Partial Differentiation Introduction & Definition

Subject- Engg.Mathematcs-I

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CONTENT OF THE TOPIC

1.Introduction to Partial Derivative, Definition, Physical Meaning and basic examples.

2. Properties of Partial Derivative and Higher order derivatives.

3.Derivative of Composite function

4.Variable to be treated as constant.

5.Homogeneous Function and Euler's Theorem.

6.Partial Derivative of Implicit function& Total Derivative.

PARTIAL DERIVATIVES

Introduction – Partial Derivative is used for functions with more than one independent variable.

If the function $Z = f(x, y)$ be a function of two variables x, y then the rate of change of z with respect to x, keeping y const. is called partial derivative of z with respect to x & it is denoted by $\frac{\partial z}{\partial x}$.

Similarly the rate of change of z with respect to y, keeping x const. is called partial derivative of z with respect to x & it is denoted by $\frac{\partial z}{\partial y}$

Thus for $Z = f(x, y)$ $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ are called First Order Partial derivatives with respect to x & y respectively.

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Thus $$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

and $$\frac{\partial z}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

In general $\frac{\partial z}{\partial x}$ $\frac{\partial z}{\partial y}$ are function of both the variables x and y so we may obtain higher derivatives

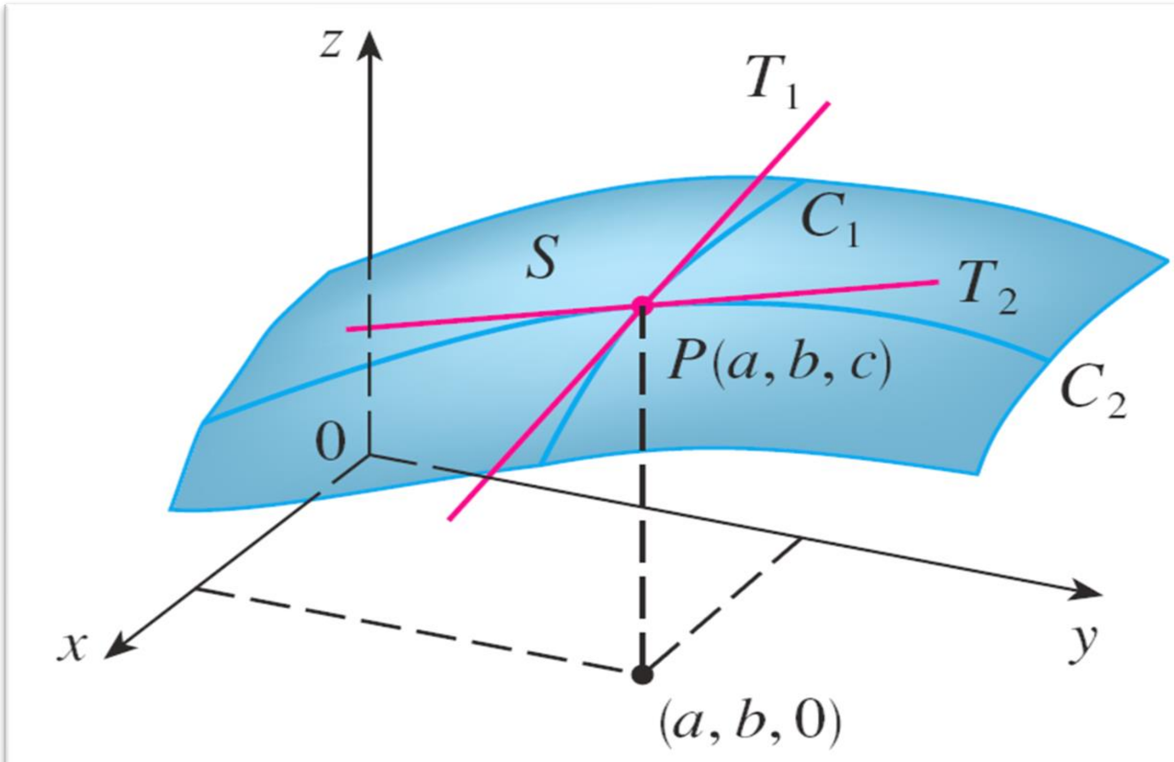
GEOMETRICAL INTERPRETATION

Geometrically z=f(x, y) represents a surface S ,so if y=k (constant) i.e. a plane parallel to X Z plane thus

z=f(x ,y) and y=k (constant) together represents a curve C which is the section of S by y=k plane.

Thus $\frac{\partial z}{\partial x}$ represents the slope of tangent to C at(x,k, z)

Similarly $\frac{\partial z}{\partial y}$ represents the slope of tangent drawn to the curve of intersection of z =f(x, y) & x=k.



If f(a,b)=c then pt P(a,b,c) lies on S

y=b intersects surface S in C1

T1 is the slope of tangent to curve C1 i.e. $\dfrac{\partial z}{\partial x}$

Similarly T2 is the slope of tangent to curve C2 i.e. $\dfrac{\partial z}{\partial y}$

Examples on Direct Partial Derivatives

Q1. If $x = r\cos\theta\ , y = r\sin\theta$ show that

$$\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = 1$$

Sol $x = r\cos\theta\ , y = r\sin\theta$ then

$(x)^2 + (y)^2 = r^2$ therefore

$$2r\frac{\partial r}{\partial x} = 2x \quad \& \quad 2r\frac{\partial r}{\partial y} = 2y$$

$$so\ \frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta \quad \& \ \frac{\partial r}{\partial y} = \frac{y}{r} = \frac{r\sin\theta}{r} = \sin\theta$$

$$thus\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = \cos^2\theta + \sin^2\theta = 1$$

Particular Example

1) If $z^3 - zx - y = 4\ then\ find\ \frac{\partial z}{\partial x}\ \&\ \frac{\partial z}{\partial y}$

Solution : Given $z^3 - zx - y = 4$----(1)

Differentiate eq(1) partially w.r.t. x keeping y as constant

$$3z^2\frac{\partial z}{\partial x} - z - x\frac{\partial z}{\partial x} = 0$$

$$(3z^2 - x)\frac{\partial z}{\partial x} = z$$

$$\frac{\partial z}{\partial x} = \frac{z}{3z^2 - x}$$

Also, differentiate eq (1) partially w.r.t. y, keeping x as constant

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$$\left(3z^2\frac{\partial z}{\partial y}-x\frac{\partial z}{\partial y}-1\right)=0$$

$$(3z^2-x)\frac{\partial z}{\partial y}=1$$

$$\frac{\partial z}{\partial y}=\frac{1}{(3z^2-x)}$$

3) 1)If $u=\frac{1}{\sqrt{x^2+y^2+z^2}}$ then find the value of

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$$

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Solution:

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[(x^2 + y^2 + z^2)^{-1/2}\right]$$

$$\frac{\partial u}{\partial x} = \frac{-1}{2}\,\frac{2x}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial u}{\partial x} = \frac{-x}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{-x}{(x^2+y^2+z^2)^{3/2}}\right)$$

