also positively correlated with the market but half as volatile and so earns a premium of only 1.5 per cent above the risk-free rate. Stock *S3* commands a return that is less than the risk-free rate, which might not make sense at first glance. After all, don't all risk averse investors require that expected returns be greater than the risk-free rate? *S3* earns less than the risk-free rate because it is negatively correlated with the market—its beta is -0.4—and therefore contributes a great deal to the diversification of the market portfolio. You can think of its low required return as commensurate with a high price to reflect its important role in diversification. A well-known example of an asset with a negative beta is gold. It's value tends to rise in times of turmoil, financial, political, or otherwise, as investors seek what they perceive to be a safe store of value.

The expression for beta above shows it can also written as

$$\rho_{jm} \frac{\sigma_j}{\sigma_m}$$

which may be more intuitive with the correlation coefficient than covariance. A security is more or less risky that the market as $\dfrac{\sigma_j}{\sigma_m}$ is greater than or less than one, but within a well diversified portfolio, this ratio is tempered by a security's correlation with the market.

Why is the covariance or correlation between the security and the market in the expression for beta? As the number of securities in a portfolio is increased, the decline in risk that we know comes about occurs because the portfolio's risk becomes more and more determined by the covariances between securities than their individual variances (often referred to as total risk). There are $\dfrac{n(n-1)}{2}$ unique covariances in an n-stock portfolio but only n variances. If n approaches the number of securities in the market, the product of the covariance matrix and the weight of each security in the market portfolio[11] yields the vector of covariances of each security with the market.

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \cdots & \sigma_{2n} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \cdots & \sigma_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \sigma_{n3} & \cdots & \sigma_{nn} \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} \sigma_{1m} \\ \sigma_{2m} \\ \sigma_{31} \\ \vdots \\ \sigma_{nm} \end{pmatrix}$$

That's the mathematical connection between a security's returns and its beta coming from diversification over many assets. It does not explain why people would hold only the market or, what amounts to the same thing, why only market risk is relevant in determining a security's return.

[11] By weight of each security in the market I mean market value weight, which is the dollar value of a company's equity divided by the dollar value of the entire market. The investment weights that make the *M* or market portfolio on the Capital Market Line in chapter 11 are market value weights, although not in the example used in that chapter.

Another way to think about beta is to assume that the random return on a security is linearly related to just one random variable, the return on the market, with of course, some error. You could call it a one-factor model. You might write

$$r_j = \alpha_j + b_j r_m + \epsilon_j$$

The slope b_j connects the return on security j to the return on the market. The portion of the security's return that cannot be accounted for by the market is the error term, ϵ_j, which you might interpret as being unique to the company and whose variance is often called *unique risk* or *unsystematic risk*.[12] Also suppose that you wanted an estimate of b_j that minimized unique risk because that estimate would give you the best fit between the security's return and the return on the market. Solve for the error term in your linear model and write its variance.

$$\begin{aligned}
\epsilon_j &= r_j - (\alpha_j + b_j r_m) \\
&= r_j - \alpha_j - b_j r_m \\
\therefore \operatorname{var}(\epsilon_j) &= \operatorname{var}(r_j - \alpha_j - b_j r_m) \\
&= \operatorname{var}(r_j - b_j r_m) \\
\therefore \sigma^2_{\epsilon_j} &= \sigma^2_j + b^2_j \sigma^2_m - 2b_j \sigma_{jm}
\end{aligned}$$

Now minimize the variance with respect to b_j

$$\begin{aligned}
\frac{d\sigma^2_{\epsilon_j}}{db_j} &= 2b_j\sigma^2_m - 2\sigma_{jm} = 0 \\
\therefore b_j &= \frac{\sigma_{jm}}{\sigma^2_m}
\end{aligned}$$

What you just did was solve for the slope in a simple linear regression of r_j on r_m.

Derivation of the CAPM. Thinking about beta as the coefficient in a one-factor model still doesn't answer why people would hold only the market or, what amounts to the same thing, why only market risk is relevant in determining a security's return. That comes from utility theory and the assumption that people are risk averse. Our proof of the CAPM follows directly from our section "Pricing a Risky Asset" in chapter 13, but now we need to impose an equilibrium or market-clearing condition, and for that, we'll will need to consider many investors and many securities.

$i = 1,2,3,\ldots, N$ investors
$j = 1,2,3,\ldots, K$ securities, excluding the risk-free security F

The uncertain end-of-period income, y_i, for any investor i is

[12] The error is independent of the return on the market.

$$
\begin{aligned}
y_i &= r_f M_i + \sum_j r_j S_{ij} \\
&= r_f W_i + \sum_j S_{ij}(r_j - r_f)
\end{aligned}
$$

where r_f is the risk-free rate, r_j is the risky return on the security j, and W_i is the investors current wealth. The investor maximizes their expected utility with respect to their dollar investment S_j in each security

$$
\text{Max } E[u(y_i)] = \text{Max } E[u(r_f W_i + \sum_j S_{ij}(r_j - r_f))] \;\forall\, i
$$

where ∀ means "for all."

There are K first-order conditions for a maximum, one for each security

$$
\frac{dE[u(y_i)]}{dS_j} = E[u'(y_i)(r_j - r_f)] = 0 \;\forall\, j
$$

Using the definition of covariance gives

$$
E[u'(y_i)]E[r_j - r_f] + \text{cov}[u'(y_i), (r_j - r_f)] = 0 \;\forall\, j
$$

or

$$
E(r_j - r_f) + \frac{\text{cov}[u'(y_i), (r_j - r_f)]}{E[u'(y_i)]} = 0 \;\forall\, j
$$

Here's where the story ends unless we make a the critical assumption that allows us to separate an investors' tastes for risk (their utility functions) from the return on the security inside the covariance operator

Assume that investors have quadratic utility or that security returns are normally distributed.

Without subjecting you to the math, either assumption allows us to write the above expression as

$$
E(r_j - r_f) + \frac{E[u''(y_i)]}{E[u'(y_i)]}\text{cov}(y_i, r_j) = 0 \;\forall\, j
$$

You'll recognize the fraction in front of the covariance operator as the coefficient of absolute risk aversion. Take the reciprocal of ARA and call it risk tolerance θ_i (makes sense, eh?)

$$
\text{Let } \theta_i = \frac{u'(y_i)}{u''(y_i)} = \frac{1}{ARA(y_i)}
$$

For the specific case of quadratic utility, $u(y) = y - \alpha y^2$, $\alpha > 0$, the risk premium can be broken down into a risk aversion part and the riskiness of the stock. To get there, note the first and second derivative of the utility function and its coefficient of absolute risk aversion

$$\theta_i E(r_j - r_f) - \text{cov}(y_{ij}, r_j) = 0 \; \forall \, i \text{ and } j$$

Now for the market-clearing condition. Aggregate (add up) over all of the investors

$$\theta_m E(r_j - r_f) - \text{cov}(y_m, r_j) = \sum_i \theta_i E(r_j - r_f) - \text{cov}(y_{ij}, r_j) = 0 \; \forall \, j$$

where θ_m is the risk tolerance of the market (a kind of average of the risk tolerance of all investors) and y_m is the income of all investors or the market. But the income of the market must be equal to the return on the market times its value

$$y_m = r_m \cdot V_m$$

so

$$\begin{aligned}
\text{if } \theta_m E(r_j - r_f) - \text{cov}(y_m, r_j) &= 0 \; \forall \, j \\
\text{then } \theta_m E(r_j - r_f) - \text{cov}(r_m \cdot V_m, r_j) &= 0 \; \forall \, j \\
\therefore \theta_m E(r_j - r_f) - V_m \, \text{cov}(r_m, r_j) &= 0 \; \forall \, j
\end{aligned}$$

Almost there. Look at the last line above. If it is true for any security j, then it must be true for a security that is the portfolio of all securities, the market

$$\begin{aligned}
\text{If } \theta_m E(r_j - r_f) - V_m \, \text{cov}(r_m, r_j) &= 0 \; \forall \, j \\
\text{then } \theta_m E(r_m - r_f) - V_m \, \text{cov}(r_m, r_m) &= 0
\end{aligned}$$

Combine the last two equations

$$\begin{aligned}
\theta_m E(r_j - r_f) &= V_m \, \text{cov}(r_m, r_j) \\
\text{and } \theta_m E(r_m - r_f) &= V_m \, \text{cov}(r_m, r_m)
\end{aligned}$$

Divide the first by the second

$$\begin{aligned}
\frac{\theta_m E(r_j - r_f)}{\theta_m E(r_m - r_f)} &= \frac{V_m \, \text{cov}(r_m, r_j)}{V_m \, \text{cov}(r_m, r_m)} \\
\therefore E(r_j - r_f) &= \frac{\text{cov}(r_m, r_j)}{\text{cov}(r_m, r_m)} E(r_m - r_f)
\end{aligned}$$

Solve for $E(r_j)$ and bingo, the CAPM

$$\begin{aligned}
E(r_j) &= r_f + \frac{\text{cov}(r_m, r_j)}{\text{var}(r_m)} E(r_m - r_f) \\
&= r_f + \beta_j \, E(r_m - r_f)
\end{aligned}$$

Time to go out and buy groceries.

EXERCISES

1. Explore a time-state preference economy inhabited by Lucy and Ricky, who have the same tastes but different endowments.

Dates:	$t = (0,1)$
State probabilities:	$(\pi_a, \pi_b) = (0.375, 0.625)$
Utility:	$u(C_0, C_a, C_b; \pi_a, \pi_b) = C_0^{1-\alpha} + \pi_a C_a^{1-\alpha} + \pi_b C_b^{1-\alpha}$, $\alpha = 0.5024$
Aggregate endowment:	$(E_0, E_a, E_b) = (100, 110, 118)$
Lucy's endowment:	$(e_0, e_a, e_b) = (40, 66, 54)$
Payoff of risky stock M:	$(X_{Ma}, X_{Mb}) = (1.10, 1.18)$
Payoff of safe bond F:	$(X_{Fa}, X_{Fb}) = (1, 1)$

a. What is the expected growth rate of the economy?

b. What is the ratio of pure state prices?

c. What are the pure state prices?

d. What are the prices of M and F?

e. What is the expected return of M and F?

f. How wealthy are Lucy and Ricky?

g. What is the consumption (Pareto) optimum?

h. What is the composition of Lucy and Ricky's portfolios at their endowment points?

i. What is the composition of Lucy and Ricky's portfolios in equilibrium?

2. Test the Capital Asset Pricing Model using the two-step protocol we discussed in class. Recall that the cross-sectional regression you need to estimate is

$$\bar{r}_j = \gamma_0 + \gamma_1 \hat{\beta}_j + \gamma_2 \hat{\sigma}^2_{\epsilon_j} + \nu_j$$

where i denotes the stock. $\bar{r}_j$ is the mean difference between the return on stock j and the risk-free rate; $\hat{\beta}_j$ is stock j's estimated beta; $\hat{\sigma}^2_{\epsilon_j}$ (unique risk) is the variance of the residual from the regression used to estimate stock i's beta; and ν_i is an error term. You can find the data in the file "Stock Returns.csv," which is the data file we used to form portfolios at the beginning of term. For your sample, use the randomly-selected 48 stocks listed in the table and returns for 72 months, 2008-01-31 through 2013-12-31. Make sure that you pull the right stocks from the file: your estimation and tests will depend on it. The date is in column 1; the return on the Market portfolio is in column 2; the 30-day return on T-Bills is in column 3; and the stock returns start in column 4.

Stock	*Ticker*	*Stock*	*Ticker*	*Stock*	*Ticker*	*Stock*	*Ticker*
4	ACO.X	73	DC.A	121	LNR	165	RON
5	ACQ	74	DDC	122	LUN	166	RUS
10	AIM	77	DSG	139	NPR.UN	167	RY
17	ATA	81	EMP.A	140	NSU	173	SLW
20	AX.UN	91	FR	145	OTC	189	TFI
23	BAM.A	96	FTT	147	PD	194	TOG
28	BDI	100	GRT.UN	150	PJC.A	196	TRP
39	BXE	108	IAG	152	POT	198	VET
49	CG	112	IMO	154	POW	199	VRX
64	CS	113	INE	155	PPL	200	VSN
70	CUS	114	IPL	158	PWF	204	WPT
72	CWT.UN	116	K	159	PWT	207	YRI

a. Do step 1 of the protocol by estimating each stock's beta, residual variance, and the mean difference of its return from the risk-free rate. Report your estimates in a table. Are there any stocks with negative betas?

b. Do step 2 of the protocol by estimating the cross-sectional regression described in the introduction to this part of the assignment. Report your results in a table and interpret the tests for hypotheses about the coefficients γ_0, γ_1, and γ_2.

c. What is the unadjusted R-squared of your regression in exercise 2? What does it mean?

ANSWERS

1. Make sure that you can explain each of your answers intuitively and in everyday language.
    a. The economy is expected to grow by 15 per cent.
    b. The ratio of pure state prices is 0.62154. Why is it less than 1?
    c. $(p_a, p_b) = (0.357467, 0.575131)$
    d. $(P_M, P_F) = (\$1.07187, \$0.932597)$. Why is the price of the riskless bond less than \$1? Could it ever be more than \$1?
    e. $(E(r_M), r_F) = (0.0728938, 0.0722741)$. Is the risk premium big or small?
    f. $\{W_{Lucy}, W_{Ricky}\} = \{\$94.6499, \$112.537\}$
    g. In equilibrium (C^*), Lucy consumes $(C_a, C_b) = (56.0842, 60.1631)$ and Ricky $(C_a, C_b) = (53.9158, 57.8369)$. How does this compare to their endowments?
    h. At the endowment, Lucy is short 150 shares of *M* and is long 231 bonds; Ricky is long 250 shares of *M* and is short 231 bonds. Who is a borrower? Who is in a riskier position and in what sense?
    i. In equilibrium, Lucy is long 50.9856 shares of *M* and has no position in bonds; Ricky is long 49.0144 shares of *M* and has no position in bonds.

2. Testing the CAPM.

   a. Time series estimates of the parameters.

