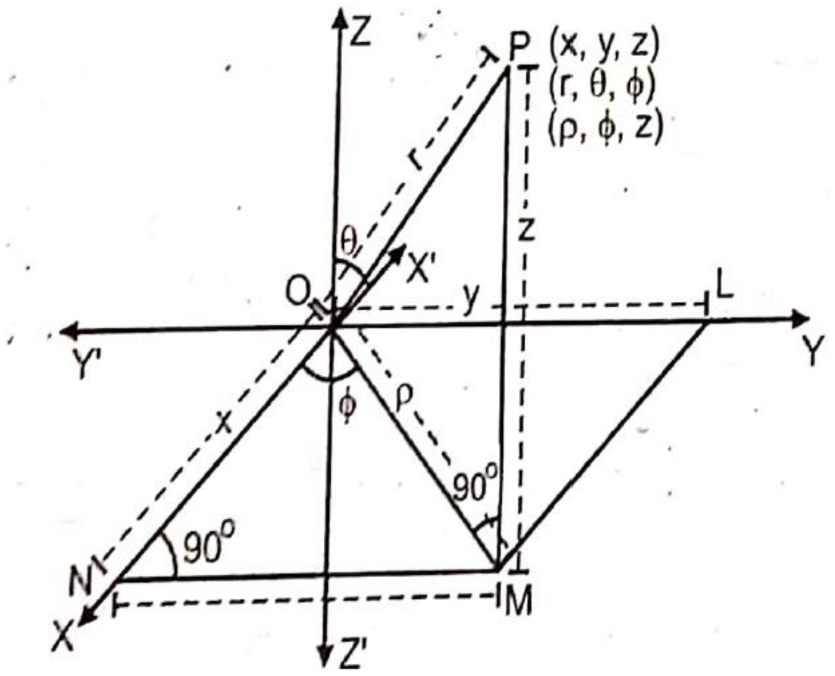
Engineering Mathematics-II

Unit - 5

Solid Geometry

❖ Co-ordinate system



In the figure, three mutually perpendicular straight lines $X'OX$, $Y'OY$, $Z'OZ$ are intersecting at the origin 'O'. OX , OY , OZ are the three co-ordinate axes and are called X , Y , Z axes respectively.

- Cartesian Co-ordinate System: Let P be any point in space. PM is perpendicular on XY plane. Let the distance $ON = x, OL = y, MP = z$. The position of the point P is described by (x, y, z) and these are called the cartesian co-ordinates of the point P.

- Spherical Polar System: Let $OP = r$. Angle made by OP with positive direction of Z-axis be denoted by θ. $\angle MOX = \phi$ which is the angle between plane POM and plane ZOX.The numbers (r, θ, ϕ) are called spherical polar co-ordinates of the point P.
- Cylindrical Co-ordinate System: Let $OM = \rho$. The numbers (ρ, ϕ, z) are called cylindrical co-ordinates of the point P.

❖ Relations between three co-ordinate systems:

- The relations between cartesian and spherical polar co-ordinates are

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$
$$x^2 + y^2 + z^2 = r^2 \ , \ \tan\phi = \frac{y}{x}$$

- The relations between cartesian and cylindrical co-ordinates are

$$x = \rho \cos\phi$$
$$y = \rho \sin\phi$$
$$z = z$$
$$x^2 + y^2 = \rho^2 \, , \tan\phi = \frac{y}{x}$$

Ex.1 Find the spherical polar co-ordinates of a cartesian point (1,1,1).

Sol.

