Excellence Cluster Universe

How to do units and scaling right (Or at least how to NOT do them wrong)

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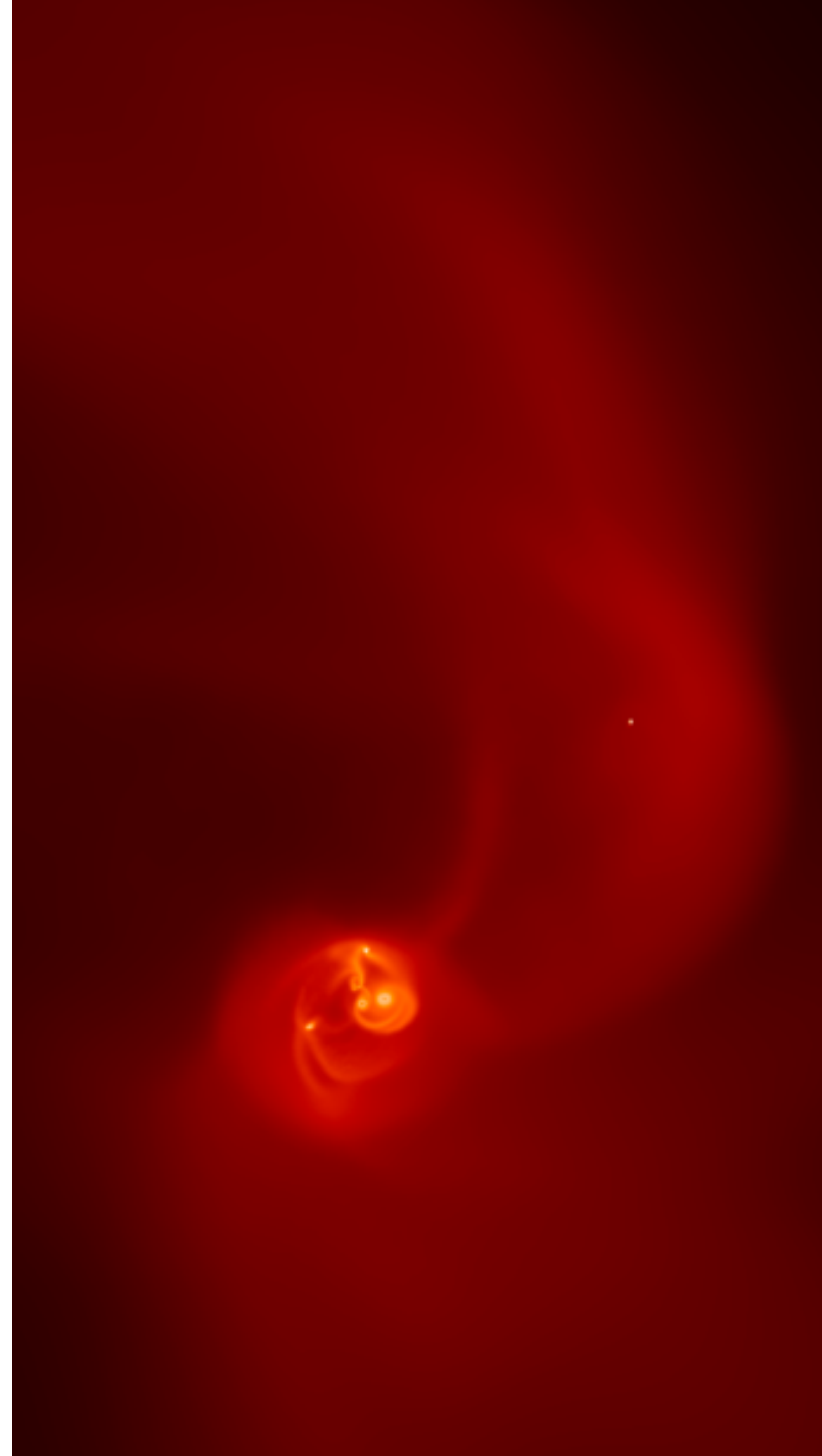

So, what's the wrong and right way to do units and why?

- Use physical units of the completely wrong scale (e.g. cm, s)

Very wrong!

- Use physical units of a more appropriate scale (e.g. pc, Myr)

Better but still wrong

- Use arbitrary dimensionless units (i.e. rely on the user to calculate any scale factors)

Correct, but more difficult than it needs to be

- Use dimensionless units where the scale factors are calculated by the code automatically

Yes!

