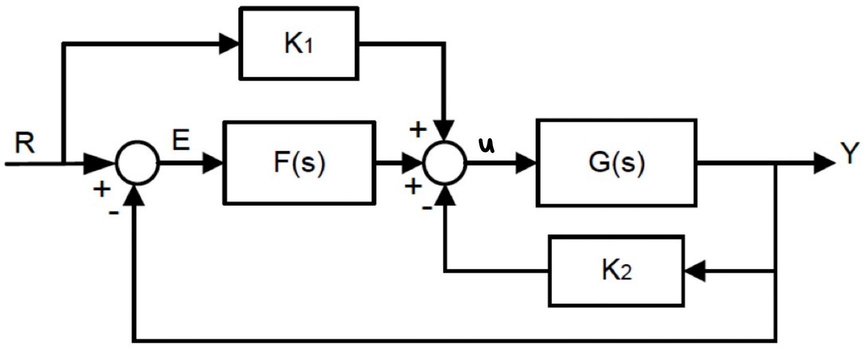
1: 10 Points

Consider the block diagram shown. For both problems give the answer assuming the system is MIMO as well as the simplified answer for a SISO system.



(a) Find the closed loop transfer function $T \triangleq \frac{Y}{R}$.

(b) Find the sensitivity function $S \triangleq \frac{E}{R}$.

a) $E(s) = R(s) - Y(s) \rightarrow Y(s) = R(s) - E(s)$

$Y(s) = G(s) \cdot u(s)$

$u(s) = F(s) \cdot E(s) + k_1 \cdot R(s) - k_2 \cdot Y(s)$

$Y(s) = G(s)\left[F(s)(R(s) - Y(s)) + k_1 \cdot R(s) - k_2 \cdot Y(s)\right]$

$Y(s) = G(s)F(s)R(s) - G(s)F(s)Y(s) + G(s)k_1R(s) - G(s)k_2Y(s)$

$Y(s) + G(s)F(s)Y(s) + G(s)k_2Y(s) = G(s)F(s)R(s) + G(s)k_1R(s)$

$Y(s)\left[1 + G(s)F(s) + G(s)k_2\right] = R(s)\left[G(s)F(s) + G(s)k_1\right]$

$$\frac{Y(s)}{R(s)} = \frac{G(s)F(s) + G(s)k_1}{1 + G(s)F(s) + G(s)k_2}$$

$$\boxed{\frac{Y(s)}{R(s)} = \frac{G(s)\left[F(s) + k_1\right]}{1 + G(s)\left[F(s) + k_2\right]}}$$

b) $Y(s) = G(s) \cdot u(s)$

$u(s) = F(s) \cdot E(s) + k_1 \cdot R(s) - k_2 \cdot Y(s)$

$E(s) = R(s) - Y(s)$

$R(s) - E(s) = G(s)\left[F(s) \cdot E(s) + k_1 \cdot R(s) - k_2 (R(s) - E(s))\right]$

$R(s) - E(s) = G(s)F(s)E(s) + G(s)k_1R(s) - G(s)k_2R(s) + G(s)k_2E(s)$

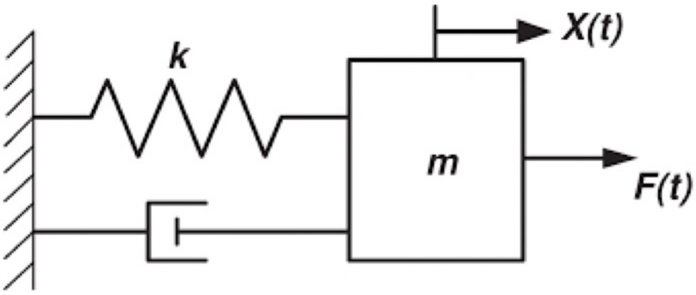
$R(s) - G(s)k_1R(s) + G(s)k_2R(s) = E(s) + G(s)F(s)E(s) + G(s)k_2E(s)$

$R(s)\left[1 - G(s)k_1 + G(s)k_2\right] = E(s)\left[1 + G(s)F(s) + G(s)k_2\right]$

$$\boxed{\frac{E(s)}{R(s)} = \frac{1 - G(s)\left[k_1 - k_2\right]}{1 + G(s)\left[F(s) + k_2\right]}}$$

2: 40 points

Consider the mass spring damper system shown. Unlike a typical MSD system the spring is nonlinear, providing a force given by $F_{\text{spring}} = -100x^3$ where x is the displacement from the spring equilibrium. In this problem assume the damping coefficient is given by $b = 10\ \frac{N \cdot s}{m}$ and the mass is given by $m = 10kg$.



(a) (10 points) Write the nonlinear differential equation that describes the motion. Linearize the differential equation about equilibrium point $x = 2\ m$.

(b) (10 points) Design a PID controller that achieves the following specifications for your linearized systems.

