Engineering Mathematics-II

Unit – 2

Applications of First Order Differential Equations

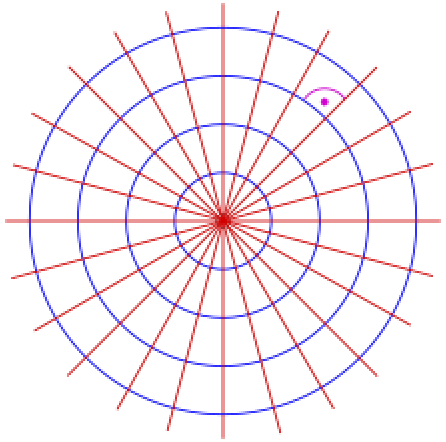
- We have discussed types of First Order Differential Equations and methods to find solution in the previous unit.In this unit we will see how these methods are useful in some applications like:

- Orthogonal Trajectories
- Newton’s law of cooling
- Simple electric circuits
- Rectilinear Motion
- Simple Harmonic Motion
- Heat Flow

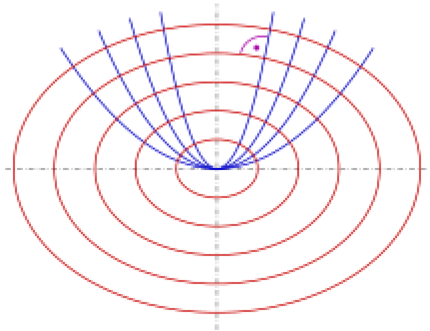
❖ Orthogonal Trajectories:

The orthogonal trajectories are the curves that are perpendicular to the family everywhere. In other words, the orthogonal trajectories are another family of curves in which each curve is perpendicular to the curves in original family.

- For example, if the given family consists of straight lines $y = mx$ passing through the origin,then the family of concentric circles $x^2 + y^2 = c^2$ represents a family of orthogonal trajectories to the given family $y = mx$.



- The orthogonal trajectories of the family of parabolas $y = Cx^2$ are the family of ellipses $x^2 + 2y^2 = d$



❖ Rules to find the equation of orthogonal trajectories for Cartesian Co-ordinates

1. Let the given equation be $f(x, y, c) = 0$, where c is parameter.

2. Differentiate $f(x, y, c) = 0$ with respect to x and eliminate parameter c. Let this differential equation be $g\left(x, y, \frac{dy}{dx}\right) = 0$

3. Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$. Let this new differential equation be $h\left(x, y, -\frac{dx}{dy}\right) = 0$. This is the differential equation of orthogonal trajectories.

4. Solve this new equation.Solution represents family of orthogonal trajectories.

❖ Rules to find the equation of orthogonal trajectories for Polar Co-ordinates

1. Let the given equation be $f(r, \theta, c) = 0$, where c is parameter.

2. Differentiate $f(r, \theta, c) = 0$ with respect to θ and eliminate parameter c. Let this differential equation be $g\left(r, \theta, \frac{dr}{d\theta}\right) = 0$

3. Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$. Let this new differential equation be $h\left(r, \theta, -r^2\frac{d\theta}{dr}\right) = 0$. This is the differential equation of orthogonal trajectories.

4.Solve this new equation.Solution represents family of orthogonal trajectories.

Ex.1 Find the orthogonal trajectories of the family of straight lines $y = mx$.

Sol.

Given $y = mx$, where m is a parameter. $\longrightarrow$ (1)

Differentiate (1) with respect to x to eliminate parameter m.

$$\frac{dy}{dx} = m$$

Substitute $m = \frac{dy}{dx}$ in equation (1)

So (1) becomes, $y = \frac{dy}{dx}x$ $\longrightarrow$ (2)

Repalcing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (2) ,we get

$$y = -\frac{dx}{dy}x \quad \therefore ydy = -xdx \quad \therefore xdx + ydy = 0 \longrightarrow (3)$$

Equation (3) is the differential equation of orthogonal trajectories.

Integrating,we get $\frac{x^2}{2} + \frac{y^2}{2} = c_1$ $\therefore$ $\boxed{x^2 + y^2 = C^2}$ This is the equation of the required orthogonal trajectories of (1)

Ex. 2 Find the orthogonal trajectories of the family of curves $r = a(1-\cos\theta)$.

Sol.