What are dimensionless units exactly?

- **Dimensionless units** are a system of units where a physical quantity can be converted to an equivalent dimensionless quantity by way of a **scaling factor**, e.g.
- For example, take the quantity a which has a scale factor of A_0, then we can convert this to an equivalent dimensionless quantity, a′ :

$$a' = \frac{a}{A_0} \quad \Longleftrightarrow \quad a = A_0\, a'$$

- Dimensionless units themselves **can always be converted back to the real, physical units** at any time
- They are different to **dimensionless constants** (such as e, π, etc..) which are truly dimensionless and are indepedent of any external system of units.

Why use dimensionless units? Floating point precision

- One of the biggest arguments for using dimensionless units is the effect of **finite floating point precision**
- The single precision floating point range is $10^{-38} - 10^{+38}$
- Many astronomers love to **continue to use cgs units** (even in simulations) even though **they're completely inappropriate for almost any astrophysical context**
- Using cgs (or SI or similar non-astronomical units) can cause floating point precision in various situations, e.g.
- Computing volumes, e.g. a parsec size box volume,

$$1\,\mathrm{pc}^3 = 2.93 \times 10^{49}\,\mathrm{m}^3 = 2.93 \times 10^{55}\,\mathrm{cm}^{-3}$$

- Computing quadrupole (or higher-order) correction terms in the gravity tree

$$\frac{1}{r^5} = \frac{1}{pc^5} = 3.57 \times 10^{-83}\,\mathrm{m}^{-5} = 3.57 \times 10^{-93}\,\mathrm{cm}^{-5}$$

Relations between dimensionless quantities

- Although we have freedom to choose a base set of dimensionless units (e.g. mass, length and time), derived quantities that use various combinations of these units must be consistent

$$r' = \frac{r}{R_0} \qquad m' = \frac{m}{M_0} \qquad t' = \frac{t}{T_0}$$

- Velocity units, for example, would be a combination of length and time units, i.e.

$$v' = \frac{v}{V_0} = v\,\frac{T_0}{R_0} \qquad \text{where} \quad V_0 = \frac{R_0}{T_0}$$

- and acceleration units :

$$a' = \frac{a}{A_0} = a\,\frac{T_0^2}{R_0} \qquad \text{where} \quad A_0 = \frac{R_0}{T_0^2}$$

Using dimensionless units to 'eliminate' physical constants

- Many interesting physical problems involves some physical constant, such as G, c, μ_0, etc..
- We can chose a set of dimensionless units such that **the physical constant in the new set of units is unity**
- Various advantages to this
  - If chosen correctly, all physical quantities will be close to unity (better for summing floating point numbers, easier to spot 'incorrect' numbers
  - Completely factors out needing to multiply by the constant (can save a little CPU time at least)

Example : Setting G = 1

- One common example of setting physical constants to unity is in gravitational problems. N-body codes would often employ a system of units that sets G = 1
- Substituting for our dimensionless units and rearranging

$$a = \frac{G\,m}{r^2} \qquad \Rightarrow \qquad \frac{R_0}{T_0^2}\,a' = \frac{G\,M_0\,m'}{R_0^2\,r'^2} \qquad \Rightarrow \qquad a' = \underbrace{\left\{\frac{G\,M_0\,T_0^2}{R_0^3}\right\}}_{G'} \frac{m'}{r'^2}$$

- The final equation in dimensionless form is similar with the constants grouped together, which we've called G′. If we wish to effectively set G′ = 1, then this imposes a constraint on one of our quantities. Traditionally, this has been the time variable :

$$\frac{G\,M_0\,T_0^2}{R_0^3} = 1 \qquad \Rightarrow \qquad T_0 = \left(\frac{R_0^3}{G\,M_0}\right)^{1/2}$$

- Typical units (in N-body and star formation problems) would select R_0 = 1pc and M_0 = 1 solar mass. What does the time unit, T_0 , come out as?

