EAS 596, Fall 2018, Homework 9

Due Friday 11/16, **9 AM**, Box outside Furnas 611

Work all problems. Unless otherwise stated, only MATLAB code will be accepted. Show all work, including any electronic or M-files you have written or adapted. Make sure your work is clear and readable - if the TA cannot read what you've written, he will not grade it. All electronic work (m-files, etc.) **must** be submitted through UBLearns and submitted by the due time. Any handwritten work may be submitted by the due time. Each problem will be graded according to the following scheme: 2 points if the solution is complete and correct, 1 point if the solution is incorrect or incomplete but was using correct ideas, and 0 points if using incorrect ideas.

1. For the following functions compute the Fourier series:

   - $f_1(x) = \sin^2(x)$
   - $f_2(x) = \begin{cases} \mathrm{x}+\pi & \text{if} \quad -\pi \leq \mathrm{x} < 0 \\ \mathrm{x} & \text{if} \quad 0 \leq \mathrm{x} < \pi \end{cases}$.

   (a) Write down the integral formula to find the coefficients of the Fourier Series for each function; use the period $-\pi \leq x \leq \pi$.

   (b) In two separate graphs (one corresponding to each function), plot the actual function and their Fourier series approximation for $N = 5$ and $N = 100$, where N is the number of terms kept in the Fourier series.

2. Another physical problem, based on PDE models, for which we can use the method of separation of variables, is the two-dimensional Laplace equation. This is a model for steady-state heat transfer in a two-dimensional body.

   Consider the rectangular domain $x \in [0, a]$ and $y \in [0, b]$. We wish to find the function $u(x, y)$ over the domain. The governing PDE is

   $$\nabla^2 u = 0$$

   with boundary conditions:

   $$u(x, 0) = 0$$
   $$u(x, b) = 5$$
   $$u(0, y) = 0$$
   $$u(a, y) = 0$$

(a) Assume the solution $u(x, y) = X(x)Y(y)$ and develop ODEs for X and Y.

(b) Using the boundary conditions, determine what must be the form of the functions $X(x)$ and $Y(y)$.

(c) Determine the coefficients and write down the complete solution for $u(x, y)$.

3. In this problem you will explore finite difference approximations to derivatives. In all cases you will be using a periodic function $f(x)$ with a periodicity of $[a, b]$ such that $f(a) = f(b)$.

(a) Using the Taylor Series expansion, verify that the second-order center finite difference approximation to the second derivative, $(f_{i+1} - 2f_i + f_{i-1})/(h^2)$ is second-order accurate. What is the exact form for the error?

(b) Let f^h_{xx} be the second-order center finite difference approximation to the second derivative. If you take second-order center finite difference approximation of f^h_{xx} it turns out you get the following:

$$f^h_{xxxx} = \frac{d^4 f}{dx^4} + \frac{h^2}{6}\frac{d^6 f}{dx^6} + \mathcal{O}(h^4).$$

Show how you can use this fact to get a 4^{th}-order approximation to the second derivative of $f(x)$. You must justify your answer.

(c) Construct a MATLAB function called `secondDer_Order2` that takes in an array of function values defined over the periodicity $[a, b]$ and the grid-spacing h. The function will return the second-order approximation to the second derivative over that range. Note that when computing the derivative, the function value at $f(a - h) = f(b - h)$ and $f(b + h) = f(a + h)$. *CAUTION: BE VERY CAREFUL AT THE END POINTS!*

(d) Construct a MATLAB function called `secondDer_Order4` that takes in an array of function values defined over the periodicity $[a, b]$ and the grid-spacing h. The function will return the fourth-order approximation to the second derivative over that range. You should be using the funtion `secondDer_Order2` in this new function.

(e) Use the two functions that you have created to compute the derivative of the function $f(x) = \sin(x)$ over the range $[0, 2\pi]$ for 10 grid points (hint: use `linspace`). Plot the calculated second derivatives and the actual derivative on a single figure.

(f) Perform an error analysis using grid points ranging from 100 to 10000 in increments of 100 (*i.e.* `[100:100:10000]`) using the function $f(x) = \sin(x)$ over the interval $[0, 2\pi]$. Show the error of your second- and fourth-order schemes as a function of grid size h on a log-log plot. What happens at small values of h for the fourth-order scheme and why?

(g) Finally, let's explore the influence of noise on the result. Let `f` be the data. You can create a function with random error of order p by using the MATLAB code
`f + 2*(h^p)*(rand(size(f))+0.5)`. Re-do the error analysis from the prior part using error orders of 2, 4, 6, and 8 and create separate figures for each error order. Provide a qualitative understanding of what the results are showing you.

4. Consider the pendulum show in Figure 1. Assume we wish to study the natural motion of the system, i.e. the term $M_t(t)$ in the figure is zero. The equation of motion of the system is given by

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin(\theta) = 0$$

where g is the acceleration due to gravity. This is a *nonlinear* ordinary differential equation for $\theta(t)$.

(a) Using a Taylor series centered at $\theta = 0$, linearize the nonlinear term in the equation of motion. That is, keep only up to first order terms in the Taylor series. (This is the "small angle" approximation.) Write down the equation of the motion of the pendulum assuming the small angle approximation. We have now *linearized* the equation. Demonstrate that this equation is now linear.

(b) Using MATLAB, e.g. to generate a plot, give an estimate of θ for which the small angle approximation ceases to be valid, say 5% error.

(c) Develop the analytical solution of the differential equation with the small angle approximation.

(d) Assume $g = 9.81$ m/s^2 and $l = 1$ m and $\theta(0) = 5^o$, $\dot{\theta}(0) = 0$. Write down the solution for the motion of the pendulum under the small angle approximation.

(e) Now rewrite this single second order differential equation as a system of first order equations.

(f) Write a MATLAB script that uses `ode45` to solve this system of differential equations using the parameters from the previous part. Plot the solution, θ, together with the analytic solution for the small angle approximation.

(g) Repeat all steps of the previous part, but now with initial conditions $\theta(0) = 45^o$, $\dot{\theta}(0) = 0$.

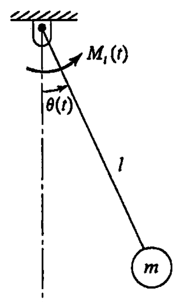

Figure 1: Pendulum for Problem 4.

5. Consider the differential equation

$$\frac{dx}{dt} = -700x - 1000e^{-t}$$

This is a stiff differential equation whose solution has a very sharp initial transient followed by a slower mode. Explicit methods will typically have a much more limited range of stability and, therefore, require much smaller timestep sizes.

(a) Write a MATLAB script that solves this differential equation using `ode45` on the interval $t = [0, 5]$, with $x(0) = 4$. How many time steps did `ode45` require?

(b) Repeat using `ode23s`, a solver more tailored towards stiff differential equations. How many time steps were required?