Unit VI
Eigen values and Eigen Vectors

Eigen values and Eigen vectors

For a given non-zero square matrix A if there exists a scalar λ and a non-zero vector X such that

$$AX = \lambda X$$

then λ is called ***eigen value*** or characteristic root and X is called ***eigen vector*** or characteristic vector.

The set of all eigen values of A is called ***spectrum*** of A.

$$\text{As} \quad AX = \lambda X$$
$$AX = \lambda IX$$
$$AX - \lambda IX = 0$$
$$(A - \lambda I)X = 0$$

which is a homogeneous system of equations. Since X is non-zero, system must have non-trivial solution and homogeneous system possesses non-trivial solution if $|A - \lambda I| = 0$, then

$$|A - \lambda I| = a_0 \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n = 0$$

is called ***characteristic equation*** of A and the root of this equation i.e. $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigen values of A.

Corresponding to every eigen value λ, there exist a corresponding eigen vector X satisfying ***matrix equation***

$$(A - \lambda I)X = 0$$

Trace of a matrix

The sum of the entries on the diagonal of a square matrix A is called trace. Thus

$$\boldsymbol{Trace\ of\ A = a_{11} + a_{22} + \cdots + a_{nn}}$$

Properties of eigen values

$$Trace\ of\ A = \lambda_1 + \lambda_2 + \ldots + \lambda_n.$$

$$|A| = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n$$

Method of finding eigen values and eigen vectors of a 3×3 matrix

Characteristic equation of A is

$$|A - \lambda I| = \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

where

$$S_1 = a_{11} + a_{22} + a_{33}$$
$$S_2 = M_{11} + M_{22} + M_{33}$$
$$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Matrix equation of A is

$$(A - \lambda I)X = \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Find the eigen values and the corresponding eigen vector of a matrix

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Characteristic equation of A is

$$|A - \lambda I| = \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

where $S_1 = a_{11} + a_{22} + a_{33}$

$$= 1 + 2 - 1 = 2$$

$$S_2 = M_{11} + M_{22} + M_{33}$$

$$= \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 0 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = -1$$

$$|A| = -2$$

$\therefore$Ch. Eq. of A is $\lambda^3 - 2\lambda^2 - \lambda - (-2) = 0$

$$\boldsymbol{\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -1}$$

Matrix equation of A is

$$(A-\lambda I)X=\begin{bmatrix}1-\lambda & 1 & -2\\ -1 & 2-\lambda & 1\\ 0 & 1 & -1-\lambda\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

For $\lambda_1 = 1$, let $X_1=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$ be corresponding eigen vector of A, then

$$\begin{bmatrix}0 & 1 & -2\\ -1 & 1 & 1\\ 0 & 1 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\therefore x_1 = 3t, x_2 = 2t\ , x_3 = t$$

$\therefore$ Eigen vector corresponding to $\lambda_1 = 1$ is $X_1=\begin{bmatrix}3\\ 2\\ 1\end{bmatrix}$

For $\lambda_2 = 2$, let $X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be corresponding eigen vector of A, then

$$\begin{bmatrix} -1 & 1 & -2 \\ -1 & 0 & 1 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore x_1 = -t, x_2 = -3t \, , x_3 = -t$$

$\therefore$ Eigen vector corresponding to $\lambda_2 = 2$ is $X_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

For $\lambda_3 = -1$, let $X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be corresponding eigen vector of A, then

$$\begin{bmatrix} 2 & 1 & -2 \\ -1 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore x_1 = -t, x_2 = 0 \; , x_3 = -t$$

$\therefore$ Eigen vector corresponding to $\lambda_3 = -1$ is $X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Method of finding eigen values and eigen vectors of a 3×3 matrix

Characteristic equation of A is

$$|A - \lambda I| = \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

where

$$S_1 = \mathrm{a}_{11} + \mathrm{a}_{22} + \mathrm{a}_{33}$$

$$S_2 = M_{11} + M_{22} + M_{33}$$

$$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Matrix equation of A is

$$(A - \lambda I)X = \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Find the eigen values and the corresponding eigen vector of a matrix

$$A = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$$

Characteristic equation of A is

$$|A - \lambda I| = \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

where $S_1 = \mathrm{a}_{11} + \mathrm{a}_{22} + \mathrm{a}_{33}$

$$= 4 + 3 + 1 = 8$$

$$S_2 = M_{11} + M_{22} + M_{33}$$

$$= \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} + \begin{vmatrix} 4 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 4 & 2 \\ -5 & 3 \end{vmatrix} = 17$$

