CMPUT 325

Lecture 2: Fun language

Fun

- List elements are seperated by a space
- In general, f((x1 ... xn)) --> x, notice the spacing for parameters
- `first( L )` returns first element of L, error if L is not a list or an empty list
- `rest( L )` returns L without the first element in L, error if L is not a list or an empty list
- Using compositions of calls to `first` (or `f`) and `rest` (or `r`), we can get any component (atom or sublist) from a given list, regardless of its depth
- For brevity, `f(f(r(L)))` can be written as `ffr(L)`
- `cons( x, L )` returns a new list K = (x y1 y2 y3 ... yn) where x is an atom and L = (y1 y2 y3 ... yn)
- In theory, there is a way to define any primitive function from scratch.
- Assume we have the following primitive functions:
    - Arithmetic: +, -, *, /, and so on
    - Comparators
    - `if`, `then`, `else`
    - `null(x)`
        - `true` iff x is an empty list
    - `atom(x)`
        - `true` iff x is an atom
    - `eq(x, y)`
        - `true` iff x and y are the same atom
        - `eq( a,a )` returns true
        - `eq( a,b )` returns false
        - `eq( (a), (a) )` returns false because the arguments are lists
- Similar idea is used to create RISC syntax.
- Remarks:
    - No notion of variable-as-storage
    - No assignment statement as in procedural languages
    - No loop constructs
    - **Recursion** as the only mechanism to define non-trivial functions
- 1-step evaluation = replacement + substitution
- Examples covered: `length( L )`, `append( L1, L2 )`, `last( L )`, `removeLast( L )`
- It makes sense to break the solution into smaller functions
- One function must do only one thing

Lecture 2: Intro to LISP

- `reverse( L )` using `append( L1, L2 )`
- Abstract data type for binary tree
    - Goal: implement a binary tree and some operations, such as inserting elements
    - Two main tasks:
        - Decide how trees are represented by lists
        - Implement an abstract data type for binary trees and the operations on them
    - User will work with trees using only these functions. The user is protected from the details of our data representation
    - Bottom up construction
- Binary tree representation
    - Empty tree: `nil`
    - Nonempty tree: `(left-subtree, node-value, right-subtree)`
    - Selectors:
        - `leftTree( Tr ) = f( Tr )`
        - `rightTree( Tr ) = f( r( r( Tr ) ) )`
        - `nodeValue( Tr ) = f( r( Tr ) )`
    - Constructors:
        - `consNilTr() = nil`
            - return an empty tree
        - `consTree(L, V, R) = cons( L, cons( V, cons( R, nil ) ) )`
            - construct tree with subtrees L, R and value V
    - Test:
        - `isEmpty( Tr ) = eq( Tr, nil )`
        - return true iff Tr is an empty tree
- Building an abstract data type
    - Functions are the only ones that need direct knowledge of our tree representation
    - Everything else can be implemented in terms of these basic functions - providing such a base set of functions is the essence of implementing an abstract data type in functional programming
- `insert` into tree
    - assume our trees contain integer values and are sorted such that every value in the left subtree < node value < all values in right subtree
    - unique values
    - `insert( Tr, int )` inserts `int` into the binary tree `Tr`

```
insert(Tr, int) =
 if isEmpty(Tr)
   then consTree(consNilTr(), Int, consNilTr())
 else if Int = nodeValue(Tr)
   then Tr
 else if Int < nodeValue(Tr)
   then consTree(insert(leftTree(Tr), Int, nodeValue, Tr),
rightTree(Tr))
 else consTree(leftTree(Tr), nodeValue(Tr), insert(rightTree(Tr),
Int))
```

LISP

- interpreted language
- case **insensitive**
- uses read-eval-print-loop (REPL) similar to a shell such as bash
    - read input
    - evaluate input
    - print result of evaluation
    - loop back to beginning
- Functions are defined by

```
(defun function-name parameter-list body)
```

