SERIES

AP

$u_n = a+(n-1)d$

$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$

$= \frac{n}{2}(a+l)$

GP

$u_n = ar^{n-1}$

$S_n = \frac{a(1-r^n)}{1-r}$ | $S_n = \frac{a(r^n-1)}{r-1}$

$S_\infty = \frac{a}{1-r}$ $\quad S_\infty$ exists $\Leftrightarrow |r|<1$

RATIO TEST

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho.$ converges $\rho<1$; diverges $\rho>1$; NO CONCLUSION $\rho=1$

P-SERIES

$\sum_{n=1}^{\infty}\frac{1}{n^p}$ $\begin{cases}\text{diverges} & 0 \le p \le 1\\ \text{converges} & p>1\end{cases}$

$\frac{d}{dx}\frac{1}{x} = -x^{-2}$

$\int \frac{1}{x}\,dx = \ln(x) + c$

POWER-SERIES

$\sum c_n(x-a)^n$

RADIUS OF CONVERGENCE:
1) At only 1 point "a" R = 0
2) In an interval: (a−h, a+h) R = h
3) Everywhere R = Infinity
Find radius by Ratio Test
$|(x + 3/2)| < 0.5 \implies R=0.5$, at center −3/2

TAYLOR-SERIES

The **Taylor series** of f at a is

$$f(a)+f'(a)(x-a)+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\cdots$$

$$=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k \qquad (1)$$

TAYLOR-THEOREM

Then $f(x) = P_n(x) + R_n(x)$

where $P_n(x) = \sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k \quad R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

for some c between a and x

We know c is between a and x, but we don't know the exact value

$$\frac{1}{1-x} = \sum_{n=0}^{\infty}x^n \qquad -1<x<1$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n \qquad -1<x<1$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n} \qquad -1<x<1$$

$$\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n} \qquad -1<x\le 1$$

$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad -\infty<x<\infty$$

$$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} \qquad -\infty<x<\infty$$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \qquad -\infty<x<\infty$$

$$\tan^{-1}x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} \qquad -1\le x\le 1$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty}nx^{n-1} \qquad -1<x<1$$

$$\frac{1}{(1-x)^3} = \frac{1}{2}\sum_{n=2}^{\infty}n(n-1)x^{n-2} \qquad -1<x<1$$

$\underset{\sim}{a}\cdot\underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$

$\perp: \underset{\sim}{a}\cdot\underset{\sim}{b} = 0$, $\quad //: \underset{\sim}{a}\times\underset{\sim}{b} = 0$

$|\underset{\sim}{a}\times\underset{\sim}{b}| = |\,|\underset{\sim}{a}||\underset{\sim}{b}|\sin\theta\,|$

Area of $\triangle ABC$: $\frac{1}{2}|\vec{AB}\times\vec{AC}| = \frac{1}{2}|\vec{BC}\times\vec{BA}|$

$c^2 = a^2+b^2-2ab\cos\theta$, c opp. of θ

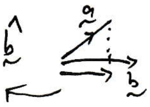
Dist: $|P_1P_2| = \left((x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2\right)^{\frac{1}{2}}$

$$\underset{\sim}{a}\times\underset{\sim}{b} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = \begin{pmatrix}a_2b_3-a_3b_2\\-(a_1b_3-a_3b_1)\\a_1b_2-a_2b_1\end{pmatrix}$$

$|\underset{\sim}{a}\cdot\hat{\underset{\sim}{b}}|$: length projection of a on b

$= \text{proj}_b\, a$

vector: $(\underset{\sim}{a}\cdot\hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$



Discriminant

	(0,0)	(0,2)
f_{xx}	-6	6
f_{yy}	-6	6
f_{xy}	0	0
D	36	36
	Max	min

CRITICAL POINT

A point (a,b) is called a ***critical point*** of f if
(i) $f_x(a,b)=0$ & $f_y(a,b)=0$; or
(ii) $f_x(a,b)$ or $f_y(a,b)$ does not exist

