1: 10 points

For a SISO system, show that the H_2 and H_∞ norms are invariant to time delays and all-pass filters, i.e. show $\|QG\|_2 = \|G\|_2$ and $\|QG\|_\infty = \|G\|_\infty$ for $Q = e^{-sT}$ and $Q = \frac{s-a}{s+a}$ with $a > 0$.

a) $Q = e^{-sT}$

$\|QG\|_2 = \|G\|_2$

$$\|e^{-sT}G\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} |e^{-j\omega T} G(j\omega)|^2 d\omega}$$

$$= \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} |\underbrace{e^{-j\omega T}}_{=1}|^2 |G(j\omega)|^2 d\omega}$$

$$= \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2 d\omega}$$

$$\boxed{= \|G\|_2}$$

$$\|e^{-sT}G\|_\infty = \max_\omega \sqrt{|e^{-j\omega T} G(j\omega)|^2}$$

$$= \max_\omega \sqrt{|G(j\omega)|^2 d\omega}$$

$$\boxed{= \|G\|_\infty}$$

b) $Q = \frac{s-a}{s+a}$

$$\left\|\frac{s-a}{s+a} G(j\omega)\right\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{s-a}{s+a}\right|^2 |G(j\omega)|^2 d\omega}$$

$$= \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} |G(j\omega)|^2 d\omega}$$

$$\boxed{= \|G\|_2}$$

$$\left\|\frac{s-a}{s+a} G(j\omega)\right\|_\infty = \max_\omega \sqrt{\left|\frac{s-a}{s+a}\right|^2 |G(j\omega)|^2}$$

$$= \max_\omega \sqrt{|G(j\omega)|^2}$$

$$\boxed{= \|G\|_\infty}$$

2: 30 points

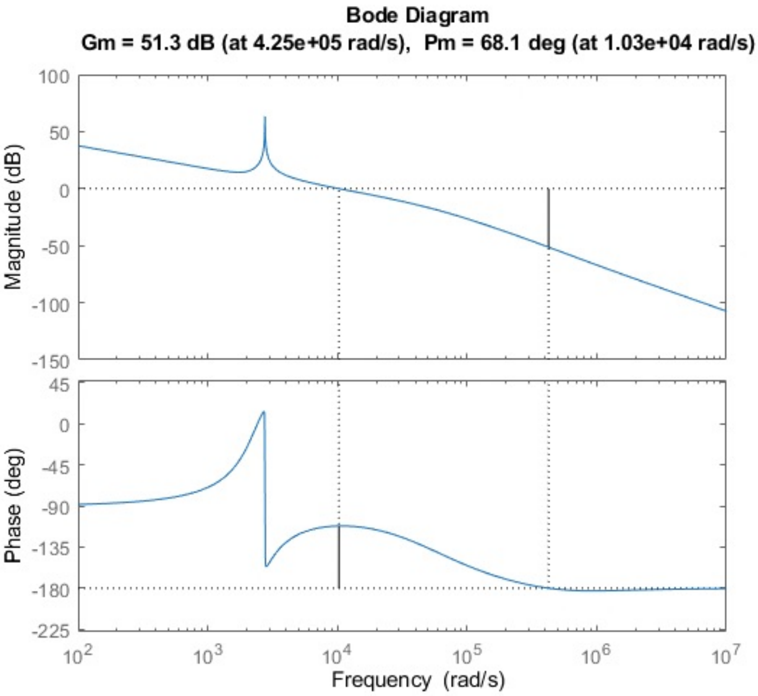
MIMO Design of an X-Y Scanning Stage

X-Y scanning stages are commonly used for atomic force microscopy (AFM) and other nanopositioning applications. While much work continues on decoupling the X and Y motion axes mechanically, the small amount of remaining coupling must be handled through controller design. For this problem we will use the following model from *Design, Fabrication, and Control of a Micro X-Y Stage with Large Ultra-thin Film Recording Media Platform* by Lu et. al. (2005)