    - Example:
        - Function definition: `(defun plus (x y) (+ x y))`
        - Function application: `(plus 3 4)`
- Lisp always interprets `(e1 e2 e3 ...)` as a function application. Use quote to "atomify" the expression
- An empty list is represented by either `()` or `nil`. Both are considered the same atom in Lisp.
- `nil` also represents false
- T represents true

Lecture 4

- `(if condition then-part else-part)` is a special function because not every block is run, unlike other functions
- `trace` to see calls and returns to specific functions
- `untrace` stops the tracing
- functions can take variable number of arguments
- `(let ((x 3) (y 4)) (* (+ x y) x))` evaluates expression, but replaces names x and y with their values 3 and 4
- `let` does not allow using one variable to define another, use `let*` instead
- eq is true iff both are equal *atoms*, runs in a single machine instruction

- `equal` is more general
- ```
  (cond (P1 S1)
        (P2 S2)
        (P3 S3)
        ...
        (T Sn)
  )
  ```
- General form of `cond` (do not use it):

  ```
  (cond (P1 S11 S12 ... S1m)
  (P2 S21 S22 ... S2m)
  ...
  (T Sn1 Sn2 ... Snm)
  )
  ```
    - If `P1` is true then evaluate `S11`, `S12`, ... and `S1m` and return the result of evaluating `S1m`
- `list`
- `caar`, `cddadr`, etc.
- `print` and `format` for printing, strings
- `random`
- Use `quote` when everything is constant
- Use `list` when some contents are the result of evaluating functions
- Use `cons`
    - for the result in recursion when you have computed a first element and the rest of a list
    - for dotted pairs
- `(car (cdr (car (cdr (cdr (cdr L)))))) = (cadr (cadddr L))`
    - max 4 levels deep
- Simple printing `(print arg)`
- Formatted printing: `(format t format-string arg1 ...)`
- `(random N)` generates a uniformly random integer from 0..N-1 if N is an integer
- `(random F)` generates a uniformly random floating point number in range [0..F)
- Accumulators
    - helper function with an extra parameter
    - the extra parameter accumulates the required result
    - Issues with simple recursion:
        - no real computation until hits the base case
  ```

        - all computation happens on return from recursion

    - Example 1: `reverse` using an accumulator

```
(defun reverse_helper (L ResultSoFar)
  (if (null L)
    ResultSoFar
    (reverse_helper (cdr L)
      (cons (car L) ResultSoFar))
    )
  )