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$$\frac{\partial^2 u}{\partial x^2} = (-x)\left(\frac{-3}{2}\right)(x^2+y^2+z^2)^{-5/2} + (-1)(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{3x^2}{(x^2+y^2+z^2)^{5/2}} - \frac{1}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$$

Similarly

$$\frac{\partial^2 u}{\partial y^2} = \frac{-x^2+2y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$$

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$$\frac{\partial^2 u}{\partial z^2} = \frac{-x^2-y^2+2z^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} =$$
$$\frac{2x^2-y^2-z^2-x^2+2y^2-z^2-x^2-y^2+2z^2}{(x^2+y^2+z^2)^{5/2}} = 0$$

3) If $u = \log(tanx + tany + tanz)$ then prove that $sin2x\,\frac{\partial u}{\partial x} + sin2y\,\frac{\partial u}{\partial y} + sin2z\,\frac{\partial u}{\partial z} = 2$

Solution : Given $u = \log(tanx + tany + tanz)$

$$\frac{\partial u}{\partial x} = \frac{sec^2x}{tanx+tany+tanz}$$

Similarly

$$\frac{\partial u}{\partial y} = \frac{sec^2y}{tanx+tany+tanz}$$

$$\frac{\partial u}{\partial z} = \frac{sec^2z}{tanx+tany+tanz}$$

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Now

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \ \frac{\partial u}{\partial z}$$

$$= \frac{\sin 2x \sec^2 x + \sin 2y \sec^2 y + \sin 2z \sec^2 z}{\tan x + \tan y + \tan z}$$

$$= \frac{\frac{2\sin x\cos x}{\cos^2 x} + \frac{2\sin y\cos y}{\cos^2 y} + \frac{2\sin z\cos z}{\cos^2 z}}{\tan x + \tan y + \tan z}$$

$$= \ \frac{2\tan x + 2\tan y + 2\tan z}{\tan x + \tan y + \tan z} = 2$$

4) If $u = \emptyset(x+ay) + \varphi(x - ay) then\ show\ that\ \frac{\partial^2 u}{\partial y^2} = a^2 \frac{\partial^2 u}{\partial x^2}$

Solution: $\frac{\partial u}{\partial y} = \emptyset'(x+ay).a + \varphi'(x-ay).(-a)$

Again differentiating w.r.t.y

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}(\emptyset'(x+ay).a + \varphi'(x+ay).(-a))$$

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$$= a.\emptyset''(x + ay).a + (-a)\varphi''(x - ay)(-a)$$

$$= a^2\left(\emptyset''(x + ay) + \varphi''(x - ay)\right)$$

Now $\frac{\partial u}{\partial x} = \emptyset'(x + ay) + \varphi'(x - ay)$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \emptyset''(x + ay). + \varphi''(x - ay)$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = a^2\frac{\partial^2 u}{\partial x^2}$$

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Partial derivative of higher order:-

Let z = f(x ,y), then $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = z_{xx}$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = z_{xy} \qquad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = z_{yx}$$

In general $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ Order of differentiation is commutative where z is a continuous function

Therefore further differentiation of any of these w .r. t either x or y or both possible.

To find the value of Parameter ‘n

Ex.1)Find the value of ‘n’ for which $u = A\, e^{-gx}\sin(nt - gx)$ satisfies the partial differential equation $\frac{\partial u}{\partial t} = m\frac{\partial^2 u}{\partial x^2}$ where A, g, m are constant.

Solution :Given $u = A\, e^{-gx}\sin(nt - gx)$

Differentiating w.r.t. ‘t ‘ keeping x constant

$$\frac{\partial u}{\partial t} = A\, e^{-gx}\cos(nt - gx)(n)$$

Now differentiating.w.r.t. ‘x’ keeping t constant

Now differentiating.w.r.t. ‘x’ keeping t constant

$$\frac{\partial u}{\partial x} = Ae^{-gx}(-g)\sin(nt - gx) + Ae^{-gx}.\cos(nt - gx)(-g)$$

Again differentiating w.r.t. x

$$\frac{\partial^2 u}{\partial x^2} = A(-g)e^{-gx}(-g).\sin(nt - gx) +$$
$$Ae^{-gx}(-g).\cos(nt - gx)\,(-g) +$$
$$Ae^{-gx}(-g)\cos(nt - gx)\,(-g) + Ae^{-gx}(-\sin(nt -$$
$$gx))(-g)(-g)$$

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$=A\, g^2 e^{-gx}[\sin(nt-gx)+\cos(nt-gx)+\cos(nt-gx)-\sin(nt-gx)]$

$$\frac{\partial^2 u}{\partial x^2}=Ag^2e^{-gx}2.\cos(nt-gx)$$

Now as $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$

$A\, e^{-gx}\cos(nt-gx)(n)=mAg^2e^{-gx}2\cos(nt-gx)$

Thus, $n=2mg^2$

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Q2 $Find\ \ 'n'$such that v= $r^n(3\cos^2\theta - 1)$

satisfies the equation $\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) +$

$\frac{1}{sin\theta}\ \frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial v}{\partial\theta}\right) = 0$

$solution$:

v= $r^n(3\cos^2\theta - 1)\ \ \therefore\frac{\partial v}{\partial r} = nr^{n-1}(3\cos^2\theta - 1)$.

$$\therefore\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) = n(n+1)v$$

Also $\frac{\partial v}{\partial\theta} = r^n[3.2\cos^1\theta(-sin\theta)]$.

$\therefore\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial v}{\partial\theta}\right) = -6r^n[-sin\theta\sin^2\theta + cos\theta.\,2sin\theta cos\theta]$

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$$\therefore \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial v}{\partial\theta}\right) = -6r^{n}[2\,cos^{2}\theta - \sin^{2}\theta]$$

$$= -6r^{n}[3\cos^{2}\theta\text{-}1] \quad = \ -6\ v$$

$$\therefore \frac{\partial}{\partial r}\left(r^{2}\ \frac{\partial v}{\partial r}\right) + \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial v}{\partial\theta}\right) = 0$$

⇨$n(n+1)v - 6v = 0$

⇨$n^{2} + n - 6 = 0\ \ , v \neq 0$

⇨n= - 3 , 2.

Example on verification of $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Ex.1) If $u = x^y + y^x$ then prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Solution : $u = x^y + y^x$-----(1)

Differentiating equation (1) partially w.r.t. y keeping ‘x’ as constant

$$\frac{\partial u}{\partial y} = x^y logx + xy^{x-1}$$

Differentiating above partially w.r.t. x keeping y as constant

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$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left[x^y logx + xy^{x-1}\right]$$

$$= x^y \frac{1}{x} + logx.\, yx^{y-1} + xy^{x-1} logy + y^{x-1}$$

$$\frac{\partial^2 u}{\partial x \partial y} = x^{y-1} + yx^{y-1} logx + xy^{x-1} logy + y^{x-1}$$

Now diff.eq(1)partially w.r.t. x keeping ‘y’ as constant

$$\frac{\partial u}{\partial x} = y\, x^{y-1} + y^x logy$$

Again diff. above w.r.t.y keeping x as constant

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$$\frac{\partial^2 u}{\partial y.\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}[y\, x^{y-1} + y^x logy]$$

= $yx^{y-1}logx + x^{y-1} + y^x\frac{1}{y} + logy.x.y^{x-1}$

=$yx^{y-1}logx + x^{y-1} + y^{x-1} + xy^{x-1}logy$ -----(B)

From (A) and (B)

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial^2 u}{\partial y\partial x}$$

Application Of P D

Partial Differentiation is used in the problems of Jacobian, vibration of string, Theory of approximation, Maxima &Minima .