Stock	*Beta*	*Var e*	*Mean r*	*Stock*	*Beta*	*Var e*	*Mean r*
ACO.X	0.41228	0.00271	0.00959	LNR	2.044810	0.022395	0.025402
ACQ	1.35137	0.03266	0.04436	LUN	3.192150	0.039504	0.011605
AIM	0.80542	0.00615	0.00395	NPR.UN	0.593695	0.002210	0.009222
ATA	1.29898	0.01759	0.02285	NSU	1.510860	0.024455	0.021134
AX.UN	1.19325	0.00600	0.01106	OTC	0.478739	0.006729	0.018707
BAM.A	0.93224	0.00293	0.00644	PD	1.973410	0.017320	0.008835
BDI	1.23943	0.00401	0.02941	PJC.A	0.294828	0.003482	0.009911
BXE	1.87595	0.02487	0.02713	POT	1.200680	0.010145	0.002114
CG	1.47118	0.07087	0.02102	POW	0.953129	0.002495	0.001665
CS	2.39506	0.01054	0.01207	PPL	0.283482	0.001619	0.015734
CUS	0.75110	0.00583	0.01675	PWF	0.904149	0.002527	0.003385
CWT.UN	1.02830	0.00396	0.00890	PWT	1.253420	0.006487	-0.004541
DC.A	1.72008	0.01044	0.00815	RON	0.285850	0.005740	-0.001115
DDC	2.15162	0.01489	0.00185	RUS	1.408770	0.004201	0.011329
DSG	0.64991	0.00470	0.01892	RY	0.666092	0.002962	0.009171
EMP.A	-0.03381	0.00237	0.00883	SLW	1.993010	0.023512	0.017771
FR	2.03960	0.02532	0.02703	TFI	1.546220	0.007561	0.023843
FTT	1.34326	0.00458	0.00454	TOG	1.275730	0.023445	0.001747
GRT.UN	1.23162	0.01582	0.01636	TRP	0.341527	0.001419	0.005771
IAG	1.10556	0.00563	0.00724	VET	0.988956	0.002080	0.013236
IMO	0.76066	0.00230	-0.00045	VRX	0.502896	0.008582	0.037474
INE	0.52022	0.00900	0.00447	VSN	0.452325	0.002240	0.010360
IPL	0.61258	0.00157	0.01963	WPT	1.486000	0.023810	0.023348
K	0.54851	0.01537	-0.01140	YRI	1.006860	0.016851	0.004841

b. Results of cross sectional regression using the estimates from part a.

Coefficient	*Estimate*	*Standard Error*	*t-statistic*	*p-value*	*CI Lower Bound*	*CI Upper Bound*
$\gamma 0$	0.00864	0.00300	2.88410	0.00600304	0.002608	0.014681
$\gamma 1$	0.00050	0.00282	0.17787	0.859621	-0.005177	0.006180
$\gamma 2$	0.28057	0.14364	1.95331	0.0570176	-0.008732	0.569871

The null hypothesis that the intercept, γ_0 = 0.00864418, is zero is rejected using its *t*-stat, *p*-value, or confidence bounds. This means that there is some constant return that is not explained by the CAPM (remember that we subtracted the risk-free rate from each side of the regression equation).

The null hypothesis for γ_1, estimated as 0.000501517, is that it should equal the average risk premium on the market, 0.00221815. γ_1's t-stat or *p*-value cannot be used to test that because those statistics test whether the coefficient is equal to zero. But the confidence bounds can be used. In our case, the null hypothesis for γ_1 is not rejected because the mean risk premium lies within the confidence bounds. This means that the slope of the Security Market Line is equal to the risk premium on the market: that expected return does change with systematic risk as it should.

The null hypothesis that γ_2 = 0.280569 is zero is not rejected. This means that whatever factors are captured by the variance of the residuals, those not contemplated by the CAPM, do not influence expected returns.

c. R-squared is 0.123421. This means that only 12 per cent of the variation in returns from one stock to the next is explained by beta and residual variance. Not impressive!

5
DERIVATIVE SECURITIES

A derivative security is one whose payoff depends on the outcome of an event. Derivative security D might be one that pays 10 cents for every point that stock index U is above 1000 on a particular date. U is called the *underlying security*. The price of D any time before that expiration date, and how it changes from day to day, ideally reflects the market's expectation of U's value on that date. D's price, of course, could just as easily reflect everyone second-guessing everyone else's forecast for U—as in Keyne's Beauty Contest—and turn out to be a poor predictor of U's value. But that is true for any financial security, not just derivatives. Either way, you can see that the market for D is a prediction market, and so the underlying security does not have to be a financial security at all. U could be any event that people have an interest in betting on, such as cumulative rainfall in a particular province as of a particular date.[13] It's not surprising that derivative securities are also called *contingent claims*. An insurance contract is a contingent claim that everyone knows.

The most common traded derivatives are forward and futures contracts and option contracts. We'll consider them here. But given that derivatives are bets on future events, there are countless variations: futures on options, options on futures, exchanges on cashflows called swaps. The list goes on and on.

Forwards and Futures

A forward contract is an agreement made now for a transaction that will take place later. It's that simple. Suppose it's January 8th and you know you will need 125 thousand euros on April 16th. You could buy the euros now at the current spot price and stuff them in a drawer until April 16th or maybe deposit them in a euro account to earn interest in the meantime.[14] You could wait until April 16th and buy them at whatever the spot price is on that day, but that's risky and you may not want to bear that risk. Or you could buy them forward. If you can find a willing seller, the two of you would enter into a contract where the seller (writer of the contract) agrees to delivery 125 thousand euros to you on April 16th and you agree to pay the price that the two of you have negotiated. That price is called the forward price. Let's say it is CAD $1.50. You have locked in the April 16th price of the euro at CAD $1.50. It doesn't matter what the spot price is on that day. Your gain on the expiration date is the difference between the spot price on that day and the forward price. If the spot price is $1.56, you'll be feeling good about yourself; if the spot price is $1.43, you'll be feeling some regret. Either way, by buying forward, you have eliminated the price risk. You would not have bought forward at $1.50 if you believed there was a good chance that

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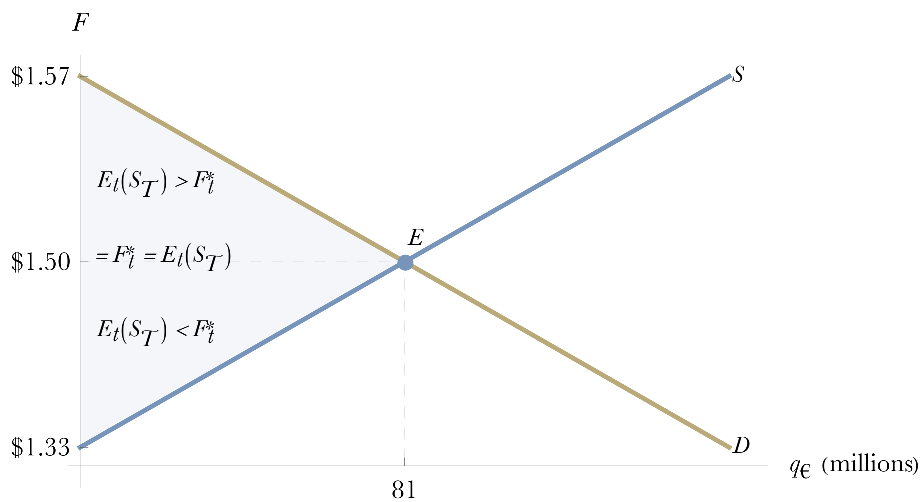
[13] There's a lot of *particulars* here.

[14] Scotiabank offers a euro account. Other Canadian banks probably do too.

the spot price on April 16th was going to be below \$1.50, and the seller would not have agreed to sell forward at \$1.50 if they believed there was a good chance the spot price was going to be above \$1.50. That's why the deal was done. Of course, your gain is the seller's loss and vice versa. It's a zero-sum game. As a matter of fact, if the spot price on April 16th ends up being \$1.52, it's okay for the seller to pay you your \$2,500 gain rather than deliver the 125 thousand euros. If you really need the euros, you can then buy them at the new spot price of \$1.52 but it's only costing you \$1.50. And what if the spot price ends up being \$1.47? Did I say it was simple?

A forward contract is a legal contract. There is offer and acceptance, and that creates obligations: you are obligated to pay CAD \$187,500 on the expiration date, and the seller is obligated to deliver the euros, however or wherever you've agreed. The contract itself has no value because no money or goods changes hands at the time it is struck, and while you or the writer might expect to gain—well, more on that below.

You may feel uneasy about the \$1.50 forward price because it is not a market price; a deal between one buyer and one seller does not a market make. If there are a number of buyers and sellers, however, communicating their offers to buy or sell, looking for the best offers, then the forward price is the one that clears the market. The supply and demand graph shows that the

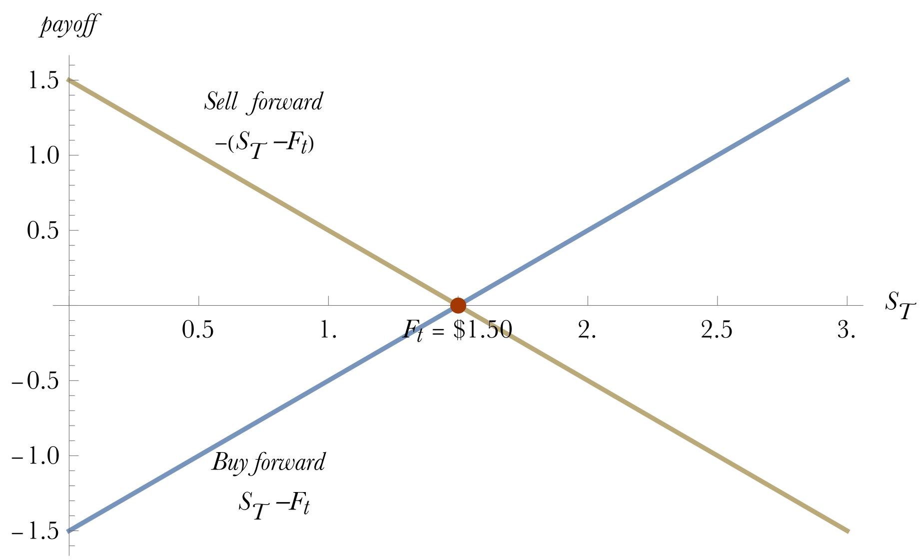


equilibrium forward price F^* at any time t should be equal to the market's expectation of what the spot price S will be in T periods, that is, at the expiration of the contract. At that price there are contracts for a total of 81 million euros.

$$F_t^* = E_t(S_T)$$

The forward buyers of the 81 million euros are people who believe the spot price in April will be at least \$1.50, and as a result, expect a non-negative gain or consumer's surplus. The forward sellers believe the opposite and so expect a non-negative producer's surplus. Their coming together resulted in the \$1.50 equilibrium price. But how can a gain be expected in a transaction that involves no real production and where no money changes hands now? It shouldn't. If the forward price didn't equal the future spot price *on average,*[15] speculators and arbitragers would exploit any pattern of overshooting or undershooting, causing the pattern and opportunity to implode. In other words, an efficient market suggests that $F_t = S_T$ on average: win a few, lose a few, but on average, flatline.

How do you know that the seller will deliver the euros? How does the seller know that you will pay? In one-on-one deals, you don't and neither do they. The risk of welshing could really put a damper on an entire market, so organized futures exchanges, such as the Chicago Mercantile Exchange or the Montreal Exchange, have safeguards in place, such as margin requirements and marking-to-market, that ensure commitments are met. As a matter of fact, buyers and sellers are anonymous. You'll never know who's supplying you those euros, and you don't need to. And that takes us to the question of the difference between a forward contract and a futures contract. A futures contract is just a standardized forward contract. Futures contracts expire at fixed dates



throughout the year, for example, the third Friday of every month for CME oil contracts, and the third Friday in March, June, September, and December for CME financial contracts, such as stock index futures. They are for a specified quantity (125,000 euros or 40,000 lbs. of pork bellies) and quality (hot-rolled coil steel), and for commodities, acceptable delivery locations. Trade is easier and works better when everyone knows what they are trading.[16]

Changes in forward prices are generally highly correlated with changes in the *current* spot prices of their underlying assets. This makes them good tools for hedging risk. Have a look at the one-month spot and forward prices for the euro for February 28, 2001 through January 31, 2007. You can find these in "Currency Spot Prices.csv" and "Currency Forward Prices.csv". Convert the 73 spot prices to 72 monthly returns in the usual way

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[15] That is, the forward price should be an unbiased predictor of the future spot price. See Froot and Thaler (1990), cited in chapter 7, for evidence.

[16] Have a look at CME euro futures at www.cmegroup.com/trading/fx/g10/euro-fx.html.

$$r_s = \frac{P_t - P_{t-1}}{P_{t-1}}$$

But convert the forward prices to "returns" this way

$$r_f = \frac{P_t - F_{t-1}}{P_{t-1}}$$

The numerator is the gain or loss on the forward position at expiration and the denominator is the price you would have paid or received had you bought or sold euros on the spot market at the beginning of the month. The reason for computing the return on the forward position this way is that, as you recall, the contract itself has no value but does result in gains and losses relative to the spot price over time.

Now compute the correlation of euro spot and forward returns. You'll find it is almost perfect at 0.9997, and as you know from your portfolio math, there's a lot of diversification to be had by combining assets that are highly correlated. For example, say that you are a Canadian exporter of goods to Europe, and every month you are paid 1 euro for your shipment. That monthly payment is a spot position in the euro. The return on this simple portfolio is

$$r_p = 1 \cdot r_s$$

and the ups and downs in return represent the exchange rate risk in converting the euro to Canadian dollars every month. You can hedge the risk by selling one euro forward every month for every euro you receive. That would lock in the conversion rate at the end of each month. The return on your portfolio would then be

$$r_p = 1 \cdot r_s - 1 \cdot r_f$$

It looks like the portfolio weights do not add up to 1 ($1 + (-1) = 0 \neq 1$), but they do. The weight of -1 in euro forwards is not a portfolio weight in the sense of a percentage of your wealth because there is no cashflow when you buy or sell forward (remember the value of the contract is zero). The weight is the percentage of the spot position being bought or sold forward, in this case, 100 per cent (euro for euro). The minus sign means sell forward. The weight is called the hedge ratio, h

$$r_p = r_s - h \cdot r_f$$

Will a hedge ratio of $h = 1$ wash away the most risk? A lot but not the most. Write the variance of the portfolio and minimize it with respect to the hedge ratio.

$$\begin{aligned} r_p &= r_s - h \cdot r_f \\ \text{var}(r_p) &= \text{var}(r_s - h \cdot r_f) \\ \therefore \sigma_p^2 &= \sigma_s^2 + h^2\sigma_f^2 - 2h\sigma_{sf} \\ \therefore \frac{d\sigma_p^2}{dh} &= 2h\sigma_f^2 - 2\sigma_{sf} = 0 \end{aligned}$$

Simplifying the last line above gives the risk-minimizing hedge ratio

$$h^* = \frac{\sigma_{sf}}{\sigma_f^2} = \rho_{sf}\frac{\sigma_s}{\sigma_f}$$

will be very close to 1—0.998 for your euro position—but in general not equal to 1 because spot and forward returns are seldom perfectly correlated and their standard deviations will differ somewhat. To see this, write $\frac{\sigma_{sf}}{\sigma_f^2}$ as $\rho_{sf}\frac{\sigma_s}{\sigma_f}$. How does that work out for your portfolio? Compute the portfolio's standard deviation unhedged ($h = 0$), naively hedged ($h = 1$) and risk-minimized ($h = h^*$). The table reports sample standard deviations, but population standard deviations would tell the same story.

