UNIT VI

APPLICATION OF TRIPLE INTEGRATION : VOLUME

Triple Integral

- Triple integration is used to find volume of solids. Volume of solids by using triple integration is given by

$$V = \iiint\limits_{V} dxdydz$$

Remarks

1) Volume in Cartesian form$= \iiint\limits_{V} dxdydz$
2) Volume in Spherical Polar system $= \iiint\limits_{V} r^2 sin\theta drd\theta d\emptyset$

Cartesian System can be converted into polar co-ordinate system by using

$x = rcos\theta \ ; y = rsin\theta \ ; dxdy = r\, drd\theta \ ; \ x^2 + y^2 = r^2$

3) Volume in Cylindrical Polar form $= \iiint\limits_{V} \rho d\rho d\emptyset dz$

Note : we convert the given example in to polar system in two cases

1) If given curve are in polar co-ordinate system

2) If the given curve equation are related to circle $x^2 + y^2 = r^2$

Steps to solve problems on Volume in Cartesian Form

Step 1 :

Decide the formula for Volume in Cartesian form $= \iiint dxdydz$

Step 2 :

Convert triple integral into double integral using limits of z

$$Volume = \iiint dxdydz = \iint_R \left\{ \int_{z=f_1(x,y)}^{f_2(x,y)} dz \right\} dxdy$$

Where R is the projection surface on the XY-plane

Step 3 :

Solve the double integral

$$V = \iint_R g(x,y)dxdy$$

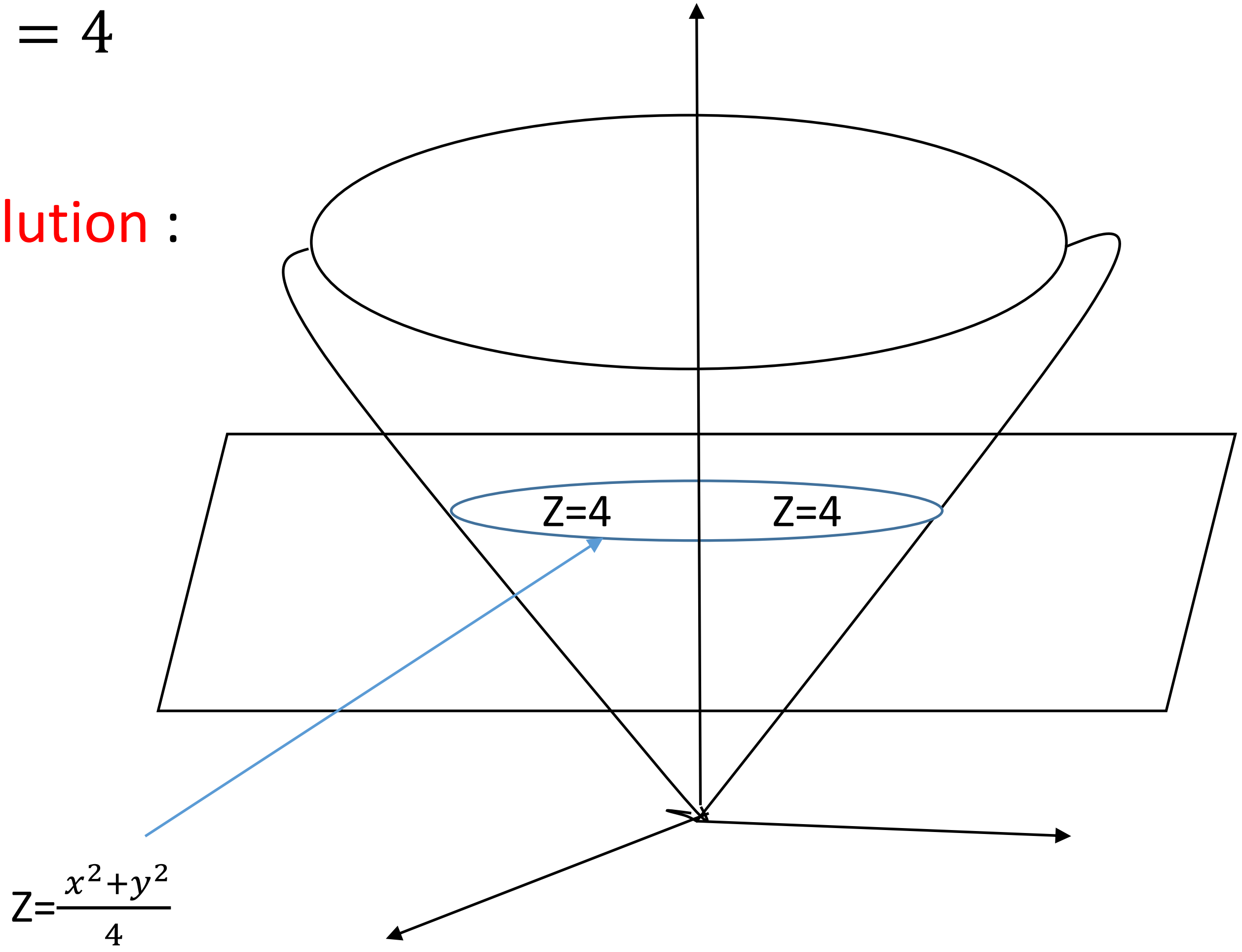
How to find Region R

$$V = \iint_R g(x, y)dxdy$$

1) Draw the given surface and take the projection on XY plane
2) From the given equation just find an equation which is free from z or eliminate z

2) Find the volume of the paraboloid of revolution $x^2 + y^2 = 4z$ cut of the plane $z = 4$

Solution :



Limits of z are

$$z = \frac{x^2+y^2}{4} \ \ to \ \ z = 4$$

Step 1 : Volume is given by

$$V = \iiint dxdydz$$

$$V = \iint \int_{z=\frac{x^2+y^2}{4}}^{4} dz\, dxdy$$

$$V = \iint [z]_{z=\frac{x^2+y^2}{4}}^{z=4} dxdy$$

$$V = \iint \left[4 - \frac{x^2+y^2}{4}\right] dxdy$$

$$V = \iint \left[4 - \frac{x^2+y^2}{4}\right] dxdy$$

Converting to polar by using

$$x^2 + y^2 = r^2$$
$$dxdy = rdrd\theta$$

$$\therefore V = \iint \left[4 - \frac{x^2 + y^2}{4}\right] dxdy$$

$$\therefore V = \iint \left[4 - \frac{r^2}{4}\right] rdrd\theta$$

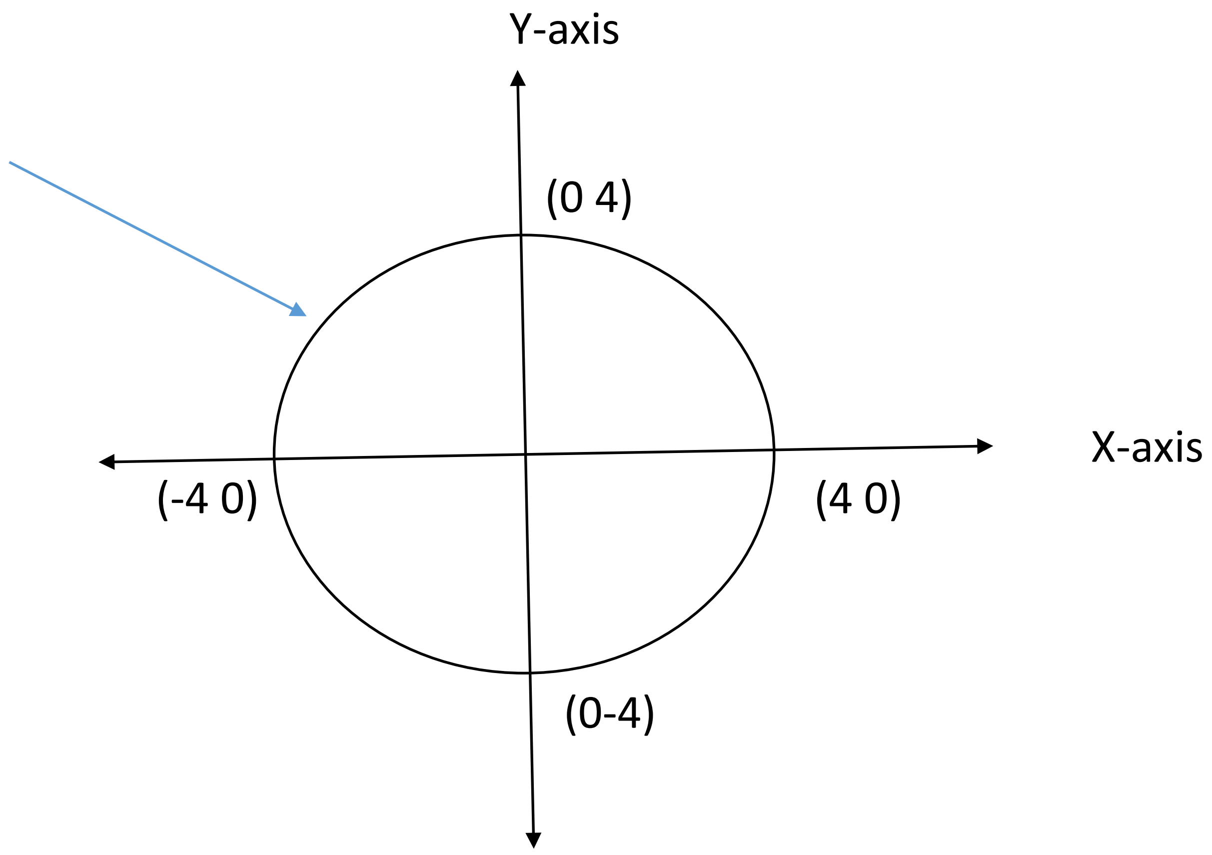
$$\therefore V = \iint \left[4r - \frac{r^3}{4}\right] drd\theta \qquad \text{--------------------------------} \quad (1)$$

Step 3 : Points of intersection

Note that paraboloid $x^2 + y^2 = 4z$ is cut off by the plane $z = 4$

$\therefore x^2 + y^2 = 4(4)$

$\therefore x^2 + y^2 = (4)^2$



Step 4 : Strip

$\therefore r = 0$ to $r = 4$

$\theta = 0$ to $\theta = {}^{\pi}/_{2}$

Step 5 : Apply Limits

From 1st

$$\therefore V = \iint \left[4r - \frac{r^3}{4}\right] dr d\theta$$

$$V = \int_0^{\pi/2} \left\{ \int_0^4 \left[4r - \frac{r^3}{4}\right] dr \right\} d\theta$$

$$V = \int_0^{\pi/2} d\theta \left\{ \int_0^4 \left[4r - \frac{r^3}{4} \right] dr \right\}$$

$$V = [\theta]_0^{\pi/2} \left\{ \left[\frac{4r^2}{2} - \frac{r^4}{16} \right]_0^4 \right\}$$

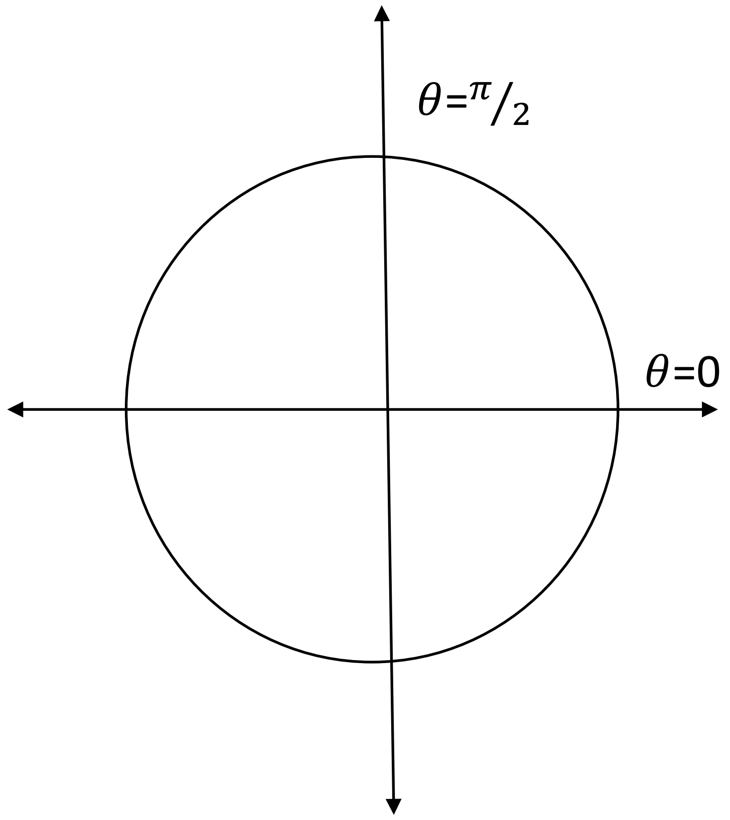
$$V = \left[\frac{\pi}{2} - 0 \right] \left\{ \frac{4(4^2)}{2} - \frac{4^4}{16} - 0 \right\}$$

