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Unit -V

MATRICES

➢ The Rank of a Matrix:

The matrix is said to be of rank $'r'$ if :
i) there exist at least one non zero minor of the order r
ii) every minor of order $(r+1)$ is equal to zero.
The rank of the matrix A is denoted by $\rho(A) = r$

If a matrix has a non $-$ zero minor of order r, then $\rho(A) \geq r$.

If a matrix has all minors of order $(r+1)$ as zeroes, then $\rho(A) \leq r$.

If A is $m \times n$ matrix then $\rho(A) \leq \min(m, n)$

Finding the Rank by reducing the matrix to Echelon Form:

Ex. 1) Find the rank of A, where $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-1} & \mathbf{3} \\ \mathbf{3} & \mathbf{4} & \mathbf{0} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{0} & \mathbf{-2} & \mathbf{7} \end{bmatrix}$

$$Consider\ , \qquad A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 0 & -1 \\ -1 & 0 & -2 & 7 \end{bmatrix}$$

$$perform\ \ R_2 - 3R_1\ and\ R_3 + R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -2 & 3 & -10 \\ 0 & 2 & -3 & 10 \end{bmatrix}$$

$$Perform\ \ R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -2 & 3 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = total\ number\ of\ non\ zero\ rows(at\ least\ one\ element\ is\ non\ zero)\ after\ reducing\ the\ matrix\ to\ echelon\ form.$

$$\boldsymbol{\rho(A) = 2}$$

Ex. 2) Find the rank of A, where $\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{-3} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{3} & \mathbf{1} & \mathbf{0} & \mathbf{2} \\ \mathbf{1} & \mathbf{1} & \mathbf{-2} & \mathbf{0} \end{bmatrix}$

Consider, $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ *Perform* $R_2 \leftrightarrow R_1$ $\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

Perform $R_3 - 3R_1$ *and* $R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

Perform $R_3 - R_2$ *and* $R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \ \boldsymbol{\rho(A) = 2}$$

➢ System of Linear Algebraic Equations:

Consider a system of $'m'$ linear equations in $'n'$ unknowns $x_1, x_2, x_3, x_4, \ldots \ldots x_n$.

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
&. \qquad\qquad . \qquad\qquad\quad . \\
&. \qquad\qquad . \qquad\qquad\quad . \\
&. \qquad\qquad . \qquad\qquad\quad . \\
&a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_n
\end{aligned}
$$

The above system of equations can ne written in compact form by using matrix

$$
\begin{bmatrix} a_{11} & a_{11} & & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ a_{31} & a_{32} & & a_{3n} \\ & . & & \\ & . & & \\ & . & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix} \qquad i.e. \quad AX = B
$$

$$
\text{where, } A = \begin{bmatrix} a_{11} & a_{11} & & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ a_{31} & a_{32} & & a_{3n} \\ & . & & \\ & . & & \\ & . & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}
$$

When the system $AX = B$ has a solution
i.e. set of values of $x_1, x_2 \dots \dots x_n$ satisfy simultaneously all m equations then system is said to be **consistant**, otherwise the system is called **inconsistant**.

Augmented Matrix $[\mathbf{A}, \mathbf{B}]$:

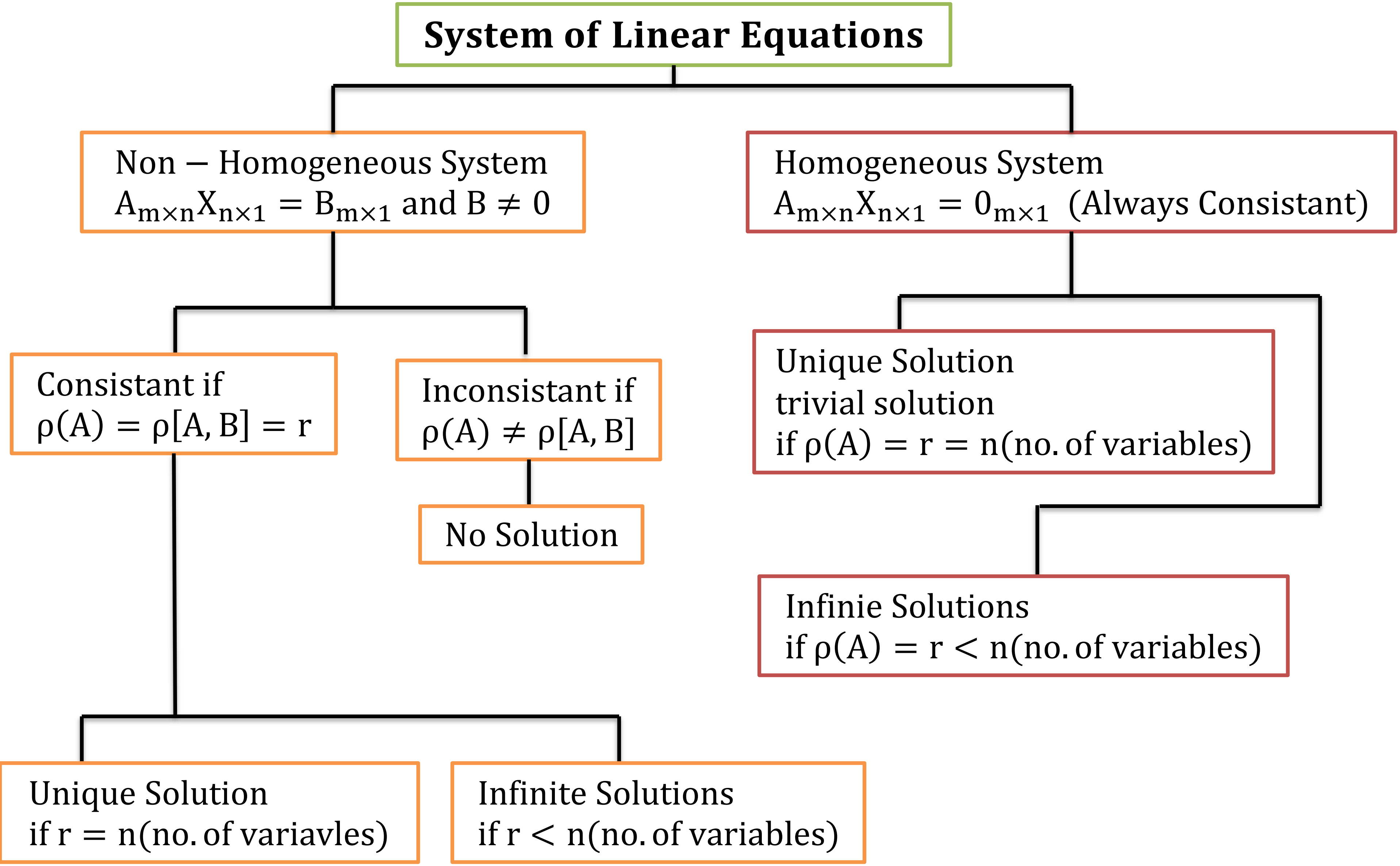
$$[A, B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & \vdots & b_2 \\ & & \ddots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & \vdots & b_m \end{bmatrix}$$