Boundary value problems, Laplace Equation, Vectors, Heat ,Wave equations & solving some problems in Electrical engineering.

PDEs can be used to describe a wide variety of phenomena such as Sound ,Heat, Electrostatics, Electrodynamics , Fluid flow,

Elasticity or Quantum Mechanics

University Questions

1) I$f \quad u = \log(x^3 + y^3 - x^2y - y^2x)$then Show that

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = -\frac{4}{(x+y)^2}$$

2) If $u = \log(x^2 + y^2)$ $, verify \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial x \partial y}$

3) If $\theta = t^n . e^{\frac{-r^2}{4t}}$ Find n such that $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$

4) Prove that at a point of surface $x^x y^y z^z =$ $c \; where \; x = y = z , \frac{\partial^2 z}{\partial x \partial y} =$

$$-(xlogex)^{-1}$$

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Partial Derivative of Composite Function

- If u is a function of r i.e. u=f(r) and r again is a function of two independent variables x, y i.e. r=g(x ,y) .Thus u becomes a composite function of x and y .

- u$\rightarrow r \rightarrow (x, y) \therefore \frac{\partial u}{\partial x} = \frac{du}{dr} . \frac{\partial r}{\partial x}$, $\frac{\partial u}{\partial y} = \frac{du}{dr} . \frac{\partial r}{\partial y}$

- u$\rightarrow r \rightarrow (x, y, z) \therefore \frac{\partial u}{\partial x} = \frac{du}{dr} . \frac{\partial r}{\partial x}$ $\frac{\partial u}{\partial y} = \frac{du}{dr} . \frac{\partial r}{y}$

$$\frac{\partial u}{\partial z} = \frac{du}{dr} . \frac{\partial r}{\partial z}$$

- $z \rightarrow x, y \rightarrow u, v$

then z is a composite function of two variables u & v

$$\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

- $l \rightarrow x, y, z \rightarrow u, v$,w

$$\therefore \frac{\partial l}{\partial u} = \frac{\partial l}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial l}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial l}{\partial x}\frac{\partial x}{\partial u}$$ &so on..

- Note: For a function of single variable ,derivative is total whereas for a function of more than one variable derivative is Partial.

Example on Composite Function

1) If $z = f(u, v) and \quad u = xcost - ysint$;

$$v = xsint + \ +ycost$$

$where\ t\ is\ a\ constant$ Then show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}$$

Solution : $z = f(u, v) and$

$u = xcost - ysint$-------(1)

$v = xsint + ycost$--------(2)

Differentiating z w.r.t. x keeping y constant

$$\left(\frac{\partial z}{\partial x}\right) = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

&

$$\left(\frac{\partial z}{\partial y}\right) = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

Differentiating equation (1) w.r.t. 'x' keeping t as constant

$$\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(xcost - ysint) = cost$$

$$\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}(xcost - ysint = -sint$$

Now $\left(\frac{\partial v}{\partial x}\right) = sint$; $\left(\frac{\partial v}{\partial y}\right) = cost$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}(cost) + \frac{\partial z}{\partial v}(sint)\text{---------(A)}$$

$$\left(\frac{\partial z}{\partial y}\right) = \frac{\partial z}{\partial u}(-sint) + \frac{\partial z}{\partial v}(cost)\text{------(B)}$$

- Multiplying (A) by x & (B) by y

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xcost\frac{\partial z}{\partial u} + x\ sint\frac{\partial z}{\partial v} - ysint\ \frac{\partial z}{\partial u}$$

$$+ ycost\frac{\partial z}{\partial v}$$

$$= \frac{\partial z}{\partial u}(xcost - usint) + \frac{\partial z}{\partial v}(\ xsint$$

$$+ ycost)$$

Thus $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}$

2) If $u = f(x-y, y-z, z-x)$ then prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$ = 0

- Solution : Let $x - y = l\,; y - z = m\,; z - x = n$

Differentiating by chain rule

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\frac{\partial n}{\partial x} \text{ ------ (A)}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}\frac{\partial n}{\partial y} \text{ --------(B)}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\frac{\partial m}{\partial z} + \frac{\partial u}{\partial n}\frac{\partial n}{\partial z} \text{ ---------(C)}$$

Since $l = x - y\,;\quad m = y - z\;\; ; n = z - x$

$$\frac{\partial l}{\partial x} = 1 \;;\; \frac{\partial m}{\partial x} = 0 \;;\; \frac{\partial n}{\partial x} = -1$$

$$\frac{\partial l}{\partial y} = -1 \;;\; \frac{\partial m}{\partial y} = 1 \;;\; \frac{\partial n}{\partial y} = 0$$

$$\frac{\partial l}{\partial z} = 0 \;;\; \frac{\partial m}{\partial z} = -1 \;;\; \frac{\partial n}{\partial z} = 1$$

Equation (A), (B),(C) becomes

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}(1) + \frac{\partial u}{\partial m}(0) + \frac{\partial u}{\partial n}(-1) \text{ ------ } (1)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}(-1) + \frac{\partial u}{\partial m}(1) + \frac{\partial u}{\partial n}(0) \text{---------} (2)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}(0) + \frac{\partial u}{\partial m}(-1) + \frac{\partial u}{\partial n}(1) \text{ ----------} (3)$$

- Adding (1),(2) ,(3) we get
- $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Example on Composite Function

Q3) If $u = f(x^2 - y^2, y^2 - z^2, z^2 - x^2)$,Prove that

$\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$

Solution : Let $x^2 - y^2 = l\,,\, y^2 - z^2 = m\,,\, z^2 - x^2 = n,$

$\therefore u = f(l, m, n)$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}.\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}.\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}.\frac{\partial n}{\partial x}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}.(2x) + \frac{\partial u}{\partial m}.(0) + \frac{\partial u}{\partial n}.(-2x)$$

similarly $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}.(2x) + \frac{\partial u}{\partial m}.(0) + \frac{\partial u}{\partial n}.(-2x)$

- similarly
- $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}$.(-2y) $+ \frac{\partial u}{\partial m}.(2y) + \frac{\partial u}{\partial n}.(0)$

$$\&\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}.(0) + \frac{\partial u}{\partial m}.(-2z) + \frac{\partial u}{\partial n}.(2z)$$

$$\therefore \frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$$

Hence Proved

Variable to be Treated As Constant

Consider the

Notation $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial r}{\partial x}\right)_\theta, \left(\frac{\partial v}{\partial y}\right)_{x,u}$ all of them have definite meaning.

