PHY F242 : QUANTUM MECHANICS-I

Practice Problems

Total Marks: 40

1. Consider the harmonic oscillator potential $V(x)$ given as: $V(x) = \frac{m\omega_0^2 x^2}{2}$ where m is the mass of the particle. Suppose the initial state of a particle in this potential is specified as

$$\phi(x,0) = \frac{1}{\sqrt{2}}[\psi_0(x) + i\psi_1(x)]$$

where ψ_i represents the i^{th} normalized stationary state of the harmonic oscillator.

a. Write an expression for $\phi(x,t)$ and $|\phi(x,t)|^2$ in terms of ψ_i's.

b. Find $< x >$ and $< p >$ as functions of t.

c. Suppose the state of a particle in a harmonic oscillator potential is described by an arbitrary superposition of the stationary states at a time instant $t = t_0$

$$\psi(x,t_0) = \Sigma_{i=0}^{n} c_i \psi_i$$

Show that the probability distribution returns to the original shape after a time interval $T = \frac{2\pi}{\omega_0}$, i.e. $|\psi(x, t_0 + T)|^2 = |\psi(x,t_0)|^2$

[2+2+2]

2. a. Show that the solution to the time-independent Schrodinger Equation ψ can always be taken to be real.

b. Consider a wavefunction ψ'_n specified as

$$\psi'_n(x) = e^{i\phi}\sqrt{\frac{2}{L}}sin(\frac{n\pi x}{L})$$

where ϕ is a constant. Show that ψ'_n is also a stationary state of the infinite square well potential system. Does this contradict the statement in (a) ?

c. Consider the potential function $V(x)$,

$$V(x) = \begin{cases} \frac{m\omega^2 x^2}{2} & x < 0 \\ \infty & x \geq 0 \end{cases}$$

Find the energy eigen-values and eigen-functions of $V(x)$.

[2+2+2]

3. Consider the creation (raising) and annihilation (lowering) operators $\hat{a}^\dagger$ and $\hat{a}$ (as defined for the harmonic oscillator potential). Let the particle (of mass m) in the harmonic oscillator potential be in the n^{th} energy eigen-state ψ_n

   a. Express $\hat{x}$ and $\hat{p}$ in terms of $\hat{a}^\dagger$ and $\hat{a}$. Use these expressions to find $< x >$ and $< p >$.

   b. Express $\hat{x}^2$ and $\hat{p}^2$ in terms of $\hat{a}^\dagger$ and $\hat{a}$. Use these expressions to find an expression for the product $\Delta x.\Delta p$. Does the system satisfy the uncertainty principle?

[2+3]

4. Consider a quantum particle released in a potential $V(x)$ described by:

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 \leq x < a \\ V_0 & a \leq x < 2a \\ \infty & x \geq 2a \end{cases}$$

   for some width a and $V_0 > 0$. Consider a stationary state with energy $E < V_0$.

   a. Obtain a transcendental equation for E.

   b. Suppose the solution of the above equation is $E = E_0$, then find the corresponding stationary state.

[2+3]

5. For a particle of mass m in an infinite square well of length l, let the initial state be described as

$$\phi(x,0) = \sqrt{\frac{1}{6}}\psi_0 + \sqrt{\frac{1}{2}}\psi_1 + \sqrt{\frac{1}{3}}\psi_2$$

where ψ_i is the i^{th} energy eigen-state of the system.

a. If a measurement of the energy is made, what are the possible observations? What are the probabilities of observing each of the energy values? Hence, find $< E >$.

b. Using the first 3 energy eigen-states, ψ_0, ψ_1 and ψ_2, construct another (normalized) state ϕ' which has the same $< E >$ and the same probabilities of measuring each energy-value, but is orthogonal to ϕ

[2+2]

6. Consider the double-delta function potential: $V(x) = -\alpha[\delta(x-a) + \delta(x+a)]$ where α is a positive constant. Consider an incoming stream of particles to the left barrier ($x = -a$). Find an expression of the reflection and transmission coefficients as seen by the potential.

Note: The transmission coefficient, here, is the ratio of the "intensity" of the wave transmitted to the right of the second barrier ($x = a$) to that of the incoming wave into the left barrier ($x = -a$). The reflection coefficient is the ratio of the "intensity" of the wave reflected from the first barrier to that of the incoming wave

[5]

7. Consider the potential $V(x)$ described as:

$$V(x) = \begin{cases} V_0 & -a \leq x \leq a \\ 0 & otherwise \end{cases}$$

where V_0 is positive. Find the transmission coefficient of the scaterring states (with energy E) for the potential barrier for:

i. $E = V_0$ ii. $E > V_0$ iii. $E < V_0$

[3+3+3]