(defun reverseAC (L)
  (reverse_helper L nil)
  )
```

    - Comparing accumulator with standard recursion
        - Standard recursion on a list
            - recurse to the end of list
            - compute result on return from recursion
            - bottom-up computation
        - Accumulators
            - accumulates results-so-far
            - computes results top-down
            - needs an extra accumulator variable for partial result
        - Questions to think about to decide whether accumulators should be used
            - top-down or bottom-up?
- Programming loops in LISP
    - In pure functional programming, we use recursion instead of loops although LISP has loop constructs (`for`, `do`, `loop`, ...)
    - break loop into two steps:
        - what to do in each run through the loop
        - how to solve the rest of the problem by recursion

Lecture 5

- Symbolic expressions (S-Expressions, s-expr, sexpr)
    - universal data structure for Lisp
    - generalization of atoms and lists
        - all atoms and lists are sexpr
        - but not all atom and lists are sexpr
    - dotted pair: `(x . y)`

- Definition

    - atom is an s-expression
    - if x1, x2, ..., xn are s-expressions then (x1 ... xn) is an s-expression
    - if x1 and x2 are s-expressions, then `( x1 . x2 )` is an s-expression (a dotted pair)
    - Examples:

        - `hello`
        - `(a b c)`
        - `(a (b) (()))`
        - `(a . b)`
        - `(a . (b . c))`
        - `(1 2 3 (4 . 5))`

- `(car (x . y))` returns x

- `(cdr (x . y))` returns y

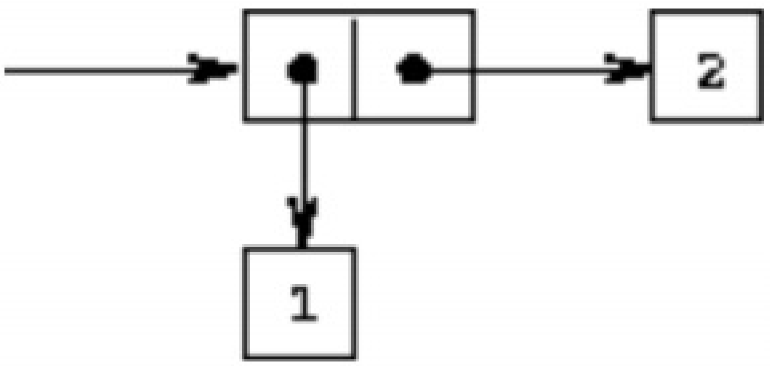
- `(car (cons 'x 'y)) = x`

- `(cdr (cons 'x 'y)) = y`

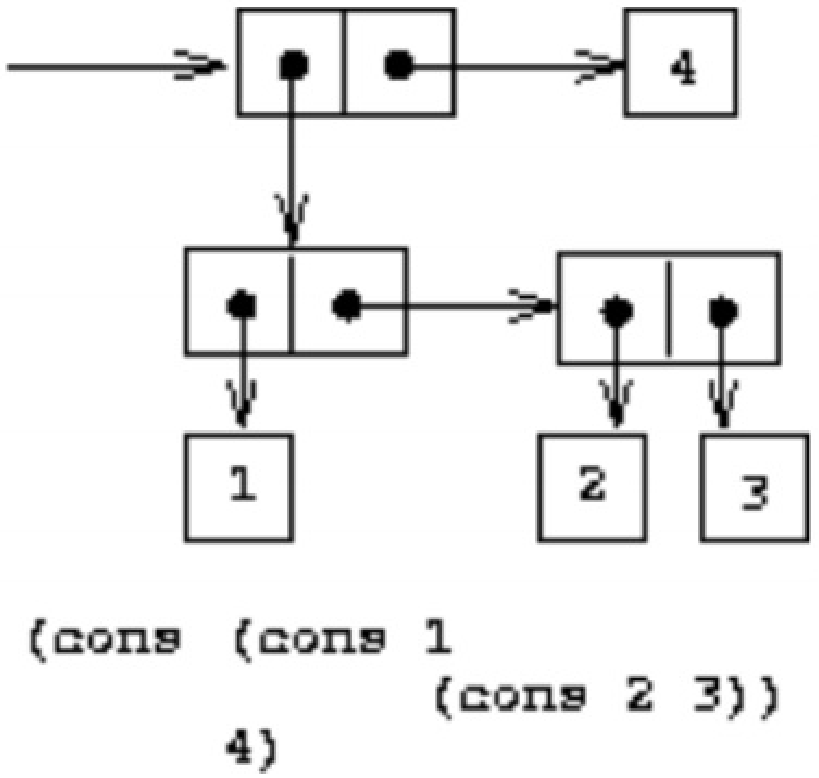
- `.` must be surrounded by whitespace: `(a.b)` is a list containing the atom `a.b`

- machine-level representations

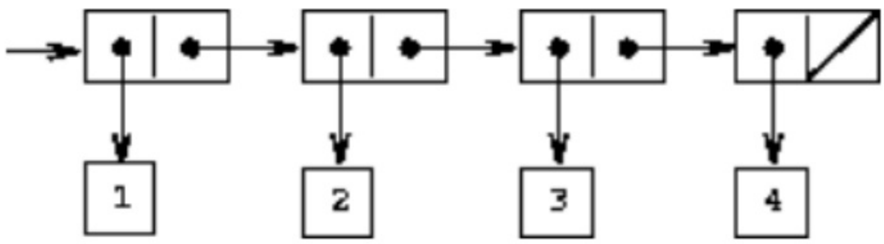
    - Example 1: `(cons 1 2)` or `(1 . 2)`



- Example 2: `( (1 . (2 . 3)) . 4 )`



- Example 3: List representation `(1 2 3 4)`



- `(a . nil)` = `(a)` because `(cdr '(a . nil))` = `(cdr ('(a)))` = `nil`
- Every list can be written as nested dotted pairs:
  `(1 . (2 . (3 . (4 . nil))))`
- Why use dotted pairs?
  - saves memory
  - simplifies direct access

Lecture 6

- Higher order functions
  - Definition: a function that takes other function(s) as input and/or produce function(s) as output
  - often used to seperate:
    - a computation pattern
    - specific repeated action

- Example 1:
    - Pattern: iterate over a list
    - Action: Compute the same function for each list element
- Example 2:
    - Pattern: reduce a list to a single result
    - Action: reduce two arguments to one
- Some typical higher order functions
    - Map - apply some function to all elements of a list
    - Reduce - apply two argument function repeatedly
    - Filter - select list elements that pass a test
    - Vector - apply many functions to one element
- `mapcar` in Lisp

