Engineering Mathematics-II

Unit - 3

Integral Calculus

- Calculus:

The word Calculus comes from Latin meaning "small stone", because it is like understanding something by looking at small pieces. It is the branch of Mathematics that deals with finding the properties of functions, by using methods based on the summation of infinitesimal differences

There are mainly two branches of calculus:

Differential Calculus

- It is a Branch of calculus concerned with the theory and applications of derivatives of functions
- Differential Calculus cuts something into small pieces to find how it changes

Integral Calculus

- It is a Branch of calculus concerned with the theory and applications of integrals of functions
- Integral Calculus joins (integrates) the small pieces together to find how much there is.

❖ Topics to be covered

- Reduction Formulae:These formulae enable us to reduce the degree of the integrand and calculate the integrals in a finite number of steps.
- Beta and Gamma Functions:Gamma is a single variable function, whereas Beta is a two-variable function. The relation between beta and gamma function will help to solve many problems in physics and mathematics.
- Differentiation under integral sign:Differentiation under the integral sign is an operation in calculus used to evaluate certain integrals. Under fairly loose conditions on the function being integrated,differentiation under the integral sign allows one to interchange the order of integration and differentiation
- Error functions:Error functions are special kind of integrals those occur often in probability, statistics, and partial differential equations.This is useful, for example, in determining the bit **error** rate of a digital communication system.

❖ Reduction formulae: Many functions are having integrals which are not directly obtainable. To find such integrals reduction formulae are used.

- Some standard reduction formulae: (Here, n and m are positive integers. n,m≥2)

1. $\int_0^{\frac{\pi}{2}} sin^n x dx = \frac{n-1}{n} . \frac{n-3}{n-2} . \frac{n-5}{n-4} \ldots \ldots \ldots . \frac{3}{4} . \frac{1}{2} . \frac{\pi}{2}$, if n is even

$= \frac{n-1}{n} . \frac{n-3}{n-2} . \frac{n-5}{n-4} \ldots \ldots \ldots . \frac{4}{5} . \frac{2}{3} . 1$, if n is odd

2. $\int_0^{\frac{\pi}{2}} cos^n x dx = \frac{n-1}{n} . \frac{n-3}{n-2} . \frac{n-5}{n-4} \ldots \ldots \ldots . \frac{3}{4} . \frac{1}{2} . \frac{\pi}{2}$, if n is even

$= \frac{n-1}{n} . \frac{n-3}{n-2} . \frac{n-5}{n-4} \ldots \ldots \ldots . \frac{4}{5} . \frac{2}{3} . 1$, if n is odd

3. $\int_0^{\frac{\pi}{2}} sin^m x cos^n x dx = \frac{\{(m-1)(m-3)........2\ or\ 1\}\{(n-1)(n-3)........2\ or\ 1\}}{(m+n)(m+n-2)(m+n-4)........2\ or\ 1} \times P$

$P = \frac{\pi}{2}$, if m and n are both even

$P = 1$, otherwise

4. $\int_0^{\pi} sin^n x dx = 2\int_0^{\frac{\pi}{2}} sin^n x dx$ for all positive integers n

5. $\int_0^{\pi} cos^n x dx = 2\int_0^{\frac{\pi}{2}} cos^n x dx$ if n is an even integer

$= \quad 0$ if n is an odd integer

6. $\int_0^{\pi} sin^m x cos^n x dx = 2\int_0^{\frac{\pi}{2}} sin^m x cos^n x dx$ if n is even , m is odd or even

$= \quad 0$ if n is an odd integer

7. $\int_0^{2\pi} \sin^n x dx = 4 \int_0^{\frac{}{2}} \sin^n x dx$, if n is an even integer

$= 0$, if n is odd

8. $\int_0^{2\pi} \cos^n x dx = 4 \int_0^{\frac{}{2}} \cos^n x dx$, if n is an even integer

$= 0$, if n is odd

9. $\int_0^{2\pi} \sin^m x \cos^n x dx = 4 \int_0^{\frac{}{2}} \sin^m x \cos^n x dx$, if m and n are both even

$= \quad 0$, otherwise

10. If $I_n = \int \sin^n x dx$ OR $I_n = \int \cos^n x dx$ where $n \geq 2$ is a positive integer.

then $I_n = \frac{n-1}{n} . I_{n-2}$

❖ Examples on reduction formulae:

1. Find $\int_0^{\frac{\pi}{2}} sin^6 x dx$

Sol.

Using formula $\int_0^{\frac{\pi}{2}} sin^n x dx = \frac{n-1}{n} . \frac{n-3}{n-2} . \frac{n-5}{n-4} \ldots\ldots\ldots . \frac{3}{4} . \frac{1}{2} . \frac{\pi}{2}$, if n is even

Here, n=6 even integer

$$\therefore \int_0^{\frac{\pi}{2}} sin^6 x dx = \frac{6-1}{6} . \frac{6-3}{6-2} . \frac{6-5}{6-4} . \frac{\pi}{2} = \frac{5}{6} . \frac{3}{4} . \frac{1}{2} . \frac{\pi}{2} = \frac{15}{96}\pi = \frac{5}{32}\pi$$

$$\boxed{\int_0^{\frac{\pi}{2}} sin^6 x dx = \frac{5}{32}\pi}$$

2. Find $\int_0^{\pi} cos^8 x dx$

Sol.