Given, $r = a(1-\cos\theta)$, where a is parameter. $\longrightarrow$ (1)

Differentiating equqtion (1) with respect to θ

$$\frac{dr}{d\theta} = a\sin\theta \quad \therefore \quad a = cosec\theta\frac{dr}{d\theta}$$

Substitute $a = cosec\theta\frac{dr}{d\theta}$ in (1)

$$r = cosec\theta\frac{dr}{d\theta}(1-\cos\theta) \longrightarrow (2)$$

Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in (2) , we get

$$r = -cosec\theta r^2\frac{d\theta}{dr}(1-\cos\theta) \;\therefore\; \frac{dr}{r} = -cosec\theta(1-\cos\theta)d\theta = \frac{-2cos^2\frac{\theta}{2}}{sin\theta} = \frac{-2cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = -\cot\frac{\theta}{2}d\theta$$

$\frac{dr}{r} = -\cot\frac{\theta}{2}d\theta$ is the differential equation of orthogonal trajectories.

Integrating,we get $\boxed{\log r = -2\log\cos\frac{\theta}{2} + \log C}$ This is the equation of the required orthogonal trajectories of (1)

Ex. 3 Find Orthogonal trajectories of family of curves $\frac{x^2}{a^2}+\frac{y^2}{b^2+\lambda}=1$ *where* λ is an arbitrary constant,whose differential equation is $x+\left(\frac{a^2-x^2}{y}\right)\frac{dy}{dx}=0.$

Sol.

Differential equation of given (original) family of curves is $x+\left(\frac{a^2-x^2}{y}\right)\frac{dy}{dx}=0. \longrightarrow$ (1)

To find orthogonal trajectories, replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (1).

$\therefore$ (1) becomes, $x-\left(\frac{a^2-x^2}{y}\right)\frac{dx}{dy}=0$ which is diff. equation of required ortho. trajectories.

We have to solve it. $\therefore \left(\frac{a^2-x^2}{y}\right)\frac{dx}{dy}=x$ $\therefore \left(\frac{a^2-x^2}{x}\right)dx=ydy$

Integrating, $\int\left(\frac{a^2-x^2}{x}\right)dx=\int ydy+C$ $\therefore \int\frac{a^2}{x}dx-\int\frac{x^2}{x}dx=\int ydy+C$

$$\boxed{a^2\log x-\frac{x^2}{2}=\frac{y^2}{2}+C}$$

This is the equation of the required orthogonal trajectories.

Ex. 4 Find Orthogonal trajectories of family of curves $r^2 = a\sin 2\theta$ if the differential equation of family of curves is $\frac{dr}{d\theta} = r\cot 2\theta$

Sol.

Differential equation of given (original) family of curves is $\frac{dr}{d\theta} = r\cot 2\theta \longrightarrow$ (1)

To find orthogonal trajectories, replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in (1).

$\therefore$(1) becomes, $-r^2\frac{d\theta}{dr} = r\cot 2\theta \;\; \therefore \tan 2\theta d\theta = -\frac{dr}{r}$ which is differential equation of required orthogonal trajectories.

Integrating it,we get $\int \tan 2\theta\, d\theta = \int -\frac{1}{r}dr + \log C$

$$-\frac{1}{2}\log\cos 2\theta = -\log r + \log C$$

$$\therefore \log r = \frac{1}{2}\log\cos 2\theta + \log C$$

$$\therefore 2\log r = \log\cos 2\theta + \log C$$

$$\therefore \log r^2 = \log(C.\cos 2\theta)$$

$$\boxed{r^2 = C.\cos 2\theta}$$

This is the equation of the required orthogonal trajectories

Ex. Find the orthogonal trajectories of the curves given by $x^2 + 2y^2 = c^2$.

Ex. Find the orthogonal trajectories of the curves given by $r^2 = a^2 \cos 2\theta$.

Ex. Find the orthogonal trajectories of the curves given by $xy = C$.

Ex. Find the orthogonal trajectories of the curves given by $r = a\,(1 + \cos\theta)$.

❖ Newton's Law of Cooling

- The law states that the rate of change of temperature of a body is proportional to the difference in temperature between that of the body itself and that of the surrounding medium.
- If θ_0 is the temperature of the surroundings and θ is the temperature of the body at any time t , then

$$\boxed{\frac{d\theta}{dt} = -k(\theta - \theta_0) \text{ ,where } k \text{ is constant.}}$$

Ex. 1 A metal ball is heated to a temperature of 100^0C and at time t=0 it is placed in water which is maintained at 40^0C. If the temperature of the ball is reduced to 60^0C in 4 minutes,find the time at which the temperature of the ball is 50^0C.