```
(defun plus1 (x) (+ x 1))
(mapcar 'plus1 '(1 2 3 4 5)) --> (2 3 4 5 6)
```

- reduce - general definition
    - input:
        - function g
        - function's identity `id`
        - a list `L = (a1 a2 ... an)`
    - compute: `(g a1 (g a2 ... (g an id) ... ))`
    - example in Lisp:
        - `(reduce '* '(2 6 4)) --> 48`
- Why define high order functions?
    - use case: a common computation pattern, where the details can vary
    - removing code duplication
    - `apply` and `funcall` tells Lisp that there is a function to be called
        - only differs in syntax, same functionality
        - `(apply function-name (arg1 ... argn))`
        - `(funcall function-name arg1 ... argn)`

Lecture 7

Lambda Functions

- get rid of named functions, why?
    - a function was the result of a higher order function
    - tried to return this newly computed function
- lambda functions are **function definitions** without names

    - Syntax: `(lambda (x1 ... xn) body)`
    - Example: `((lambda (x y) (+ x y)) 5 3)`
    - lambda function application
        - to apply a lambda function

        ```
        ((lambda (x1 ... xn) body) a1 ... an)
        x1 ... xn are formal arguments of function
        a1 ... an are actual parameters for which we want to
        evaluate the function
        ```

- Lisp-1 and Lisp-2
    - Lisp-1 systems: "values" and functions in the same namespace
    - Lisp-2 systems: in seperate spaces
    - Common Lisp standard requires Lisp-2
    - if we have a variable bound to a function, we need to tell SBCL this is a function to be called
    - Consequence:
        - Working with lambda functions is much messier in Lisp-2 systems than in Lisp-1
    - never quote a lambda expression in Lisp-2
- `function`
    - syntax: `(function arg)`
    - purpose: evaluates lambda function given by `arg`
    - takes lambda function as its argument
    - returns function definition in an internal format used by SBCL
    - compiles it and returns an internal representation of the compiled code
    - representation is called **closure**
    - next, the function in the closure can be called in an application
    - use `funcall` or `apply` for the application
- `function` vs `funcall` vs `apply`
    - `function` takes as argument a *function definition* and returns an internal representation of that definition
    - does NOT apply the function
    - `funcall` and `apply` are for *function application*
- applying lambda functions
    - use `funcall` and `apply` as usual by giving the whole lambda function as an argument

Lambda Calculus

- Intro to lambda calculus
    - formal, abstract language
    - all functions are defined without giving them names
    - Lisp is based on lambda calculus but adds a large language on top of it

- formal language with only four concepts:

  ```
  [identifier]  := a | b | ...
  [function]    := (lambda (x) [expression])
  [application] := ([expression] [expression])
  [expression]  := [identifier] | [application] | [function]
  ```

  - `identifier`: corresponds to an atom
  - `function`: is a lambda function definition
  - `expression`: can be an arbitrary lambda expression. It plays the role of the body in the function definition.
    - corresponds to sexpr in Lisp
  - `application`: both function and arguments can be any expressions
    - usually, first expression will evaluate to a lambda function and the second argument will evaluate to the arguments for the lambda function
- all valid expressions defined by this language are called lambda expressions
- lambda expressions can represent any computation

- Unary vs N-ary functions
  `[function] := (lambda (x) [expression])`
  - we only have unary functions - functions that take **one** parameter
  - any **n-ary function** (function with n arguments) can be defined using **a series of unary functions**
  - consequence:
    - to understand the model of computation for general functional programming
- Curried functions
  - Goal: define an n-ary functions by a series of unary functions
  - can solve this by using higher order functions
  - main idea:
    - series of n unary function applications
    - each application processes one argument
    - the application produces a new function which has this argument hardcoded
  - Intuition
    - Example: `(plus 5 2)` is a function with two args
    - `(plus5 2)` is a function with one argument, the "add 5" is hardcoded into the new function `plus5`
    - function takes only the first argument, and produces as result a new function
    - this new function now takes the second argument and produces a new function
    - the function that takes the last argument will have other argument

values "hardcoded"
    - each function is computed on the fly by all the previous function applications

  - splitting up the two argument function

```
(lambda (x y) (if  (< x y) x y))

is equivalent to

(lambda (x) (lambda (y) (if (< x y) x y)))
```

Lecture 8

Reductions in Lambda Calculus

- Goal: reduce a lambda expression to its **simplest possible form**

- This process is called *operational semantics* of lambda calculus

- In lambda calculus, computation is **the process of reductions from one expression to another expression**

- Example:

```
((lambda (x) (x 2)) (lambda (z) (+ z 1))) → (+ 2 1)
```

- Shorthand notation in lambda calculus
  (λx (+ x 1)) for