$$V = 8\pi$$

For Symmetry

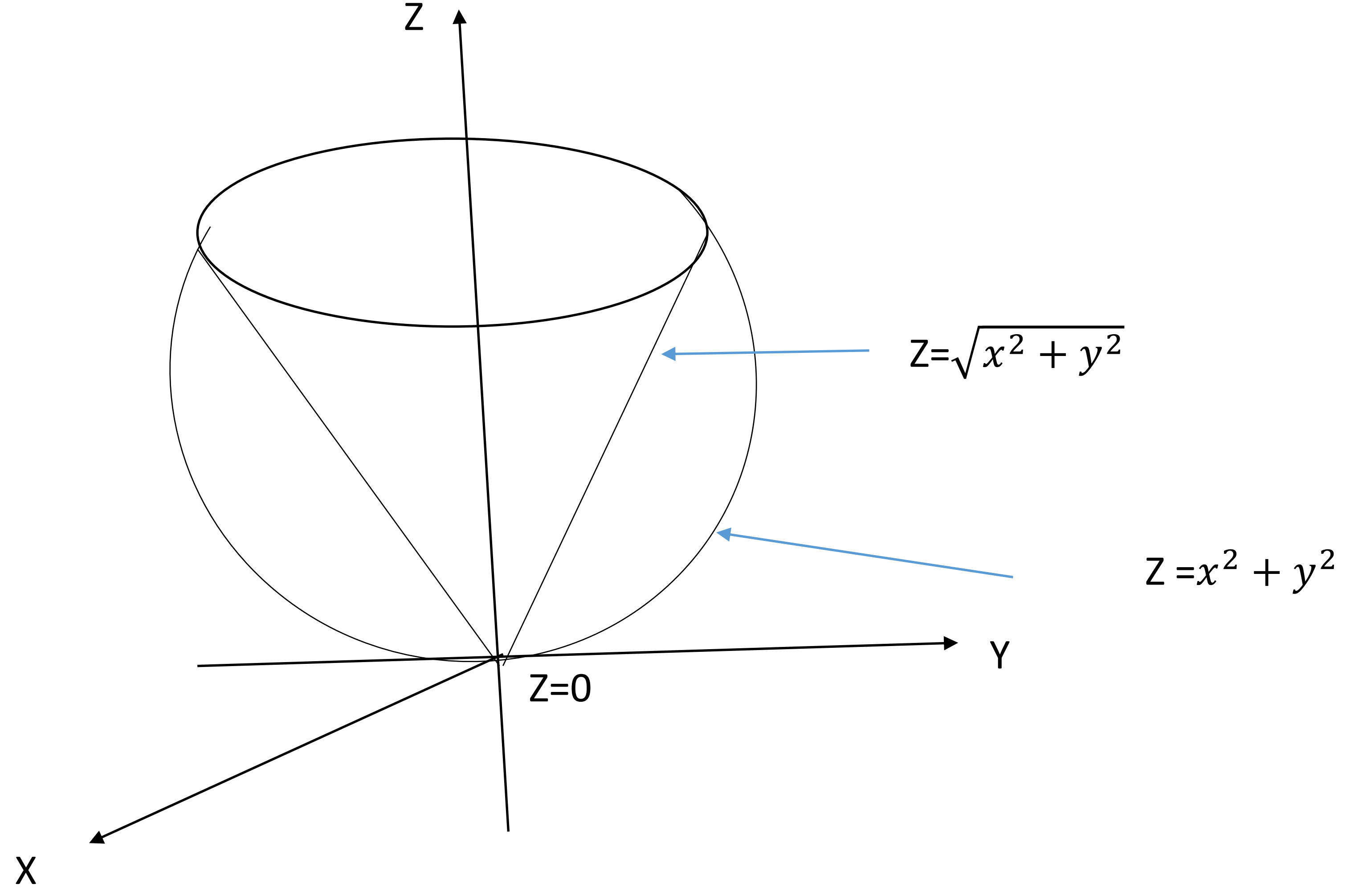
$$V = 4 \times 8\pi$$

$$V = 32\pi$$



3) Find the volume of the region enclosed by the cone $z = \sqrt{x^2 + y^2}$ and paraboloid plane $z = x^2 + y^2$

Solution :



Limit of z are $\quad z = 0 \quad to\, z = \sqrt{x^2 + y^2}$

Step 1 : Volume is given by

$$V = \iiint dxdydz$$

$$V = \iint \int_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} dz\, dxdy$$

$$V = \iint [z]_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} dxdy$$

$$V = \iint \left[\sqrt{x^2 + y^2} - x^2 + y^2\right] dxdy$$

Converting to polar by using

$$x^2 + y^2 = r^2$$
$$dxdy = rdrd\theta$$

$$\therefore V = \iint \left[\sqrt{r^2} - r^2\right] rdrd\theta$$

$$\therefore V = \iint [r - r^2]\, rdrd\theta$$

$$\therefore V = \iint [r^2 - r^3]\, drd\theta \quad \text{-------------------------------} \; (1)$$

Step3 : Points of intersection

Cone : $z=\sqrt{x^2+y^2}$ and paraboloid : $z=x^2+y^2$

$\therefore\ x^2+y^2=\sqrt{x^2+y^2}$

$\therefore\ \sqrt{x^2+y^2}\sqrt{x^2+y^2}=\sqrt{x^2+y^2}$

$\therefore\ \sqrt{x^2+y^2}=1$

Squaring both side

$x^2+y^2=1$ ------ circle

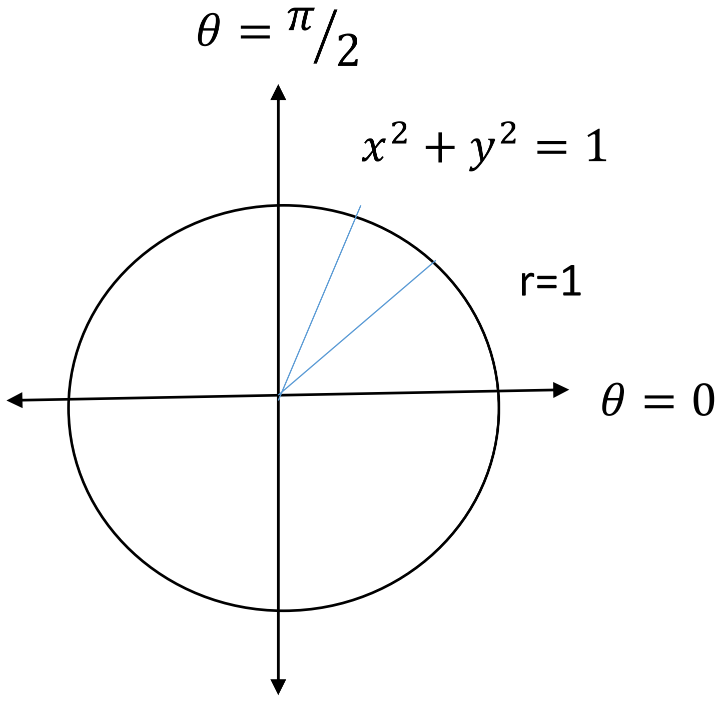
i.e.

$r^2=1$

$\mathrm{r}=1$

$\therefore\ \mathrm{r}=0\ \ to\ \ r=1$

$\therefore\ \theta=0\ \ to\ \ \theta={}^{\pi}/_{2}$



Step 4 : Strip

$\therefore\ \mathrm{r} = 0\ \ to\ \ r = 1$

$\therefore\ \theta = 0\ \ to\ \ \theta = \pi/2$

Step 5 : Apply Limits

From 1st

$$\therefore V = \iint [r^2 - r^3]\, dr d\theta$$

$$V = \int_0^{\pi/2} \left\{ \int_0^1 [r^2 - r^3]\, dr \right\} d\theta$$

$$V = \int_0^{\pi/2} d\theta \left\{ \int_0^1 [r^2 - r^3]\, dr \right\}$$

$$V = [\theta]_0^{\pi/2} \left\{ \left[\frac{r^3}{3} - \frac{r^4}{4} \right]_0^1 \right\}$$

$$V = [\theta]_0^{\pi/2} \left\{ \left[\frac{r^3}{3} - \frac{r^4}{4} \right]_0^1 \right\}$$

$$V = \left[\frac{\pi}{2} - 0 \right] \left\{ \frac{(1^3)}{3} - \frac{1^4}{4} - 0 \right\}$$

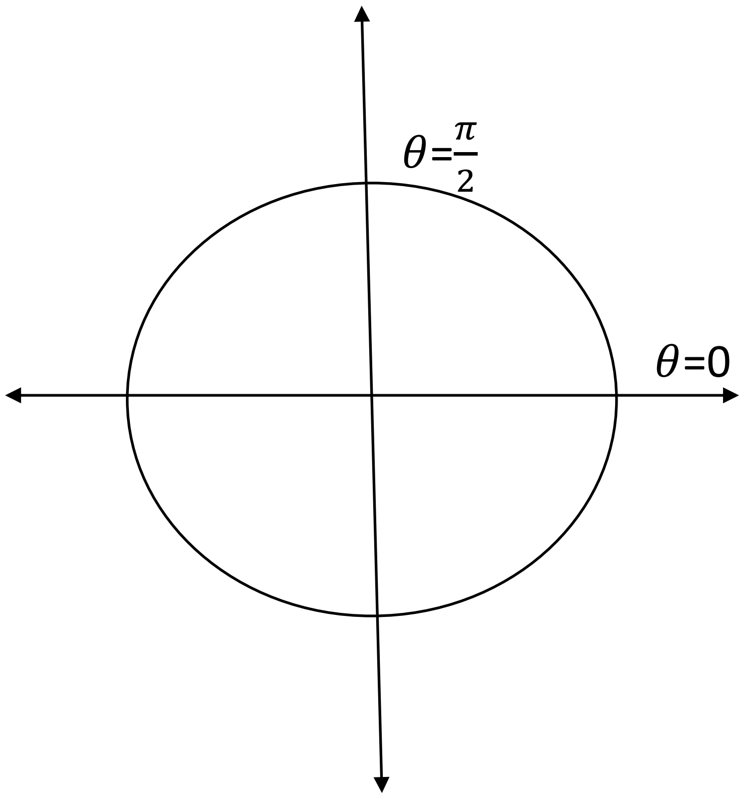
$$V = \left[\frac{\pi}{2} \right] \left\{ \frac{1}{12} \right\}$$

$$V = \frac{\pi}{24}$$

For Symmetry

$$V = 4 \times \frac{\pi}{24}$$

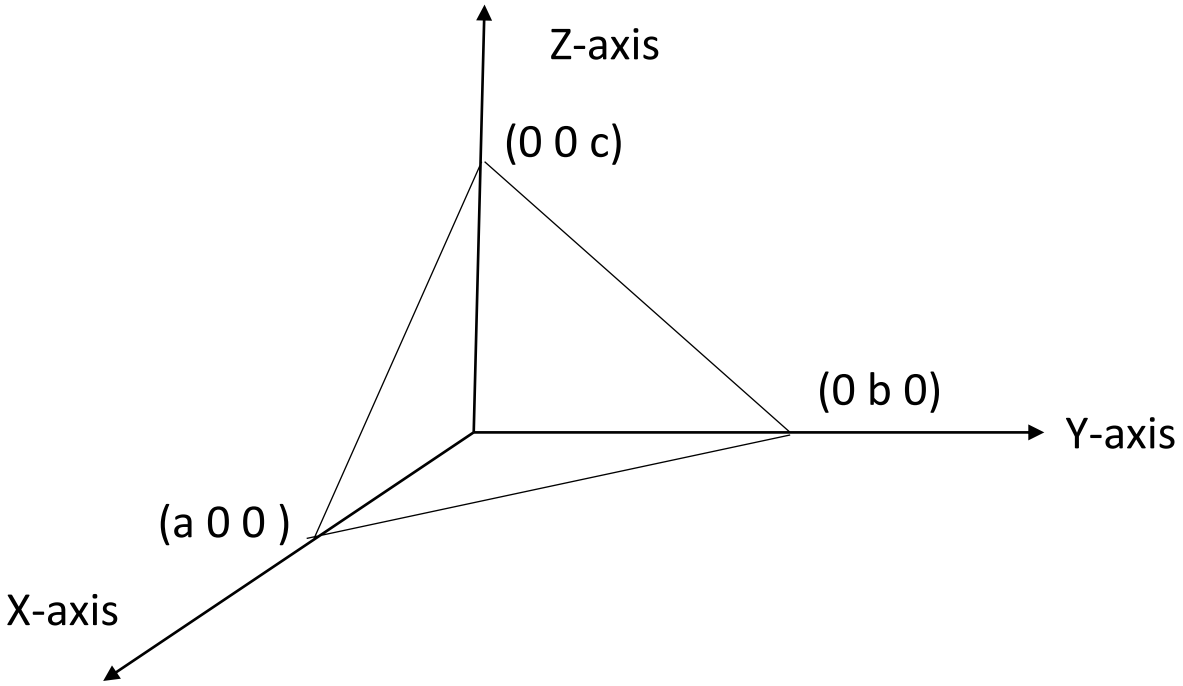
$$V = \frac{\pi}{6}$$



4) Find the volume of the tetrahedron bounded by co-ordinate plane and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Solution :

