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Vector properties

u, v, w vectors, $c \& d \in R$

1) $u + v = v + u$
2) $u + (v+w) = (u+v)+w$
3) $u + 0 = u = 0 + u$
4) $u + (-u) = 0$
5) $c(du) = (cd)u$
6) $c(u+v) = cu + cv$
7) $(c+d)u = cu + du$
8) $1u = u$

Thm 3.1.6

1) $a \cdot b = b \cdot a$
2) $a \cdot (b+c) = a \cdot b + a \cdot c$
3) $(c_1 a) \cdot (c_2 b) = c_1 c_2 (a \cdot b)$
4) Not associative

Linear span

Let $S = \{u_1, u_2, \dots, u_k\} \in R^n$

The set of all linear combination of $u_1, u_2, \dots, u_k$ is span of S

e.g. show span$\{(1,0,1),(1,1,0),(0,1,1)\} = R^3$

ans: Exist $a, b, c \in R$: $(x,y,z) \in R^3$

$= a(1,0,1) + b(1,1,0) + c(0,1,1)$

$\left[\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 0 & 1 & 1 & y \\ 1 & 0 & 1 & z \end{array}\right] \to [\ ]$ see consistent or not

★ If REF no zero row, always consistent

Thm 3.2.7

Let $S = \{u_1, u_2, \dots, u_k\} \subseteq R^n$

If $k < n$, span$(S) \neq R^n$

Thm 3.2.9

Let $S = \{u_1, u_2, \dots, u_k\} \subseteq R^n$

1) $0 \in$ span(S)
2) For any $v_1, v_2, \dots, v_r \in$ span(S)
   $c_1 v_1 + c_2 v_2 + \dots + c_r v_r \in$ span(S)

Thm 3.2.10

span$(S_1) \subseteq$ span(S_2) iff each u_i is a LC of v_1, v_2, where $S_1 = \{u_1, u_2 \dots\}$ $S_2 = \{v_1, v_2 \dots\}$

e.g. span$(S_1) \subseteq$ span(S_2)

$\Rightarrow \left[\begin{array}{ccc|c|c|c} v_1 & v_2 & v_3 & u_1 & u_2 & u_3 \\ \text{col} & \text{col} & \text{col} & \text{col} & \text{col} & \text{col} \end{array}\right]$

see consistent or not

Redundant vector Thm 3.2.12

span$\{u_1, u_2, \dots, u_{k-1}\} =$ span$\{u_1, u_2, \dots u_{k-1}, u_k\}$ if u_k is LC of $u_1, u_2, \dots, u_{k-1}$

Subspace

Let V be subset of R^n

V is subspace of R^n if $V =$ span(S) where $S = \{u_1, \dots u_k\}$ $u_1, \dots, u_k \in R^n$

i.e.

1) Must contain zero vector
2) closed under addition & multiplication
   i.e. $\forall \vec{u}, \vec{v} \in V$, $c, d \in R$, $c\vec{u} + d\vec{v} \in V$

Thm 3.3.6 Sol^n set of homo eq^n is a subspace with n var in R^n

Linear Independence

$c_1 u_1 + c_2 u_2 + \dots + c_k u_k = 0$

If $c_1 = 0 = c_2 = c_k$ only trivial sol^n, then LI.

Thm 3.4.4

Let $S = \{u_1, u_2 \dots u_k\} \subseteq R^n$, $k \geq 2$

1) S is L.Dependent iff at least 1 vector $\in S$ can be written as LC of other vectors in S
2) S is L.Inde iff no vectors in S can be written as LC of other vectors in S.