$$|A| = 10$$

$\therefore$Ch. Eq. of A is $\lambda^3 - 8\lambda^2 + 17\lambda - 10 = 0$

$$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 5$$

Matrix equation of A is

$$(A-\lambda I)X = \begin{bmatrix} 4-\lambda & 2 & -2 \\ -5 & 3-\lambda & 2 \\ -2 & 4 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda_1 = 1$, let $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be corresponding eigen vector of A, then

$$\begin{bmatrix} 3 & 2 & -2 \\ -5 & 2 & 2 \\ -2 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore x_1 = 8t, x_2 = 4t , x_3 = 16t$$

$\therefore$ Eigen vector corresponding to $\lambda_1 = 1$ is $X_1 = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$

For $\lambda_2 = 2$, let $X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be corresponding eigen vector of A, then

$$\begin{bmatrix} 2 & 2 & -2 \\ -5 & 1 & 2 \\ -2 & 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore x_1 = -9t, x_2 = -9t \, , x_3 = -18t$$

$\therefore$ Eigen vector corresponding to $\lambda_2 = 2$ is $X_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

For $\lambda_3 = 5$, let $X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be corresponding eigen vector of A, then

$$\begin{bmatrix} -1 & 2 & -2 \\ -5 & -2 & 2 \\ -2 & 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore x_1 = 0, x_2 = 12t \, , x_3 = \, 12t$$

$\therefore$ Eigen vector corresponding to $\lambda_3 = 5$ is $X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Find the eigen values and the corresponding eigen vector of a matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$

Characteristic equation of A is

$$|A - \lambda I| = \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

where $S_1 = \mathrm{a}_{11} + \mathrm{a}_{22} + \mathrm{a}_{33}$

$$= 2 + 3 + 4 = 9$$

$$S_2 = M_{11} + M_{22} + M_{33}$$

$$= \begin{vmatrix} 3 & 2 \\ 3 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = 15$$

$$|A| = 7$$

$\therefore$Ch. Eq. of A is $\lambda^3 - 9\lambda^2 + 15\,\lambda - 7 = 0$

$$\lambda_1 = 7\,, \lambda_2 = 1 = \lambda_3$$

Matrix equation of A is

$$(A-\lambda I)X=\begin{bmatrix}2-\lambda & 1 & 1\\ 2 & 3-\lambda & 2\\ 3 & 3 & 4-\lambda\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

For $\lambda_1=7$, let $X_1=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$ be corresponding eigen vector of A, then

$$\begin{bmatrix}-5 & 1 & 1\\ 2 & -4 & 2\\ 3 & 3 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\therefore x_1=6t, x_2=12t\ , x_3=18t$$

$\therefore$ Eigen vector corresponding to $\lambda_1=7$ is $X_1=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$

For $\lambda_2 = 1 = \lambda_3$, let $X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be corresponding eigen vector of A, then

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Consider the equation $x_1 + x_2 + x_3 = 0$

Putting arbitrary values for 2 variables

$$x_2 = s\,,\, x_3 = t,\, x_1 = -x_2 - x_3 = -s - t$$

$\therefore$ Eigen vector corresponding to $\lambda_2 = 1 = \lambda_3$ is

$$\begin{bmatrix} -s - t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Find the eigen values and the corresponding eigen vector of a matrix

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Characteristic equation of A is

$$|A - \lambda I| = \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

where $S_1 = a_{11} + a_{22} + a_{33}$

$$= -2 + 1 + 0 = -1$$

$$S_2 = M_{11} + M_{22} + M_{33}$$

$$= \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix} = -21$$

$$|A| = 45$$

$\therefore$Ch. Eq. of A is $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$$\lambda_1 = 5 , \lambda_2 = -3 = \lambda_3$$

Matrix equation of A is

$$(A-\lambda I)X=\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & -\lambda\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

For $\lambda_1 = 5$, let $X_1=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$ be corresponding eigen vector of A, then

$$\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\therefore x_1 = 8t, x_2 = 16t\ , x_3 = -8t$$

$\therefore$ Eigen vector corresponding to $\lambda_1 = 5$ is $X_1=\begin{bmatrix}1\\ 2\\ -1\end{bmatrix}$

For $\lambda_2 = -3 = \lambda_3$, let $X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be corresponding eigen vector of A, then