Tetrahedron is the diagram which has three faces which are triangles in the three co-ordinate planes as shown in fig



Note : For Tetrahedron solve triple integration by using Dirichlet's theorem

$$\iiint x^{l-1} y^{m-1} z^{n-1} dxdydz = \frac{\Gamma l \ \Gamma m \ \Gamma n}{\Gamma(l+m+n+1)}$$

Now

$$V = \iiint dxdydz$$

put and Differentiate

$$\frac{x}{a} = u \qquad \therefore \ x = au \quad \rightarrow \qquad dx = adu$$

$$\frac{y}{b} = v \qquad \therefore \ \mathrm{y} = bv \rightarrow \qquad dy = bdv$$

$$\frac{z}{c} = w \qquad \therefore \ \mathrm{z} = cw \rightarrow \qquad dz = cdw$$

$$V = \iiint adu \, bdv \, cdw$$

$$V = \iiint adu\, bdv\, cdw$$

$$V = abc \iiint du\, dv\, dw$$

Write

$$V = abc \iiint u^0 v^0 w^0 du\, dv\, dw$$

$$V = abc \iiint u^{1-1} v^{1-1} w^{1-1} du\, dv\, dw$$

By Dirichlet's theorem

$$\iiint x^{l-1} y^{m-1} z^{n-1} dxdydz = \frac{\Gamma l\ \ \Gamma m\ \Gamma n}{\Gamma(l+m+n+1)}$$

Here $l = 1$; $m = 1$; $n = 1$

$$V = abc\,\frac{\Gamma 1\ \Gamma 1\ \Gamma 1}{\Gamma(1+1+1+1)}$$

$$V = abc\,\frac{\Gamma 1\ \Gamma 1\ \Gamma 1}{\Gamma(4)}$$

We know $\Gamma n = (n-1)!\ and\ 0! = 1$

$$V = abc\,\frac{0!\,0!\,0!}{3!}$$

$$V = \frac{abc}{6}$$

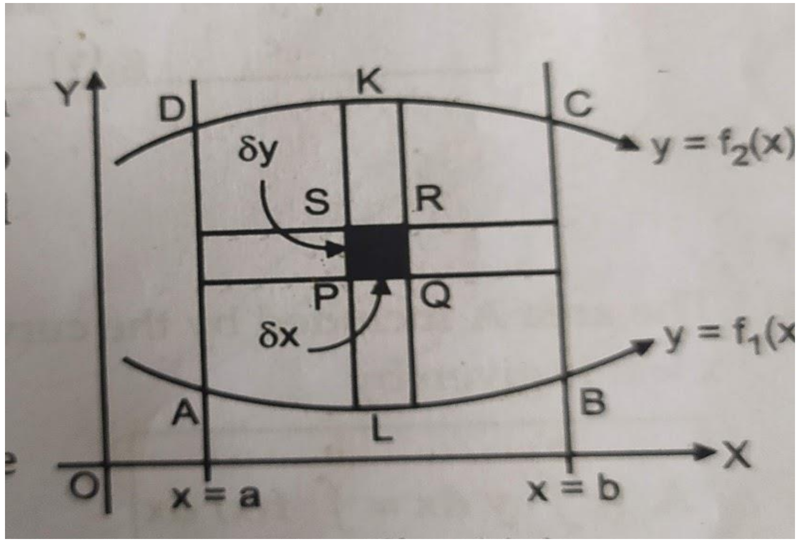
AREA,VOLUME,MEAN AND ROOT MEAN SQUARE VALUES

AREA ENCLOSED BY PLANE CURVES EXPRESSED IN CARTESIAN COORINATES

Consider the area enclosed by the curves y = $f_1(x)$ and y = $f_2(x)$ and the ordinates x= a ,x= b (a$< b$)

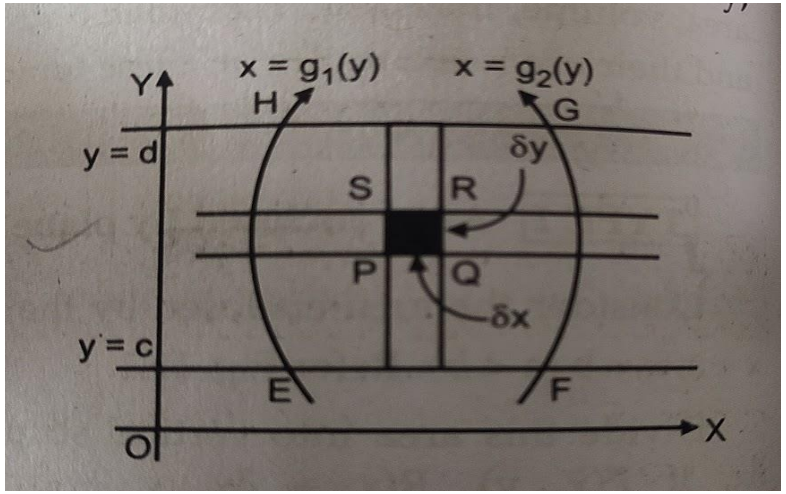
here δx *remains same and* δy *varies from* y = $f_1(x)$ to y = $f_2(x)$

Area =$\int_a^b dx \int_{f_1(x)}^{f_2(x)} dy$



Consider the area enclosed by the curves x = $g_1(y)$ and x = $g_2(y)$ and the ordinates y=c,y=d(c$< d$)
here δy *remains same and* δx *varies from*

Area $= \int_c^d dy \int_{g_1(y)}^{g_2(y)} dx$



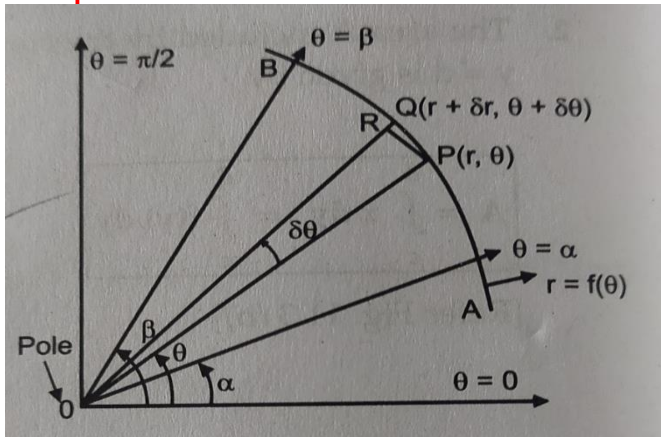
Note :

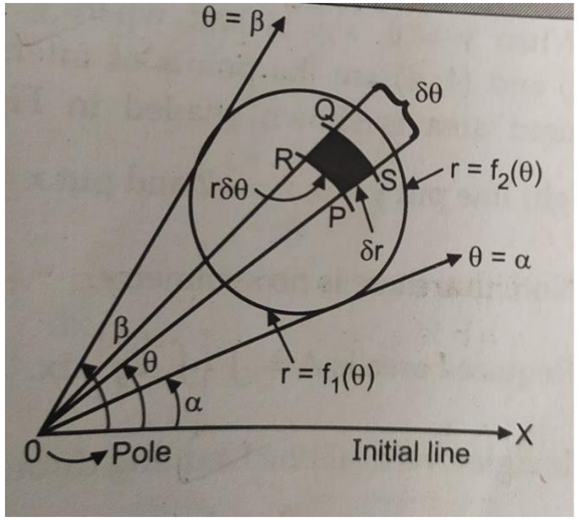
1) A = $\int_a^b y \ dx = \int_a^b f(x)dx$
2) A = $\int_c^d x\, dy = \int_c^d f(y)\, dy$

Area enclosed by plane curves expressed in polar co-ordinates

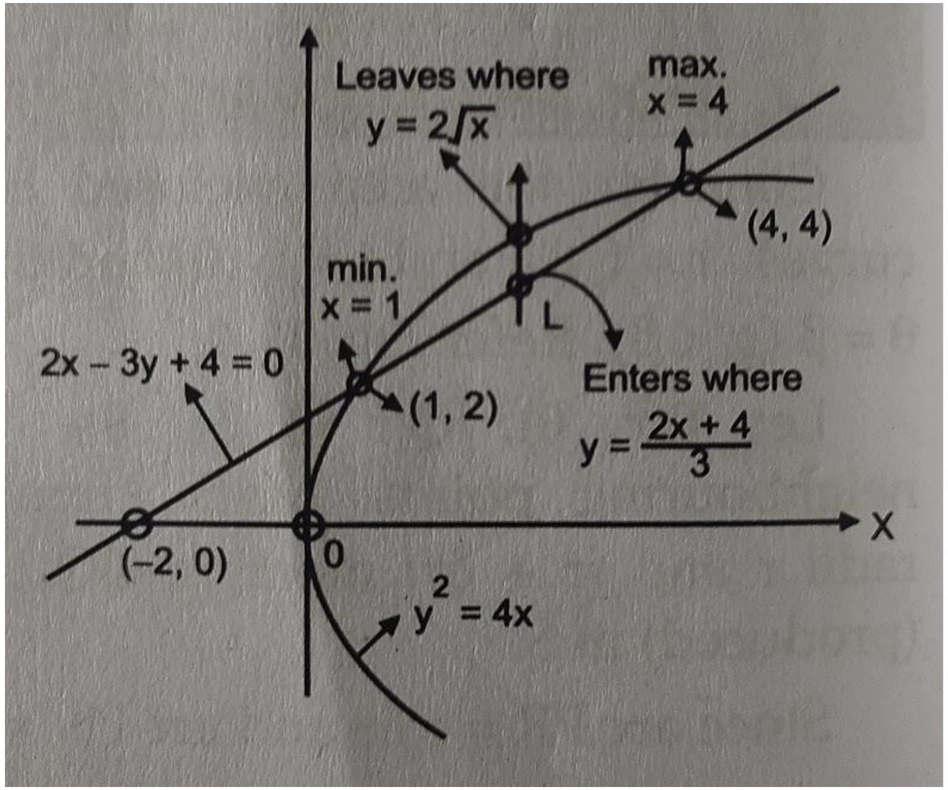
Area = $\int_\alpha^\beta \left\{\int_{f_1(\theta)}^{f_2(\theta)} r\, dr\right\} d\theta$

Area = $\frac{1}{2}\int_\alpha^\beta r^2 d\theta$





Ex.1 Find the area between the curves y^2= 4x and 2x-3y+4 = 0.



Sol. y^2= 4x and 2x-3y+4 = 0

∴ $\frac{y^2}{2}$ = 2x ∴ $\frac{y^2}{2}$ -3y + 4 = 0 or y^2 -6y +8 =0

(y-4)(y-2)=0 $\therefore y = 2, y = 4 \implies x = 1, x = 4$

$(4,4), (1,2)$ are pts of intersection

Limits of $y = \frac{2x+4}{3}$ to$y = 2\sqrt{x}\,; limits\ of\ x, x = 1\ to\ x = 4$

$$A = \int_1^4 \left\{\int_{\frac{2x+4}{3}}^{2\sqrt{x}} dy\right\} dx = \int_1^4 [y]_{\frac{2x+4}{3}}^{2\sqrt{x}} dx$$

$$= \int_1^4 \left[2\sqrt{x} - \frac{2x+4}{3}\right] dx = \frac{1}{3}$$

Ex. Find the total area included between the two cardioides r = a $(1+\cos\theta)$ *and* r = a $(1 - \cos\theta)$.