Sol. Let θ_0 be the temperature of the surroundings and θ be the temperature of the body at time t minute,then the differential equation is given by,

$$\frac{d\theta}{dt} = -k(\theta - \theta_0) \quad \therefore \frac{d\theta}{\theta - 40} = -kdt \longrightarrow \quad (1)$$

Integrating (1) we get, $-kt = \log_e(\theta - 40) + \log_e C \longrightarrow$ (2)

Note:All Logarithms are found for the base e

At t=0 minute, θ=100 substituting in (2) , we get

$$-k.0 = \log_e(100-40) + \log_e C \therefore\ 0 = \log_e 60 + \log_e C \therefore \log_e C = -\log_e 60$$

Substitute $\log_e C = -\log_e 60$ in (2)

$$-kt = \log_e(\theta - 40) - \log_e 60 \quad \therefore -kt = \log\frac{(\theta-40)}{60} \longrightarrow \quad (3)$$

At t=4 minutes, θ=60 substitute in (3)

$$-4k = \log\frac{20}{60} = \log\frac{1}{3} \quad \therefore k = \frac{1}{4}\log_e 3$$

Now put θ=50 and $k = \frac{1}{4}\log_e 3$ in (3) and find t.

$$\therefore (3) \text{ becomes,} \left(-\frac{1}{4}\log 3\right)t = \log\frac{1}{6} \quad \therefore t = -4\frac{\log\frac{1}{6}}{\log 3} = 4\frac{\log 6}{\log 3} = 6.5\ minutes.$$

$$\boxed{t = 6.5 \text{ minutes}}$$

Ex.2 A body originally at 80^0C cools down to 60^0C in 20 minutes,the temperature of the air being 40^0C. What will be the temperature of the body after 40 minutes from the original ?

Sol.

Let θ_0 be the temperature of the surroundings and θ be the temperature of the body at time t minute,then the differential equation is given by,

$$\frac{d\theta}{dt} = -k(\theta - \theta_0) \quad \therefore \frac{d\theta}{\theta - 40} = -kdt \longrightarrow (1)$$

Integrating (1) we get, $-kt = \log(\theta - 40) + \log C \longrightarrow (2)$

At t=0 minute, θ=80 substituting in (2) , we get

$$-k.0 = \log(80 - 40) + \log C \quad \therefore 0 = \log 40 + \log C \quad \therefore \log C = -\log 40$$

Substitute $\log C = -\log 40$ in (2) , $\therefore -kt = \log(\theta - 40) - \log 40 \quad \therefore -kt = \log\frac{(\theta-40)}{40} \longrightarrow (3)$

At t=20 minutes, θ=60 $\quad \therefore -20k = \log\frac{(60-40)}{40} = \log\frac{20}{40} = \log\frac{1}{2} \quad \therefore k = -\frac{1}{20}\log\frac{1}{2} = \frac{1}{20}\log 2$

Now put t=40 and $k = \frac{1}{20}\log 2$ in (3) and find θ.

$$-40\frac{1}{20}\log 2 = \log\frac{(\theta - 40)}{40} \quad \therefore \frac{(\theta - 40)}{40} = e^{\log\frac{1}{4}} = \frac{1}{4} \therefore \theta - 40 = 10$$

$$\boxed{\theta = 50^0C}$$

Ex.3 A body of temperature 100^0C is placed in a room whose temperature is 20^0C and cools upto 60^0C in 5 minutes. By Newton's law of cooling the differential equation is $\frac{d\theta}{dt} = -\left(\frac{1}{5}\log_e 2\right)(\theta - 20)$. Find the temperature after 8 minutes.

Sol.

$$\frac{d\theta}{dt} = -k(\theta - \theta_0) = -\left(\frac{1}{5}\log_e 2\right)(\theta - 20) \quad \therefore k = \frac{1}{5}\log_e 2$$

$$\int \frac{d\theta}{(\theta - 20)} = -\left(\frac{1}{5}\log_e 2\right)\int dt + C \quad \therefore \log(\theta - 20) = -\left(\frac{1}{5}\log_e 2\right)t + logC \longrightarrow \quad (1)$$

Put $t = 5\ and\ \theta = 60^0$ $\quad \therefore \log(60 - 20) = -\left(\frac{1}{5}\log_e 2\right)5 + C \quad \therefore \log 40 = -\log_e 2 + \log c$

$$\therefore 40 = \frac{c}{2} \quad \therefore c = 80$$

Put t=8 and c=80 in (3) $\quad \therefore \log(\theta - 20) = -\left(\frac{1}{5}\log_e 2\right)8 + log80 = \log\frac{80}{2^{\frac{8}{5}}}$

$$(\theta - 20) = \frac{80}{2^{\frac{8}{5}}} \quad \therefore \theta = 46.4^0\text{C}$$

Ex. Water at temperature 100^0C cools in 10 minutes to 60^0C in a room temperature of 20^0C. Find when the temperature will be 30^0C ?