Thm 3.4.7

$S = \{u_1 \dots u_k\} \subseteq R^n$

if $k > n$, S is L.dependent

Thm 3.4.10

$u_1, u_2, \dots u_k$ be L.Inde vectors in R^n

If u_{k+1} not LC of $u_1, u_2, \dots, u_k$, then $u_1, u_2, \dots, u_k, u_{k+1}$ are L.Inde

Vector space

V is vector space if either $V = R^n$ or V is a subspace of R^n

Bases:

1) S is Linearly inde.
2) S spans V.

e.g. show $S = \{(1,2,1),(2,9,0),(3,3,4)\}$ is basis for R^3

: $a(1,2,1) + b(2,9,0) + c(3,3,4) = (0,0,0)$

$\Rightarrow$ show only trivial sol^n $a = b = c = 0 \Rightarrow$ Lin Inde

show S spans $R^3 \Rightarrow$ REF S no zero row

Thm 3.5.11

Let S be basis for vector space $V \subseteq R^n$, $|S| = k$

Let $v_1, v_2, \dots v_r \in V$

1) $v_1, v_2, \dots, v_r$ are linearly depen. vectors in V iff $(v_1)_S, (v_2)_S, \dots, (v_r)_S$ are lin. dep. in R^k
   The other way Lin. Inde also true
2) span$\{v_1, v_2, \dots, v_r\} = V$ iff span$\{(v_1)_S, (v_2)_S, \dots, (v_r)_S\} = R^k$

Size of bases (Thm 3.6.1)

V is vector space with basis of k vectors

1) Any subset of V > k vectors $\Rightarrow$ Lin Depen
2) Any subset of V < k vectors cannot span V

Dimension

number of vectors in a basis for V.

dimen(zero space) $= 0$, basis for $\{0\} = \phi$

Thm 3.6.7 equivalent statement:

V be vector space of dimen k. S subset of V

1) S is basis for V
2) S is L.Inde and $|S| = k$
3) S spans V and $|S| = k$

To check if S is basis for V, check 2 of ~~above~~

(i) S is L.Inde, (ii) S spans V, (iii) $|S| = k$

Thm 3.6.9

U be subspace of vectorspace V.

dimen$(U) \leq$ dim(V)

if $U \neq V$, dim$(U) <$ dim(V)

Transition Matrix

Q is TM from T to S : $Q[w]_T = [w]_S$

Let $S = \{(1,0,-1),(0,-1,0),(1,0,2)\} = \{u_1, u_2, u_3\}$

$T = \{(1,1,1),(1,1,0),(-1,0,0)\} = \{v_1, v_2, v_3\}$

TM from S to T:

$$\begin{array}{c} \left[\begin{array}{ccc|c|c|c} 1 & 1 & -1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 2 \end{array}\right] \\ \begin{array}{cccccc} T & T & T & S & S & S \end{array} \end{array}$$

$\xrightarrow{G.E} \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{array}\right]$

$[u_1]_T$ $[u_2]_T$ $[u_3]_T$

TM from S to T:

$\begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix}$

Let w such that $(w)_S = (2,-1,2)$

$\therefore (w)_T = P(w)_S$

$= \begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = (2,-1,-3)$

★ TM. S to T $=$ (TM. T to S)$^{-1}$

Thm 3.7.5

S and T be 2 bases for vector space

P is TM from S to T

1) P is invertible
2) P^{-1} is TM from T to S

Row space

row space of A : span$\{r_1, r_2, \dots r_m\} \subseteq R^n$

$A = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix}$

Column space

$A = [c_1 \ c_2 \dots c_n]$

col space of A : span$\{c_1, c_2, \dots, c_n\} \subseteq R^m$

Col space of A = row space of A^T, vice versa

Thm 4.1.7

If Matrix A & B row equivalent, row space of A = row space of B.

Not true for col. space.

Basis of Row space

$A = [\ ] \xrightarrow{G.J.E} R = \begin{bmatrix} 2 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ } Span of non-zero rows are the row space.

$\therefore \{(2,2,1,1,1),(0,0,1,1,3)\}$ is basis for row space of A.

Basis of column space

$\begin{bmatrix} 1 & 4 & 7 & 10 & 13 \\ 2 & 5 & 8 & 11 & 14 \\ 3 & 6 & 9 & 12 & 15 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 4 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

basis: P P P

Put as ~~col vector, eg.~~

Thm 4.1.11

$[\ ] \to [P \ P \ P]$ pivot column in Result is Lin. Inde/Dep

Mean respective col in original also Lin. Inde/dep.