Using formula $\int_0^{\pi} cos^n x dx = 2\int_0^{\frac{\pi}{2}} cos^n x dx$ if n is an even integer

$$\int_0^{\pi} cos^8 x dx = 2\int_0^{\frac{\pi}{2}} cos^8 x dx$$ Here n=8 , even integer

Now using formula, $\int_0^{\frac{\pi}{2}} cos^n xdx = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}\ldots\ldots\ldots.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$, if n is even

$$\int_0^{\pi} cos^8 xdx = 2\int_0^{\frac{\pi}{2}} cos^8 xdx = 2\frac{7}{8}.\frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \frac{35}{128}\pi$$

$$\boxed{\int_0^{\pi} cos^8 xdx = \frac{35}{128}\pi}$$

3. Find $\int_0^{2\pi} sin^6 xcos^4 xdx$

Sol.

Using formula, $\int_0^{2\pi} sin^m xcos^n xdx = 4\int_0^{\frac{\pi}{2}} sin^m xcos^n xdx$, if m and n are both even

Here, m=6 and n=4 $\therefore$ $\int_0^{2\pi} sin^6 xcos^4 xdx = 4\int_0^{\frac{\pi}{2}} sin^6 xcos^4 xdx$

Now using formula, $\int_0^{\frac{\pi}{2}} sin^m xcos^n xdx = \frac{\{(m-1)(m-3)\ldots\ldots..2\ or\ 1\}\{(n-1)(n-3)\ldots\ldots.2\ or\ 1\}}{(m+n)(m+n-2)(m+n-4)\ldots\ldots..2\ or\ 1} \times \frac{\pi}{2}$

$$\int_0^{2\pi} sin^6 xcos^4 xdx = 4\frac{(6-1)(6-3)(6-5)(4-1)(4-3)}{(6+4)(6+4-2)(6+4-4)(6+4-6)(6+4-8)} \times \frac{\pi}{2}$$

$$\int_0^{2\pi} \sin^6 x \cos^4 x\,dx = 4\,\frac{5.3.1.3.1}{10.8.6.4.2} \times \frac{\pi}{2} = \frac{3}{128}\pi$$

$$\boxed{\int_0^{2\pi} \sin^6 x \cos^4 x\,dx = \frac{3}{128}\pi}$$

4. Find $\int_0^{\frac{\pi}{4}} \cos^2(2x)\,dx$

Sol.

Substitute $2x = t$ $\quad \therefore 2dx = dt \quad \therefore dx = \frac{dt}{2}$

x	0	$\frac{\pi}{4}$
t	0	$\frac{\pi}{2}$

$$\int_0^{\frac{\pi}{4}} \cos^2(2x)\,dx = \int_0^{\frac{\pi}{2}} \cos^2(t)\,\frac{dt}{2} = \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^2(t)\,dt = \frac{1}{2}\,\frac{(2-1)}{2}\,\frac{\pi}{2} = \frac{\pi}{8}$$

$$\boxed{\int_0^{\frac{\pi}{4}} \cos^2(2x)\,dx = \frac{\pi}{8}}$$

5. If $I_n = \int_0^{\frac{\pi}{4}} tan^n x dx$ and $I_n = \frac{1}{n-1} - I_{n-2}$ then I_4 is equal to...

Sol. Given, $I_n = \frac{1}{n-1} - I_{n-2}$ $\quad \therefore I_4 = \frac{1}{4-1} - I_{4-2} = \frac{1}{3} - I_2$ $\quad \therefore I_4 = \frac{1}{3} - \left(\frac{1}{2-1} - I_{2-2}\right)$

$$\therefore I_4 = \frac{1}{3} - \left(\frac{1}{1} - I_0\right) = \frac{1}{3} - (1 - I_0) \longrightarrow (1)$$

Now, $I_n = \int_0^{\frac{\pi}{4}} tan^n x dx$ $\quad \therefore I_0 = \int_0^{\frac{\pi}{4}} tan^0 x dx$ $\quad \therefore I_0 = \int_0^{\frac{\pi}{4}} 1 dx = (x)_0^{\frac{\pi}{4}} = \frac{\pi}{4}$

Substitute $I_0 = \frac{\pi}{4}$ in equation (1)

$$I_4 = \frac{1}{3} - \left(1 - \frac{\pi}{4}\right) = \frac{1}{3} - 1 + \frac{\pi}{4} = -\frac{2}{3} + \frac{\pi}{4}$$

$$\boxed{I_4 = -\frac{2}{3} + \frac{\pi}{4}}$$

Solve: If $I_n = \int_0^{\frac{}{3}} cos^n x dx$ $\; and \;$ $I_n = \frac{\sqrt{3}}{n\, 2^n} + \left(\frac{n-1}{n}\right) I_{n-2}$ then find I_2.