Non − Homogeneous System of Equations:

For the system of equations $AX = B$ if **matrix B is not a null matrix** then the system $AX = B$ is known as non homogeneous system of equations.

Homogeneous System of Equations:

For the system of equations $AX = B$ if **matrix B is a null matrix** then the system $AX = B$ is known as homogeneous system of equations.
i.e. $\mathbf{AX} = \mathbf{0}$ (0 is null matrix).


System of Linear Equations
Non − Homogeneous System
$A_{m\times n}X_{n\times 1} = B_{m\times 1}$ and $B \neq 0$
Homogeneous System
$A_{m\times n}X_{n\times 1} = 0_{m\times 1}$ (Always Consistant)
Consistant if
$\rho(A) = \rho[A, B] = r$
Inconsistant if
$\rho(A) \neq \rho[A, B]$
No Solution
Unique Solution
trivial solution
if $\rho(A) = r = n$(no. of variables)
Infinie Solutions
if $\rho(A) = r < n$(no. of variables)
Unique Solution
if $r = n$(no. of variavles)
Infinite Solutions
if $r < n$(no. of variables)

Ex. 1) Is the following system of equations consistant ? If so solve it.

$$\mathbf{x + y + z = 3}$$
$$\mathbf{x + 2y + 3z = 4}$$
$$\mathbf{x + 4y + 9z = 6}$$

Solution: Given system of equations in matrix form can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} \qquad i.e.\ AX = B$$

Consider augmented matrix

$$[A, B] = \begin{bmatrix} 1 & 1 & 1 & | & 3 \\ 1 & 2 & 3 & | & 4 \\ 1 & 4 & 9 & | & 6 \end{bmatrix}$$

Perform $R_2 - R_1$ and $R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & | & 3 \\ 0 & 1 & 2 & | & 1 \\ 0 & 3 & 8 & | & 3 \end{bmatrix}$$

Perform $R_3 - 3R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & | & 3 \\ 0 & 1 & 2 & | & 1 \\ 0 & 0 & 2 & | & 0 \end{bmatrix}$$

$\rho(A) = 3$ $\qquad$ $\rho[A, B] = 3$

$\therefore\ \rho(A) = \rho[A, B]$ $\qquad$ $\therefore$ given system is consistant.

$\rho(A) = \rho[A, B] = 3 = \text{no. of variables}$

$\therefore$ The given system has unique solution

$by\ R_3, \quad 2z = 0$

$by\ R_2, \quad y + 2z = 1$

$by\ R_1, \quad x + y + z = 3$

$\therefore z = 0, y = 1, x = 2$

Hence the solution is, $\mathbf{x = 2, \quad y = 1, \quad z = 0}$

Ex. 2) Is the following system of equations consistant ? if so solve it.

$$\boldsymbol{x + y + z = 6}$$
$$\boldsymbol{x - y + 2z = 5}$$
$$\boldsymbol{3x + y + z = 8}$$
$$\boldsymbol{2x - 2y + 3z = 7}$$

Solution: Given system of equations in matrix form can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \\ 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 8 \\ 7 \end{bmatrix}$$

Consider augmented matrix,

$$[A,B]=\begin{bmatrix}1 & 1 & 1 & | & 6\\ 1 & -1 & 2 & | & 5\\ 3 & 1 & 1 & | & 8\\ 2 & -2 & 3 & | & 7\end{bmatrix}$$

Perform, $R_2 - R_1, R_3 - 3R_1$ and $R_4 - 2R_1$

$$\sim\begin{bmatrix}1 & 1 & 1 & | & 6\\ 0 & -2 & 1 & | & -1\\ 0 & -2 & -2 & | & -10\\ 0 & -4 & 1 & | & -5\end{bmatrix}$$

Perform $R_3 - R_2$ and $R_4 - 2R_2$

$$\sim\begin{bmatrix}1 & 1 & 1 & | & 6\\ 0 & -2 & 1 & | & -1\\ 0 & 0 & -3 & | & -9\\ 0 & 0 & -1 & | & -3\end{bmatrix}$$

