

EAS596

Homework 1

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1. In the xy plane, mark all nine of these linear combinations:

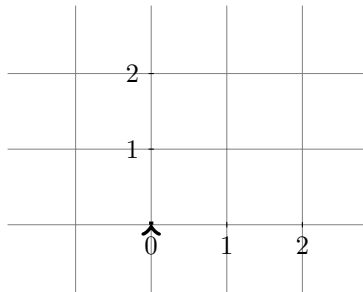
$$y = c \begin{bmatrix} 2 \\ 1 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{with} \quad c = 0, 1, 2 \quad \text{and} \quad d = 0, 1, 2$$

Solution :

$$y = \begin{bmatrix} 2c \\ c + d \end{bmatrix}$$

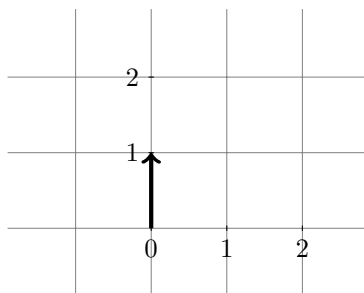
(a) When c,d = 0,0

$$y = \begin{bmatrix} 2c \\ c + d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



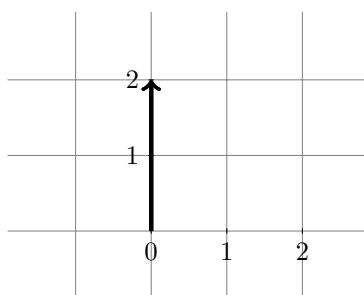
(b) When c,d = 0,1

$$y = \begin{bmatrix} 2c \\ c + d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



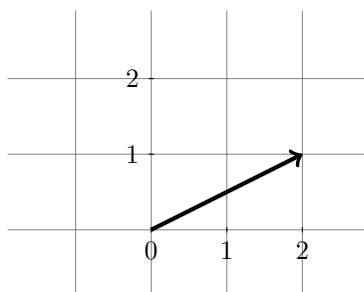
(c) When $c, d = 0, 2$

$$y = \begin{bmatrix} 2c \\ c + d \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$



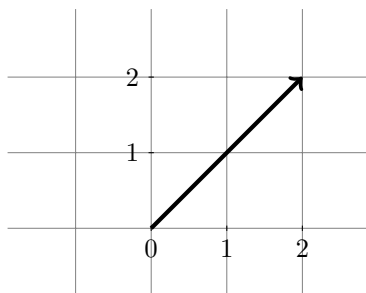
(d) When $c, d = 1, 0$

$$y = \begin{bmatrix} 2c \\ c + d \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



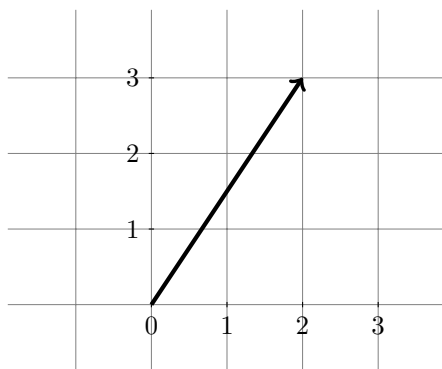
(e) When $c, d = 1, 1$

$$y = \begin{bmatrix} 2c \\ c + d \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$



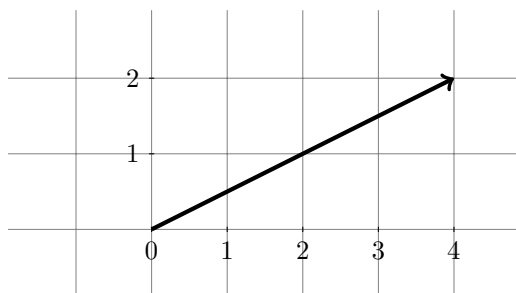
(f) When $c, d = 1, 2$

$$y = \begin{bmatrix} 2c \\ c + d \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$



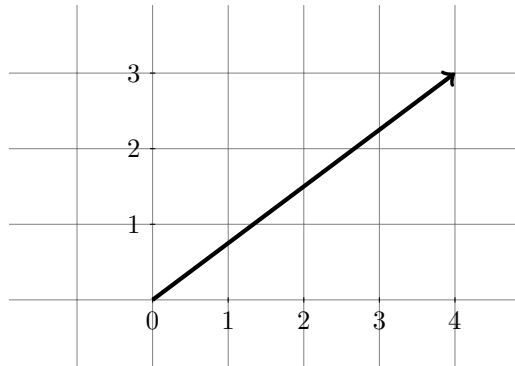
(g) When $c, d = 2, 0$

$$y = \begin{bmatrix} 2c \\ c + d \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$