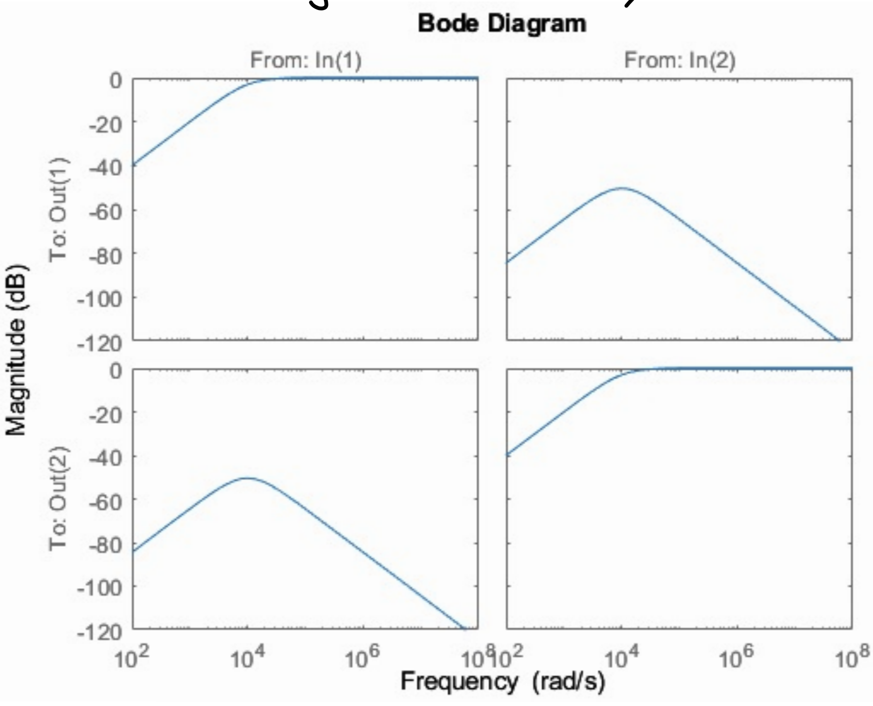
$$G(s) = 20\frac{(2\pi 440)^2}{s^2 + 2(0.0009)2\pi 440 s + (2\pi 440)^2}\begin{bmatrix} 1 & 0.006 \\ 0.006 & 1 \end{bmatrix}$$

(a) First design a controller using a SISO assumption (ignore cross coupling). Use *loopsyn* to design a controller that achieves (attempts anyway) a loopshape of $L = \frac{10000}{s}$ for one of the diagonal entries of G. Use the margin command to plot the achieved loop shape. Now make a diagonal DIDO controller with the designed K on the diagonal and plot the magnitude of the sensitivity function of the DIDO system.