Perform $R_4 - \frac{1}{3}R_3$

$$\sim\begin{bmatrix}1 & 1 & 1 & | & 6\\ 0 & -2 & 1 & | & -1\\ 0 & 0 & -3 & | & -9\\ 0 & 0 & 0 & | & 0\end{bmatrix}$$

$\rho(A) = 3$ $\qquad$ $\rho[A,B] = 3$

$\rho(A) = \rho[A,B] = 3 =$ no. of variables

$\therefore$ given system is consistant and has unique solution

$by\ R_3,\qquad -3z = -9 \qquad\qquad \therefore z = 3$

$by\ R_2,\qquad -2y + z = -1 \qquad\qquad \therefore -2y + 3 = -1, \qquad y = 2$

$by\ R_1,\qquad x + y + z = 6 \qquad\qquad \therefore x + 2 + 3 = 6, \qquad x = 1$

Hence the solution is $\mathbf{x = 1}$, $\mathbf{y = 2}$, $\mathbf{z = 3}$

Ex. 3) By considering the ranks of relevant matrices, examine for consistency the system of equations:

$$\begin{aligned} \mathbf{2x - y - z} &= \mathbf{2} \\ \mathbf{x + 2y + z} &= \mathbf{2} \\ \mathbf{4x - 7y - 5z} &= \mathbf{2} \end{aligned}$$

And solve them if consistent.

Solution: Given system of equations in matrix form can be written as

$$\begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ 4 & -7 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \qquad i.e.\ AX = B$$

Consider augmented matrix,

$$[A, B] = \begin{bmatrix} 2 & -1 & -1 & | & 2 \\ 1 & 2 & 1 & | & 2 \\ 4 & -7 & -5 & | & 2 \end{bmatrix}$$

perform $R_2 \leftrightarrow R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & | & 2 \\ 2 & -1 & -1 & | & 2 \\ 4 & -7 & -5 & | & 2 \end{bmatrix}$$

perform $R_2 - 2R_1$ and $R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & | & 2 \\ 0 & -5 & -3 & | & -2 \\ 0 & -15 & -9 & | & -6 \end{bmatrix}$$

Perform $R_3 - 3R_2$

$$\sim \begin{bmatrix} 1 & 2 & 1 & | & 2 \\ 0 & -5 & -3 & | & -2 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$\rho(A) = 2$ $\qquad$ $\rho[A, B] = 2$

$\therefore\ \rho(A) = \rho[A, B]$ $\qquad$ $\therefore$ given system is consistent

$\rho(A) = \rho[A, B] = 2 <$ no. of variables.

$\therefore$ system has infinite number of solutions.

$by\ R_2, \qquad -5y - 3z = -2 \qquad\qquad y = \dfrac{2-3z}{5}$

$by\ R_1, \qquad x + 2y + z = 2$

$x = 2 - 2y - z, \qquad x = 2 - \dfrac{2-3z}{5} - z \qquad\qquad x = \dfrac{6+z}{5}$

$Let\ z = t$

Hence the solution set is $\mathbf{x = \dfrac{6+t}{5}}, \qquad \mathbf{y = \dfrac{2-3t}{5}}, \qquad \mathbf{z = t}$

Ex. 4) Examine for consistancy and if consistant then solve it

$$\begin{aligned}\mathbf{2x - 3y + 5z = 1}\\ \mathbf{3x + y - z = 2}\\ \mathbf{x + 4y - 6z = 1}\end{aligned}$$

Solution: Given system of equations in matrix form can be written as

$$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 1 & -1 \\ 1 & 4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Consider augmented matrix,

$$[A,B] = \begin{bmatrix} 2 & -3 & 5 & | & 1 \\ 3 & 1 & -1 & | & 2 \\ 1 & 4 & -6 & | & 1 \end{bmatrix}$$

Perform $R_3 \leftrightarrow R_1$

$$\sim \begin{bmatrix} 1 & 4 & -6 & | & 1 \\ 3 & 1 & -1 & | & 2 \\ 2 & -3 & 5 & | & 1 \end{bmatrix}$$

$Perform, R_2 - 3R_1 and\ R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 4 & -6 & | & 1 \\ 0 & -11 & 17 & | & -1 \\ 0 & -11 & 17 & | & -1 \end{bmatrix}$$

Perform $R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 4 & -6 & | & 1 \\ 0 & -11 & 17 & | & -1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$\rho(A) = 2$ $\qquad$ $\rho[A,B] = 2$

$\therefore\ \rho(A) = \rho[A,B]$

$\therefore \rho(A) = \rho[A,B] = 2 <$ no. of variables

$\therefore$ system is consistant and system has infinte number of solutions.

$by\ R_2, \qquad -11y + 17z = -1$ $\qquad \therefore y = \dfrac{1 + 17z}{11}$

$by\ \ R_1, \qquad x + 4y - 6z = 1$

$$x = 1 - 4\left(\frac{1 + 17z}{11}\right) + 6z \qquad \therefore x = \frac{7 - 2z}{11}$$

$Let, \qquad z = t$

Hence the solution set is, $\mathbf{x = \dfrac{7 - 2t}{11}}, \qquad \mathbf{y = \dfrac{1 + 17t}{11}}, \qquad \mathbf{z = t}$

Ex. 5) Examine for consistancy and if consistant then solve it

$$\mathbf{x + 2y + 3z = 0}$$
$$\mathbf{4x + 5y + 6z = 2}$$
$$\mathbf{7x + 8y + 9z = 3}$$

$Solution$:

$System\ is\ inconsistant.$

Ex. 5) Investigate for what values of λ and μ , the system of simultaneous equations:

$$\mathbf{2x - y + 3z = 2}$$

Solution: Given system of equations in matrix form can be written as

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 2 \\ 5 & -1 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ \mu \end{bmatrix}$$