Hedging the Euro, 2001-02-28 to 2007-01-31 (72 months)

Hedge ratio	*Expected return*	*Standard deviation*	*%Δ Standard deviation*
$h = 0$	0.001623	0.025275	-
$h = 1$	0.000235	0.000604	-0.976121
$h^* = 0.998118$	0.000238	0.000602	-0.003121

A one-for-one hedge gets rid of 98 per cent of the euro risk. A risk-minimizing hedge improves that by only three-tenths of one per cent. The farther away that h^* is from 1, the greater the reduction in risk over a naive hedge. You notice also that hedging has affected the portfolio's mean return in this case. A forward position is not necessarily zero gain.

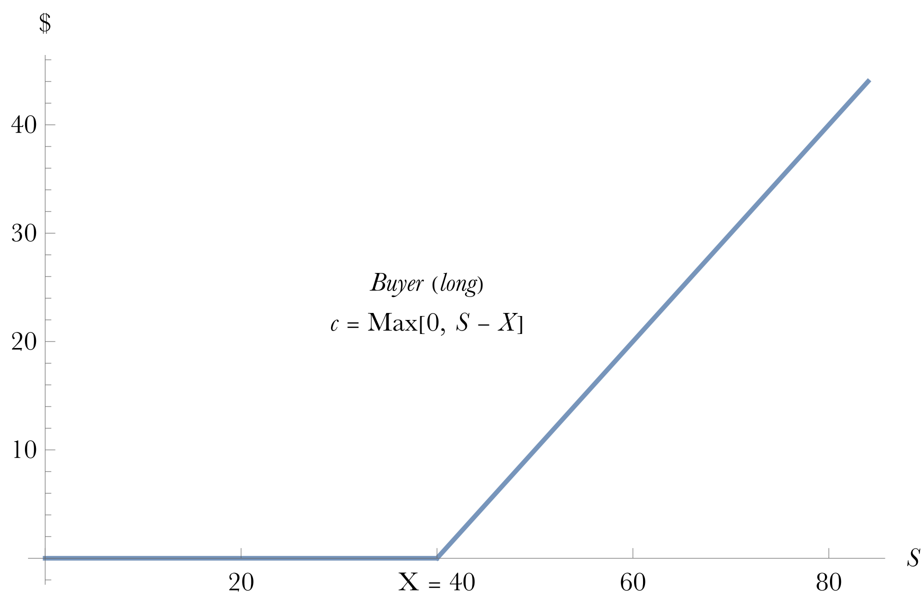
Options

An option contract gives its owner either the right to buy or the right to sell an underlying asset at a specified price (the exercise price) on a specified date (the expiration date).[17] The difference between an option contract and a forward contract can be found in the words *owner* and *right*. To be the owner of something you must have paid for it or inherited it, in which case it was already paid for. In paying the option price (called the *option premium*) the buyer has fulfilled their contractual obligation up front. The seller or writer of the contract, however, is on the hook until the expiration date when they must satisfy their side of the deal if the buyer exercises.

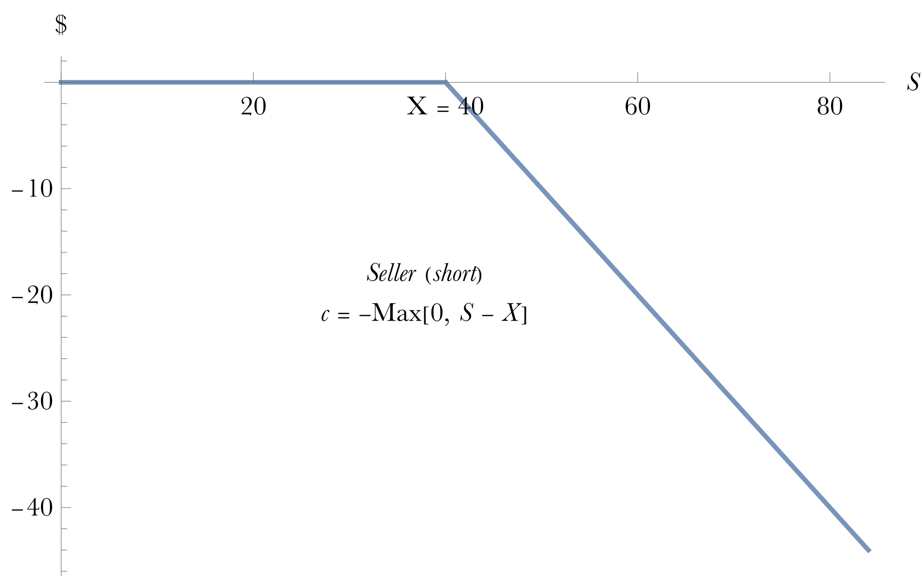
A call option is the right to buy. The next diagram shows the payoff c at maturity of a call option with an exercise price of $X = \$40$ on a stock whose value at maturity is S. The diagram does not include the price paid, whatever that is, for the option today. If S is greater than X the owner exercises their right to buy, and their payoff is $S - X$; otherwise, the owner lets the option die unexercised and the payoff is \$0. The payoff to the buyer of an option, unlike the payoff from having bought something forward, can never be negative because the price paid confers a right,

[17] The exercise price is also called the strike price and the expiration date, the maturity date.

and people will only exercise a right when there is more to gain than by not exercising it.[18] The seller loses when the buyer gains. It is a zero-sum game just like it is with forward contracts. Don't



feel bad that the seller loses if the option is exercised; they received the option premium up front.



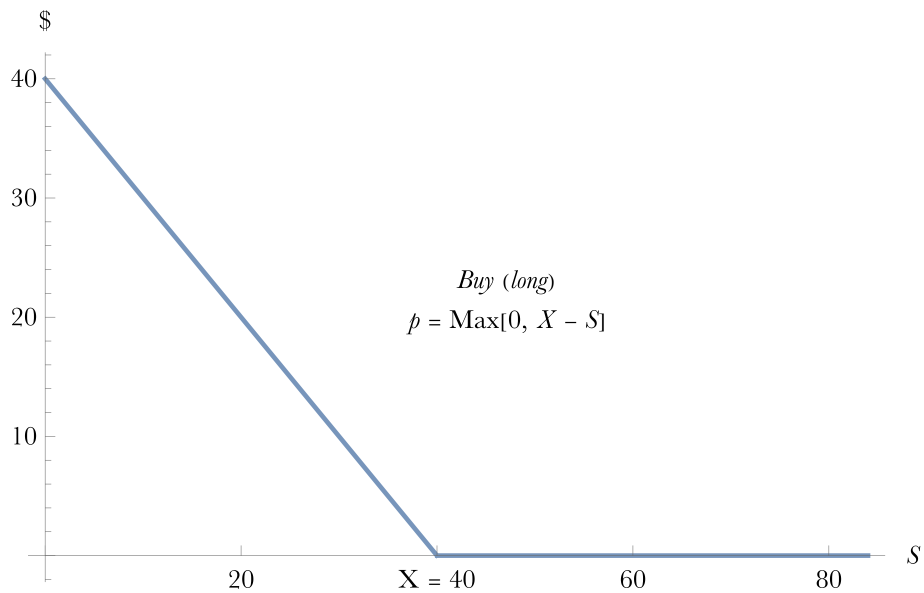
A put option is the right to sell. It can act as insurance or a hedge if you own the underlying asset, allowing you to sell the asset for X no matter the asset's price. The price of the put is like an insurance premium. The seller (writer) of the put is in the position of insurer because they are obligated to pay the exercise price (pay a claim) to the owner who exercises their put.[19] The pay-

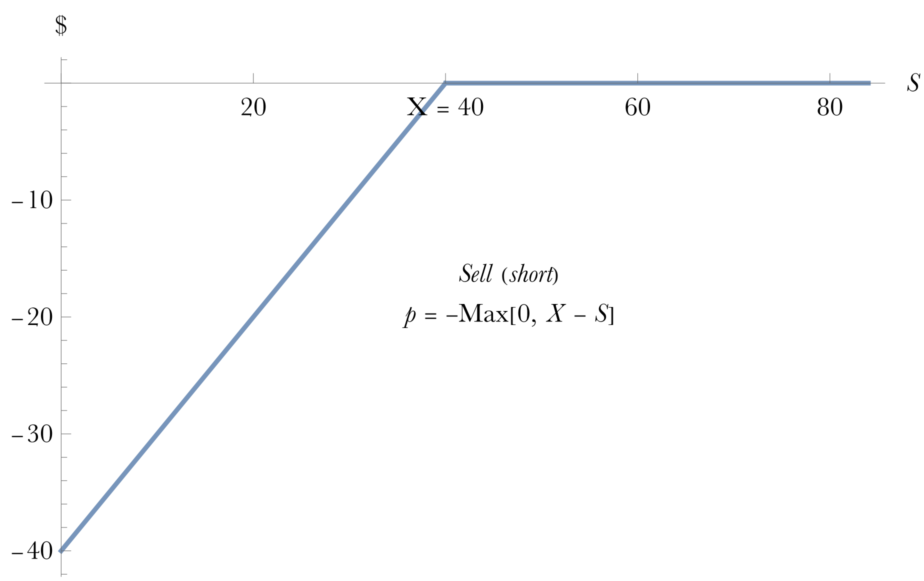
---

[18] It is correct to say buy option contracts or sell options contracts. Is it correct to say buy forward contracts or sell forward contracts?

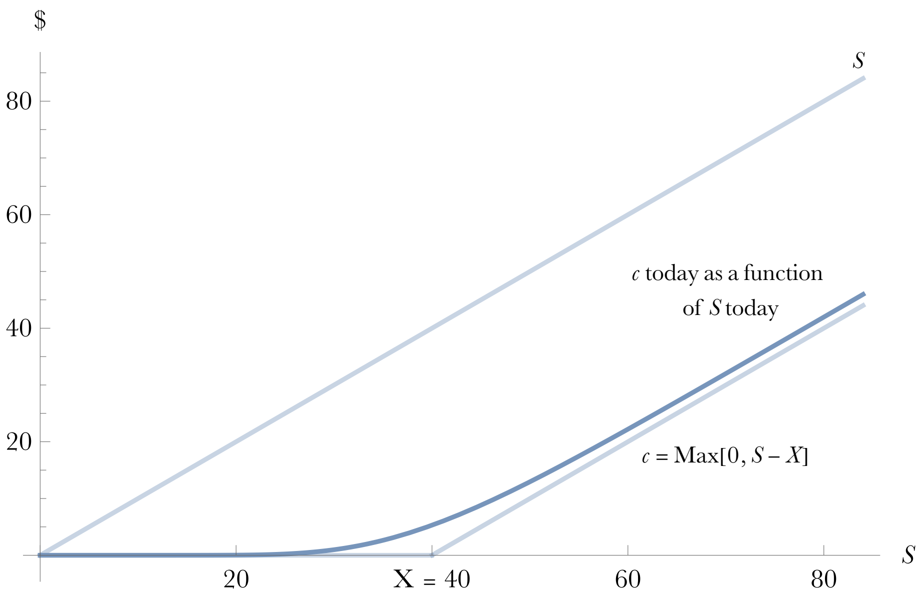
[19] Compare hedging by buying a put with selling forward.

offs to the buyers and sellers of puts, as with calls and forwards, are a zero-sum game as shown in the payoff diagrams.





Determinants of the price of a call option. What does the payoff of a call option at maturity tell us about its price today? The next diagram shows the upper and lower bounds for the value of a call at any time *t* before maturity.The value of a call today can never be greater than the value of the underlying asset because the call gives its owner nothing more than the right to own the asset. For a call option on a stock, the owner of the stock has something that the owner of the call does not: voting rights. The value of a call today must also be greater than its payoff at



maturity, Max[0, S - X], because with time remaining there is a chance that the price of the underlying asset will rise further. Putting the two together, at any time t before maturity, the value of the call c must be

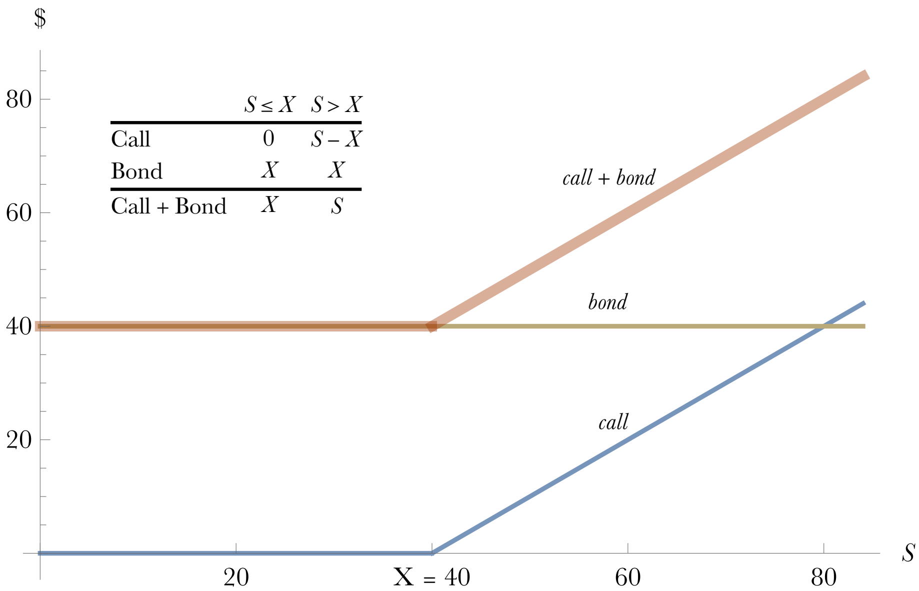
$$\text{Max}[0, S - X\,] \leq c_t \leq S_t$$

This also means that a call's value will be higher the longer the time remaining and the higher the stock price today, the idea being that, if asset price changes follow a random walk, a higher price today is more likely to end with an even higher price than would a lower starting price. Because a call option's payoff is never less than zero, it's value today is greater the riskier the underlying asset, where risk can be measured by the variance of the underlying asset's price changes. Finally, it's pretty obvious that a call's value will be lower the higher the exercise price since that is the cost of exercising but increasing in the rate of interest since an amount equal to the exercise price can earn interest for its owner up to the expiration date.

$$c = f\left(\overset{+}{S}, \overset{-}{X}, \overset{+}{r}_f, \overset{+}{T}, \overset{+}{\sigma}_{\Delta S}\right)$$

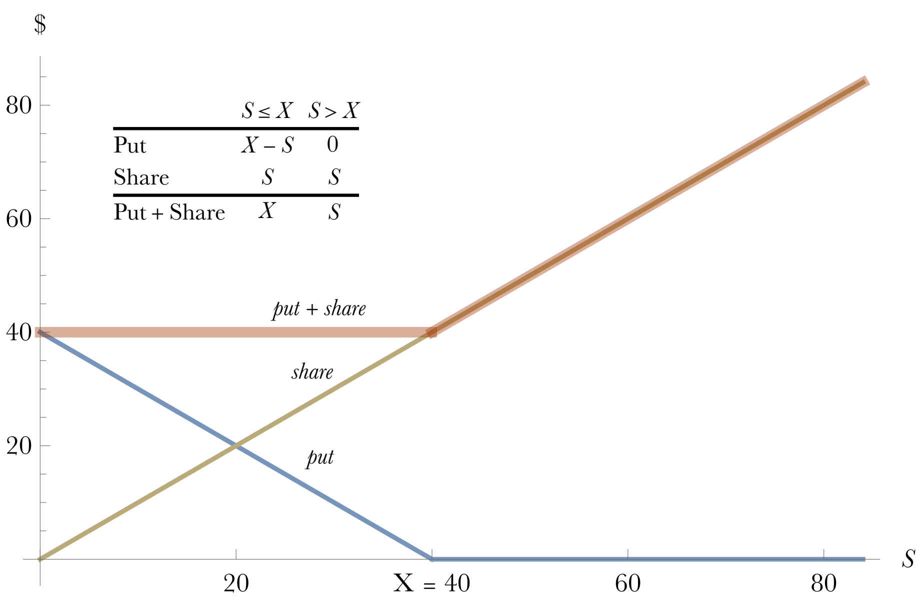
See "The Black-Scholes Option Pricing Model" later in this chapter to see how to compute the call prices.