Ex. A copper ball is heated to a temperature of 100^0C. Then at time t=0 it is placed in water which is at temperature 30^0C. At the end of 3 minutes ,the temperature of a ball reduced to 70^0C.Find the time at which the temperature of a ball drops to 31^0C.

❖ Simple Electric Circuits

Consider a circuit mde up of Resistance R (ohm Ω),Inductance L (Henry H),Capacitance C (Farad F) and voltage source with Electromotive force E volt,then

- Current is the rate of flow of charge given by $i = \frac{dq}{dt}$ where q is the electric charge in Coulomb.
- Voltage drop across resistace $R = Ri$
- Voltage drop across inductance $L = L\frac{di}{dt}$
- Voltage drop across Capacitance $C = \frac{q}{C}$

❖ Differential equation of L-R circuit

Consider a circuit containing resistance R and inductance L in series with e.m.f. E. According to Kirchhoff's law,the sum of voltage drop across R and L is equal to E.

$$\therefore Ri + L\frac{di}{dt} = E$$

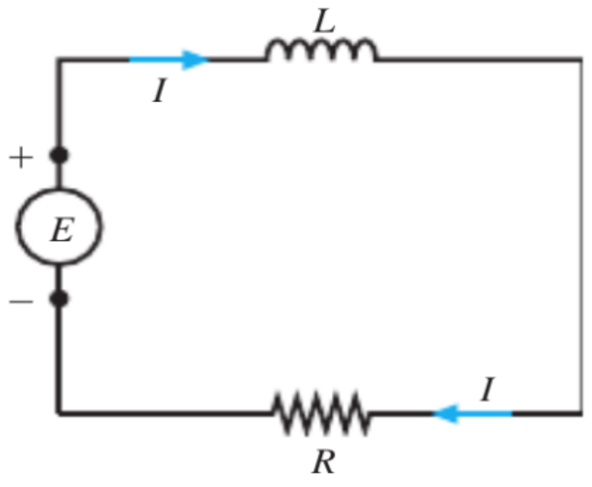


Figure 5. An *LR* circuit.

Dividing by L throughout,we get

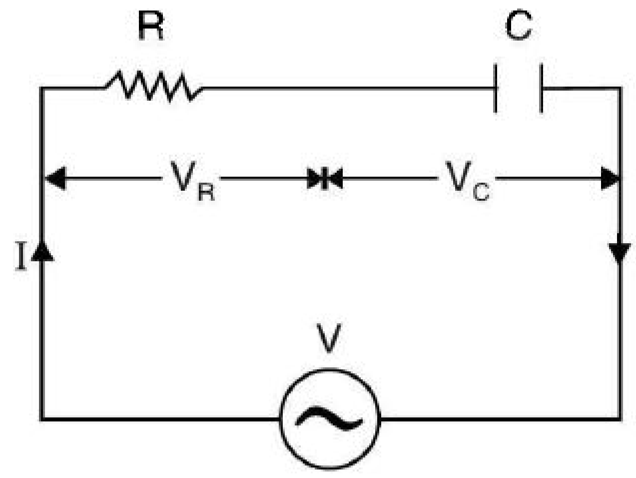
$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L} \longrightarrow (1)$$

Equation (1) is linear differential equation. $I.F. = e^{\int \frac{R}{L}dt} = e^{\frac{Rt}{L}}$ Solving further, we get

$$\boxed{i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)}$$

❖ Differential equation of R-C circuit

Consider a circuit containing resistance R and capacitance C in series with voltage V. According to Kirchhoff's law, the sum of voltage drop across R and C is equal to V.



$$\therefore\ Ri + \frac{q}{C} = V$$

Since, $i = \frac{dq}{dt}$, $\quad R\frac{dq}{dt} + \frac{q}{C} = V$

Dividing by R throughout, $\frac{dq}{dt} + \frac{q}{RC} = \frac{V}{R} \longrightarrow (1)$

Equation (1) is linear differential equation.Solving this ,we get

$$\boxed{q = VC + Ae^{-\frac{t}{RC}}, where\ A\ is\ arbitrary\ constant}$$

- Current i can be found by finding the value of constant A and then integrating equation with respect to time t.

Ex. 1 In R-L circuit in series with constant voltage source E, current i is given by $i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$ Find maximum current i_{max} .

Sol.

Current at any time t is given by $i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right) \longrightarrow (1)$

i_{max} will be obtained at $t = \infty$. Hence substitute $i = i_{max}$ and $t = \infty$ in (1).

$$\therefore (1) becomes, i_{max} = \frac{E}{R}(1 - e^{-\infty}) = \frac{E}{R}(1 - 0) = \frac{E}{R}$$

$$\boxed{i_{max} = \frac{E}{R}}$$

Ex. 2 In R-L circuit in series with constant e.m.f. E, current i is given by $i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$ Find the time required to reach the current to half of its theoretical maximum.