Solve: Find $\int_0^{\pi} sin^5\left(\frac{x}{2}\right) dx$

Solve: Find $\int_{-\pi/2}^{\pi/2} sin^4\theta cos^2\theta d\theta$

Hint: $sin^4\theta cos^2\theta$ is an even function.

$$\therefore \int_{-\pi/2}^{\pi/2} sin^4\theta cos^2\theta d\theta = 2\int_0^{\pi/2} sin^4\theta cos^2\theta d\theta$$

Now use the formula, $\int_0^{\frac{}{2}} sin^m x cos^n x dx = \frac{\{(m-1)(m-3)\ldots\ldots\ldots 2\ or\ 1\}\{(n-1)(n-3)\ldots\ldots\ldots 2\ or\ 1\}}{(m+n)(m+n-2)(m+n-4)\ldots\ldots\ldots 2\ or\ 1} \times \frac{\pi}{2}$

❖ Gamma Functions:

- Gamma function is a single variable function.
- It is used to determine time based occurrences, such as life span of electronic component.
- It is used to compute the average kinetic energy per molecule for the classical gas at temperature T.

Gamma function of n>0 is denoted by $\Gamma(n)$ and is defined as definite integral,

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

❖ Properties of Gamma function:

1. $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$

2. $\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1} dx$

3. $\int_0^\infty e^{-ky} y^{n-1}\, dy = \frac{\Gamma(n)}{k^n}$

4. $\Gamma(n+1) = \mathrm{n}\Gamma(n)$, in general
   $\Gamma(n+1) = \mathrm{n}!$, if n is a positive integer

5. $\Gamma(p).\Gamma(1-p) = \frac{\pi}{\sin p\pi} \quad if\ 0 < p < 1$

6. $\Gamma(n) = 1$

7. $\Gamma(0) = \infty$

8. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Types of examples	Solution
1. Integral of the form $\int_0^\infty x^n e^{-ax^m} dx$	Substitute $ax^m = t$ so that given integral reduces to Gamma function.
2. Integral of the form $\int_0^\infty \frac{x^a}{a^x} dx \;\; OR \;\; \int_0^\infty x^a a^{-x} dx$	Substitute $a^x = e^t \;\; OR \;\; a^{-x} = e^{-t}$ so that given integral reduces to Gamma function
3. Integral of the form $\int_0^1 x \log x \, dx \quad OR \quad \int_0^1 x \log \frac{1}{x} dx$	Substitute $\log x = -t \;\; OR \;\; \log \frac{1}{x} = t$ and given integral reduces to Gamma function

❖ Examples of Gamma function

1. Find $\Gamma(9)$

Sol.

We have, $\Gamma(n+1) = \text{n!}$,if n is a positive integer

$$\Gamma(9) = \Gamma(8+1) = 8!$$

2. Find $\Gamma\left(\frac{3}{2}\right)$

Sol.

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right)$$

Using $\Gamma(n+1) = \mathrm{n}\Gamma(n)$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ we get

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

$$\boxed{\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}}$$

3. Find $\Gamma\left(\frac{9}{2}\right)$

Sol.

$$\Gamma\left(\frac{9}{2}\right) = \Gamma\left(\frac{7}{2}+1\right) = \frac{7}{2}\Gamma\left(\frac{7}{2}\right) = \frac{7}{2}\Gamma\left(\frac{5}{2}+1\right)$$

$$= \frac{7}{2}.\frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{7}{2}.\frac{5}{2}\Gamma\left(\frac{3}{2}+1\right) = \frac{7}{2}.\frac{5}{2}.\frac{3}{2}\Gamma\left(\frac{3}{2}\right)$$

$$= \frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\Gamma\left(\frac{1}{2}+1\right) = \frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi} = \frac{105}{16}\sqrt{\pi}$$

$$\boxed{\Gamma\left(\frac{9}{2}\right) = \frac{105}{16}\sqrt{\pi}}$$

4. Find $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$

Sol.

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)=\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)$$

Now using $\Gamma(p).\Gamma(1-p)=\dfrac{\pi}{\sin p\pi}\ \ if\ 0<p<1$

Here $p=\frac{1}{4}$

$$\therefore \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)=\frac{\pi}{\sin\frac{1}{4}\pi}=\frac{\pi}{\sin\frac{\pi}{4}}=\frac{\pi}{1/\sqrt{2}}=\sqrt{2}\pi$$

$$\boxed{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)=\sqrt{2}\pi}$$

5. Find $\int_0^\infty e^{-t}t^{\frac{3}{2}}\,dt$

Sol.

