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Unit -I
Differential Calculus

- Rolle’s theorem
- Mean value theorems
- Taylor’s series and Maclaurin’s series
- Expansion of functions using standard expansions
- Indeterminate forms
- L’Hospital’s rule
- Evaluation of limits and applications

Applications

1) Look at the calendar listing the precise time for sunset each day. One notices that around the precise date in the summer when sunset is the latest, the precise hour changes very little from day to day in the vicinity of the precise date. This is an illustration of Rolle's theorem because near a point where the derivative vanishes, the function changes very little.

2) Image processing has lots of calculus and its used in many different fields such as medical imaging (CT scans, MRI, diagnosing sick patients), industrial automation (quality control, bubble detection etc .)

3) Taylor's series can be used by computer programs to compute values of sine, cosine, and tangent or logarithm functions

4) Continuous compounding interest rates encountered everyday in *investments, different types of bank accounts, or when paying credit cards bills, mortgages, etc.* L'Hospital's Rule is used to prove that the compound interest rate equation through continuous compounding equals Part

Unit -I Differential Calculus

- **Rolle's Theorem:**

Statement: Let f(x) be a function defined on closed interval $[a, b]$. If

i) f(x) is continuous function on $[a, b]$

ii) f(x) is differntiable on (a, b)

iii) f(a)=f(b), then there exist at least one point c $\epsilon(a\ , b)$ such that f'(c)=0

Ex.1) Verify Rolle′s Theorem for the function;
$f(x)=x^2\,(1-x)^2, x\in[0,1]$

Solution:

The given function can be written as $(x)=x^4-2x^3+x^2$ which is polynomial function of degree 4

$\therefore f(x)$ *is continious as well as differntiable on*$[0,1]$
Also $f(0)=f(1)=0$

$\therefore$*All conditions of Rolle′s theorem are varified*
$\therefore$*ther exit at least one pont* $c\epsilon(0,1)$ *such that* $f'(c)=0$

Now we have to find such c, $f'(x)=4x^3-6x^2+2x$
$\therefore f'(c)=4c^3-6c^2+2c=0$
$\therefore 2(2c^2-3c+1)=0$ $\quad\quad 2c(c-1)(c-\frac{1}{2})=0$
$\therefore c=0,=1,c=\frac{1}{2}$

$c=\frac{1}{2}\in(0,1)$ *such that* $f'(c)=0$

Lagranges Mean Value Theorem

Statement: If a function f(x) is

i)Continuous in the closed interval $[a, b]$

ii)Differntiable in the open interval (a, b)

Then there exist at least one point x=c in(a, b)such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Some deductions from Mean Value Theorem

Consider a function f (x) which satisfies the conditions of the LMVT in (a, b) .

i) If $f'(x) = 0$ for every $x \in (a, b)$, then f (x) is constant in that interval.

ii) If $f'(x) > 0$ for every $x \in (a, b)$, then f x is strictly increasing function in $[a, b]$.

iii) If $f'(x) < 0$ for every $x \in (a, b)$, then f x is strictly decreasing function in $[a, b]$.

$\boldsymbol{Ex.\ 1)\ Verify\ LMVT\ for\ the\ function\ f\ x = \log x\ in\ [1, e]}$

Solution:

The given function f (x) = log x is continuous in $[1, e]$and differentiable in(1, e) .

Hence all conditions of LMVT are satisfied.

Therefore there exist at least one point c in (1, e) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$=\frac{f(e)-f(1)}{e-1}$$

$$= \frac{1}{c} = \frac{\log e - \log 1}{e-1}$$

$$=\frac{1-0}{e-1} = \frac{1}{e-1}$$

c=e-1

here c=e-1 lies in (1,e)

Hence,LMVT is verified.

Q.3 Prove that $\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$ $\quad$ *where* $0 < a < b < 1$

Hence show that $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$

Solution:

Let f(x)=$\tan^{-1} x$ in $[a, b]$, 0<a<b<1

$\therefore f(x) \textit{ is continuous in } [a, b]$

f'(x)=$\frac{1}{1+x^2}$, in (a,b)

$\therefore f(x)$ *is differentiable* in (a,b)

$\therefore$ *By LMVT*, $\exists c \in (a, b)$ such that

$f'(c) = \frac{f(b)-f(a)}{b-a}$

$\therefore \frac{1}{1+c^2} = \frac{\tan^{-1} b - \tan^{-1} a}{b-a}$

we have $0 < a < b < 1$ *and* $a < c < b$

$\therefore 1 + a^2 < 1 + c^2 < 1 + b^2$

$$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$$

$$\therefore \frac{1}{1+b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b-a} < \frac{1}{1+a^2}$$