Given, $x = 1 , y = 1 , z = 1$

We have to find spherical polar co-ordinates r, θ, ϕ

Using $x^2 + y^2 + z^2 = r^2$, we get

$1^2 + 1^2 + 1^2 = r^2$ $\qquad \therefore\ r^2 = 3$ $\qquad \therefore r = \sqrt{3}\ , (r > 0)$

Now, $z = r \cos\theta$

$$\therefore\ \cos\theta = \frac{z}{r} = \frac{1}{\sqrt{3}} \qquad \therefore\ \theta = \cos^{-1}\frac{1}{\sqrt{3}} = 54.74^0$$

We have, $\tan\phi = \frac{y}{x}$ $\qquad \therefore\ \tan\phi = \frac{1}{1} = 1$ $\qquad \therefore \phi = 45^0$

Thus the spherical polar co-ordinates are

$$\boxed{(r, \theta, \phi) = \left(\sqrt{3}, 54.74^0, 45^0\right)}$$

Ex.2 Find the cylindrical co-ordinates of a cartesian point $(3,4,-5)$.

Sol.

Given, $x = 3\,, y = 4\,, z = -5$

We have to find cylindrical co-ordinates ρ, ϕ, z.

Using $x^2+y^2 = \rho^2$, we get

$3^2+4^2 = \rho^2$ $\quad\therefore \rho^2 = 25$ $\quad\therefore \rho=5\,, (\rho > 0)$

We have, $\tan\phi = \frac{y}{x}$ $\quad\therefore \tan\phi = \frac{4}{3}$ $\quad\therefore \phi = \tan^{-1}\frac{4}{3} = 53.8^0$

For cylindrical co-ordinates , $z = z$ $\quad\therefore z = -5$

Thus the cylindrical co-ordinates are

$$\boxed{(\rho, \phi, z) = (5, 53.8^0, -5)}$$

Ex.3 Find the cartesian co-ordinates of spherical polar co-ordinates $\left(2, \frac{\pi}{3}, \frac{\pi}{4}\right)$

Sol.

Given, $r = 2, \theta = \frac{\pi}{3}, \phi = \frac{\pi}{4}$

We have, $x = r \sin\theta \cos\phi$

$$\therefore x = 2 \sin\frac{\pi}{3} \cos\frac{\pi}{4}$$

$$\therefore x = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$y = r \sin\theta \sin\phi$

$$\therefore y = 2 \sin\frac{\pi}{3} \sin\frac{\pi}{4}$$

$$\therefore y = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$z = r \cos\theta$

$$\therefore z = 2 \cos\frac{\pi}{3}$$

$$\therefore z = 2\left(\frac{1}{2}\right) = 1$$

Thus the cartesian co-ordinates are

$$\boxed{(x, y, z) = \left(\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}, 1\right)}$$

Solve: Find the spherical polar co-ordinates and cylindrical co-ordinates of the cartesian point $(-1,2,-3)$.

Solve: Find cartesian co-ordinates of the cylindrical co-ordinates $(\sqrt{2}, 45^0, 1)$.

- Relation between direction ratios and direction cosines of a line

If $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$ are the direction cosines and a, b, c are the direction ratios of a line, then

- $l = \frac{a}{\sqrt{a^2+b^2+c^2}}$, $m = \frac{b}{\sqrt{a^2+b^2+c^2}}$, $n = \frac{c}{\sqrt{a^2+b^2+c^2}}$
- $l^2 + m^2 + n^2 = 1$
- If the line AB joins the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ then direction ratios of AB are $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

Ex. What are the d.c.s of the line equally inclined to the axes?

Sol. Line equally inclined to the axes. (i.e. line makes equal angles with X, Y, Z axes).

