Engineering Mathematics-II

Unit - 4

Curve tracing

❖ Topics to be covered

1. Curve tracing: Graphical representation of mathematical function makes easy to understand the function. We are going to study the general principles of curve plotting to know the approximate shape of the curve.

Method of curve tracing depends upon the representation of equation in :

- Cartesian form
- Polar form
- Parametric form

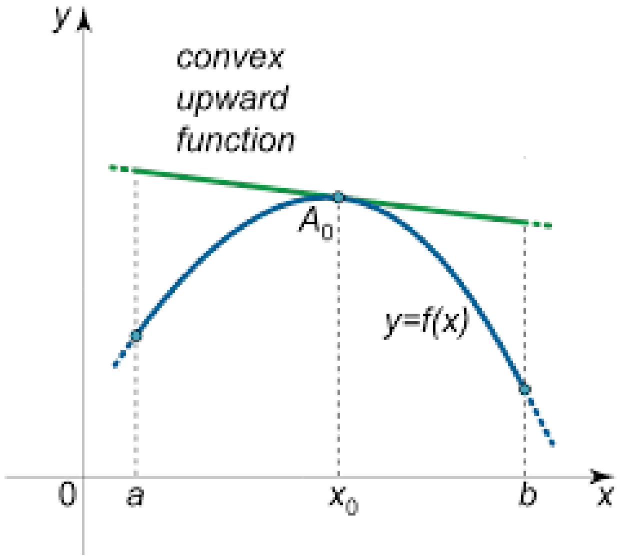
2. Rectification of curves: The procedure to measure the length of the arc of the curve is called Rectification.

We will come across the formulae to find length of arc of the curve having equation in Cartesian , Polar and Parametric form.

❖ Important terms:

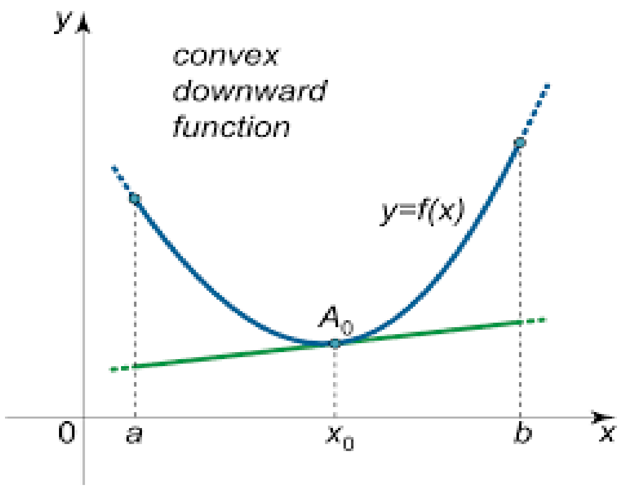
1. Convex Upwards:

If the portion of the curve on both sides of A_0 lies below the tangent at A_0 then the curve is convex upwards (Or concave downwards)



2. Convex Downwards:

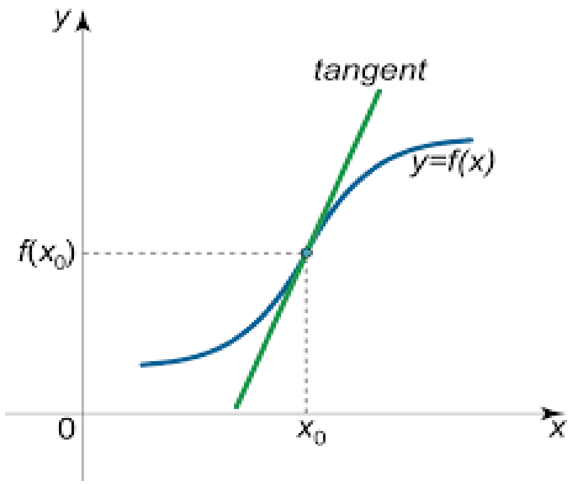
If the portion of the curve on both sides of A_0 lies above the tangent at A_0 then the curve is convex downwards (Or concave upwards)



3. Singular points: Singular points are unusual points on the curve. Those are point of inflexion,double point, multiple point,cusp,node or conjugate point.

4. Point of inflexion:

The point that separates the convex part of a continuous curve from the concave part is called the point of inflexion. In the figure, point $(x_0, f(x_0))$ is point of inflexion.