Consider augmented matrix

$$[A, B] = \begin{bmatrix} 2 & -1 & 3 & | & 2 \\ 1 & 1 & 2 & | & 2 \\ 5 & -1 & \lambda & | & \mu \end{bmatrix}$$

Perform $R_2 \leftrightarrow R_1$

$$\sim \begin{bmatrix} 1 & 1 & 2 & | & 2 \\ 2 & -1 & 3 & | & 2 \\ 5 & -1 & \lambda & | & \mu \end{bmatrix}$$

Perform $R_2 - 2R_1$ and $R_3 - 5R_1$

$$\sim \begin{bmatrix} 1 & 1 & 2 & | & 2 \\ 0 & -3 & -1 & | & -2 \\ 0 & -6 & \lambda - 10 & | & \mu - 10 \end{bmatrix}$$

perform $R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 1 & 2 & | & 2 \\ 0 & -3 & -1 & | & -2 \\ 0 & 0 & \lambda - 8 & | & \mu - 6 \end{bmatrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & 0 & \lambda-8 & \mu-6 \end{array}\right]$$

(i) if $\rho(A) \neq \rho[A, B]$ then system has no solution

$\therefore$ *Let,* $\lambda = 8$, *then* $\rho(A) = 2$

Let , $\mu \neq 6$, *then* $\rho[A, B] = 3$

(ii) If $\rho(A) = \rho[A, B] =$ no. of variables, then system has unique solution

$\therefore$ *let,* $\lambda \neq 8$ *and* μ *may have any value* then
$\rho(A) = \rho[A, B] = 3$

(iii) if $\rho(A) = \rho[A, B] <$ no of variables, then system has infinte no. of solutions.

$\therefore$ Let, $\lambda = 8$ and $\mu = 6$ then
$\rho(A) = \rho[A, B] = 2 <$ no. of variables

Ex. 6) Investigate for what value of k the equations

$$\mathbf{x + y + z = 1}$$
$$\mathbf{2x + y + 4z = k}$$
$$\mathbf{4x + y + 10z = k^2}$$

have infinite number of solutions ? Hene find solutions.

Solution: Given system of equations in matrix form can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 4 & 1 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ k \\ k^2 \end{bmatrix}$$

Consider augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & k \\ 4 & 1 & 10 & k^2 \end{array}\right]$$

perform $\mathrm{R}_2 - 2\mathrm{R}_1$ and $\mathrm{R}_3 - 4\mathrm{R}_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & k-2 \\ 0 & -3 & 6 & k^2-4 \end{array}\right]$$

perform $\mathrm{R}_3 - 3R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & k-2 \\ 0 & 0 & 0 & k^2-3k+2 \end{array}\right]$$

$$\rho(A) = 2$$

if $\rho(A) = \rho[A, B] <$ number of variables, then system has infinite number of solutions

$\therefore$ consider $k^2 - 3k + 2 = 0$ then $\rho(A) = \rho[A, B] = 2 <$ number of variables

$\therefore$ *we get,* $k = 1 \text{ or } k = 2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & -1 & 2 & | & k-2 \\ 0 & 0 & 0 & | & k^2-3k+2 \end{bmatrix}$$

$For, \quad k = 1$

$$\begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & -1 & 2 & | & -1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

we get, $\quad -y + 2z = -1 \qquad \therefore \; y = 1 + 2z$

$x + y + z = 1 \qquad \therefore \; x = 1 - (1 + 2z) - z \qquad \therefore \; x = -3z$

$Let, \quad z = t$

$\therefore$ **Solution set is** $\mathbf{x = -3t}, \quad \mathbf{y = 1 + 2t}, \quad \mathbf{z = t}$

For, $\quad$ k = 2

$$\begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & -1 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

we get,

$-y + 2z = 0 \qquad \therefore y = 2Z$

$x + y + z = 1$

$\therefore \; x = 1 - (2z) - z \qquad \therefore \; x = 1 - 3z$

$Let \;\; z = t$

$\therefore$ **Solution set is** $\mathbf{x = 1 - 3t}, \quad \mathbf{y = 2t}, \quad \mathbf{z = t}$

Ex. 7) Examine for non − trivial solution, the following set of equations and solve

$$\mathbf{x - 2y + z + 3w = 0}$$
$$\mathbf{3x + y + z + w = 0}$$
$$\mathbf{2x - 4y + 2z + 6w = 0}$$
$$\mathbf{4x - y + 2z + 4w = 0}$$

Solution: Given system of equations in matrix form can be written as

$$\begin{bmatrix} 1 & -2 & 1 & 3 \\ 3 & 1 & 1 & 1 \\ 2 & -4 & 2 & 6 \\ 4 & -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Consider coefficient matrix

$$A = \begin{bmatrix} 1 & -2 & 1 & 3 \\ 3 & 1 & 1 & 1 \\ 2 & -4 & 2 & 6 \\ 4 & -1 & 2 & 4 \end{bmatrix}$$

perform $R_2 - 3R_1, R_3 - 2R_1$ and $R_4 - 4R_1$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 7 & -2 & -8 \\ 0 & 0 & 0 & 0 \\ 0 & 7 & -2 & -8 \end{bmatrix}$$

perform $R_4 - R_2$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 7 & -2 & -8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here, $\rho(A) = 2 <$ number of variables