Put-call parity. Payoff diagrams can be used to establish a relationship between the value of calls and puts on the same underlying asset and having the same exercise prices and maturities. The diagram below shows the payoff at maturity of a call option and a riskless pure discount bond that, conveniently, has a face value equal to the exercise price.



	$S \leq X$	$S > X$
Call	0	$S - X$
Bond	X	X
Call + Bond	X	S

A put and a share gives the same distribution of payoffs.



	$S \leq X$	$S > X$
Put	$X - S$	0
Share	S	S
Put + Share	X	S

If the distribution of payoffs of the two portfolios is the same at maturity, their values must be the same today. This is put-call parity.

$$c + \mathrm{PV}(X) = p + S$$

The Black-Scholes Option Pricing Model. The Black-Scholes equation yields the price of a European call or put on a non-dividend paying stock.

$$c = S\,\mathrm{N}(d_1) - Xe^{-r_f T}\mathrm{N}(d_2)$$

where N(·) is the standard normal cumulative distribution and

$$d_1 = \frac{\ln\left(\frac{S}{X}\right) + \left(r_f + \frac{1}{2}\sigma^2\right)T}{\sqrt{\sigma^2 T}}$$

$$d_2 = d_1 - \sqrt{\sigma^2 T}$$

For example, letting

$S = \$36$: current stock price
$X = \$40$: exercise price of the call
$r_f = 0.10$: annualized risk-free interest rate under continuous compounding
$\sigma^2 = 0.15$: Annual variance of the continuous return on the stock
$T = 0.5$: Time to expiration as a fraction of a year, e.g., 0.37

the value of a call on the stock will be \$3.09. Here is how the prices of the call and a put with the same parameters exercise price vary with the price of the underlying stock. The value of the put is inferred using put-call parity.

S	*c*	*p*	*d1*	*N(d1)*	*d2*	*N(d2)*
\$0	\$0.0000	\$38.0492	-∞	0.00000	-∞	0.00000
\$12	\$0.0000	\$26.0492	-4.07678	0.00002	-4.35064	0.00001
\$24	\$0.1567	\$14.2059	-1.54577	0.06108	-1.81963	0.03441
\$36	\$3.0895	\$5.1387	-0.06522	0.47400	-0.33908	0.36728
\$48	\$11.2345	\$1.2837	0.98525	0.83775	0.71139	0.76158
\$60	\$22.2107	\$0.2598	1.80005	0.96407	1.52619	0.93652
\$72	\$33.9987	\$0.0478	2.46580	0.99317	2.19194	0.98581
\$84	\$45.9594	\$0.0086	3.02868	0.99877	2.75482	0.99706

EXERCISES

1. You are the proud owner a Canadian company that exports all of its production to Poland. Every month you receive Polish zloty (PLN) as your customers pay you. You could say that you have a portfolio that is comprised of just one asset, Polish zloty. It is a spot position in Polish zloty. The return on the portfolio is the return on the PLN.

$$r_p = r_s$$

where Rs is the monthly percentage change in the price of PLN.

You face foreign exchange risk because you have to sell those Polish zloty for Canadian dollars as they come in. You may not like that risk, and one way to hedge it is to sell Polish zloty forward. If you know that you are going to receive one PLN at the end of the month, then today you could arrange to sell it the moment you receive it at a price that is agreed upon today. That price is called a forward price, and selling forward (called writing a contract) means that you have taken a short position in the forward market to offset the variation in your long position in the spot market (don't you love the investment jargon?). The return on your hedged portfolio becomes

$$r_p = r_s - h \cdot r_f$$

where r_f is the monthly percentage gain or loss on your forward position, and h is the fraction of your spot position that you have hedged. h is called the hedge ratio.[20] h is 0 if your portfolio is unhedged. h is 1 if you sell exactly one PLN forward for every one you will receive; this is sometimes called a naive hedge. It is also possible to calculate h to minimize the variance of a portfolio's return (something that you will do here if we haven't already done it in class). This is called a risk-minimizing hedge. A practical question in risk management is whether a risk-minimizing hedge lowers risk more than a naive hedge, particularly when the hedge ratio is a forecast to be applied to future returns.

You can find spot currency prices in “Currency Spot Prices.csv” and one-month forward prices in “Currency Forward Prices.csv.” The first 72 months (2001-02-28 to 2007-01-31) is your in-sample. You’ll use this time period to compute the risk-minimizing hedge ratio. You’ll use the remaining 72 months (2007-02-28 to 2013-01-31) as your out-sample to see how your risk-minimizing hedge performs when it is used as a forecast.

[20] h looks like an investment weight (you get h of r_f) but it really isn’t because you do not invest in a forward contract. You haven’t put up any money to buy or haven’t received any to sell.

a. Fill in the tables. The risk-minimizing hedge ratio is the same in both.

In-Sample Hedging Performance of Zloty Forwards
2001-02-28 to 2007-01-31 (T = 72 months)

	Mean Return	σ	$\%\Delta\sigma$
Unhedged (h = 0)			-
Naive hedge (h = 1)			
Risk-minimizing hedge ($h = h^*$)			

Hold-out Sample Hedging Performance of Zloty Forwards
2007-02-28 to 2013-01-31 (T = 72 months)

	Mean Return	σ	$\%\Delta\sigma$
Unhedged (h = 0)			-
Naive hedge (h = 1)			
Risk-minimizing hedge ($h = h^*$)			

b. Plot the returns of your unhedged and risk-minimizing hedged portfolios for both your in-sample and hold-out sample. Put the dates on the horizontal axis and returns on the vertical.

c. Discuss your results. Make sure to mention the risk-minimizing hedge ratio; how much risk is reduced by using a naive hedge; the extra risk reduction of the risk-minimizing hedge; and whether hedging affected average return. Is the risk-minimizing hedge ratio always less than 1? Must the risk-minimizing hedge always out-perform a naive hedge? Will hold-out sample hedging performance always be better or worse than in-sample performance?

2. Draw the payoff diagram for a long butterfly:

$$1 \text{ Call } (X = \$7) - 2 \text{ Calls } (X = \$14) + 1 \text{ Call } (X = \$21)$$

3. Show that you can create a forward contract with a forward price of \$8 by combining a call option and a put option, both exercisable at \$8.

4. Use the Black-Scholes option pricing model to value European calls and puts with an exercise price of \$27 and which expire in 0.75 years. The variance of the underlying stock is 0.10, and the risk-free rate is 0.03. Tip: Both Excel (Mac and Windows) and Numbers (Mac) have cumulative Normal distribution functions.

   a. Fill in the table.

S	*c*	*p*	*d1*	*N(d1)*	*d2*	*N(d2)*
\$0						
\$9						
\$18						
\$27						
\$36						
\$45						
\$54						

   b. Plot the value of the call along with its upper bound *S* and lower bound.

1. Hedging Polish zloty.

    a. Descriptive statistics.

$h^* = 0.978639$

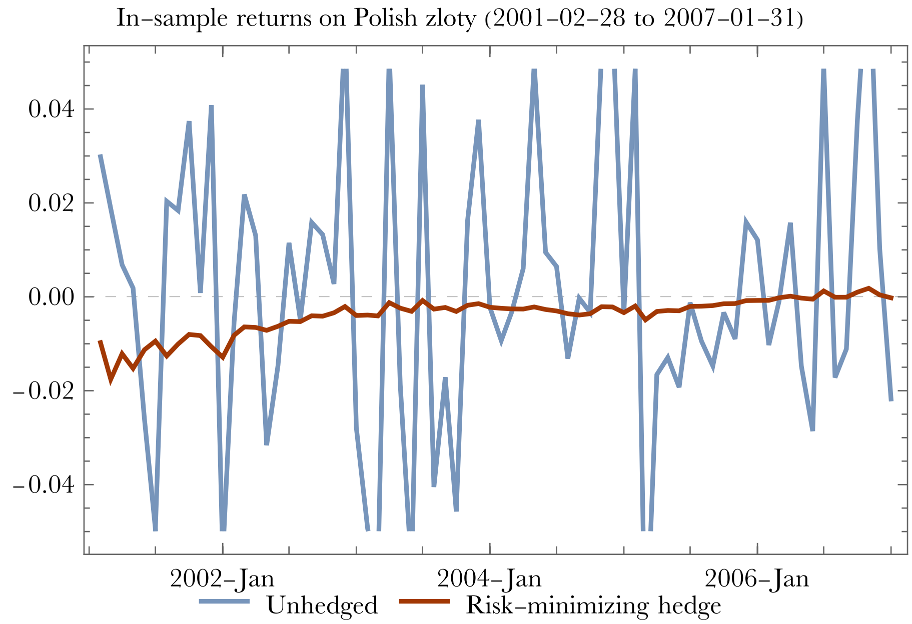
In-Sample Hedging Performance of Zloty Forwards
2001-02-28 to 2007-01-31 (T = 72 months)

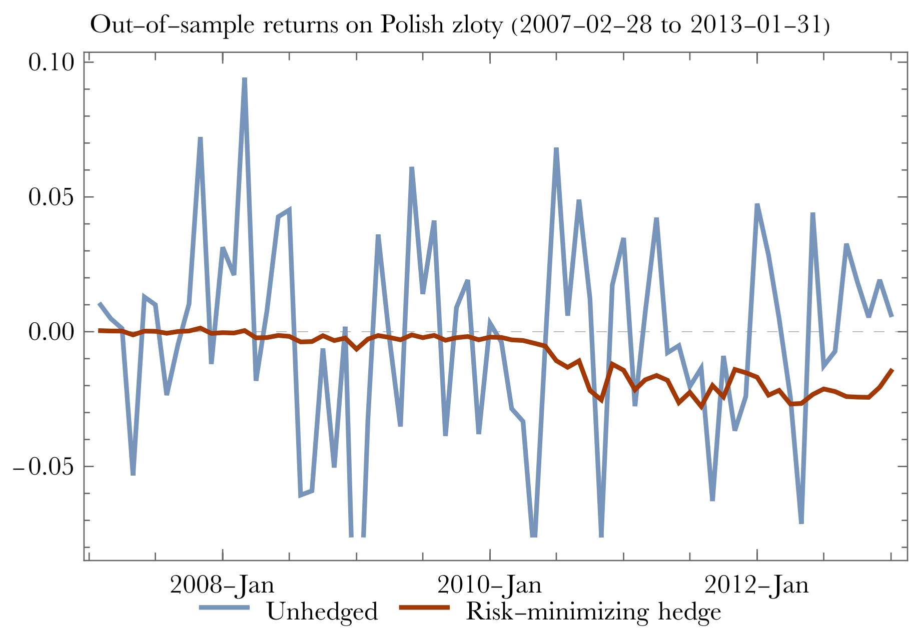
	Mean Return	σ	$\%\Delta\sigma$
Unhedged ($h = 0$)	0.00139	0.03099	-
Naive hedge ($h = 1$)	-0.00411	0.00405	-0.86945
Risk-minimizing hedge (h^*)	-0.00399	0.00399	-0.01385

Hold-out sample Hedging Performance of Zloty Forwards
2007-02-28 to 2013-01-31 (T = 72 months)

	Mean Return	σ	$\%\Delta\sigma$
Unhedged ($h = 0$)	-0.00190	0.03947	-
Naive hedge ($h = 1$)	-0.00982	0.00988	-0.74958
Risk-minimizing hedge ($h = h^*$)	-0.00965	0.00971	-0.01757

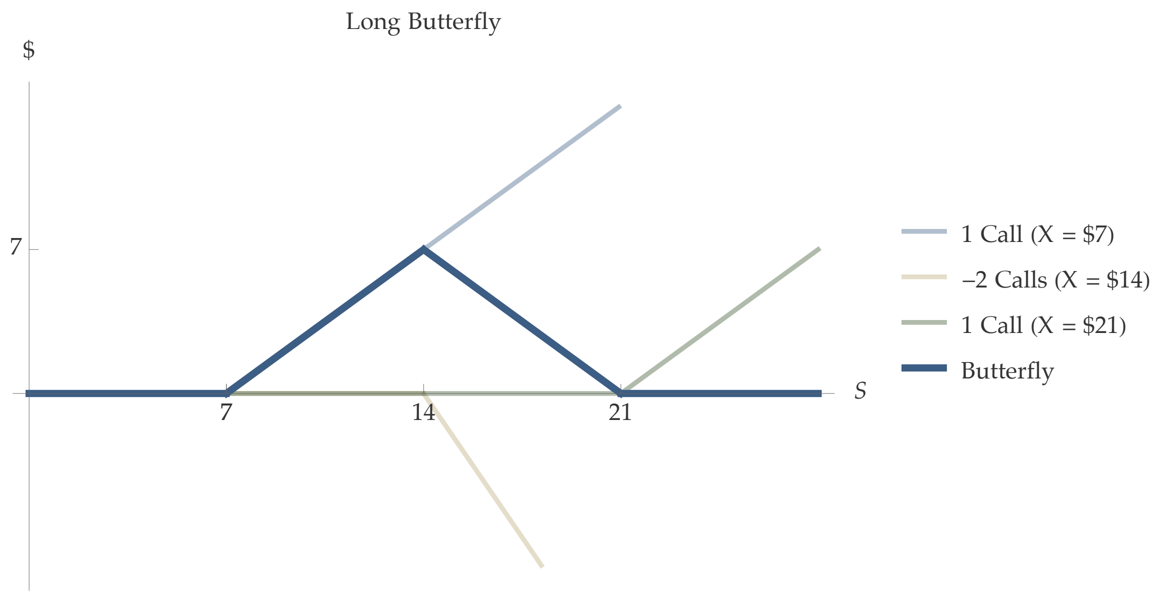
b. Give your plots enough detail so that they tell most of the story on their own.