Sol.

Current at any time t is given by $i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right) \longrightarrow (1)$

In Ex. 1 we have found theoretical maximum $i_{max} = \frac{E}{R}$

Half of the theoretical maximum = $\frac{1}{2} i_{max} = \frac{1}{2}\frac{E}{R}$

Now we have to find time t for $i=\frac{1}{2}\frac{E}{R}$ $\quad \therefore$ Substitute $i=\frac{1}{2}\frac{E}{R}$ in equation (1)

So (1) becomes, $\frac{1}{2}\frac{E}{R} = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right) \quad \therefore \frac{1}{2} = 1 - e^{-\frac{Rt}{L}} \quad \therefore e^{-\frac{Rt}{L}} = 1 - \frac{1}{2} = \frac{1}{2}$

$$\therefore \quad \frac{1}{e^{\frac{Rt}{L}}} = \frac{1}{2} \qquad \therefore e^{\frac{Rt}{L}} = 2 \qquad \therefore \frac{Rt}{L} = \log 2$$

$$\boxed{t = \frac{L}{R}\log 2}$$

Ex.3 The charge q on the plate of condenser of capacity C through a resistance R by steady voltage V is given by $q = CV\left(1 - e^{\frac{-t}{RC}}\right)$. Find the current flowing through the plate.

Sol.

Given, $q = CV\left(1 - e^{\frac{-t}{RC}}\right) \longrightarrow (1)$

Current $i = \frac{dq}{dt}$ Hence, differentiating equation (1) with respect to t,we get

$$i = \frac{dq}{dt} = CV\left(0 + e^{\frac{-t}{RC}}\frac{1}{RC}\right) = CVe^{\frac{-t}{RC}}\frac{1}{RC} = \frac{V}{R}e^{\frac{-t}{RC}}$$

$$\boxed{i = \frac{V}{R}e^{\frac{-t}{RC}}}$$

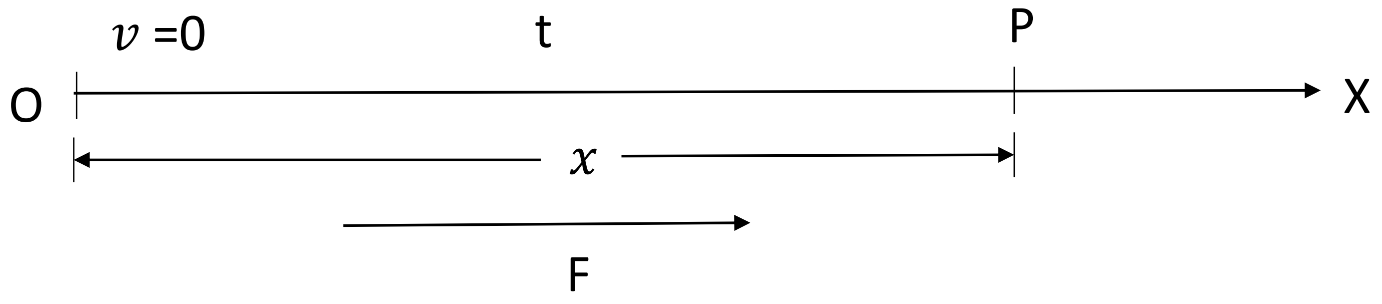
Solve: In R-C circuit,charge q as function of time t is given by $q = e^{-3t} - e^{-6t}$. Find the time for the charge to be maximum.

Solve: A circuit containing resistance R and inductance L in series with voltage E.The differential equation for current i is $L\frac{di}{dt} + Ri = E$. Given L=640 H , R=250 ohm , E=500 volts. Find integrating factor of differential equation.

❖ Rectilinear Motion

- Rectilinear motion is a motion of a body along a straight line.
- When we require only one co-ordinate axis along with time to describe the motion of a particle it is said to be in linear motion or rectilinear motion.
- Following are the rectilinear motion examples:

1. The use of elevators in public places is an example of rectilinear motion.
2. Gravitational forces acting on objects resulting in free fall is an example of rectilinear motion.
3. Kids sliding down from a slide is a rectilinear motion.
4. The motion of planes in the sky is a rectilinear motion.
5. parade of soldiers
6. A train moving along a straight line



Let’s consider a body of mass ‘m’ start moving from a fixed point ‘O’ along a straight line OX under the action of force ‘F’.