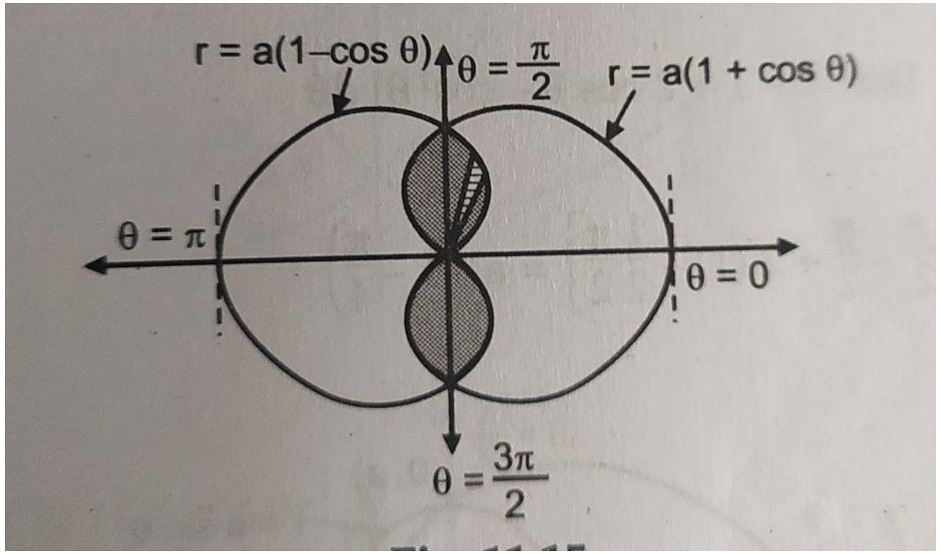
Sol. Area =$4\int_0^{\frac{\pi}{2}} \left\{\int_0^{a\,(1-\cos\theta)} r dr\right\} d\theta$

$$= 4\int_0^{\frac{\pi}{2}} \frac{a^2\,(1-\cos\theta)^2}{2}\, d\theta$$

$$= 2a^2 \int_0^{\frac{\pi}{2}} (1 - 2\cos\theta + \cos^2\theta)\, d\theta$$

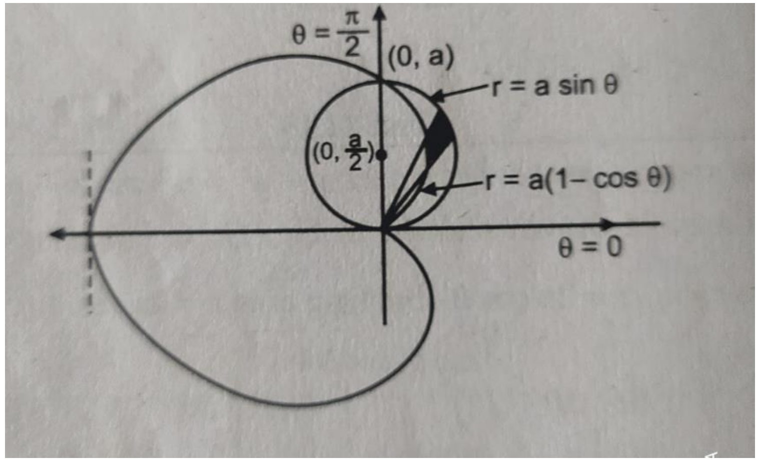
$$=2a^2 \left[\frac{\pi}{2} - 2 + \frac{1}{2}\frac{\pi}{2}\right]$$

$$=2a^2 \left[\frac{3\pi}{4} - 2\right]$$



Ex.3 .find by double integration the area insie the circle r = asinθ *and outside the cardioide* r = a (1 − cosθ).

Ans. $a^2\left(1 - \frac{\pi}{4}\right)$



$*$VOLUME OF SOLIDS :

The volume of a solid as a triple integral is given by

Volume = $\iiint_v \; dxdydz$

If $\rho = f(x, y, z)$ $is\ the\ density\ of\ the\ solid\ at\ the\ point\ P(x, y, z), then\ the\ mass\ of\ the\ solid\ is$

$Mass$ = $\iiint_v \; \rho dxdydz$ = $\iiint_v \; f(x, y, z)\ dxdydz$

In spherical polar system V = $\iiint r^2 \sin\theta \; dr d\theta d\varphi$

In cylindrical polar system V = $\iiint \rho \; d\rho \; d\varphi \; dz$

Ex.1. prove that the volume bounded by the cylinder $y^2 = x$, $x^2 = y$ and the planes $z = 0, x + y + z = 2$ is $\frac{11}{30}$.

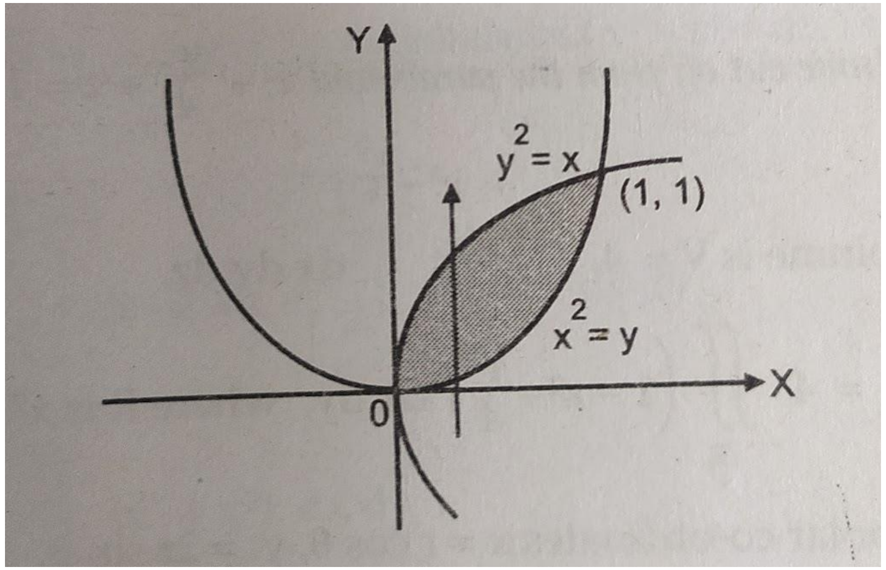
Sol. Volume = V = $\iiint dx\,dy\,dz = \iint dx\,dy \int_0^{2-x-y} dz$

V = $\iint_R (2 - x - y)\,dx\,dy$, R is region in xoy plane.

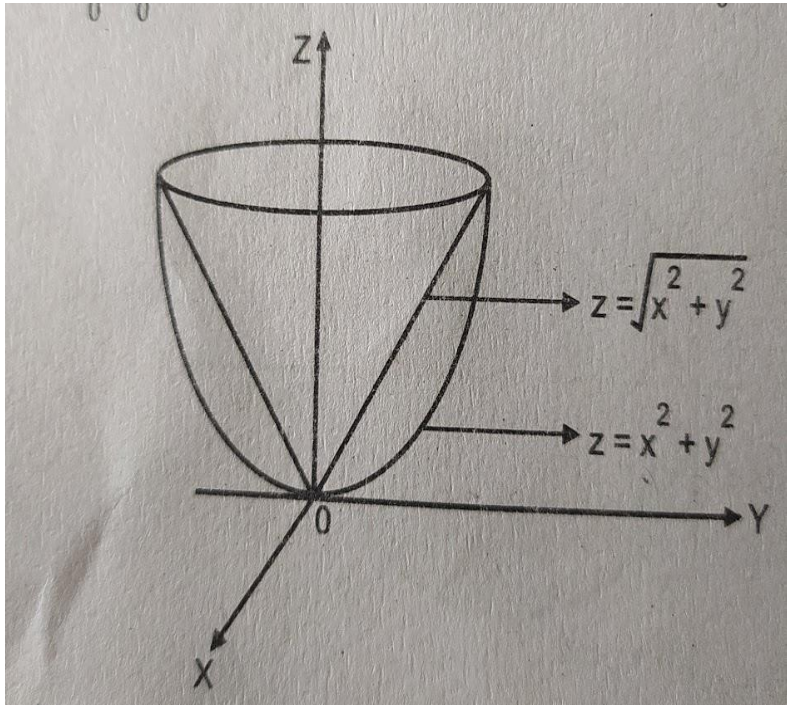
V = $\int_0^1 \int_{x^2}^{\sqrt{x}} (2 - x - y)\,dy\,dx$

V = $\int_0^1 (2y - xy - \frac{y^2}{2})_{x^2}^{\sqrt{x}}\ dx = \int_0^1 \left[\left(2\sqrt{x} - x^{3/2} - \frac{x}{2}\right) - \left(2x^2 - x^3 - \frac{x^4}{2}\right)\right] dx$

$= \frac{11}{30}$



Ex.2. find the volume of the region enclosed by the cone z = $\sqrt{x^2+y^2}$ and paraboloid z = x^2+y^2.



Sol. V = $\iint \int_{x^2+y^2}^{\sqrt{x^2+y^2}} dx\, dy\, dz$

= $\iint_R \left[\sqrt{x^2+y^2} - x^2 + y^2\right] dxdy$

Where R is $\sqrt{x^2+y^2} = x^2+y^2$ or x^2+y^2=1

Transforming to polar coordinates,

V = 4 $\int_0^{\frac{\pi}{2}} \int_0^1 (r - r^2) r\, dr\, d\theta$ = 4 $\left(\frac{\pi}{2}\right)\left(\frac{r^3}{3} - \frac{r^4}{4}\right)_0^1$=$2\pi\frac{1}{12}$=$\frac{\pi}{6}$

Ex. Find the volume of region bouned by paraboloid $x^2 + y^2$=2z
And cylinder or $x^2 + y^2$=4 .
Sol. Use cylindrical polar co-ordinates,

x = $\rho \ \cos\varphi, \quad y = \ \rho \sin\varphi, z = z, \quad x^2 + y^2$=$\rho^2$, $dxdydz = \rho \ d\rho \ d\varphi \ dz$

V = 4 $\iiint dxdydz$

= 4 $\iiint \rho \ d\rho \ d\varphi \ dz$

= 4 $\int_0^{\frac{\pi}{2}} d\varphi \int_0^2 \rho d\rho \int_0^{\rho^2/2} dz$

= $4\frac{\pi}{2} \int_0^2 \rho \frac{\rho^2}{2} d\rho = \ \pi \left[\frac{\rho^4}{4}\right]_0^2$= 4π

Mean an Root mean Square values:

$*$Mean square value of y = f(x) over (a,b) is M.S. of y = $\frac{\int_a^b y^2 dx}{\int_a^b dx} = \frac{\int_a^b [f(x)]^2 dx}{\int_a^b dx}$

$*$Mean square value of z = f(x,y) over an area A = $z_m = \frac{\iint_A f(x,y) dxdy}{\iint_A dxdy}$

$*$Mean square value of u = f(x,y,z) over a region of volume V = $u_m = \frac{\iiint f(x,y,z)\, dx\, dy\, dz}{\iiint dx\, dy\, dz}$

$*$ Root mean square value (R.M.S. value):

R.M.S. value of y = $\sqrt{\frac{\int_c^{c+p} y^2 dx}{\int_c^{c+p} dx}}$

Ex. *Find the M.V. of* $x^2y^2z^2$ *over the positive octant of the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Sol. M.V. $= \dfrac{\iiint x^2y^2z^2\,dx\,dy\,dz}{\iiint dx\,dy\,dz}$ $\qquad \because \iiint dx\,dy\,dz = \frac{3}{4}\pi abc.$

put x= $ar\sin\theta\cos\emptyset$, $y = br\sin\theta\sin\emptyset, z = \ cr\cos\theta \ \ dxdydz = abc\, r^2\sin\theta d\theta d\emptyset dr$

$$\text{M.V.} = \frac{\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 a^2b^2c^2\sin^5\theta\cos^2\theta\sin^2\emptyset\cos^2\emptyset r^8 d\theta d\emptyset dr}{\frac{1}{8}\cdot\frac{4}{3}\pi abc} = \frac{a^2b^2c^2}{315}$$

Centre of Gravity

- If $m_{1,}\ m_2, m_3, \ldots \ldots . m_n$ are the point masses situated at the points $(x_1\ , y_1, z_1)$, $(\ x_2, y_2\ , z_2\ ) \ldots \ldots \ldots (x_n, y_n, z_n)$ respectively and $(\bar{x}\ , \bar{y}, \bar{z})$ are the coordinates of centre of gravity of the system then
- $\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$; $\bar{y} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}$; $\bar{z} = \frac{\sum_{i=1}^{n} m_i z_i}{\sum_{i=1}^{n} m_i}$

Where,

$m_{1,} + m_2 + m_3 + \cdots \ldots . + m_n = \sum_{i=1}^{n} m_i$

$m_1 x_1 + m_2 x_2 + \ldots \ldots . . + m_n\, x_n = \sum_{i=1}^{n} m_i x_i$ .....etc

- Instead of discrete masses, if the mass distribution is continuous (i.e., rigid body) then
- $\bar{x} = \frac{\int x\, dm}{\int dm}$;
- $\bar{y} = \frac{\int y\, dm}{\int dm}$;
- $\bar{z} = \frac{\int z\, dm}{\int dm}$

where dm is an element of the distributed mass of the body . $(\bar{x}, \bar{y}, \bar{z})$ can be considered as centre of gravity of mass distribution.