- Zero steady-state error to a step reference.
- A settling time of $t_s \leq 4\ s$.
- A rise time of $t_r \leq 1.5\ s$.

Describe how you achieved the objectives and use the *stepinfo* command to demonstrate your success. HINT: You might find the command *controlSystemDesigner* to be useful to have Matlab perform the design for you.

(c) (20 points) Construct the nonlinear dynamics in Simulink and apply the controller designed in part (b) alongside the linearized model from part (a). Plot the step response of the closed loop system for a 0.1 m step from $x_0 = 2\ m$ to $x_F = 2.1\ m$ for the linearization and the nonlinear system on the same graph. Be sure to include an image of your Simulink model in the solution. HINT: You will have to consider how to apply the equilibrium conditions correctly to make the linear and nonlinear simulations line up.

a) $m\ddot{x}(t) + b\dot{x}(t) + kx(t) = F(t)$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \qquad x_2 = 2m \qquad F = kx$$

$$100x^3 = kx$$

$$100x^2 = k$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{F}{m} - \frac{k}{m}x_1 - \frac{b}{m}x_2 \end{bmatrix} \quad \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F$$

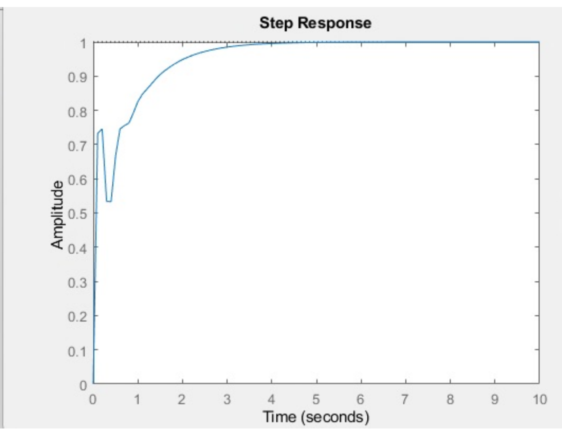
$$\frac{\partial F}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{300x^2}{m} & -\frac{b}{m} \end{bmatrix} \qquad \frac{\partial F}{\partial u} = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}$$

$$\frac{\partial F}{\partial x} = \begin{bmatrix} 0 & 1 \\ -120 & -1 \end{bmatrix}$$

$$\boxed{\dot{x} = \begin{bmatrix} 0 & 1 \\ -120 & -1 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{10} \end{bmatrix} F}$$

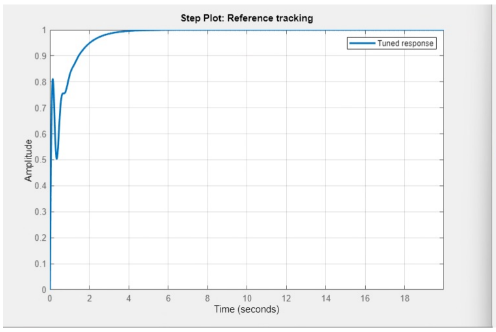
b)

```
A = [0 1; -120 -1];
B = [0; 0.1];
C = [1 0];
D = 0;
[b, a] = ss2tf(A, B, C, D);
G = tf(b, a)
kp = 1003;
ki = 2570;
kd = 98;
H = tf(pid(kp, ki, kd));
stepinfo(feedback(G*H,1))
t = 0:0.1:10;
step(feedback(G*H,1), t)
```



$$\frac{Y}{R} = \frac{\left(k_p + \frac{k_i}{s} + k_d s\right) G(s)}{1 + \left(\left(k_p + \frac{k_i}{s} + k_d s\right) G(s)\right)}$$

```
    RiseTime: 1.4505
TransientTime: 2.7686
SettlingTime: 2.7686
 SettlingMin: 0.9003
 SettlingMax: 0.9987
   Overshoot: 0
  Undershoot: 0
        Peak: 0.9987
    PeakTime: 4.9867
```



Controller Parameters

	Tuned
Kp	1003.8093
Ki	2570.1547
Kd	98.0129
Tf	n/a

Performance and Robustness

	Tuned
Rise time	1.45 seconds
Settling time	2.77 seconds
Overshoot	0 %
Peak	0.999
Gain margin	Inf dB @ NaN rad/s
Phase margin	62.7 deg @ 17.5 rad/s
Closed-loop stability	Stable

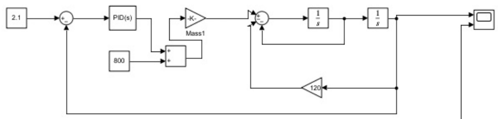
Close

zero-ss: decrease Kp to decrease overshoot → reducing time to settle

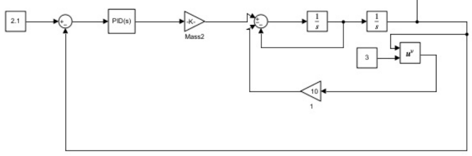
rise time: increased response speed which increased Kd to decrease rise time