After some time ‘t’ let the particle reach at point ‘P’. Distance OP=x

- Velocity $(v) = \frac{dx}{dt}$
- Acceleration $(a) = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$

$$(a) = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}.v$$

$$\text{Acceleration } (a) = \frac{dv}{dt} = \frac{d^2x}{dt^2} = v.\frac{dv}{dx}$$

According to Newton’s second law of motion Force$(F) = ma = m\frac{dv}{dt} = mv\frac{dv}{dx} = m\frac{d^2x}{dt^2}$

Ex.1 A body of mass m , falling from rest is subjected to the force of gravity and an air resistance proportional to the square of velocity 'kv^2'. Find the equation of motion of a body and also obtain the relation between velocity and displacement if $mg = ka^2$.

Sol. A body is subjected to 2 types of forces.

1) The force of gravity acting downwards = mg
2) An air resistance acting upwards = $-kv^2$ (It is negative since it opposes the motion of a body)

∴ Net force on the body = $F = mg - kv^2 = ka^2 - kv^2$

$$mv\frac{dv}{dx} = k(a^2 - v^2)$$ This is equation of motion.

Separating the variables and integrating, $\int \frac{v}{a^2-v^2} dv = \int \frac{k}{m} dx + C$

$$-\frac{1}{2}\int \frac{-2v}{a^2 - v^2} dv = \int \frac{k}{m} dx + C$$

Using formula, $\int \frac{f'(x)}{f(x)} dx = \log f(x)$, we get

$$-\frac{1}{2}\log(a^2 - v^2) = \frac{kx}{m} + C \longrightarrow \quad (1)$$

A body is falling from rest, so substitute $x = 0\,, v = 0$ in (1)

$$-\frac{1}{2}\log(a^2) = C$$

Back substitute $-\frac{1}{2}\log(a^2) = C$ in (1)

$$-\frac{1}{2}\log(a^2 - v^2) = \frac{kx}{m} - \frac{1}{2}\log(a^2)$$

$$-\frac{1}{2}\log(a^2 - v^2) + \frac{1}{2}\log(a^2) = \frac{kx}{m}$$

$$\frac{1}{2}\log\left(\frac{a^2}{a^2 - v^2}\right) = \frac{kx}{m}$$

$$\boxed{\frac{2kx}{m} = \log\left(\frac{a^2}{a^2 - v^2}\right)}$$

This is the desired relation between velocity and displacement

Ex.2 A particle of unit mass moves in a horizontal straight line OA with an acceleration $\frac{k}{r^3}$ at a distance r and directed towards O. if initially the particle was at rest at $r = a$ and equation of motion is $v\frac{dv}{dr} = -\frac{k}{r^3}$ then find the relation between r , v.

Sol.

Given,equation of motion is $v\frac{dv}{dr} = -\frac{k}{r^3}$ ⟶ (1)

Separating the variables,we get $vdv = -\frac{k}{r^3}dr$

Integrating both sides, $\int vdv = \int -\frac{k}{r^3}dr + C$, where C is constant of integration.

$$\therefore \frac{v^2}{2} = \frac{k}{2r^2} + C \longrightarrow (2)$$

Given, initially the particle was at rest at $r = a$ i.e. at $r = a$, $v = 0$

Put $r = a$, $v = 0$ in (2). So (2) becomes, $0 = \frac{k}{2a^2} + C$ $\therefore$ $C = -\frac{k}{2a^2}$

Back substituting $C = -\frac{k}{2a^2}$ in (2) , $\frac{v^2}{2} = \frac{k}{2r^2} - \frac{k}{2a^2}$

$$\boxed{v^2 = \mathrm{k}\left(\frac{1}{r^2} - \frac{1}{a^2}\right)}$$

This is the desired relation between r and v.

Ex. 3 A body of mass m falls from rest under gravity in a fluid whose resistance to motion at any instant is mkv where k is constant.The differential equation of motion is $\frac{dv}{dt} = g - kv$. Find the terminal velocity.

Sol.

Terminal velocity is attained when the weight of the body exactly balances the resistance to motion.If V is the terminal velocity ,then $mg = mkV$

$$\boxed{V = \frac{g}{k}}$$

Solve: A particle moving in a straight line with acceleration $k\left(x + \frac{a^4}{x^3}\right)$ directed towards the origin.The equation of motion is $v\frac{dv}{dx} = -k\left(x + \frac{a^4}{x^3}\right)$. If it starts from rest at a distance $x = a$ from the origin then find the relation between velocity v and displacement x.