A) Centre of gravity of an arc

- Let , the mass is distributed in the form of curve $y = f(x)$, 'ds' be an elementary arc at the point $P(x, y)$. If ρ is density at the point $P(x, y)$ then the mass of this element is

  $dm = \rho\, ds$

If $(\bar{x}, \bar{y})$be centre of gravity of arc AB , then

$$\bar{x} = \frac{\int x\, dm}{\int dm} ; \bar{y} = \frac{\int y\, dm}{\int dm} \quad \text{or} \quad \bar{x} = \frac{\int x\, \rho\, ds}{\int \rho\, ds} ; \bar{y} = \frac{\int y\, \rho\, ds}{\int \rho\, ds}.$$

If ρ is constant then

$$\bar{x} = \frac{\int x\, ds}{\int ds} ; \bar{y} = \frac{\int y\, ds}{\int ds}$$

Note

1) If $y = f(x)$ then $ds = \sqrt{1 + (\frac{dy}{dx})^2}$ .dx

2) If x $= f(y)$ then $ds = \sqrt{1 + (\frac{dx}{dy})^2}$ .dy

3) If $x = f_1(t), y = f_2(t)$ then

$$ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}$$

4) If $r = f(\theta)$, then

$$ds = \sqrt{r^2 + (\frac{dr}{d\theta})^2} \,.d\theta$$

5) If θ $= f(r)$, then

$$ds = \sqrt{1 + r^2(\frac{d\theta}{dr})^2} \,.dr$$

B) Centre of gravity of Plane Lamina

- If $(\bar{x}, \bar{y})$be coordinates of centre of gravity of plane lamina bounded by the curve C and , 'ρ' is density at the point P(x, y), then

$$dm = \rho\, dA$$

and

$$\bar{x} = \frac{\int x\, dm}{\int dm}\,;$$

$$\bar{y} = \frac{\int y\, dm}{\int dm}, \ (dA = dx\, dy\,)$$

- $\bar{x} = \dfrac{\iint_R x\rho \, dx \, dy}{\iint_R \rho \, dx \, dy}$;
- $\bar{y} = \dfrac{\iint_R y\rho \, dx \, dy}{\iint_R \rho \, dx \, dy}$

If ρ is constant then

- $\bar{x} = \dfrac{\iint_R x \, dx \, dy}{\iint_R \, dx \, dy}$;
- $\bar{y} = \dfrac{\iint_R y \, dx \, dy}{\iint_R \, dx \, dy}$,

Where R is region bounded by the curve C or lamina

Centre of gravity of solid

- If ($\bar{x}$, $\bar{y}, \bar{z}$) be coordinates of centre of gravity of the solid which encloses volume V.

If ρ is density at the point P(x, y, z) then

$$dm = \rho dv = \rho \, dx \, dy \, dz$$

Hence, $\bar{x} = \dfrac{\iiint_V x \, \rho \, dx \, dy \, dz}{\iiint_V \rho \, dx \, dy \, dz}$

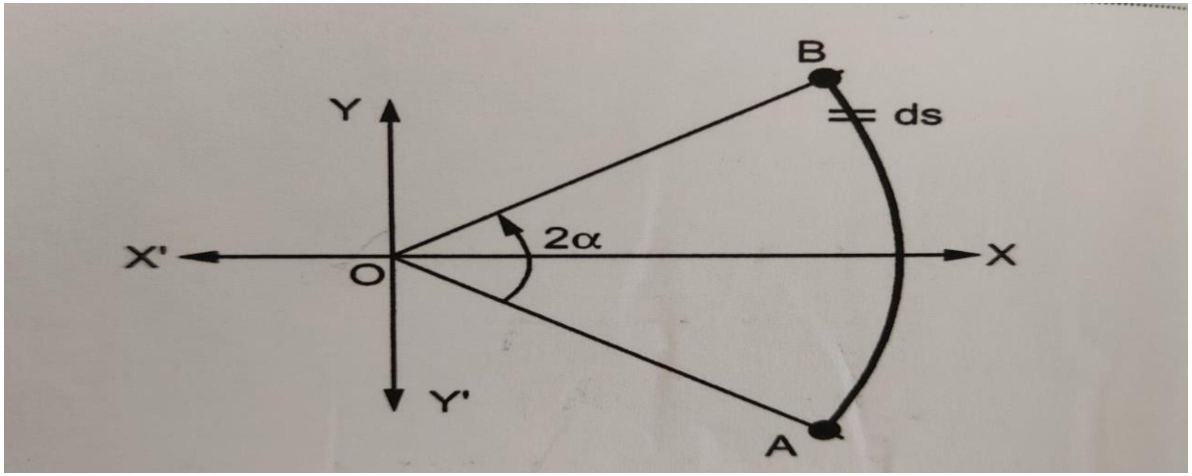
$$\bar{y} = \frac{\iiint_V y \, \rho \, dx \, dy \, dz}{\iiint_V \rho \, dx \, dy \, dz}$$

$$\bar{z} = \frac{\iiint_V z \, \rho \, dx \, dy \, dz}{\iiint_V \rho \, dx \, dy \, dz}$$

.

- **Q.1) Find the C.G. of the arc of a uniform sector of a circle of radius 'a' angle at the centre being 2α. Deduce the same to semicircle**
- Solution: Let the equation of circle be $x^2 + y^2 = a^2$

Parametric equations : $x = a\ cos\theta\ , y = a\ sin\theta$



- X-axis bisecting central angle of sector .

By symmetry C.G. of arc AB lies on X-axis .

i.e., $\bar{y}$ = 0

and $\bar{x} = \frac{\int x\, \rho\, ds}{\int \rho\, ds} = \frac{\int x\ \ ds}{\int\ ds}$ (ρ is constant)

$$s = a\,\theta$$

$$ds = a\,d\theta$$

Hence, $\int x\, ds = 2\int_0^{\alpha} a\,cos\theta\,.\,d\theta$

$= 2a^2 sin\alpha$

- $\int ds = 2\int_0^{a} a \; d\theta$

  $= 2\, a\, \alpha$

Hence, $\bar{x} = \frac{2a^2 \sin \alpha}{2a\ \alpha}$

$= \frac{a \ \sin \alpha}{\alpha}$

As for semicircle $\alpha = \frac{\pi}{2}$

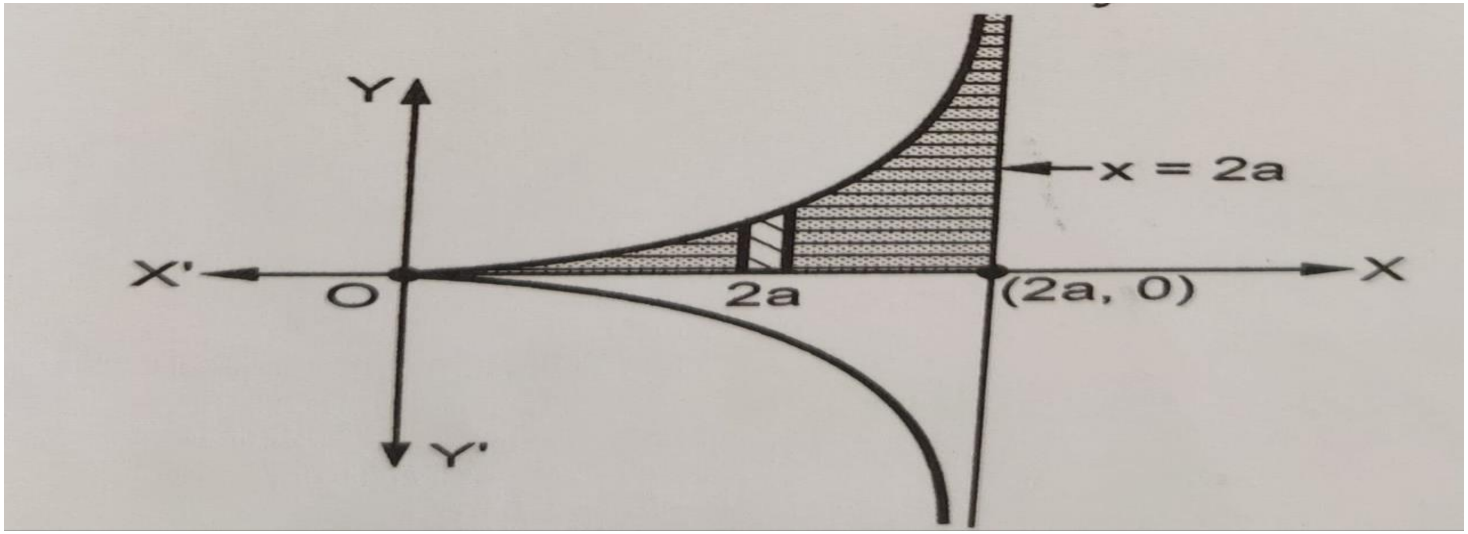
$\bar{x} = \frac{2a}{\pi}$, $\bar{y} = 0$

- **Q.2) Find the centroid of the area bounded by $y^2(2a - x) = x^3$ and its asymptote.**
- Solution:

$$y^2(2a - x) = x^3$$

The curve is cissoid as shown in the figure .

The curve is symmetrical about X-axis. Hence , $\bar{y}$ = 0



- $\bar{x} = \frac{\iint x\,dx\,dy}{\iint dx\,dy} = \frac{N}{D}$; ……………………………………(1)

N = $\int_0^{2a} . \int_0^{y} x\,dx\,dy = \int_0^{2a} x\ (y-0)\,dx$

$= \int_0^{2a} xy\,dx = \int_0^{2a} x\left(\frac{x^{\frac{3}{2}}}{\sqrt{2a-x}}\right) dx$

i.e, $N = \int_0^{2a} x\left(\frac{x^{\frac{3}{2}}}{\sqrt{2a-x}}\right) dx;$

Put $x = 2a\,sin^2\theta$

$dx = 4a\,sin\theta\,cos\theta\,d\theta$ Limits :

x	0	$2a$
θ	0	$\frac{\pi}{2}$

- $\rightarrow N = \int_0^{\pi/2} \frac{2a\, sin^2\theta.(2a sin^2\theta)^{3/2}}{\sqrt{2a-2a\, sin^2\theta}} 4a sin\theta cos\theta\, d\theta$

$$= \frac{(2a)^{\frac{5}{2}}}{(2a)^{\frac{1}{2}}} \int_0^{\frac{\pi}{2}} \frac{sin^5\theta\ 4a\ sin\theta\ cos\theta}{cos\theta} d\theta$$

$$= (2a)^2(4a) \int_0^{\frac{\pi}{2}} sin^6\theta\ d\theta$$

$$= 16a^3 \left(\frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}\right)$$

$$= \frac{5\pi a^3}{2}$$ ....(2)

- And

D = $\int_0^{2a} . \int_0^{y} dx\, dy = \int_0^{2a} y\, dx$

=$\int_0^{2a} \frac{x^{\frac{3}{2}}}{\sqrt{2a-x}} dx$ ..... Put $x = 2a\, sin^2\theta$

$= 8a^2 \int_0^{\frac{\pi}{2}} sin^4\theta\; d\theta$

$= 8a^2 (\frac{3}{4} . \frac{1}{2} . \frac{\pi}{2})$

= $\frac{3\pi a^2}{2}$ ....(3)

- From equations (1), (2), & (3)
- $\bar{x} = \dfrac{\iint x\,dx\,dy}{\iint dx\,dy} = \dfrac{N}{D}$; ……

$$= \frac{\frac{5\pi a^3}{2}}{\frac{3\pi a^2}{2}}$$

$$= \frac{5a}{3}$$

Hence, Centre of gravity is $(\bar{x}\ ,\bar{y}) = (\frac{5a}{3},\ 0)$

- **Q.3) Find the centroid of the region in the first octant bounded by $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$; $(a>0, b>0, c>0)$**

- Solution: Let , the centroid be $(\bar{x}, \bar{y}, \bar{z})$

$$\bar{x} = \frac{\iiint_V x\,\rho\,dx\,dy\,dz}{\iiint_V \rho\,dx\,dy\,dz}$$

$$\bar{y} = \frac{\iiint_V y\,\rho\,dx\,dy\,dz}{\iiint_V \rho\,dx\,dy\,dz}$$

$$\bar{z} = \frac{\iiint_V z\,\rho\,dx\,dy\,dz}{\iiint_V \rho\,dx\,dy\,dz}$$

Put , $x = au, y = bv, z = cw, \rho = constant$

Hence, $dx\,dy\,dz = abc\,du\,dv\,dw$

And $u + v + w = 1$

- $\rightarrow \iiint x \, dx \, dy \, dz \;=\; \iiint au \, du \, dv \, dw$

$$= a^2bc \iiint u^{2-1} \, v^{1-1} \, w^{1-1} \, du \, dv \, dw$$

$$= a^2bc \; \frac{\lceil 2 \;\; \lceil 1 \;\; \lceil 1}{\lceil \overline{1+2+1+1}}$$

$$= a^2bc \; \frac{1}{4!} = \frac{a^2bc}{24}$$

Similarly , $\iiint y \, dx \, dy \, dz \;=\; \dfrac{b^2ac}{24}$;

$$\iiint z \, dx \, dy \, dz \;=\; \frac{c^2ab}{24}.$$

- Also, $\iiint dx\,dy\,dz = Volume\ of\ tetrahedron = \frac{abc}{6}$

Hence, $\overline{x} = \frac{\frac{a^2bc}{24}}{\frac{abc}{6}} = \frac{a}{4}$

Similarly , $\overline{y} = \frac{b}{4}$; $\bar{z} = \frac{c}{4}$.

i.e., C.G. is $(\overline{x}, \overline{y}, \overline{z}) = (\frac{a}{4}, \frac{b}{4}, \frac{c}{4})$

Moment of Inertia

- The moment of inertia is a physical quantity which describes how easily a body can be rotated about given axis.
- It is the property of matter which resists change in its state for motion
- The larger the inertia , the greater force that is required to bring some change in it’s velocity, in the given amount of time.
- Definition:
- Let, the mass m be situated at a point P which is at distance r from a line then the product mr^2 is called the moment of Inertia of the mass m about the line or the axis.