```
(lambda (x) (+ x 1))
```

- In lambda calculus, we do not need any of the primitive functions

  - numbers can be represented by lambda expressions

- Beta reduction

  - function application
  - written as $\rightarrow^{\beta}$
  - rule:

    - given an expression `(( lambda (x) body ) a)`, reduce it to body
    - replace all occurences of x in body by a

  - Example:

    - `((lambda (x) (x (x 1))) 5)` $\rightarrow^{\beta}$ `(5 (5 1))`

    - `( lambda (x) (x x x x x)) (1 2 3 4 5)` $\rightarrow^{\beta}$
      `((1 2 3 4 5) (1 2 3 4 5) (1 2 3 4 5) (1 2 3 4 5) (1 2 3 4 5))`
      `x = (1 2 3 4 5)`

- Alpha reduction

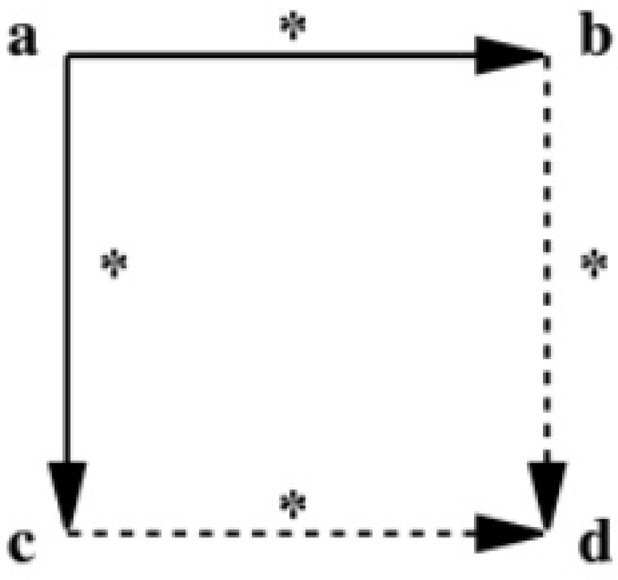
    - $\rightarrow^{\alpha}$ means renaming variables
    - Intuition: changing the name of local variables in a function does not change the meaning
    - *name conflict* between arguments:
        - `(defun f(x x) (- x x))`
        - this gives a compile-time error in SBCL: variable x occurs more than once in a lambda expression
    - Example: `(lambda (x) (+ x y))`
        - x is **bound** in the scope of `( lambda (x) ...)`
        - y is **free**
        - x can be renamed to anything except y
        - y cannot be renamed
    - Free vs bound variables
        - depend on their scopes
        - like global and local variables
    - In lambda calculus, a bound variable's name can be replaced by another if the latter does not cause any name conflict
        - use a new variable name
        - called alpha reduction
        - without alpha reduction, direct substitution does not always work
- **Note**: Perform alpha reduction first!
    - $((\lambda x (\lambda z (x z))) z)$
    - rename the z in $(\lambda z \dots)$
        - $((\lambda x (\lambda u (x u))) z)$
    - now the bound variable is called u and will not conflict with the argument z
    - finally replace x by z in body
        - $(\lambda (u) (z u))$
- Scope of variables and beta-reduction
    - scope of a variable should be preserved by variable renaming to ensure that reduction is correct
        - $((\lambda x (\lambda z (x z))) z) \rightarrow^{\beta} (\lambda u (z u))$
        - where u is some new variable
        - exercise: fill in the steps
    - correct beta reductions can always be achieved by renaming (alpha-reduction), if needed
    - beta-reduction using direct substitution
- Summary of reductions

- one β-reduction corresponds to a one-step function application
- direct substitution does not always work, variables may need to be renamed before the substitution step
- the substitution of the formal variable by the argument must be done carefully to avoid name conflicts
- α-reduction renames function arguments
- after using such renaming where necessary, a simple substitution in the body gives a correct beta-reduction
- to be safe can always use α-reduction with names for bound variables

Lecture 9

Normal form, order of reduction and Church Rosser Theorem

- **Normal form**: a lambda expression that cannot be reduced further (by beta reduction)
- not all lambda expressions have a normal form
    - apply a sequence of reductions and the same expression will be obtained
- `( ( lambda (x) (x x)) (lambda (z) (z z)) )` does not have a normal form
- order of reduction
    - Normal order reduction (NOR):
        - evaluate leftmost outermost application
    - Applicative order reduction (AOR):
        - evaluate leftmost innermost application
- tie-breaking rules:
    - choose the leftmost one for either order when dealing with multiple choices
- Efficiency
    - AOR is generally more efficient
    - NOR terminates more often, AOR might get into infinite reduction
- Church Rosser theorem
    - Church and Rosser proved two important properties of reductions and normal forms
        - If A reduces to B and A reduces to C then there exists an expression D such that B reduces to D and C reduces to D
        - If A has a normal form E then there is a **NOR** from A to E
    - Remarks
        - If A reduces to B and A reduces to C then there exists an expression D such that B reduces to D and C reduces to D

- 

- No matter what reduction strategies are used initially to get from A to B and C, there is always a way to converge from both B and C back to the same expression D
- There is always at most one normal form
- If A has a normal form E then there is an NOR from A to E
- NOR guarantees termination if the given expression has a normal form
    - can be inefficient
- Theorem does not tell us if a normal exists

Lecture 10: Interpreter based on context and closure

- implement an interpreter for a Lisp-like language based program, only consisting of lambda functions
- interpreter based on two concepts: **context** and **closure**
- Language
    - variables
    - constant expressions: `(quote e)`
    - arithmetic
    - relations and logic
    - primitives for s-expr:
        - `car`, `cdr`, `cons`, `atom`, `null`
    - `if`
    - lambda functions
    - function call
    - simple block `(let (x1.e1) ... (xk.ek) e)`
    - (optional) recursive block
    - `(letrec (x1 . e1) ... (xk . ek) e)`, instead of the Lisp way to simplify
- Why not just use β-reductions?

    - determine the scope of each parameter
    - detect potential name conflicts