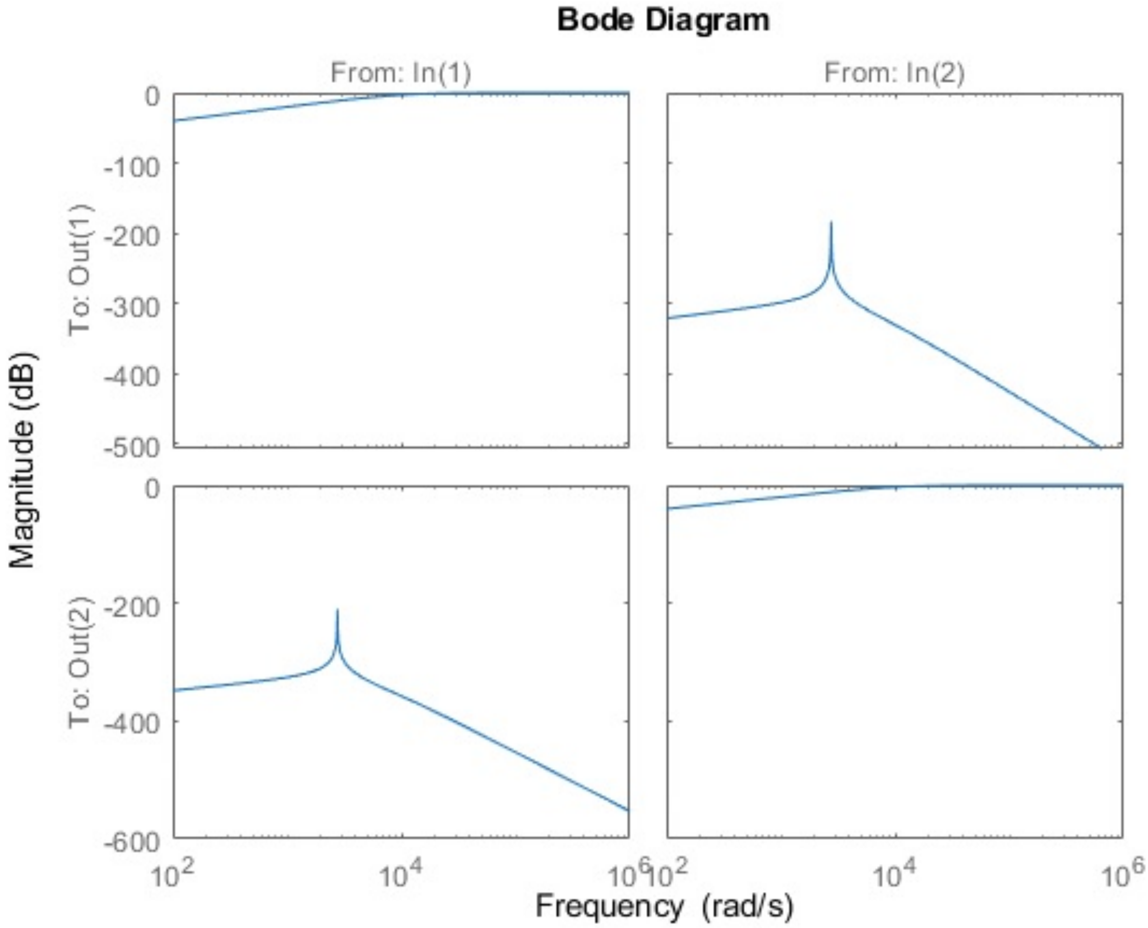
Achieved Loop Shape



Magnitude of Sensitivity Function

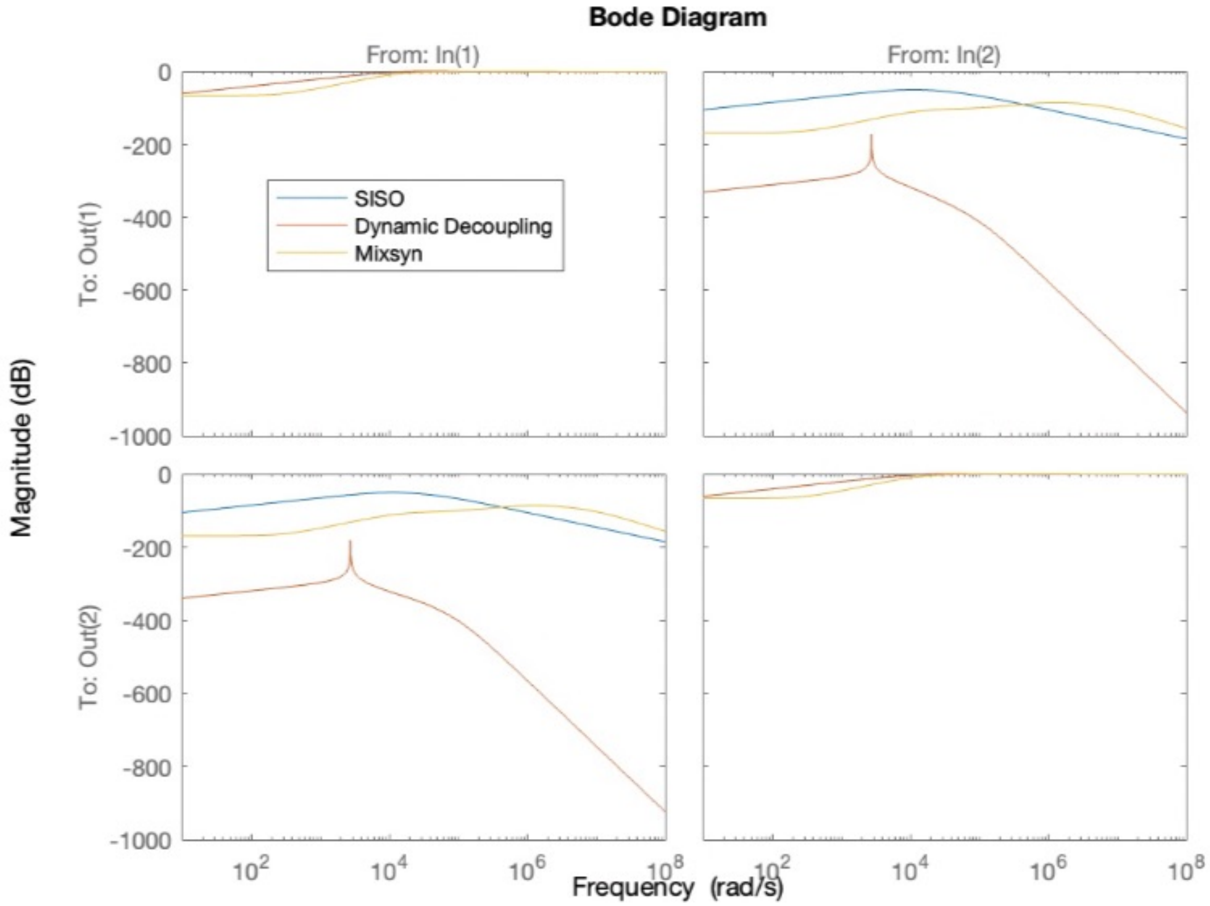


(b) Now let's incorporate dynamic decoupling to better control the system. Repeat the design of Part a using dynamic decoupling and plot the magnitude of the sensitivity function of the DIDO system.