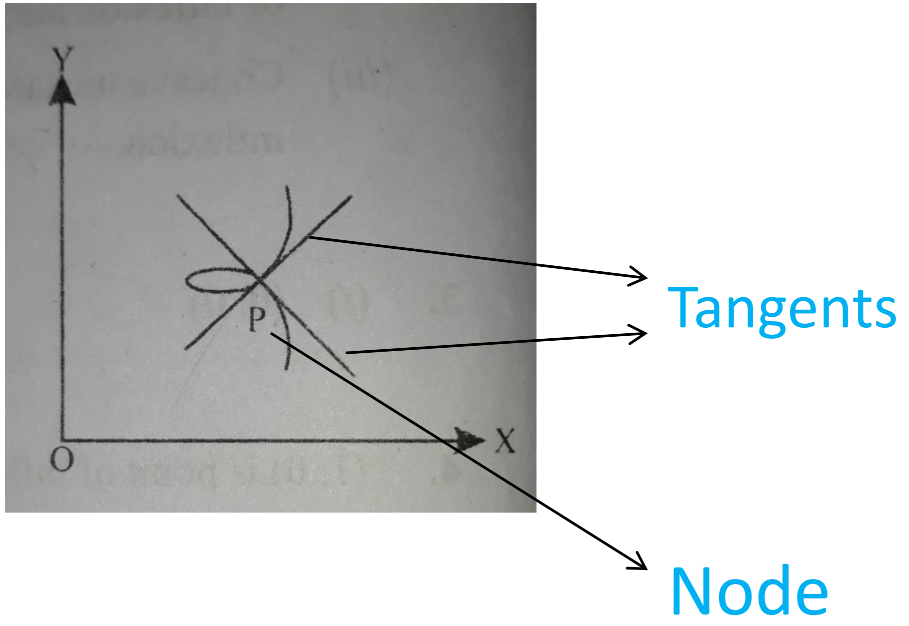


5. Multiple point: A point through which more than one branches of a curve pass is called multiple point of the curve.

6. Double point: A point through which two branches of a curve pass is called double point of the curve.

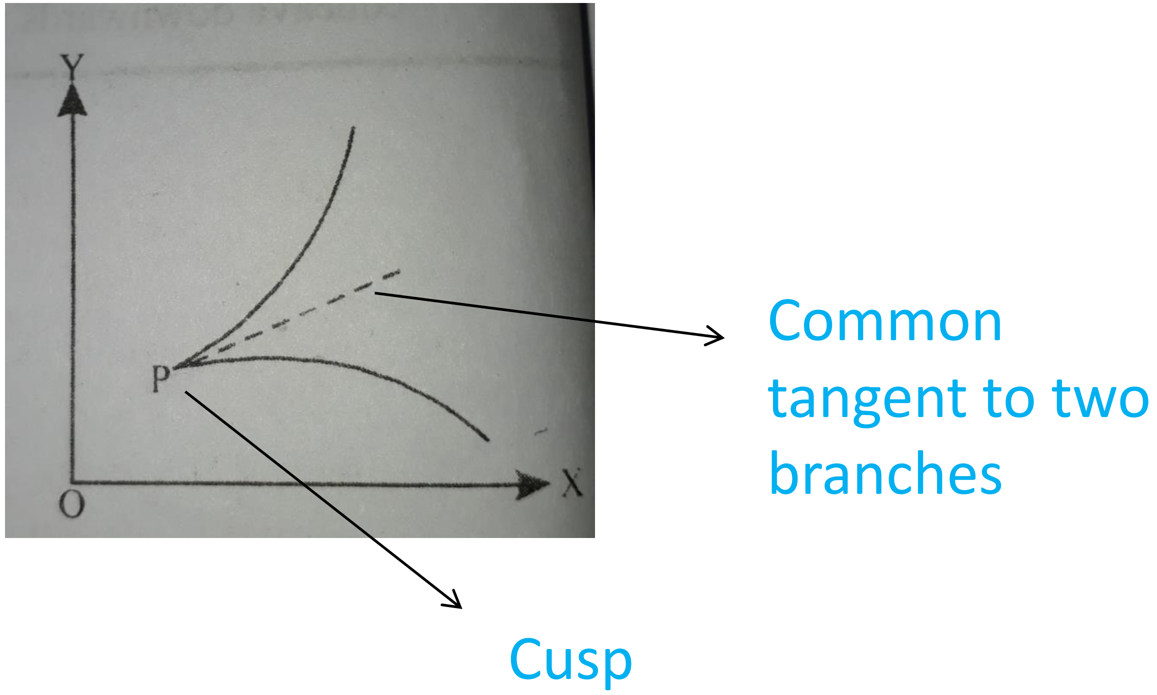
7. Node:

A double point is called node if the branches of curve passing through it are real and the tangents at the common point of intersection are not coincident(tangents are distinct).



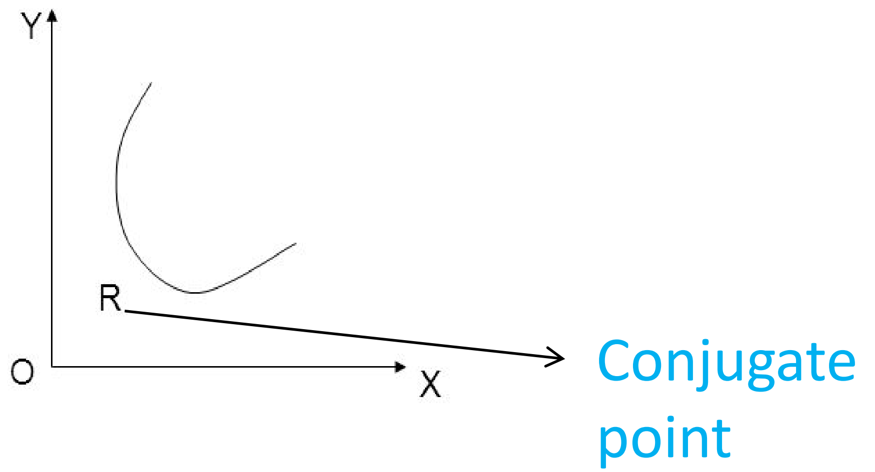
8. Cusp:

A double point is called cusp if the tangents at it to the two branches of the curve are coincident.



9. Conjugate point(Isolated point):

R is called conjugate point if there exists a neighbourhood of point R in which no real points of the curve exist. Point R is also called as isolated point.



❖ Tracing of cartesian curves

Following are the rules which help in tracing cartesian curves

Rule 1: Symmetry

(a) In the equation of curve,if the powers of of y are even everywhere,then curve is symmetrical about x-axis. For example, $y^2 = 4ax$

(b) In the equation of curve,if the powers of x are even everywhere,then curve is symmetrical about y -axis. For example, $x^2 = 4ay$

(c) In the equation of curve,if the powers of x and y both are even everywhere,then curve is symmetrical about both the axes. For example, $x^2 + y^2 = 1$

(d) If the equation of the curve remains unchanged after replacing x by $-x$ and y by $-y$ simultaneously, then curve is symmetric in opposite quadrants. For example, $y = x^3$

(e) If the equation of the curve remains unchanged when x is changed to y and y is changed to x,then curve is symmetrical about the line $y = x$. For example, $xy = c^2$

(f) If the equation of the curve remains unchanged when x is changed to $-y$ and y is changed to $-x$,then curve is symmetrical about the line $y = -x$.

Rule 2 : Points of intersection

(a) Find whether the curve passes through the origin. If equation of curve is satisfied by point $(0,0)$ then it passes through origin.

(b) Find points of intersection with co-ordinate axes.

To find the intersection with X-axis, put $y = 0$

To find the intersection with Y-axis, put $x = 0$

Find tangents at these points if necessary.

Rule 3 : Tangents at the origin

- If the curve passes through origin then tangent at the origin can be obtained by equating the lowest degree terms taken together in the equation to zero.

Rule 4 : Asymptotes

Asymptotes are the tangents to the curve at infinity.
(a) To find the equation of asymptote parallel to X-axis, equate the coefficient of highest degree terms in x to zero.
(b) To find the equation of asymptote parallel to Y-axis, equate the coefficient of highest degree terms in y to zero.

Rule 5 : Special points

Find $\frac{dy}{dx}$

(a) If $\left(\frac{dy}{dx}\right)_{P(x_{1,}y_1)} = 0$ then tangent at point $P(x_{1,}y_1)$ is parallel to X-axis.
(b) If $\left(\frac{dy}{dx}\right)_{P(x_{1,}y_1)} = \infty$ then tangent at point $P(x_{1,}y_1)$ is parallel to Y-axis.

Rule 6 : Region of absence of the curve

Find the region where either x or y becomes imaginary.

Ex.1 Trace the curve $y^2(2a-x)=x^3$

Sol.

Given equation can be written as $y^2=\frac{x^3}{2a-x}$ $\longrightarrow$ (1)

- Even powers of y everywhere, hence curve is symmetric about X-axis.
- Point (0,0) satisfies equation of curve, therefore curve passes through origin (0,0).
- To find tangent at origin, equate lowest degree terms in equation to zero.

Equation of the curve can be written as $\underbrace{2ay^2}_{\text{Degree 2}}-\underbrace{y^2x}_{\text{Degree 3}}-\underbrace{x^3}_{\text{Degree 3}}=0$

Lowest degree term is $2ay^2$ $\therefore 2ay^2=0$ $\therefore y=0$ This is an eqn. of X-axis

Hence, X-axis is tangent to the curve at origin.

- To find intersection with X-axis, put $y=0$ in (1) $\therefore\ x^3=0$ $\therefore\ x=0$

To find intersection with Y-axis, put $x=0$ in (1) $\therefore\ y^2=0$ $\therefore\ y=0$

Thus the curve meets the co-ordinate axis only at (0,0).