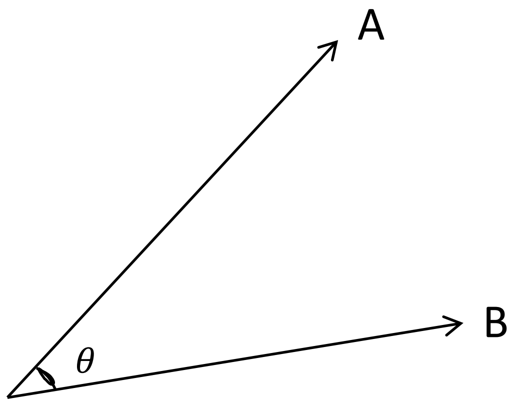
So, d.c.s of a line are also equal. $\therefore l = m = n$

Also we have, $l^2 + m^2 + n^2 = 1$

$\therefore l^2 + l^2 + l^2 = 1$ $\therefore 3l^2 = 1$ $\therefore l^2 = \frac{1}{3}$ $\therefore l = \pm\frac{1}{\sqrt{3}}$

$$\boxed{l = m = n = \pm\frac{1}{\sqrt{3}}}$$

❖ Angle between two lines:



- If $(l_1, m_1, n_1)$and (l_2, m_2, n_2) are the direction cosines of the two lines inclined at an angle θ, then $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$

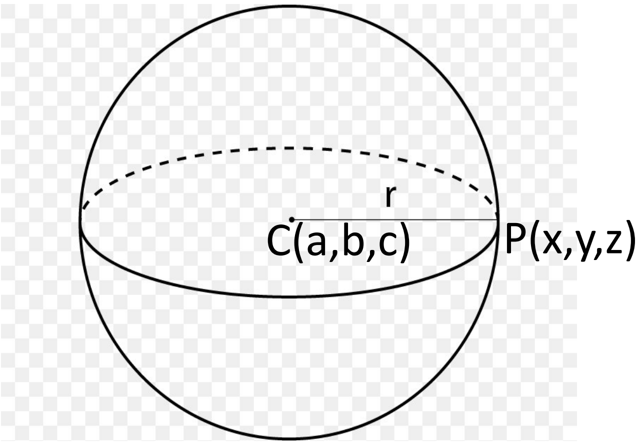
- If direction ratios of two lines are $a_1, b_1, c_1$and a_2, b_2, c_2 then $\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{{a_1}^2+{b_1}^2+{c_1}^2}\sqrt{{a_2}^2+{b_2}^2+{c_2}^2}}$

❖ **The plane:** A plane is a surface such that if any two points are taken on it, then the straight line joining them lies wholly in the surface.

- The general form of the equation of plane is $ax + by + cz + d = 0$ where a, b, c are direction ratios of normal to the plane.
- Consider two planes, $a_1 x + b_1 y + c_1 z + d_1 = 0$ and $a_2 x + b_2 y + c_2 z + d_2 = 0$. d.r.s of normals to the planes are (a_1, b_1, c_1) and (a_2, b_2, c_2).Angle between two planes is angle between their normals. If θ is angle between normals then angle between planes is

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{{a_1}^2+{b_1}^2+{c_1}^2}\sqrt{{a_2}^2+{b_2}^2+{c_2}^2}}$$

❖ Sphere:



In geometry, a sphere is a solid, that is absolutely round in shape defined in three-dimensional space (XYZ space).

- A sphere is a symmetrical object.
- All the surface points of sphere are equidistant from center and that constant distance is called radius of the sphere.
- A sphere has only curved surface, no flat surface, no edges and no vertices.

(1) Centre-radius form: Let $P(x, y, z)$ be any point on the sphere, $C(a, b, c)$ be the centre and r be the radius, then the equation of the sphere in centre-radius form is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

(2) General form: Equation of the sphere in the general form is

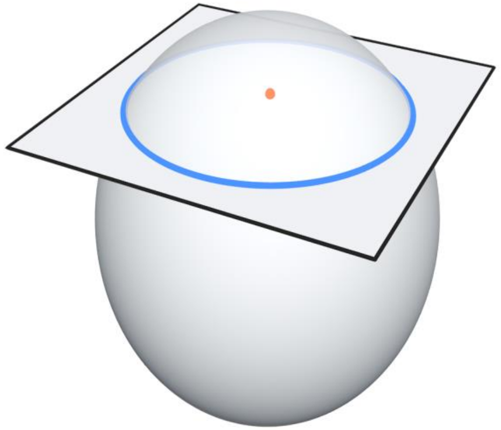
$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ whose centre is $(-u, -v, -w)$ and radius is $\sqrt{u^2 + v^2 + w^2 - d}$