Comparing with $\int_0^\infty e^{-x}x^{n-1}dx=\Gamma(n)$

$$n-1=\frac{3}{2}\quad \therefore n=\frac{3}{2}+1=\frac{5}{2}$$

$$\therefore \int_0^\infty e^{-t}t^{\frac{3}{2}}\,dt=\Gamma\left(\frac{5}{2}\right)=\Gamma\left(\frac{3}{2}+1\right)$$

$$=\frac{3}{2}\Gamma\left(\frac{3}{2}\right)=\frac{3}{2}\Gamma\left(\frac{1}{2}+1\right)$$

$$=\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}=\frac{3}{4}\sqrt{\pi}$$

$$\boxed{\int_0^\infty e^{-t}t^{\frac{3}{2}}\,dt=\frac{3}{4}\sqrt{\pi}}$$

6. Find the value of integral $\int_0^\infty \sqrt[4]{x} e^{-\sqrt{x}}\, dx$

Sol.

x	0	∞
t	0	∞

Substitute $\sqrt{x} = t \quad \therefore x = t^2 \quad \therefore dx = 2t dt$

$$\int_0^\infty \sqrt[4]{x} e^{-\sqrt{x}}\, dx = \int_0^\infty x^{\frac{1}{4}} e^{-\sqrt{x}}\, dx = \int_0^\infty (t^2)^{\frac{1}{4}} e^{-t}\, 2t dt$$

$$= 2\int_0^\infty (t)^{\frac{1}{2}}.\, t e^{-t}\, dt = 2\int_0^\infty (t)^{\frac{3}{2}} e^{-t}\, dt$$

Comparing with $\int_0^\infty e^{-x} x^{n-1} dx = \Gamma(n)$

$$n - 1 = \frac{3}{2} \quad \therefore n = \frac{5}{2}$$

$$\int_0^\infty \sqrt[4]{x} e^{-\sqrt{x}}\, dx = 2\int_0^\infty (t)^{\frac{3}{2}} e^{-t}\, dt = 2\Gamma\left(\frac{5}{2}\right) = 2.\frac{3}{2}.\frac{1}{2}.\sqrt{\pi} = \frac{3}{2}\sqrt{\pi}$$

$$\boxed{\int_0^\infty \sqrt[4]{x} e^{-\sqrt{x}}\, dx = \frac{3}{2}\sqrt{\pi}}$$

7. Find $\int_0^\infty \frac{x^4}{4^x}dx$

Sol.

Substitute $4^x = e^t$

$$\therefore \log 4^x = \log e^t$$

$$\therefore x\log 4 = t \quad \therefore x = \frac{t}{\log 4}$$

$$\therefore dx = \frac{1}{\log 4}dt$$

x	0	∞
t	0	∞

$$\int_0^\infty \frac{x^4}{4^x}dx = \int_0^\infty \frac{t^4}{(\log 4)^4}e^{-t}\frac{1}{\log 4}dt = \frac{1}{(\log 4)^5}\int_0^\infty e^{-t}\,t^4 dt \qquad n-1=4 \quad \therefore n=5$$

$$= \frac{1}{(\log 4)^5}\Gamma(5) = \frac{1}{(\log 4)^5}.4! = \frac{24}{(\log 4)^5}$$

$$\boxed{\int_0^\infty \frac{x^4}{4^x}dx = \frac{24}{(\log 4)^5}}$$

8. Find the value of integral $\int_0^1 \frac{dx}{\sqrt{x \log\left(\frac{1}{x}\right)}}$

Sol.

Substitute $\log\left(\frac{1}{x}\right) = t$ $\qquad \therefore \ \log x = -t$

$\therefore \ x = e^{-t}$ $\qquad \therefore \ dx = -e^{-t} dt$

x	0	1
t	∞	0

$$\int_0^1 \frac{dx}{\sqrt{x \log\left(\frac{1}{x}\right)}} = \int_\infty^0 \frac{-e^{-t} dt}{\sqrt{t.e^{-t}}} = \int_0^\infty \frac{e^{-t} dt}{\sqrt{t.e^{-t}}} = \int_0^\infty e^{-t} e^{\frac{t}{2}} t^{\frac{-1}{2}} \, dt = \int_0^\infty e^{\frac{-t}{2}} t^{\frac{-1}{2}} dt$$

Using $\int_0^\infty e^{-ky} y^{n-1} \, dy = \frac{\Gamma(n)}{k^n}$ $\qquad$ Here, $k = \frac{1}{2}$ and $n - 1 = \frac{-1}{2}$ $\ \therefore n = \frac{1}{2}$

$$\int_0^1 \frac{dx}{\sqrt{x \log\left(\frac{1}{x}\right)}} = \int_0^\infty e^{\frac{-t}{2}} t^{\frac{-1}{2}} dt = \frac{\Gamma\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{\left(\frac{1}{2}\right)^{\frac{1}{2}}} = \sqrt{2\pi}$$

$$\boxed{\int_0^1 \frac{dx}{\sqrt{x \log\left(\frac{1}{x}\right)}} = \sqrt{2\pi}}$$

Solve: Find the value of the integral $\int_0^\infty x^9 e^{-2x^2} dx$ Hint: Substitute $2x^2 = t$

Solve: Find the value of the integral $\int_0^\infty \frac{x^2}{2^x} \mathrm{dx}$ Hint: Substitute $2^x = e^t$

Solve: Find the value of the integral $\int_0^1 \frac{dx}{\sqrt{-\log x}}$ Hint: Substitute $\log x = -t$

❖ Beta function

- Beta function is two-variable function.
- Beta function is used to determine average time of completing selected task in time management problems.