Thm 4.1.16

Lin system $Ax = b$ is consistent iff b lies in col space of A

Thm 4.2.1

Row space & col space of a Matrix same dimension

Rank

- No. of non-zero row or no. of pivot column of A.
- Rank(A) is dimen of its row space (or col space)
- Rank$(0) = 0$ Rank$(I_n) = n$ (identity)
- For $m \times n$ Matrix A, rank$(A) \leq$ min(m, n)
- If rank$(A) =$ min(m,n), A is full rank
- Full rank iff det $\neq 0$
- rank(A) = rank(A^T)
- $Ax = b$ consistent iff A and $[A|b]$ same rank.

Null space, Nullity

- Sol^n space of homogeneous linear system $Ax = 0$ is null space of A.
- Nullity of A is dim of null space of A

$\therefore$ Nullity(A) = dim (null space of A)
= number of arbi param
= no. of non-pivot col

A consistent $Ax = b$ has only 1 sol^n iff nullspace of $A = \{0\}$

because

Thm 4.2.8

Multiply matrices, rank go down

$A = m \times n$, $B = n \times p$

rank$(AB) \leq$ min(rank(A), rank(B))

Thm 4.3.4

Rank + nullity = number of columns.

Thm 4.3.6

$Ax = b$ has general sol^n

x = (a general sol^n for $Ax = 0$) + (1 particular sol^n to $Ax = b$)

Length

$\vec{u} = (a,b,c)$, $\vec{v} = (d,e,f)$

$\|\vec{u}\| = \sqrt{a^2 + b^2 + c^2} = \sqrt{\vec{u} \cdot \vec{u}}$

$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \sqrt{(a-d)^2 + (b-e)^2 + (c-f)^2}$

Angle

$\vec{u} - \vec{v}$

$\theta = \cos^{-1}\left(\dfrac{\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2}{2\|\vec{u}\| \|\vec{v}\|}\right) = \cos^{-1}\Big(\underbrace{\dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}}_{[-1,1]}\Big)$

Thm 5.1.5

1) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$, $\vec{w} \cdot (\vec{u} + \vec{v}) = \vec{w} \cdot \vec{u} + \vec{w} \cdot \vec{v}$
3) $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$
4) $\|c\vec{u}\| = $ abs$(c) \|\vec{u}\|$, because len > 0
5) $\vec{u} \cdot \vec{u} \geq 0$, $\vec{u} \cdot \vec{u} = 0$ iff $\vec{u} = 0$

Orthogonality

1) $\vec{u}$ and $\vec{v}$ orthogonal if $\vec{u} \cdot \vec{v} = 0$

Orthonormal : orthogonal + length = 1

Normalizing to len 1 preserves orthogonality.

Thm 5.2.4 orthogonality implies Lin. Independence.

Standard Matrix: transformation for standard vectors.

i.e. $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right)$ S.M. is $\left[T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \ T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \ T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]$

and $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = [S.M.] \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Commute
A & B commute if $AB = BA$

Scalar Matrix
- Diagonal and all diagonal entries same
$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Upper △
- square, $a_{ij} = 0$ when $i > j$
$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix}$
- Sum & product of 2 upper △ is also upper △.
- Inverse of upper △ also upper △

Lower: $a_{ij} = 0$ when $i < j$

Power
$A^0 = I \quad A^m A^n = A^{m+n}$
$A^{-n} = (A^{-1})^n$

Transpose
~~Thm 2.2.22~~
1) $(A^T)^T = A$
2) $(A+B)^T = A^T + B^T$ size must fit
3) If c scalar, $(cA)^T = cA^T$
4) $(AB)^T = B^T A^T$

} Thm 2.2.22

5) $\det(A^T) = \det(A)$ (Thm 2.5.10)

Identical Rows/col
- $\det = 0$ (2.5.12)

e.g. $V = \{(a,b,c,d) \in \mathbb{R}^4 \mid a+c=0,\ b+c+2d=0\}$
$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 2 \end{bmatrix}$ REF rank 2. Rank is codimension of V.
$\dim V + \operatorname{codim} V = \dim E$
$= 4$ (in this case)
$\therefore \dim V = 4 - 2 = 2$