Hence , $\frac{1}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{1}{1+a^2}$ $\dots\dots..(2)$

Let a=1 ,b =$\frac{4}{3}$

$\therefore$Equation(2) becomes,

$$\frac{\frac{4}{3} - 1}{1 + \left(\frac{4}{3}\right)^2} < \tan^{-1}\frac{4}{3} - \tan^{-1} 1 < \frac{\frac{4}{3} - 1}{1 + 1}$$

$$\frac{3}{25} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{1}{6}$$

Hence $\boldsymbol{\frac{\pi}{4} + \frac{3}{25} < tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}}$

Ex. 2) Verify LMVT for the function f (x) = x - x^3 in [-2, 1]

Solution:
We know that

i) f (x) is polynomial function in x is continuous function in$[-2,1]$

ii) f(x) is differntiable in (-2,1) and f ′ (x) = 1 - 3x^2

Hence all the conditions of LMVT are satisfied.

Therefore there exist at least one point c∈ (-2,1) such that f ′(c) =$\frac{f(b)-f(a)}{b-a}$

1-3c^2 = $\frac{f(1)-f(-2)}{1-(-2)}$ 1-3c^2= $\frac{0+6}{3}$

1-3c^2 = 2 c^2= 1 c=± 1

Clearly the values of c = -1 lie in (-2,1). Hence LMVT verified

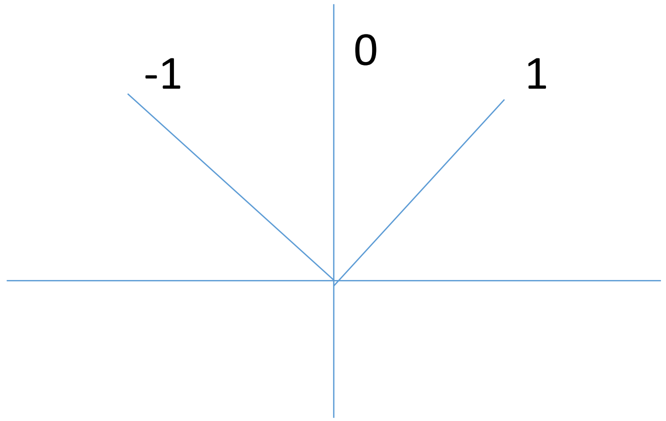
Ex.) Verify LMVT for the function f(x) = $|x|$, x ∈ [-**1**, **1**]

Solution:
Here we note that f(x) = $|x|$ is continuous in $[-1,1]$

but the f (x) is not differntiable at x = 0 ∈ (-1,1)

∴ f(x) is not differntiable in -1,1

Thus second condition of LMVT is not satisfied



∴ LMVT is not applicable to f (x) in $[-1,1]$

Expansion of Function

Power Series: a series of the form fg

f(x)=$\sum_{n=0}^{\infty} b_n(x-a)^n$= b_0+$b_1(x-a)^1$+$b_2(x-a)^2$+$b_3(x-a)^3$+…….

Where b_0, b_1, b_2, b_3,…..and a are constants and x varies around a, is called a power series of f(x) in powers of (x-a).

For a= 0 it becomes

f(x) = $\sum_{n=0}^{\infty} b_n x^n$= b_0+b_1x+b_2x^2+b_3x^3+…….+b_nx^n+…. is called a power series of f(x) in powers of x.

Taylor's Theorem :

Let f(x) be a function defined in $[a, a+h]$ such that

i) f(x),f'(x),f''(x),……$f^{(n-1)}$(x) be continuous in $[a, a+h]$

ii) $f^{(n)}$(x) exists, ∀ x $\epsilon(a, a+h)$

Then there exists at least one number $\theta(0<\theta<1)$ *such that*

$$f(a+h) = f(a)+ hf'(a)+\frac{h^2}{2!}f''(a)+......+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(x) +R_n$$

where R_n *is the remainder after n terms*

By Taylors theorem : If the remainder $R_n \to 0$ $as\ n \to \infty, then\ the\ Taylors\ assumes\ the\ form\ of\ Taylors\ series$
And it is given by.