Beta function of m>0 ,n>0 is denoted by $\beta(m,n)$ and defined as definite integral

$$\beta(m,n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}dx \;\; , m > 0, n > 0$$

❖ Properties of Beta function:

1. Relation between Beta and Gamma function

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

2. $\beta(m,n) = \beta(n,m)$

3. $\beta(m,n) = 2\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta . cos^{2n-1}\theta \; d\theta$

4. $\int_0^{\frac{\pi}{2}} sin^p\theta . cos^q\theta \; d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

5. $\beta(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\; dx$

6. $\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}} . \Gamma(2m)$

This is called duplication formula for Gamma function.

Ex. 1 Find $\int_0^1 x^{\frac{1}{2}}(1-x)^{\frac{5}{2}}dx$

Sol.

Comparing given integral with $\int_0^1 x^{m-1}\,(1-x)^{n-1}dx = \beta(m,n)$ we get

$$m-1=\frac{1}{2}\quad ,n-1=\frac{5}{2}\qquad \therefore m=\frac{3}{2}\quad ,n=\frac{7}{2}$$

$$\boxed{\therefore \int_0^1 x^{\frac{1}{2}}(1-x)^{\frac{5}{2}}dx=\beta\left(\frac{3}{2},\frac{7}{2}\right)}$$

Ex.2 Find $\int_0^{\frac{\pi}{2}} sin^{\frac{3}{2}}\theta cos^4\theta d\theta$

Sol.

Comparing given integral with $\int_0^{\frac{\pi}{2}} sin^p\theta . cos^q\theta\ d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)$ we get

$$p=\frac{3}{2}\ ,q=4\qquad \therefore \frac{p+1}{2}=\frac{5}{4}\ ,\frac{q+1}{2}=\frac{5}{2}$$

$$\boxed{\therefore \int_0^{\frac{\pi}{2}} sin^{\frac{3}{2}}\theta cos^4\theta d\theta=\frac{1}{2}\beta\left(\frac{5}{4},\frac{5}{2}\right)}$$

Ex.3 Find $\int_0^{\infty} \frac{x^8}{(1+x)^{24}} dx - \int_0^{\infty} \frac{x^{14}}{(1+x)^{24}} dx$

Sol.

Given integral can be written as

$$\int_0^{\infty} \frac{x^{9-1}}{(1+x)^{9+15}} dx - \int_0^{\infty} \frac{x^{15-1}}{(1+x)^{15+9}} dx$$

Using $\int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} \, dx = \beta(m,n)$ above integral becomes

$$\beta(9,15) - \beta(15,9)$$

Now,using the property $\beta(m,n) = \beta(n,m)$ we get

$$\beta(9,15) - \beta(15,9) = \beta(9,15) - \beta(9,15) = 0$$

$$\boxed{\int_0^{\infty} \frac{x^8}{(1+x)^{24}} dx - \int_0^{\infty} \frac{x^{14}}{(1+x)^{24}} dx \ = 0}$$

Ex.4 The integral $\int_3^7 \sqrt{(x-3)(7-x)}\, dx$ by using substitution $x = 4t + 3$ reduces to...

Sol.

Note: The appropriate substitution to reduce the integral $\int_a^b (x-a)^m (b-x)^n\, dx$ to Beta integral is $x = (b-a)t + a$

Substitute $x = 4t + 3$ $\therefore dx = 4dt$

x	3	7
t	0	1

$$\int_3^7 \sqrt{(x-3)(7-x)}\, dx = \int_0^1 \sqrt{(4t+3-3)(7-4t-3)}\; 4dt$$

$$\int_0^1 \sqrt{(4t)(4-4t)}\; 4dt \qquad = 16 \int_0^1 \sqrt{(t)(1-t)}\; dt$$

$$\boxed{16 \int_0^1 t^{\frac{1}{2}} (1-t)^{\frac{1}{2}}\, dt}$$

Ex.5 Find $\beta\left(\frac{1}{4},\frac{3}{4}\right)$

Sol.

Using $\beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ we get

$$\beta\left(\frac{1}{4},\frac{3}{4}\right)=\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}+\frac{3}{4}\right)}=\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma(1)}=\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)$$

Now using $\Gamma(p).\Gamma(1-p)=\frac{\pi}{\sin p\pi}$

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)=\frac{\pi}{\sin\frac{1}{4}\pi}=\frac{\pi}{1/\sqrt{2}}=\sqrt{2}\,\pi$$

$$\boxed{\beta\left(\frac{1}{4},\frac{3}{4}\right)=\sqrt{2}\,\pi}$$

Solve: Find the value of the integral $\int_0^{\infty} \frac{x^3+x^2}{(1+x)^7}\, dx$

Hint: use the formula $\beta(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$

Solve : Find the value of the integral $\int_0^2 x(8-x^3)^{\frac{1}{3}}\, dx$ Hint: Substitute $x^3 = 8t$

Solve: Find $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta$ Hint: use the formula $\int_0^{\frac{\pi}{2}} sin^p\theta . cos^q\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

Solve: Reduce the given integral $\int_5^9 (x-5)^{\frac{1}{4}}(9-x)^{\frac{1}{4}}\, dx$ to Beta function integral.

