

PHY F242 : QUANTUM MECHANICS-I

Assignment-1 Solutions

1. given, $\psi_0 = Axe^{-ax^2/2}, x \in (-\infty, \infty)$

i. For normalization, $\int_{-\infty}^{\infty} |\psi_0|^2 dx = 1 \implies \int_{-\infty}^{\infty} \psi_0^* \psi_0 dx = \int_{-\infty}^{\infty} \psi_0^2 dx = 1$ (since ψ_0 is real)

$$\implies \int_{-\infty}^{\infty} A^2 x^2 e^{-ax^2} dx = 2A^2 \int_0^{\infty} x^2 e^{-ax^2} dx = 1$$

Substituting $ax^2 = t$, we get $\frac{A^2}{a\sqrt{a}} \int_0^{\infty} \sqrt{t} e^{-t} dt = \frac{A^2}{a\sqrt{a}} \int_0^{\infty} t^{\frac{3}{2}-1} e^{-t} dt = \frac{A^2}{a\sqrt{a}} \Gamma(\frac{3}{2}) = \frac{A^2}{2a} \sqrt{\frac{\pi}{a}} = 1$

$$\implies A^2 = 2a\sqrt{\frac{a}{\pi}} \implies A = \sqrt{2a} \left(\frac{a}{\pi}\right)^{1/4}$$

ii. $\hat{T} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}, \langle T \rangle = \int_{-\infty}^{\infty} \psi_0^* \hat{T} \psi_0 dx = -\frac{\hbar^2}{2m} \int_{-\infty}^{\infty} \psi_0 \frac{\partial^2 \psi_0}{\partial x^2} dx$

$$\frac{\partial \psi_0}{\partial x} = Ae^{-ax^2/2} - Aax^2 e^{-ax^2/2}$$

$$\frac{\partial^2 \psi_0}{\partial x^2} = A(a^2 x^3 - 3ax)e^{-ax^2/2}$$

$$\begin{aligned} \langle T \rangle &= -\frac{A^2 \hbar^2}{2m} \int_{-\infty}^{\infty} (x)(a^2 x^3 - 3ax)e^{-ax^2} dx = \frac{A^2 \hbar^2}{2m} \int_{-\infty}^{\infty} (3ax^2 - a^2 x^4)e^{-ax^2} dx = \\ &= \frac{A^2 \hbar^2}{m} \int_0^{\infty} (3ax^2 - a^2 x^4)e^{-ax^2} dx = \frac{A^2 \hbar^2}{m} \int_0^{\infty} (3ax^2 e^{-ax^2} dx - \int_0^{\infty} a^2 x^4 e^{-ax^2} dx). \end{aligned}$$

Substituting $ax^2 = t$, we get

$$\int_0^{\infty} 3ax^2 e^{-ax^2} dx = \frac{3}{4} \sqrt{\frac{\pi}{a}} \text{ and } \int_0^{\infty} a^2 x^4 e^{-ax^2} dx = \frac{3}{8} \sqrt{\frac{\pi}{a}}$$

$$\langle T \rangle = \frac{A^2 \hbar^2}{m} \left(\frac{3}{4} \sqrt{\frac{\pi}{a}} - \frac{3}{8} \sqrt{\frac{\pi}{a}} \right) = \frac{3A^2 \hbar^2}{8m} \sqrt{\frac{\pi}{a}} = \frac{3\hbar^2 a}{4m}$$

Final Answer:

i. $A = \sqrt{2a} \left(\frac{a}{\pi}\right)^{1/4}$

ii. $\langle T \rangle = \frac{3\hbar^2 a}{4m}$

$$2. \frac{d\langle p \rangle}{dt} = \frac{d}{dt} \int_{-\infty}^{\infty} \psi^* \frac{\hbar}{i} \frac{\partial \psi}{\partial x} dx = \frac{\hbar}{i} \int_{-\infty}^{\infty} \frac{\partial}{\partial t} (\psi^* \frac{\partial \psi}{\partial x}) dx =$$

$$\frac{\hbar}{i} \int_{-\infty}^{\infty} (\frac{\partial \psi^*}{\partial t} \frac{\partial \psi}{\partial x} + \psi^* \frac{\partial}{\partial x} \frac{\partial \psi}{\partial t}) dx$$