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots\dots + \frac{h^n}{n!}f^{(n)}(a) + \dots.$$

Different forms of Taylors series :

i)The expansion of f(x+h) in ascending powers of h:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \dots\dots + \frac{h^n}{n!}f^{(n)}(x) + \dots.$$

ii) The expansion of f(x+h) in ascending powers of x :

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \dots\dots + \frac{x^n}{n!}f^{(n)}(h) + \dots.$$

iii) The expansion of f(x) in ascending powers of (x-a) :

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots\dots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \dots.$$

which is also known as the expansion of f(x) at x=a.

Maclaurin's Series : The $\mathit{Taylors\ series}$ of f(x) at a = 0 is known as Maclaurin's series.
It is given by

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots\ldots + \frac{x^n}{n!}f^{(n)}(0) + \ldots.$$

It is used to find the expansion of some standard functions.

i) Expansion of e^x

Let f(x)= e^x $\therefore$ $f(0) = e^0 = 1$

f'(x)=e^x $\therefore$ f'(0)=1

f''(x)= f'''(x)=……. $f^{(n)}$(x)= e^x and

f''(0)= f'''(0)=……. $f^{(n)}$(0)= 1

Substituting all these values in Maclaurin's series we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots\ldots$$

$$\Rightarrow e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

ii) **Expansion of sinx** :

f(x)= sinx $\therefore$ $f(0) = 0$

f'(x)= $cosx$ $\therefore$ $f(0) = 1$

f''(x)= $-$sinx $\therefore$ $f(0) = 0$

f'''(x)= -$cosx$ $\therefore$ $f(0) = -1$

Substituting all these values in Maclaurin's series we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots\ldots$$

$$\text{Sinx} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots$$

iii) **Expansion of log (1+x):**

Standard Functions Series:

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1. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = \sum_{r=0}^{\infty} \frac{x^r}{r!}$.

2. $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \ldots\ldots = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!}$.

3. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r+1}}{(2r+1)!}$

4. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{(2r)!}$.

5. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17}{315} x^7 + \ldots$

6. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots = \sum_{r=0}^{\infty} \frac{x^{2r+1}}{(2r+1)!}$.

7. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\ldots = \sum_{r=0}^{\infty} \frac{x^{2r}}{(2r)!}$.

8. $\tanh x = x - \frac{x^3}{3} + \frac{2}{15} x^5 - \frac{17}{315} x^7 + \ldots\ldots$

9. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots\ldots = \sum_{r=1}^{\infty} \frac{(-1)^{r-1} x^r}{r}$.

10. $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \ldots\ldots = -\sum_{r=1}^{\infty} \frac{x^r}{r}$.

11. $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots$ Binomial series.

12. $\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots$ for $|x| < 1$.

13. $\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots\ldots$

Ex.1) **Prove that log (secx) =** $\frac{x^2}{2}+\frac{1}{3}\frac{x^4}{4}+\frac{2}{15}\frac{x^6}{6}+\cdots$

Sol.Let y = log(secx)

Then $\frac{dy}{dx}=tanx$

5. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17}{315}x^7 + \ldots$

Integrating both sides wrt x bet. Limits 0 to x,we get

$$[\log(secx)]_0^x = \left[\frac{x^2}{2}+\frac{1}{3}\frac{x^4}{4}+\frac{2}{15}\frac{x^6}{6}+\cdots\right]_0^x$$

$$[\log(secx)-0] = \frac{x^2}{2}+\frac{1}{3}\frac{x^4}{4}+\frac{2}{15}\frac{x^6}{6}+\cdots$$

$$\therefore \log(secx) = \frac{x^2}{2}+\frac{1}{3}\frac{x^4}{4}+\frac{2}{15}\frac{x^6}{6}+\cdots$$

Ex. 2) Expand $cos^{-1}\left[\frac{x-x^{-1}}{x+x^{-1}}\right]$ in ascending powers of x.

