1: 10 points

Zeros found through the solution to the generalized eigenvalue problem are known as *invariant* zeros because they are unchanged by state feedback $u = Kx$. To see this, show that

$$\text{rank}\left(\begin{bmatrix} A + BK - sI & B \\ C + DK & D \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}\right).$$

to simplify, if MIMO

$$\begin{bmatrix} A+BK-SI & B \\ C+DK & D \end{bmatrix} = \begin{bmatrix} A-sk & B \\ C & D \end{bmatrix}\begin{bmatrix} I & 0 \\ k & I \end{bmatrix} \quad \begin{matrix} (A-sk)I + BK & (A-sk)\cdot 0 + B\cdot I \\ C\cdot I + D\cdot k & C\cdot 0 + D\cdot I \end{matrix}$$

according to google and piazza
the rank of the multiplied matrix is equal to the rank of A if B is of full rank

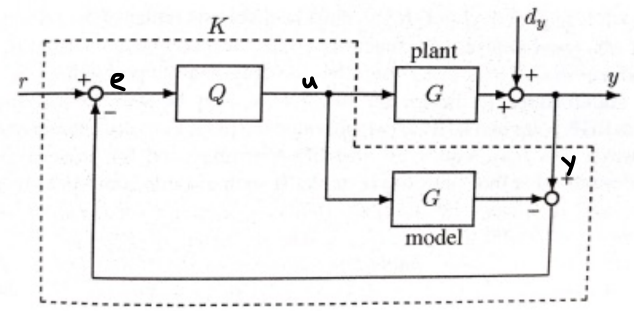
since

$$r\begin{bmatrix} A+BK-SI & B \\ C+DK & D \end{bmatrix}$$ is full, it is equal to $$r\begin{bmatrix} A-sk & B \\ C & D \end{bmatrix}$$ as $$r\begin{bmatrix} I & 0 \\ k & I \end{bmatrix}$$ is full because of I's

2: 20 Points

Internal Stability

(a) The control structure shown below is called Internal Model Control (IMC). Show that the IMC structure is internally unstable if Q is unstable. Here assume that the model G is identical to the plant G.



$u = Qe$

$e = r - y \rightarrow r - GQr$

perfect control at model = plant and $Q = \text{plant}^{-1}$

$u = Qr$ perfect

$e = r - GQr$

$e = (1 - GQ)r$

$r = (1 - GQ)^{-1} e$

$u = Q(1 - GQ^{-1})e$

$K = Q(1 - GQ^{-1})$

$\downarrow$

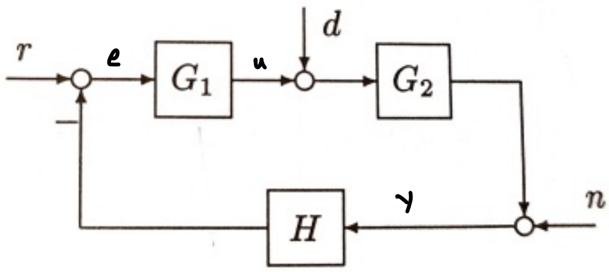
$Q = (1 + KG^{-1})K$ or $K(1 + GK)^{-1}$

$u = Ke$

$K = \frac{u}{e}$

Q must be stable for IMC to be stable

(b) Recalling that a feedback system is internally stable $\iff$ all closed loop transfer functions are stable, find the conditions for internal stability of the feedback system shown. How do those simplify if $H(s)$ and $G_1(s)$ are both stable? You can assume that the systems are SISO throughout.