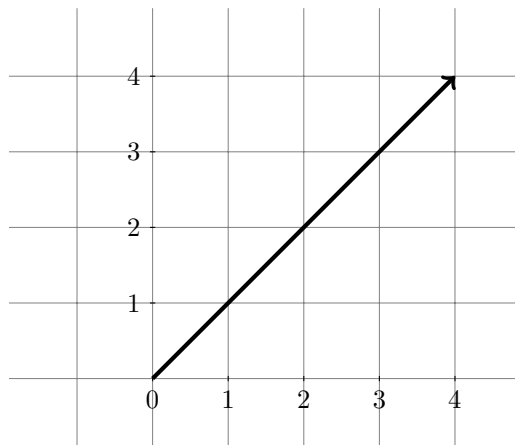
(h) When $c, d = 2, 1$

$$y = \begin{bmatrix} 2c \\ c + d \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$



(i) When $c, d = 2, 2$

$$y = \begin{bmatrix} 2c \\ c + d \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$



2. (a) Find vector v and w so that $v + w = (4, 5, 6)$ and $v - w = (2, 5, 8)$.
Solution: We have

$$v + w = (4, 5, 6) \tag{1}$$

$$v - w = (2, 5, 8) \tag{2}$$

Adding (1) and (2), we get

$$\begin{aligned} 2v &= (4 + 2, 5 + 5, 6 + 8) = (6, 10, 14) \\ \implies v &= \left(\frac{6}{2}, \frac{10}{2}, \frac{14}{2} \right) \implies v = (3, 5, 7) \end{aligned} \tag{3}$$

Now subtracting (2) from (1), we get

$$\begin{aligned} (v + w) - (v - w) &= (4, 5, 6) - (2, 5, 8) \\ \implies 2w &= (2, 0, -2) \\ \implies w &= (1, 0, -1) \end{aligned}$$

2. (b) This is a equation with 2 unknown numbers and an equal number of equations to find those numbers.

This answer is assuming just vectors as unknown but if we solve it assuming variables for each elements of the vectors v and w , we would get 6 unknowns with 6 equations.

3. (a) Find unit vectors u_1 and u_2 in the directions of $v = (3, 1)$ and $w = (2, 1, 2)$.

Solution: A unit vector in the direction of u_1 is $\left(\frac{u_1}{\|u_1\|}\right) = \left(\frac{(3,1)}{\sqrt{3^2+1^2}}\right) = \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$

And a unit vector in the direction of u_2 is $\left(\frac{u_2}{\|u_2\|}\right) = \left(\frac{(2,1,2)}{\sqrt{2^2+1^2+2^2}}\right) = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$

3. (b) Find unit vectors U_1 and U_2 that are perpendicular to u_1 and u_2 .

Solution: Let $U_1 = (a, b)$ be the unit vector perpendicular to u_1 . So we have

$$\|(a, b)\| = 1 \implies a^2 + b^2 = 1 \quad (4)$$

$$\text{and, } (a, b) \cdot \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right) = \|(a, b)\| * \left\|\left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)\right\| \cos \theta$$

$$\cos \theta = 0 \quad \text{because } \theta = 90^\circ$$

$$\implies 3a + b = 0 \implies b = -3a \quad (5)$$

Solving equation (4) and (5), we get $a = \pm(\frac{1}{\sqrt{10}})$ and $b = \mp(\frac{3}{\sqrt{10}})$

So $U_1 = (\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}})$

Now, let $U_2 = (x, y, z)$ be the unit vector perpendicular to u_2 . So let's can find a vector in the x-y plane which is perpendicular to u_2 . Let $z = 0$

$$\|(x, y, z)\| = 1 \implies x^2 + y^2 + z^2 = 1 \implies x^2 + y^2 = 1 \quad (6)$$

$$\text{and, } (x, y, z) \cdot (1, 0.5, 1) = 0 \implies x + 0.5y = 0 \quad (7)$$

Solving equation (6) and (7), we get $x = \pm(\frac{1}{\sqrt{5}})$ and $b = \mp(\frac{2}{\sqrt{5}})$