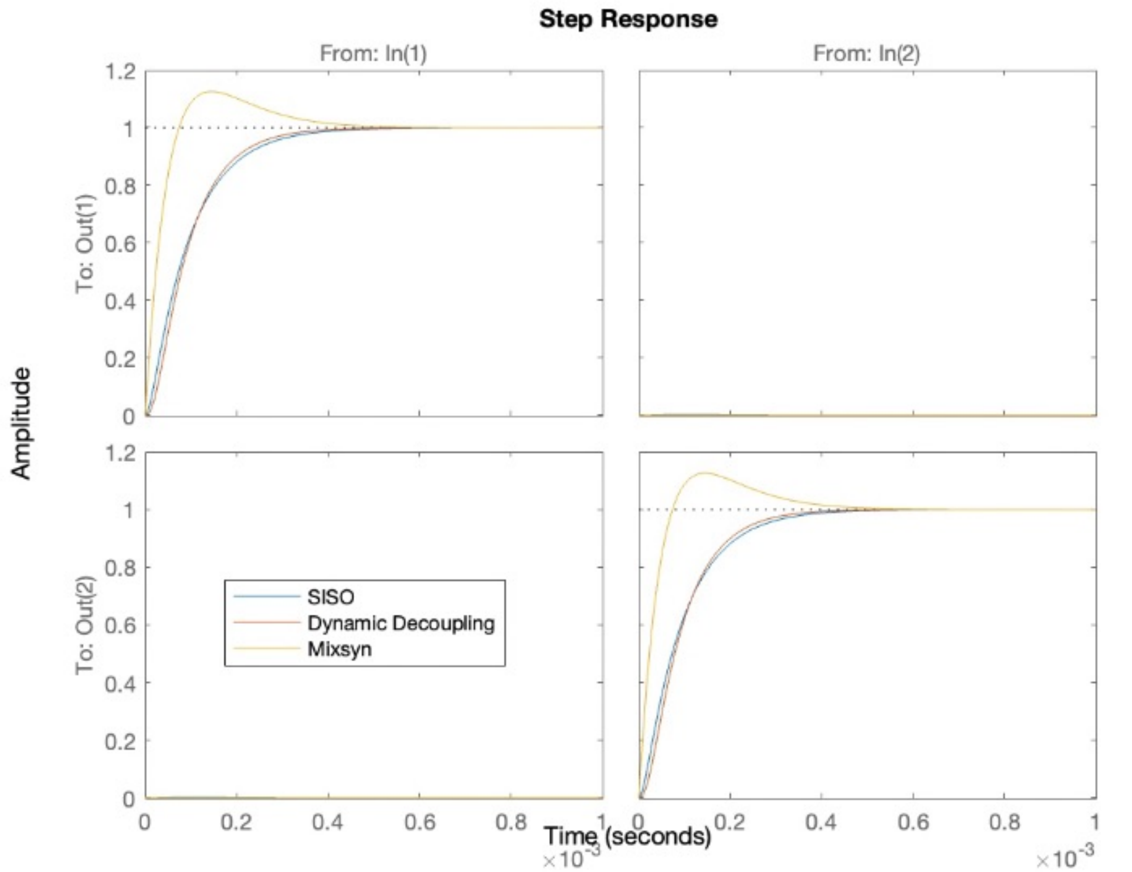


(c) Finally, design a controller for the DIDO system using *mixsyn* that uses a diagonal 2nd order performance weight with sensitivity peak of 2, low frequency sensitivity gain of 1/1000, and sensitivity crossover of 10000 rad/s. Find the sensitivity function and plot the sensivity function magnitude on the same plot as the other designs. Also, plot the step response of the closed loop system for all three designs.

Magnitude Sensitivity Function



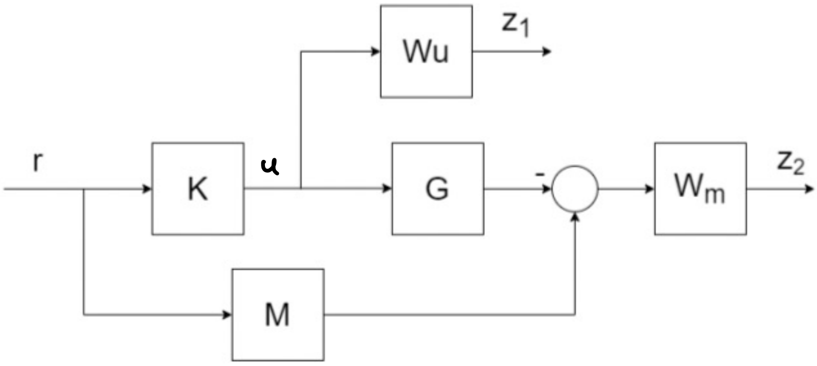
Step Response of Closed Loop



3: 30 points

Generalized Plants

Inverse plant models are often used for feedforward control, but for various reasons (e.g., model uncertainty / RHP zeros) they can be difficult to find. The H_∞ model matching problem can be used to find the "best" inverse approximation to a given plant. The block diagram is shown below, with W_u weighting the controller usage and W_m weighting the model mismatch error. Here M is the model that we are trying to match; for plant inversion, we will try to match $M = I$.

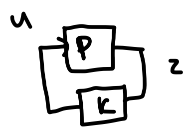


(a) (Written) Find the generalized plant P.