Replacing the expression for $\frac{\partial \psi}{\partial t}$ and $\frac{\partial \psi^*}{\partial t}$ from the Schrodinger's equation into the equation above, we get

$$\frac{d\langle p \rangle}{dt} = \frac{\hbar}{i} [\int_{-\infty}^{\infty} \frac{i\hbar}{2m} (\psi^* \frac{\partial^3 \psi}{\partial x^3} - \frac{\partial^2 \psi^*}{\partial x^2} \frac{\partial \psi}{\partial x}) dx + \int_{-\infty}^{\infty} \frac{i}{\hbar} (V \psi^* \frac{\partial \psi}{\partial x} - \psi^* \frac{\partial (V \psi)}{\partial x}) dx]$$

Upon repeated use of the integral by parts rule and using the fact that the value of $\psi, \psi^*, \frac{\partial \psi}{\partial x}$ and $\frac{\partial \psi^*}{\partial x}$ all become 0 as $x \rightarrow \pm\infty$, we see that the first integral becomes 0

$$\Rightarrow \frac{d\langle p \rangle}{dt} = \int_{-\infty}^{\infty} (V \psi^* \frac{\partial \psi}{\partial x} - \psi^* \frac{\partial (V \psi)}{\partial x}) dx = \int_{-\infty}^{\infty} (V \psi^* \frac{\partial \psi}{\partial x} - \psi^* V \frac{\partial \psi}{\partial x} - \psi^* \frac{\partial V}{\partial x} \psi) dx = \int_{-\infty}^{\infty} \psi^* (-\frac{\partial V}{\partial x}) \psi dx = \langle -\frac{\partial V}{\partial x} \rangle$$

$$3. \text{ i. For Normalization, } \int_{-1}^1 A^2 (x - x^3)^2 dx = A^2 \int_{-1}^1 (x^2 + x^6 - 2x^4) dx = 1$$

$$\Rightarrow A^2 (\frac{2}{3} + \frac{2}{7} - \frac{4}{5}) = A^2 \frac{16}{105} = 1 \Rightarrow A^2 = \frac{105}{16} \Rightarrow A = \sqrt{\frac{105}{16}}$$

$$\text{ii. } \langle x \rangle = A^2 \int_{-1}^1 x (x - x^3)^2 dx = 0 \text{ (odd integral)}$$

$$\langle x^2 \rangle = A^2 \int_{-1}^1 x^2 (x - x^3)^2 dx = \int_{-1}^1 (x^4 + x^8 - 2x^6) dx = A^2 (\frac{2}{5} + \frac{2}{9} - \frac{4}{7}) =$$

$$A^2 \frac{16}{315} = \frac{1}{3}$$

$$\text{iii. } \frac{\partial \psi_0}{\partial x} = A(1 - 3x^2) \frac{\partial^2 \psi_0}{\partial x^2} = -6Ax$$

$$\langle p \rangle = A^2 \int_{-1}^1 (x - x^3)(1 - 3x^2) dx = 0 \text{ (odd integral)}$$

$$\langle p^2 \rangle = 6A^2 \hbar^2 \int_{-1}^1 (x^2 - x^4) dx = 6A^2 \hbar^2 (\frac{2}{3} - \frac{2}{5}) = \frac{8A\hbar^2}{5} = \frac{21\hbar^2}{2}$$

$$\text{iv. } \sigma_x = \sqrt{\frac{1}{3}}, \sigma_p = \sqrt{\frac{21}{2}} \hbar, \sigma_x * \sigma_p = \sqrt{\frac{7}{2}} \hbar \approx 1.87\hbar > 0.5\hbar \Rightarrow$$

consistent with uncertainty principle