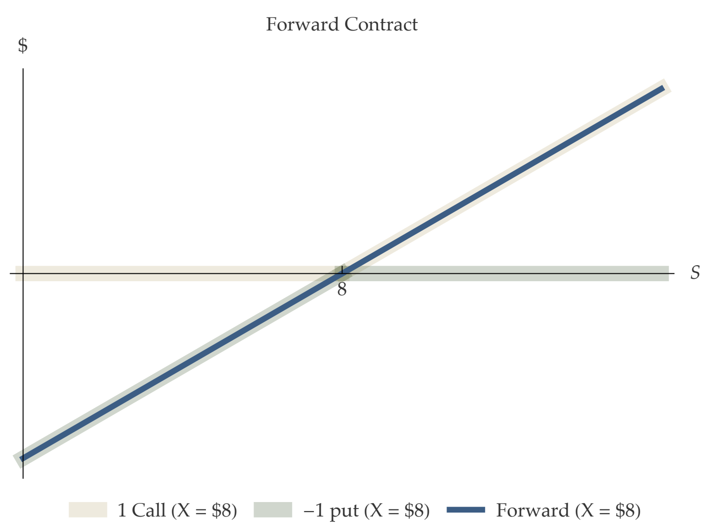


c. Discussion left to you to raise in class, tutorials, yoga class, family gatherings, weddings, and such.

2. Quick, hand me that can of Raid.



3. Don't you hate when someone begins or ends a sentence with "going forward"?

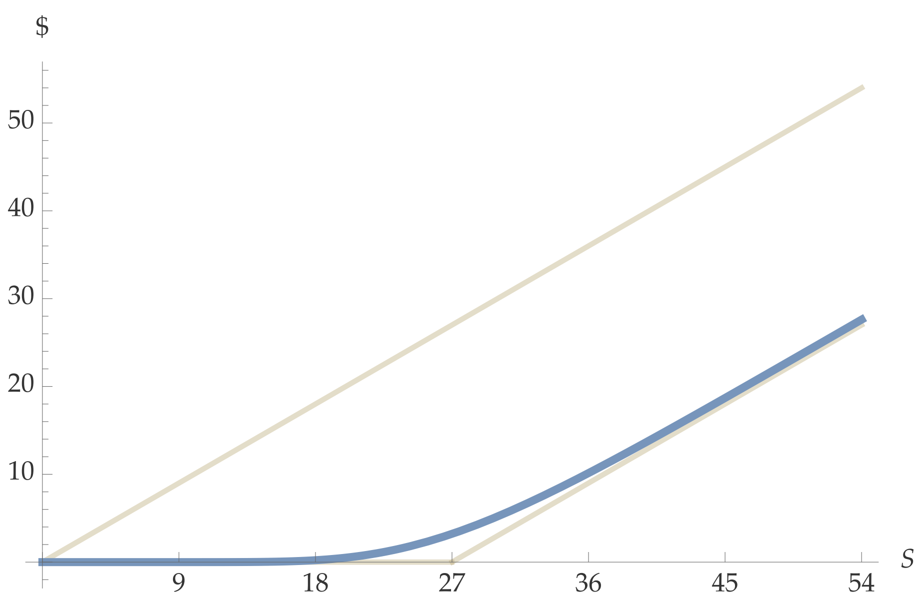


4. Prices of European call and put options for $X = \$27$, $T = 0.75$, $r = .03$, and var(S) = 0.10.

a. The table.

S	c	p	$d1$	$N(d1)$	$d2$	$N(d2)$
\$0	\$0.000000	26.399300	-∞	0.000000	-∞	0.000000
\$9	\$0.000041	17.399300	-3.792480	0.000075	-4.066340	0.000024
\$18	\$0.218228	8.617510	-1.261460	0.103572	-1.535320	0.062352
\$27	\$3.218080	2.617360	0.219089	0.586710	-0.054772	0.478160
\$36	\$10.140300	0.539575	1.269560	0.897878	0.995694	0.840301
\$45	\$18.692200	0.091492	2.084360	0.981436	1.810500	0.964891
\$54	\$27.615100	0.014400	2.750100	0.997021	2.476240	0.993361

b. The plot.



6
ARBITRAGE

The Law of One Price: perfect substitutes sell for the same price.

Examples of Arbitrage in your Own Backyard

Big Coke, Little Coke

So, you want to deal in Coke? On March 18, 2021 the price of a two-litre bottle of Coca Cola at Métro was $3 and the price of a one-litre bottle was $1.50. You're not surprised. The big bottle has to sell for twice the price of the bottle half its size because all anyone cares about is the sugar water inside.[21] Perfect substitutes must sell for the same price or, as in the case of Big Coke and Little Coke, a constant price multiple equal to the measure of some substitutable part, such as volume of liquid. If they didn't, everyone would buy the relatively cheaper one, and their prices would adjust until the ratio was two-to-one again. This is the *Law of One Price*. The force of excess demand and excess supply are extreme for perfect substitutes, turning the equilibrium price ratio of 2:1 into a super magnet. At $2.97 Big Coke would fly off the shelves while bottles of Little Coke would collect dust.

There's more behind the perfect substitutes super magnet. Suppose that late in the afternoon on March 18 the price of Little Coke rose to $1.75 but Big Coke was still selling for $3. How could you make a quick $1,000? You'd borrow 4,000 bottles of Little Coke from my friend Agnes and sell them right away for $7,000.[22] You'd then buy 2,000 bottles of Big Coke for $6,000 and give them to Agnes in repayment of the loan. You're $1,000 smackers ahead without having to put up any money of your own or take on any risk. Profit with no investment and no risk: that's arbitrage. Who wouldn't jump on that? Sounds too good to be true.

Arbitrage is part of the force of excess demand and excess supply, except that arbitrageurs, unlike consumers who buy the cheaper good and pass over the other, are both buyers and sellers at the same time: they buy the under-priced good with the money raised by short-selling the over-priced substitute. Within the limits of the cost of producing the two goods, there is no right price for either one in an absolute sense. It's all relative. A price ratio that differs from 2:1 for Big Coke

[21] Spotting perfect substitutes isn't easy. That only the soft drink itself matters to consumers is an assumption I'm making for the purpose of this example. It doesn't stand up as a fact: the deal for the one-litre bottles was buy-two-for-$3. A single one-litre bottle was $1.69. On the same day, the price for half a litre was $2.49. What gives? The size of the container matters too because of the different ways people consume Coke. Someone who knows that they won't finish two litres in a week (it will go flat in the fridge) may be willing to pay a premium for one litre. Someone who wants to knock back a single serving (gag) may be willing to pay an even bigger premium for 500 ml. My example assumes that people have enough room in their fridges and a big enough thirst that they're fine buying at least one two-litre bottle.

[22] Agnes has a warehouse. She moves all kinds of goods. Doesn't advertise, though, strictly word-of-mouth.

and Little Coke means that one or both prices is “wrong”. How would you arbitrage Big Coke and Little Coke if the price ratio was 9:4?

How fast prices are driven back to equilibrium is an indication of the efficiency of a market. In an active market in which information is widely available and both goods plentiful, you would expect arbitrage opportunities to be few, fleeting, and the profits puny because greedy arbitrageurs are always on standby. In a perfectly *efficient market,* a world concocted by economists, arbitrage opportunities never exist. The *absence* of arbitrage profits is what tells us that the prices of perfect substitutes are in equilibrium and is known as *the no arbitrage condition*.[23]

How About a Burger with that Coke?

The US dollar hovered around C$1.28 in December 2020. A Big Mac in Canada cost C$6.77 and US$5.66 in the United States. Money tastes better than Big Macs, so this is what I did. I borrowed 1,000 Big Macs in Montreal from my friend Eugene who owns a franchise. I put them into a big thermal knapsack and hitchhiked down Highway 15 to the Saint-Bernard-de-Lacolle - Champlain US border crossing, entered the United States, and hitch another ride, this time to Plattsburg, where I sold the whole sack of burgers to hungry SUNY students for US$5,660.[24] I immediately hitched a ride back to Montreal, exchanged my US dollars for C$7,245, paid back Eugene the $6,770 I owed him and deposited my $475 arbitrage profit in the bank. Not bad for half a day’s work, considering that at the time I was suspended without pay from Concordia University for threatening a student who handed in a poorly written report.

Exchange rates should be determined by the value of goods that can be bought with the local currency, a relationship called *purchasing power parity*.

$$\text{exchange rate} = \frac{\text{price of basket of goods in Canada}}{\text{price of basket of goods in United States}}$$

A better way to say this is, exchange rates should adjust to equalize the prices of goods across countries

$$\text{price of basket of goods in Canada} = \text{exchange rate} \times \text{price of basket of goods in United States}$$

A Big Mac is far from representing a basket of goods, but an exchange rate of $1.28 implies that a Big Mac should cost 28 per cent more in Canada than in the United States. In December 2020, a the burger cost only 20 per cent more in Canada

[23] Wait a minute. If prices are always in equilibrium because of the very threat of arbitrage, how are arbitrageurs able to make a living? Do they quit and become baristas? And then, if no one is looking for arbitrage opportunities because they don’t expect any, what keeps keeps the prices in equilibrium in the first place? Something is missing here.

[24] None of the friendly people who gave me a ride appreciated the smell.

$$\frac{\$6.77}{\$5.66} \approx 1.196113$$

implying that the Canadian dollar was undervalued by about 7 per cent

$$\frac{1.196113}{1.28} \approx 0.934$$

Being undervalued (or overvalued if that were the case) suggests that the Canadian dollar will adjust over the long run to parity, whatever long run means. Easy arbitrage, where the arbitrageur enjoys burger aroma therapy, a day outdoors, and gets to make new friends on the side of the road, is presumably part of the push towards parity. But everyone who has travelled abroad has compared prices and knows that purchasing power parity does not hold. Many goods and services are not traded across international borders (Big Macs and haircuts), and factors of production, particularly labour, are not mobile either; that is why Big Macs are cheaper in poorer countries where wages are low.

Interest Rate Parity

Suppose that the interest rate is three per cent in Canada and five per cent in France. What do you make of the difference? Both rates are for one year and are risk-free. They are rates that you would earn on safe government bonds.

Invest \$1 million for a year here at home and your return is \$1,030,000. If instead you invest the \$1 million in France, you'll first have to buy Euros because French government bonds are priced in Euros. Say that the spot exchange rate for the Euro is \$1.3021. One million Canadian dollars will buy 767,990 Euros. Those Euros, invested at the five per cent French risk-free rate, will return €806,390 at the end of the year.

One million dollars is guaranteed to grow to C\$1,030,000 or €806,390. What would you do? Invest in Canada or France? It's not clear because if you invest in France, you will have to sell the Euros for Canadian dollars at the end of the year if you want to buy smoked meat in Montreal. The decision would be easy if you knew today what the value of the Euro was going to be a year from now. But you do not, and that uncertainty about the exchange rate makes the French option risky even though the return on French government bonds is risk-free. If the Euro doesn't budge, you'll bring home \$1,050,000 (obvious, right?). If it falls by two cents, \$1,033,872, which is \$3,872 more than you'd earn in Canada. But if it falls by three cents, you'll be left with \$1,025,808, and it would have been better to have kept your money in Canada. Most people don't like risk; they need to be compensated to bear it. And if you're like most people, you might not be willing to invest in France even if you thought there was little chance of the Euro depreciating by three cents.

Ignore risk for now—we can do that in a theory course—and pretend that all you care about is earning as much as possible (what do we call that kind of attitude towards risk?). What would the spot price of the Euro have to be one year from now for you to break even? It would have to be \$1.2773 because selling 806,389.68 Euros for \$1.2773 gives \$1,030,000.

$$\$1{,}030{,}000 = \$1.2771 \times €806{,}389.68$$

It probably doesn't surprise you that $1.2773 is about two per cent less than the current Euro spot price of $1.3021. The interest rate in France is two percentage points higher than in Canada, so the Euro would have to depreciate by about two per cent to cancel France's interest rate advantage.

The total return on $1 million invested in Canada at r_d = three per cent (*d* for domestic) is

$$\$1{,}000{,}000(1 + r_d) = \$1{,}000{,}000(1.03) = \$1{,}030{,}000$$

Or you can buy C$1 million worth of Euros

$$\frac{\$1{,}000{,}000}{p_0} = \frac{\$1{,}000{,}000}{\$1.3021} = €767{,}990.17$$

where p_0 is the exchange rate or Canadian dollar spot price of the Euro, and invest them in France at r_f = five per cent (*f* for foreign) for a total return of

$$€767{,}990.17(1 + r_f) = €767{,}990.17(1.05) = €806{,}389.68$$

Then sell the Euros in a year at an as yet unknown spot price, p_1,

$$€806{,}389.68\, p_1 = €767{,}990.17(1 + r_f)p_1$$

To calculate what the spot price a year from now would have to be to break even, equate the return in Canada with the return in France

$$\$1{,}000{,}000(1 + r_d) = \frac{\$1{,}000{,}000}{p_0}(1 + r_f)\,p_1$$
$$\therefore p_1 = p_0\frac{1 + r_d}{1 + r_f} = \$1.3021 \times \frac{1.03}{1.05} = \$1.2773$$

The calculation is done using three known current prices: two interest rates and the spot price of the Euro. If all three prices are equilibrium prices in the supply-equals-demand sense, then every investor must have already decided to place their money here or there. Equilibrium means that everyone is content keeping their money wherever they have decided to keep it, and because of that, it must also mean that the market expects to break-even, $E(p_1) = \$1.2773$. If this weren't true, people would still be moving their money around to exploit some perceived advantage and the three prices would still be in flux; they would not be equilibrium prices. The equilibrium relationship between the interest rates in two countries and the exchange rate is known as *uncovered interest rate parity*.

$$(1 + r_d) = \frac{1}{p_0}(1 + r_f)\,E(p_1)$$

The left-hand side is the sure total return on one unit of the home country currency invested at home, and the right-hand side is the expected total return on one unit of the home country currency invested abroad.

The interpretation is neat. If the interest rate in France is higher than the interest rate in Canada, the market must be expecting the Euro to depreciate ($1.2773 is less than $1.3021). If France's interest rate is lower than Canada's, a Euro appreciation is expected. The difference between the two interest rates is a one-year market forecast of the exchange rate! To see the relationship in percentage terms (the expected return on the currency), put the two interest rates on one side of the equal sign and the spot prices on the other, and then subtract one from both sides

$$\frac{1+r_d}{1+r_f} - 1 = \frac{E(p_1)}{p_0} - 1 \equiv E(r_{cur})$$

The difference in interest rates

$$\frac{1+r_d}{1+r_f} - 1 = \frac{1.03}{1.05} - 1 = -0.0190476 \approx r_d - r_f = -0.02$$

predicts the expected percentage change in the exchange rate, $\frac{E(p_1)}{p_0} - 1$, a depreciation of 1.9 per cent. The simple difference, $r_d - r_f = -2$ per cent will give good approximation when the two interest rates are not too far apart. You may earn two per cent more on French government bonds but on average you will lose that when you convert Euros back to Canadian dollars.