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Consider the equation $x_1 + 2x_2 - 3x_3 = 0$

Putting arbitrary values for 2 variables

$$x_2 = s\ ,\ x_3 = t,\ x_1 = -2x_2 + 3x_3 = -2s + 3t$$

$\therefore$ Eigen vector corresponding to $\lambda_2 = 1 = \lambda_3$ is

$$\begin{bmatrix} -2s + 3t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -2s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} 3t \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

Find the eigen values and the corresponding eigen vector of a matrix

$$A = \begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$$

- Ch. Poly. $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$
- Eigen values: $1,1,1$
- Eigen Vector: $\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$

Find the eigen values and the corresponding eigen vector of a matrix

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 3 & -2 & 3 \\ 0 & 3 & 0 \end{bmatrix}$$

Find the eigen values and the corresponding eigen vector of a matrix

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

Cayley Hamilton Theorem

Every square matrix satisfies its own characteristic equation

Ch. eqn. of A is

$$|A - \lambda I| = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0$$

By Cayley Hamilton Theorem

$$a_0A^n + a_1A^{n-1} + \cdots + a_nI = 0$$

Q1)Verify Cayley Hamilton theorem for the matrix A and use it to find A^4 and A^{-1}

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Characteristic equation of A is

$$|A - \lambda I| = \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

where $S_1 = \mathrm{a}_{11} + \mathrm{a}_{22} + \mathrm{a}_{33}$

$$= 1 + 1 + 1 = 3$$

$$S_2 = M_{11} + M_{22} + M_{33}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 3$$

$$|A| = 1$$

$\therefore$Ch. Eq. of A is $\lambda^3 - 3\,\lambda^2 + 3\lambda - 1 = 0$

$\therefore$ By Cayley Hamilton's Theorem

$$A^3 - 3\,A^2 + 3A - I = 0$$

$$\because A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \therefore A^2 = A \cdot A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A^3 = A \cdot A^2 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A^3 - 3\,A^2 + 3A - I = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$+3\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore$ Cayley Hamilton's Theorem is verified

For finding A^4 consider,

$$A(A^3 - 3\,A^2 + 3A - I) = A(0) = 0$$

$$A^4 - 3\,A^3 + 3A^2 - A = 0$$

$$\therefore A^4 = 3\,A^3 - 3A^2 + A$$

$$\therefore A^4 = 3A^3 - 3\,A^2 + A = 3\begin{bmatrix}1 & 0 & 3\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$-3\begin{bmatrix}1 & 0 & 2\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}+\begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 4\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

For finding A^{-1} consider,

$$A^{-1}(A^3 - 3\,A^2 + 3A - I\ ) = A^{-1}(0) = 0$$

$$A^2 - 3\,A + 3I - A^{-1} = 0$$

$$\therefore A^{-1} = A^2 - 3A + 3I$$

$$\therefore A^{-1} = A^2 - 3A + 3I = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$-3\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Q2)Verify Cayley Hamilton theorem for the matrix A and use it to find A^{-1}

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

Characteristic equation of A is

$$|A - \lambda I| = \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

where $S_1 = \mathrm{a}_{11} + \mathrm{a}_{22} + \mathrm{a}_{33}$

$$= 0 + 2 + 1 = 3$$

$$S_2 = M_{11} + M_{22} + M_{33}$$

$$= \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -8$$

$$|A| = -2$$

$\therefore$Ch. Eq. of A is $\lambda^3 - 3\,\lambda^2 - 8\lambda + 2 = 0$

As $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ $\quad A^2 = A \cdot A = \begin{bmatrix} 7 & 4 & 5 \\ 11 & 8 & 11 \\ 4 & 6 & 10 \end{bmatrix}$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 19 & 20 & 31 \\ 41 & 38 & 57 \\ 36 & 26 & 36 \end{bmatrix}$$

$$A^3 - 3A^2 - 8A + 2I = \begin{bmatrix} 19 & 20 & 31 \\ 41 & 38 & 57 \\ 36 & 26 & 36 \end{bmatrix} - 3\begin{bmatrix} 7 & 4 & 5 \\ 11 & 8 & 11 \\ 4 & 6 & 10 \end{bmatrix}$$

$$-8\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Cayley Hamilton’s Theorem is verified

$$A^{-1}(A^3 - 3A^2 - 8A + 2I) = A^{-1}(0) = 0$$

$$A^2 - 3A - 8I + 2A^{-1} = 0$$

$$\therefore A^{-1} = \frac{1}{2}(-A^2 + 3A + 8I)$$

$$\therefore A^{-1} =$$

$$\frac{1}{2}\left\{-\begin{bmatrix} 7 & 4 & 5 \\ 11 & 8 & 11 \\ 4 & 6 & 10 \end{bmatrix} + 3\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix}$$