$f(x,y)$, $\quad (a,b)$
$f_x(a,b) = 0 = f_y(a,b)$

$D = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2$

- $D>0$: $f_{xx}>0$ → Min; $f_{xx}<0$ → Max
- $D<0$ → Saddle Point
- $D=0$ → No Conclusion

Logistic:

$D = sN$, where s is small no.

$$\boxed{\frac{dN}{dt} = (B-sN)N = BN - sN^2}$$

$\frac{dN}{dt} = (B-sN)N$

$\int \frac{1}{N(B-sN)}\,dN = \int 1\,dt$

$\int \left(\frac{1}{BN} + \frac{s}{B(B-sN)}\right)dN = \int 1\,dt$

$\therefore \frac{1}{B}\ln(N) - \frac{1}{B}\ln(B-sN) = t + C$

$\frac{1}{B}\ln\left(\frac{N}{B-sN}\right) = t + c$

$\frac{N}{B-sN} = ke^{Bt}$, where $k = e^{Bc}$

Let N_0 be initial pop. at $N(0)$

$\therefore \frac{N}{B-sN} = ke^{Bt} = \frac{N(0)}{B-sN(0)}e^{Bt}$

Rearranging,

$$\boxed{N(t) = \frac{B}{s+\left(\frac{B}{N(0)}-s\right)e^{-Bt}}}$$

PARTIAL DERIVATIVE

By: Gan Chin Yao I 2017 Dec

Let $k(y) = f(a,y)$

The ***partial derivative*** of f wrt y at (a,b) is $k'(b)$
denoted by $\left.\frac{\partial f}{\partial y}\right|_{(a,b)}$ $\quad f_y(a,b)$

$f_{xx} = (f_x)_x = \frac{\partial^2 f}{\partial x^2} \qquad f_{xy} = (f_x)_y = \frac{\partial^2 f}{\partial y\partial x}$

$f_{yx} = (f_y)_x = \frac{\partial^2 f}{\partial x\partial y} \qquad f_{yy} = (f_y)_y = \frac{\partial^2 f}{\partial y^2}$

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$D_{\mathbf u}f(a,b) = f_x(a,b)\,u_1 + f_y(a,b)\,u_2$$

where $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a ***unit*** vector

SMALL CHANGE

$df = D_u f(a,b)\cdot dt$

$\Delta f \approx df$

$\Delta f = f(Q) - f(P)$, where Q is final pt, $f(P)$ is initial

$df \approx f(Q) - f(P)$

$\frac{dc}{dt}$: $\frac{\Delta\text{ Points/location}}{\Delta\text{ How much you move}}$

GRADIENT VECTOR

$\nabla f = f_x\mathbf{i} + f_y\mathbf{j} = \begin{pmatrix}f_x\\f_y\end{pmatrix}$

$|\nabla f(a,b)| = \left|\begin{pmatrix}f_x\\f_y\end{pmatrix}\right|$ ☆ $= \sqrt{f_x^2+f_y^2}$

$$\nabla f(a,b)\bullet\mathbf{u} = D_{\mathbf u}f(a,b)$$

$= |\nabla f(a,b)|\,|\mathbf{u}|\cos\theta$

$= |\nabla f(a,b)|\cos\theta \quad 0 \le \theta \le \pi$

MAX & MIN

$D_u f(a,b)$ is Postive & Max when $\cos\theta = 1$ i.e., $\theta = 0$

$D_u f(a,b)$ is Negative & Min when $\cos\theta = -1$ i.e., $\theta = 180$

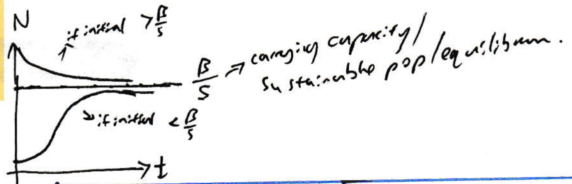
$\nabla f(a,b)$, u, $\theta = 0$: At (a,b), the function f increases most rapidly when u is in the direction $\nabla f(a,b)$

u, $\nabla f(a,b)$, $\theta = 180°$: At (a,b), the function f decreases most rapidly when u is in the direction $-\nabla f(a,b)$