To say that the market expects the Euro to depreciate to $1.2773 does not mean that each of us expects that. Greed keeps the parity in uncovered interest rate parity. You are part of the reason that the market expects the Euro to depreciate to $1.2773. Say that you believe that the Euro will not fall by as much. Your forecast is $1.28. Being the gutsy individualist that you are, you speculate by borrowing $1 million in Canada at three per cent and use it to buy French government bonds paying five. Your expected profit is $2,178.79. Let $\hat{p}_1 = \$1.28$ be your forecast for the Euro, which replaces the expected price implied by the market

$$\begin{aligned}
\$1{,}000{,}000(1+r_d) &< \frac{\$1}{p_0}(1+r_f)\,\hat{p}_1 \\
\$1{,}000{,}000(1.03) &< \frac{\$1}{p_0}(1.05)\,\frac{\$1.28}{\$1.3021} \\
\$1{,}030{,}000 &< \$1{,}032{,}178.7881
\end{aligned}$$

The left-hand side is what you owe on your loan and the right is the expected return on your investment. The difference is your expected profit. Hope things work out for you. If the Euro is above $1.2773 in a year, you'll be laughing; if it is below, you'll be crying. There may be other speculators who predict that the Euro will be even higher than $1.28. They sit above you on the demand curve for Euros. Still others believe that it will fall below $1.2773, and they speculate by borrowing Euros at five per cent to invest in Canada at three. They are on the lower part of the supply curve for Euros. Together, all of you speculators are at least partly responsible for equilibrium interest rates levels and the current equilibrium exchange rate.

Where is arbitrage in this story? If there is a forward market for currencies, the equilibrium forward price must be equal to the expected future spot price implied by uncovered interest rate parity[25]

$$f_0 = E(p_1)$$

making *covered interest rate parity* is possible. If the forward price of the Euro is not equal to the expected future spot price a riskless profit is possible—just like printing money.

$$(1 + r_d) = \frac{1}{p_0}(1 + r_f) f_0$$

Suppose that the forward price is temporarily out of whack, say, f_0 = \$1.25. A forward price of \$1.25 means that you can contract now to buy or sell Euros in one year at \$1.25 no matter what the spot price ends up being. You would borrow 767,990 Euros at five per cent to invest in Canada at three per cent and at the same time buy 806,390 Euros forward at \$1.25 because that is how many Euros you will need to pay off your loan. That leaves you a guaranteed profit of \$22,012.90. Follow the inequality.

$$\$1{,}000{,}000(1 + r_d) > \frac{\$1}{p_0}(1 + r_f) f_0$$

$$\$1{,}000{,}000(1.03) > \frac{\$1{,}000{,}000}{\$1.3021} \times 1.05 \times \$1.25$$

$$\$1{,}000{,}000(1.03) > €767{,}990 \times 1.05 \times \$1.25$$

$$\$1{,}030{,}000 > \$1{,}007{,}987.0978$$

Earn \$30,000 with a loan that cost about \$8,000. The two investments, Canadian government bonds or French government bonds, should be perfect substitutes for all investors when a forward currency market exists—not just for hypothetical risk neutral investors—because they yield the same distribution of payoffs, in this case, a risk-free payoff. If the forward exchange rate is \$1.25, they do not, and an attractive profit opportunity exists. Of course, because everyone knows that the forward rate should be \$1.2773 just by looking at the interest rates and the spot price, it should not be surprising that these sorts of opportunities are rare. The absence of arbitrage profit tells us that markets are in equilibrium.

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[25] As with the interest rates and spot exchange rate I am not using a * to indicate equilibrium values. It should be clear from the context.

The Term Structure of Interest

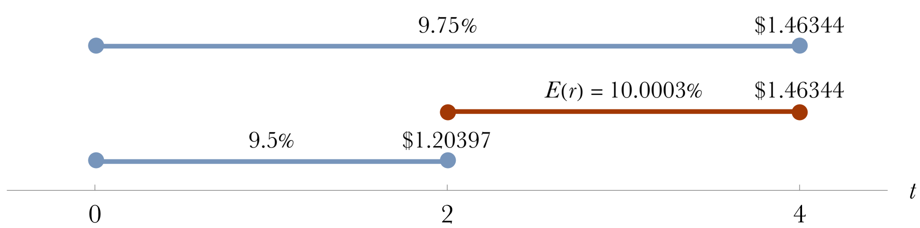
Suppose the risk-free, four-year spot rate of interest is 9.75 per cent. It is the rate that you might have earned many years ago on an insured deposit, such as a guaranteed investment certificate or a treasury bond, if you were willing to tie up your money for the full four years. One dollar will grow to \$1.46344 in four years if the interest is compounded semi-annually.

$$\$1\left(1+\frac{0.0975}{2}\right)^{2\times 4} = \$1.46344$$

What if the risk-free, two-year spot rate of interest was 9.5 per cent? You could invest or lend one dollar for two years and earn \$1.20397.

$$\$1\left(1+\frac{0.095}{2}\right)^{2\times 2} = \$1.20397$$

Which would you choose: \$1.46344 in four years or \$1.20397 in two? The two investments can't be compared because they don't pay off at the same time. You might say, "It depends on what I think I could earn in years 3 and 4 if I were to invest for two years and then reinvest my principal and interest for another two years." But you can't know now what the two-year spot rate of interest will be in two year's time. The two-year investment rolled over for another



two is risky, while the four-year investment is not. We'll have to fudge this by assuming that you and all other investors are risk neutral, which implies that all anyone cares about is earning as much money as possible at the end of four years, regardless of the risk. Now the investments can be compared.

The two investments are equivalent if we expect the future two-year spot rate for years 3 and 4, E(r), to be just a hair over 10 per cent.

$$\left(1+\frac{0.095}{2}\right)^{2\times 2}\left(1+\frac{E(r)}{2}\right)^{2\times 2} = \left(1+\frac{0.0975}{2}\right)^{2\times 4} = \$1.46344$$

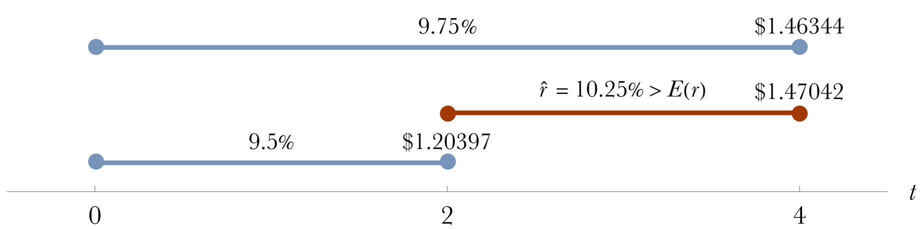
$$\Longrightarrow \; E(r) = 0.100003$$

We know three things: the short-term spot interest rate is 9.5 per cent; the long-term spot rate is 9.75 per cent; and people act on their expectations. If the two interest rates are equilibrium interest rates, in the supply-equals-demand sense as always, then money is not being shifted from short-term to long-term or vice versa. Those who expect the future two-year spot rate to be higher than

10 per cent have invested short term; those who expect it to be lower than 10 per cent have invested or lent long term. And those dollars that lie at the two intersection of the supply and demand schedules represent investors who are indifferent between investing short term or long term. They must expect the short-term investment rolled over to yield the same return as the long-term investment. That's why the break-even rate, the equilibrium expected future spot interest rate (the word *future* is redundant I think), is 10.0003 per cent.

This is neat because the two spot interest rates, which exist and with which we can transact, tell us what the market expects will be the short-term spot rate in two year's time, in this case, 10.0003 per cent. You can now say why the long-term interest rate is higher than the short-term rate: the market expects the short-term rate to rise in the future. If the four-year rate was lower than the two-year rate, the market must be expecting the short-term rate to fall. This is known as the *Expectations Theorem of the Term Structure*. The term structure or yield curve refers to the shape of the graph of interest rates on y-axis against their terms, two years or four on the x-axis. In our example, the yield curve is upward sloping, which is usual, but there is nothing unusual about a flat or even a downward-sloping yield curve since the difference between the short and long rates is determined by expectations.

If you thought that the two-year spot rate in years 3 and 4 would climb to 10.25 per cent from its current 9.5 per cent, then you would take the chance by investing short-term and rolling over the investment for another term because the expected total return is 1.47042, and that's



more than the $1.46344 you get by investing long term.

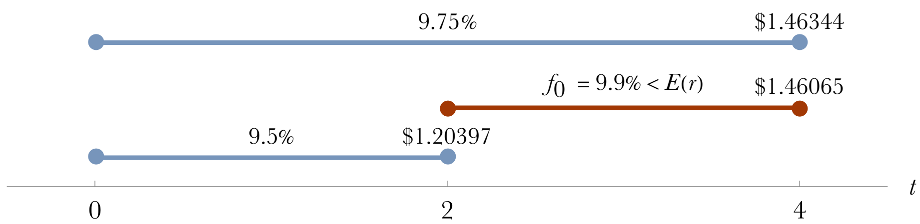
$$\$1\left(1+\frac{0.095}{2}\right)^{2\times 2}\left(1+\frac{0.1025}{2}\right)^{2\times 2}=\$1.47042$$

What if you don't have a dollar to lend (darn those video lottery terminals)? No problem, borrow it at the four-year rate. At the end of four years you will owe $1.46344 which you are expecting to be able to repay from a return of $1.47042 for an expected profit of $0.00697287 or $6.97 per $1,000. If the future two-year interest rate ends up even higher, your smile will be bigger; but you'll be in trouble if it doesn't climb high enough. What would your speculative strategy be if you expected the future two-year spot rate to be 9.8 per cent?

If a forward market for borrowing and lending exists, then the forward rate of interest must be equal to the expected future spot rate implied by the expectations theorem, 0.100003,

$$f_0 = E(r)$$

because if it wasn't, you and everyone would arbitrage the term structure back into equilibrium. To see this, suppose that, by some blip, the actual two-year forward interest rate is 9.9 per cent instead of its equilibrium value that I claim should be 10.0003 per cent. You would borrow one dollar for two years at 9.5 per cent and invest it for four years at 9.75 per cent. You already know that at the end of four years you will receive \$1.46344. But how will you repay the loan after just two years? That's \$1.20397 you need to come up with. The answer is to borrow forward. This



means that today you arrange a two-year loan of \$1.20397 to start in year 3, at the forward rate of 9.9 per cent. What you have done is rolled over or refinanced your original loan at a rate that is fixed in advance in your favour. At the end of year 4, you will owe \$1.46065 for sure.

$$\$1\left(1+\frac{0.095}{2}\right)^{2\times 2}\left(1+\frac{0.099}{2}\right)^{2\times 2} = \$1.46065$$

Earn \$1.4634 and repay \$1.46065 for an arbitrage profit of \$0.00279382 per dollar played or \$2,793.82 per \$1 million. Too good to be true? Pretty much, or at least too fleeting to be captured, because if you can do it, so can everyone else. The efficient markets hypothesis tells us that arbitrage profits should not exist! That is why the forward price is always equal the implied expected future spot price: $f_0 = E(r)$.

Capital Structure Irrelevance

Consider the distribution of company earnings shown in the table.

	Recession	*Normal*	*Boom*
π	0.125	0.500	0.375
Operating income	\$180	\$380	\$500

For the earnings to have a market value, a claim or claims on it must be traded in a market

$$V = PV(\text{expected operating income}) = PV(\$400)$$

If the probability distribution of the earnings is expected to stay the same forever, its present value is

$$V = \frac{E(\text{expected operating income})}{r_A}$$

where r_A is the required return assets because I figure there are real assets generating the possible earnings.[26] (Time for a digression: Isn't it enough to assume that the expected payoff, rather than its probability distribution, is constant over time for it to be valued as a perpetuity? The answer is yes but it is worth reminding ourselves that r_A must then also account for expected changes, if any, in the probability distribution. The required return compensates investors for the time value of money and risk as compared to the probability distributions of earnings coming from all other real assets from now to eternity, and which may be expected to change over time. Yep, there's a lot packed into r_A.)

Suppose that the market value is \$3,200. Then the required return must be 12.5 per cent.

$$r_A = \frac{E(\text{expected operating income})}{V} = \frac{\$400}{\$3{,}200} = 0.125$$

Most textbook treatments would take the required return of 12.5 per cent as given and then compute compute value, but I prefer to infer the required return from value because we can see value—market prices—but we cannot observe required return. Either way, though, expected payoffs in the numerator are not observable. Sigh, so much we don't know.

There's nothing in the calculation that you didn't already know, except that maybe it occurred to you that the value of the assets has nothing to do with the types of claims on them—who owns them or how they are financed. Let's prove it.

Suppose the real assets belong to an all-equity company U that has 100 shares outstanding. Payoff is now operating income. Every year all of the earnings are paid as dividends.

Company U	*Recession*	*Normal*	*Boom*
π	0.125	0.500	0.375
Operating income	\$180	\$380	\$500
Earnings per share	\$1.80	\$3.80	\$5.00

The share price is \$32, and the required return on equity is equal to the required return on the assets because the company is un-levered.

$$r_{E_U} = r_{A_U} = \frac{E(EPS_U)}{P_U} = \frac{\$4}{\$32} = 0.125$$

Another company, L, has the same assets and earnings distribution as U, and for now is also un-levered with 100 shares outstanding. The bosses at L think that they can increase the company's value by borrowing at the risk-free interest rate of 10 per cent to buy back some of its shares.[27] Maybe the reasoning is that paying out a portion of its earnings as interest at 10 per cent is less costly than paying out a portion at a rate of 12.5 per cent, which is also called the *cost of*

[26] Required return is an expected return. The expectations operator E is customarily omitted when a return is used as a discount rate: r_A rather than $E(r_A)$. We're assuming that the first payoff is a year from now.

[27] How is it that L can borrow at the risk-free rate if its earnings are risky? It can as long as it does not borrow more than \$1,800. Can you say why?

equity, and that the "savings" must go to the remaining outstanding shares. If that were true, company L would be worth more than company U. But that can't be because L's assets and earnings are the same as U.

