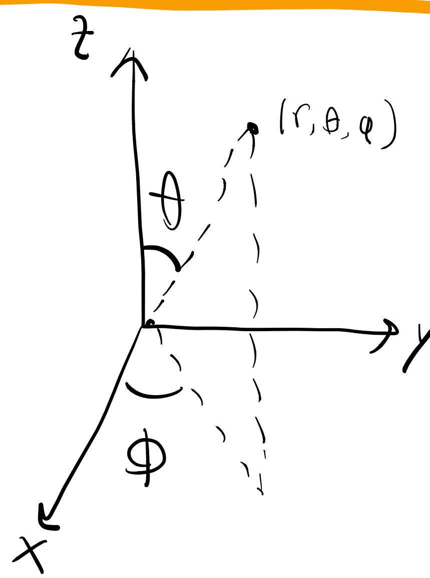
Reduced 4-body (Spherical)

Step 0 : L



$$T = \frac{1}{2} m_s v^2$$

$$= \frac{1}{2} m_s \left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\varphi}^2\right)$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$
$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\hat{\mathbf{r}}}$$
$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\left(\frac{d\hat{\mathbf{r}}}{dr}\frac{d\mathbf{r}}{dt} + \frac{d\hat{\mathbf{r}}}{d\theta}\frac{d\theta}{dt} + \frac{d\hat{\mathbf{r}}}{d\phi}\frac{d\phi}{dt}\right)$$
$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\left(0 + (\cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}})\,\dot{\theta} + (-\sin\theta\sin\phi\hat{\mathbf{x}} + \sin\theta\cos\phi\hat{\mathbf{y}})\,\dot{\phi}\right)$$
$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\left(\hat{\theta}\dot{\theta} + \dot{\phi}\sin\theta\hat{\phi}\right)$$

http://web.physics.ucsb.edu/~fratus/phys103/Disc/disc_notes_3_pdf.pdf

Deriving v^2

1. Use $\bar{r} = r\hat{r}$

$$\bar{v} = \frac{d}{dt}(r\hat{r}) = \dot{r}\hat{r} + r\dot{\hat{r}}$$

2. Alternatively use $d\bar{r}$ from below:

$$v = \dot{\bar{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\varphi}\hat{\varphi}$$

$$\Rightarrow v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2$$

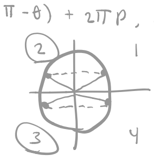
$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

$(\theta$ and $-\theta) + 2\pi \cdot p$, $p \in \mathbb{Z}$

but we only need $+\theta$, $p = 0$

$$r = \sqrt{x^2+y^2+z^2}$$
$$\theta = \arccos\left(\frac{z}{r}\right) \in [0 ; \pi] \checkmark$$
$$\varphi = \arctan\left(\frac{y}{x}\right) \in \left[-\frac{\pi}{2} ; \frac{\pi}{2}\right] \text{ NB}$$

$(\theta$ and $\pi - \theta) + 2\pi p$, $p \in \mathbb{Z}$



$x < 0 \Rightarrow Q3 + Q4$

$\rightarrow x < 0 \Rightarrow \pi - \arcsin\left(\frac{y}{x}\right)$

$$V = -G m_s \sum_i \frac{M_i}{|\bar{r} - \bar{r}_i|}$$

$$L = \frac{1}{2} m_s\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right) + G m_s \sum_i \frac{M_i}{|\bar{r} - \bar{r}_i|}$$

$$\frac{\partial \bar{r}}{\partial r} = \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z}$$
$$\left|\frac{\partial \bar{r}}{\partial r}\right| = 1$$

$$\frac{\partial \bar{r}}{\partial \theta} = r\cos\theta\cos\varphi\,\hat{x} + r\cos\theta\sin\varphi\,\hat{y} - r\sin\theta\,\hat{z}$$
$$\left|\frac{\partial \bar{r}}{\partial \theta}\right| = r$$

$$\frac{\partial \bar{r}}{\partial \varphi} = -r\sin\theta\sin\varphi\,\hat{x} + r\sin\theta\cos\varphi\,\hat{y}$$
$$\left|\frac{\partial \bar{r}}{\partial \varphi}\right| = r\sin\theta$$

$$x\hat{x} + y\hat{y} + z\hat{z} = \bar{r}(r,\theta,\varphi) = r\sin\theta\cos\varphi\,\hat{x} + r\sin\theta\sin\theta\,\hat{y} + r\cos\theta\,\hat{z}$$