- If we put $x = 2a$ in eqn.(1) then $y = \infty$

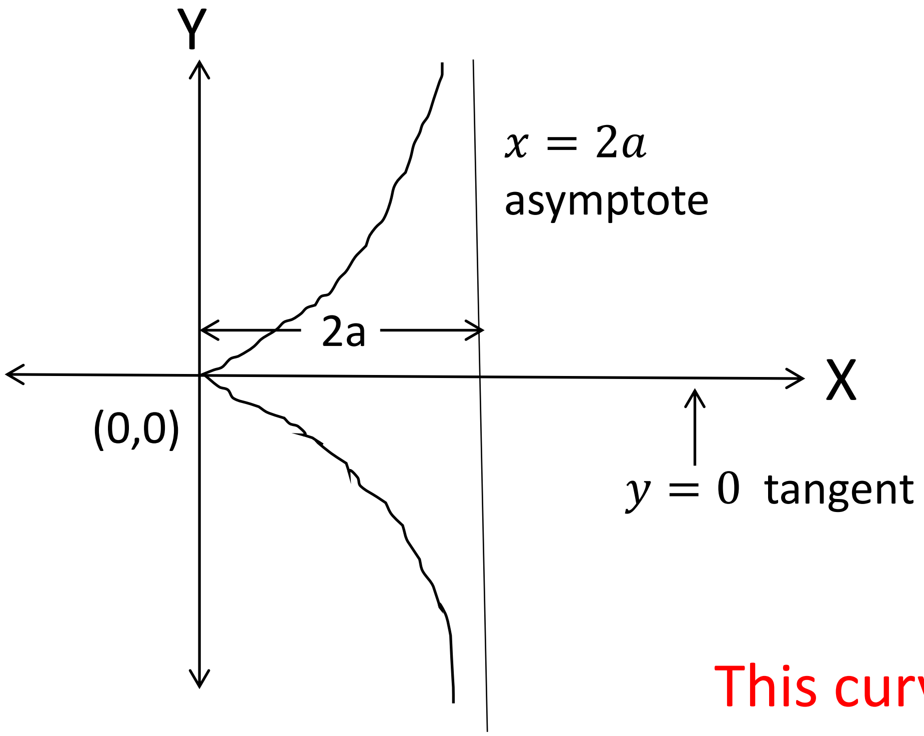
  Thus line $x = 2a$ is asymptote

- In eqn. (1) ,if we take $x < 0$ then numerator $x^3 < 0$. So we get $y^2 < 0$ which is imaginary.

  Also in eqn. (1) , if we take $x > 2a$ then denominator $(2a - x) < 0$. So again we get $y^2 < 0$ which is imaginary.

  Therefore curve does not exist for $x < 0$ and $x > 2a$.

We get rough sketch of the curve as shown below.



This curve is known as " The Cissoid of Diocle."

Ex. 2 Trace the curve $xy^2 = a^2(a-x)$

Sol.

Given equation can be written as $y^2 = \frac{a^2(a-x)}{x}$ $\longrightarrow$ (1)

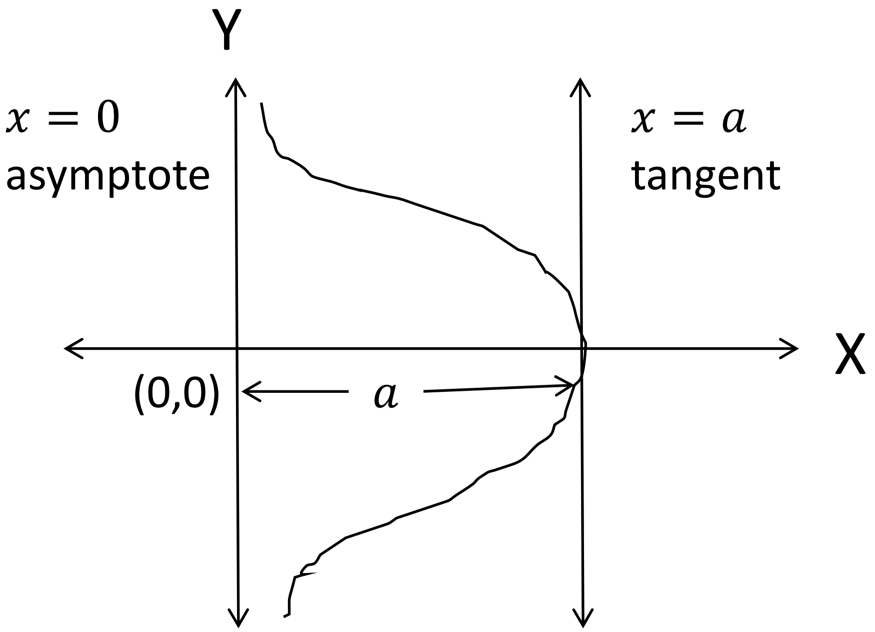
- Even powers of y everywhere, hence curve is symmetric about X-axis.
- Curve does not pass through the origin.
- To find intersection with X-axis, put $y = 0$ in (1)

  $\therefore a^2(a-x) = 0 \qquad \therefore x = a$

  Hence,curve cuts the X-axis at point $(a, 0)$.

  Line $x = a$ becomes tangent to the curve.
- If we put $x = 0$ in eqn. (1) , then $y = \infty$. Thus line $x = 0$ (Y-axis) is asymptote
- For $x < 0$ in (1) , y^2 becomes imaginary. Also if $x > a$ in (1) , then y^2 becomes imaginary.

  Therefore curve does not exist for $x < 0$ and $x > a$.



Ex.3 Find the region of absence for the curve represented by the equation $x^2 = \frac{a^2y^2}{a^2-y^2}$

Sol.

Given equation is $x^2 = \frac{a^2y^2}{a^2-y^2}$

If $y > a$, then x^2 becomes negative.

Also if $y < -a$, then x^2 becomes negative.

So, the region of absence for the curve is

$$\boxed{y > a \text{ and } y < -a}$$

Ex. 4 Find the equation of asymptotes parallel to Y-axis to the curve $x^2y^2 = a^2(y^2 - x^2)$

Sol.

Given equation can be written as

$$x^2y^2 = a^2y^2 - a^2x^2$$

$$\therefore\ a^2y^2 - x^2y^2 = a^2x^2$$

$$\therefore\ y^2(a^2 - x^2) = a^2x^2$$

$$\therefore\ y^2 = \frac{a^2x^2}{a^2 - x^2}$$

Consider denominator $a^2 - x^2 = 0$. So asymptotes are

$$\boxed{x = a\,, x = -a}$$

Solve: Trace the curve $y^2(a^2 - x^2) = a^3x$

Solve: Find the symmetry of the curve $x^2y^2 = a^2(y^2 - x^2)$

Solve: Find the region of absence for the curve $y^2 = \frac{x^2(a-x)}{a+x}$

❖ Tracing of polar curves

In polar co-ordinates,the curve is represented by the equation $r = f(\theta)$

In polar system,origin is known as pole, X-axis is known as initial line and Y-axis is called as a line perpendicular to initial line.

Following are the rules which help in tracing polar curves

Rule 1 : Symmetry

(a) If the equation of the curve remains unchanged by changing θ to $-\theta$, it will be symmetrical to the initial line.

(b) If the equation of the curve remains unchanged by changing r to $-r$, it will be symmetrical to the pole.

(c) If the equation of the curve remains unchanged by changing θ to $-\theta$ and r to $-r$ simultaneously, the curve is symmetrical to the Y-axis.

(d) If the equation of the curve remains unchanged by changing θ to $\pi - \theta$, the curve is symmetrical to the Y-axis.

Rule 2: Pole and tangents

- Put $r = 0$ in the equation of the curve. If we get at least one real value of θ for $C = 0$,the curve passes through the pole (origin).
- Such value of θ gives the tangent at the pole.