Diagonalizable
1) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 2) $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ 3) $\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$ 4) $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ 5) $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

Not diagonalizable Matrix:
1) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ 2) $\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$

Orth. diagonalizable: $\begin{bmatrix} 1 & 3 \\ 3 & 8 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Not orthogonally diagonalizable: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Inverse
iff $\det() \neq 0$ (2.5.19)
$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

A & B invertible, c is scalar $\neq 0$:
1) cA is invertible, $(cA)^{-1} = \frac{1}{c}A^{-1}$
2) A^T is invertible $(A^T)^{-1} = (A^{-1})^T$
3) A^{-1} is invertible $(A^{-1})^{-1} = A$
4) AB is invertible $(AB)^{-1} = B^{-1}A^{-1}$
$\Rightarrow (A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} \cdots A_2^{-1} A_1^{-1}$
5) A^n is invertible $(A^n)^{-1} = (A^{-1})^n$

} Thm 2.3.9

Find inverse:
$(A \mid I) \longrightarrow (I \mid A^{-1})$

Singular:
- iff $\det = 0$
- Means RREF at least one zero row
e.g. $\begin{bmatrix} \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \det = 0$
- If A is singular, AC is singular (2.4.14)

Adjoints
$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A)$ (Thm 2.5.25)
$= \det(A^{-1})\operatorname{adj}(A)$

e.g. $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$

$\operatorname{adj}(A) = \begin{bmatrix} \begin{vmatrix} -1 & 0 \\ 0 & 3 \end{vmatrix} & - \begin{vmatrix} 0 & 0 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} \\ - \begin{vmatrix} -1 & 1 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} & - \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix} & - \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix} \end{bmatrix}^{T}$ ☆

Crammer Rule:
$\underbrace{\begin{bmatrix} 1 & 1 & 3 \\ 2 & -2 & 2 \\ 3 & 9 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix}}_{B}$

$x = \frac{1}{\det(A)}\begin{bmatrix} \det(A_1) \\ \vdots \\ \det(A_n) \end{bmatrix} \to X, Y, Z$

$X = \frac{1}{\det(A)}\det(A_1)$, $Y = \frac{1}{\det(A)}\det(A_2)$

$X = \frac{1}{\det(A)}\begin{vmatrix} \boxed{0} & 1 & 3 \\ \boxed{4} & -2 & 2 \\ \boxed{3} & 9 & 0 \end{vmatrix}$ $Y = \frac{1}{\det(A)}\begin{vmatrix} 1 & \boxed{0} & 3 \\ 2 & \boxed{4} & 2 \\ 3 & \boxed{3} & 0 \end{vmatrix}$

$Z = \frac{1}{\det(A)}\begin{vmatrix} 1 & 1 & \boxed{0} \\ 2 & -2 & \boxed{4} \\ 3 & 9 & \boxed{3} \end{vmatrix}$

Thm 5.2.8.1
(special case of 5.2.15 when $\vec{w}$ is contained in V)

Let $S = \{\vec{u}_1, \ldots, \vec{u}_k\}$ be orthogonal basis for V.
For any $\vec{w} \in V$,
$\vec{w} = \frac{\vec{w}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{w}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2 + \cdots + \frac{\vec{w}\cdot\vec{u}_k}{\vec{u}_k\cdot\vec{u}_k}\vec{u}_k$

(Thm 5.2.8.2) for orthonormal basis, when denominator = 1

Thm 5.2.15.1
V be subspace of $\mathbb{R}^n$, $\{\vec{u}_1, \ldots, \vec{u}_k\}$ is orthogonal basis for V. ☆
For any $\vec{w} \in \mathbb{R}^n$
$\frac{\vec{w}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{w}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2 + \cdots + \frac{\vec{w}\cdot\vec{u}_k}{\vec{u}_k\cdot\vec{u}_k}\vec{u}_k$ is the projection of $\vec{w}$ onto V.