Sol. Let y= $\cos^{-1}\left[\frac{x-x^{-1}}{x+x^{-1}}\right]$

cosy = $\frac{x^2-1}{x^2+1}$

Put x= $\tan\theta$

cosy = $\frac{tan^2\theta-1}{tan^2\theta+1}$= - cos2$\theta$ $= cos(\pi - 2\theta)$

$\Rightarrow$ $y = \pi - 2\theta$= $\pi - 2\tan^{-1} x$

$y = \pi$- 2 $\left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right]$

Ex.3) Expand $(1+x)^x$ *in a series* upto the term containing x^4

Sol. Let y = $(1+x)^x$

Taking log on both sides,

Logy = x log (1+x)

Using standard expansion for log (1+x)

Logy = x $\left[x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right]$

= $\left[x^2-\frac{x^3}{2}+\frac{x^4}{3}-\frac{x^5}{4}+\cdots\right]$

= z (say)

y = e^z =1+ $z+\frac{z^2}{2!}+\frac{z^3}{3!}$ +……

= 1 +$\left[x^2-\frac{x^3}{2}+\frac{x^4}{3}-\frac{x^5}{4}+\cdots\right]+\frac{1}{2}\left[x^2-\frac{x^3}{2}+\frac{x^4}{3}\right]^2+\frac{1}{6}\left[x^2-\frac{x^3}{2}\right]^3$ +…

Neglecting the higher powers of x we get ,

= 1 + $x^2-\frac{x^3}{2}+\frac{5}{6}x^4-\frac{3x^5}{4}+\cdots$

Ex.3)Expand $(1+x)^{\frac{1}{x}}$ upto the term containing x^2

Ans. e$\left[1-\frac{x}{2}+\frac{11}{24}x^2+\cdots.\right]$

Ex.5) Expand $\sqrt{1+sinx}$ upto x^6.

Sol. Let y =$\sqrt{1+sinx}$

$$=\sqrt{\left(sin^2\frac{x}{2}+cos^2\frac{x}{2}\right)+2sin\frac{x}{2}\,cos\frac{x}{2}}$$

$$=\sqrt{(sin\frac{x}{2}+cos\frac{x}{2})^2}$$

$$= sin\frac{x}{2}+cos\frac{x}{2}$$

$$=\left[\left(\frac{x}{2}\right)-\frac{1}{3!}\left(\frac{x}{2}\right)^3+\frac{1}{5!}\left(\frac{x}{2}\right)^5-\cdots\right]+\left[1-\frac{1}{2!}\left(\frac{x}{2}\right)^2+\frac{1}{4!}\left(\frac{x}{2}\right)^4-\cdots\right]$$

$$=1+\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{48}+\frac{x^4}{384}+\frac{x^5}{3840}-\ldots\ldots.$$

Ex.6) Show that $e^{e^x} = e\left(1 + x + x^2 + \frac{5}{6}x^3 + \frac{5}{8}x^4 + \cdots\right)$

Sol.

By exponential series,

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \ldots\ldots = 1 + y$$

Where $y = x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\ldots$

$$\therefore\ e^{e^x} = e^{1+y} = e.\, e^y$$

$$= e\left(1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} \ldots\ldots\right)$$

$$= e\left[1 + \left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\right) + \frac{1}{2}\left(x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)^2 + \frac{1}{6}\left(x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)^3 + \frac{1}{24}(x + \ldots)^4 \ldots\right]$$

$$= e\left[1 + \left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\right) + \frac{x^2}{2}\left(1 + \frac{x}{2} + \frac{x^2}{6} + \ldots\right)^2 + \frac{x^3}{6}\left(1 + \frac{x}{2} + \frac{x^2}{6} + \ldots\right)^3 + \frac{x^4}{24}(1 + \ldots)^4 \ldots\right]$$

$$= e\left[1 + x + \left(\frac{1}{2} + \frac{1}{2}\right)x^2 + \left(\frac{1}{6} + \frac{1}{2} + \frac{1}{6}\right)x^3 + \left(\frac{1}{24} + \frac{7}{24} + \frac{1}{4} + \frac{1}{24}\right)x^4 + \cdots\right]$$

$$e^{e^x} = e\left[1 + x + x^2 + \frac{5}{6}x^3 + \frac{5}{8}x^4 + \cdots\right]$$

Ex.7) Expand log $\left(\mathbf{1}+\boldsymbol{x}+\boldsymbol{x}^{2}+\boldsymbol{x}^{3}\right)$ upto a term in $\boldsymbol{x}^{8}$.