Rule 3: Prepare table showing values of r for different values of θ.

Rule 4 : Angle between the radius vector and tangent

Find $\tan \emptyset = r\frac{d\theta}{dr}$ where $\emptyset$ is an angle between the radius vector and tangent.

Rule 5 : Find region of absence of the curve

Ex.1 Trace the curve $r^2 = a^2 \cos 2\theta$

Sol.

- Points (a),(b),(c) of symmetry are satisfied for the given equation. Hence, it is symmetric about pole, initial line and Y-axis.

- Put $r = 0$ in the given equation.

$\therefore \cos 2\theta = 0$

$$\therefore \quad 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \dots$$

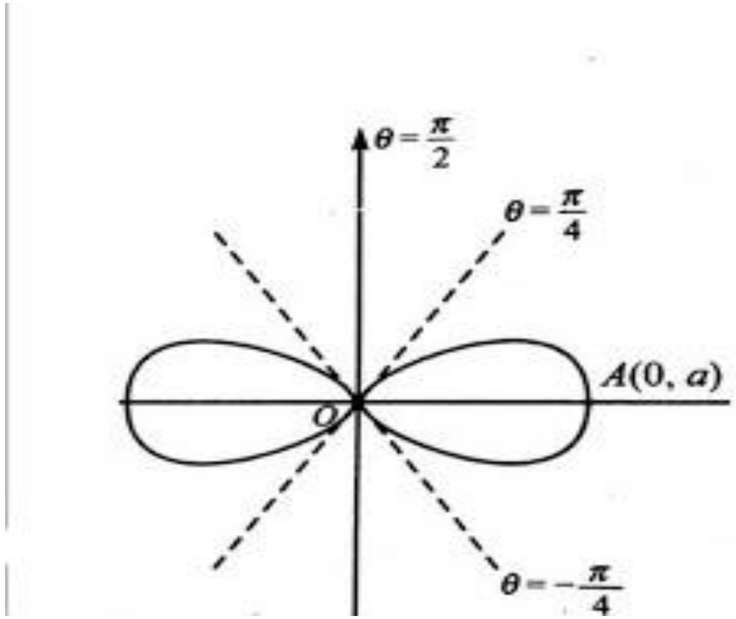
$$\therefore \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \dots$$

These values of θ represent tangent at pole.

- Following table shows values of r for different values of θ.

θ	0	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$	2π
r	a	0	0	$-a$	0	0	a

- As θ increases from $\frac{\pi}{4}$ to $\frac{3\pi}{4}$, r^2 becomes negative and r becomes imaginary. Hence curve does not exist in the region $\frac{\pi}{4} < \frac{3\pi}{4}$.
  Similarly, curve does not exist in the region $\frac{5\pi}{4} < \frac{7\pi}{4}$.
- Rough sketch of the curve is shown below



This curve is known as "Lemniscate of Bernoulli".

Ex.2 Trace the curve $r = a(1 + \cos\theta)$

Sol.

- The equation of the curve remains unchanged by changing θ to $-\theta$, hence it will be symmetrical to the initial line.

- Put $r = 0$ in the given equation.

$\therefore \cos\theta = -1 \qquad \therefore \theta = \pi$

$\theta = \pi$ (initial line itself) represents tangent at pole.

Following table shows values of r for different values of θ.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	$2a$	a	0	a	$2a$

- Angle between radius vector and tangent is $\tan\emptyset = r\frac{d\theta}{dr}$

Given, $r = a(1 + \cos\theta)$

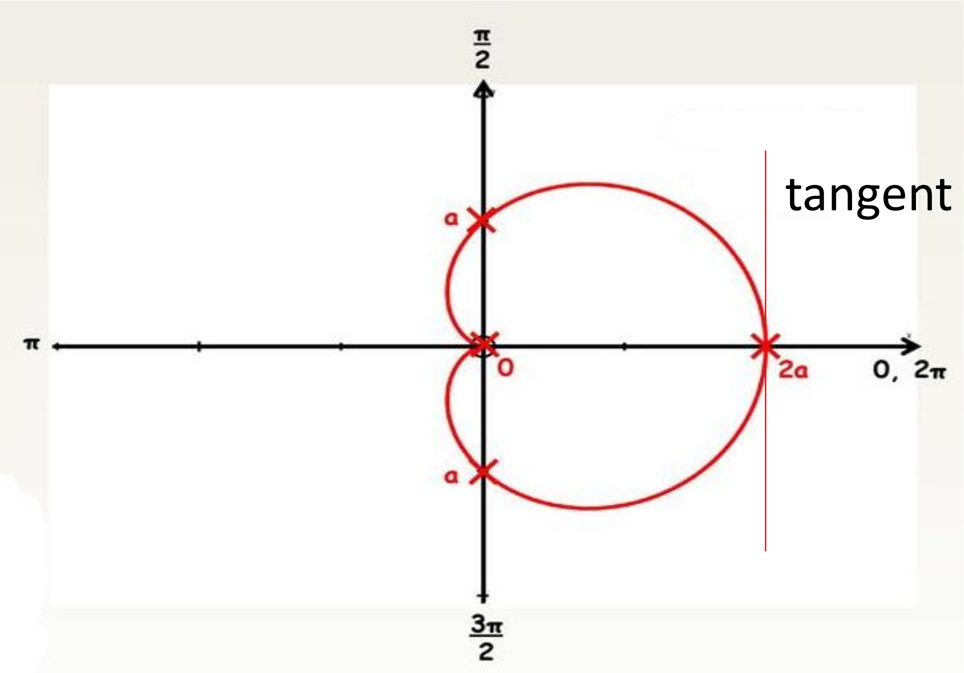
$\therefore \frac{dr}{d\theta} = -a\sin\theta$

$$\therefore \tan\emptyset = -\frac{a(1+\cos\theta)}{a\sin\theta} = -\frac{2cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = -\cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

$$\therefore \emptyset = \frac{\pi}{2}+\frac{\theta}{2}$$

At $\theta = 0$, $\emptyset = \frac{\pi}{2}$

So, tangent is perpendiculat to the initial line at $(2a, 0)$



This Heart-shaped curve is known as 'Cardioide'

❖ Rose Curves:

Polar curves of the form $r = a\sin n\theta$ or $r = a\cos n\theta$ where n is positive odd or even integer.

- Symmetry: same as that for the polar curves.
- Put $r = 0$ and find corresponding values of θ. These values of θ give tangent at the pole.
- The maximum numerical value of $\sin n\theta$ and $\cos n\theta$ is 1.Hence, the maximum numerical value of $a\sin n\theta$ and $a\cos n\theta$ is $'a'$. Therefore entire curve lies within the circle of radius $'a'$.
- For drawing loops of $r = a\sin n\theta$ or $r = a\cos n\theta$ each quadrant is divided into ‘n’ equal parts.
- $r = a\sin n\theta$ or $r = a\cos n\theta$ consists of ‘n’ equal loops if n is odd and consists of ‘2n’ equal loops if n is even.
- If n is odd, draw loops in two sectors alternately keeping two sectors vacant. If n is even, draw loops in two sectors consecutively.
- For $r = a\sin n\theta$, first loop is drawn along $\theta = \frac{\pi}{2n}$

  For $r = a\cos n\theta$, first loop is drawn along $\theta = 0$

Ex.1 Trace the curve $r = a\sin 3\theta$

Sol.