(3) Intercept form: Equation of the sphere which cuts-off intercepts a, b, c from X, Y, Z axes respectively is $x^2 + y^2 + z^2 - ax - by - cz = 0$

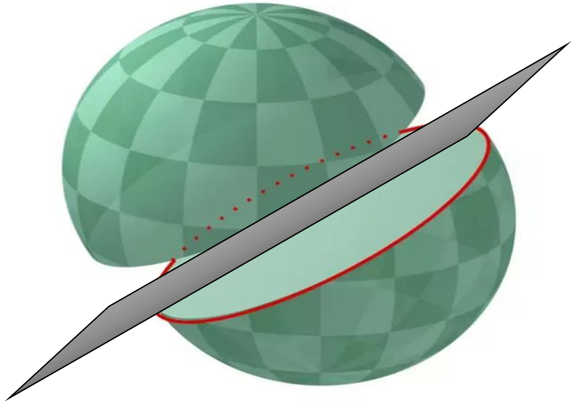
(4) Diameter form: Let points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ form diameter PQ and $R(x, y, z)$ be any point on the sphere, then equation of the sphere in diameter form is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$$

- Intersection of sphere by a plane gives circle. If $S = 0$ is sphere and $U = 0$ is plane ,then circle of intersection is $\boxed{S = 0, U = 0}$



- If plane intersects sphere through centre of sphere,then intersection is 'Great Circle'.Centre and radius of Great circle are same as those of given sphere



(5) Intersection of two spheres : The intersection of two spheres is a circle. If $S_1 = 0$ and $S_2 = 0$ are equations of spheres , then

$$\boxed{\begin{matrix} S_1 = 0 \\ S_2 = 0 \end{matrix}}$$

Together represent a circle.

(6) Sphere through a circle:

Consider, sphere $S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

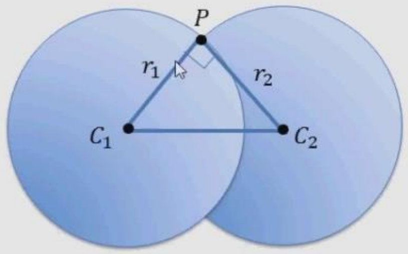
plane $U \equiv lx + my + nz - p = 0$

Intersection of sphere S and plane U is a circle. The family of spheres passing through this circle is given by $S + \lambda U = 0$

(7) Orthogonal spheres:

Orthogonal Spheres

Two spheres are said to be orthogonal if the tangent planes to the two spheres at the point of intersection are at right angles



$$(C_1C_2)^2 = r_1^2 + r_2^2$$

Let $x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$
and $x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$
be equations of two spheres.

The condition for the two spheres to be orthogonal is $2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2$

(8) Tangent plane: Consider the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$. The equation of tangent plane of the sphere at point (x_1, y_1, z_1) is given by

$$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$$

Solve: Find the equation of the sphere whose centre is $(2, -3, 1)$ and radius is 5.

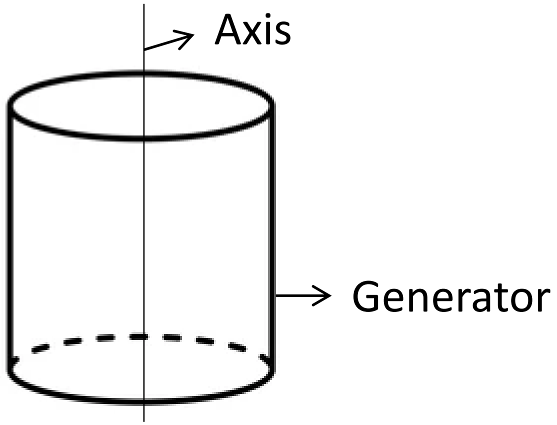
Solve: Find the equation of the sphere with centre $(2, -2, 3)$ and passing through $(7, -3, 5)$.
Hint: Distance between given two points is a radius.$\sqrt{(7-2)^2 + (-3+2)^2 + (5-3)^2} = \sqrt{25+1+4} = \sqrt{30} = radius$. Now use centre-radius form.