- Consider a body of mass m which consists of infinite number of small particles . Let, their masses be $m_{1,}\ m_{2,}\ m_{3,}$.... Situated at $r_{1,}\ r_{2,}\ r_{3,}$ respectively then $\mathrm{M.I.} = \sum mr^2$

- If the mass is continuosly distributed over body.

Consider an elementary particle of mass dm at a distance p from the axis then M.I. of the whole body is

$$\mathrm{M.I.} = \int p^2\, dm$$

- Moment of Inertia of an arc
- $M.I. = \int p^2 \rho\, ds$

(where $dm = \rho\, ds$)

Moment of Inertia of a plane Lamina

- Consider a plane lamina R bounded by the curve C .
- If ρ is density at the point P(x, y) then $dm = \rho\, dx\, dy$
- If p is the distance of this elementary mass from the axis , the M.I. about this axis is

M.I. = $\iint_R \rho\ p^2 dx\ dy$

- **The moment of inertia of the lamina about X-axis is**

$\mathrm{M.I.} = \iint \rho\, y^2\, dx\, dy$ (p=y)

- **The moment of inertia of the lamina about Y-axis is**

$\mathrm{M.I.} = \iint \rho\, x^2\, dx\, dy$ (p=x)

- **The moment of inertia in polar coordinates is**

M.I. = $\boldsymbol{\iint_R \rho\, p^2 r\, d\theta\, dr}$

Moment of Inertia of Solid

- Moment of Inertia of Solid
- Consider a solid of volume V and ρ is density at the point

P(x, y, z) then

$dm = \rho\ dx\ dy\ dz$

- The moment of inertia of solid which is at distance p from the axis is
- $M.I. = \iiint_V \rho\, p^2 dx\, dy\, dz$

- The Moment of Inertia about X-axis is

$$M.I. = \iiint \rho(y^2 + z^2)dx\,dy\,dz$$

$$(\therefore p = \sqrt{(y^2 + z^2)})$$

- The Moment of Inertia about Y -axis is

$$M.I. = \iiint \rho(x^2 + z^2)dx\,dy\,dz$$

$$(\therefore p = \sqrt{(x^2 + z^2)})$$

- The Moment of Inertia about Z-axis is

$$M.I. = \iiint \rho(y^2 + x^2)dx\,dy\,dz$$

$$(\therefore p = \sqrt{(y^2 + x^2)})$$

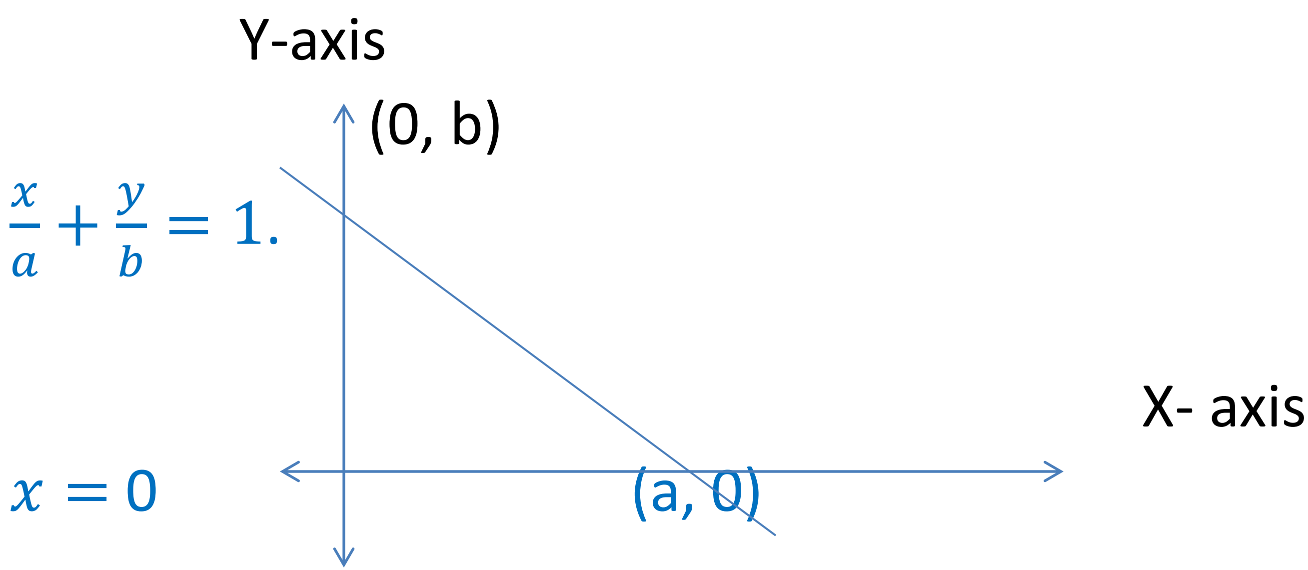
- Find the M.I. about the X-axis of the area enclosed by the lines $x = 0, \frac{x}{a} + \frac{y}{b} = 1.$

- **Solution:** $M.I. = \iint_A \rho\, p^2\, dx\, dy$

M.I. about the X-axis is

$M.I. = \iint_A \rho\, y^2\, dx\, dy$ (p = y)..........................(1)

Where A is area as shown in figure



- Consider small area dx dy at a distance y from X-axis
- From equation (1)
- $M.I.= \rho \int_{y=0}^{b} . \int_{0}^{\frac{a}{b}(b-y)} y^2 \, dx \, dy$

$$=\rho \int_0^b y^2 \, (x)_{0}^{\frac{a}{b}(b-y)} \, \mathrm{dy}$$

$$= \rho \int_0^b y^2 \, \frac{a}{b}(b-y)\mathrm{dy}$$

$$= \rho\frac{a}{b} \int_0^b (by^2 - y^3)dy$$

$$= \rho\frac{a}{b} \left(b\frac{y^3}{3} - \frac{y^4}{4}\right)_0^b$$

$$=\rho\frac{a}{b}\left(b\,\frac{b^3}{3}-\frac{b^4}{4}\right)$$

$$= \rho\,\frac{a}{b}\;b^4\left(\frac{1}{3}-\frac{1}{4}\right)$$

$$\text{M.I.} = \rho\,\frac{a}{12}\;b^3$$

Now, mass m of the area is

M = ρ × area of the triangle OAB

$$= \rho\,\frac{ab}{2}$$

- Hence , $\rho = \frac{2M}{ab}$
- Hence,

$$\text{M.I.} = \rho\,\frac{a}{12}\;b^3$$

$$= \frac{2M}{ab}\,\frac{a}{12}\;b^3$$

$$\text{M.I.} = \frac{b^2\,M}{6}$$

- Find the Centroid of gravity of the area bounded by $y^2 = x \text{ and } x + y = 2$
- Find the moment of inertia about the line $\theta = \frac{\pi}{2}$ of the area enclosed by
  $r = a(1 + cos\theta)$.
- Find the moment of inertia of a sphere about a diameter.

DOUBLE INTEGRATION.

Representation of Area as a Double Integral : Consider the region bounded by , y = f (x) , x = a , x = band the X – axis.

Let P (x,y) and Q (x+δx , x + δx) be two adjacent points on the curve.

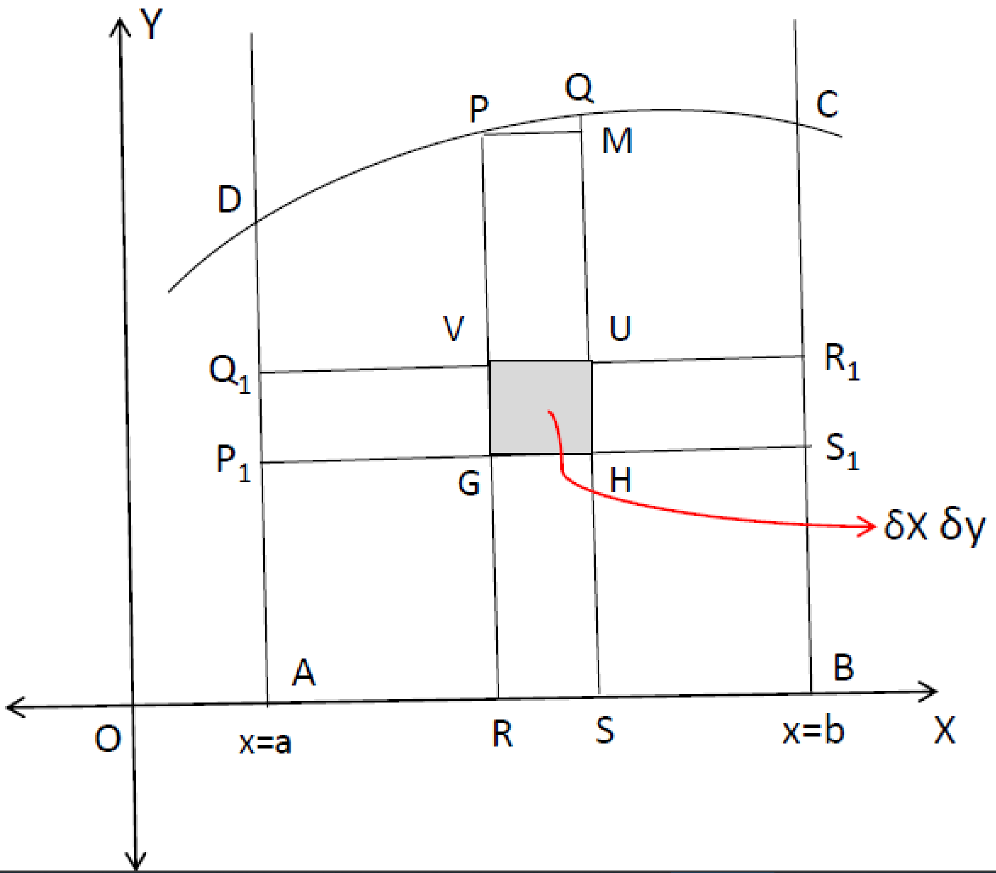
Area ABCD can be considered as sum of infinite number of inscribed rectangles like PMRS.

Expression for the area ABCD is

$$A = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} y.\delta x$$

Which is expressed in the integral notation as

A = $\int_a^b y\,dx$ **OR** $\int_a^b f(x)\,dx$



Properties of Double Integrals :

1. $\iint_R k\, f(x,y)\,dA = k \iint_R f(x,y)\,dA$ .where k is constant.

2. $\iint_R [\, \mathbf{f(x,y)} \pm g(x,y)]\, dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$

3. If $R = R_1 \cup R_2$ and $R_1 \cap R_2 = \emptyset$ then

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

Evaluation Of Double Integrals :

Double integrals over a region R may be evaluated by two successive integrations as follows :

1. Suppose that R can be expressed as x = a , x = b ,

Y = f_1 (x) and y = f_2 (x) then

$$\boxed{I = \iint\limits_{R} f(x,y)\, dy\, dx = \int_a^b \left\{ \int_{f_1(x)}^{f_2(x)} f(x,y)\, dy \right\} dx} \quad \text{----------------} \quad (\mathrm{I})$$

We first integrate the inner integral w.r.t. y keeping x as constant between the limits $y=f_1(x)$, $y=f_2(x)$ then the resulting expression w.r.t. x between the limits x=a ,x=b.We then get the value of double integral (I)

2. Suppose that R can be expressed as y = c , y = d ,

$x = f_1(y)$ and $x = f_2(y)$ then

$$\boxed{I = \iint f(x,y)\,dx\,dy = \int_c^d \left\{ \int_{f_1(y)}^{f_2(y)} f(x,y)\,dx \right\} dy} \text{ ----------------- (II)}$$

We first integrate the inner integral w.r.t. x keeping y as constant between the limits $x = f_1(y)$, $x = f_2(y)$ then the resulting expression w.r.t. y between the limits y = c , y = d.We then get the value of double integral (II)

3. Suppose that R can be expressed as x = a , x = b , y = c , y = d ,

then

$$\boxed{I = \int_a^b \left\{ \int_c^d f(x,y)dy \right\} dx \quad \textbf{OR} \quad \int_c^d \left\{ \int_a^b f(x,y)\,dx \right\} dy}$$

Where a , b , c , d are constants then order of integration must be clearly specified.