❖ Heat Flow

Principles involved in the problems of heat conduction are:

- Heat flows from a higher temperature to a lower temperature.
- According to Fourier's law of heat conduction,the rate of heat flow across an area is proportional to the area and to the temperature gradient.
  If q(cal/sec.) is the quantity of heat that flows across a slab of area $A(cm^2)$ and thickness δx in one second, where the difference of temperature at the faces is δT , then

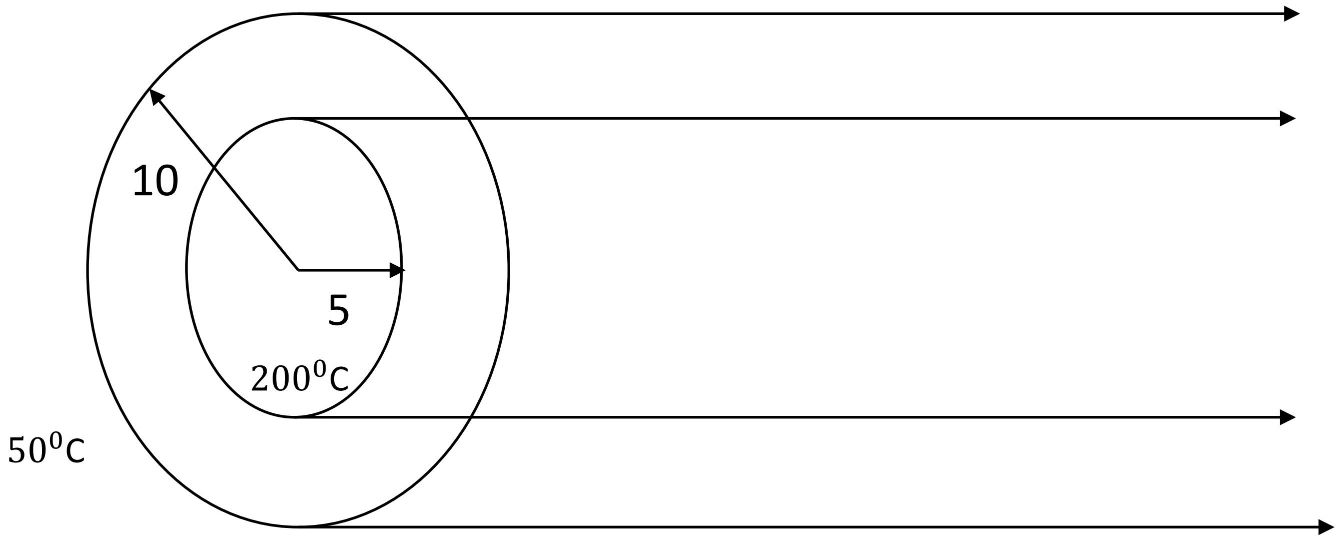
$$q \propto A\frac{dT}{dx}$$

$$q = -kA\frac{dT}{dx}$$, where k is constant called thermal conductivity.

Negative sign indicates that temperature T decreases as x increases.

Ex.1 A long hollow pipe has an inner diameter of 10 cm. and outer diameter of 20 cm. The inner surface is kept at 200^0C and outer surface at 50^0C. The thermal conductivity k =0.12. Find the amount of heat loss Qcal/sec.

Sol.



Let Qcal/sec.be the quantity of heat flowing through a surface of the pipe having radius x cm and length(l) 1 cm.Then the area of the surface $A = 2\pi x l = 2\pi x$

The differential equation of heat conduction is $Q = -kA\frac{dT}{dx} = -k2\pi x\frac{dT}{dx}$

$$\therefore dT = \frac{-Q}{2\pi k}\frac{dx}{x}$$

Integrating we get, $T = \frac{-Q}{2\pi k}\log_e x + C \longrightarrow$ (1)