Hint: Refer Note given in example 4.

❖ Differentiation Under integral Sign (DUIS)

- By using DUIS technique,the definite integral converts into an ordinary differential equation.The solution of this equation gives value of the definite integral.

We have to discuss two rules of DUIS

Rule 1: Integral with limits as constants

If $I(\alpha) = \int_a^b f(x,\alpha)dx$, where a and b are constants and α is a parameter, then

$$\frac{dI}{d\alpha} = \int_a^b \frac{\partial}{\partial\alpha}\{f(x,\alpha)\}dx \,,$$

Rule 2: Leibnitz's Rule: Integrals with limits as functions of parameter(Either both the limits are functions of parameter or one of the limits is function of parameter

If $I(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)dx$, where a and b are functions of parameter α , then

$$\frac{dI}{d\alpha} = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial\alpha}\{f(x,\alpha)\}\, dx \; + f(b,\alpha)\frac{db}{d\alpha} - f(a,\alpha)\frac{da}{d\alpha}$$

Ex.1 If $\phi(a) = \int_0^{\infty} \frac{e^{-x}}{x}(1 - e^{-ax})dx$, $a > -1$ then by DUIS rule, find $\frac{d\phi}{da}$

Sol.

Here, $f(x,a) = \frac{e^{-x}}{x}(1 - e^{-ax})$, a is parameter and limits of integration are constants.

Hence, by rule 1 of DUIS

$$\frac{d\phi}{da} = \int_0^{\infty} \underline{\frac{\partial}{\partial a}\left\{\frac{e^{-x}}{x}(1 - e^{-ax})\right\}} dx$$

To find above partial derivative with respect to a, treat x as constant

$$\frac{d\phi}{da} = \int_0^{\infty} \left\{\frac{e^{-x}}{x}(0 + xe^{-ax})\right\} dx = \int_0^{\infty} \frac{e^{-x}}{x} xe^{-ax} dx = \int_0^{\infty} e^{-x}e^{-ax} dx$$

$$= \int_0^{\infty} e^{-(a+1)x} dx$$

$$\boxed{\frac{d\phi}{da} = \int_0^{\infty} e^{-(a+1)x} dx}$$

Ex. 2 If $\phi(a) = \int_0^1 \frac{x^a - 1}{\log x}\, dx$; $a \geq 0$ then by DUIS rule, find $\frac{d\phi}{da}$

Sol.

Here, $f(x, a) = \frac{x^a - 1}{\log x}$, a is parameter and limits of integration are constants.

Hence, by rule 1 of DUIS

$$\frac{d\phi}{da} = \int_0^1 \frac{\partial}{\partial a}\left(\frac{x^a - 1}{\log x}\right) dx$$

To find partial derivative with respect to a, treat x as constant

$$\therefore \frac{d\phi}{da} = \int_0^1 \left(\frac{x^a \log x - 0}{\log x}\right) dx = \int_0^1 x^a dx$$

$$\boxed{\frac{d\phi}{da} = \int_0^1 x^a dx}$$

Ex.3 If $\phi(\alpha)= \int_0^\infty \frac{e^{-x}\sin\alpha x}{x}\,dx$ then by DUIS rule find $\frac{d\phi}{d\alpha}$

Sol.

Here, $f(x,\alpha) = \frac{e^{-x}\sin\alpha x}{x}$, α is parameter and limits of integration are constants.

Hence, by rule 1 of DUIS

$$\frac{d\phi}{d\alpha} = \int_0^\infty \frac{\partial}{\partial\alpha}\left(\frac{e^{-x}\sin\alpha x}{x}\right)dx$$

To find partial derivative with respect to α , treat x as constant

$$\frac{d\phi}{d\alpha} = \int_0^\infty \left(\frac{e^{-x}x\cos\alpha x}{x}\right)dx \qquad = \int_0^\infty e^{-x}\cos\alpha x\,dx$$

$$\boxed{\frac{d\phi}{d\alpha} = \int_0^\infty e^{-x}\cos\alpha x\,dx}$$

Ex.4 If $\phi(a) = \int_0^1 \frac{x^a - x^b}{\log x} dx$, $a > 0, b > 0$ then by DUIS rule, find $\frac{d\phi}{da}$

Sol.

Here, $f(x, a) = \frac{x^a - x^b}{\log x}$, a is parameter and limits of integration are constants.

Hence, by rule 1 of DUIS

$$\frac{d\phi}{da} = \int_0^1 \frac{\partial}{\partial a}\left(\frac{x^a - x^b}{\log x}\right) dx$$

To find partial derivative with respect to a, treat x and b as constants

$$\therefore \frac{d\phi}{da} = \int_0^1 \left(\frac{x^a \log x - 0}{\log x}\right) dx = \frac{d\phi}{da} = \int_0^1 x^a dx$$

$$\boxed{\frac{d\phi}{da} = \int_0^1 x^a dx}$$

Ex.5 If $\phi(a) = \int_a^{a^2} \log(ax)\ dx$ then by DUIS rule 2 find $\frac{d\phi}{da}$

Sol.