Here, n=3. (odd)

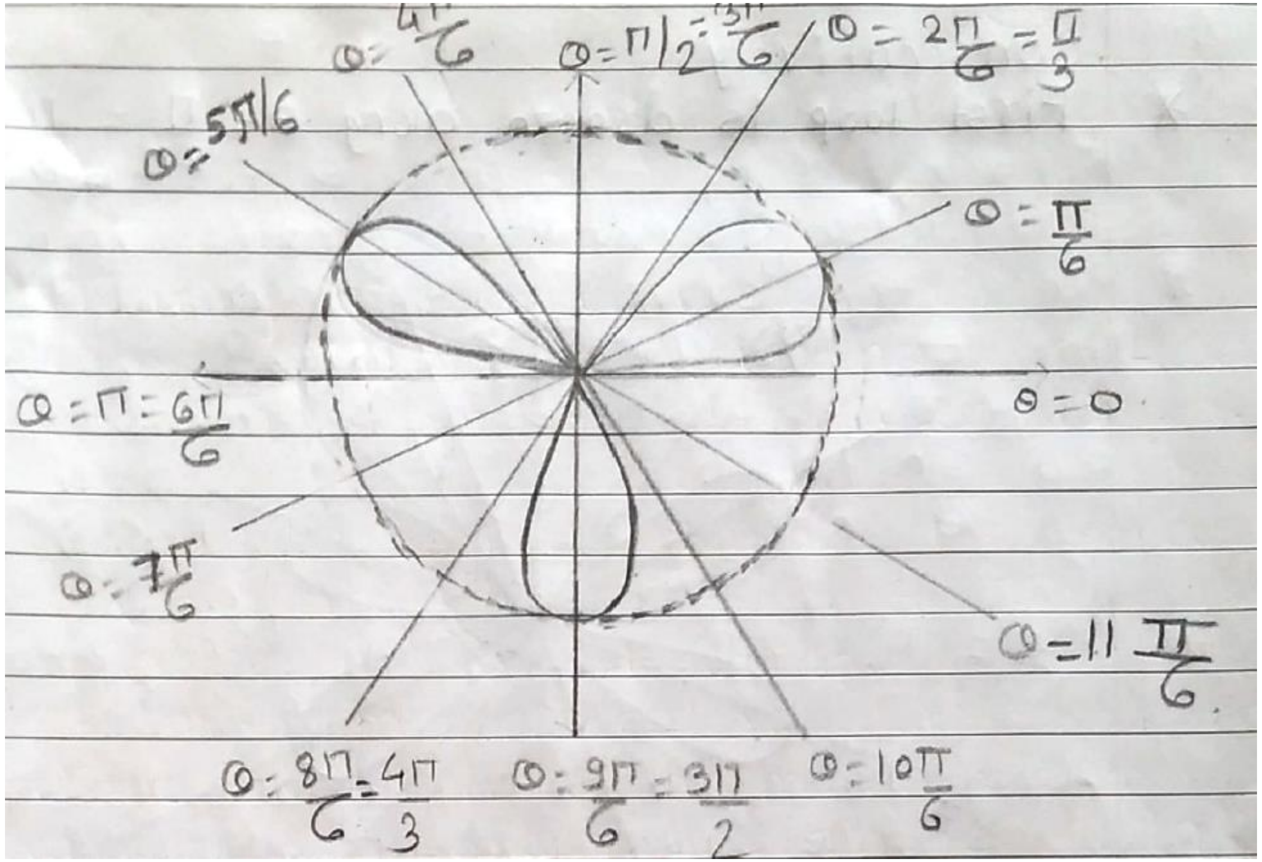
- Replacing r by $-r$ and θ by $-\theta$ simultaneously, equation remains unchanged.Hence curve is symmetrical about the Y-axis.
- Put $r = 0$ in the equation. $\therefore$ $\sin 3\theta = 0.$ We have, $\sin n\pi = 0$ for $n = 0,1,2,3 \ldots$

$$\therefore 3\theta = n\pi \quad \therefore \theta = \frac{n\pi}{3},\ n = 0,1,2,3 \ldots$$

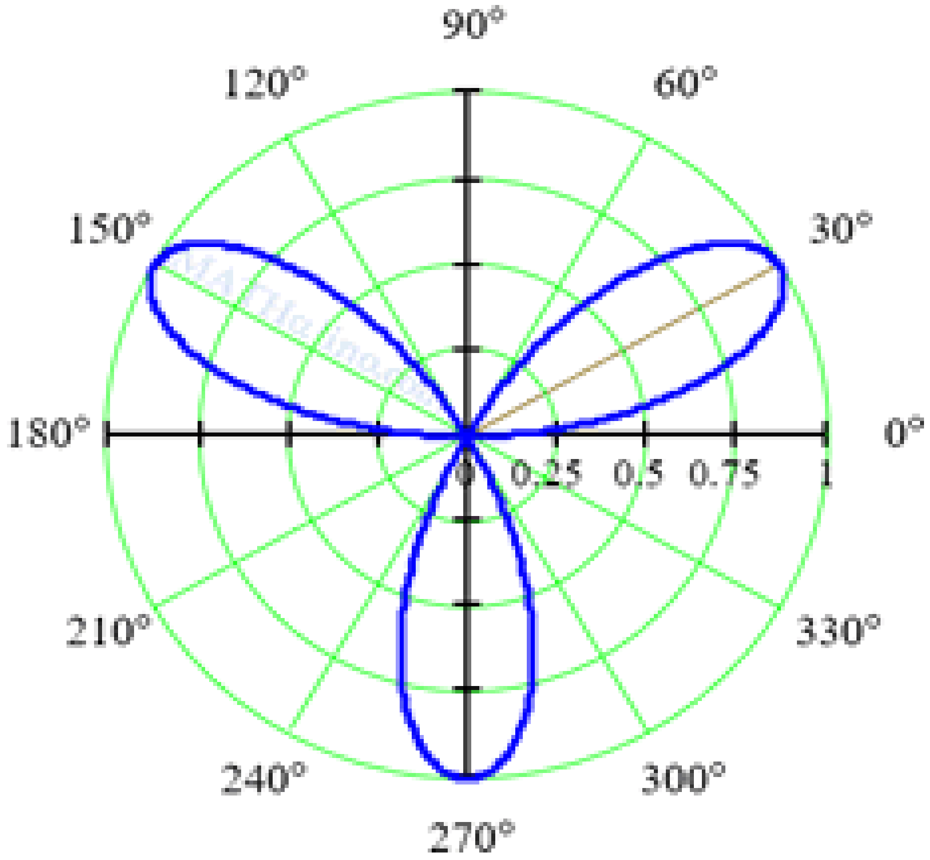
$$\therefore \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{3}, \frac{4\pi}{3} \ldots$$

Curve passes through the pole and above values of θ represent tangents at pole.

- Entire curve lies within the circle of radius a.
- n=3 , so each quadrant is divided into 3 equal parts.
- n=3 is odd. Hence, curve consists of 3 equal loops and loops are drawn in 2 sectors alternately keeping two sectors vacant.
- First loop is drawn along $\theta = \frac{\pi}{2n}$ i.e. $\theta = \frac{\pi}{2\times 3} = \frac{\pi}{6}$



θ = 4π/6
θ = π/2 = 3π/6
θ = 2π/6 = π/3
θ = 5π/6
θ = π/6
θ = π = 6π/6
θ = 0
θ = 7π/6
θ = 11 π/6
θ = 8π/6 = 4π/3
θ = 9π/6 = 3π/2
θ = 10π/6



90°
120°
60°
150°
30°
180°
0°
0
0.25
0.5
0.75
1
210°
330°
240°
300°
270°

Ex.2 Trace the curve $r = a\cos 2\theta$

Sol.