Settling time: increased response time and that increased Ki

c) Linearized



Nonlinear



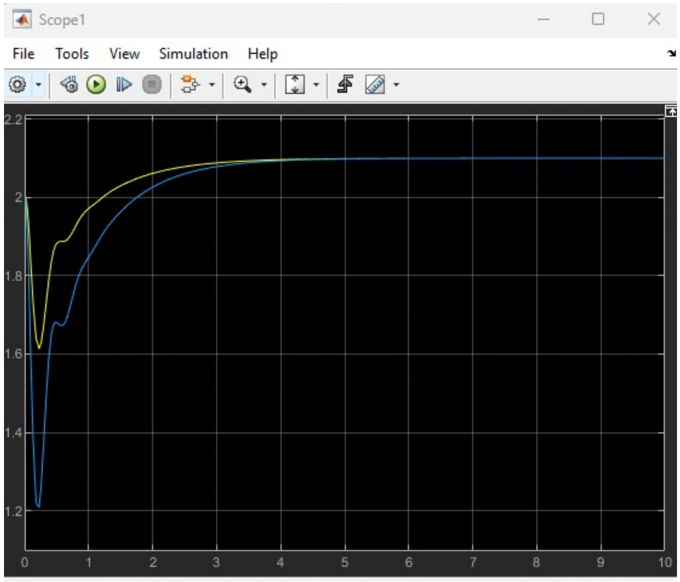
nonlinear

$$\ddot{x} = -10x^3 - \dot{x} + \frac{1}{10}F(t)$$

linear

$$\partial \dot{x}_1 = -1202 x_1 - \partial \dot{x}_1 + \frac{1}{10}\partial u$$

$$\partial \dot{x}_1 = 2x_2$$



3: 20 points

Consider the LTI system given by

$$\dot{x} = \begin{bmatrix} -5 & 1 & 3 \\ 2 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0.5 & 0.25 & -0.375 \end{bmatrix} x.$$

(a) Is the system controllable? Observable? Lyapunov stable?

(b) Find a minimal realization. Design a controller for the minimal realization that achieves a time constant of 1 millisecond.

a) $P = [B \;\; AB \;\; A^2B]$

$$= \begin{bmatrix} 2 & -10 & 54 \\ 0 & 4 & -20 \\ 0 & 0 & 16 \end{bmatrix}$$

rank 3 ✓

$\boxed{\text{controllable}}$

$$AB = \begin{bmatrix} -5 & 1 & 3 \\ 2 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -10 \\ 4 \\ 0 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} -5 & 1 & 3 \\ 2 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix}^2 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 54 \\ -20 \\ 16 \end{bmatrix}$$

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0.25 & -0.375 \\ -2 & -1 & 1.5 \\ 8 & 4 & -6 \end{bmatrix}$$

rank 1

$\boxed{\text{not observable}}$

$$CA = \begin{bmatrix} 0.5 & 0.25 & -0.375 \end{bmatrix} \begin{bmatrix} -5 & 1 & 3 \\ 2 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix}$$

$$= [-2 \;\; -1 \;\; 1.5]$$

$$CA^2 = \begin{bmatrix} 0.5 & 0.25 & -0.375 \end{bmatrix} \begin{bmatrix} -5 & 1 & 3 \\ 2 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix}^2$$

$$= [8 \;\; 4 \;\; -6]$$

$\lambda - IA$

$$\begin{bmatrix} \lambda+5 & -1 & -3 \\ -2 & \lambda & 0 \\ 0 & -4 & \lambda \end{bmatrix}$$

$\lambda = -4, -3, 2$

$\boxed{\text{lyap unstable}}$

b) $G(s) = C(sI-A)^{-1}B = [0.5 \;\; 0.25 \;\; -0.375] \left(s \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -5 & 1 & 3 \\ 2 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

$$M^{-1} = \begin{bmatrix} -2 & -1 & 1.5 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$\hat{A} = M^{-1}AM$ matlab $\begin{bmatrix} -4 & 0 & 0 \\ 0 & 0 & 4 \\ -1 & 1.5 & -1 \end{bmatrix}$

$\hat{B} = M^{-1}B$ matlab $\begin{bmatrix} -4 \\ 0 \\ 0 \end{bmatrix}$

$\hat{C} = CM = [-0.25 \;\; 0 \;\; 0]$

$$\boxed{\begin{aligned} \dot{x} &= -4x - 4u \\ y &= -0.25x \end{aligned}}$$

values from Ao, Ac...

$\tau = 0.001s$

P controller

$$\frac{k_p \frac{1}{s+4}}{1 + k_p \frac{1}{s+4}} = \frac{k_p}{s+4+k_p}$$

$0.001 = \tau = -\frac{1}{-4-k}$ $\qquad \frac{1}{4+k} = 0.001$

$\boxed{k_p = 996}$