$z_1 = W_u \cdot u \quad = W_u \cdot K \cdot r$

$z_2 = W_m(-G \cdot u + M \cdot r)$

$v = r$



$$\begin{bmatrix} z_1 \\ z_2 \\ v \end{bmatrix} = \begin{bmatrix} 0 & W_u \\ MW_m & -GW_m \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r \\ u \end{bmatrix}$$

$$\boxed{P = \left[\begin{array}{c|c} 0 & W_u \\ MW_m & -GW_m \\ \hline 1 & 0 \end{array}\right]}$$

(b) **(Written)** Find the closed loop transfer function N such that $z = Nw$.

$$z = Nw$$

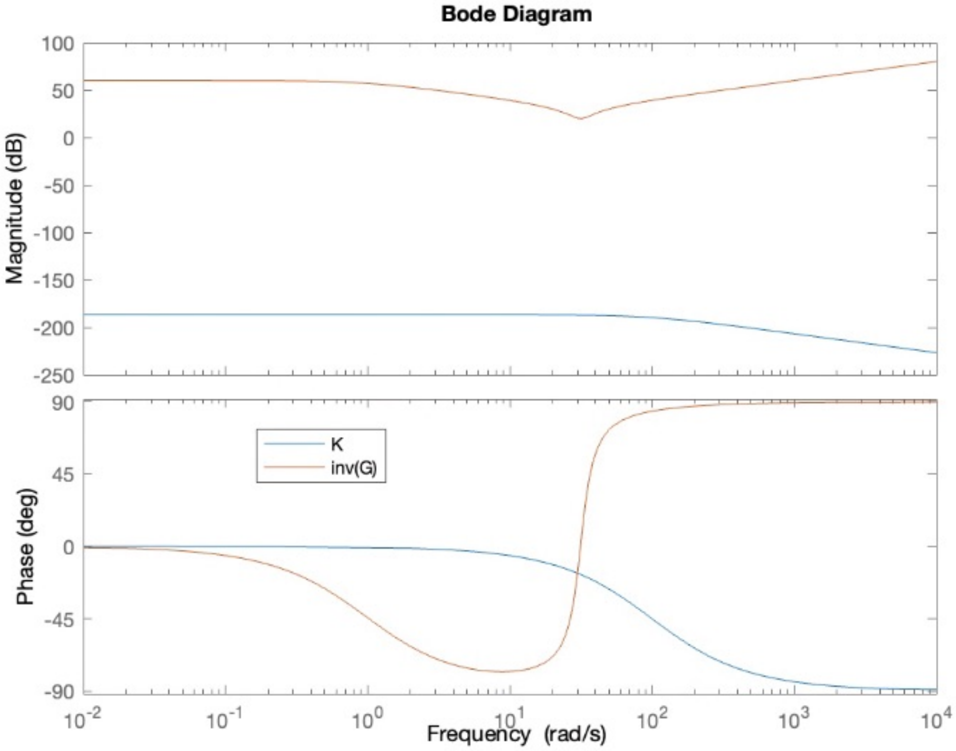
$$N = P_{11} + P_{12} K (I - P_{22} K)^{-1} P_{21}$$

$$= \begin{bmatrix} 0 \\ MW_m \end{bmatrix} + \begin{bmatrix} W_u \\ -GW_m \end{bmatrix} K (1-0)^{-1} [1]$$

$$\boxed{= \begin{bmatrix} W_u \cdot K \\ MW_m K - GW_m K \end{bmatrix}}$$

(c) (Matlab) Consider the plant $G(s) = \frac{s+1}{s^2+10s+1000}$ along with weights $W_u = 1/100$ and $W_m = \frac{s+10{,}000}{0.01s+1}$. Here W_m is a low pass filter that determines the frequency range over which the match is emphasized. Now solve the optimal inversion problem using the generalized plant above (use *hinfsyn*) and provide the Bode plot of the resulting K and G^{-1} on the same plot AND the Bode plot of $L = GK$.

K and G^{-1}



L=GK