Malthus: $B(t)$: per capita birth rate
$D(t)$: per capita death rate

$\frac{dN}{dt} = (B-D)N$

$N(t) = N(0)e^{kt}$, $k = B-D$



Harvesting: E: Amt fish we catch (constant)/yr

$$\boxed{\frac{dN}{dt} = BN - sN^2 - E = (-s)(N-B_1)(N-B_2)}$$

B_1 & B_2 sol^n to

where B_1 & B_2 are sol^n to $\frac{dN}{dt} = 0$

Equilibrium: $\frac{dN}{dt} = 0$ $\qquad B_1 + B_2 = \frac{B}{s}$

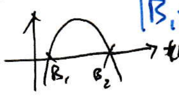
1) 2 sol^n - discriminant > 0 $\qquad B_1B_2 = \frac{E}{s}$
2) 1 sol^n - " = 0
3) No sol^n - " < 0

carrying capacity

discriminant: $B^2 - 4sE$ $\frac{dN}{dt}$

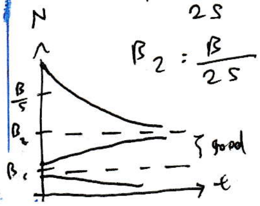
case 1): $B^2 - 4sE > 0$
$E < \frac{B^2}{4s}$

$\boxed{B_1 < B_2 < \frac{B}{s}}$



$B_1 = \frac{B}{2s} - \left(\frac{1}{2s}\right)\sqrt{B^2-4sE}$

$B_2 = \frac{B}{2s} + \left(\frac{1}{2s}\right)\left(\sqrt{B^2-4sE}\right)$



B_2: stable
B_1: NOT

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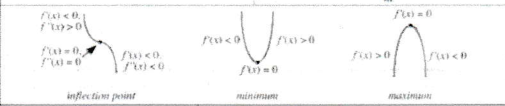
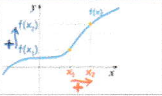
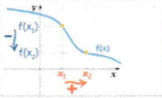
Trigonometric Functions	$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\tan x) = \sec^2 x$
	$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$	$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
Exponential Functions	$\frac{d}{dx}(e^x) = e^x$		$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x \ln a}) = a^x \ln a$
Logarithmic Functions	$\frac{d}{dx}(\ln x) = \frac{1}{x}$		$\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e$
Implicit Functions	$\frac{d}{dx} f(y) = \frac{d}{dy} f(y) \times \frac{dy}{dx}$		Note: whenever there is a y function, there will be a **dy/dx** function in the answer.
Inverse trigonometric functions	$\frac{d}{dx} \sin^{-1} f(x) = \frac{f'(x)}{\sqrt{1-x^2}}$	$\frac{d}{dx} \cos^{-1} f(x) = -\frac{f'(x)}{\sqrt{1-x^2}}$	$\frac{d}{dx} \tan^{-1} f(x) = \frac{f'(x)}{1+f(x)^2}$
Parametric Differentiation	$\frac{dy}{dt} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx} \times \frac{dx}{dt}$		When given parametric eqn, just find dy/dx and the points from there
Tangent and normal	$y - y_1 = m(x - x_1)$		$y - y_1 = -\frac{1}{m}(x - x_1)$
First derivative test			
Second derivative test	$\frac{d^2y}{dx^2} = 0$ *(use table form to determine nature)*	$\frac{d^2y}{dx^2} > 0$	$\frac{d^2y}{dx^2} < 0$
Increasing functions		For increasing functions, $f'(x) > 0$. If **concave downwards**, $f''(x) < 0$. If **concave upwards**, $f''(x) > 0$. Shape: **n** shape. *Strictly increasing*: No flat points allowed.	
Decreasing functions		For decreasing functions, $f'(x) < 0$. If **concave downwards**, $f''(x) < 0$. If **concave upwards**, $f''(x) > 0$. Shape: **u** shape. *Strictly decreasing*: No flat points allowed.	

