24-677 Modern Control Theory
Project 2

Exercise 1:

1. **Determine Controllability and Observability**

   a. **Controllable and Observable**

a. $\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$

$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x$

$M = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ $\det(M) = -1$, $-1 \neq 0$ $\boxed{\text{controllable}}$

$N = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -2 \\ 2 & 5 & 4 \end{bmatrix}$ $\det(N) = 1$, $1 \neq 0$ $\boxed{\text{observable}}$

   b. **Controllable and Observable**

b. $\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} u$

$y = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x$

$\lambda I - A = 0$ $\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 2 & \lambda+1 \end{bmatrix}$ $\lambda = 0, 1, -2$ MATLAB

for $\lambda = 0$ check $(-A \,\vdots\, B)$ and $\left(\frac{I - A}{C}\right)$

$r\left[\begin{array}{ccc|cc} 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 \end{array}\right] = 2 \neq 0$ controllable $\boxed{\text{controllable}}$

$r\left[\begin{array}{ccc} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -2 & 2 \\ \hline 1 & 0 & 1 \end{array}\right] = 3 \neq 0$ observable $\boxed{\text{observable}}$

for $\lambda = 1$ check $(I - A \,\vdots\, B)$ and $\left(\frac{I - A}{C}\right)$

$r\left[\begin{array}{ccc|cc} 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & -2 & 2 & 0 & 0 \end{array}\right] = 3 \neq 0$ controlldble

$r\left[\begin{array}{ccc} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -2 & 2 \\ \hline 1 & 0 & 1 \end{array}\right] = 3 \neq 0$ observable

for $\lambda = 2$ check $(-2I - A \,\vdots\, B)$ and $\left(\frac{-2I - A}{C}\right)$

$r\left[\begin{array}{ccc|cc} -2 & -1 & 0 & 0 & 1 \\ 0 & -2 & -1 & 1 & 0 \\ 0 & -2 & -1 & 0 & 0 \end{array}\right] = 3 \neq 0$ controlldble

$r\left[\begin{array}{ccc} -2 & -1 & 0 \\ 0 & -2 & -1 \\ 0 & -2 & -1 \\ \hline 1 & 0 & 1 \end{array}\right] = 3 \neq 0$ observable

   c. **Controllable and Not Observable**

c. $\dot{x} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} u$

$y = \begin{bmatrix} 2 & 2 & 1 & 3 & -1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} x$

$\lambda = 2$ check $\hat{B}^2$ and $\hat{C}^2$

$\hat{B}^2$ $r\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} = 3$ controllable $\boxed{\text{controllable}}$ $\boxed{\text{NOT observable}}$

$\hat{C}^2$ $r\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} = 2$ NOT observable

$\lambda = 1$ check $\hat{B}^2$ and $\hat{C}^2$

$\hat{B}^2$ $r\begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = 2$ controllable

$\hat{C}^2$ $r\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = 1$ NOT observable

2. State Space Realizations

2. state space realization

$$G(s) = \begin{bmatrix} \frac{1}{s} & \frac{s+3}{s+1} \\ \frac{1}{s+3} & \frac{s}{s+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{s+3-(s+1)}{s+1} \\ \frac{1}{s+3} & \frac{s-(s+1)}{s+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{2}{s+1} \\ \frac{1}{s+3} & \frac{-1}{s+1} \end{bmatrix}$$

$$G(\infty) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \qquad d = s(s+1)(s+3) = \frac{1}{s^3+4s^2+3s}$$

$$G_{sp} = d \begin{bmatrix} s^2+4s+3 & 2s^2+6s \\ s^2+s & -s^2-3s \end{bmatrix}$$

$$N_1(s) = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \quad N_2(s) = \begin{bmatrix} 4 & 6 \\ 1 & -3 \end{bmatrix} \quad N_3(s) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -4 & 0 & -3 & 0 & 0 & 0 \\ 0 & -4 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 4 & 6 & 3 & 0 \\ 1 & -1 & 1 & -3 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

3. State Feedback Control Matrix K

```
%Part 3
a = [1 1 -2; 0 1 1; 0 0 1];
b = [1;0;1];
p = [0 0 0];
K = acker(a,b,p)
```

```
K = 1×3
     1     5     2
```

4. Observer Matrix L

```
%Part 4
a = [-2 4; -3 9];
b = [0;1];
c = [3 1];
p = [-0.5+0.5i -0.5-0.5i];
L = place(a',c',p)
```

```
L = 1×2
    1.4653    3.6042
```

Exercise 2:

1. Longitudinal Velocities Controllability

```
%Part 1
f = 0.025;
g = 9.81;
m = 1000;
Iz = 3004.5;
Ca = 20000;
lf = 1.18;
lr = 0.82;
syms F_input x y wheel_angle psi x_dot y_dot psi_dot X Y s

V = [2, 5, 8];
A = [0 1 0 0
     0 -4*Ca/(m*x_dot) 4*Ca/m -2*Ca*(lf-lr)/(m*x_dot)
     0 0 0 1
     0 -2*Ca*(lf-lr)/(Iz*x_dot) 2*Ca*(lf-lr)/Iz  -2*Ca*(lf^2+lr^2)/(Iz*x_dot)];
B = [0 0
      2*Ca/m 0
      0 0
      2*Ca*lf/Iz 0];

for i = 1:length(V)
    x_dot = V(i);
    A_sub = subs(A);
    B_sub = subs(B);
    AB = A_sub * B_sub;
    A2B = A_sub^2 * B_sub;
    A3B = A_sub^3 * B_sub;
    M = horzcat(B_sub,AB,A2B,A3B);
    r = rank(M);
    fprintf("Velocity: %d Rank: %d\n", V(i), r)
end
fprintf("Controllable for All Velocities")
```

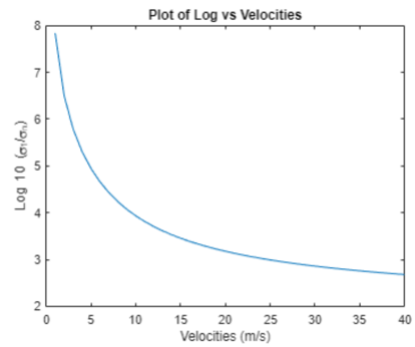
```
Velocity: 2 Rank: 4
Velocity: 5 Rank: 4
Velocity: 8 Rank: 4

Controllable for All Velocities
```

2. Plots Longitudinal Velocities

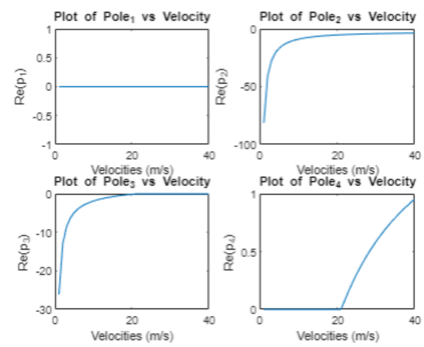
a.

```
%Part 2
V = 1:1:40;
sing_val = [];
for i = 1:40
    x_dot = i;
    A = [0 1 0 0
     0 -4*Ca/(m*x_dot) 4*Ca/m -2*Ca*(lf-lr)/(m*x_dot)
     0 0 0 1
     0 -2*Ca*(lf-lr)/(Iz*x_dot) 2*Ca*(lf-lr)/Iz  -2*Ca*(lf^2+lr^2)/(Iz*x_dot)];
    A_sub = subs(A);
    B_sub = subs(B);
    AB = A_sub * B_sub;
    A2B = A_sub^2 * B_sub;
    A3B = A_sub^3 * B_sub;
    M = horzcat(B_sub,AB,A2B,A3B);
    [~,S,~] = svd(M);
    sing_val = [sing_val log10(S(1,1)/S(4,4))];
end
figure()
plot(V,sing_val)
xlabel("Velocities (m/s)")
ylabel("Log 10 (\sigma_1/\sigma_n)")
title("Plot of Log vs Velocities")
```



b.

```
%Part 2
V = 1:1:40;
sing_val = [];
for i = 1:40
    x_dot = i;
    A = [0 1 0 0
     0 -4*Ca/(m*x_dot) 4*Ca/m -2*Ca*(lf-lr)/(m*x_dot)
     0 0 0 1
     0 -2*Ca*(lf-lr)/(Iz*x_dot) 2*Ca*(lf-lr)/Iz  -2*Ca*(lf^2+lr^2)/(Iz*x_dot)];
    eigs_vel = eig(A);
    poles1(i:i) = real(eigs_vel(1:1));
    poles2(i:i) = real(eigs_vel(2:2));
    poles3(i:i) = real(eigs_vel(3:3));
    poles4(i:i) = real(eigs_vel(4:4));
end
subplot(2,2,1);
plot(V,poles1)
xlabel("Velocities (m/s)")
ylabel("Re(p_1)")
title("Plot of Pole_1 vs Velocity")
subplot(2,2,2)
plot(V,poles2)
xlabel("Velocities (m/s)")
ylabel("Re(p_2)")
title("Plot of Pole_2 vs Velocity")
subplot(2,2,3)
plot(V,poles3)
xlabel("Velocities (m/s)")
ylabel("Re(p_3)")
title("Plot of Pole_3 vs Velocity")
subplot(2,2,4)
plot(V,poles4)
xlabel("Velocities (m/s)")
ylabel("Re(p_4)")
title("Plot of Pole_4 vs Velocity")
```



c. The conclusion that can be made about the overall controllability and stability of the system based on these two plots is that the system seems to be more controllable for the higher longitudinal velocities in regards to the lateral direction. This may be attested due to the SVD values or ratios where the smaller the value, the less the system is flawed. So, in the case of the tractor, it will be easier to control at high speeds in the lateral direction. The poles graph shows that the system will be more unstable as the velocity increases. Along with that, we see the system is unstable at velocity near 20.

