+++ title = '$(\mathbf{AB})^\mathrm{T} = \mathbf{B}^\mathrm{T}\mathbf{A}^\mathrm{T}$' date = 2024-06-23T18:23:07+08:00 +++

Notation

$[\mathbf{M}]_{ij}$ denotes the entry at the i^th row, j^th column of matrix $\mathbf{M}$.

Matrix Product and Transpose

The product between generic matrices $\mathbf{X}$ and $\mathbf{Y}$ can be written concisely as such:

$$[\mathbf{XY}]_{ij} = \sum_{k=1}^{n} [\mathbf{X}]_{ik}[\mathbf{Y}]_{kj}$$

Likewise, transpose of $\mathbf{X}$ is written as such:

$$\left[\mathbf{X}^\mathrm{T}\right]_{ij} = [\mathbf{X}]_{ji}$$

Proving the Theorem

$$\begin{aligned}
\left[(\mathbf{AB})^\mathrm{T}\right]_{ij} &= [\mathbf{AB}]_{ji} \\
&= \sum_{k=1}^{n} [\mathbf{A}]_{jk}[\mathbf{B}]_{ki} \\
&= \sum_{k=1}^{n} [\mathbf{B}]_{ki}[\mathbf{A}]_{jk} \\
&= \sum_{k=1}^{n} \left[\mathbf{B}^\mathrm{T}\right]_{ik}\left[\mathbf{A}^\mathrm{T}\right]_{kj} \\
&= \left[\mathbf{B}^\mathrm{T}\mathbf{A}^\mathrm{T}\right]_{ij}
\end{aligned}$$

Extension

$$\begin{aligned}
(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3 \ldots \mathbf{A}_{N-2}\mathbf{A}_{N-1}\mathbf{A}_N)^\mathrm{T} &= (\mathbf{A}_1(\mathbf{A}_2\mathbf{A}_3 \ldots \mathbf{A}_{N-2}\mathbf{A}_{N-1}\mathbf{A}_N))^\mathrm{T} \\
&= (\mathbf{A}_2\mathbf{A}_3 \ldots \mathbf{A}_{N-2}\mathbf{A}_{N-1}\mathbf{A}_N)^\mathrm{T}\mathbf{A}_1^\mathrm{T} \\
&= (\mathbf{A}_2(\mathbf{A}_3 \ldots \mathbf{A}_{N-2}\mathbf{A}_{N-1}\mathbf{A}_N))^\mathrm{T}\mathbf{A}_1^\mathrm{T} \\
&= (\mathbf{A}_3 \ldots \mathbf{A}_{N-2}\mathbf{A}_{N-1}\mathbf{A}_N)^\mathrm{T}\mathbf{A}_2^\mathrm{T}\mathbf{A}_1^\mathrm{T} \\
&= (\mathbf{A}_3(\ldots \mathbf{A}_{N-2}\mathbf{A}_{N-1}\mathbf{A}_N))^\mathrm{T}\mathbf{A}_2^\mathrm{T}\mathbf{A}_1^\mathrm{T} \\
&= (\ldots \mathbf{A}_{N-2}\mathbf{A}_{N-1}\mathbf{A}_N))^\mathrm{T}\mathbf{A}_3^\mathrm{T}\mathbf{A}_2^\mathrm{T}\mathbf{A}_1^\mathrm{T} \\
&= \cdots \\
&= (\mathbf{A}_{N-1}\mathbf{A}_N)^\mathrm{T}\mathbf{A}_{N-2}^\mathrm{T} \ldots \mathbf{A}_3^\mathrm{T}\mathbf{A}_2^\mathrm{T}\mathbf{A}_1^\mathrm{T} \\
&= \mathbf{A}_N^\mathrm{T}\mathbf{A}_{N-2}^\mathrm{T}\mathbf{A}_{N-1}^\mathrm{T} \ldots \mathbf{A}_3^\mathrm{T}\mathbf{A}_2^\mathrm{T}\mathbf{A}_1^\mathrm{T}
\end{aligned}$$