Example : Setting G = 1

$$T_0 = 4.7 \times 10^{14}\,\mathrm{s} = 14.91\,\mathrm{Myr}$$

- This has a knock-on effect on any other unit that has time as a dimension
- e.g. Velocity unit

$$V_0 = \frac{R_0}{T_0} = 0.065\,\mathrm{km\,s^{-1}} = 0.067\,\mathrm{pc\,Myr^{-1}}$$

Another example : Setting $k_b/m_h = 1$

- In hydrodynamics, if we wish to convert from internal energy to temperature, we must use the Boltzmann constant, k_b, and the mass of a hydrogen atom, m_h.
- Similar to setting G = 1, we can set the collection of constants in the sound speed equation to unity to set an appropriate unit for temperature, i.e.

$$c^2 = \gamma \frac{k_b\, T}{\bar{m}} \quad \Rightarrow \quad V_0^2\, {c'}^2 = \gamma\, \frac{k_b\, \theta_0\, T'}{m_h\, \bar{\mu}} \quad \Rightarrow \quad {c'}^2 = \gamma\, \underbrace{\left\{ \frac{k_b\, \theta_0}{m_h\, V_0^2} \right\}}_{1} \frac{T'}{\bar{\mu}}$$

$$\frac{k_b\, \theta_0}{m_h\, V_0^2} = 1 \quad \Rightarrow \quad \theta_0 = V_0^2\, \frac{m_h}{k_b}$$

- Using the same typical star formation units we employed for the G = 1 example, we fine that

$$\theta_0 = 0.51\,\mathrm{K}$$

Units in GANDALF

- We created a 'SimUnit' class in GANDALF to hold and compute all required scaling factors
- Each required unit class inherits from this base class

```
class SimUnit
{
 public:

  SimUnit();
  virtual ~SimUnit() {};

  virtual DOUBLE SIUnit(string) = 0;
  virtual string LatexLabel(string) = 0;
  DOUBLE OutputScale(string);

  DOUBLE inscale;
  DOUBLE inSI;
  DOUBLE outcgs;
  DOUBLE outscale;
  DOUBLE outSI;
  string inunit;
  string outunit;
};
```

```
class LengthUnit: public SimUnit
{
 public:
  LengthUnit() : SimUnit() {};
  DOUBLE SIUnit(string);
  string LatexLabel(string);

};

class MassUnit: public SimUnit
{
 public:
  MassUnit() : SimUnit() {};
  DOUBLE SIUnit(string);
  string LatexLabel(string);

};

etc..
```

Units in GANDALF

- An all-encompasing class called 'SimUnits' (there's an extra 's') which then holds everything in one place :

```
class SimUnits
{
 public:

  SimUnits();
  ~SimUnits();

  void SetupUnits(Parameters *);
  void OutputScalingFactors(Parameters *);

  int dimensionless;                    ///< Are we using dimensionless units?
  bool ReadInputUnits;                  ///< Are input units read from snapshot?


  // Instances of all unit classes
  //-----------------------------------------------------------------------
  LengthUnit r;                         ///< Length unit
  MassUnit m;                           ///< Mass unit
  TimeUnit t;                           ///< Time unit
  VelocityUnit v;                       ///< Velocity unit
  AccelerationUnit a;                   ///< Acceleration unit
  DensityUnit rho;                      ///< Density unit
  etc..
};
```

Input units vs Output units

- GANDALF is designed to handle simulataneously an input and output set of units
- For example, maybe you are reading in initial conditions in one set of units (e.g. pcs, Myr) but want to output in a different set (e.g. au, yr)
- However, in most cases, you will generate initial conditions with the same set of units OR create initial conditions on the fly
- Your choice of output units will often be set in the parameters file

```
#---------------------------
# Simulation units variables
#---------------------------
Use physical units                             : dimensionless = 0
Length units                                   : routunit = pc
Mass units                                     : moutunit = m_sun
Time units                                     : toutunit = myr
Velocity units                                 : voutunit = km_s
Density units                                  : rhooutunit = g_cm3
Temperature units                              : tempoutunit = K
Specific internal energy units                 : uoutunit = J_kg
Angular velocity unit                          : angveloutunit = rad_s
```

Computing the scaling factors

- In order to simpify the calculation of scaling factors, **we calculate everything in the same set of units internally**. We use **SI units** (although we could have chosen cgs or another set if we wished)

$$R_0 = \underbrace{R_{\mathrm{outscale}}}_{\text{Converts code units to requested output units}} \times \overbrace{R_{\mathrm{outSI}}}^{\text{Converts requested output units to SI units}}$$

- If we have a different set of input and output units, then both of these should be consistent with each other, i.e.