Here, n=2. (even)

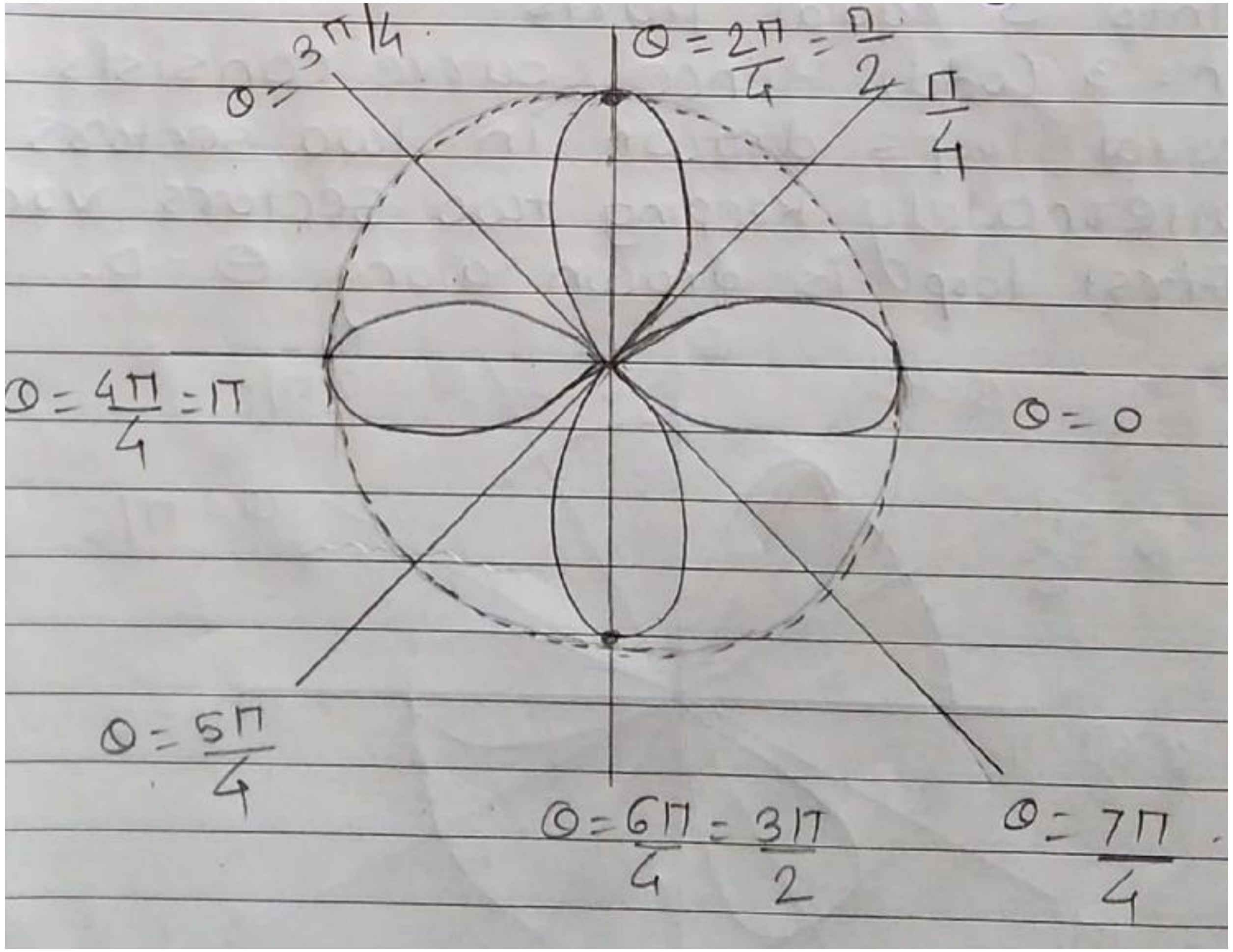
- Repalcing θ by $-\theta$ equation remains unchanged. Hence curve is symmetrical about the X-axis.(initial line)
- Put $r = 0$ in the equation. $\therefore\ \cos 2\theta = 0.$ We have, $\cos(2n+1)\frac{\pi}{2} = 0\ , n = 0,1,2,3\ldots$

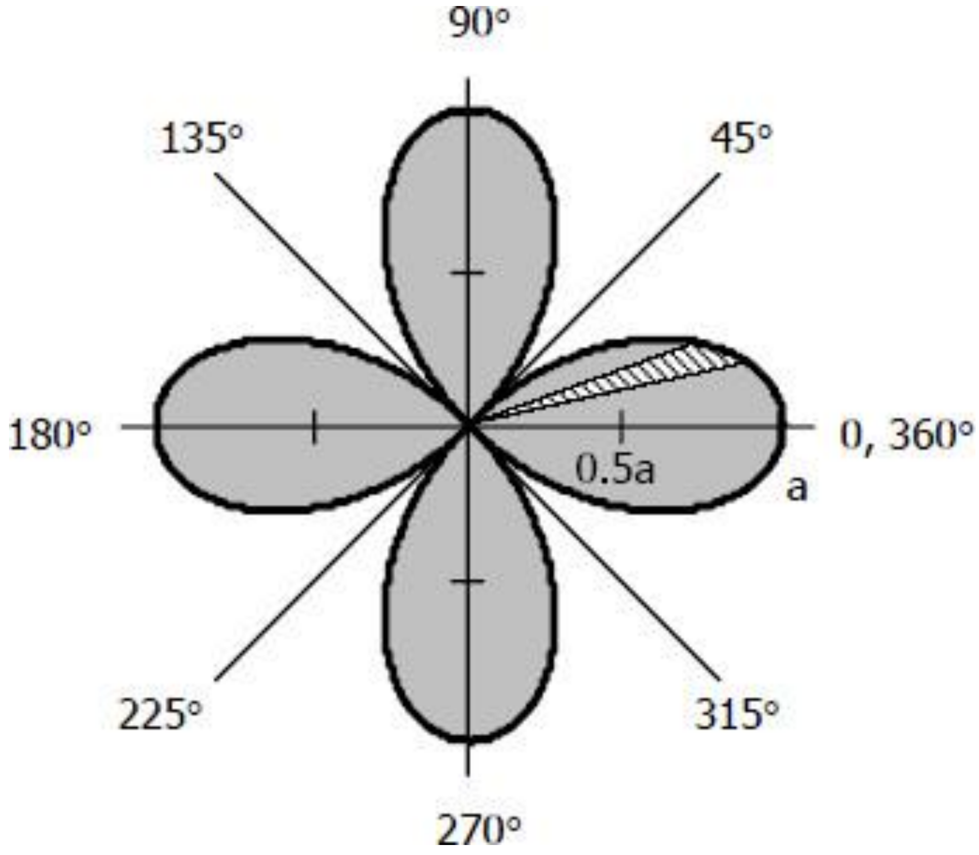
$\therefore\ 2\theta = (2n+1)\frac{\pi}{2} \quad \therefore\ \ \theta = (2n+1)\frac{\pi}{4}, n = 0,1,2,3\ldots$

$$\therefore\ \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}\ldots$$

Curve passes through the pole and above values of θ represent tangents at pole.