Solve:Find the equation of the sphere on the join of $(2, -3, 1)$ and $(1, -2, -1)$ as diameter.
Hint: use formula (4)

Solve: Find the equation of tangent plane to the sphere $x^2 + y^2 + z^2 - 4x + 2y = 4$ at the point $(4, -2, 2)$. Hint: Use formula in (8)

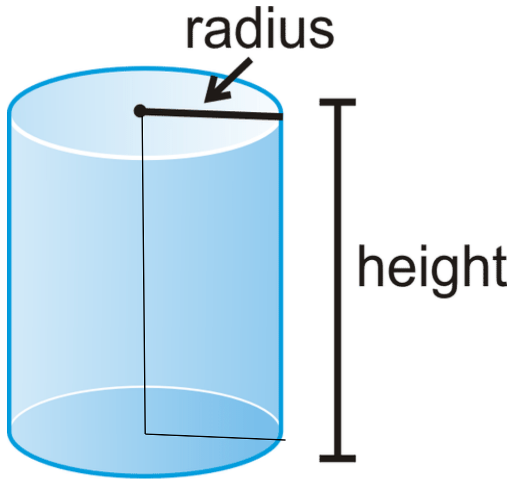
❖ Cylinder:

- In mathematics, a cylinder is a three-dimensional solid that holds two parallel bases joined by a curved surface, at a fixed distance. These bases are normally circular in shape (like a circle) and the center of the two bases are joined by a line segment, which is called the axis. The perpendicular distance between the bases is the height
- A cylinder is a surface generated by a straight line which is parallel to a fixed straight line. Fixed line is axis of the cylinder and moving line is generator of the cylinder.

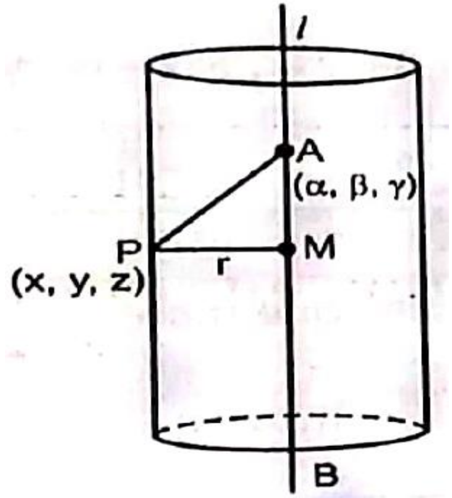


❖ Right Circular Cylinder:

- a cylinder with the bases circular and with the axis joining the two centers of the bases perpendicular to the planes of the two bases is called right circular cylinder.
- Water pipes , torches etc. have right circular cylindrical shape.



- To find the equation of right circular cylinder, three parameters are required:
  1) Radius of the cylinder 2) One point on the axis 3) Direction cosines of the axis
- How to find equation of right circular cylinder whose radius is r and axis is the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$



Let $A(\alpha,\beta,\gamma)$ be a fixed point on axis AB and $P(x,y,z)$be any point on the cylinder. PM =Radius=r

$$\therefore distance\ AP=\sqrt{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}$$

D.r.s of AP are $x-\alpha, y-\beta, z-\gamma$. D.r.s of axis are $l, m,$ n.

hence,D.c.s of axis are $\frac{l}{\sqrt{l^2+m^2+n^2}}, \frac{m}{\sqrt{l^2+m^2+n^2}}, \frac{n}{\sqrt{l^2+m^2+n^2}}$

AM =projection of AP on axis $AB=\frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{\sqrt{l^2+m^2+n^2}}$

In $\Delta\ PMA, AP^2=AM^2+PM^2$

$$\therefore (x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=\left\{\frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{\sqrt{l^2+m^2+n^2}}\right\}^2+r^2$$

This is the required equation of the right circular cylinder.