4. **Suppose that R can be expressed as x = a , x = b , y = c , y = d ,**

As in (3)and integrand is separable i.e. f (x , y) = u(x) v (y)

then $\boxed{I = \int_{x=a}^{b} \int_{y=c}^{d} f(x,y)dy\ dx \quad \textbf{OR} \quad \int_{x=a}^{b} u(x)dx \, . \int_{y=c}^{d} v\,(y)dy}$

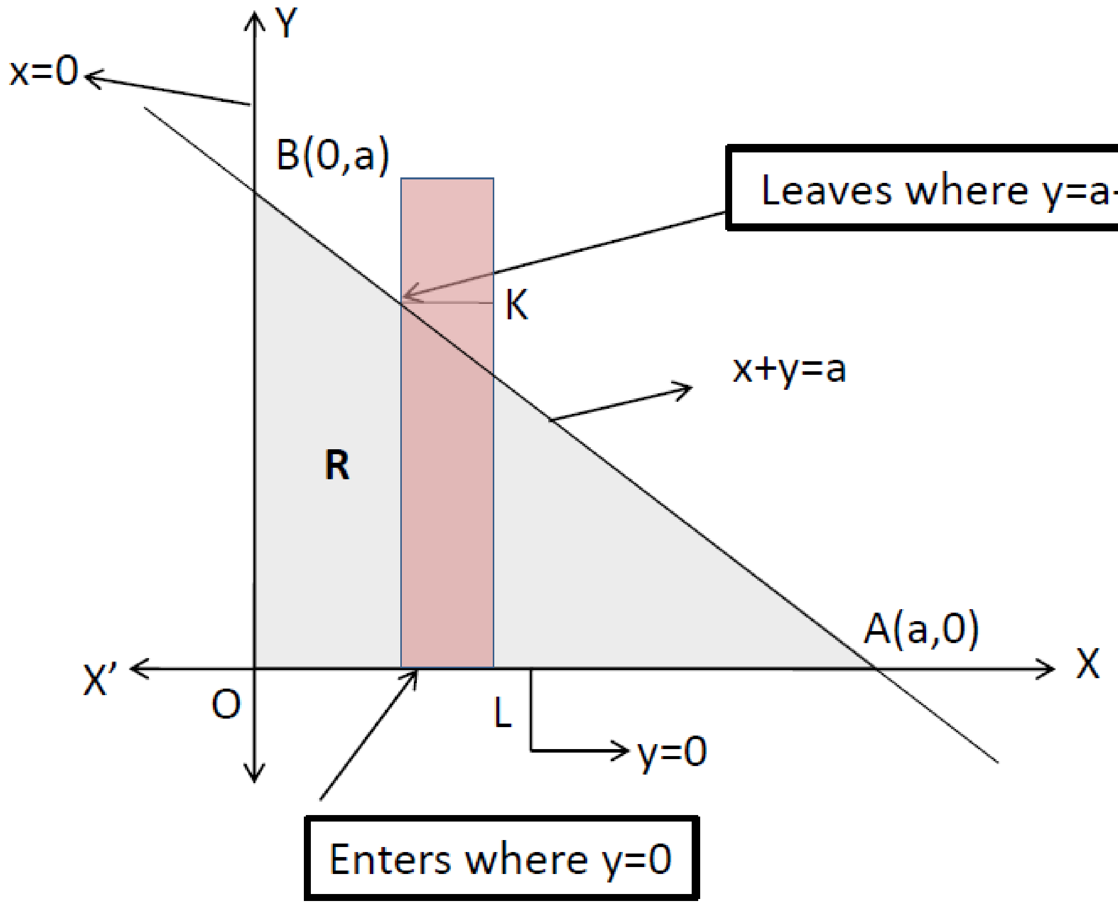
Determining The Limits Of Integration :

To evaluate $\iint_R f(x,y)dx\,dy$

over the region given by $x = 0, y = 0$ *and* $x + y = a$

Method - I : Integrating w.r.t. y then w.r.t. x

- Draw the region R bounded by $x = 0, y = 0$ and $x + y = a$.Here R is $\triangle$ OAB.
- We have I = $\int\{\int f\,(x,y)dy\}\,dx$.Since we are integrating first w.r.t. y, always imagine a vertical strip LK anywhere in the region R.



- **To find the limits for y :** Lower end of the strip enters the region R where y = o and upper end leaves the region r where y = a – x.
- **To find the limits for x:** Move the strip in horizontal direction from left to right coverin the entire region R.

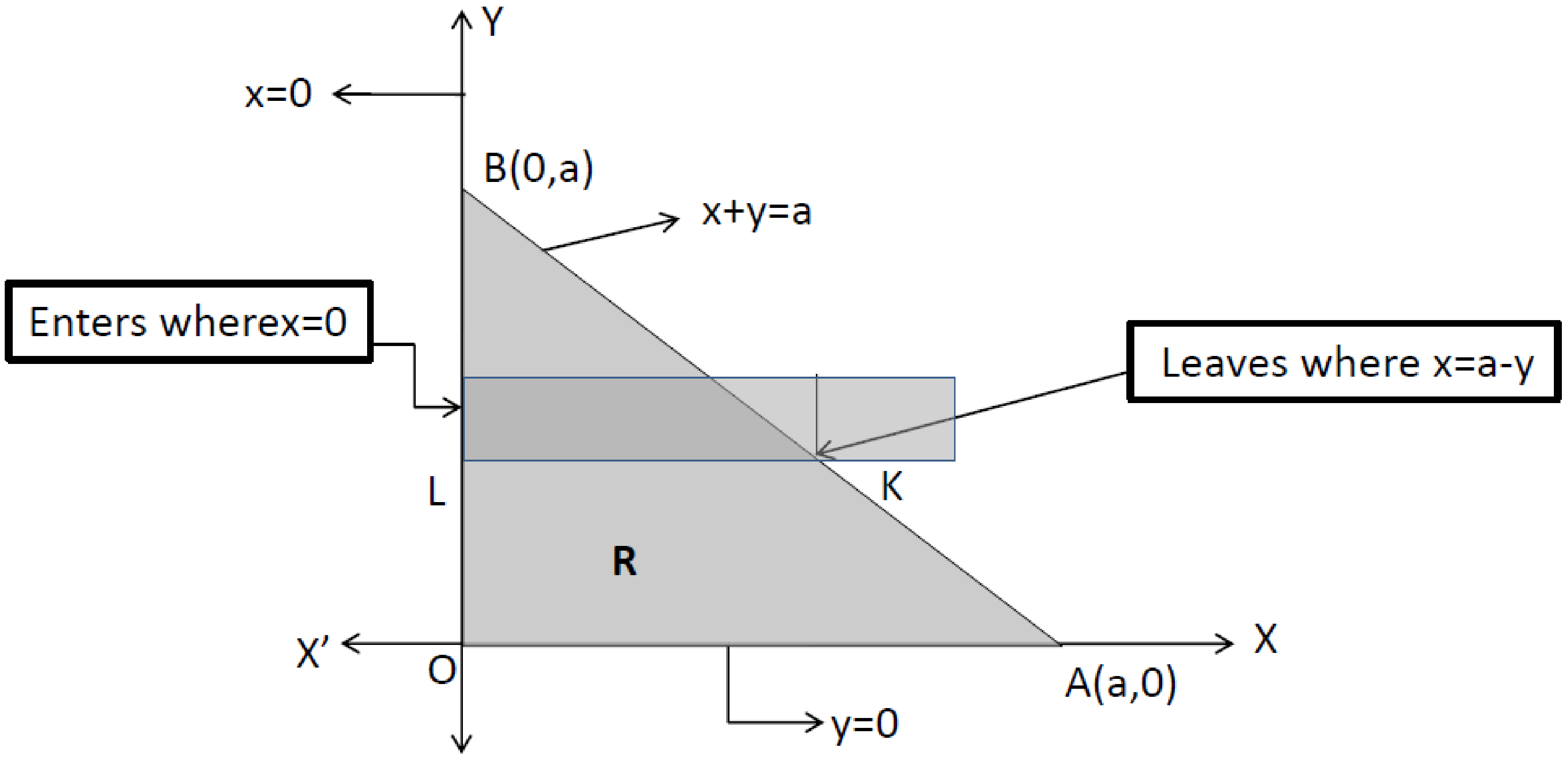
  Therefore x varies from x = 0 to x = a

There with complete limits

$$I = \int_0^a \left\{ \int_0^{a-x} f(x, y) dy \right\} dx$$

Method - II : Integrating w.r.t. x then w.r.t. y

- Draw the region R bounded by $x = 0, y = 0$ and $x + y = a$.Here R is $\triangle$ OAB.
- We have I = $\int \{ \int f\ (x, y)\ dx \}\ dy$.Since we are integrating first w.r.t. x, always imagine a horizontal strip LK anywhere in the region R.



- **To find the limits for x:** Left end of the strip enters the region R where $x = o$ and right end leaves the region R where $x = a - y$
- **To find the limits for y:** Move the strip in vertical direction from bottom to top covering the entire region R.

 Therefore y varies from $y = 0$ to $y = a$

Therefore with complete limits

$$I = \int_0^a \left\{ \int_0^{a-y} f(x,y)\,dx \right\} dy$$

Problems on double integrations are mainly divided inyo following types.

- Problems on direct evaluation of double integrals.
- Problems on integrals when limits are not provided.
- Problems onchange of order of integration.

Q1) Evaluate

$$\int_0^1 \int_0^y xy\,dx\,dy$$

Sol:Since limits of inner integral are func's of y

integrate 1st w.r.t. x

$\therefore$

$$\int_0^1 \left[\int_0^y yx\,dx \right] dy = \int_0^1 \left[\frac{x^2}{2} \right]_0^y y\,dy = \int_0^1 \frac{y^3}{2} dy = \left[\frac{y^4}{8} \right]_0^1 = \frac{1}{8}$$

Q2) Evaluate $\int_0^1 \int_0^{1-x} (x+y)\,dx\,dy$

Sol:Since limits of inner integral are func's of x

$\therefore$ integrate 1st w.r.t. y

$$\int_0^1 \left[\int_0^{1-x} (x+y)\,dy \right] dx = \int_0^1 \left[xy + \frac{y^2}{2} \right]_0^{1-x} dx$$

$$= \int_0^1 [x(1-x) + \frac{(1-x)^2}{2}]\,dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3} + \frac{(1-x)^3}{2.3} \right]_0^1 = \frac{1}{3}$$

Q3) Evaluate $\int_0^1\int_0^1 \frac{dxdy}{(1+x^2)(1+y^2)}$

Sol:Since both the limits of both integrals are constants & variables can be seperated $\therefore$ double integrals is a product of two single integrals

$$= \int_0^1 \frac{1}{(1+x^2)}dx \int_0^1 \frac{1}{(1+y^2)}dy$$

$$= \left[\tan^{-1} x\right]_0^1 \left[\tan^{-1} y\right]_0^1 = \frac{\pi}{4}.\frac{\pi}{4}$$

$$= \frac{\pi^2}{16}$$

Q4) Evaluate $\int_0^1 dx \int_1^\infty e^{-y} y^x \log y dy$

Sol:Here it is advantageous to integrate w.r.t. x 1st . Since both the limits are constants ,we can just interchange the order of integration

$$= \int_1^\infty e^{-y} \log y dy \int_0^1 y^x dx$$

$$= \int_1^\infty e^{-y} \log y \frac{(y-1)}{\log y} dy = \int_1^\infty e^{-y}(y-1) dy$$

$$= \int_{\frac{1}{e}}^0 (-du) = [-u]_{\frac{1}{e}}^0 = \frac{1}{e}$$

$$ye^{-y} = u \therefore \left(e^{-y} - ye^{-y}\right) dy = du$$

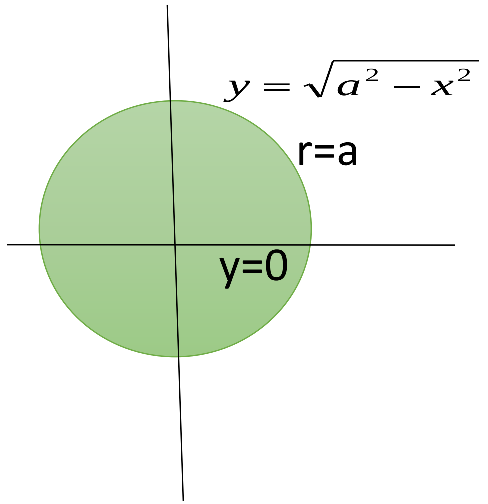
$$\therefore (y-1)e^{-y} dy = -du$$

y	1	∞
u=y e^{-y}	e^{-1}	0

Q5) Evaluate $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-x^2-y^2}\,dxdy$

