nota si $i=1,2,\ldots,q$ y como $\mu=\lambda$

$$\left.\frac{\partial f(\pi)}{\partial T_i}\right|_{\pi=\lambda} = \frac{1-\psi_q}{tr(\Sigma)} .$$

Por otra parte, si $i=q+1,\ldots,p$

$$\left.\frac{\partial f(\pi)}{\partial T_i}\right|_{\pi=\lambda} = \frac{-\psi_q}{tr(\Sigma)} .$$

Considerando que $\Sigma = 2\Lambda^2 = 2\begin{pmatrix}\lambda_1^2 & 0 & \cdots & 0\\ 0 & \lambda_2^2 & \cdots & 0\\ \vdots & \vdots & \ddots & \\ 0 & 0 & & \lambda_p^2\end{pmatrix}$

$$D' = \left.\left(\frac{\partial f(\pi)}{\partial T_1},\ldots,\frac{\partial f(\pi)}{\partial T_p}\right)\right|_{\pi=\lambda}$$

$$= \left(\underset{1}{\frac{1-\psi_q}{tr(\Sigma)}},\ldots,\underset{q}{\frac{1-\psi_q}{tr(\Sigma)}},\underset{q+1}{-\frac{\psi_q}{tr(\Sigma)}},\ldots,\underset{p}{-\frac{\psi_q}{tr(\Sigma)}}\right)$$

$$= \frac{1}{tr(\Sigma)}\left(1-\psi_q,\ldots,1-\psi_q,-\psi_q,\ldots,-\psi_q\right).$$

De forma que $D'\Sigma D =$

$$\frac{2}{(tr(\Sigma))^2}\left(1-\psi_q,\ldots,1-\psi_q,-\psi_q,\ldots,-\psi_q\right)\begin{pmatrix}\lambda_1^2 & 0 & \cdots & 0\\ 0 & \lambda_2^2 & \cdots & 0\\ \vdots & \vdots & \ddots & \\ 0 & 0 & & \lambda_p^2\end{pmatrix}\begin{pmatrix}1-\psi_q\\ \vdots\\ 1-\psi_q\\ -\psi_q\\ \vdots\\ -\psi_q\end{pmatrix}$$

$$D' \mathbf{\Sigma} D = \left[(1-\psi_q)^2\lambda_1^2+\cdots+(1-\psi_q)^2\lambda_q^2+\psi_q^2\lambda_{q+1}^2+\cdots+\psi_q^2\lambda_p^2\right]$$

$$\times \frac{2}{(tr(\mathbf{\Sigma}))^2}$$

$$= \frac{2}{(tr(\mathbf{\Sigma}))^2} \times \left[(1-\psi_q)^2(\lambda_1^2+\cdots+\lambda_q^2)+\psi_q^2(\lambda_{q+1}^2+\cdots+\lambda_p^2)\right]$$

$$= \frac{2}{(tr(\mathbf{\Sigma}))^2} \times \left[\psi_q^2(\lambda_1^2+\cdots+\lambda_p^2)-2\psi_q(\lambda_1^2+\cdots+\lambda_q^2)+(\lambda_1^2+\cdots+\lambda_q^2)\right].$$

Sea $\beta = \dfrac{\lambda_1^2+\cdots+\lambda_q^2}{\lambda_1^2+\cdots+\lambda_p^2}$, $\quad \lambda_1^2+\cdots+\lambda_q^2 = \beta(\lambda_1^2+\cdots+\lambda_p^2)$,

entonces

$$D' \mathbf{\Sigma} D = \frac{2(\lambda_1^2+\cdots+\lambda_p^2)}{(tr(\mathbf{\Sigma}))^2} \times \left[\psi_q^2-2\psi_q\beta+\beta\right].$$

Así, para encontrar la distribución asintótica de $\hat{\psi}_q = f(l_1,\ldots,l_p)$, usamos los teoremas DAUP y TEAN para obtener el siguiente

TEOREMA (Distribución Asintótica de $\hat{\psi}_q$)

Para $q \leq p$, Sean $\psi_q = \dfrac{\lambda_1+\cdots+\lambda_q}{\lambda_1+\cdots+\lambda_p}$ y $\hat{\psi}_q = \dfrac{l_1+\cdots+l_q}{l_1+\cdots+l_p}$

La proporción de la variabilidad de los datos explicada por las primeras q componentes principales y su estimación muestral. Entonces, cuando el tamaño de muestra n crece a $+\infty$.

$$\sqrt{n-1}\left(\hat{\psi}_q - \psi_q\right) \xrightarrow{d} N_1(0, \omega^2), \quad \cdots (u)$$

donde

$$\omega^2 = \frac{2(\lambda_1^2 + \cdots + \lambda_p^2)}{(tr(\Sigma))^2} \times \left\{ \psi_q^2 - 2\psi_q \beta + \beta \right\}$$

y $$\beta = \frac{\lambda_1^2 + \cdots + \lambda_q^2}{\lambda_1^2 + \cdots + \lambda_p^2}.$$

Con este resultado, podemos construir un intervalo de confianza a nivel $(1-\alpha) \times 100\%$ para el verdadero valor de la proporción ψ_q.

Usando (u); si n es suficientemente grande

$$\mathbb{P}\left(\left| \frac{\hat{\psi}_q - \psi_q}{\omega / \sqrt{n-1}} \right| \leq Z_{1-\alpha/2} \right) \overset{\text{aprox}}{\approx} 1-\alpha, \quad \cdots (v)$$

donde $\alpha \in (0,1)$ es un tamaño de error (nivel de significancia) y $Z_{1-\alpha/2}$ es el cuantil de una distribución $N(0,1)$ correspondiente a probabilidad $1-\alpha/2$:

$$\int_{-\infty}^{z_{1-\alpha/2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx = 1-\alpha/2 .$$

$$\begin{aligned}
1-\alpha &\overset{\text{aprox}}{\approx} \mathbb{P}\left(\left| \frac{\hat{\psi}_q - \psi_q}{w/\sqrt{n-1}} \right| \le z_{1-\alpha/2} \right) \\
&= \mathbb{P}\left(-z_{1-\alpha/2} \le \frac{\hat{\psi}_q - \psi_q}{w/\sqrt{n-1}} \le z_{1-\alpha/2} \right) \\
&= \mathbb{P}\left(-z_{1-\alpha/2} \le \frac{\psi_q - \hat{\psi}_q}{w/\sqrt{n-1}} \le z_{1-\alpha/2} \right) \\
&= \mathbb{P}\left(\hat{\psi}_q - w/\sqrt{n-1} \times z_{1-\alpha/2} \le \psi_q \le \hat{\psi}_q + w/\sqrt{n-1} \times z_{1-\alpha/2} \right).
\end{aligned}$$

Entonces, a nivel $(1-\alpha)\times 100\%$ de confianza ψ_q debe estar en el intervalo

$$\left(\hat{\psi}_q - w/\sqrt{n-1} \times z_{1-\alpha/2} \,,\ \hat{\psi}_q + w/\sqrt{n-1} \times z_{1-\alpha/2} \right).$$