Thm 5.2.15.2 same but orthonormal basis

Gram-Schmidt Process (Thm 5.2.19)
Turns basis to orthogonal basis
Let $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ be basis for V
$\vec{v}_1 = \vec{u}_1$
$\vec{v}_2 = \vec{u}_2 - \frac{\vec{u}_2\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1$
$\vec{v}_3 = \vec{u}_3 - \frac{\vec{u}_3\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 - \frac{\vec{u}_3\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2$
$\vdots$
Then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is orthogonal basis for V.

Least square sol^n
A vector $\vec{u} \in \mathbb{R}^n$ is LSS to $Ax = b$
iff $\|\vec{b} - A\vec{u}\| \le \|\vec{b} - A\vec{v}\|\ \forall \vec{v} \in \mathbb{R}^n$
i.e. find $\vec{u}$ that minimize $\|\vec{b} - A\vec{u}\|$

Thm 5.3.8
$\vec{u} \in \mathbb{R}^n$ is LSS to $Ax = b$
iff $\vec{p} = A\vec{u}$ is best approx. of $\vec{b}$ onto column space of A
iff $\vec{p} = A\vec{u}$ is projection of $\vec{b}$ onto column space of A.

e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ $V = \operatorname{span}\{[1\ 0\ 1], [0\ 1\ 1]\}$
After calc, projection of b onto V is $\vec{p} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$
$\therefore$ by Thm 5.3.8, $x = \begin{bmatrix} x \\ y \end{bmatrix}$ is LSS to $Ax = b$
iff $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

Thm 5.3.10
$\vec{u}$ is LSS to $Ax = b$ iff $\vec{u}$ is a sol^n to $A^T A x = A^T b$

Orthogonal Matrices
Matrix A if $A^{-1} = A^T$

Thm 5.4.6
square A. Equivalent:
1) A is orthogonal
2) Rows of A form orthonormal basis for $\mathbb{R}^n$
3) Col. of A form orthonormal basis for $\mathbb{R}^n$

Thm 5.4.7
S and T are orthonormal bases
P is TM from S to T.
1) P is orthogonal
2) P^T is TM from T to S.

To find eigenvector: 1st find eigenvalue. Then solve $(\lambda I - A)\vec{x} = 0$. $\vec{x}$ is all possible eigenvector

Eigenvalues, Eigenvectors
A is square Matrix order n.
$\vec{u} \in \mathbb{R}^n$, $\vec{u} \neq 0$ is eigenvector of A
if $A\vec{u} = \lambda\vec{u}$, for scalar λ
scalar λ is eigenvalue of A, can be 0.

Find eigenvalues
λ is eigenvalue of A
iff $A\vec{u} = \lambda\vec{u}$ for non-zero col vector $\vec{u} \in \mathbb{R}^n$
iff $\lambda\vec{u} - A\vec{u} = 0$ iff $(\lambda I - A)\vec{u} = 0$
iff linear system $(\lambda I - A)x = 0$ has non trivial sol^n iff $\det(\lambda I - A) = 0$ (Thm 6.1.11)
(bcos eigenvector $\neq 0$, $\therefore x \neq 0$, $\therefore$ must have non-trivial sol^n)
$P^{-1}BP = D$, P is matrix that diagonalize B, D is a diagonal Matrix.
λ eigenvalue of $A \Rightarrow \lambda$ e.v. A^T

Thm 6.1.9 △ Matrix eigenvalues diagonal entry

Characteristic polynomial
$\det(\lambda I - A)$
char. eqn: $\det(\lambda I - A) = 0$ (λ is var)

Thm 6.1.8. equivalence
1) A is invertible
2) Linear system $Ax = 0$ has only trivial sol^n
3) RREF of A is Identity
4) A can be expressed as product of elementary Matrices
5) Rows of A form basis for $\mathbb{R}^n$
6) Col of A form basis for $\mathbb{R}^n$
7) $\det(A) \neq 0$
8) $\operatorname{rank}(A) = n$ 9) 0 is not eigenvalue of A
(bcos $A\vec{x} = \lambda\vec{x} = 0\vec{x}$)

Eigenspace
λ is eigenvalue of A $\longrightarrow$ Null space of $(\lambda I - A)$
Sol^n space of $(\lambda I - A)x = 0$, denoted E_λ

Diagonalizable
if there exists an invertible Matrix P such that $P^{-1}AP$ is diagonal Matrix
P is said to diagonalize A. zero matrix diagonalizable

Thm 6.2.3
A is square order n. A is diagonalizable iff A has n linearly independent eigenvector.