- These notations arise when there are more than one functional relations given.
- $\left(\frac{\partial u}{\partial x}\right)_y$ means express u in terms of x & y and differentiate

u partially w . r. t. x keeping y as constant using elimination method.

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$\left(\frac{\partial r}{\partial x}\right)_{\theta}$ means express r in terms of x & θand differentiate r partially w . r. t. x keeping θas constant using elimination method.

$\left(\frac{\partial v}{\partial y}\right)_{x,u}$ means express v in terms of x , y & u and differentiate v partially w . r. t. y keeping x ,u as constant using elimination method.

Note

In case the elimination of certain variables is not possible or difficult then we interpret these notations as follows :

1) For a given notation variables in denominator are to be treated as independent variables & remaining variables as dependent .For e.g. $(\frac{\partial u}{\partial z})_{x,v}$ x,z,v are independent variables & u, z, y are dependent variables if there are two equations in 5 variables u,v,x,y,z .

2 In these cases we differentiate all the given equations w.r.t. one of the independent variables keeping other independent variables as constants. Therefore we get simultaneous equations in terms of number of derivatives.

3 We eliminate other derivatives to get the required derivative.

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Examples on variable to be treated as constant

Q1 If $x^2 = au + bv, \;\; y^2 = au - bv$ then show that

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u$$

solution: $x^2 = au + bv, \;\; y^2 = au - bv$ $\quad \left(\frac{\partial x}{\partial u}\right)_v = \frac{a}{2x}$ $\quad \left(\frac{\partial y}{\partial v}\right)_u = -\frac{b}{2y}$

Since $x^2 + y^2 = 2au \qquad x^2 - y^2 = 2bv$

$$u = \frac{x^2 + y^2}{2a} \left(\frac{\partial u}{\partial x}\right)_y = \frac{x}{a}$$

$$v = \frac{x^2 - y^2}{2b} \quad \left(\frac{\partial v}{\partial y}\right)_x = -\frac{y}{b}$$

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- $\therefore \left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{x}{a} \quad \frac{a}{2x} = \frac{1}{2}$

- $\left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u = \left(-\frac{y}{b}\right)\left(-\frac{b}{2y}\right) = \frac{1}{2}$

2) $If\ x = \frac{r}{2}\left(e^{\theta} + e^{-\theta}\right); y = \frac{r}{2}\left(e^{\theta} - e^{-\theta}\right)$

then prove that $\left(\frac{\partial x}{\partial r}\right)_{\theta} = \left(\frac{\partial r}{\partial x}\right)_{y}$

Solution : Given $x = \frac{r}{2}\left(e^{\theta} + e^{-\theta}\right); y = \frac{r}{2}\left(e^{\theta} - e^{-\theta}\right)$

$\therefore x = rcosh\theta\ ; y = rsinh\theta$

Differentiating w.r.t. r keeping θ as constant

$\therefore \left(\frac{\partial x}{\partial r}\right)_{\theta} = cosh\theta$ ----(1)

Now we express r as a function of x and y

$x = rcosh\theta\ ; y = rsinh\theta$

$\therefore\ x^2 - y^2 = r^2cosh^2\theta - r^2sinh^2\theta = r^2\ (1) = r^2$

$$\therefore\ r^2 = x^2 - y^2$$

Differentiating r with respect to x

keeping y constant

$$2r\left(\frac{\partial r}{\partial x}\right)_y = 2x \therefore \left(\frac{\partial r}{\partial x}\right)_y = \frac{x}{r} = \frac{r\cosh\theta}{r} = \cosh\theta$$

Thus from (1) and (2)

$$\left(\frac{\partial x}{\partial r}\right)_\theta = \left(\frac{\partial r}{\partial x}\right)_y$$

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3) $x = utanv$; $y = usecv, then\ prove\ that$:

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial v}{\partial x}\right)_y = \left(\frac{\partial u}{\partial y}\right)_x \left(\frac{\partial v}{\partial y}\right)_x$$

- Solution : Given $x = utanv; y = usecv$

We have to find u,v are functions of x,y

i.e.we have to find $u\ = f(x, y); v = f(x, y)$

squaring x ,y and subtracting

- $y^2 - x^2 = \ u^2sec^2v - u^2tan^2v$
- $\qquad = \ u^2[sec^2v - tan^2v] = u^2$
- $y^2 - x^2 = u^2$

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$differnetiate\ u^2$ w.r.t.x keeping y as constant

$$u^2 = \ y^2 - x^2$$

$$2u\,\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\ (y^2 - x^2)$$

$$2u\,\frac{\partial u}{\partial x} = -2x$$

$$\left(\frac{\partial u}{\partial x}\right)_y = \frac{-2x}{2u} = \frac{x}{u} = \ -\left[\frac{utanv}{u}\right]$$

$\left(\frac{\partial u}{\partial x}\right)_y = \ -tanv$ -----(1)

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differentiate u^2 w.r.t. y , keeping x as constant

$$2u\left(\frac{\partial u}{\partial y}\right)_x = \frac{\partial}{\partial y}(y^2 - x^2) = 2y$$

$$\left(\frac{\partial u}{\partial y}\right)_x = \frac{y}{u} = \frac{usecv}{u} = secv \text{-----(2)}$$

Now to find v as function of x,y

$$\frac{x}{y} = \frac{utanv}{usecv} = \frac{sinv}{cosv\left(\frac{1}{cosv}\right)} = sinv$$

$$\frac{x}{y} = sinv \text{-------(I)}$$

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Differentiate eq(1) w.r.t.x, keeping y as constant

$$\frac{1}{y}\frac{\partial}{\partial x}(x) = \frac{\partial}{\partial x}(sinv)$$

$$\frac{1}{y} = cosv \; \left(\frac{\partial v}{\partial x}\right)_y$$

$$\left(\frac{\partial v}{\partial x}\right)_y = \frac{1}{ycosv} \text{ -----(3)}$$

Differentiate (I) w.r.t.y keeping x as constant

$$x.\frac{\partial}{\partial y}\left(\frac{1}{y}\right) = \frac{\partial}{\partial y}(sinv)$$

$$\frac{-x}{y^2} = cosv.\left(\frac{\partial v}{\partial y}\right)_x$$

$$\left(\frac{\partial v}{\partial y}\right)_x = \frac{-x}{y^2 cosv}\text{-----}(4)$$

From eq(1) and (3) we get,

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial v}{\partial x}\right)_y = -tanv\frac{1}{ycosv} = \frac{-sinv}{ycos^2v}$$

And from eq(2) and(4) ,we get

$$\left(\frac{\partial u}{\partial y}\right)_x \left(\frac{\partial v}{\partial y}\right)_x = secv.\frac{-x}{y^2cosv} = \frac{-x}{y^2cos^2v}$$