Here, $f(x, a) = \log(ax)$, a is parameter and limits of integration are functions of parameter a.

Hence, by rule 2 of DUIS

$$\frac{d\phi}{da} = \int_a^{a^2} \frac{\partial}{\partial a}\{\log(ax)\}dx + f(a^2, a)\frac{d(a^2)}{da} - f(a, a)\frac{da}{da}$$

$$\frac{d\phi}{da} = \int_a^{a^2} \frac{\partial}{\partial a}\{\log(ax)\}dx + 2a\log(a^3)\ - \log(a^2)$$

$$\boxed{\frac{d\phi}{da} = \int_a^{a^2} \frac{\partial}{\partial a}\{\log(ax)\}dx + 2a\log(a^3)\ - 2\log(a)}$$

Ex.6 If $f(x) = \int_a^x (x-t)^2 G(t)dt$, where a is constant and x is parameter then by DUIS rule 2 find $\frac{df}{dx}$

Sol.

Here, $f(t,x) = (x-t)^2 G(t)$ where t is variable, x is parameter and a is constant.
Upper limit of the integral is function of parameter .

Hence, by rule 2 of DUIS

$$\frac{df}{dx} = \int_a^x \frac{\partial}{\partial x}\{(x-t)^2 G(t)\}dt + f(x,x)\frac{dx}{dx} - f(a,x)\frac{da}{dx}$$

$$\frac{df}{dx} = \int_a^x \frac{\partial}{\partial x}\{(x-t)^2 G(t)\}dt + 0.\frac{dx}{dx} - (x-t)^2 G(t).0$$

$$\boxed{\frac{df}{dx} = \int_a^x \frac{\partial}{\partial x}\{(x-t)^2 G(t)\}dt}$$

Ex.7 Using DUIS rule,find the value of integral $\phi(\alpha)= \int_0^\infty \frac{e^{-2x}\sin\alpha x}{x}\,dx$ with $\frac{d\phi}{d\alpha} = \frac{2}{\alpha^2+4}$

Sol.

Given, $\frac{d\phi}{d\alpha} = \frac{2}{\alpha^2+4}$ $\quad\therefore d\phi = \frac{2}{\alpha^2+4}d\alpha$

Integrating both the sides,we get

$\int d\phi = \int \frac{2}{\alpha^2+4}d\alpha + C$ where C is constant of integration.

$$\phi(\alpha) = \tan^{-1}\left(\frac{\alpha}{2}\right) + C \longrightarrow (1)$$

$$\phi(0) = \tan^{-1}\left(\frac{0}{2}\right) + C = 0 + C = C \qquad \therefore C = \phi(0) \longrightarrow (2)$$

Now, we have $\phi(\alpha)= \int_0^\infty \frac{e^{-2x}\sin\alpha x}{x}\,dx$

$\therefore \phi(0)= \int_0^\infty \frac{e^{-2x}\sin 0}{x}\,dx = \int_0^\infty 0dx = 0$ $\quad\therefore$ (2) becomes, $C = 0$

Back substitute $C = 0$ in equation (1)

$$\boxed{\phi(\alpha) = \tan^{-1}\left(\frac{\alpha}{2}\right)}$$

Solve: If $F(t) = \int_t^{t^2} e^{tx^2} dx$ then by DUIS rule 2 find $\frac{dF}{dt}$

Solve: Using DUIS rule find the value of the integral $\phi(a) = \int_0^\infty \frac{e^{-x}}{x}(1 - e^{-ax})dx$, $a > -1$

with $\frac{d\phi}{da} = \frac{1}{a+1}$ Hint: To find the value of constant C , substitute $a = 0$ in $\phi(a)$

Solve: If $I(a) = \int_0^\infty \frac{e^{-x} - e^{-ax}}{x \sec x} dx$, $a > 0$ then by rule of DUIS find $\frac{dI}{da}$ Hint: Use rule 1 of DUIS

Solve: If $\phi(a) = \int_0^a \frac{\log(1+ax)}{1+x^2} dx$, then by DUIS rule 2 find $\frac{d\phi}{da}$

❖ Error Function

- Error function of x is denoted by $\operatorname{erf}(x)$ and given by the integral

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$$

- Complementary error function of x is denoted by $\operatorname{erfc}(x)$ and given by the integral

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du$$

❖ Properties of error functions

1. $\operatorname{erf}(\infty) = 1$

2. $\operatorname{erf}(0) = 0$

3. $\operatorname{erf}(x) + \operatorname{erfc}(x) = 1$

4. Error function is an odd function

$\therefore\ \operatorname{erf}(-x) = -\operatorname{erf}(x)$

5. $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots \right]$

6. Alternate definitions:

- $\operatorname{erf}(x) = \frac{1}{\sqrt{\pi}} \int_0^{x^2} e^{-t} t^{-\frac{1}{2}} dt$
- $\operatorname{erfc}(x) = \frac{1}{\sqrt{\pi}} \int_{x^2}^{\infty} e^{-t} t^{-\frac{1}{2}} dt$