- Entire curve lies within the circle of radius a.
- n=2 , so each quadrant is divided into 2 equal parts.
- n=2 is even. Hence, curve consists of $2n = 2\times 2 = 4$ equal loops are drawn in 2 sectors consecutively.
- First loop is drawn along $\theta = 0$





Solve: Trace the curve $r = a + b\cos\theta$, $a > b$. (Curve is known as Pascal's Limacon)

Solve: Trace the curve $r = a\cos 3\theta$

Solve: Trace the curve $r = a\sin 4\theta$

❖ Tracing of parametric curves

Let the parametric equations of the curve are $x = f(t)\,, y = g(t)$

Rule 1: Symmetry

(a) If $f(t)$ is even function of t and $g(t)$ an odd function,then curve is symmetrical to X –axis.
(b) If $f(t)$ is odd function of t and $g(t)$ an even function,then curve is symmetrical to Y –axis.
(c) If both $f(t)$ and $g(t)$ are odd functions then curve is symmetrical in opposite quadrants.
(d) By replacing t by $\pi - t$, if y remains unchanged but x has equal and opposite values, then the curve is symmetrical to Y –axis.

Rule 2: Origin

Put $x = 0$ in the equation of the curve. If we obtain $y = 0$ for some value of t, then curve passes through the origin.

Rule 3 : Special points

Find those points where $\frac{dy}{dx} = 0$ or $\frac{dy}{dx} = \infty$. Use formula $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

Rule 4: Make a table of values of t, x, y.

Ex.1 Trace the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. (This curve is called Astroid or Star-shaped curve)

Sol.

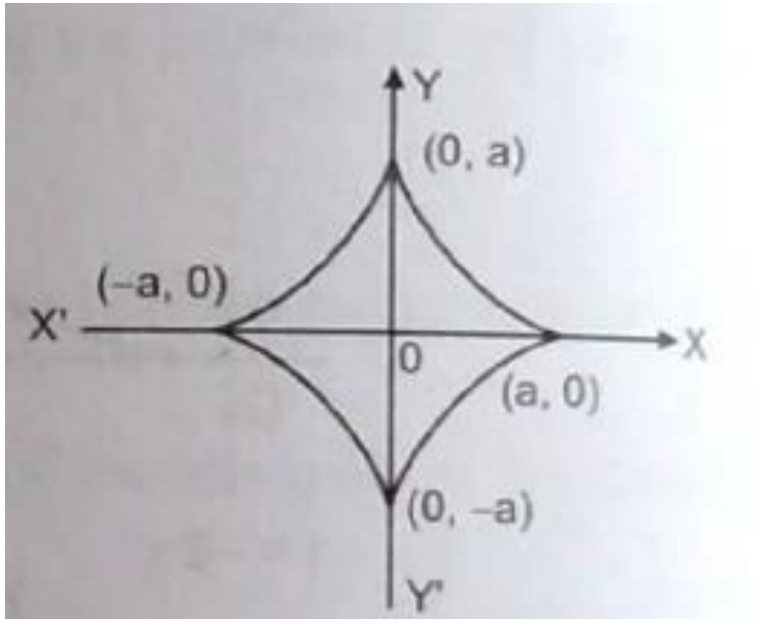
The parametric equations of the Astroid are $x = acos^3t$, $y = asin^3t$

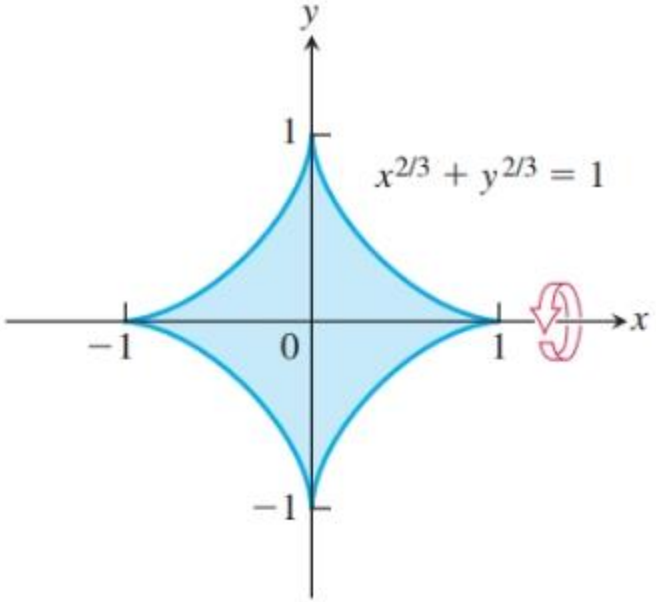
$x = acos^3t$ is even function and $y = asin^3t$ is an odd function.

- Therefore, curve is symmetric about X-axis.
- Replacing t by $\pi - t$, y remains unchanged but x changes sign. So, curve is symmetric about Y-axis also.
- The table of values of t, x, y is as follows:

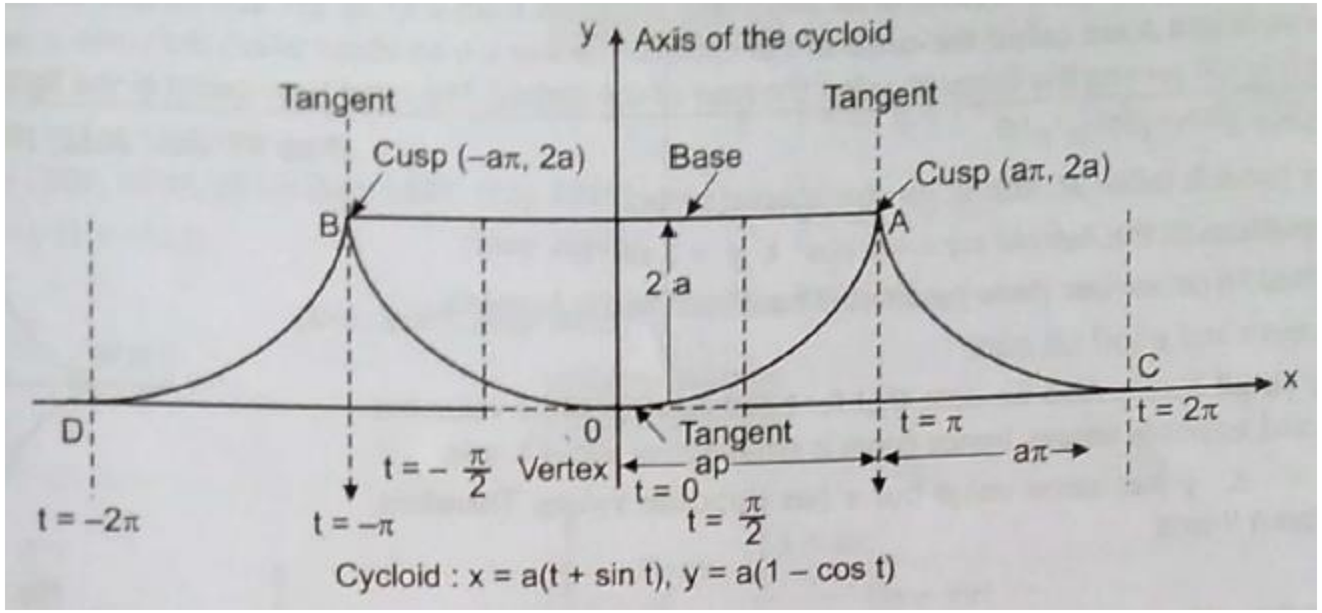
t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	a	0	$-a$	0	a
y	0	a	0	$-a$	0

- As t ranges from 0 to $\frac{\pi}{2}$, x decreases and y increases.