This shows that $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial v}{\partial x}\right)_y = \left(\frac{\partial u}{\partial y}\right)_x \left(\frac{\partial v}{\partial y}\right)_x$

4) If $u = 2x + 3y; v = 3x - 2y$ then find the value of

$i)$ $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v \left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u$

$ii)$ $\left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_x = \frac{13}{4}$

$iii)$ $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{4}{13}$

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Solution : given $u = 2x + 3y$----(1)

$v = 3x - 2y$---(2)

We have to find

$y \to f(x, v); v \to f(y, u)$

$u \to f(x, y); x \to f(u, v)$

Differentiate eq(1) w.r.t.x keeping y as constant

$\left(\frac{\partial u}{\partial x}\right)_y = 2$----(I)

To find $y \to f(x, v)$ From eq(2)

$y = \frac{nx - v}{m}$

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$$\therefore \left(\frac{\partial y}{\partial v}\right)_x = \left(\frac{-1}{2}\right) \text{----(II)}$$

To find $v \rightarrow f(y, u)$ from eq(1)

$x = \frac{u-3y}{2}$ put in eq(2)

$$v = n\left[\frac{u-3y}{2}\right] - 2y$$

$$\left(\frac{\partial v}{\partial y}\right)_u = \frac{3}{2}(-3) - 2 = \frac{-3^2}{2} - 2$$

$$\left(\frac{\partial v}{\partial y}\right)_u = -\left[\frac{\left(2^2+3^2\right)}{2}\right] = \left(\frac{-13}{2}\right) \text{----(III)}$$

To find $x \to f(u, v)$

We do, $m\ x\ eq(1) + n\ x\ eq(2)$

$$2u = 2^2 x + 6y$$

$$3v = 3^2 x - 6y$$

$$2u + 3v = 13x$$

$$x = \frac{2u+3v}{13}$$

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{2}{13} \text{------(IV)}$$

From (I),(II),(III),(IV)

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v \left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u = 2\left(\frac{-1}{2}\right)\left(\frac{-13}{2}\right)\frac{2}{13} = 1$$

from(II)and(III)

ii) $\left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_x = \frac{13}{4}$

from(I) and(IV)

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{4}{13}$$

University Questions

1) If $z = f(u, v)\ ; u = x^2 - 2xy - y^2 \quad and \quad v = y$

Show that $(x+y)\frac{\partial z}{\partial x} + (x-y)\frac{\partial z}{\partial y} = (x-y)\frac{\partial z}{\partial v}$

2) If $x = e^u tanv \quad and\ y = e^u secv$ find the value of

$$\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right)$$

3) If $z = e^{ax+by} f(ax - by)\ , prove\ that \quad b\frac{\partial z}{\partial x} +$

$a\frac{\partial z}{\partial y} = 2abz$

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4) If $x = \frac{r}{2}\left(e^{\theta} + e^{-\theta}\right)$ & $y = \frac{r}{2}\left(e^{\theta} - e^{-\theta}\right)$

Prove that

$(\partial x/\partial r)_{\theta} = (\partial r/\partial x)_{y}$

5) If $ux + vy = 0$ & $\frac{u}{x} + \frac{v}{y} = 1$ *prove that*

$$\left(\frac{\partial u}{\partial x}\right)_y - \left(\frac{\partial v}{\partial y}\right)_x = \frac{x^2+y^2}{y^2-x^2}$$

UNIT III: Partial Differentiation

Topic: Euler's Theorem

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HOMOGENEOUS FUNCTIONS

A function z of two variables x and y is said to be *homogeneous function of degree 'n'* if it is possible to express z in the form

$$\mathbf{z = x^n . f\left(\frac{y}{x}\right) \quad OR \quad z = y^n . f\left(\frac{x}{y}\right)}$$

Euler's Theorem : If z is a *homogeneous* expression of two variables x and y *of degree n then ,*

$$x\frac{\partial \mathbf{z}}{\partial \mathbf{x}} + \mathbf{y}\frac{\partial \mathbf{z}}{\partial \mathbf{y}} = \mathbf{n\,z}$$

$$z = x^n f\left(\frac{y}{x}\right) \Rightarrow \frac{\partial z}{\partial x} = x^n f\,'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + nx^{n-1} f\left(\frac{y}{x}\right)$$

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$$\therefore x\frac{\partial z}{\partial x} = -yx^{n-1}f'\left(\frac{y}{x}\right) + nx^n f\left(\frac{y}{x}\right) - - - (1)$$

$$\frac{\partial z}{\partial y} = x^n f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$y\frac{\partial z}{\partial y} = yx^{n-1}f'\left(\frac{y}{x}\right) - - - - (2)$$

Adding (1) and (2)

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = -yx^{n-1}f'\left(\frac{y}{x}\right) + nx^n f\left(\frac{y}{x}\right) + yx^{n-1}f'\left(\frac{y}{x}\right)$$

$$\text{Hence } x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx^n f\left(\frac{y}{x}\right) = nz$$

Example On Euler's Theorem

1) $If\ T = sin\left[\frac{xy}{x^2+y^2}\right] + \sqrt{x^2 + y^2} + \frac{x^2y}{x+y}$ then find the value of $x\frac{\partial T}{\partial x} + y\frac{\partial T}{\partial y}\ at\ (3,4\ )$

Solution : $T = sin\left[\frac{xy}{x^2+y^2}\right] + \sqrt{x^2 + y^2} + \frac{x^2y}{x+y}$

Let T = U +V + W

Where $U = sin\left[\frac{xy}{x^2+y^2}\right]$; $V = \sqrt{x^2 + y^2}$; $W = \frac{x^2y}{x+y}$

Put $x = xt; y = yt; in\ U, V, W\ we\ get$

$$U = sin\left[\frac{xt\ .yt}{(xt)^2 + (yt)^2}\right] = t^0\ sin\left[\frac{xy}{x^2+y^2}\right]$$

$$V = \sqrt{(xt)^2 + \ (yt)^2} \ = \ t^1\sqrt{x^2 + \ y^2} = t\ V$$

$$W = \ \frac{xt^{2\cdot}yt}{xt + yt} = \ t^2\ \frac{x^2y}{x+y} = \ t^2w$$

Thus U, V, W are homogeneous functions of degree 0,1,2 respectively

Thus by Euler's theorem

$$x\frac{\partial U}{\partial x}+y\frac{\partial U}{\partial y}=nU=0.U=0 \text{------}(1)$$

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}=nV=1.V=V \text{---------}(2)$$

$$x\frac{\partial W}{\partial x}+y\frac{\partial W}{\partial y}=nW=2.w=2W \text{------}(3)$$

Adding,We get

$$x\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial x}+\frac{\partial W}{\partial x}\right)+y\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial y}\right)=0+1V+2W \text{------}(4)$$

But as T = U +V + W

- Differentiating T w.r.t. x & y

$$\therefore \frac{\partial T}{\partial x} = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial x} + \frac{\partial W}{\partial x}$$

$$\frac{\partial T}{\partial y} = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial y}$$

Equation (4) becomes

$$x.\frac{\partial T}{\partial x} + y.\frac{\partial T}{\partial y} = V + 2W$$

$$\therefore x.\frac{\partial T}{\partial x} + y.\frac{\partial T}{\partial y} = \sqrt{x^2 + y^2} + 2\,\frac{x^2 y}{x+y}$$

At (x , y) = (3,4)

$$x.\frac{\partial T}{\partial x} + y.\frac{\partial T}{\partial y} = (5) + 2\left(\frac{36}{7}\right) = \frac{107}{7}$$

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Deductions From Euler's Theorem :

(1) If z is a *homogeneous* expression of two variables x and y *of degree n then ,*

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)\,z$$

(2) If z is a *homogeneous* function of two variables x and y of degree n and z = f(u) then ,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\,\frac{f(u)}{f'(u)}$$

$$(3)\quad x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=g(u)[g'(u)-1]$$

Where $g(u)=n\,\frac{f(u)}{f'(u)}$

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HOMOGENEOUS FUNCTIONS OF THREE VARIABLES.