Sol. Let f(x) = log $(1+x+x^{2}+x^{3})$

$$= \log\left[\frac{\left(1+x+x^{2}+x^{3}\right)(1-x)}{(1-x)}\right]$$

$$= \log\left[\frac{1-x^{4}}{1-x}\right]$$

$$= \log\left(1-x^{4}\right) - \log(1-x)$$

$$= \left[-x^{4}-\frac{x^{8}}{2}\right]-\left[-x-\frac{x^{2}}{2}-\frac{x^{3}}{3}-\frac{x^{4}}{4}-\frac{x^{5}}{5}-\frac{x^{6}}{6}-\frac{x^{7}}{7}-\frac{x^{8}}{8}-\cdots\right]$$

$$= x+\frac{x^{2}}{2}+\frac{x^{3}}{3}-\frac{3x^{4}}{4}+\frac{x^{5}}{5}+\frac{x^{6}}{6}+\frac{x^{7}}{7}-\frac{3x^{8}}{8}\ldots$$

Ex .7) Prove that log $\left(1+x+x^{2}+x^{3}+x^{4}\right)=x+\frac{x^{2}}{2}+\frac{x^{3}}{3}+\frac{x^{4}}{4}-\frac{4}{5}x^{5}+\ldots\ldots$

Ex. 8) Expand 40 + $\mathbf{53(x-2)+19(x-2)^2+2(x-2)^3}$ in ascending powers at x.

Sol. $-6+x+7x^2+2x^3$

Ex. 8) Using Taylor's theorem express 7 + $(x+2)^2+3(x+2)^3+(x+2)^4$ in ascending powers at x.

Sol. : let $f(x+h) = 7 + (x+h)^2+3(x+h)^3+(x+h)^4$

$$f(x)=7+x+3x^3+x^4$$

$$f(x+h) = f(h)+ xf'(h)+\frac{x^2}{2!}f''(h)+\frac{x^3}{3!}f'''(h)+\ldots$$

Put $h=2, then$

$$f(x+2) = f(2)+ xf'(2)+\frac{x^2}{2!}f''(2)+\frac{x^3}{3!}f'''(2)+\frac{x^4}{4!}f^{iv}(2)+\ldots \qquad (1).$$

$f(x)=7+x+3x^3+x^4$	$f(2)=49$
$f'(x)=1+9x^2+4x^3$	$f'(2)$=69
$f''(x)$ = $18x+12x^2$	$f''(2)$ =84
$f'''(x)$ = 18+24x	$f'''(2)$ =66
$f^{iv}(x)$ = 24	$f^{iv}(x)$ = 24
$f^{v}(x)$=0	

Substituting in (1) we have

$$f(x+2) = 49+ x.69+\frac{x^2}{2!}.84+\frac{x^3}{3!}.66+\frac{x^4}{4!}.24+0\ldots$$

$$f(x+2) = 49+ 69x+42x^2+11x^3+x^4$$

Ex. 9) Using Taylor's series express 5 + 4$(x-1)^2 - 3(x-1)^3 + (x-1)^4$ in ascending powers at x.

Sol. : let $f(x+h)$ = 5 + 4$(x-1)^2 - 3(x-1)^3 + (x-1)^4$

Put h = -1

By Taylor's theorem ,

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \frac{x^3}{3!}f'''(h) + \frac{x^4}{4!}f^{iv}(h) + \ldots \ldots$$

$f(x) = 5 + 4x^2 - 3x^3 + x^4$ $\qquad$ $f(-1) = 13$

$f'(x) = 8x - 9x^2 + 4x^3$ $\qquad$ $f'(-1)$=-21

$f''(x)$ = 8-18$x + 12x^2$ $\qquad$ $f''(-1)$ =38

$f'''(x)$ = -18+24x $\qquad$ $f'''(-1)$ =-42

$f^{iv}(x)$ = 24 $\qquad$ $f^{iv}(-1)$ = 24

$$\therefore \text{ we get } f(x+h) = f(-1) + xf'(-1) + \frac{x^2}{2!}f''(-1) + \frac{x^3}{3!}f'''(-1) + \frac{x^4}{4!}f^{iv}(-1) + \ldots$$

$$= 13 - 21x + 19x^2 - 7x^3 + x^4$$

Ex.10) $\boldsymbol{Expand\ \ x^4 - 3x^3 + 2x^2 - x + 1\ \ in\ power\ of\ (x-3)}$.