Diagonalization of matrix

Given square matrix AA of order n having n linearly independent eigen vectors can be written as ,

$$D = P^{-1}AP$$

where DD is a diagonal matrix called as **spectral matrix** having eigen values of AA as entries on diagonal and PP is non-singular matrix called as **modal matrix** having eigen vectors of AA as columns.

Let PA be a 3×3 matrix with $\boldsymbol{X_1}, \boldsymbol{X_2}, \boldsymbol{X_3}$ be the eigen vectors corresponding to eigen values $\boldsymbol{\lambda_1}, \boldsymbol{\lambda_2}, \boldsymbol{\lambda_3}$ then,

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad , \qquad P = [X_1 \quad X_2 \quad X_3].$$

Q.1) Find the modal matrixP P such that P^{-1}APAP is a diagonal matrix where

$$A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Characteristic equation ofA A is,

$$|A - \lambda I| = \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

Where, $S_1 = a_{11} + a_{22} + a_{33}$

$$= 1 + 2 + 3 = 6$$

$$S_2 = M_{11} + M_{22} + M_{33}$$

$$= \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 6 \\ 1 & 2 \end{vmatrix}$$

$$= 6 + 3 - 4 = 5$$

$$|A| = -12 .$$

$\therefore$Ch. Eq. of A is $\lambda^3 - 6\lambda^2 + 5\lambda - 12 = 0$

$$\boldsymbol{\lambda_1 = -1, \lambda_2 = 3, \lambda_3 = 4.}$$

Matrix equation of A is,

$$(A - \lambda I)X = \begin{bmatrix} 1-\lambda & 6 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda_1 = -1$, let $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be corresponding eigen vector of A, then

$$\begin{bmatrix} 2 & 6 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore x_1 = -3t, x_2 = t, x_3 = 0$$

$\therefore$ Eigen vector corresponding to $\lambda_1 = -1$ is $X_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$

For $\lambda_2 = 3$, let $X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be corresponding eigen vector of A, then

$$\begin{bmatrix} -2 & 6 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore x_1 = 2t, x_2 = t, x_3 = 0$ For $\lambda_2 = 3$,

$\therefore$ Eigen vector corresponding to $\lambda_2 = 3$ is $X_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

For $\lambda_3 = 4$, let $X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be corresponding eigen vector of A, then

$$\begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore x_1 = 2t, x_2 = t, x_3 = 0$

$\therefore$ Eigen vector corresponding to $\lambda_3 = 4$ is $X_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$\therefore Modal\, matrix\ P = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$ *Such that ,*

$$P^{-1}AP = \begin{bmatrix} {}^{-1}/_{5} & {}^{2}/_{5} & {}^{1}/_{20} \\ 0 & 0 & {}^{-1}/_{4} \\ {}^{1}/_{5} & {}^{3}/_{5} & {}^{1}/_{5} \end{bmatrix} \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = D$$

$$\therefore \mathrm{P} = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -4 & 0 \end{bmatrix}$$

Q.2) Find a matrix P such that $P^{-1}AP$ is a diagonal matrix where

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Characteristic equation of A is,

$$|A - \lambda I| = \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

Where, $S_1 = a_{11} + a_{22} + a_{33}$

$$= 1 + 2 - 1 = 2$$

$$S_2 = M_{11} + M_{22} + M_{33}$$

$$= \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 0 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}$$

$$= -3 - 1 + 3 = -1$$

$$|A| = -2\ .$$

$\therefore$Ch. Eq. of A is $\lambda^3 - 2\lambda^2 - \lambda - (-2) = 0$

$$\boldsymbol{\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -1.}$$

Matrix equation of A is,

$$(A - \lambda I)X = \begin{bmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda_1 = 1$, let $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be corresponding eigen vector of A, then

$$\begin{bmatrix} 0 & 1 & -2 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore x_1 = 3t, x_2 = 2t, x_3 = t$$

$\therefore$ Eigen vector corresponding to $\lambda_1 = 1$ is $X_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