A function u = f(x , y , z) is said to be homogeneous function of three variables x , y , z of degree 'n' if it is possible to express u in the form

$$\mathbf{u = x^n . f\left(\frac{y}{x}, \frac{z}{x}\right) \quad OR \quad u = y^n . f\left(\frac{x}{y}, \frac{z}{y}\right)}$$

Example On Deduction (1)

1) $If\ z = x^n f\left(\frac{y}{x}\right) + y^{-n}\emptyset\left(\frac{x}{y}\right)$ then prove that

$x^2 z_{xx} + 2xy\, z_{yy} + x.z_x + y.z_y = n^2 z$

Solution: Given $z = x^n f\left(\frac{y}{x}\right) + y^{-n}\emptyset\left(\frac{x}{y}\right)$

$Z = U + V$ --------(1)

Where $U = x^n f\left(\frac{y}{x}\right)$; $V = y^{-n}.\emptyset\left(\frac{x}{y}\right)$

$\therefore$ U and V are homogeneous functions in x & y of degree n and –n respectively

$\therefore$ By Euler's theorem

$$x\,U_x + y\,U_y = nU \text{ ------(2)}$$

$$x\,V_x + y\,V_y = -nV \text{ -------(3)}$$

And

$$x^2U_{xx} + 2xyU_{xy} + y^2U_{yy} = n\,(n-1)U \text{ -------(4)}$$

$$x^2V_{xx} + 2xyV_{xy} + y^2V_{yy} = -n\,(-n-1)V$$

$$= n(n+1) \text{ -------(5)}$$

Adding (2) ,(3) ,(4) and (5) we get

$$x^2(U_{xx}+V_{xx})+2xy\left(U_{xy}+V_{xy}\right)$$
$$+\quad y^2\left(U_{yy}+V_{yy}\right)+x(U_x+V_x)+y\left(U_y+V_y\right)$$
$$=n(n-1)U+n(n+1)V+nU-nV$$
$$=n^2U-nU+n^2V+nV+nU-nV$$
$$=n^2(U+V)\text{-------}(6)$$

Now from equation (1)

$Z=U+V$

Differentiating Z w.r.t. x & y

$Z_x = U_x + V_x$ and $Z_y = U_y + V_y$

$$\therefore Z_{xy} = U_{xy} + V_{xy}; Z_{xx} = U_{xx} + V_{xx};$$
$$Z_{xy} = U_{yy} + V_{yy};$$

$$x^2 Z_{xx} + 2xy\, Z_{xy} + y^2 Z_{yy} + x\, Z_x + y Z_y = n^2 Z$$

Equation (6) becomes

EXAMPLES On Deduction (2)

EX 1) : If $x = \mathrm{e^u tanv}$ and $\mathrm{y = \ e^u \sec v}$ find the value of

$$\left(\mathrm{x}\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \mathrm{y}\frac{\partial \mathrm{u}}{\partial \mathrm{y}}\right)\left(\mathrm{x}\frac{\partial \mathrm{v}}{\partial \mathrm{x}} + \mathrm{y}\frac{\partial \mathrm{v}}{\partial \mathrm{y}}\right) = 0$$

Eliminating v between the given relation

$$y^2 - \mathrm{x}^2 = \mathrm{e}^{2\mathrm{u}} = \mathrm{z}\ (\mathrm{say})$$

Then $z = x^2\left(\frac{y^2}{x^2} - 1\right) = x^2 f\left(\frac{y}{x}\right)$ $\ \ and\ also\ z = f(u) = e^{2u}$

$\Rightarrow z\ is\ a\ Homogeneous\ function\ of\ x\ and\ y\ of\ degree\ 2.$

$\therefore$ from deduction of Euler$'$s theorem

$$x\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \mathrm{y}\frac{\partial \mathrm{u}}{\partial \mathrm{y}} = n\ \frac{f(u)}{f'(u)} = 2\frac{e^{2u}}{2e^{2u}} = 1\ ---(1)$$

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Now eliminating u between the given relation

$$\frac{y}{x} = \frac{\sec v}{\tan v} = \operatorname{cosec} v = z \text{ (say)}, \text{Here } z = x^0 \left(\frac{y}{x}\right), \text{also } z = g(v) = \operatorname{cosec} v$$

$\Rightarrow$ *z is a Homogeneous function of x and y of degree* 0.

$\therefore$ from deduction of Euler's theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = n\ \frac{g(v)}{g'(v)} = 0 \times \frac{cosec\ v}{-cot^2 v} = 0\ ---(2)$$

$\therefore$ from (1)and (2)we get the required result.

$$\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) = 1 \times 0 = 0$$

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2) $If\ u = \ sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)\ then\,prove\ that$

$$2x\ \frac{\partial u}{\partial x} + 2y\ \frac{\partial u}{\partial y} = \tan u$$

Solution : Given $u = \ sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ --------(1)

Put $x = xt : y = yt\ in\ equation\ (1) we\ get$

$$u = \ sin^{-1}\left(\frac{xt\ + yt}{\sqrt{xt} + \ \sqrt{yt}}\right) = \ sin^{-1}\left[t^{1/2}\left(\frac{x + y}{\sqrt{x} + \ \sqrt{y}}\right)\right]$$

Is not homogeneous function

But $f(u) = \sin u = t^{1/2}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right) = t^{1/2}u$ is homogeneous function in x & y of degree ½

Thus, by Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = \frac{1}{2}\frac{\sin u}{\cos u}$$

$$\therefore 2x\frac{\partial u}{\partial x} + 2y\frac{\partial u}{\partial y} = \tan u$$

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EX : If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, Show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (\,1 - 4sec^2 u\,)\sin 2u$$

Given $\tan u = \frac{x^3+y^3}{x-y} = x^2\left[\frac{1+y^3/x^3}{1-y/x}\right] = x^2 f\left(\frac{y}{x}\right) = z(say)$

Also z = f (u) = tan u

Homogeneous function of x and y of degree 2.