$\therefore$ System has infinite number of solutions by R_2, $7y - 2z - 8w = 0$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 7 & -2 & -8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$by\ R_1$, $x - 2y + z + 3w = 0$

$$\therefore y = \frac{2z + 8w}{7}$$

$$\therefore\ x = 2y - z - 3w$$

$$\therefore x = 2\left(\frac{2z + 8w}{7}\right) - z - 3w$$

$$\therefore x = \frac{-3z - 5w}{7}$$

Let $z = t$ and $w = s$

$\therefore$ **Solutiion set is** $\mathbf{x = \frac{-3t - 5s}{7}}$, $\mathbf{y = \frac{2t + 8s}{7}}$, $\mathbf{z = t}$, $\mathbf{w = s}$

Ex. 8) For different value of k, discuss the nature of solution of the following eqs

$$\mathbf{x + 2y - z = 0}$$
$$\mathbf{3x + (k + 7)y - 3z = 0}$$
$$\mathbf{2x + 4y + (k - 3)z = 0.}$$

Solution: Given system of equations in matrix form can be written as

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & k+7 & -3 \\ 2 & 4 & k-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Consider coefficient matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & k+7 & -3 \\ 2 & 4 & k-3 \end{bmatrix}$$

perform $\mathrm{R_2 - 3R_1}$ and $\mathrm{R_3 - 2R_1}$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & k+1 & 0 \\ 0 & 0 & k-1 \end{bmatrix}$$

If $\mathrm{k \neq \pm 1}$, then $\mathrm{\rho(A) = 3 =}$ number of variables

$\therefore$ system has unique solution which is trivial solution if $\mathrm{k \neq \pm 1}$
i.e. $\mathrm{x = 0, y = 0}$ and $\mathrm{z = 0}$

If $\mathrm{k = \pm 1}$, then $\mathrm{\rho(A) = 2 <}$ number of variables

$\therefore$ System has infinite number solutions if $\mathrm{k = \pm 1}$

➢ LINEAR DEPENDENT AND INDEPEDENT VECTORS:

Vector:

We define n dimentional vector as an ordered set of n elements $x_1, x_2, x_3, \dots, x_n$ and is denoted by

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$, $X = [x_1 \quad x_2 \quad x_3 \quad \cdot \quad \cdot \quad , \quad x_n]$. The element $x_1, x_2, x_3, \dots x_n$ are called componant of x.

Linear Dependent Vectors:

Let $X_1, X_2, X_3, \dots, X_n$ be n vectors of the same order
If ther exist n scalars $c_1, c_2, c_3, \dots, c_n$, **not all zero**
such that $c_1X_1 + c_2X_2 + c_3X_3 + \cdots + c_nX_n = 0$
then these n vectors are called as linearly dependent.

Linear Independent Vectors:

Let $X_1, X_2, X_3, \dots, X_n$ be n vectors of the same order
These n vectors are said to be linearly independent if
$c_1X_1 + c_2X_2 + c_3X_3 + \cdots + c_nX_n = 0$, then
$\mathbf{c_1 = c_2 = c_3 = \cdots = c_n = 0}$

Ex. 1) Examine for linear dependence or independence of vectors $(2,-1,3,2),(1,3,4,2)$ and $(3,-5,2,2)$. Find the reletion between them if dependent.

Solution: Let, givern vectors be $X_1=(2,-1,3,2), X_2=(1,3,4,2)$ and $X_3=(3,-5,2,2)$

Consider linear combination $\quad c_1X_1+c_2X_2+c_3X_3=0$

$$c_1(2,-1,3,2)+c_2(1,3,4,2)+c_3(3,-5,2,2)=(0,0,0,0)$$

$$i.e, \quad 2c_1+c_2+3c_3=0$$

$$-c_1+3c_2-5c_3=0$$

$$3c_1+4c_2+2c_3=0$$

$$2c_1+2c_2+2c_3=0$$

In matrix form, $\begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & -5 \\ 3 & 4 & 2 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$Let, \qquad A=\begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & -5 \\ 3 & 4 & 2 \\ 2 & 2 & 2 \end{bmatrix} \qquad \underset{perform\ R_{12}}{\sim} \begin{bmatrix} -1 & 3 & -5 \\ 2 & 1 & 3 \\ 3 & 4 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 3 & -5 \\ 2 & 1 & 3 \\ 3 & 4 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

Perform, $R_2 + 2R_1, R_3 + 3R_1, R_4 + 2R_1$

$$\sim \begin{bmatrix} -1 & 3 & -5 \\ 0 & 7 & -7 \\ 0 & 13 & -13 \\ 0 & 8 & -8 \end{bmatrix}$$

perform, $R_3 - \frac{13}{7}R_2$ and $R_4 - \frac{8}{7}R_2$

$$\sim \begin{bmatrix} -1 & 3 & -5 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2 <$ no of variables

$\therefore$ system has infinte number of solutions

by R_2, $\quad 7c_2 - 7c_3 = 0 \qquad c_2 = c_3$

by R_1, $\quad -c_1 + 3c_2 - 5c_3 = 0 \qquad c_1 = 3c_2 - 5c_3 \qquad \therefore\ c_1 = -2c_3$

Let, $\quad c_3 = t$

$\therefore\ c_1 = -2t, \quad c_2 = t, \quad c_3 = t$

we can find c_1, c_2 and c_3 are not all zero

$\therefore$ **the given vectors $\mathbf{X_1}, \mathbf{X_2}$ and $\mathbf{X_3}$ are linearly dependents**.

Relation: $c_1X_1 + c_2X_2 + c_3X_3 = 0$

$$-2tX_1 + tX_2 + tX_3 = 0 \qquad Let, \qquad t = 1$$

$we\ get, \qquad X_2 + X_3 = 2X_1$

Ex. 2)Examine for linear dependence or independence of vectors $\boldsymbol{X_1 = (2, 2, 7, -1), X_2 = (3, -1, 2, 4), X_3 = (1, 1, 3, 1)}$. Find relation if dependent.

