PHY F242 : QUANTUM MECHANICS-I

Assignment-1 Solutions

1. given, $\psi_0 = Axe^{-ax^2/2}, x \in (-\infty, \infty)$

   i. For normalization, $\int_{-\infty}^{\infty} |\psi_0|^2 dx = 1 \implies \int_{-\infty}^{\infty} \psi_0^* \psi_0 dx = \int_{-\infty}^{\infty} \psi_0^2 dx = 1$ (since ψ_0 is real)

   $\implies \int_{-\infty}^{\infty} A^2 x^2 e^{-ax^2} dx = 2A^2 \int_0^{\infty} x^2 e^{-ax^2} dx = 1$

   Substituting $ax^2 = t$, we get $\frac{A^2}{a\sqrt{a}} \int_0^{\infty} \sqrt{t} e^{-t} dt = \frac{A^2}{a\sqrt{a}} \int_0^{\infty} t^{\frac{3}{2}-1} e^{-t} dt = \dfrac{A^2}{a\sqrt{a}} \Gamma(\frac{3}{2}) = \dfrac{A^2}{2a} \sqrt{\dfrac{\pi}{a}} = 1$

   $\implies A^2 = 2a\sqrt{\dfrac{a}{\pi}} \implies A = \sqrt{2a}\ (\dfrac{a}{\pi})^{1/4}$

   ii. $\hat{T} = -\dfrac{\hbar^2}{2m}\dfrac{\partial^2}{\partial x^2}, \langle T \rangle = \int_{-\infty}^{\infty} \psi_0^* \hat{T} \psi dx = -\dfrac{\hbar^2}{2m} \int_{-\infty}^{\infty} \psi_0 \dfrac{\partial^2 \psi_0}{\partial x^2} dx$

   $\dfrac{\partial \psi_0}{\partial x} = Ae^{-ax^2/2} - Aax^2 e^{-ax^2/2}$

   $\dfrac{\partial^2 \psi_0}{\partial x^2} = A(a^2x^3 - 3ax)e^{-ax^2/2}$

   $\langle T \rangle = -\dfrac{A^2\hbar^2}{2m} \int_{-\infty}^{\infty} (x)(a^2x^3 - 3ax)e^{-ax^2} dx = \dfrac{A^2\hbar^2}{2m} \int_{-\infty}^{\infty} (3ax^2 - a^2x^4)e^{-ax^2} dx = \dfrac{A^2\hbar^2}{m} \int_0^{\infty} (3ax^2 - a^2x^4)e^{-ax^2} dx = \dfrac{A^2\hbar^2}{m} \int_0^{\infty} (3ax^2 e^{-ax^2} dx - \int_0^{\infty} a^2x^4) e^{-ax^2} dx.$

   Substituting $ax^2 = t$, we get

   $\int_0^{\infty} 3ax^2 e^{-ax^2} dx = \dfrac{3}{4}\sqrt{\dfrac{\pi}{a}}$ and $\int_0^{\infty} a^2 x^4 e^{-ax^2} dx = \dfrac{3}{8}\sqrt{\dfrac{\pi}{a}}$

   $\langle T \rangle = \dfrac{A^2\hbar^2}{m}(\dfrac{3}{4}\sqrt{\dfrac{\pi}{a}} - \dfrac{3}{8}\sqrt{\dfrac{\pi}{a}}) = \dfrac{3A^2\hbar^2}{8m}\sqrt{\dfrac{\pi}{a}} = \dfrac{3\hbar^2 a}{4m}$

   Final Answer:
   i. $A = \sqrt{2a}\ (\dfrac{a}{\pi})^{1/4}$
   ii. $\langle T \rangle = \dfrac{3\hbar^2 a}{4m}$

2. $\frac{d\langle p\rangle}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty}\psi^*\frac{\hbar}{i}\frac{\partial\psi}{\partial x}dx = \frac{\hbar}{i}\int_{-\infty}^{\infty}\frac{\partial}{\partial t}(\psi^*\frac{\partial\psi}{\partial x})dx =$
$\frac{\hbar}{i}\int_{-\infty}^{\infty}(\frac{\partial\psi^*}{\partial t}\frac{\partial\psi}{\partial x} + \psi^*\frac{\partial}{\partial x}\frac{\partial\psi}{\partial t})dx$
Replacing the expression for $\frac{\partial\psi}{\partial t}$ and $\frac{\partial\psi^*}{\partial t}$ from the Schrodinger's equation into the equation above, we get
$\frac{d\langle p\rangle}{dt} = \frac{\hbar}{i}[\int_{-\infty}^{\infty}\frac{i\hbar}{2m}(\psi^*\frac{\partial^3\psi}{\partial x^3} - \frac{\partial^2\psi^*}{\partial x^2}\frac{\partial\psi}{\partial x})dx + \int_{-\infty}^{\infty}\frac{i}{\hbar}(V\psi^*\frac{\partial\psi}{\partial x} - \psi^*\frac{\partial(V\psi)}{\partial x})dx]$
Upon repeated use of the integral by parts rule and using the fact that the value of $\psi, \psi^*, \frac{\partial\psi}{\partial x}$ and $\frac{\partial\psi^*}{\partial x}$ all become 0 as $x \to \pm\infty$, we see that the first integral becomes 0
$\implies \frac{d\langle p\rangle}{dt} = \int_{-\infty}^{\infty}(V\psi^*\frac{\partial\psi}{\partial x} - \psi^*\frac{\partial(V\psi)}{\partial x})dx = \int_{-\infty}^{\infty}(V\psi^*\frac{\partial\psi}{\partial x} - \psi^*V\frac{\partial\psi}{\partial x} - \psi^*\frac{\partial V}{\partial x}\psi)dx = \int_{-\infty}^{\infty}\psi^*(-\frac{\partial V}{\partial x})\psi dx = \langle -\frac{\partial V}{\partial x}\rangle$

3. i. For Normalization, $\int_{-1}^{1}A^2(x-x^3)^2dx = A^2\int_{-1}^{1}(x^2+x^6-2x^4)dx = 1$
$\implies A^2(\frac{2}{3}+\frac{2}{7}-\frac{4}{5}) = A^2\frac{16}{105} = 1 \implies A^2 = \frac{105}{16} \implies A = \sqrt{\frac{105}{16}}$
ii. $\langle x\rangle = A^2\int_{-1}^{1}x(x-x^3)^2dx = 0$ (odd integral)
$\langle x^2\rangle = A^2\int_{-1}^{1}x^2(x-x^3)^2dx = \int_{-1}^{1}(x^4+x^8-2x^6)dx = A^2(\frac{2}{5}+\frac{2}{9}-\frac{4}{7}) = A^2\frac{16}{315} = \frac{1}{3}$
iii. $\frac{\partial\psi_0}{\partial x} = A(1-3x^2)$ $\frac{\partial^2\psi_0}{\partial x^2} = -6Ax$
$\langle p\rangle = A^2\int_{-1}^{1}(x-x^3)(1-3x^2)dx = 0$ (odd integral)
$\langle p^2\rangle = 6A^2\hbar^2\int_{-1}^{1}(x^2-x^4)dx = 6A^2\hbar^2(\frac{2}{3}-\frac{2}{5}) = \frac{8A\hbar^2}{5} = \frac{21\hbar^2}{2}$
iv. $\sigma_x = \sqrt{\frac{1}{3}}, \sigma_p = \sqrt{\frac{21}{2}}\hbar, \sigma_x * \sigma_p = \sqrt{\frac{7}{2}}\hbar \approx 1.87\hbar > 0.5\hbar \implies$ consistent with uncertainty principle