Ex.1 If $\operatorname{erf}(ax) = \frac{2}{\sqrt{\pi}} \int_0^{ax} e^{-u^2} du$ then find $\frac{d}{dx} \operatorname{erf}(ax)$

Sol.

$$\frac{d}{dx}\operatorname{erf}(ax) = \frac{d}{dx}\left[\frac{2}{\sqrt{\pi}} \int_0^{ax} e^{-u^2} du\right] = \frac{2}{\sqrt{\pi}} \frac{d}{dx} \int_0^{ax} e^{-u^2} du$$

Here, x is parameter ,u is variable and upper limit ax is the function of the parameter

Hence, by rule 2 of DUIS

$$= \frac{2}{\sqrt{\pi}}\left[\int_0^{ax} \frac{\partial}{\partial x} e^{-u^2} du + e^{-a^2x^2} \frac{d}{dx}(ax) - e^{-0} \frac{d}{dx}(0)\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[0 + ae^{-a^2x^2} - 0\right] = \frac{2}{\sqrt{\pi}} ae^{-a^2x^2}$$

$$\boxed{\frac{d}{dx}\operatorname{erf}(ax) = \frac{2}{\sqrt{\pi}} ae^{-a^2x^2}}$$

Ex.2 If $\operatorname{erfc}(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_{\sqrt{t}}^{\infty} e^{-u^2} du$ then find $\frac{d}{dt} \operatorname{erfc}(\sqrt{t})$

Sol.

$$\frac{d}{dt} \operatorname{erfc}(\sqrt{t}) = \frac{d}{dt} \left\{ \frac{2}{\sqrt{\pi}} \int_{\sqrt{t}}^{\infty} e^{-u^2} du \right\}$$

Here, t is parameter , u is variable and limits of integral are functions of parameter.

Hence by Rule 2 of DUIS,

$$\frac{d}{dt} \operatorname{erfc}(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \left[\int_{\sqrt{t}}^{\infty} \frac{\partial}{\partial t} e^{-u^2} du + e^{-\infty} \frac{d}{dt} \infty - e^{-\sqrt{t}^2} \frac{d}{dt} \sqrt{t} \right]$$

$$= \frac{2}{\sqrt{\pi}} \left[\int_{\sqrt{t}}^{\infty} 0 + 0 - e^{-t} \frac{1}{2\sqrt{t}} \right] = \frac{-e^{-t}}{\sqrt{\pi t}}$$

$$\boxed{\frac{d}{dt} \operatorname{erfc}(\sqrt{t}) = \frac{-e^{-t}}{\sqrt{\pi t}}}$$

Ex.3 If $\frac{d}{dx}erf(ax) = \frac{2a}{\sqrt{\pi}}e^{-a^2x^2}$ then find $\frac{d}{dx}erfc(ax)$

Sol.

We have, $\text{erf}(ax) + \text{erfc}(ax) = 1$

$$\therefore \frac{d}{dx}[\text{erf}(ax) + \text{erfc}(ax)] = \frac{d}{dx}(1) = 0$$

$$\frac{d}{dx}\text{erf}(ax) + \frac{d}{dx}\text{erfc}(ax) = 0$$

$$\therefore \frac{d}{dx}\text{erfc}(ax) = -\frac{d}{dx}\text{erf}(ax) = -\frac{2a}{\sqrt{\pi}}e^{-a^2x^2}$$

$$\boxed{\frac{d}{dx}\text{erfc}(ax) = -\frac{2a}{\sqrt{\pi}}e^{-a^2x^2}}$$

Ex.4 If $\frac{d}{da}erfc(ax) = -\frac{2x}{\sqrt{\pi}}e^{-a^2x^2}$ then find $\frac{d}{da}erf(ax)$

Ans: $\frac{d}{da}erf(ax) = \frac{2x}{\sqrt{\pi}}e^{-a^2x^2}$ (Same as that of the Ex.3)

Ex.5 If $\operatorname{erfc}(a) = \frac{2}{\sqrt{\pi}} \int_a^\infty e^{-u^2} du$ then by using substitution $x + a = u$ find the integral $\int_0^\infty e^{-(x+a)^2} dx$ in terms of $\operatorname{erfc}(a)$

Sol.

We have to find $\int_0^\infty e^{-(x+a)^2} dx$

Substitute $x + a = u$ $\quad \therefore dx = du$

x	0	∞
u	a	∞

$$\therefore \int_0^\infty e^{-(x+a)^2} dx = \int_a^\infty e^{-u^2} du$$

$$= \frac{\sqrt{\pi}}{2} \operatorname{erfc}(a)$$

Given, $\operatorname{erfc}(a) = \frac{2}{\sqrt{\pi}} \int_a^\infty e^{-u^2} du$

$$\therefore \frac{\sqrt{\pi}}{2} \operatorname{erfc}(a) = \int_a^\infty e^{-u^2} du$$

$$\boxed{\int_0^\infty e^{-(x+a)^2} dx = \frac{\sqrt{\pi}}{2} \operatorname{erfc}(a)}$$