$Solution$: Consider linear combination

$$c_1X_1 + c_2X_2 + c_3X_3 = 0$$

$$c_1(2, 2, 7, -1) + c_2(3, -1, 2, 4) + c_3(1, 1, 3, 1) = (0, 0, 0, 0)$$

$$2c_1 + 3c_2 + c_3 = 0$$
$$2c_1 - c_2 + c_3 = 0$$
$$7c_1 + 2c_2 + 3c_3 = 0$$
$$-c_1 + 4c_2 + c_3 = 0$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 2 & -1 & 1 \\ 7 & 2 & 3 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad Let, \qquad A = \begin{bmatrix} 2 & 3 & 1 \\ 2 & -1 & 1 \\ 7 & 2 & 3 \\ -1 & 4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 2 & -1 & 1 \\ 7 & 2 & 3 \\ -1 & 4 & 1 \end{bmatrix}$$

perform, R_{14}

$$\sim \begin{bmatrix} -1 & 4 & 1 \\ 2 & -1 & 1 \\ 7 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

perform $R_2 + 2R_1, R_3 + 7R_1$ and $R_4 + 2R_1$

$$\sim \begin{bmatrix} -1 & 4 & 1 \\ 0 & 7 & 3 \\ 0 & 30 & 10 \\ 0 & 11 & 3 \end{bmatrix}$$

perform, $R_3 - \frac{30}{7}R_2, R_4 - \frac{11}{7}R_2$

$$\sim \begin{bmatrix} -1 & 4 & 1 \\ 0 & 7 & 3 \\ 0 & 0 & -\frac{20}{7} \\ 0 & 0 & -\frac{12}{7} \end{bmatrix}$$

perform $R_4 - \frac{12}{20}R_3$

$$\sim \begin{bmatrix} -1 & 4 & 1 \\ 0 & 7 & 3 \\ 0 & 0 & -\frac{20}{7} \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 3 =$ number of variables.

$\therefore$ system has unique solution

$\therefore$ we get, trivial solution

i.e. $c_1 = 0, \quad c_2 = 0, \quad c_3 = 0$

$\therefore$ **vectors X_1, X_2 and X_3 are linearly independent**.

Ex. 3) Examine for linear dependence or independenace of vectors $X_1 = (1, -1, 1), X_2 = (2, 1, 1), X_3 = (3, 0, 2)$. Find relation if dependent.

$Ex.$ 4)**Examine for linear dependence or independenace of vectors $X_1 = (1, 2, 3), X_2 = (2, -2, 6)$. *Find relation if dependent*.**

$Ex.$ 5)**Examine for linear dependence or independenace of vectors $X_1 = (3, 1, -4)^T, X_2 = (2, 2, -3)^T, X_3 = (0, -4, 1)^T$ *Find relation if dependent*.**

➢ Linear transformation:

The relation $\mathbf{Y} = \mathbf{AX}$ where A is square matrix expresses $y_1, y_2, y_3, \dots y_n$ in terms of $x_1, x_2, x_3, \dots x_n$ is called linear transformation.

$'A'$ is called matrix of transformation. $|A|$ is known as modulus of transformation.

If $|A| = 0$, then linear tansformation $Y = AX$ is called singular transformation.
If $|A| \neq 0$, then linear transformation $Y = AX$ is called non $-$ singular transformation.

For a non $-$ singular transformation $Y = AX$, A^{-1} exist and we can write the **inverse transformation**,
$\mathbf{X} = \mathbf{A^{-1}Y}$

If a transformation from $(x_1, x_2, \dots x_n)$ to $(y_1, y_2, \dots, y_n)$ is given by $Y = AX$ and another transformation from $(y_1, y_2, \dots, y_n)$ to $(z_1, z_2, \dots, z_n)$ is given by $Z = BY$, then transformation from $(x_1, x_2, \dots x_n)$ to $(z_1, z_2, \dots, z_n)$ is given by
$Z = BY = B(AX) = (BA)X$
i.e. $\mathbf{Z} = (\mathbf{BA})\mathbf{X}$.

Ex. 1) Given the transformation $\mathbf{Y} = \begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{2} \\ \mathbf{1} & \mathbf{0} & \mathbf{-2} \end{bmatrix} \begin{bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \\ \mathbf{x_3} \end{bmatrix}$. **Find the co – ordinates** $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})$ **in X corresponding to** $(\mathbf{1}, \mathbf{2}, \mathbf{-1})$ **in Y.**

Solution: $\quad Y = AX, \quad i.e.\ AX = Y$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

This is non – homogeneous system of equations

perform R_{12}

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 1 \\ 1 & 0 & -2 & -1 \end{bmatrix}$$

perform $R_2 - 2R_1$ and $R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -1 & -3 & -3 \\ 0 & -1 & -4 & -3 \end{bmatrix}$$

perform $R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -1 & -3 & -3 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$\rho(A) = 3 = \rho[A, B] =$ number of variables

System has unique solution.

by $R_3, \quad -x_3 = 0$

by $R_2, \quad -x_2 - 3x_3 = -3$

by $R_1, \quad x_1 + x_2 + 2x_3 = 2$

$x_3 = 0$

$x_2 = 3$

$x_1 = -1$

Hence $(-1, 3, 0)$ corresponds to $(1, 2, -1)$ iin Y.