Ex.1 Find equation of the right circular cylinder of radius 2 and axis is the line $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{2}$

Sol.

Here, $\alpha=1,\ \beta=2,\ \gamma=3,\ l=2,\ m=1,\ n=2\ ,r=2$

Using $(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=\left\{\frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{\sqrt{l^2+m^2+n^2}}\right\}^2+r^2$

We get, $(x-1)^2+(y-2)^2+(z-3)^2=\left\{\frac{2(x-1)+1(y-2)+2(z-3)}{\sqrt{2^2+1^2+2^2}}\right\}^2+2^2$

$$\boxed{(x-1)^2+(y-2)^2+(z-3)^2=\left\{\frac{2x+y+2z-10}{3}\right\}^2+4}$$

This is the required equation of RCC.

Ex.2 Find the radius of right circular cylinder whose axis is $\frac{x-2}{2}=\frac{y-1}{1}=\frac{z}{3}$ and which passes through the point $(0,0,3)$.

Sol. Here, $\alpha=2,\ \beta=1,\ \gamma=0,\ l=2,\ m=1,\ n=3$

Using $(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=\left\{\frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{\sqrt{l^2+m^2+n^2}}\right\}^2+r^2$

We get, $(x-2)^2+(y-1)^2+(z-0)^2=\left\{\frac{2(x-2)+1(y-1)+3(z-0)}{\sqrt{2^2+1^2+3^2}}\right\}^2+r^2$

Cylinder passes through the point $(0,0,3)$. Hence,above equation is satisfied by $(0,0,3)$

$$\therefore (0-2)^2+(0-1)^2+(3-0)^2=\left\{\frac{2(0-2)+1(0-1)+3(3-0)}{\sqrt{2^2+1^2+3^2}}\right\}^2+r^2$$

$$\therefore 14=\frac{16}{14}+r^2 \qquad \therefore\ r^2=14-\frac{16}{14}=\frac{180}{14}=\frac{90}{7}$$

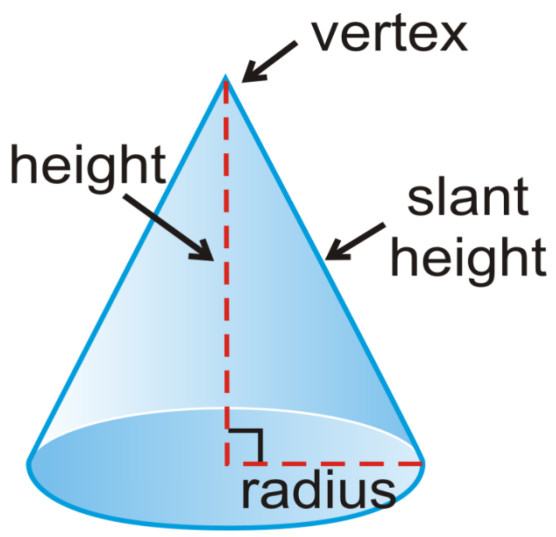
$$\boxed{r=\sqrt{\frac{90}{7}}}$$

Solve: Find the equation of the right circular cylinder of radius 3 whose axis is the line $\frac{x-1}{2}=\frac{y-3}{2}=\frac{z-5}{-1}$

Solve:Find the equation of the right circular cylinder of radius 2 whose axis is the line $\frac{x-1}{2}=\frac{y}{3}=\frac{z-3}{1}$