By $\text{Euler}'\text{s theorem}$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\,\frac{f(u)}{f'(u)} = 2\,\frac{\tan u}{\sec^2 u}$$

$$= 2\sin u\cos u = g(u)say - -(1)$$

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$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=g(u)[g'(u)-1],\ \text{Where } g(u)=n\ \frac{f(u)}{f'(u)}$$

$$=\sin 2u\,[2\cos 2u-1]$$

$$=\sin 2u\,[2(1-2\sin^2 u)-1]$$

$$=\sin 2u\,[1-4\sin^2 u]$$

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EX : If $T = \sin\left(\frac{xy}{x^2+y^2}\right) + \sqrt{x^2+y^2} + \frac{x^2 y}{x+y}$ then

find $x\frac{\partial T}{\partial x} + y\frac{\partial T}{\partial y}$

Let $T = u + v + w$ Then

$$x\frac{\partial T}{\partial x} + y\frac{\partial T}{\partial y} = \left(x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y}\right) + \left(x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y}\right) + \left(x\frac{\partial W}{\partial x} + y\frac{\partial W}{\partial y}\right) - -(1)$$

$$u = \sin\left(\frac{xy}{x^2+y^2}\right) \Rightarrow \sin^{-1} u = \frac{xy}{x^2+y^2}$$

$$= \frac{x^2(y/x)}{x^2\left(1+\frac{y^2}{x^2}\right)} = x^0 f\left(\frac{y}{x}\right) = z$$

and also $\sin^{-1} u = z$

u is homogeneous function of x, y order 0.

By Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\ \frac{f(u)}{f'(u)} = 0 \ \text{-}-(1)$$

$$v = \sqrt{x^2 + y^2} = x\sqrt{1 + \frac{y^2}{x^2}} = xf\left(\frac{y}{x}\right)$$

It is homogeneous function of x, y, of order 1

By Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\,v = 1.\sqrt{x^2 + y^2} = \sqrt{x^2 + y^2} - -(2)$$

$$w = \frac{x^2 y}{x+y} = \frac{x^3(y/x)}{x(1+y/x)} = x^2 f\left(\frac{y}{x}\right)$$

So w is homogeneous function of x ,y of order 2.

By Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\,v = 2.\frac{x^2 y}{x+y} - -(3)$$

Putting the values from (1) , (2) and (3)

$$x\frac{\partial T}{\partial x} + y\frac{\partial T}{\partial y} = 0 + \sqrt{x^2 + y^2} + \frac{2x^2 y}{x+y} = \sqrt{x^2 + y^2} + \frac{2x^2 y}{x+y}$$

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Example On Deduction (3)

- $If\ u = cosec^{-1}\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$

Put $x = xt$; $y = yt\ in\ u$,we get

$$u = cosec^{-1}\sqrt{\frac{(xt)^{1/2}+(yt)^{1/2}}{(xt)^{1/3}+(yt)^{1/3}}}$$

$$u = cosec^{-1}\sqrt{\frac{t^{1/2}\left(x^{1/2}+y^{1/2}\right)}{t^{1/3}\left(x^{1/3}+y^{1/3}\right)}}$$

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$$u = cosec^{-1}\sqrt{\frac{t^{1/4}\sqrt{x^{1/2}+y^{1/2}}}{t^{1/6}\sqrt{x^{1/3}+y^{1/3}}}}$$

$$u = cosec^{-1}\left[t^{1/2}\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}\right]$$

U is not homogeneous function

But $f(u) = cosecu = t^{1/2}\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$

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is homogeneous function in x & y of degree ½

Thus, by Euler's theorem on homogeneous function, we get

$x^2u_{xx} + 2xy\, u_{xy} + \ y^2u_{yy} = G(u)[G'(u) - 1\,]$ -----(1)

Where $G(u) = \ \frac{n\,f(u)}{f'(u)} = \ \frac{1}{12}.\frac{cosec\,u}{-cosecu.\cot u} = \ \frac{-1}{12}\ sec^2u$

Equation (1) becomes

$x^2u_{xx} + 2xy\, u_{xy} + \ y^2u_{yy} = \ \frac{-1}{12}\ tanu\left[\frac{-1}{12}\ sec^2u - 1\right]$

$$= \frac{1}{144} \tan u \, [sec^2u + 12]$$

$$= \frac{1}{144} \tan u \ \ [tan^2u + 1 + 12]$$

$$= \ \ \frac{1}{144} \tan u \ \ [tan^2u + 13]$$

TOTAL DERIVATIVE

If $u = f(x\,,\; y)$, where $x = \emptyset(t)$ and $y = \psi(t)$ then

u can be expressed as function of single variable t.

and derivative of u w.r.t. t is ordinary differential coefficient $\frac{du}{dt}$

This $\frac{du}{dt}$ is called as Total Derivative.

If we put $t = x$ in (1) then

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

Similarly if we put $t = y$ in (1) then

$$\frac{du}{dy} = \frac{\partial u}{\partial x}\frac{dx}{dy} + \frac{\partial u}{\partial y}$$

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DIFFERENTIATION OF IMPLICIT FUNCTION

If $z = f(x, y) = 0$, y is function of x then

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\frac{dx}{dx} + \frac{\partial z}{\partial y}\frac{dy}{dx} \quad \text{i.e.} \quad \frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} - - - (1)$$

Since $z = f(x, y) = 0 \Longrightarrow \frac{dz}{dx} = 0$

$$\therefore \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} = 0 \Longrightarrow \frac{dy}{dx} = -\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{p}{q}$$

as $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x\ \partial y}$, $s = \frac{\partial^2 z}{\partial x}$, $t = \frac{\partial^2 z}{\partial y}$

or $p = \frac{\partial f}{\partial x}$, $q = \frac{\partial f}{\partial y}$, $r = \frac{\partial^2 f}{\partial x\ \partial y}$, $s = \frac{\partial^2 f}{\partial x}$, $t = \frac{\partial^2 f}{\partial y}$

Differentiating once more we get

$$\therefore \quad \frac{d^2y}{dx^2} = -\left[\frac{q^2r - 2pqs + p^2t}{q^3}\right]$$

This can also be remembered as

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{\det\begin{bmatrix} r & s & p \\ s & t & q \\ p & q & 0 \end{bmatrix}}{q^3}$$

EXAMPLES.