$$\frac{\partial \bar{r}}{\partial r} \Big/ \left|\frac{\partial \bar{r}}{\partial r}\right| = \hat{r} = \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z}$$
$$\frac{\partial \bar{r}}{\partial \theta} \Big/ \left|\frac{\partial \bar{r}}{\partial \theta}\right| = \hat{\theta} = \cos\theta\cos\varphi\,\hat{x} + \cos\theta\sin\varphi\,\hat{y} - \sin\theta\,\hat{z}$$
$$\frac{\partial \bar{r}}{\partial \varphi} \Big/ \left|\frac{\partial \bar{r}}{\partial \varphi}\right| = \hat{\varphi} = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}$$

$$d\bar{r} \quad \sum \frac{\partial \bar{r}}{\partial x_i} dx_i = \sum \left|\frac{\partial \bar{r}}{\partial x_i}\right| \frac{\frac{\partial \bar{r}}{\partial x_i}}{\left|\frac{\partial \bar{r}}{\partial x_i}\right|} \cdot dx_i = \sum \left|\frac{\partial \bar{r}}{\partial x_i}\right| dx_i\,\hat{x}_i = d\bar{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\varphi\,\hat{\varphi}$$

step 1: q_i

$$p_r = \frac{\partial L}{\partial \dot{r}} = m_s \dot{r}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m_s r^2 \dot{\theta}$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m_s r^2 \sin^2\theta \, \dot{\varphi}$$

step 2: $\dot{q}_i = \dot{q}_i(\bar{q}, \bar{p}, t)$

$$\dot{r} = \frac{p_r}{m_s}$$

$$\dot{\theta} = \frac{p_\theta}{m_s r^2}$$

$$\dot{\varphi} = \frac{p_\varphi}{m_s r^2 \sin^2\theta}$$

$$\Rightarrow T = \frac{1}{2} m_s \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\varphi}^2\right)$$

$$= \frac{1}{2} m_s \left(\frac{p_r^2}{m_s^2} + r^2 \frac{p_\theta^2}{m_s^2 r^4} + r^2 \sin^2\theta \frac{p_\varphi^2}{m_s^2 r^4 \sin^4\theta}\right)$$

$$= \frac{p_r^2}{2m_s} + \frac{p_r^2}{2m_s r^2} + \frac{p_\varphi^2}{2m_s r^2 \sin^2\theta}$$

$$\Rightarrow L = \frac{p_r^2}{2m_s} + \frac{p_r^2}{2m_s r^2} + \frac{p_\varphi^2}{2m_s r^2 \sin^2\theta} + G m_s \sum_i \frac{M_i}{|\bar{r} - \bar{r}_i|}$$

step 3: H

$$H = \sum p_i \dot{q} - L$$

$$H = \frac{p_r^2}{m_s} + \frac{p_\theta^2}{m_s r^2} + \frac{p_\varphi^2}{m_s r^2 \sin^2\theta} - \frac{p_r^2}{2m_s} - \frac{p_r^2}{2m_s r^2} - \frac{p_\varphi^2}{2m_s r^2 \sin^2\theta} - G m_s \sum_i \frac{M_i}{|\bar{r} - \bar{r}_i|}$$

$$= \frac{p_r^2}{2m_s} + \frac{p_\theta^2}{2m_s r^2} + \frac{p_\varphi^2}{2m_s r^2 \sin^2\theta} - G m_s \sum_i \frac{M_i}{|\bar{r} - \bar{r}_i|}$$

Step 4: Hamilton's eqns of motion

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$\dot{r} = \frac{p_r}{m_s}$$

$$\dot{\theta} = \frac{p_\theta}{m_s r^2}$$

$$\dot{\varphi} = \frac{p_\varphi}{m_s r^2 \sin^2\theta}$$

$|\bar{r} - \bar{r}_i|$:

$$d_i = |\bar{r} - \bar{r}_i| = \sqrt{(x-x_i)^2 + (y-y_i)^2 + (z-z_i)^2}$$

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

$$(x-x_i)^2 = (r\sin\theta\cos\varphi - r\sin\theta_i\cos\varphi_i)^2$$
$$= r^2\sin^2\theta\cos^2\varphi + r_i^2\sin^2\theta_i\cos^2\varphi_i - 2rr_i\sin\theta\sin\theta_i\cos\varphi\cos\varphi_i$$

$$(y-y)^2 = r^2\sin\theta^2\sin\varphi^2 + r_i^2\sin^2\theta_i\sin\varphi_i - 2rr_i\sin\theta\sin\theta_i\sin\varphi\sin\varphi_i$$

$$(z-z_i)^2 = \sqrt{r^2\cos^2\theta + r_i^2\cos^2\theta_i - 2rr_i\cos\theta\cos\theta_i}$$