For $\lambda_2 = 2$, let $X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be corresponding eigen vector of A, then

$$\begin{bmatrix} -1 & 1 & -2 \\ -1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore x_1 = -t, x_2 = -3t, x_3 = -t$$

$\therefore$ Eigen vector corresponding to $\lambda_2 = 2$ is $X_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

For $\lambda_3 = -1$, let $X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be corresponding eigen vector of A, then

$$\begin{bmatrix} 2 & 1 & -2 \\ -1 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore x_1 = -t, x_2 = 0, x_3 = -t$

$\therefore$ Eigen vector corresponding to $\lambda_3 = -1$ is $X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\therefore Modal\ matrix\ P = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ *Such that ,*

$$P^{-1}AP = \begin{bmatrix} 1/2 & 0 & -1/2 \\ -1/3 & 1/3 & 1/3 \\ -1/6 & -1/3 & 7/6 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} = D$$

$$\therefore \mathrm{P} = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

∴Ch. Eq. of A is $\lambda^3 - 8\lambda^2 + 17\lambda - 10 = 0$

$$\boldsymbol{\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 5.}$$

Matrix equation of A is,

$$(A - \lambda I)X = \begin{bmatrix} 4-\lambda & 2 & -2 \\ -5 & 3-\lambda & 2 \\ -2 & 4 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda_1 = 1$, let $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be corresponding eigen vector of A, then

$$\begin{bmatrix} 3 & 2 & -2 \\ -5 & 2 & 2 \\ -2 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore x_1 = 8t, x_2 = 4t, x_3 = 16t$$

∴ Eigen vector corresponding to $\lambda_1 = 1$ is $X_1 = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$

For $\lambda_2 = 2$, let $X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be corresponding eigen vector of A, then

$$\begin{bmatrix} 2 & 2 & -2 \\ -5 & 1 & 2 \\ -2 & 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore\ x_1 = -9t, x_2 = -9t, x_3 = -18t$$

$\therefore$ Eigen vector corresponding to $\lambda_2 = 2$ is $X_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

For $\lambda_3 = 5$, let $X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be corresponding eigen vector of A, then

$$\begin{bmatrix} -1 & 2 & -2 \\ -5 & -2 & 2 \\ -2 & 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore x_1 = 0, x_2 = 12t, x_3 = 12t$

$\therefore$ Eigen vector corresponding to $\lambda_3 = 5$ is $X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\therefore Modal\ matrix\ P = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$ *Such that* ,

$$P^{-1}AP = \begin{bmatrix} -1 & -1 & 1 \\ 3 & 2 & -2 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} = D$$

$$\therefore \mathrm{P} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

Q. Find the modal matrix P such that $P^{-1}AP$ is a diagonal matrix where

1) $A = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$

2) $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 1 \end{bmatrix}$

3) $A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Quadratic form

A homogenous polynomial of the second degree in any number of variables is called a quadratic form.

A quadratic form

$$Q(X) = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3$$

can be written in matrix form as

$$Q(X) = X'AX = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

where $a_{ij} = a_{ji} (i \neq j)$

The quadratic form Q(X)= $X'AXX'AX$ can be reduce to another quadratic form

$$Q(x) = X'AX = c_1x_1^2 + c_2x_2^2 + \cdots + c_rx_r^2$$

by non-singular transformation $X = PY$, then the reduced quadratic form $Q'(x) = Y'BY$is called **Canonical form** or **sum of the square form**.

In this case, matrix BB will be a diagonal matrix.

The number of positive terms in canonical form is called **index** and is denoted by pp , the rank r of A OR BB is called **rank** of the quadratic form.

The number of negative terms in the canonical form = r-p$\boldsymbol{r-p}$

The difference between positive terms and negative terms called as **signature** of quadratic form , it is denoted by s

$$\therefore \boldsymbol{S = p - (r - p) = 2p - r}$$

To reduce the given quadratic form $Q(x) = X'AX$ to canonical or sum of the square form $Q'(x) = Y'BY$ and to find matrix PP of the linear transformation $X = PY$, consider $A = IA$.

By performing identical row and column transformation on matrix AA on L.H.S. to obtain diagonal matrix B B, while perform only corresponding row transformation on prefactor matrix I I on R.H.S. Thus we get $B = P'A$, then $P = (P')'$.

