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Jacobians

If $u = f(x, y)$ and $v = g(x, y)$ then Jacobian of u, v w.r.t x, y denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ and is defined as

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

The Jacobian of u, v, w w.r.t x, y, z is given by

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

If $J = \frac{\partial(u,v)}{\partial(x,y)}$ then $\frac{\partial(u,v)}{\partial(x,y)}$ is denoted by J^* or J'.

If $J \neq 0$ the $J\,J' = 1$.

Chain Rule of Jacobians

If x, y be functions of u, v and u, v functions of r, s such that

$$\left.\begin{array}{ll} x = \phi_1(u, v); & y = \phi_2(u, v) \\ u = \psi_1(r, s); & v = \psi_2(r, s) \end{array}\right\}$$

Then

$$\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(r,s)} = \frac{\partial(x,y)}{\partial(r,s)}$$

Jacobian of Implicit Functions

If u, v are functions of x, y and f, g be implicit functions of u, v, x, y such that $f(u, v, x, y) = 0$ and $g(u, v, x, y) = 0$ then

$$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\frac{\partial(f,g)}{\partial(x,y)}}{\frac{\partial(f,g)}{\partial(u,v)}}$$

Similarly,

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\frac{\partial(f,g,h)}{\partial(x,y,z)}}{\frac{\partial(f,g,h)}{\partial(u,v,w)}}$$

Partial derivative of Implicit Function By Using Jacobian

$$\frac{\partial u}{\partial x} = \frac{-\frac{\partial(f_1, f_2)}{\partial(x, v)}}{\frac{\partial(f_1, f_2)}{\partial(u, v)}}$$

$$\frac{\partial u}{\partial x} = \frac{-\frac{\partial(f_1, f_2)}{\partial(u, x)}}{\frac{\partial(f_1, f_2)}{\partial(u, v)}}$$

Functional Dependence

Let $u = f(x, y)$ and $v = g(x, y)$ be given differentiable functions, then u and v are said to be functionally dependent if either u is a function of v or v is a function of u.

Two functions u and v are functionally dependent if their Jacobian is zero, i.e.,

$$\frac{\partial(u,v)}{\partial(x,y)} = 0$$

If $\frac{\partial(u,v)}{\partial(x,y)} \neq 0$ then u and v are said to be functionally independent.

Examples :

EX.1: Given $u = x^2 - y^2$ and $v = 2xy$. Calculate $\frac{\partial(u,v)}{\partial(x,y)}$.

Solution: Given: $u_x = 2x, \quad u_y = -2y, \ v_x = 2y, \ v_y = 2x.$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4\begin{vmatrix} x & -y \\ y & x \end{vmatrix} = 4(x^2 + y^2)$$

EX.2: If $x = uv$ and $y = \frac{u+v}{u-v}$, Find $\frac{\partial(u,v)}{\partial(x,y)}$.

Solution: Here x,y are functions of u,v. Hence it is easier to find $J = \frac{\partial(x,y)}{\partial(u,v)}$ and then we shall apply the formula $J.\ J'=1$ to obtain $J' = \frac{\partial(u,v)}{\partial(x,y)}$.

$$\frac{\partial x}{\partial u} = v,\ \frac{\partial x}{\partial v} = u,\ \frac{\partial y}{\partial u} = \frac{u-v-(u+v)}{(u-v)^2} = \frac{-2v}{(u-v)^2},\ \frac{\partial y}{\partial v} = \frac{u-v+(u+v)}{(u-v)^2} = \frac{2u}{(u-v)^2}$$

$$\therefore J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{-2v}{(u-v)^2} & \frac{2u}{(u-v)^2} \end{vmatrix}$$

$$= \frac{uv}{(u-v)^2} \begin{vmatrix} 1 & 1 \\ -2 & 2 \end{vmatrix} = \frac{4uv}{(u-v)^2}$$

Hence, $J' = \frac{\partial(u,v)}{\partial(x,y)} = \frac{(u-v)^2}{4uv}$

EX.3: If $x = u(1-v), y = uv$, show that $JJ' = 1$

Solution: We first find J: Here x,y→ u,v.

Let $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u - uv + uv = u$

Next, we shall obtain $J' = \frac{\partial(u,v)}{\partial(x,y)}$ by the method of implicit function. From given relation, $f_1 = x - u + uv, f_2 = y - uv$

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\frac{\partial(f_1,f_2)}{\partial(x,y)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}$$

Here, $\frac{\partial(f_1,f_2)}{\partial(x,y)} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$ **and** $\frac{\partial(f_1,f_2)}{\partial(u,v)} = \begin{vmatrix} -1+v & u \\ -v & -u \end{vmatrix} = u$

Putting in equation (1) the values of these Jacobians, we have

$J' = \frac{1}{u}$.