Hyperbolic

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2} \qquad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1 - e^{-2x}}{1 + e^{-2x}}$$

Exponential Functions	$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$	$\int f'(x)a^{f(x)}\,dx = \frac{e^{f(x)}}{\ln a} + c$
Trigonometric functions	$\int \sin x\,dx = -\cos x + c$	$\int \cos x\,dx = \sin x + c$
	$\int \sec^2 x\,dx = \tan x + c$	$\int \sec x \tan x\,dx = \sec x + c$
	$\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$	$\int \operatorname{cosec}^2 x\,dx = -\cot x + c$
	$\int \tan x\,dx = \ln\lvert\sec x\rvert + c$	$\int \cot x\,dx = \ln\lvert\sin x\rvert + c$
	$\int \sec x\,dx = \ln\lvert\sec x + \tan x\rvert + c$	$\int \operatorname{cosec} x\,dx = -\ln\lvert\operatorname{cosec} x + \cot x\rvert + c$
Trigonometric identities	$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + c$	$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + c$
	$\int \tan^2 x\,dx = \int \sec^2 x - 1\,dx = \tan x - x + c$	$\int \cot^2 x\,dx = \int \operatorname{cosec}^2 x - 1\,dx = -\cot^2 x - x + c$
Factor Formulae	$\sin P + \sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$	$\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$
	$\cos P + \cos Q = 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}$	$\cos P - \cos Q = -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}$
Involving fractions	$\int \frac{1}{(px+q)^2 + a^2}\,dx = \frac{1}{ap}\tan^{-1}\left(\frac{px+q}{a}\right) + c$	$\int \frac{1}{\sqrt{a^2 - (px+q)^2}}\,dx = \frac{1}{p}\sin^{-1}\left(\frac{px+q}{a}\right) + c$
	$\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert$	$\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert$
Partial Fractions	$\frac{px+q}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}$	$\frac{px+qx+r}{(ax+b)(cx+d)^2} = \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{(cx+d)^2}$
	$\frac{px^2+qx+r}{(ax+b)(x^2+c)} = \frac{A}{ax+b} + \frac{Bx+C}{x^2+c}$	
Integration by substitution	1. Change all x to u. 2. Change du to dx 3. Integrate (should be easy) 4. Change all u back to x.	Note: Only use integration by substitution if the **question says so.**
Integration by parts	$\int_a^b u\frac{dv}{dx}\,dx = [uv]_a^b - \int_a^b v\frac{du}{dx}$ • Choice of function to be differentiated u follows the order: **LIATE** (logarithmic, inverse trigo, algebraic, trigo, exponential) • It can be applied twice on certain integrals.	Can be used for: • Integrating **simple functions** (logarithmic and inverse trigo functions) such as $\ln x$, $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ • Product of **different types of functions.** $uv - \int(v\frac{du}{dx})dx$, u: LIATE

$$\sin(A \pm B) \equiv \sin A\cos B \pm \cos A\sin B$$
$$\cos(A \pm B) \equiv \cos A\cos B \mp \sin A\sin B$$
$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$
$$\sin 2A \equiv 2\sin A\cos A$$
$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$$
$$\tan 2A \equiv \frac{2\tan A}{1 - \tan^2 A}$$
$$\sin P + \sin Q \equiv 2\sin\tfrac{1}{2}(P+Q)\cos\tfrac{1}{2}(P-Q)$$
$$\sin P - \sin Q \equiv 2\cos\tfrac{1}{2}(P+Q)\sin\tfrac{1}{2}(P-Q)$$
$$\cos P + \cos Q \equiv 2\cos\tfrac{1}{2}(P+Q)\cos\tfrac{1}{2}(P-Q)$$
$$\cos P - \cos Q \equiv -2\sin\tfrac{1}{2}(P+Q)\sin\tfrac{1}{2}(P-Q)$$

COOLING

$$\frac{dT}{dt} = k(T - T_0)$$

$$T(t) - T_0 = (T(0) - T_0)e^{kt}$$

C: $T(t) > T_0$
H: $T(t) < T_0$

$\frac{dT}{dt} = k(T - T_0)$ → T of obj, → of medium

$\int \frac{1}{T-T_0}\,dT = \int k\,dt$

$\ln(T - T_0) = kt + c$

$T - T_0 = \cancel{ke}\,e^{kt+c}$

$\therefore T(t) - T_0 = Ae^{kt}$, where $A = e^c$

when $t = 0$, $T(0) - T_0 = A$

$\therefore T(t) - T_0 = (T(0) - T_0)e^{kt}$

Heating:

$\frac{dT}{dt} = k(T_0 - T)$

$\int \frac{1}{T_0 - T}\,dT = \int k\,dt$

$-\ln(T_0 - T) = kt + c$

$T_0 - T = e^{-kt-c}$

$\therefore T - T_0 = Ae^{kt}$, where $A = e^c$

θ (0°)	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
0 (0°)	0	1	0
$\frac{\pi}{6}$ (30°)	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$ (45°)	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$ (60°)	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$ (90°)	1	0	–

DROPPING

$$m\frac{dv}{dt} = mg - bv$$

$$ve^{\frac{bt}{m}} = \int ge^{\frac{bt}{m}}\,dt = g\frac{m}{b}e^{\frac{bt}{m}} + c$$

1ST ORDER ODE

$$\frac{dy}{dx} + p(x)y = Q(x)$$

$$ye^{\int p(x)dx} = \int Q(x)e^{\int p(x)dx}\,dx$$

Integrating factor: $e^{\int p(x)dx}$

DECAY

$$x(t) = x(0)e^{kt}$$

$$x(t) = x(0)e^{-\frac{\ln 2}{\tau}t}$$

$\frac{dx}{dt} = kx$

$\int \frac{1}{x}\,dx = \int k\,dt$

$\ln(x) = kt + c$

$x = e^{kt+c}$

$x = Ae^{kt}$, where $A = e^c$

when $t = 0$, $x(t) = x(0)$

$\boxed{\therefore x(t) = x(0)e^{kt}}$

$x(\tau) = \frac{1}{2}x(0)$, where τ = half-life

$\therefore \frac{1}{2}x(0) = x(0)e^{k\tau}$

$\frac{1}{2} = e^{k\tau}$

$\ln(\frac{1}{2}) = k\tau$

$k = -\frac{\ln 2}{\tau}$

Hence, $x(t) = x(0)e^{-\frac{\ln 2}{\tau}t}$

2ND ORDER ODE

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0$$

λ_1 & λ_2 are sol^n.

$\boxed{A = -(\lambda_1 + \lambda_2) \quad B = \lambda_1\lambda_2}$

3 types:

1) 2 Distinct roots.
i.e. $y = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}$

2) 1 Distinct root
i.e. $y = C_1e^{\lambda x} + C_2xe^{\lambda x}$

3) 2 complex root
i.e. $y = e^{ax}(C_1\cos bx + C_2\sin bx)$
roots are $(a + bi)$ and $(a - bi)$
$\lambda_1 \qquad \lambda_2$

Reverse engineer: $A = -(\lambda_1 + \lambda_2) = \boxed{-2a}$
$B = \lambda_1\lambda_2 = \boxed{a^2 + b^2}$

For $\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0$

e.g. Solve $y'' + y' - 2y = 0$

$\boxed{\text{Let } y = e^{\lambda x}}$

$\therefore y' = \lambda e^{\lambda x}$

$y'' = \lambda^2e^{\lambda x}$

Hence, $\lambda^2e^{\lambda x} + \lambda e^{\lambda x} - 2e^{\lambda x} = 0$

$e^{\lambda x}(\lambda^2 + \lambda - 2) = 0$

$\lambda = 1, -2.$ (2 distinct)

Sol^n: $y(x) = C_1e^{(1)(x)} + C_2e^{(-2)(x)}$

Salt: $\frac{dQ}{dt}$ = Inflow $- \frac{Q}{\text{number}} \times \%$ → outflow rate

Amt salt

proportion of salt

BERNOULLI

$$y' + p(x)y = q(x)y^n$$

$y'y^{-n} + p(x)yy^{-n} = q(x)$

$y'(1-n)y^{-n} + (1-n)y^{-n}p(x)y = (1-n)q(x)$

Let $z = y^{1-n}$, then $\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$

$\boxed{\therefore z' + (1-n)p(x)z = (1-n)q(x)}$

↑ 1st order ODE