Sol. By Taylor's theorem ,

$$f(x) = f(a) + (x\text{-}a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots\ldots$$

$$f(x) = f(3) + (x\text{-}3)f'(3) + \frac{(x-3)^2}{2!}f''(3) + \ldots\ldots \qquad (1)$$

$f(x) = x^4 - 3x^3 + 2x^2 - x + 1$ $\qquad$ $f(3) = 16$

$f'(x) = 4x^3 - 9x^2 + 4x - 1$ $\qquad$ $f'(3)$=38

$f''(x)$ = $12x^2$ $-18x + 4$ $\qquad$ $f''(3)$ =58

$f^{iv}(x)$ = 24 $\qquad$ $f^{iv}(3)$ = 24

$f^{v}(x)$=0 $\qquad$ $f^{v}(3)$=0

Substituting in (1) ,we have

$$f(x) = 16 + 38(x\text{-}3) + \frac{58(x-3)^2}{2!} + \frac{54(x-3)^3}{3!} + 24\frac{(x-3)^4}{4!} + 0$$

$$f(x) = 16 + 38(x-3) + 29\,(x-3)^2 + 9(x-3)^3 + (x-3)^4$$

Ex.11) $\boldsymbol{Expand\ 3x^3 - 2x^2 + x - 4\ in\ powers\ of\ (x+2)\ using}$ **Taylors theorem.**

Sol. . By Taylor's theorem ,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots\ldots$$

$f(x) = 3x^3 - 2x^2 + x - 4$ $\qquad$ $f(-2) = -38$

$f'(x) = 9x^2 - 4x + 1$ $\qquad$ $f'(-2) = 45$

$f''(x) = 18x - 4$ $\qquad$ $f''(-2) = 40$

$f''(x) = 18$ $\qquad$ $f''(-2) = 18$

$$f(x) = -38 + 45(x+2) - 20\,(x+2)^2 + 3(x+2)^3$$

Ex.12) Use Taylor's theorem to $\sqrt{25.15}$

Sol.Let $f(x+h) = \sqrt{x+h} = \sqrt{25.15} = \sqrt{25+0.15}$

$\therefore$ $f(x) = \sqrt{x}$ and h = 0.15 , x =25

$f'(x) = \frac{1}{2\sqrt{x}}$, $f''(x) = -\frac{1}{4}x^{-3/2}$, $f'''(x) = \frac{3}{8}x^{-5/2}$

$\therefore$ By Taylor's theorem

$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots$

$= \sqrt{x} + \frac{h}{2}\frac{1}{\sqrt{x}} - \frac{h^2}{8(\sqrt{x}^3)} + \frac{h^3}{16(\sqrt{x}^5)}$

Put x= 25 and h= 0.15

Let $f(x+h) = \sqrt{25.15} = 5 + \frac{0.15}{2} \times \frac{1}{5} - \frac{0.15^2}{8(5)^3} + \frac{0.15^3}{16(5)^5} = \sqrt{25.15} = 5.01478$

INDETERMINATE FORMS

Let f(x) and g(x) be any two functions of x such that f(a) = 0 and g(a) = 0,then the ratio $\frac{f(x)}{g(x)}$ is said to assume The indeterminate form $\frac{0}{0}$ at x= a.

There are seven indeterminate forms, $\frac{0}{0}$, $\frac{\infty}{\infty}, 0 \times \infty,\ \infty - \infty,\ \infty^0, 1^\infty,\ 0^0$

*L'Hospital's Rule :

Let f(x) and g(x) be functions of x such that f(a) = 0 and g(a) = 0,

i.e. $\lim_{x \to a} f(x) = 0 \quad and \quad \lim_{x \to a} g(x) = 0$

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ provided derivative of f(x) and g(x) exists.

$= \frac{f'(a)}{g'(a)}$ (g'(a) $\neq 0$).

Important Formulae of limits :

1) $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

2) $\lim\limits_{x \to 0} \frac{\tan x}{x} = 1$

3) $\lim\limits_{x \to 0} \frac{\sin^{-1} x}{x} = 1$

4) $\lim\limits_{x \to 0} (1 + x)^{1/x} = e$

5) $\lim\limits_{x \to 0} \frac{a^{x}-1}{x} = \log a$

6) $\lim\limits_{x \to 0} \frac{e^{x}-1}{x} = 1$

1) Evaluate $\lim\limits_{x \to 0} \dfrac{e^{ax}-e^{-ax}}{\log(1+bx)}$

Let ,

$$L = \lim_{x \to 0} \frac{e^{ax}-e^{-ax}}{\log(1+bx)} \qquad \ldots\ldots\left(\frac{0}{0}\, form\right)$$