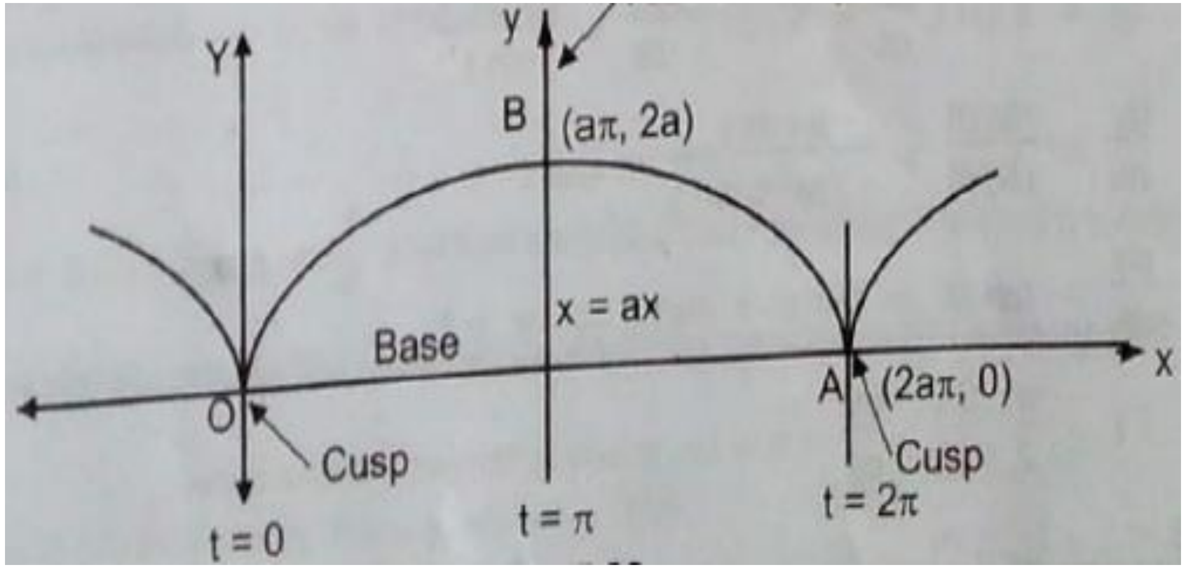


Ex.2 Graph of the curve $x = a(t + \sin t)$, $y = a(1 - \cos t)$ which is symmetrical about Y-axis



This curve is known as Cycloid

Ex.3 Graph of the curve $x = a(t - \sin t)$, $y = a(1 - \cos t)$ which is symmetrical about Y-axis



This curve is known as Cycloid

❖ Rectification: Rectification is the process to measure the length of arc of plane curves.

Equation of the curve	Formula to determine arc length
1) Cartesian curve $y = f(x)$	$S = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$
2) Cartesian curve $x = f(y)$	$S = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$
3) Parametric curve $x = f(t)\,, y = g(t)$	$S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$
4) Polar curve $r = f(\theta)$	$S = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$
5) Polar curve $\theta = f(r)$	$S = \int_{r_1}^{r_2} \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\, dr$

Ex.1 Find the arc length AB where $A(a, 0)$, $B(0, a)$ are any two points on the circle $x^2 + y^2 = a^2$, using $1 + \left(\frac{dy}{dx}\right)^2 = \frac{a^2}{a^2 - x^2}$

Sol.

Here, x varies from a to 0.

Using formula, $S = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$$\therefore\ S = \int_a^0 \sqrt{\frac{a^2}{a^2 - x^2}}\, dx = \int_a^0 a \sqrt{\frac{1}{a^2 - x^2}}\, dx$$

$$= a\left[\sin^{-1}\left(\frac{x}{a}\right)\right]_a^0 \qquad = a\left[\sin^{-1} 0 - \sin^{-1} 1\right]$$

$$= a\left(0 - \frac{\pi}{2}\right) = -\frac{\pi a}{2}$$

Arc length= $\frac{\pi a}{2}$ (Length is always positive)

Ex.2 Integral for calculating the length of arc of parabola $y^2 = 4x$,cut off by the line $3y = 8x$ is...

Sol.

Given, $3y = 8x \qquad \therefore y = \dfrac{8x}{3} \qquad \therefore \ y^2 = \dfrac{64x^2}{9} \longrightarrow$ (1)

Also given parabola $y^2 = 4x \longrightarrow$ (2)

Comparing equations (1) and (2) ,we get

$$\frac{64x^2}{9} = 4x \qquad \therefore \frac{64x^2}{9} - 4x = 0 \qquad \therefore x\left(\frac{64}{9}x - 4\right) = 0$$

$\therefore x = 0$ and $\dfrac{64}{9}x - 4 = 0$

$\therefore x = 0$ and $\dfrac{64}{9}x = 4$

$\therefore x = 0$ and $x = \dfrac{36}{64} = \dfrac{9}{16}$

Required integral is

$$\boxed{\int_0^{\frac{9}{16}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$

Ex.3 Find the length of arc of upper part of the curve $x = t^2 , y = t\left(1 - \frac{t^2}{3}\right)$ where t varies from 0 to $\sqrt{3}$, using $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (1 + t^2)^2$

Sol.

Equation of the curve is given in parametric form.

Arc length is given by formula, $S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$\therefore S = \int_0^{\sqrt{3}} \sqrt{(1 + t^2)^2}\, dt$$

$$\therefore S = \int_0^{\sqrt{3}} (1 + t^2)\, dt \qquad = \left[t + \frac{t^3}{3}\right]_0^{\sqrt{3}}$$

$$\therefore S = \left(\sqrt{3} + \frac{\left(\sqrt{3}\right)^3}{3}\right) = \sqrt{3} + \frac{3\sqrt{3}}{3} = \sqrt{3} + \sqrt{3} = 2\sqrt{3}$$

Required arc length is $2\sqrt{3}$

Solve: Find the length of arc of upper part of loop of the curve $3y^2 = x(x-1)^2$ from $(0,0)$ to $(1,0)$ using $1 + \left(\frac{dy}{dx}\right)^2 = \frac{(3x+1)^2}{12x}$ Hint: Use formula (1) in the table of rectification

Solve: Find the length of upper half of the cardioide $r = a(1+\cos\theta)$ where θ varies from 0 to π , using $r^2 + \left(\frac{dr}{d\theta}\right)^2 = 2a^2(1+\cos\theta)$ Hint: Use formula (4) in the table of rectification

Solve: Find the length of the arc of Astroid $x = acos^3\theta$, $y = asin^3\theta$ in the first quadrant between two consecutive cusps, where θ varies from 0 to $\frac{\pi}{2}$,

using $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 9a^2sin^2\theta cos^2\theta.$ Hint: Use formula (3) in the table of rectification