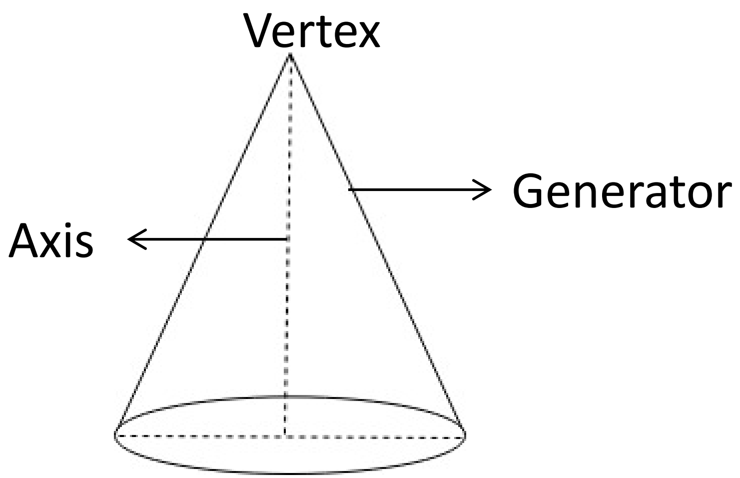
❖ Cone:

- A cone is a three-dimensional shape in geometry that narrows smoothly from a flat base (usually circular base) to a point called the apex or vertex.
- The distance from the vertex of the cone to the base is the height of the cone. The circular base has measured value of radius. And the length of the cone from vertex to any point on the circumference of the base is the slant height.



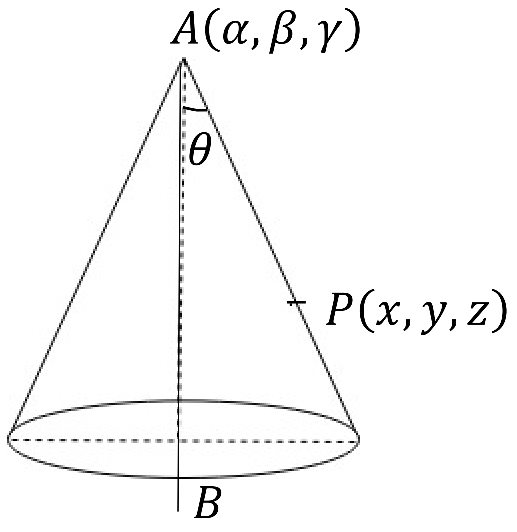
❖ Right Circular Cone:

- A right circular cone is asurface generated by a straight line which passes through a fixed point and makes constant angle with a fixed straight line through the vertex.
- The fixed point is called Vertex, the fixed line is called Axis of the cone and angle is known as Semi-vertical angle



Ex. Ice-cream cones, a clown’s cap, a tent etc. have right circular cone shape.

- To find the equation of right circular cone, three parameters are required:
  1) Vertex 2) Direction ratios of axis 3) Semi-vertical angle
- How to find the equation of right circular cone whose vertex is at (α, β, γ),semi-vertical angle θ and axis is the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$.



Let AB be the axis of the cone with direction ratios l, m, n. Let $P(x, y, z)$ be any point on the cone and $A(\alpha, \beta, \gamma)$ be the vertex.

Direction ratios of line AP are $x - \alpha, y - \beta, z - \gamma$

Let θ be an angle between axis AB and line AP.

$$\therefore \cos\theta = \frac{l(x-\alpha) + m(y-\beta) + n(z-\gamma)}{\sqrt{(l^2+m^2+n^2)}\sqrt{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}}$$

Simplifying this, we get required equation of right circular cone.

Ex.1 Find Equation of Right Circular cone with vertex at (1,2,3) and axis of cone $\frac{x-1}{2}=\frac{y-2}{-1}=\frac{z-3}{4}$ and semi-Vertical angle 60^0.

Sol.