Say that L borrows $1,600 to buy back half of its shares, and will pay $160 in interest every year in perpetuity. L's cashflow will look like this.[28]

Company L	*Recession*	*Normal*	*Boom*
π	0.125	0.500	0.375
Operating income	$180	$380	$500
Interest	$160	$160	$160
Net earnings	$20	$220	$340
Earnings per share	$0.40	$4.40	$6.80

You can prove that $V_L = V_U$ by showing that $V_L > V_U$ is impossible. This is called *proof by contradiction*. It's nifty. Assume that company L's value is $3,600 after having borrowed $1,600 and retiring 50 shares. The price of L's shares is now $40 on expected earnings per share of $4.80. This implies that the return on equity is 12 per cent even though the return on the identical assets of U is still 12.5 per cent.

$$P_L = \frac{\$3{,}600 - \$1{,}600}{50} = \$40$$

$$r_{E_L} = \frac{E(EPS_L)}{P_L} = \frac{\$4.80}{\$40} = 0.12$$

But you wouldn't pay $40 for a share of L when you can get exactly the same payoff for only $32 by buying two shares of U and borrowing $32 at 10 per cent to pay for one of them? And neither

Two shares of U + borrowing	*Recession*	*Normal*	*Boom*
π	0.125	0.500	0.375
Operating income	$180	$380	$500
EPS_U x 2	$3.60	$7.60	$10.00
Interest on loan to buy one share	$3.20	$3.20	$3.20
Payoff	$0.40	$4.40	$6.80

would anyone else. The value of company L must be equal to the value of company U, and the price of a share of L must equal $32, the price of a share of U. If L's price were $40, arbitrageurs would short sell it and use the proceeds to buy one share of U for $32, pocketing $8, and borrow $32 at 10 per cent to buy a second share to replicate the EPS of L. It is a perpetual, self-financing

[28] Check my work. I failed accounting class seven times. No, I think it was eight times.

short position that returns an immediate riskless profit. With all the clever arbitrageurs getting in on this action, the price of L's shares will be driven down to $32 in the blink of an eye.

In your corporate finance course you learned that, in the absence of tax deductibility of interest on debt, the value of a firm is independent of its capital structure. This is Miller and Modigliani's Proposition I.

$$V_L = V_U$$

If capital structure is irrelevant, the return on equity must depend on the level of debt. At its true price of $32, the required return on L's equity is 15 per cent not 12.

$$r_{E_L} = \frac{E(EPS_L)}{P_L} = \frac{\$4.80}{\$32} = 0.15$$

Taking on debt subjects shareholders to financial risk (the burden of having to pay $160 a year in interest no matter what the company's earnings), so the required return on equity increases just enough to cancel the increase in earnings per share that leverage affords. You can see this by thinking of the return on a company's assets as the return on a portfolio of equity and debt, except that r_A is fixed (12.5 per cent in our case), and it is r_E that responds to changes in financial risk when the portfolio weights change in order to keep r_A constant.

$$r_A = w_E \times r_E + w_D \times r_D$$

where the weights are the market values of equity and debt to company value, $\frac{E}{V}$ and $\frac{D}{V}$. Rearranging gives Miller and Modigliani's Proposition II, which says that the required return on equity is proportional to a company's debt-equity ratio.

$$r_E = r_A + \frac{D}{E}\left(r_A - r_D\right)$$

The Arbitrage Principle in an Time-State Model

The discussion here follows Hal Varian's article "The Arbitrage Principle in Financial Economics," which appeared in the fall 1987 issue of the *Journal of Economic Perspectives*. Using the time-state preference approach is a nice way of illustrating the no arbitrage condition because TSP is all about patterns or distributions of payoffs: two patterns of payoffs that are the same are perfect substitutes and therefore must have the same value.

Asset Pricing

Consider a time-state economy like the one in chapter 4. There are two periods, $t = \{0, 1\}$, with 0 denoting the present and 1 the future. But instead of two states there are S possible states of nature at time 1, $s = \{1, 2, \ldots, S\}$, and K financial assets, $k = \{1, 2, \ldots, K\}$. Each asset k pays x_{sk} dollars in state s. The S x K payoff matrix is

$$\mathbf{X} = \begin{pmatrix} x_{11} & \cdots & x_{1K} \\ \vdots & \ddots & \vdots \\ x_{S1} & \cdots & x_{SK} \end{pmatrix}$$

A row in **X** is the vector of payoffs for all the assets in state s, and a column is the vector of payoffs for asset k for the states. The assets can be combined into portfolios by taking a position of q_k units in each asset k. q_k can be positive (a long position), negative (a short position), or zero (no position). **C** is the vector of possible dollar payoffs, say, for a portfolio, for the states,

$$\mathbf{C} = \mathbf{X} \cdot \mathbf{q}$$

$$\begin{pmatrix} C_1 \\ \vdots \\ C_S \end{pmatrix} = \begin{pmatrix} x_{11} \cdots & x_{1K} \\ \vdots \; \ddots & \vdots \\ x_{S1} \cdots & x_{SK} \end{pmatrix} \begin{pmatrix} q_1 \\ \vdots \\ q_K \end{pmatrix}$$

In chapter 4 we referred to **C** a number of equivalent ways: a distribution of future consumption, income, or payoffs.[29] You can add to that list a distribution of future wealth because it is a final payoff; **C** occurs in the last period, the second period of a two-period world. No matter what you call it, what is important is that **C** is all that investors care about, and what they are choosing is the pattern or distribution of payoffs across states for a date in the future.

The No Arbitrage Condition

Two portfolios that offer the same pattern of payoffs must have the same value. Otherwise, investors would short sell the higher-valued one and use the proceeds to buy the lower-valued one, earning a profit equal to the difference in the values. An arbitrage portfolio (a short position in one plus a long position in the other) that pays zero in every state but has a positive value now (an arbitrage profit or "free lunch") or pays a positive amount in some states and zero in all others but zero value (cost or investment) now should not be possible. This is the *no-arbitrage condition.* It can be stated in the mathematics of the time-state model as

$$\text{If } \mathbf{X} \cdot \mathbf{q} \geq \mathbf{0} \text{ then } \mathbf{P}^\top \cdot \mathbf{q} \geq 0$$

Not as intuitive as it is in words, is it? The expression on the left, $\mathbf{X} \cdot \mathbf{q} \geq \mathbf{0}$, is the S-vector of payoffs, each of which is zero or positive.[30] The expression on the right, $\mathbf{P}^\top \cdot \mathbf{q} \geq 0$, is the cost, value or price of that pattern of payoffs, which can also be written as $\sum_k P_k q_k$, where P_k is the price of

[29] *Distribution* when it is understood that we are including the corresponding probabilities. If only the C's, then *pattern* or *outcomes for* is better.

[30] The boldface zero, **0**, is a vector of zeros.

asset k in the portfolio. The no-arbitrage condition says that any portfolio that does not pay less than zero in any state (the expression on the left) has to be worth at least zero (the expression on the right). If a portfolio pays zero in every state, its value must be zero.

It can be shown that the existence of a complete set of Arrow-Debreu prices (pure state prices) is a necessary and sufficient condition for the no-arbitrage condition to hold. A complete set of Arrow-Debreu prices means that the payoff of one dollar in any state has a market which, in turn, means that the pattern of payoffs of any asset or portfolio can be replicated using Arrow-Debreu securities. The prices of assets must therefore be the sum of the values of their state payoffs, $P_K = \sum_s p_s X_{sk}$, where the lower case p is a Arrow-Debreu price. If state prices exists, then there cannot be arbitrage opportunities; if there are no arbitrage opportunities, then state prices must exist.

Put-Call Parity as an Example of the No Arbitrage Condition

This example needs no explanation. What are the states? What are the two portfolios? Why are they perfect substitutes?

	$S \leq X$	$S > X$		$S \leq X$	$S > X$
Call	0	S - X	Put	X - S	0
Bond	X	X	Share	S	S
Call + Bond	X	S	Put + Share	X	S

Arbitrage Pricing Theory

In discussing the CAPM in chapter 4, we considered a statistical interpretation of a stock's beta as the slope coefficient of the regression of the stock's return on the market's. This required modelling the random return on the stock as a linear function of the random return on the market plus random error that turns out to be idiosyncratic risk.

$$r_j = \alpha_j + b_j r_m + \epsilon_j$$

This setup implies that the return on every stock depends on one common risk factor, the market. But the return on the market is not really a factor because it is the return on a portfolio of all stocks. The market in a sense stands in for all the primary risk factors, such as surprises in inflation. What if the returns of risky assets were modelled directly on the primary risk factors? The return generating process with K factors might look like this

$$r_i = a_i + b_{i1} f_1 + b_{i2} f_2 + \ldots + b_{iK} f_K + \epsilon_i$$
$$r_i = a_i + \sum_k^K b_{ik} f_k + \epsilon_i$$

for $i = 1, \ldots, N$ stocks whose residuals are independent of one another and $k = 1, \ldots, K$ independent factors.

Some restrictions on the process:

$E(f_k) = 0$	All factors have zero mean
$E(\epsilon_i) = 0$	All residuals have zero mean
$E(\epsilon_i \epsilon_j) = 0$	The residuals are uncorrelated
$E(f_k f_m) = 0$	The factors are uncorrelated
$E(f_k \epsilon_i) = 0$	The factors are uncorrelated with the residuals

Start by considering a single-factor model in a strange world without idiosyncratic risk (no residual)

$$r_i = a_i + b_i f$$

The expected return on asset i is a_i because factor f is assumed is have a mean of zero.

$$E(r_i) = a_i$$

b_i is the sensitivity of r_i to factor f. The no arbitrage principle tells us that any two assets with the same b must have the same expected return a. A relationship to price assets with different non-zero sensitivities, $b_i \neq b_j$, can be built by forming a portfolio of the two assets. Invest w_i in asset i and $w_j = 1 - w_i$ in asset j. The return on the portfolio is

$$r_p = w_i \left(a_i + b_i f\right) + \left(1 - w_i\right) \left(a_j + b_j f\right)$$

Convince yourself that the equation can be written as

$$r_p = w_i \left(a_i - a_j\right) + a_j + \left[w_i \left(b_i - b_j\right) + b_j\right] f$$

What does the investment in asset i have to be for the portfolio to be riskless? Because the risk is factor f, choose w_i so that the coefficient of f is zero. $w_i = \dfrac{b_j}{b_j - b_i}$ does the trick. The return on the two-asset portfolio must then be equal to the return on a risk-free asset.

$$r_p \equiv r_f = \frac{b_j}{b_j - b_i} \left(a_i - a_j\right) + a_j$$

The equation above is the no-arbitrage condition for the expected returns on any two assets i and j because a_i and a_j are their expected returns. The values of a_i and a_j are not independent of one another but must be such that the right-hand side is equal to the risk-free rate. This can be seen more clearly by rewriting the condition as

$$\frac{a_i - r_f}{b_i} = \frac{a_j - r_f}{b_j}$$

Now that's more intuitive. It says that the excess expected return per unit of risk factor sensitivity must be equal for all assets?

Rewrite the condition once more, this time solving for the expected return on asset i, $E(r_i) \equiv a_i$; but instead of any asset j on the right-hand side, choose an unnamed asset that has unit sensitivity, $b = 1$, to factor f[31]

$$\frac{a_i - r_f}{b_i} = \frac{a - r_f}{b}$$
$$a_i - r_f = b_i \frac{a - r_f}{b}$$
$$a_i = r_f + b_i \left(a - r_f\right) \because b = 1$$

Letting $\lambda = a - r_f$ be the excess expected return on the asset with unit sensitivity to f, the expected return on any asset can be written as

$$a_i = r_f + b_i \lambda$$

Looks kind of like the CAPM, doesn't it? Except that we have a risk premium on a factor rather than the market portfolio. Expanded to K factors, the model becomes Arbitrage Pricing Theory.

$$a_i = r_f + \sum_k^K b_{ik} \lambda_k$$

where λ_k is the risk premium for an asset that has unit sensitivity to factor f_k. The risk factors that have been used to test APT empirical include unexpected changes in inflation, GNP (industrial production), investor confidence (the default premium on corporate bonds), and the yield curve (expectations about future interest rates).

It's not a problem if a riskless asset is not available because, as we saw, a riskless asset can be created from a portfolio of any two risky assets. In that case, λ_0, the implied risk-free rate, replaces r_f in the APT equation. When idiosyncratic risk (the ϵ's) is reintroduced, the arbitrage implied by APT is no longer riskless because there is no guarantee that all idiosyncratic risk can be diversified away.

The CAPM and APT are theories in the everyday sense. It is less clear if they are in the scientific sense. A scientific theory must be *testable* and *falsifiable*. Richard Roll made the case that the CAPM is not testable and, therefore, that all of the empirical tests that had been done were invalid.[32] His argument rests on two statements. The first is that any mean-variance efficient port-

[31] In much the same way that the beta of the market portfolio is 1, except that the beta of the market is 1 because it is the sensitivity to itself.

[32] His famous analysis is known as Roll's Critique.

folio p—one that sits on an efficient frontier—implies the linear CAPM relation between the expected return on a security and its beta if p is used as a proxy for the market portfolio.

$$E(r_j) = r_f + \beta_{jp}\left[E(r_p) - r_f\right]$$

This follows directly from portfolio math and requires none of the assumptions of the CAPM. For any proxy for the market, testing the CAPM is equivalent to testing the mean-variance efficiency of the proxy. If the proxy is assumed to be mean-variance efficient, the CAPM is a tautology! Roll's second statement is that the true market portfolio cannot be observed, as it includes rare Beatles records, baseball cards, and your chalet in the Alps. Since testing the CAPM is equivalent to testing the mean-variance efficiency of the market portfolio and the market portfolio is unobservable, the CAPM is untestable. Professor Roll ruffled a lot of tail feathers with that one. APT is subject to the criticism that it is not falsifiable because the factors are not specified. It may be possible to find factors that fit the data well after the fact (data mining) or to dismiss a failure to fit the data with the excuse that the wrong factors had been used (maybe we should have used factor Y instead of factor X). If the chosen factors are mean-variance portfolios, then APT is subject to Roll's Critique for the same reason as the CAPM.

Reading for Your Interest

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The interactive version of the index.

https://www.economist.com/big-mac-index.

You can download the Big Mac Index here:

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Exchange rates at The Bank of Canada:

https://www.bankofcanada.ca/rates/exchange/daily-exchange-rates-lookup/

Exercises

1. Big Coke is a one-litre bottle, Little Coke a half. Suppose they are perfect substitutes. What would you have to do to earn a $100 arbitrage profit if today Big Coke costs $3.60 and Little Coke $1.60?
2. On July 30, 2021 the US dollar was C$1.25, a Big Mac in Montreal cost C$5.69 and US$3.99 in Burlington, Vermont. What does Purchasing Power Parity have to say about the value of the Canadian dollar? What would you have to do to earn an arbitrage profit as near as possible to C$10,000?
3. The US dollar spot price is C$1.24 and the one-year forward price C$1.27. The one-year risk-free rate of interest is 4.5 per cent in Canada and 3.2 per cent in the United States. Describe the arbitrage profit that can be earned for a investment of C$1 million.
4. The three-year risk-free interest rate is 7.6 per cent and the five-year rate is 6.1 per cent. The two-year forward rate for a loan or investment commencing in year 4 is 4.2 per cent. Describe the arbitrage profit that can be earned for a investment of $1 million.
5. [coming soon] MM and arbitrage.
6. What are the states in put-call parity? the portfolios? Why are the portfolios perfect substitutes?
7. Thinking about APT, what would arbitrageurs do if $\frac{a_i - r_f}{b_i} < \frac{a_j - r_f}{b_j}$?