$$R_0 = R_{\mathrm{outscale}} \times R_{\mathrm{outSI}} = R_{\mathrm{inscale}} \times R_{\mathrm{inSI}}$$

Length and mass units

- As discussed earlier, we select the length and mass units directly so these are trivial and are set as 1.0 each. We also need to compute the SI and cgs factors

```
// Length units
//-----------------------------------------------------------
r.inunit   = params->stringparams["rinunit"];
r.outunit  = params->stringparams["routunit"];
r.inSI     = r.SIUnit(params->stringparams["rinunit"]);
r.outSI    = r.SIUnit(params->stringparams["routunit"]);
r.outcgs   = 100.0*r.outSI;
r.outscale = 1.0;
r.inscale  = r.outscale*r.outSI/r.inSI;
```

$$R_{\text{inscale}} = \frac{R_{\text{outscale}} \times R_{\text{outSI}}}{R_{\text{inSI}}}$$

Mass units

- As discussed earlier, we select the length and mass units directly so these are trivial and are set as 1.0 each

```
// Mass units
//-------------------------------------------------------
m.inunit   = params->stringparams["minunit"];
m.outunit  = params->stringparams["moutunit"];
m.inSI     = m.SIUnit(params->stringparams["minunit"]);
m.outSI    = m.SIUnit(params->stringparams["moutunit"]);
m.outcgs   = 1000.0*m.outSI;
m.outscale = 1.0;
m.inscale  = m.outscale*m.outSI/m.inSI;
```

Time units

- The time units are computed to ensure G = 1 as described earlier

```
// Time units
//-----------------------------------------------------------------------
t.inunit   = params->stringparams["tinunit"];
t.outunit  = params->stringparams["toutunit"];
t.inSI     = t.SIUnit(params->stringparams["tinunit"]);
t.outSI    = t.SIUnit(params->stringparams["toutunit"]);
t.inscale  = pow(r.inscale*r.inSI,1.5)/sqrt(m.inscale*m.inSI*G_const);
t.inscale  /= t.inSI;
t.outscale = pow(r.outscale*r.outSI,1.5)/sqrt(m.outscale*m.outSI*G_const);
t.outscale /= t.outSI;
t.outcgs   = t.outSI;
```

$$T_0 = \left(\frac{R_0^3}{G\,M_0}\right)^{1/2} \quad \Rightarrow \quad T_{\rm outscale} \times T_{\rm outSI} = \left(\frac{R^3_{\rm outscale} \times R^3_{\rm outSI}}{G_{\rm SI}\,M_{\rm outscale} \times M_{\rm outSI}}\right)^{1/2}$$

$$T_{\rm outscale} = \left(\frac{R^3_{\rm outscale} \times R^3_{\rm outSI}}{G_{\rm SI}\,M_{\rm outscale} \times M_{\rm outSI}}\right)^{1/2} \frac{1}{T_{\rm outSI}}$$

Converting initial conditions and input parameters to code units

- Converting initial conditions and/or parameters is simply a case of using the inscale/outscale variables.
- e.g. if you input velocities in km/s, then

```
Velocity units                                  : voutunit = km_s
```

- Then all velocity quantities are scaled using

$$v' = \frac{v(\mathrm{km\_s})}{V_{\mathrm{outscale}}}$$

- If you have a parameter that is in either SI or cgs, then also divide by the SI/cgs factor

$$rho' = \frac{rho(\mathrm{g\,cm^{-3}})}{rho_{\mathrm{outscale}} \times rho_{\mathrm{outcgs}}}$$

Code units vs output units

- GANDALF only uses these dimensionless units internally for computing quantities internally
- When generating output, such as snapshot files, all quantities are converted back to physical units (specified by the outscale selections)

$$v(\mathrm{km\_s}) = V_{\mathrm{outscale}} \times v'$$

- For example, when outputting to file :

```
outfile << part.r[0]*simunits.r.outscale << "   "
        << part.v[0]*simunits.v.outscale << "   "
        << part.m*simunits.m.outscale << "   "
        << part.h*simunits.r.outscale << "   "
        << part.rho*simunits.rho.outscale << "   "
        << part.u*simunits.u.outscale << "   "
        << endl;
```

So, some basic rules about chosing units for your code

- Obviously, choose some set of units that is representive of your problem. Having a nice dimensionless framework means **nothing if you still chose strange units**
- **And that's about it!**