Orthogonal Matrix:

A square matrix A is said to be orthogonal if $AA' = A'A = I$

If A is orthogonal then $A^{-1} = A'$

Ex. 1) Show that $\mathbf{A} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$ **is orthogonal matrix.**

Solution:

$$Consider\ , \qquad AA' = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} cos^2\theta & 0 & -\sin\theta\cos\theta + \cos\theta\sin\theta \\ 0 & 1 & 0 \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & 0 & sin^2\theta + cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$AA' = I, \qquad \therefore$ A is orthogonal.

Ex. 2) Determine the values of a, b, c when $\begin{bmatrix} \mathbf{0} & \mathbf{2b} & \mathbf{c} \\ \mathbf{a} & \mathbf{b} & \mathbf{-c} \\ \mathbf{a} & \mathbf{-b} & \mathbf{c} \end{bmatrix}$ **is othogonal.**

Solution: If A is othogonal then it requires $\mathrm{AA}' = \mathrm{I}$

$$AA' = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix} \begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix}$$

$$= \begin{bmatrix} 4b^2 + c^2 & 2b^2 - c^2 & -2b^2 + c^2 \\ 2b^2 - c^2 & a^2 + b^2 + c^2 & a^2 - b^2 - c^2 \\ -2b^2 + c^2 & a^2 - b^2 - c^2 & a^2 + b^2 + c^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we get, $\quad 4b^2 + c^2 = 1, \quad 2b^2 - c^2 = 0, \quad a^2 + b^2 + c^2 = 1, \quad a^2 - b^2 - c^2 = 0$

we get, $\quad a = \pm\frac{1}{\sqrt{2}}, \quad b = \pm\frac{1}{\sqrt{6}}, \quad c = \pm\frac{1}{\sqrt{3}}$

Ex. 3) Verify whether the matrix A is orthogonal or not. If so find $\mathbf{A^{-1}}$.

$$\mathbf{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Solution:

Consider,

$$AA' = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}+0+\frac{1}{2} & 0+0+0 & \frac{1}{2}+0-\frac{1}{2} \\ 0+0+0 & 0+1+0 & 0+0+0 \\ \frac{1}{2}+0-\frac{1}{2} & 0+0+0 & \frac{1}{2}+0+\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

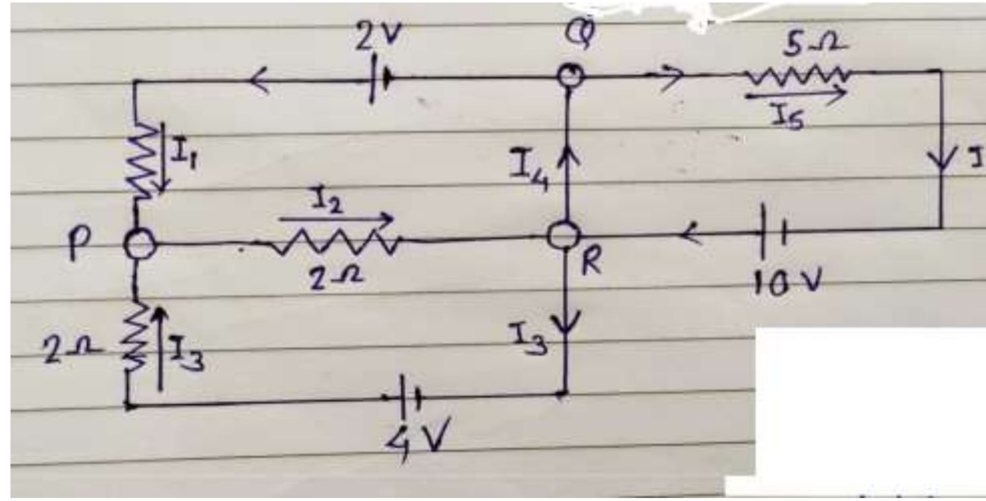
$\therefore AA' = I$ $\quad\quad\quad\quad$ $\therefore$ A is Orthogonal Matrix

As matrix A is orthogonal, $A^{-1} = A'$

$$\therefore \quad A^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

➢ Applications of System of Linear Equations:

Ex. 1) Find the currents in the circuit *shown in figure*.



Solution:

Applying KCL at the nodes P, Q, and R

$I_1 + I_3 = I_2$

$I_4 = I_1 + I_5$

$I_2 + I_5 = I_3 + I_4$

Applying KVL to three loops $\quad I_1 + 2I_2 = 2 \quad 2I_2 + 2I_3 = 4 \quad 5I_5 = 10$

from last equation $\mathrm{I}_5 = 2$

we get,

$I_1 - I_2 + I_3 = 0$
$-I_1 + I_4 = 2$
$-I_2 + I_3 + I_4 = 2$
$I_1 + 2I_2 = 2$
$I_2 + I_3 = 2$

In matrix form,

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

Consider Augmented matrix

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 2 \\ 0 & -1 & 1 & 1 & 2 \\ 1 & 2 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 2 \\ 0 & 0 & -2 & 3 & 6 \\ 0 & 0 & 0 & 4 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$by\ R_4, \qquad I_4 = \frac{5}{2}$