ANSWERS

1. Big Coke is overpriced because it costs more than twice as much as Little Coke. Or you could say that Little Coke is underpriced because costs less than half of that of Big Coke. An arbitrage profit of 40 cents a litre is possible. To earn $100, you would short sell 250 bottles of Big Coke, bringing in $900, and immediately buy 500 bottles of Little Coke at a cost of $800 to cover your short position. Burp.

2. The C$5.69 cost of a Big Mac in Canada and US$3.99 in the United States implies that the exchange rate should be 1.42607. The relative difference between this and the actual exchange rate, 1.25, suggests the Canadian dollar is overvalued by 14.1 per cent, and is therefore expected to depreciate. For now, there's a possible arbitrage profit of C$0.7025 per burger if you borrow them in the United States and sell them here. To earn C$10,000, borrow 14,235 burgers in Burlington, Vermont and sell them in Montreal for C$80,997.20. Pay back your burger loan of US$56,797.70 (C$70,997.10), leaving you with $10,000.10. Hungry?

3. The one-year forward price of the US dollar is higher than the expected future spot price implied by the difference in interest rates. Borrow C$1 million here and invest it in riskless US government bonds for one year. Sell the proceeds forward for C$1,056,967.74 to cover the loan and lock in a profit of C$11,967.74.

4. The forward rate of interest is higher than the expected future spot rate implied by the current short-term and long-term spot rates. Borrow $1 million for five years at 6.1 per cent and invest it for three years at 7.6 per cent. Roll over the investment in advance by investing forward for another two years at 4.2 per cent for a total return of $1,359,211.65 at the end of 5 years. This will cover the amount owing on the loan and leave you with a profit of $8,757.14, which you can use to buy me a Rolex.

5. [coming soon] MM and arbitrage.

6. For you to think about.

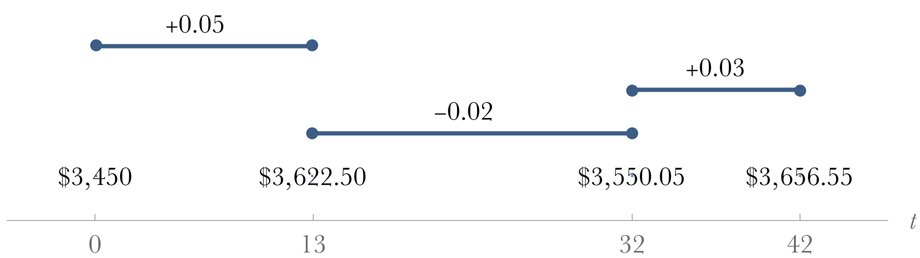
7. For you to think about.

Appendices

RETURN

You bought a stock when its price was \$3,450 and vaguely recall selling it 42 months later for \$3,656.55. You weren't interested in tracking its price, but are sure there is a notebook in the desk where you jotted it down after 13 months and then again after 32. That's okay, you say to yourself. Life's short and you had better things to do, right? But now you decide to do some bean counting.

No problem computing the *holding period return* because you know the price you paid and the price you sold it for: a gain of \$206.552 or 5.987 per cent. But wait, you can't find the notebook or a record of any of the prices on paper. Why? Think, think. Ah, that's because you wrote them with a blue Sharpie on the forearm of someone you were trying to impress at Osheaga 2018 while listening to City and Colour performing "Sleeping Sickness". Time to turn to the most reliable source of knowledge you know—the Internet—where you stumble upon a website run by someone named Webster who is apparently obsessed with numbers and keeps an eclectic archive of data (websterlikestocountstuffbecauseshehasnolife.com). What luck! Webster has a record of the return on your stock for the three periods. What are the chances of that happening?[33]



So now all you have in front of you is three percentages: 0.05, -0.02, and 0.03. Let's see, a dollar grows by five per cent to \$1.05, which falls by two per cent to \$1.029, and that grows by three per cent to \$1.05987. Holy Sweet Mother of Molasses! That's the 5.987 per cent you calculated with the prices that you thought you had but never really had![34]

$$1 + HPR_{42} = (1 + 0.05)(1 - 0.02)(1 + 0.03) = 1.05987$$

[33] A little higher than two students working independently coming up with the same random samples for the portfolio simulation in assignment 1. Don't think I won't check.

[34] Like having flashbacks to the 70s. Man, I miss Tangerine Dream and T-Rex.

You're feeling really good about yourself, thinking of changing your name to Einstein. But then the phone rings. It's your old high school friend, Nigel. He never calls unless he has something to brag about.

"Hey Francesca, what's up?"

"Not much."

"I remember you talking about an investment a couple of years ago, so I thought I'd dive in too."

"Great." Should I say my phone is dying.

"I earned 6.3 per cent over 45 months. I couldn't believe that I earned a higher return than you, considering that I have no experience."

"How do you know what my investment earned?"

"Osheaga. 5.987 per cent. It took me an hour to scrub those prices off my arm."

The devil is a squirrel. But then you got to thinking.

"Wait. You can't compare your 6.3 per cent to my 5.987 per cent. You held onto your stock for 45 months. I cashed in after only 42. We need to compare our returns using a common time period, like a year.

"Do I divide my return by 3.75 years and you divide yours by 3.5?"

"No, you have answer the question, What interest rate compounded for 3.75 years grows to 6.3 per cent? I do the same for mine.

"You do it Ms Spreadsheet."

"Okay, I'll email the answer to you along with a formula."

POSTSCRIPT

Francesca was happy to find that her annualized holding period return was 1.675 per cent while Nigel's was 1.643. While you're here, why don't you write a formula for converting a holding period return to its equivalent in any frequency: yearly, quarterly. monthly, weekly, daily, minutely (didn't even know that was a word). Put it right here, kiddo.

Your formula here

VECTORS AND MATRICES

Let's work with these.

v is a 3 x 1 vector $$\mathbf{v} = \begin{pmatrix} 3 \\ -2 \\ 8 \end{pmatrix}$$

m is a 3 x 2 matrix $$\mathbf{m} = \begin{pmatrix} 5 & 9 \\ 13 & 6 \\ 4 & -12 \end{pmatrix}$$

1 is a 3 x 1 vector of 1s $$\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

s is a 2 x 2 square matrix $$\mathbf{s} = \begin{pmatrix} -4 & 9 \\ 11 & -6 \end{pmatrix}$$

Transposing means swapping rows and columns. **v** becomes 1 x 3 and **m** 2 x 3. The transpose of a vector or matrix is usually denoted by $\top$ or $'$ (prime).

$$\mathbf{v}^\top = (3 \quad -2 \quad 8)$$

$$\mathbf{m}^\top = \begin{pmatrix} 5 & 13 & 4 \\ 9 & 6 & -12 \end{pmatrix}$$

Dot product is multiplication. You can multiply a vector by a vector, a matrix by a matrix, and a vector by a matrix. You do this by multiplying each row element by the corresponding column element and summing the products. Order matters: the dimensions of the vectors and matrices being multiplied need to be compatible in that the number of columns in the vector or matrix on the left must equal the number of rows in the one on the right.

(1 x **3**) · (**3** x 2) is (1 x 2) but (2 x **3**) · (**3** x 1) is (2 x 1).

$$\mathbf{v}^\top \cdot \mathbf{m} = (3 \quad -2 \quad 8) \cdot \begin{pmatrix} 5 & 9 \\ 13 & 6 \\ 4 & -12 \end{pmatrix} = (21 \quad -81)$$

$$\mathbf{m}^\top \cdot \mathbf{v} = \begin{pmatrix} 5 & 13 & 4 \\ 9 & 6 & -12 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ 8 \end{pmatrix} = \begin{pmatrix} 21 \\ -81 \end{pmatrix}$$

Excel's function for computing dot products is MMULT.

What about addition and subtraction? Too easy. Add or subtract the corresponding elements. The dimensions must be the same, which means, for example, that you can't add a vector to a matrix.

The *inverse* of a square matrix is like the reciprocal of a number. A number multiplied by its reciprocal is 1, as in $6 \times \frac{1}{6} = 1$. A matrix multiplied by its inverse gives the identity matrix $\mathbf{I}$, a square matrix with 1s along the main diagonal and 0s elsewhere.

The inverse of $\mathbf{s}$ is
$$\mathbf{s}^{-1} = \begin{pmatrix} \frac{2}{25} & \frac{3}{25} \\ \frac{11}{75} & \frac{4}{75} \end{pmatrix}$$

Check:
$$\mathbf{s} \cdot \mathbf{s}^{-1} = \begin{pmatrix} -4 & 9 \\ 11 & -6 \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{25} & \frac{3}{25} \\ \frac{11}{75} & \frac{4}{75} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

Excel's function for computing matrix inverses is MINVERSE.

Exercises

Compare your answers with your classmates.

1. Compute and interpret $\mathbf{v}^{\top} \cdot \mathbf{1}$.
2. Compute and interpret $\mathbf{v}^{\top} \cdot \mathbf{v}$.
3. Compute $\mathbf{m} \cdot \mathbf{m}^{\top}$.
4. Compute $\mathbf{m}^{\top} \cdot \mathbf{m}$.
5. Is $\mathbf{s}^{-1} \cdot \mathbf{s} = \mathbf{I}$?

STATISTICS

This is not so much a review of statistics as it is a review of statistical expectation, $E(\cdot)$, the probability-weighted average that is at the heart of statistics that describe things that occur with seeming randomness.

Let x and y be random variables whose possible outcomes are indexed by i and probability of occurring π_i. Here are $n = 9$ observations on x and y.

$$x = \{34, 30, -5, 18, 11, 33, 6, 19, 12\} \text{ and } y = \{17, 14, -5, 5, -7, 8, 11, -9, -2\}$$

With no more than this meagre information about outcomes and no information at all about likelihood, we describe the distributions as best we can using samples such as these, and assign in place of probabilities the same weight $\frac{1}{n}$ to each observation.

Expected value

Expected value or mean is a probability-weighted average of possible outcomes of x. It is a measure of central value, and, as everybody knows, is estimated by the good old arithmetic average.

$$E(x) = \sum_i \pi_i x_i = \pi^\top \cdot \mathbf{x}$$

$E(x) = 17.\dot{5}$ and $E(y) = 3.\dot{5}$. Sample means are traditionally denoted with over-lines, as in $\bar{x}$, but I use $E(x)$ in these notes because I refer to it more often as expected value, and it should be obvious whenever a mean has been estimated from sample data.

Variance

Variance or its square root, standard deviation, is a measure of how spread out the possible outcomes are from their mean. It is the most common measure of risk in financial economics. The expression in words is expected squared deviation from the mean. Notice I didn't write the expression using summation notation? The expectations operator, $E(\cdot)$, does the job elegantly. Get used to seeing it that way.

$$\text{Var}(x) \equiv \sigma_x^2 = E\left[\left(x - E(x)\right)^2\right]$$

$\sigma_x^2 = 153.58$ and $\sigma_y^2 = 82.2469$. You will also see variance written as σ_{xx}, especially when it appears in a matrix. When you check my variance calculations for x and y, you'll find that I report population statistics rather than sample statistics even though we are working with samples. While dividing by n rather than n-1 biases the statistics downward a little, dividing by n is a reminder that these are averages. I do the same for covariance. When we do formal statistical tests, we'll divide by n-1.

Covariance

Covariance is a measure of how one random variable moves against another. Does the value of x tend to be high (above its mean) when the value of y is high (also above its mean) or low (below its mean)? Covariance is the expected product of the deviations from their means of two random variables.

$$\text{Cov}(x, y) \equiv \sigma_{xy} = E\left[\left(x - E(x)\right)\left(y - E(y)\right)\right]$$

That σ_{xy} is positive, 67.6914, means that x and y tend to move in the same direction. The sign of covariance, positive, zero, or negative, tells you the direction of co-movement, but its magnitude doesn't tell you anything about the strength of the relationship. Does 67.6914 mean that x and y move in the same direction most of the time or just more often than not? You cannot know because covariance depends on the dimensions of the random variables. x could be millimetres of precipitation per month and y bushels of wheat. Their covariance would be in units millimetres x bushels. You cannot say that the covariance of 67.6914 for x and y (mm x bu) is stronger than a covariance of 43.986 for w and z (˚C x BP), the temperature outside and my blood pressure.

Correlation coefficient

The correlation coefficient is covariance normalized to lie between -1 and +1, making it dimensionless. This is done by dividing covariance by the product of the variables' standard deviations.

$$\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

The closer that ρ_{xy} is to +1 or -1, the stronger the positive or negative co-movement; the two are more in sync. ρ_{xy} = 0.60229 is a stronger positive relationship than the 0.5472 correlation between temperature and my blood pressure.

Exercises

No solutions provided for these. Compare notes among yourselves!

1. Show that $E(a + bx) = a + bE(x)$, where a and b are constants. Piece of cake, eh? Why is this important?

2. Show that variance can be written as $E(x^2) - \left[E(x)\right]^2$: the mean of the square minus the square of the mean.

3. Show that covariance can be written as $E(xy) - E(x)E(y)$: the mean of the product minus the product of the means.

RULES FOR TAKING DERIVATIVES

The first derivative of a function f is represented by $'$(prime), $\frac{df}{dx} \equiv f'(x)$. These are derivatives that come up a lot in economics. I'm sure you remember them fondly.

Constant rule:	if $f(x) = c$ then $f'(x) = 0$
Constant multiple rule:	if $g(x) = cf(x)$ then $g'(x) = cf'(x)$
Power rule:	if $f(x) = x^n$ then $f'(x) = n\,x^{n-1}$
Addition rule:	if $h(x) = f(x) \pm g(x)$ then $h'(x) = f'(x) \pm g'(x)$
Product rule:	if $h(x) = f(x)g(x)$ then $h'(x) = f'(x)g(x) + f(x)g'(x)$
Quotient rule:	if $h(x) = \frac{f(x)}{g(x)}$ then $h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$
Chain rule:	if $h(x) = f(g(x))$ then $h'(x) = f'(g(x))g'(x)$
Exponential rules:	if $f(x) = a^x$ then $f'(x) = \ln(a)\,a^x$
	if $f(x) = e^x$ then $f'(x) = e^x$
	if $f(x) = a^{g(x)}$ then $f'(x) = \ln(a)\,a^{g(x)}g'(x)$
	if $f(x) = e^{g(x)}$ then $f'(x) = e^{g(x)}g'(x)$
Logarithm rules:	if $f(x) = \log_a(x)$ then $f'(x) = \frac{1}{\ln(a)\,x}$
	if $f(x) = \ln(x)$ then $f'(x) = \frac{1}{x}$
	if $f(x) = \log_a(g(x))$ then $f'(x) = \frac{g'(x)}{\ln(a)\,g(x)}$
	if $f(x) = \ln(g(x))$ then $f'(x) = \frac{g'(x)}{g(x)}$