Sol: Region is bounded by y=0 & $y=\sqrt{a^2-x^2}$ or $y^2+x^2=a^2$ Transforming to polar coordinates

$$I=\int_0^{\frac{\pi}{2}}\int_0^a e^{-r^2}\,rdrd\theta$$



$$I=\int_0^{\frac{\pi}{2}} d\theta\left(-\frac{1}{2}\right)\int_0^a e^{-r^2}(-2rdr)=-\frac{1.\pi}{2.2}\left[e^{-r^2}\right]_0^a$$

$$I=\frac{\pi}{4}\left[1-e^{-a^2}\right]$$

Q6) Evaluate $$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sin\left\{\frac{\pi}{a^2}\left(a^2-x^2-y^2\right)\right\} dxdy$$

$$y=\sqrt{a^2-x^2}$$

Q7) Evaluate $$\int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2-y^2}} \log_e\left(x^2+y^2\right) dxdy$$

Double Integration when limits are not provided

Q1) Evaluate $\iint \frac{xy}{\sqrt{1-y^2}}\,dxdy$ over +ive quadrant of circle $x^2+y^2=1$

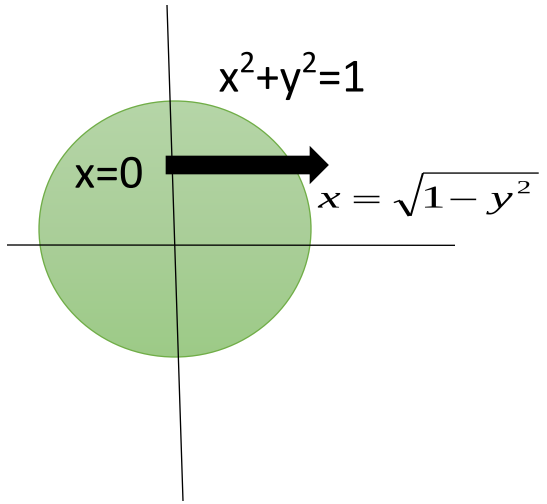
Sol: Region is bounded by x=0 & $x=\sqrt{1-y^2}$ or $x^2+y^2=1$

$$I=\int_0^1 \frac{y}{\sqrt{1-y^2}}\left[\int_{x=0}^{x=\sqrt{1-y^2}} x\,dx\right]dy$$

$$I=\int_0^1 \left[\frac{x^2}{2}\right]_0^{\sqrt{1-y^2}} \frac{y}{\sqrt{1-y^2}}\,dy$$

$$I=\frac{1}{2}\int_0^1 (1-y^2)\frac{y}{\sqrt{1-y^2}}\,dy$$

$$I=\frac{1}{2}\int_0^1 y\sqrt{1-y^2}\,dy$$



Put y=sinθ ,dy=cos θd θ

$$I=\frac{1}{2}\int_0^{\pi/2} \sin\theta\cos^2\theta\,d\theta=\frac{1}{2}.\frac{1}{3}.=\frac{1}{6}$$

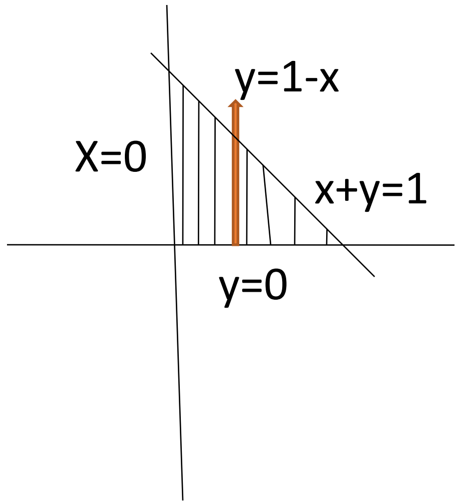
Q2) Evaluate $\iint_R xy \ dxdy$ over the region bounded by x=0,y=0,x+y=1

Sol: Region is bounded by x=0,y=0 , x+y=1

$$I=\int_0^1 x[\int_{y=0}^{y=1-x} y \ dy]dx=\int_0^1 x\frac{1}{2}\left[y^2\right]_0^{1-x} dx$$

$$I=\frac{1}{2}\int_0^1 x(1-x)^2 dx=\frac{1}{2}\int_0^1 (x-2x^2+x^3)dx$$

$$I=1/24$$



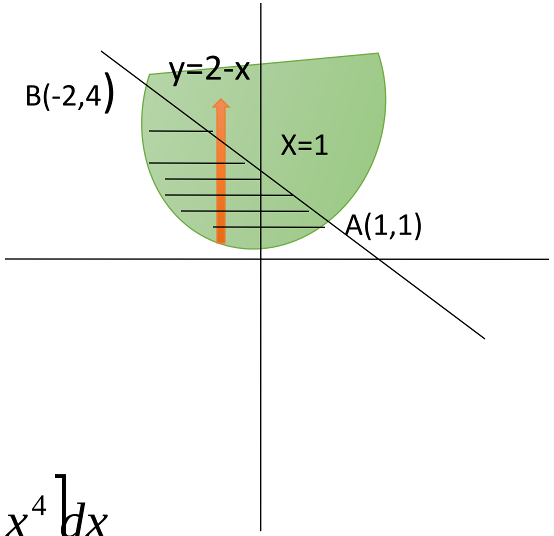
Q3 Evaluate $\iint_R ydxdy$ where R is annulus between $y=x^2$ & $x+y=2$

Sol: Points of intersection of parabola $y=x^2$ and line $x+y=2$ is given by

$x+x^2=2$ or $x+x^2-2=0$

Or $(x+2)(x-1)=0$, therefore $x=-2,1$

For $x=-2$, $y=2-x=4$ & for $x=1$, $y=2-x=1$



$$I=\int_{-2}^{1}\int_{y=x^2}^{y=2-x} ydxdy$$

Integrating w.r.t. y first $I=\int_{-2}^{1}\left[\frac{y^2}{2}\right]_{x^2}^{2-x}dx=\frac{1}{2}\int_{-2}^{1}\left[(2-x)^2-x^4\right]dx$

$$I=\frac{1}{2}\int_{-2}^{1}\left[4-4x+x^2-x^4\right]dx$$

$$I=\frac{1}{2}\left[4x-4\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^5}{5}\right]_{-2}^{1}=\frac{1}{2}\left[21-\frac{33}{5}\right]=\frac{36}{5}$$

Q4 Evaluate $\iint_R \frac{1}{x^4 + y^2}\, dxdy$ where R is annulus between $y \geq x^2$, $x \geq 1$

Q5) Evaluate $\int \int x^2 y^2 dxdy$ over +ive quadrant of circle $x^2+y^2=1$

Change of order of integration

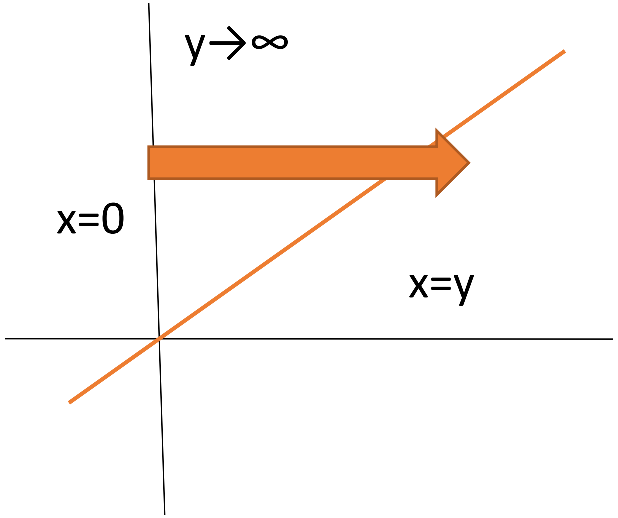
Q1

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dxdy = 1$$

Sol: Limits y=x to y=∞ & x=0 to x=∞, Integrating w.r.t. y is difficult

So change the order of integration

$$I = \int_0^\infty \frac{e^{-y}}{y}\left[\int_0^y dx\right]dy$$



Now integrating w.r.t. x first

$$I_1 = \int_0^x dx = [x]_0^y = y$$

$$I = \int_0^\infty \frac{e^{-y}}{y}.ydy$$

$$I = -\left[e^{-y}\right]_0^\infty = -\left[e^{-\infty} - e^0\right] = -[0-1] = 1$$

Q2 Evaluate by changing order of integration $\int_0^{\infty}\int_0^{x} xe^{\frac{-x^2}{y}}\,dy\,dx$

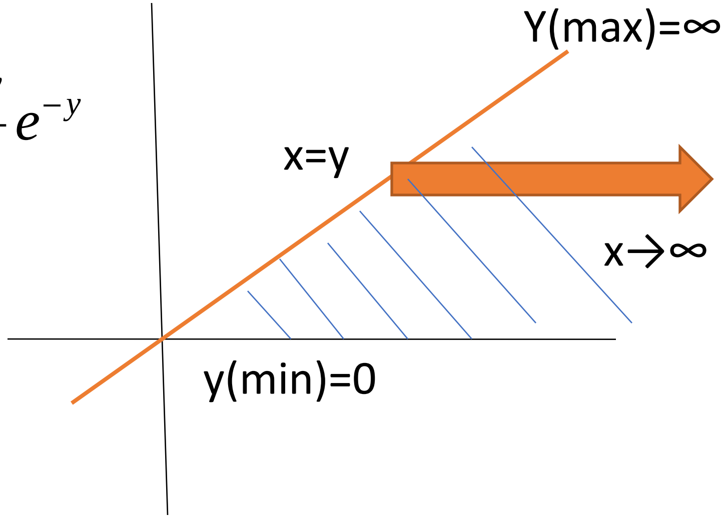
Sol: Limits y=0 to y=x & x=0 to x=∞, Integrating w.r.t. y is difficult

So change the order of integration, Now integrating w.r.t. x first

$$I=\int_0^{\infty}\left[\int_y^{\infty} xe^{\frac{-x^2}{y}}\,dx\right]dy$$

Put $x^2=t$, $2xdx=dt$ &
lim changes y^2 to ∞

$$I_1=\int_y^{\infty} e^{-\frac{x^2}{y}}\frac{1}{2}(2xdx)=\frac{1}{2}\left[\frac{e^{-\frac{t}{y}}}{-\frac{1}{y}}\right]_{y^2}^{\infty}=-\frac{y}{2}\left[e^{-\infty}-e^{-y}\right]=\frac{y}{2}e^{-y}$$



$$I=\frac{1}{2}\int_0^{\infty} ye^{-y}dy=\frac{1}{2}\left[y(-e^{-y})-1(e^{-y})\right]_0^{\infty}$$

$$I=\frac{1}{2}\left[-e^{-y}(y+1)\right]_0^{\infty}=\frac{1}{2}\left[-e^{-\infty}+e^{0}(0+1)\right]=\frac{1}{2}[0+1]=\frac{1}{2}$$

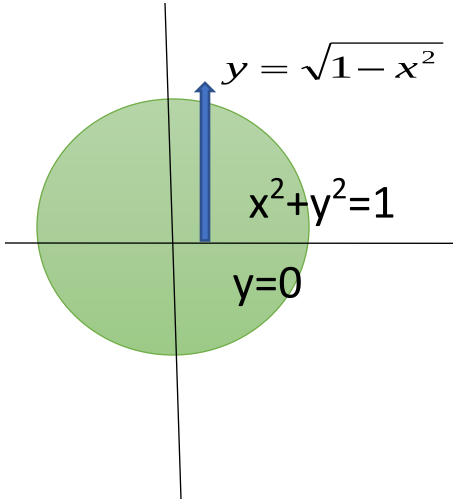
Q3) Evaluate $\int_0^1 \int_0^{\sqrt{1-y^2}} \frac{\cos^{-1} x}{\sqrt{1-x^2-y^2}\sqrt{1-x^2}} dxdy$

Sol: Here limits are y=0 & $x = \sqrt{1-y^2}$ but int w.r.t. x is difficult to solve so we change the order of integration

$$I = \int_0^1 \frac{\cos^{-1} x}{\sqrt{1-x^2}} \left[\int_{y=0}^{y=\sqrt{1-x^2}} \frac{1}{\sqrt{1-x^2-y^2}} dy \right] dx$$

Let $1-x^2 = a^2$

$$I_1 = \int_0^{\sqrt{1-x^2}} \frac{dy}{\sqrt{(1-x^2)-y^2}} = \int_0^{\sqrt{a^2}} \frac{dy}{\sqrt{a^2-y^2}} = \left[\sin^{-1} \frac{y}{a} \right]_0^a = \frac{\pi}{2}$$



$$I = \int_0^1 \frac{\cos^{-1} x dx}{\sqrt{1-x^2}} . \frac{\pi}{2} = \frac{\pi}{2} \int_0^{\pi/2} t dt$$

Put $\cos^{-1}x=t$

$$\therefore \frac{-1}{\sqrt{1-x^2}} dx = dt$$

$$I = \frac{\pi}{2} \left[\frac{t^2}{2} \right]_0^{\pi/2} = \frac{\pi^3}{16}$$