$by\ R_3, \qquad -2I_3 + 3I_4 = 6$

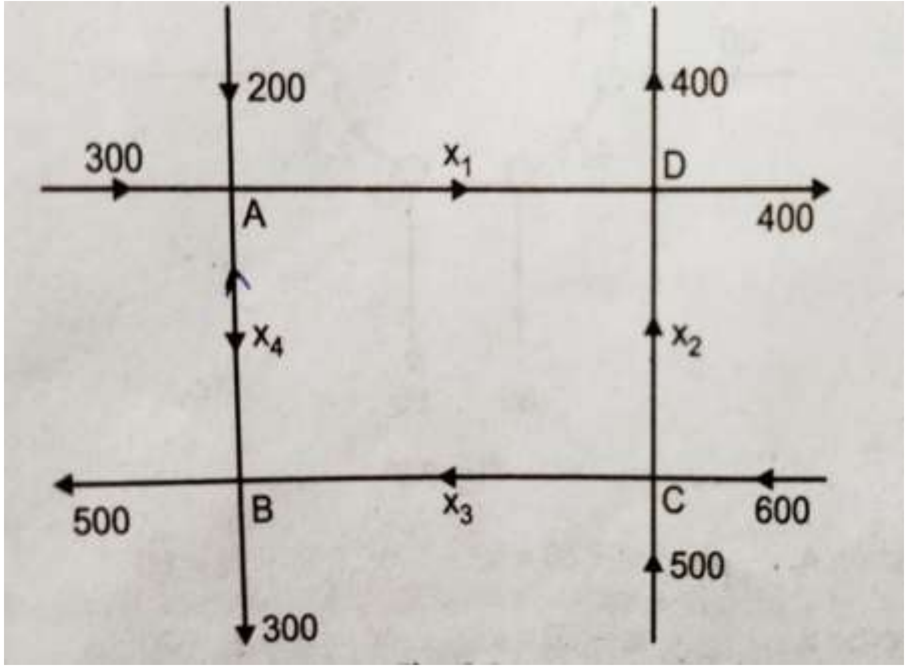
$by\ R_2, \qquad -I_2 + I_3 + I_4 = 2$

$by\ R_1, \qquad I_1 - I_2 + I_3 = 0$

$we\ get, \qquad I_4 = \frac{5}{2}$

$I_3 = \frac{3}{4}, \qquad I_2 = \frac{5}{4}, \qquad I_1 = \frac{1}{2}$

Ex. 2) Solve the following traffic problem



Solution:

At junction A, $x_1 + x_4 = 300 + 200$

At junction B, $x_3 + x_4 = 500 + 300$

At junction C, $x_2 + x_3 = 600 + 500$

At junction D, $x_1 + x_2 = 400 + 400$

The system of equation is,

$$
\begin{aligned}
x_1 + x_4 &= 500\\
x_3 + x_4 &= 800\\
x_2 + x_3 &= 1100\\
x_1 + x_2 &= 800
\end{aligned}
$$

Matrix form is,

$$
\begin{bmatrix} 1 & 0 & 0 & 1\\ 0 & 0 & 1 & 1\\ 0 & 1 & 1 & 0\\ 1 & 1 & 0 & 0 \end{bmatrix}
\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4 \end{bmatrix}
=
\begin{bmatrix} 500\\ 800\\ 1100\\ 800 \end{bmatrix}
\qquad i.e.\ AX = B
$$

$$
[A, B] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 500\\ 0 & 0 & 1 & 1 & 800\\ 0 & 1 & 1 & 0 & 1100\\ 1 & 1 & 0 & 0 & 800 \end{array}\right]
\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 500\\ 0 & 1 & 0 & -1 & 300\\ 0 & 0 & 1 & 1 & 800\\ 0 & 0 & 0 & 0 & 0 \end{array}\right]
$$

Here $\rho(A) = 3 = \rho[A, B] <$ number of variables.

system has infinite number of solutions

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & | & 500 \\ 0 & 1 & 0 & -1 & | & 300 \\ 0 & 0 & 1 & 1 & | & 800 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

by R_3, $\quad x_3 + x_4 = 800$

Let, $\quad x_4 = t$

by R_2, $\quad x_2 - x_4 = 300$

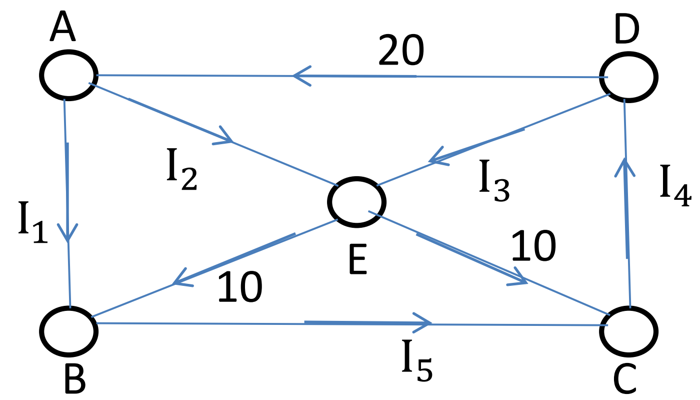
by R_1, $\quad x_1 + x_4 = 500$

$$x_3 = 800 - t$$
$$x_2 = 300 + t$$
$$x_1 = 500 - t$$

Take $t = 100$

we get, $\quad \mathbf{x_1 = 400}, \quad \mathbf{x_2 = 400}, \quad \mathbf{x_3 = 700}, \quad \mathbf{x_4 = 100}$

Ex. 3) Find the currents in the various branches of the following network, Shown in figure.



Solution:

At node A, $20 = I_1 + I_2$

At node B, $10 + I_1 = I_5$

At node C, $10 + I_5 = I_4$

At node D, $I_4 = I_3 + 20$

At node E, $I_2 + I_3 = 10 + 10$

i. e.

$I_1 + I_2 = 20$

$I_1 - I_5 = -10$

$I_4 - I_5 = 10$

$I_3 - I_4 = -20$

$I_2 + I_3 = 20$

In Matrix Form,

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix} = \begin{bmatrix} 20 \\ -10 \\ 10 \\ -20 \\ 20 \end{bmatrix}$$