When x=5 , T=200 $\therefore$ equation (1) becomes,

$$200 = \frac{-Q}{2\pi k}\log_e 5 + C \longrightarrow (2)$$

When x=10 , T=50 $\therefore$ equation (1) becomes,

$$50 = \frac{-Q}{2\pi k}\log_e 10 + C \longrightarrow (3)$$

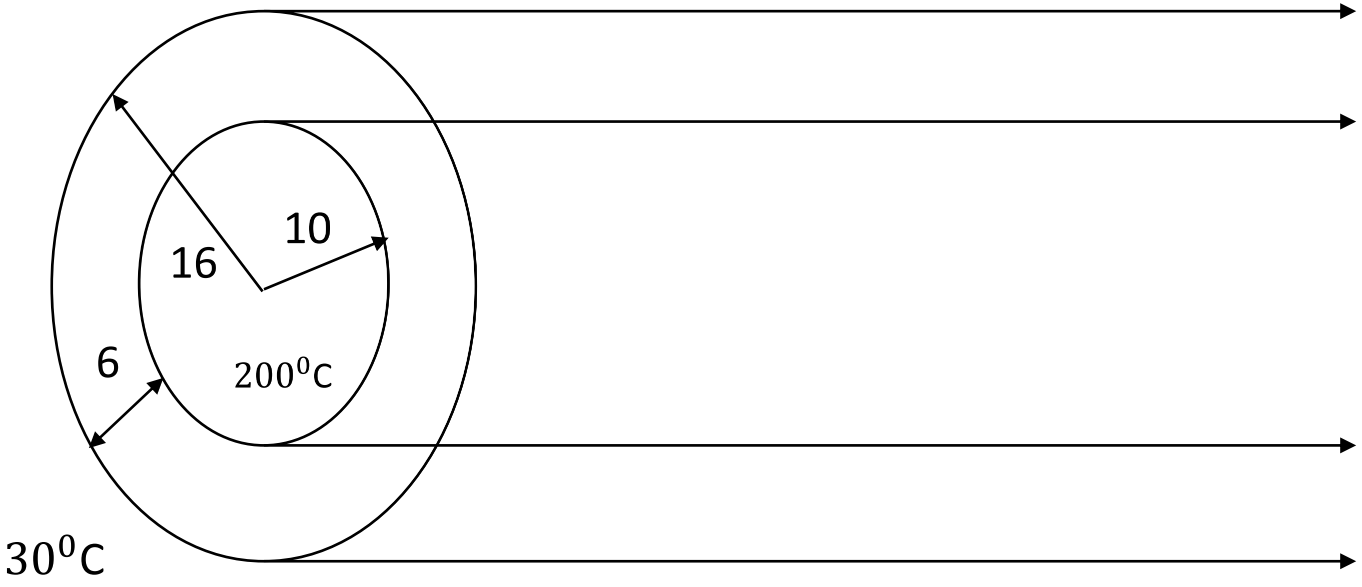
Subtract equation (3) from equation (2) so that constant C will be eliminated.

$$150 = \frac{Q}{2\pi k}(\log_e 10 - \log_e 5) = \frac{Q}{2\pi k}\log_e \frac{10}{5} = \frac{Q}{2\pi k}\log_e 2$$

$$\boxed{Q = \frac{150 \times (2\pi k)}{\log_e 2}}$$

Ex. 2 A steam pipe 20 cm. in diameter is protected with covering 6 cm.thick for which $k = 0.0003$ The inner surface is at 200^0C and outer surface of the covering is at 30^0C. The diffrrential equation of heat conduction is $dT = \frac{-Q}{2\pi k}\frac{dx}{x}$ Find the amount of heat loss Qcal/sec.

Sol.



Let Qcal/sec.be the quantity of heat flowing through a surface of the pipe having radius x cm and length(l) 1 cm.Then the area of the surface $A = 2\pi xl = 2\pi x$

The diffrrential equation of heat conduction is $dT = \frac{-Q}{2\pi k}\frac{dx}{x}$

Integrating, $-\int_{200}^{30} dT = \frac{Q}{2\pi k}\int_{10}^{16}\frac{dx}{x}$

$$(200-30) = \frac{Q}{2\pi k}(\log_e 16 - \log_e 10)$$

$$170=\frac{Q}{2\pi k}\log_e\left(\frac{16}{10}\right)=\frac{Q}{2\pi k}\log_e 1.6$$

$$\boxed{Q=\frac{170\times 2\pi k}{\log_e 1.6}}$$

Solve: A pipe 20 cm in diameter contains steam at 150^0C and is protected with a covering 5 cm thick for which $k=0.0025$. If the temperature of the outer surface of the covering is 40^0C , find the amount of heat loss Qcal/sec.

Solve: The differential equation for the heat loss Q per unit time from a spherical shell with thermal conductivity k,inner radius r_1,outer radius r_2,inner temperature T_1,outer temperature $T_2$is $Q=-k(4\pi r^2)\frac{dT}{dr}$. Find the temperature T of spherical shell of radius r.

Hint: Given $Q=-k(4\pi r^2)\frac{dT}{dr}$ $\therefore dT=\frac{-Q}{4\pi k r^2}dr$ which is variable separable form.

Integrate both the sides to find temperature T.

❖Simple Harmonic Motion

The motion of a particle moving along a straight line with an acceleration whose direction is always towards a fixed point on the line and whose magnitude is proportional to the distance from the fixed point is called simple harmonic motion

In the diagram, a weight is attached to one end of a spring. The other end of the spring is connected to a support. If the system is left at rest at the equilibrium position then there is no net force acting on the mass. However, if the mass is displaced from the equilibrium position, the spring exerts a restoring elastic force that obeys Hooke's law.

$F = -kx$, where **F** is the restoring elastic force exerted by the spring , k is the spring constant and **x** is the displacement from the equilibrium position