$$\Rightarrow d_i = \sqrt{r^2 + r_i^2 - 2rr_i\left[\sin\theta\sin\theta_i(\cos\varphi\cos\varphi_i + \sin\varphi\sin\varphi_i) + \cos\theta\cos\theta_i\right]}$$
$$= \sqrt{r^2 + r_i^2 - 2rr_i\left[\sin\theta\sin\theta_i\cos(\varphi-\varphi_i) + \cos\theta\cos\theta_i\right]}$$

$$\frac{\partial}{\partial r}\frac{1}{d_i} = \frac{-r + r_0\left[\sin\theta\sin\theta_0\cos(\varphi-\varphi_0) + \cos\theta\cos\theta_0\right]}{\left(r^2 + r_i^2 - 2rr_i\left[\sin\theta\sin\theta_i\cos(\varphi-\varphi_i) + \cos\theta\cos\theta_i\right]\right)^{3/2}}$$

$$\frac{\partial}{\partial\theta}\frac{1}{d_i} = \frac{rr_i\left[\cos\theta\sin\theta_i\cos(\varphi-\varphi_i) - \sin\theta\cos\theta_i\right]}{\left(r^2 + r_i^2 - 2rr_i\left[\sin\theta\sin\theta_i\cos(\varphi-\varphi_i) + \cos\theta\cos\theta_i\right]\right)^{3/2}}$$

$$\frac{\partial}{\partial\varphi}\frac{1}{d_i} = \frac{-rr_i\sin\theta\sin\theta_i\sin(\varphi-\varphi_i)}{\left(r^2 + r_i^2 - 2rr_i\left[\sin\theta\sin\theta_i\cos(\varphi-\varphi_i) + \cos\theta\cos\theta_i\right]\right)^{3/2}}$$

$$\dot{p}_r = \frac{p_\theta^2}{m_s r^3} + \frac{p_\varphi^2}{m_s r^3\sin^2\theta} + Gm_s\sum m_i \frac{-r + r_0\left[\sin\theta\sin\theta_0\cos(\varphi-\varphi_0) + \cos\theta\cos\theta_0\right]}{\left(r^2 + r_i^2 - 2rr_i\left[\sin\theta\sin\theta_i\cos(\varphi-\varphi_i) + \cos\theta\cos\theta_i\right]\right)^{3/2}}$$

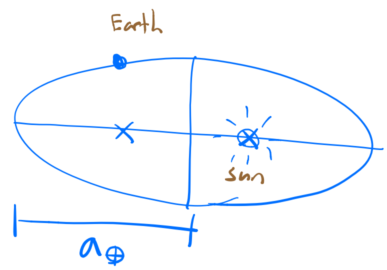
$$\dot{p}_\theta = \frac{-p_\varphi^2}{m_s r^2\sin^2\theta\tan\theta} + Gm_s\sum m_i \frac{rr_i\left[\cos\theta\sin\theta_i\cos(\varphi-\varphi_i) - \sin\theta\cos\theta_i\right]}{\left(r^2 + r_i^2 - 2rr_i\left[\sin\theta\sin\theta_i\cos(\varphi-\varphi_i) + \cos\theta\cos\theta_i\right]\right)^{3/2}}$$

$$\dot{p}_\varphi = Gm_s\sum m_i \frac{-rr_i\sin\theta\sin\theta_i\sin(\varphi-\varphi_i)}{\left(r^2 + r_i^2 - 2rr_i\left[\sin\theta\sin\theta_i\cos(\varphi-\varphi_i) + \cos\theta\cos\theta_i\right]\right)^{3/2}}$$

R4B Non-Dimensionalized

Proposed characteristic units:

Length: $a_\oplus \approx 1.5 \times 10^8$ km



Time: $\frac{1}{\omega_\oplus} = \sqrt{\frac{a_\oplus^3}{G M_\odot}} = \sqrt{\frac{a_\oplus^3}{\mu_\odot}} \approx 58$ days $\approx 5 \times 10^6$ s

Speed: $a_\oplus \omega_\oplus = \sqrt{\frac{\mu_\odot}{a_\oplus}}$

$\approx 2.57 \times 10^6$ km/days

≈ 29.7 km/s

In Bsc project with Earth-Moon system we had:

$$\text{unit length} = 3.85 \times 10^5\,\text{km}, \tag{3.39}$$

$$\text{unit time} = 4.3484\,\text{days}, \tag{3.40}$$

$$\text{unit velocity} = 1.025\,\text{km/s}. \tag{3.41}$$