3. Velocity Observabilities

a.

```
%Part 3
C = [1 0 0 0
     0 0 1 0];
V = [2, 5, 8];
for i = 1:length(V)
    x_dot = V(i);
    A_sub = subs(A);
    C_sub = subs(C);
    CA = C_sub * A_sub;
    CA2 = C_sub * A_sub^2;
    CA3 = C_sub * A_sub^3;
    M = vertcat(C_sub,CA,CA2,CA3);
    r = rank(M);
    fprintf("Velocity: %d Rank: %d\n", V(i), r)
end
fprintf("Observable for All Velocities")
fprintf("The conclusion about this is that if we only consider\nthe heading angle, " + ...
    "the rank is not technically full for Q,\nmeaning that the system is not observable.")
```

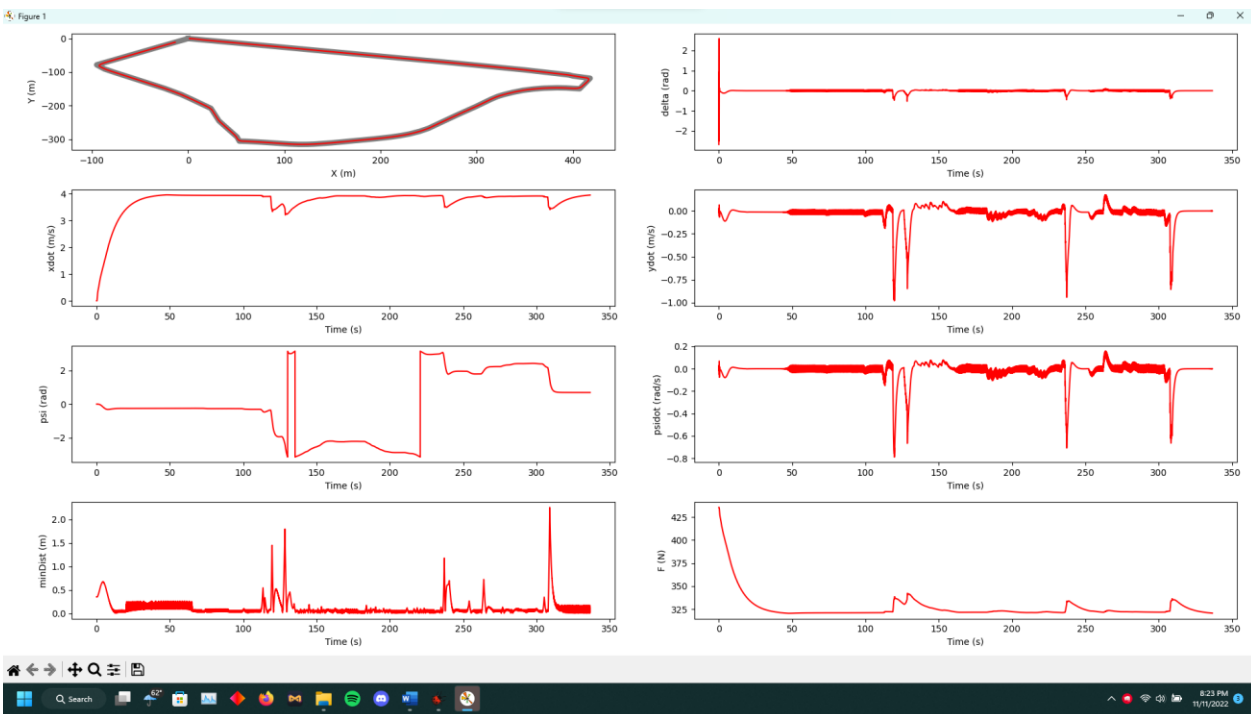
```
Velocity: 2 Rank: 4
Velocity: 5 Rank: 4
Velocity: 8 Rank: 4

Observable for All Velocities

The conclusion about this is that if we only consider
the heading angle, the rank is not technically full for Q,
meaning that the system is not observable.
```

4. **Pole Placement**
    a. **poles = np.array([-5, -4, -0.5, 0])**
        i. Tractor is too fast and completely misses the road, picked random numbers that were relatively "small"
    b. **poles = np.array([-15,-25,-2,0])**
        i. Worked perfect with new poles, picked farther away from axis for more accuracy and a slightly slower response
    c. **poles = np.array([-15,-25,-1,0])**
        i. Wanted to see if the third pole had a drastic impact, changed value slightly but system performed better
5. **Reconstruct the Error Derivative States**
    a. We can find the error derivatives by taking the difference between the current error and the previous step's error and taking the entire thing and dividing it by the time we take during each step.

Exercise 3:




```
Evaluating...
Score for completing the loop: 30.0/30.0
Score for average distance: 30.0/30.0
Score for maximum distance: 30.0/30.0
Your time is  336.544
Your total score is : 100.0/100.0
total steps:  336544
maxMinDist:  2.252381138711125
avgMinDist:  0.12227835341023865
```