Algo 6.2.4 diagonalize a Matrix
1) Find all distinct eigenvalues (solve characteristic eqn)
2) For each eigenvalue λ_i, find a basis S_{λ_i} for eigenspace
3) Let $S = S_{\lambda_1} \cup S_{\lambda_2} \cup S_{\lambda_3} \cdots \cup S_{\lambda_k}$
a) If $|S| < n$ A is not diagonalizable (A is order n)
b) If $|S| = n$ A is diagonalizable, $P = [\vec{u}_1\ \vec{u}_2 \cdots \vec{u}_n]$ where $S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ is invertible Matrix that diagonalizes A.

Thm 6.2.7
If A has n distinct eigenvalues then A is diagonalizable

Orthogonal diagonalization
Orthogonal diagonalizable if $\exists$ orthogonal Matrix P (i.e. $P^{-1} = P^T$) such that $P^T A P$ is diagonal Matrix
Orthogonal diagonalizable iff A is symmetric (6.3.4) i.e. $A^T = A$

Algo 6.3.5 orthogonally diagonalize Matrix
Same as 6.2.4, but step 2 after find basis S_{λ_i}, use Gram-Schmidt transform to orthonormal basis.
NOT orthogonal

Linear Transform
$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = [\text{standard Matrix}]\begin{bmatrix} x \\ y \end{bmatrix}$

Thm 7.1.4
$T: \mathbb{R}^n \to \mathbb{R}^m$
1) $T(0) = 0$ (2) If $\vec{u}_1, \ldots, \vec{u}_k \in \mathbb{R}^n$, then
$T(c_1\vec{u}_1 + \cdots + c_k\vec{u}_k) = c_1 T(\vec{u}_1) + \cdots + c_k T(\vec{u}_k)$

Thm 7.1.11
$T \circ S$ is also L. Transformation if S and T are L.T.

Range
$T: \mathbb{R}^n \to \mathbb{R}^m$
Range, $R(T) = \{T(\vec{u}) \mid \vec{u} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$

Thm 7.2.4
$T: \mathbb{R}^n \to \mathbb{R}^m$, A is standard Matrix for T.
Then $R(T)$ = col space of A.

Ranks
$T: \mathbb{R}^n \to \mathbb{R}^m$
$\operatorname{rank}(T)$ is dim of $R(T)$ (range)
If A is standard Matrix for T,
$\operatorname{rank}(T) = \dim(R(T))$
$= \dim(\text{col. space of } A)$ (7.2.4)
$= \operatorname{rank}(A)$

Kernels
is the set of vectors in $\mathbb{R}^n$ whose image is 0 vector in $\mathbb{R}^m$.
$\operatorname{Ker}(T) = \{\vec{u} \mid T(\vec{u}) = \vec{0}\} \subseteq \mathbb{R}^n$
= null space of standard Matrix (7.2.9)

Nullities
$\operatorname{Nullity}(T) = \dim(\operatorname{Ker}(T))$
$= \dim(\text{null space of } A)$ (7.2.9) ↗ standard Matrix
$= \operatorname{nullity}(A)$

7.2.12 Dimension Thm
$T: \mathbb{R}^n \to \mathbb{R}^m$
$\operatorname{rank}(T) + \operatorname{nullity}(T) = n$, not m.

Determinant
1) $\det(A^T) = \det(A)$ (2.5.10)
2) Multiply row by scalar, det also multiply (2.5.15) (row only)
3) Row swap = $-\det$ (2.5.15)
4) ⊕ or ⊖ operation det no change (2.5.15)
5) $\det(AB) = \det(A)\det(B) = \det(BA)$ (2.5.22.2)
6) $\det(A^{-1}) = \frac{1}{\det(A)}$ (2.5.22.3)
7) $\det(A^n) = \det(A)^n$ (8) $\det(cA) = c^n \det(A)$ (2.5.22)