Q.1) Reduce the following quadratic form to the "sum of the squares form". Find the corresponding linear transformation. Also find the index and signature

$$Q(x) = 2x_1^2 + 9x_2^2 + 6x_3^2 + 8x_1x_2 + 8x_2x_3 + 6x_1x_3$$

In matrix form as,

$$Q(x) = X'AX = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & 4 & 3 \\ 4 & 9 & 4 \\ 3 & 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Consider, $A = IA$

$$\begin{bmatrix} 2 & 4 & 3 \\ 4 & 9 & 4 \\ 3 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$Apply\ \ R_2 - 2R_1\ , R_3 - \frac{3}{2}R_1$ on L.H.S. of matrix A & matrix I

$$\begin{bmatrix} 2 & 4 & 3 \\ 0 & 1 & -2 \\ 0 & -2 & {}^{3}/_{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ {}^{-3}/_{2} & 0 & 1 \end{bmatrix} A$$

Apply similar column operations only on L.H.S. of matrix A

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & {}^{3}/_{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ {}^{-3}/_{2} & 0 & 1 \end{bmatrix} A$$

Apply, $R_3 + 2R_2$ on prefactor of A & to the L.H.S. of matrix A

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & {}^{-5}/_{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ {}^{-11}/_{2} & 2 & 1 \end{bmatrix} A$$

Apply similar column operations on L.H.S. of matrix A

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & {}^{-5}/_{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ {}^{-11}/_{2} & 2 & 1 \end{bmatrix} A$$

$$\therefore B = P'A$$

$$P = (P')' = \begin{bmatrix} 1 & -2 & {}^{-11}/_{2} \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ The canonical form (sum of squares) is,

$$Q' = Y'BY$$

$$\therefore Q' = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & {}^{-5}/_{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\therefore Q' = 2x_1^2 + x_2^2 - \frac{5}{2} x_3^2 \ .$$

$$\therefore The\ rank\ of\ the\ qudratic\ form\ (r) = 3$$

$$\therefore The\ index\ of\ the\ qudratic\ form\ (p) = 2$$

$$\therefore Signature\ (s) = 2p - r = 2(2) - 3$$

$$\therefore s = 1$$

Q.2) Reduce the following quadratic form to the "sum of the squares form". Find the corresponding linear transformation. Also find the index and signature

$$Q(x) = 6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_1x_3$$

In matrix form as,

$$Q(x) = X'AX = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Consider, $A = IA$

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$Apply\ \ R_2 + \frac{1}{3}R_1\ ,R_3 - \frac{1}{2}R_1$ on L.H.S. of matrix A & matrix I

$$\begin{bmatrix} 6 & -2 & 2 \\ 0 & 7/3 & -1/3 \\ 0 & -1/3 & 7/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix} A$$

Apply similar column operations on L.H.S. of matrix A

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 7/3 & -1/3 \\ 0 & -1/3 & 7/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix} A$$

Apply, $R_3 + \frac{1}{7} R_2$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 7/3 & -1/3 \\ 0 & 0 & 16/7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -2/7 & 1/7 & 1 \end{bmatrix} A$$

Apply similar column operations on L.H.S. of matrix A

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 7/3 & 0 \\ 0 & 0 & 16/7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -2/7 & 1/7 & 1 \end{bmatrix} A$$

$$\therefore B = P'A$$

$$P = (P')' = \begin{bmatrix} 1 & 1/3 & -2/7 \\ 0 & 1 & 1/7 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ The canonical form (sum of squares) is,

$$Q' = Y'BY$$

$$\therefore Q' = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 7/3 & 0 \\ 0 & 0 & 16/7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\therefore Q' = 6x_1^2 + \frac{7}{3} x_2^2 + \frac{16}{7} x_3^2 \ .$$

$\therefore$ *The rank of the qudratic form* $(r) = 3$

$\therefore$ *The index of the qudratic form* $(p) = 3$

$\therefore$ *Signature* $(s) = 2p - r = 2(3) - 3$

$$\therefore s = 3$$

Q 3) Reduce the following quadratic form to the "sum of the squares form". Find the corresponding linear transformation. Also find the index and signature

$$\mathrm{Q(x)} = 6x^2 + 3y^2 + 14z^2 + 4yz + 18xz + 4xy$$

Q 4) Reduce the following quadratic form to the "sum of the squares form". Find the corresponding linear transformation. Also find the index and signature

$$Q(x) = 3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3 + 2x_1x_3$$