$u = d + G_1(r - H(n + G_2 u))$

$\hookrightarrow u(I + G_1 G_2 H) = d + G_1 r - G_1 H n$

$u = \dfrac{d + G_1 r - G_1 H n}{I + G_1 G_2 H}$

$e = r - H(n + G_2(d + G_1 e))$

$e(1 + HG_1G_2) = r - Hn - HG_2 d$

$e = \dfrac{r - Hn - HG_2 d}{1 + HG_1G_2}$

$y = n + G_2(d + G_1(r - Hy))$

$y(1 + G_1G_2H) = n + G_2 d + G_2 G_1 r$

$y = \dfrac{n + G_2 d + G_2 G_1 r}{1 + G_1 G_2 H}$

for stability - internal

- $G_1(I + G_1G_2H)^{-1}H$
- $G_1(I + HG_1G_2)^{-1}$
- $I - H(I + G_1G_2H)^{-1}G_1G_2$
- $(I + G_2HG_1)^{-1}G_2$
- $I - (I + G_1HG_2)^{-1}G_1HG_2$
- $(I + G_1G_2H)^{-1}G_1G_2$
- $(I + HG_1H_2)^{-1}H$
- $H(I + HG_1G_2)^{-1}G_2$

if G_1 and H are stable only check

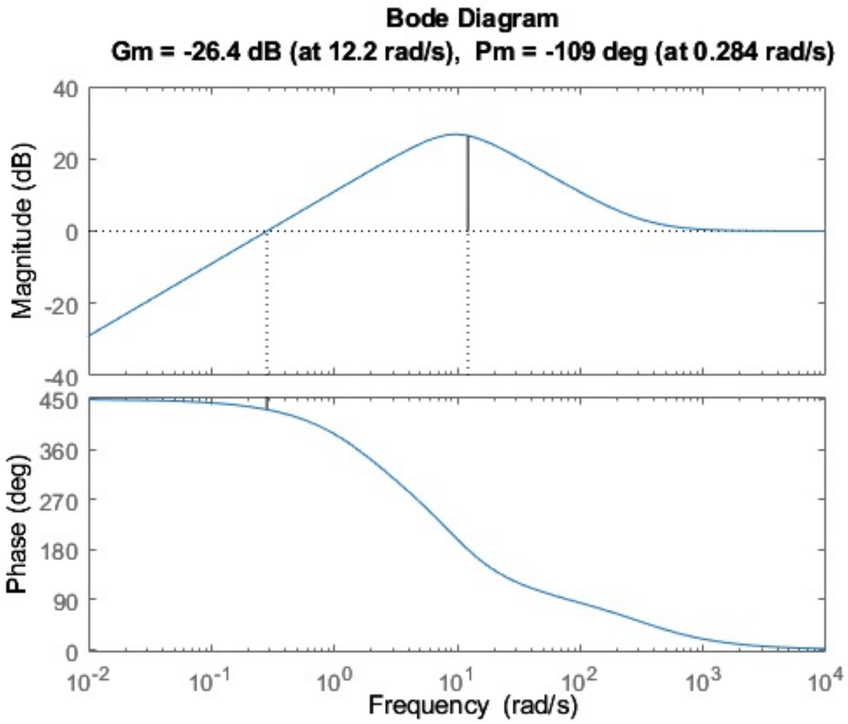
$(I + G_1G_2H)^{-1}G_2$

3: 20 points

Consider the plant model

$$G(s) = \frac{s-1}{s(s-2)}.$$

(a) Based on the RHP poles and zeros of the system, and assuming a conventional loop shape, our rules of thumb suggest the bandwidth must be both < 0.5 rad/s and > 1 rad/s. That seems like a problem, but let's see what a synthesis routine can do with this. Use loopsyn to design a controller given a target loopshape of $L_d = \frac{10}{s}$ and plot the achieved sensitivity function and step response. What are the gain and phase margins?



(b) Compute the minimum peak of the sensitivity function. Does your result from Part a match your computation?

min peak $= 20\log_{10}$ (Sensitivity function)

$$\sqrt{\frac{|1+2|^2}{|1-2|^2}} = 3$$

$20\log_{10}(3)$

$= 9.54$ dB

Results dont match Part a which is expected due to the mismatch bandwidth requirement.