So $U_2 = (\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}, 0)$

4. (a) If $\|v\| = 5$ and $\|w\| = 3$, what are the smallest and the largest values of $\|v - w\|$

Solution: Using law of cosines we have

$$\|v - w\|^2 = \|v\|^2 + \|w\|^2 - 2 * \|v\| * \|w\| \cos \theta$$

Plugging in the values we get, $\|v - w\| = \sqrt{5^2 + 3^2 - 2 * 5 * 3 \cos \theta}$

So, the largest value of $\|v - w\|$ will be when $\cos \theta$ is -1 and smallest when $\cos \theta = 1$. Therefore the largest value is $\sqrt{5^2 + 3^2 + 2 * 5 * 3} = \sqrt{64} = 8$. And the smallest value is $\sqrt{5^2 + 3^2 - 2 * 5 * 3} = \sqrt{4} = 2$

4. (b) What are the smallest and largest values of $v \cdot w$?

Solution: We know that $v \cdot w = \|v\| \|w\| \cos \theta \implies v \cdot w = 15 \cos \theta$

The maximum and minimum values of $\cos \theta$ is 1 and -1. So the maximum and minimum values of $v \cdot w$ is 15 and -15 respectively.

5. Consider a parallelogram with sides given by v and w . Show that $\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$.

Solution:

Using law of cosines we can write

$$\begin{aligned} \|v + w\|^2 + \|v - w\|^2 &= (\|v\|^2 + \|w\|^2 + 2\|v\| \|w\| \cos \theta) + (\|v\|^2 + \|w\|^2 - 2\|v\| \|w\| \cos \theta) \\ \implies \|v + w\|^2 + \|v - w\|^2 &= 2\|v\|^2 + 2\|w\|^2 \end{aligned}$$

6. Find a combination $x_1 w_1 + x_2 w_2 + x_3 w_3$ that gives the zero vector. Are these vectors independent or dependent?

$$w_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, w_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, w_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

$$\textbf{Solution: } x_1 w_1 + x_2 w_2 + x_3 w_3 = x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 4x_2 + 7x_3 = 0 \tag{8}$$

$$2x_1 + 5x_2 + 8x_3 = 0 \tag{9}$$

$$3x_1 + 6x_2 + 9x_3 = 0 \tag{10}$$

We see that $w_1 + w_3 = 2w_2$, which means that these are dependent vectors. It also means that there will be infinitely many solutions to the equations above. So, let's find one of those solutions.

(9)-2*(8) gives,

$$-3x_2 - 6x_3 = 0 \tag{11}$$

$$\implies x_2 = -2x_3$$

So, if $x_3 = 1$, then $x_2 = -2$, and putting these values in (8), we get $x_1 = 1$. So, one of the value of (x_1, x_2, x_3) is (1,-2,1)

7. Let the following matrices be defined.

$$A = \begin{bmatrix} 4 & 7 \\ 1 & 2 \\ 5 & 6 \end{bmatrix}, B = \begin{bmatrix} 4 & 3 & 7 \\ 1 & 2 & 7 \\ 2 & 0 & 4 \end{bmatrix}, C = \begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix}, D = \begin{bmatrix} 9 & 4 & 3 & -6 \\ 2 & -1 & 7 & 5 \end{bmatrix}, E = \begin{bmatrix} 1 & 5 & 8 \\ 7 & 2 & 3 \\ 4 & 0 & 6 \end{bmatrix},$$

$$F = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 7 & 3 \end{bmatrix}, G = \begin{bmatrix} 7 & 6 & 4 \end{bmatrix}$$

Answer the following regarding these matrices:

(a) What are the dimensions of the matrices?

Solution: $\dim(A) = (3,2)$, $\dim(B) = (3,3)$, $\dim(C) = (3,1)$, $\dim(D) = (2,4)$, $\dim(E) = (3,3)$, $\dim(F) = (2,3)$, $\dim(G) = (1,3)$

(b) Identify the square, column and row matrices.