    - implement variable renaming (α-reductions)
    - implement direct substitutions
    - it is possible to do it, but not very efficient
    - main problem:
        - need to check all the above repeatedly after each substitution step

- new idea
    - key idea: *delay* the subsitutions by using contexts and closures
    - this technique is used in real Lisp interpreters
    - also helps with understanding compilation

- **Context**
    - a context is *a list of bindings*
      $[n_1 \rightarrow v_1 \dots n_k \rightarrow v_k]$
        - where n_i are identifiers, v_i are expressions (value that the variable is bound to)
        - v_i can also be a closure, represents the state of an incomplete evaluation
    - a context is used to keep record of lookup name bindings
    - a context can be *extended* when a new binding $n \rightarrow v$ is created in a function application

- **evaluation with a context**
    - always begin with an empty context
    - compare with other programming languages
        - may have global variables already bound to values
    - in the middle of evaluating an expression, the context is usually non-empty
    - substitutions are delayed to the point where the value of a variable is really needed for the evaluation to continue
    - variables are left as-is
    - a variable is bound as "needed"...
        - if binding can be found in context

- **Closure**
    - pair `[f, CT]`
    - `f` is a lambda function
    - CT is a (possibly empty) context
    - when function `f` is applied we know the parameters and body
    - we get values for the variables **from the context** which are used in the body of `f`

- **Function application in a context - Algorithm**
    - when interpretation of a program starts, the context is empty

    - when a function is applied:
        - evaluate the arguments in the current context
        - evaluate the functional part in the current context
        - extend the context
            - bind the parameter names to the evaluated arguments
            - add these bindings to current context to form the next context
        - evaluate the body of the function in this extended context
- **Implementation of context for interpreter**
    - define a data structure to represent a context
    - two lists, name list and value list
    - both lists are in sync
        - for each name there is a corresponding value stored in the same location in the other list
    - name list is a list of lists of atoms
    - value list is a list of lists of symbolic expressions
- example
    - name list `((x y) (z) (w s))`
    - value list `((1 2) ((lambda (x) (* x x)))`
    - list of three sublists
        - corresponds to three (nested) lambda function applications
    - in previous notation, this implements the context
- compare context and closure model with runtime execution model of programs
    - compare to runtime model of a programming language
    - a runtime model has:
        - a call stack for all active functions
        - one stack frame for each function
    - the names are compiled away, replaced by relative addresses on the stack
    - in our model, we still keep the names in the context
    - we will soon see a similar compilation for Lisp, also compiling away names, in the SECD machine
- **name lookup for context**
    - search for a name in a context
    - walk synchronously over both name and value lists
    - if a name is found:
        - s-expr in the same position in the value list is its binding
    - name lookup function `assoc(x, n, v)`
        - name list n

            - value list v
            - name to lookup stored in x
            - linear search implementation

Lecture SECD machine

- programming language implementations:
    - interpreter
    - compiler to assembly, real machine code
    - compiler to virtual machine
- SECD is a virtual machine that runs simple Lisp programs
- virtual machines
    - not real hardware, but has its own set of instructions
    - to run, implement on a real machine
- SECD usage
    - compiler from Lisp to SECD machine code
    - execute the compiled code on an abstract cmachine
    - main differences to interpreter
        - once compiled, code can be executed many times
        - code optimization is possible
- SECD Machine
    - consists of four stacks
        - s: Stack used for evaluation of expressions
        - e: Environment stores the current value list
        - c: Control stores the machine instructions
        - d: Dump stores *suspended* invocation context
- SECD operations and functions
    - NIL: push a nil pointer
    - LD: load from environment
    - LCD: load constant
    - LDF: load function
    - AP: apply function
    - RTN: return
    - SEL: select in if statement
    - JOIN: rejoin main control
    - builtin funtions: +, *, ATOM, CAR, CONS, EQ, etc.
- definition of SECD operations
    - each operation is defined by its effect on the four stacks
        - s e c d ---> s' e' c' d'
    - representation of a stack s

- s-expression with dot notation
- note: for bevity, in this topic spaces around the '.' are omitted as required in Lisp
- first position `(car s)` = top of the stack
- push onto stack s `---> (e.s)`
- pop from the stack `(e.s) ---> s`