$\therefore JJ' = u.\frac{1}{u} = 1$

Ex.4: If $x = u + v + w$, $y = u^2 + v^2 + w^2$, $z = u^3 + v^3 + w^3$ then show that $\frac{\partial u}{\partial x} = \frac{vw}{(u-v)(u-w)}$

Solution : Here

$$f_1 = x - u - v - w$$

$$f_2 = y - u^2 - v^2 - w^2$$

$$f_3 = z - u^3 - v^3 - w^3$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\frac{\partial(f_1, f_2, f_3)}{\partial(x, v, w)}}{\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)}} = \frac{\begin{vmatrix} 1 & -1 & -1 \\ 0 & -2v & -2w \\ 0 & -3v^2 & -3w^2 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & -1 \\ -2u & -2v & -2w \\ -3u^2 & -3v^2 & -3w^2 \end{vmatrix}}$$

$$\frac{\partial u}{\partial x} = \frac{vw}{(u-v)(v-w)}$$

Ex.5: Ascertain whether the following functions are functionally dependent, if so find the relation between them

$$u = \frac{x+y}{1-xy}, v = \tan^{-1} x + \tan^{-1} y.$$

Solution: $\frac{\partial u}{\partial x} = \frac{(1-xy)-(x+y)(-y)}{(1-xy)^2} = \frac{1-xy+xy+y^2}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$

$$\frac{\partial u}{\partial y} = \frac{(1-xy)-(x+y)(-x)}{(1-xy)^2} = \frac{1-xy+xy+x^2}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2}$$

$\frac{\partial v}{\partial x} = \frac{1}{1+x^2} + 0 = \frac{1}{1+x^2}, \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

Thus, $J = \frac{\partial(u,v)}{\partial(x,y)} = 0$, Hence u and v are functionally dependent.

Relation between u and v: we have

$$v = \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1} u$$

$$\therefore \quad v = \tan^{-1} u$$

Error and Approximations

Let $z = f(x,y)$ and dx, dy be small changes in x and y respectively then the corresponding change in f is given by

$$df = f\,(x + dx,\, y + dy\,) - f(x,y)$$

$$= \frac{dy}{dx}\,dx + \frac{df}{dy}\,dy$$

dx, dy, df is called as actual errors in x, y, f resp.

$\frac{dx}{x}, \frac{dy}{y}, \frac{df}{f}$ is called as relative errors in x, y, f resp.

$\frac{100dx}{x}, \frac{100dy}{y}, \frac{100df}{f}$ is called as % errors in x, y, f resp.

EX: In calculating volume of right circular cylinder, errors of 2 % and 1% are found in measuring height and base radius respectively. Find the percentage error in circulating volume of the cylinder.

Solution : If r and h are base radius and height of the right circular cylinder, its volume V is given by;

$$V = \pi r^2 h$$

Taking log of both sides,

$$\log V = \log \pi + 2 \log r + \log h$$

Taking differentials , we have

$$\frac{dV}{V} = 2\frac{dr}{r} + \frac{dh}{h}$$

$$\frac{100dV}{V} = 2\left(\frac{100dr}{r}\right) + \left(\frac{100dh}{h}\right) = 2(1) + (2) = 4\%$$

% error in V = $\frac{100dV}{V}$ = 4%

EX : Find the possible percentage error in computing the parallel resistance r of three resistances r_1, r_2, r_3 from the formula $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$ if r_1, r_2, r_3 are each in error by plus 1.2 %.

Solution : Here $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$. (I)

Differentiating , we get

$$-\frac{1}{r^2}dr = -\frac{1}{{r^2}_1}dr_1 - \frac{1}{{r^2}_2}dr_2 - \frac{1}{{r^2}_3}dr_3$$

$$\frac{1}{r}\left(\frac{100dr}{r}\right) = \frac{1}{r_1}\left(\frac{100dr_1}{r_1}\right) + \frac{1}{r_2}\left(\frac{100dr_2}{r_2}\right) + \frac{1}{r_3}\left(\frac{100dr_3}{r_3}\right)$$

$$= \frac{1}{r_1}(1.2) + \frac{1}{r_2}(1.2) + \frac{1}{r_3}(1.2) = (1.2)\left[\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right]$$

$$= 1.2\left(\frac{1}{r}\right) \text{ from (I)}$$

$\frac{100dr}{r}$ = 1.2 % = the % error in r.

EX : Find the percentage error in the area of an ellipse when an error of 1% is mode in measuring its major and minor axes.

Solution : If A is area and 2a and 2b are the major and minor axes of the ellipse having equation $\frac{x^2}{a^2} + \frac{y^2}{b^2}$ = 1 then

A = π a b

Taking log of both sides, log A = log π + log a + log b

Differentiating, we have

$$\frac{dA}{A} = 0 + \frac{da}{a} + \frac{db}{b}$$

$$\frac{100\ dA}{A} = \frac{100\ da}{a} + \frac{100\ db}{b}$$

Given that percentage errors $\frac{100\ da}{a}$ and $\frac{100\ db}{b}$ each are equal to 1, hence

$$\frac{100\ dA}{A} = 1 + 1 = 2$$

Percentage error in the area A = 2% .