Solution:

Square Matrices = B, E

Row Matrices = G

Column Matrices = C

(c) What are the values of the elements: a_{12} , b_{23} , d_{32} , e_{22} , f_{12} , g_{12}

Solution: $a_{12} = 7$, $b_{23} = 7$, d_{32} = Does not exist, $e_{22} = 2$, $f_{12} = 0$, $g_{12} = 6$

d Perform the following operations or, if not well defined, explain why:

$$(i) E + B = \begin{bmatrix} 5 & 8 & 15 \\ 8 & 4 & 10 \\ 6 & 0 & 10 \end{bmatrix}$$

(ii) $A + F$ = Cannot perform operation because for addition the dimensions of both the matrices should match.

$$(iii) B - E = \begin{bmatrix} 3 & -2 & -1 \\ -6 & 0 & 4 \\ -2 & 0 & -2 \end{bmatrix}$$

(iv) $A \times B$ = Cannot perform operation as matrix-multiplication needs the inner dimensions of the matrices to be the same, here they are different, 2 and 3 respectively.

$$(v) B \times A = \begin{bmatrix} 54 & 76 \\ 41 & 53 \\ 28 & 38 \end{bmatrix}$$

$$\text{(vi)} \quad D^T = \begin{bmatrix} 9 & 2 \\ 4 & -1 \\ 3 & 7 \\ -6 & 5 \end{bmatrix}$$

$$\text{(vii)} \quad E \times B = \begin{bmatrix} 25 & 13 & 74 \\ 36 & 25 & 75 \\ 28 & 12 & 52 \end{bmatrix}$$

$$\text{(viii)} \quad C^T = [3 \quad 6 \quad 1]$$

(ix) $A \times C$ = Operation cannot be performed because inner dimension of the matrices E and B does not match.

(x) $I \times B = B$. Identity matrix multiplied(cross-product) to any matrix yields the same matrix.

$$\text{(xi)} \quad E^T \times E = \begin{bmatrix} 1 & 7 & 4 \\ 5 & 2 & 0 \\ 8 & 3 & 6 \end{bmatrix} \times \begin{bmatrix} 1 & 5 & 8 \\ 7 & 2 & 3 \\ 4 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 66 & 19 & 53 \\ 19 & 29 & 46 \\ 53 & 46 & 109 \end{bmatrix}$$

$$\text{(xii)} \quad C^T \times C = [3 \quad 6 \quad 1] \times \begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix} = 46.$$

8. Let the following matrices be defined.

$$A = \begin{bmatrix} 5 & 6 & 6 & 8 \\ 2 & 2 & 2 & 8 \\ 6 & 6 & 2 & 8 \\ 2 & 3 & 6 & 7 \end{bmatrix}, B = \begin{bmatrix} 17 & -9 & 12 & 16 \\ 17 & 8.75 & -11.75 & -16 \\ -4 & -2.25 & 2.75 & 4 \\ 1 & 0.75 & -0.75 & -1 \end{bmatrix}, C = \begin{bmatrix} -17 & -9 & 12 & 16 \\ 17 & 8.75 & -11.75 & -16 \\ -4 & -2.25 & 2.75 & 4 \\ 1 & 0.75 & -0.75 & -1 \end{bmatrix}$$

Which matrix, B or C is an inverse to matrix A?

Solution: Inverse of a matrix multiplied to itself yields an identity matrix. So the matrix, which when multiplied with A produces an identity matrix is the inverse of A. Let's see.

$$A \times B = \begin{bmatrix} 5 & 6 & 6 & 8 \\ 2 & 2 & 2 & 8 \\ 6 & 6 & 2 & 8 \\ 2 & 3 & 6 & 7 \end{bmatrix} \times \begin{bmatrix} 17 & -9 & 12 & 16 \\ 17 & 8.75 & -11.75 & -16 \\ -4 & -2.25 & 2.75 & 4 \\ 1 & 0.75 & -0.75 & -1 \end{bmatrix} = \begin{bmatrix} 171 & 0 & 0 & 0 \\ 68 & 1 & 0 & 0 \\ 204 & 0 & 1 & 0 \\ 68 & 0 & 0 & 1 \end{bmatrix}$$

$$A \times C = \begin{bmatrix} 5 & 6 & 6 & 8 \\ 2 & 2 & 2 & 8 \\ 6 & 6 & 2 & 8 \\ 2 & 3 & 6 & 7 \end{bmatrix} \times \begin{bmatrix} -17 & -9 & 12 & 16 \\ 17 & 8.75 & -11.75 & -16 \\ -4 & -2.25 & 2.75 & 4 \\ 1 & 0.75 & -0.75 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So, we found that C is the inverse of A.