$\therefore$ *Applying L'hospital Rule , we get*

$$= \lim_{x \to 0} \frac{ae^{ax}-e^{-ax}(-a)}{\frac{1}{(1+bx)}(b)}$$

$$= \lim_{x \to 0} \frac{ae^{ax}+ae^{-ax}}{\left(\frac{b}{1+bx}\right)}$$

$$= \frac{a+a}{b}$$

$$= \frac{2a}{b}$$

2) Evaluate $\lim\limits_{x \to 0} \frac{(1+x)^n - 1}{x}$

Sol. Let L = $\lim\limits_{x \to 0} \frac{(1+x)^n - 1}{x}$ .... $\left(\frac{0}{0}\ form\right)$

$\therefore Applying\ \ L'hospital\ Rule\ ,we\ get$

L = $\lim\limits_{x \to 0} \frac{n(1+x)^{n-1}}{1}$

= $n(1+0)^{n-1}$

= n

3) Evaluate $\lim\limits_{x\to 0} \dfrac{xe^x - \log(1+x)}{x^2}$

Sol. Let L = $\lim\limits_{x\to 0} \dfrac{xe^x - \log(1+x)}{x^2}$ .... $\left(\frac{0}{0}\ form\right)$

By $L'hospital\ Rule$,

L = $\lim\limits_{x\to 0} \dfrac{(xe^x + e^x) - \frac{1}{1+x}}{2x}$ .... $\left(\frac{0}{0}\ form\right)$

L = $\lim\limits_{x\to 0} \dfrac{(xe^x + e^x + e^x) + 1/(1+\mathrm{x})^2}{2}$

L = $\dfrac{1+1+1}{2}$

= $\dfrac{3}{2}$

Ex.4) If $\lim_{x\to 0}\frac{\sin 2x+\text{p}\sin x}{x^3}$ is finite then find the value of p and hence the value of limit.

Sol. $\lim_{x\to 0}\frac{\sin 2x+\text{p}\sin x}{x^3}$

$$= \lim_{x\to 0}\frac{\sin 2x+\text{p}\sin x}{x^3}$$

$$= \lim_{x\to 0}\frac{sinx}{x}\times\frac{2cosx+p}{x^2}$$

$$= \lim_{x\to 0}\frac{2cosx+p}{x^2} \qquad \left(\because \lim_{x\to 0}\frac{sinx}{x}=1\right)$$

Here the denominater being zero for x = 0 and numerator becomes 2+p. Therefore,if the limit is to be finite, The numerator must be zero for x = 0.this requires

$$2+p=0 \qquad \Longrightarrow \qquad p=-2$$

With this value of p ,required limit

$$= \lim_{x\to 0}\frac{2cosx-2}{x^2} \qquad \left(form\ \frac{0}{0}\right)$$

$$= \lim_{x\to 0}\left(-\frac{sinx}{x}\right) \qquad \text{by L'hospital rule}$$

=-1

$$\therefore\ p=-2\ and\ limit=-1$$

Ex.5) Evaluate $\lim_{x \to 0} \frac{\log tanx}{logx}$

Solutin: $\lim_{x \to 0} \frac{\log tanx}{logx}$ $\left(form\ \frac{\infty}{\infty}\right)$

$\therefore Applying\ \ L'hospital\ Rule\ , we\ get$

$$= \lim_{x \to 0} \frac{\frac{1}{\tan x}\ sec^2 x}{\frac{1}{x}} \qquad \left(form\ \frac{\infty}{\infty}\right)$$

$$= \lim_{x \to 0} \frac{x}{\sin x \cos x} \qquad \left(form\ \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2x}{\sin 2x}$$

$$= 1$$

Ex.6) Evaluate $\lim\limits_{x\to 0} sinxlogx$.

Sol. $\lim\limits_{x\to 0} sinxlogx \qquad (form\ 0\times\infty)$

$$= \lim_{x\to 0} \frac{logx}{cosecx} \qquad \left(form \frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0} \frac{{}^{1}/_{x}}{-cosecxcotx} = -\lim_{x\to 0} \frac{sin^2x}{xcosx} \qquad \left(form \frac{0}{0}\right)$$

$$= -\lim_{x\to 0} \frac{2sinxcosx}{cosx-xsinx} \qquad (by\ Lhospital\ rule)$$

$$= -\left(\frac{0}{1-0}\right) = 0$$

Ex.7) Evaluate $\boldsymbol{\lim_{x\to 1}\left[\frac{x}{x-1}-\frac{1}{logx}\right]}$