EX : If $u = x \log xy$ and $x^3 + y^3 + 3xy = 0$ then find $\frac{du}{dx}$

Given $u = x \log xy$ and $x^3 + y^3 + 3xy = 0$

u is composite function of x

$$\therefore \quad \frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx} - - - (1)$$

$$u = x \log xy \ \text{ and } \ f(x\,,y) \equiv x^3 + y^3 + 3xy = 0$$

$$\frac{\partial u}{\partial x} = x\left(\frac{1}{xy}\right) y + \log xy = 1 + \log xy - - - (2)$$

$$\frac{\partial u}{\partial y} = x\left(\frac{1}{xy}\right) x = \frac{x}{y} - - - \quad (3)$$

$$\frac{dy}{dx} = -\frac{p}{q} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{3x^2+3y}{3y^2+3x} \quad - - - (4)$$

Putting the values from (2) , (3) and (4) in (1)

$$\frac{du}{dx} = 1 + \log xy + \frac{x}{y}\left(-\frac{3x^2 + 3y}{3y^2 + 3x}\right)$$

$$= 1 + \log xy - \left(\frac{x^3 + xy}{y^3 + xy}\right)$$

$$= \log xy + \frac{y^3 + xy - x^3 - xy}{y^3 + xy}$$

$$\therefore \quad \frac{du}{dx} = \log xy + \frac{y^3 - x^3}{y^3 + xy}$$

EX: If $(\cos x)^y = (\sin y)^x$ then find $\frac{dy}{dx}$

EXAMPLES ON TOTAL DERIVATIVES

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Ex : If $x^2y - e^z + x\sin z = 0$ and $x^2 + y^2 + z^2 = a^2$

Then evaluate $\frac{dy}{dx}$ *and* $\frac{dx}{dz}$

Let $f(x,y,z) = x^2y - e^z + x\,\text{sin}z = 0$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$0 = (2xy + \sin z)dx + (x^2)dy + (-e^z + x\cos z)dz \text{---} (1)$$

Let $\emptyset(x,y,z) = x^2 + y^2 + z^2 - a^2 = 0$

$$d\emptyset = \frac{\partial \emptyset}{\partial x}dx + \frac{\partial \emptyset}{\partial y}dy + \frac{\partial \emptyset}{\partial z}dz$$

$$0 = x\,dx + y\,dy + z\,dz - - - -(2)$$

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From (1) and (2) by Cramer's rule for dx , dy , dz

$$\frac{dx}{y(-e^z + x\cos z) - zx^2} = \frac{dy}{z(2xy + \sin z) - x(-e^z + x\cos z)}$$
$$= \frac{dz}{x^3 - y(2xy + \sin z)}$$

$$\frac{dy}{dx} = \frac{z(2xy + \sin z) - x(-e^z + x\cos z)}{y(-e^z + x\cos z) - zx^2}$$

$$\frac{dx}{dz} = \frac{y(-e^z + x\cos z) - zx^2}{x^3 - y(2xy + \sin z)}$$

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2) If $u = \text{xlog}(xy)$ and $x^3 + y^3 + 3xy = 0$ then find $\frac{du}{dx}$ at (1,1)

Solution: Given $u = x\log(xy) = x[logx + logy]$

&

$x^3 + y^3 + 3xy = 0$

By differentiating u w.r.t.x by chain rule

$\frac{du}{dx} = \frac{\partial u}{\partial x} . \frac{dx}{dx} + \frac{\partial u}{\partial y} . \frac{dy}{dy}$ ------(1)

$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} x[logx + logy] = 1 + logx + logy$

And

$$\frac{\partial u}{\partial y} = \frac{x}{y}$$

Eq(1) becomes

$$\frac{du}{dx} = [1 + logx + logy] + \frac{x}{y}.\frac{dy}{dx} \text{ ------(2)}$$

As $f = x^3 + y^3 + 3xy - 1$

We have $\frac{dy}{dx} = -\left[\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\right] = -\left[\frac{3x^2+3y}{3y^2+3x}\right]$=

$$= -\left[\frac{x^2+y}{y^2+x}\right]$$

$eq(2)$ becomes

$$\frac{du}{dx} = [1 + logx + logy] + \frac{x}{y}$$

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$$=[1 + logx + logy] - \frac{x}{y}.\left[\frac{x^2+y}{y^2+x}\right]$$

$$\left(\frac{du}{dx}\right)_{(1,1)} = (1 + 0 + 0) - \frac{1}{1}\left[\frac{1+1}{1+1}\right]$$

$$\left(\frac{du}{dx}\right)_{(1,1)} = 1-1$$

$$\left(\frac{du}{dx}\right)_{(1,1)} = 0$$

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3) If $f(x, y) = 0; \emptyset(x, z) = 0$ then prove that

$$\frac{\partial \emptyset}{\partial x}\frac{\partial f}{\partial y}\frac{dy}{dz} = \frac{\partial f}{\partial x}\frac{\partial \emptyset}{\partial z}$$

Solution : Given

$f(x, y) = 0; \emptyset(x, z) = 0$

$f \to x, y \to x \;\&\; \emptyset \to x, z \to x$

Means $f \to x \;\&\; y \to x$

$$\emptyset \rightarrow x \,\&\, z \rightarrow x$$

Thus by chain rule

$$df = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

$$df = \frac{\partial \emptyset}{\partial x}\frac{dx}{dx} + \frac{\partial \emptyset}{\partial y}\frac{dy}{dx}$$

$$f = 0 \; and \; \emptyset = 0$$

Thus df = 0 and d$\emptyset = 0$

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$$\therefore\ \frac{dy}{dx} = -\left[\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\right] \text{ And } \frac{dz}{dx} = -\left[\frac{\frac{\partial \emptyset}{\partial x}}{\frac{\partial \emptyset}{\partial z}}\right]$$

$$\frac{\left(\frac{dy}{dx}\right)}{\left(\frac{dz}{dx}\right)} = \frac{-\left[\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\right]}{-\left[\frac{\frac{\partial \emptyset}{\partial x}}{\frac{\partial \emptyset}{\partial z}}\right]}$$

Therefore $\frac{dy}{dz} = \frac{\frac{\partial f}{\partial x}.\frac{\partial \emptyset}{\partial z}}{\frac{\partial \emptyset}{\partial x}.\frac{\partial f}{\partial y}}$

And $\frac{\partial \emptyset}{\partial x}\frac{\partial f}{\partial y}\frac{dy}{dz} = \frac{\partial f}{\partial x}\frac{\partial \emptyset}{\partial z}$

3) Find $\frac{dz}{dx}$ $if\ z = x^2y\ \&x^2 + xy + y^2 = 1$

Solution : $z = x^2y\ \&x^2 + xy + y^2 = 1$

Differentiate z w.r.t x ,keeping y as constant and then differentiate z w.r.t y,keeping x as constant

$\frac{\partial z}{\partial x} = 2xy$ & $\frac{\partial z}{\partial y} = x^2$

As $f(x, y) = x^2 + xy + y^2 - 1$

$\frac{dy}{dx} = -\left[\frac{2x+y}{x+2y}\right]$

By chain rule $z \to x, y \to x$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\frac{dx}{dx} + \frac{\partial z}{\partial y}\frac{dy}{dx} = (2xy)(1) + (x^2)\left(-\left[\frac{2x+y}{x+2y}\right]\right)$$

Thus $\frac{dz}{dx} = 2xy - x^2\left[\frac{2x+y}{x+2y}\right]$

Ex : If $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$

Then prove that

$$\frac{dx}{bny - cmz} = \frac{dy}{clz - anx} = \frac{dz}{amx - bly}$$

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References:-

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   B. V. Raman
2) Advanced Engineering Mathematics -
   H. K. Dass
3) Advanced Engineering Mathematics --