Here, $\alpha = 1, \beta = 2, \gamma = 3,\ l = 2, m = -1, n = 4,\ \theta = 60$

Using, $\cos\theta = \frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{\sqrt{(l^2+m^2+n^2)}\sqrt{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}}$

We get, $\cos 60 = \frac{2(x-1)-1(y-2)+4(z-3)}{\sqrt{(4+1+16)}\sqrt{(x-1)^2+(y-2)^2+(z-3)^2}}$

$$\frac{1}{2} = \frac{2x-y+4z-12}{\sqrt{21}\sqrt{(x-1)^2+(y-2)^2+(z-3)^2}}$$

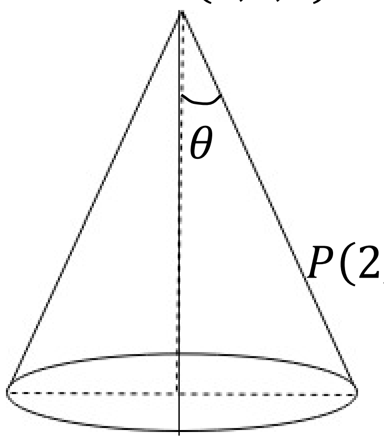
Squaring both sides, $\frac{1}{4} = \frac{(2x-y+4z-12)^2}{21\,[(x-1)^2+(y-2)^2+(z-3)^2]}$

$$\boxed{21\,[(x-1)^2+(y-2)^2+(z-3)^2] = 4(2x-y+4z-12)^2}$$

This is the required equation of right circular cone.

Ex.2 Find Semi- vertical angle of right circular cone which passes through the point (2,1,3) with vertex at (1,1,2) and axis parallel to the line $\frac{x-2}{2}=\frac{y-1}{-4}=\frac{z+2}{3}$.

Sol.



Let θ be the angle between line AP and axis AB.

Direction ratios of line AP are (2-1,1-1,3-2)=(1,0,1)

Direction ratios of axis are 2,-4,3.

Using formula for angle between two lines,we get

$$\cos\theta=\frac{(2)(1)+(-4)(0)+(3)(1)}{\sqrt{2^2+(-4)^2+3^2}\sqrt{1^2+0^2+1^2}}=\frac{5}{\sqrt{29}\sqrt{2}}$$

$$\boxed{\therefore\ \cos\theta=\frac{5}{\sqrt{58}}}$$

Solve: Find semi vertical angle of right circular cone which passes through the point (1,1,2) and has its axis as Line $6x=-3y=4z$ and vertex origin. Hint: Axis line can be written as $\frac{6x}{12}=\frac{-3y}{12}=\frac{4z}{12}\therefore\ \frac{x}{2}=\frac{y}{-4}=\frac{z}{3}$. Thus, direction ratios of axis are 2,-4,3. Vertex is origin (0,0,0).Now find semi-vertical angle as done in Ex.2.

Solve: Find the equation of right circular cone with vertex at $(0,0,0)$ and axis of cone $\frac{x}{2} = \frac{y}{-1} = \frac{z}{2}$ with semi vertical angle $\frac{\pi}{4}$

Solve: Find the direction ratios of axis of right circular cone which passes through the point (1,1,2) and has its axis as Line $2x = -y = 4z$ and vertex origin.

Solve: Find the equation of right circular cone with vertex at $(0, 0, 0)$, axis is the z axis and semi vertical angle 45^0. Hint: Axis is Z-axis. Hence direction ratios of axis are 0,0,1.Now solve as done in Ex.1.

Solve: Find the semi- vertical angle of right circular cone with vertex at (0,0,2) direction ratios of generator are 0, 3, -2 and axis is z-axis. Hint: direction ratios of Z-axis are (0,0,1)=(a_1, b_1, c_1) and direction ratios of generator are (0,3,-2)=(a_2, b_2, c_2). Find angle between axis and generator by using $\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1{}^2+b_1{}^2+c_1{}^2}\sqrt{a_2{}^2+b_2{}^2+c_2{}^2}}$