Sol. $\lim_{x\to 1}\left[\frac{x}{x-1}-\frac{1}{logx}\right]$ $(form\ \infty-\infty)$

$$= \lim_{x\to 1}\left[\frac{xlogx-x+1}{(x-1)logx}\right] \qquad \left(form\ \frac{0}{0}\right)$$

$$= \lim_{x\to 1}\left[\frac{(1+logx)-1}{\frac{(x-1)}{x}+logx}\right] \qquad (by\ Lhospital\ rule)$$

$$= \lim_{x\to 1}\left[\frac{logx}{1-\frac{1}{x}+logx}\right] \qquad \left(form\ \frac{0}{0}\right)$$

$$=\lim_{x\to 1}\left[\frac{\left(\frac{1}{x}\right)}{\frac{1}{x^2}+\frac{1}{x}}\right] \qquad (by\ Lhospital\ rule)$$

$$= \frac{1}{2}$$

Ex.8) Evaluate $\lim_{x \to \frac{\pi}{2}} secx^{cotx}$

Sol. $\lim_{x \to \frac{\pi}{2}} secx^{cotx}$

Ex.9) Evaluate $\boldsymbol{\lim_{x \to \frac{\pi}{2}} cosx^{cosx}}$

Sol. Let L = $\lim_{x \to \frac{\pi}{2}} cosx^{cosx}$ $(form 0^0)$

Log L= $\lim_{x \to \frac{\pi}{2}}$ cosx log(cosx) $(form 0 \times \infty)$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\log(cosx)}{secx} \quad \left(form \ \frac{\infty}{\infty}\right)$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{(-sinx/cosx)}{(\sec x \ tanx)}$$

= $\lim_{x \to \frac{\pi}{2}}$ (-cosx) = 0

L = e^0 = 1

Ex.10) Solve $\boldsymbol{\lim_{x\to 0}\left(\frac{2^x+3^x}{2}\right)^{1/x}}$

Sol. Let L= $\lim_{x\to 0}\left(\frac{2^x+3^x}{2}\right)^{1/x}$ $\qquad (form 1^{\infty})$

Log L = $\lim_{x\to 0}\frac{1}{x}.\log\left(\frac{2^x+3^x}{2}\right)$ $\qquad (form \infty \times 0)$

= $\lim_{x\to 0}\frac{\log\left(\frac{2^x+3^x}{2}\right)}{x}$ $\quad form\left(\frac{0}{0}\right)$

$= \lim_{x\to 0}\frac{\left(\frac{2}{2^x+3^x}\right)\left(\frac{2^x log2+3^x log3}{2}\right)}{1}$ $\qquad (by\ L'hospital\ rule)$

= $\lim_{x\to 0}\frac{2^x log2+3^x log3}{2^x+3^x}$

= $\frac{log2+log3}{2}$ = $\frac{1}{2}log6 = log\sqrt{6}$

L = $e^{log\sqrt{6}}$ = $\sqrt{6}$

Ex.11) Evaluate $\lim\limits_{x\to\infty}\left[\dfrac{a^{1/x}+b^{1/x}+c^{1/x}}{3}\right]^{x}$

Sol. Put $\frac{1}{x} = y, then\ y \to 0\ as\ x \to \infty\ and$

$$\lim_{x\to\infty}\left[\frac{a^{1/x}+b^{1/x}+c^{1/x}}{3}\right]^{x} = \lim_{y\to 0}\left[\frac{a^{y}+b^{y}+c^{y}}{3}\right]^{1/y}$$

Let $\quad L = \lim\limits_{y\to 0}\left[\dfrac{a^{y}+b^{y}+c^{y}}{3}\right]^{1/y}$

$$\log L = \lim_{y\to 0}\frac{1}{y}\ log\left[\frac{a^{y}+b^{y}+c^{y}}{3}\right] = \lim_{y\to 0}\frac{log\left[\frac{a^{y}+b^{y}+c^{y}}{3}\right]}{y} \qquad ....\left(\frac{0}{0} form\right)$$

Apply $L'hospital\ rule$

$$=\lim_{y\to 0}\frac{\left(\frac{3}{a^{y}+b^{y}+c^{y}}\right)\left(\frac{a^{y}loga+b^{y}logb+c^{y}logc}{3}\right)}{1}$$

$$= \lim_{y\to 0}\frac{a^{y}loga+b^{y}logb+c^{y}logc}{a^{y}+b^{y}+c^{y}} = \lim_{y\to 0}\frac{loga+logb+logc}{1+1+1}$$

$$=\frac{logabc}{3} = \log abc^{1/3}$$

$$\therefore L = abc^{1/3}$$