EAS 595 PROB HW2

Prof: Ehsan Esfahani

Abhishek Kumar
UB# 50291056

1. Chapter 2 Problems.

3. P(Fischer wins the match) follows a geometric distribution (sum), hence

ⓐ $$P(\text{Fischer wins the match}) = \sum_{n=1}^{10} \underbrace{(0.4)}_{\text{wins}} \underbrace{(0.3)^{n-1}}_{\text{draws}}$$

$$= \frac{0.4(1-0.3^9)}{1-0.3} = \frac{0.4}{0.7} \times (1-0.3^9)$$

$$= 0.5714173$$

ⓑ For $n < 10$, n matches are played if $(n-1)$ matches are drawn and 1 match is won by either Fischer or Spassky. And $n=10$ matches are played only when 9 matches are drawn and does not depend on the result of the 10th match.

So $$P_N(n) = P_N(N=n) = \begin{cases} (0.7)(0.3)^{n-1} & n = 1, 2, \ldots, 8, 9, \\ (0.3)^9 & n = 10, \\ 0, & \text{otherwise} \end{cases}$$

(18) X will be equiprobable for $2^a, 2^{a+1}, \ldots 2^b$, so the PMF of X will be

$$P_X(x) = \begin{cases} \dfrac{1}{b-a+1} & x = 2^k \text{ and } a \leq k \leq b, \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore E[X] = \sum_{k=a}^{b} \frac{1}{b-a+1} \cdot 2^k = \frac{1}{b-a+1} \cdot \sum_{k=a}^{b} 2^k$$

$$= \frac{1}{b-a+1} \left(2^a + 2^{a+1} + 2^{a+2} + \cdots + 2^b \right)$$

$$= \frac{2^a}{b-a+1} \left(1 + 2 + 2^2 + \cdots + 2^{b-a} \right)$$

$$= \frac{2^a}{b-a+1} \times \frac{1 \cdot \left(1 - 2^{b-a+1}\right)}{(1-2)} = \frac{2^a}{b-a+1} \times \left(2^{b-a+1} - 1\right)$$

$$\therefore E[X^2] = \sum_{k=a}^{b} \frac{1}{b-a+1} \cdot 2^{2k} = \frac{1}{b-a+1} \sum_{k=a}^{b} \left(4^a + 4^{a+1} + \cdots 4^b\right)$$

$$= \frac{1}{b-a+1} \cdot 4^a \left(1 + 4 + 4^2 + \cdots + 4^{b-a} \right)$$

$$= \frac{4^a}{b-a+1} \times \frac{\left(1 - 4^{b-a+1}\right)}{1-4} = \frac{4^a}{b-a+1} \frac{\left(4^{b-a+1} - 1\right)}{3}$$

$$\therefore Var(X) = E[X^2] - (E[X])^2$$

$$= \left[\frac{4^a}{b-a+1} \times \frac{(4^{b-a+1}-1)}{3}\right] - \left[\frac{2^a}{b-a+1} \times (2^{b-a+1}-1)\right]^2$$

(21) For $P(tail) = p$, the PMF of the count n, the number of tosses till the 1st tail appears is a geometric distribution

Hence,

$$P_N(N=n) = p(1-p)^{n-1}$$

Here, for a fair coin, $p = \frac{1}{2}$

$$\therefore P_N(N=n) = \frac{1}{2} \times \left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^n$$

And the amount ~~he~~ received at count n is 2^n

$\therefore$ The expected amount to be received is $= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \cdot 2^n$

$$= \sum_{n=1}^{\infty} 1$$

Oh!! Infinite. I would pay as much as he asked and I had at that time. :)

(36) To Show,

$$P_{X,Y,Z}(x,y,z) = P_X(x) P_{Y|X}(y|x) P_{Z|Y,X}(z|y,x)$$

$\Rightarrow$ We can see this as a chain rule where 1^{st} X happens with prob $= P_X(x)$, then Y happens given X has happend with prob $P_{Y|X}(y|x)$, and likewise further.

(a) $$P_{X,Y,Z}(x,y,z) = P(X=x) \cdot \underbrace{P(Y=y, Z=z \mid X=x)}_{y,z \text{ happens given } X \text{ has occured}}$$

$$= P(X=x) \cdot P(Y=y \mid X=x) \cdot P(Z=z \mid X=x, Y=y)$$

$$= P_X(x) \cdot P_{Y|X}(y|x) \cdot P_{Z|X,Y}(z|x,y)$$

(b) Multiplication rule states that:

Assuming that all of the conditioning events have positive probability, we have

$$P(\cap_{i=1}^{n} A_i) = P(A_1) P(A_2|A_1) P(A_3|A_1 \cap A_2) \cdots \cdots P(A_n \mid \cap_{i=1}^{n-1} A_i)$$

which can be viewed as one event occuring after other events have occured and its multiplication gives the final event that has occured.

Similar to the multiplication rule, we have here:

$$P(X=x)\, P(Y=y \mid X=x)\, P(Z=z \mid X=x, Y=y) =$$

$$P_{X,Y,Z}(x,y,z)$$

and can be seen as a special case of multiplication rule.

(c) The generalization is:

$$P_{X_1,X_2,\ldots X_n}(x_1,x_2,\ldots x_n) = P_{X_1}(x_1)\cdot P_{X_2|X_1}(x_2|x_1)\ldots\ldots$$

$$\ldots P_{X_n|X_1,X_2\ldots X_{n-1}}(x_n|x_1,x_2,\ldots x_{n-1})$$

2. Chapter 3.

(8) a). From total probability theorem, we have

$$F_X(x) = P(X \leq x) = p\,P(Y \leq x) + (1-p)\,P(Z \leq x)$$
$$= p\,F_Y(x) + (1-p)\,F_Z(x)$$

∴ Differentiating both sides, we get

$$f_X(x) = p\,f_Y(x) + (1-p)\,f_Z(x)$$

b) $$f_X(x) = \begin{cases} p\lambda e^{\lambda x} & \text{if } x < 0 \\ (1-p)\lambda e^{-\lambda x} & \text{if } x \geq 0 \end{cases}$$

for $x < 0$, we have

$$\therefore F_X(x) = P(X < x)$$

$$= \int_{-\infty}^{x} p\lambda e^{\lambda x} = \frac{p\cancel{\lambda}}{\cancel{\lambda}} e^{\lambda x}\Big|_{-\infty}^{x}$$

$$= pe^{\lambda x}$$

and for $x \geq 0$, we have

$$F_X(x) = P(X < x) = \int_{-\infty}^{x} f_X(x) = \int_{-\infty}^{0} f_X(x) + \int_{0}^{x} f_X(x)$$

$$= \int_{-\infty}^{0} p\lambda e^{\lambda x} + \int_{0}^{x} (1-p)\lambda e^{-\lambda x}$$

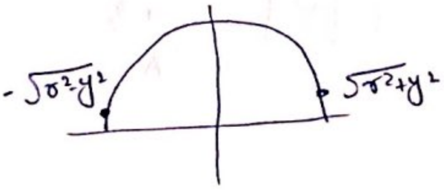
$$= p + \frac{(1-p)\lambda}{-\lambda} e^{-\lambda x}\Big|_{0}^{x}$$

$$= p + (p-1)(e^{-\lambda x} - 1) = \cancel{p} + pe^{-\lambda x} - \cancel{p} - e^{-\lambda x} + 1$$

$$= 1 - e^{-\lambda x}(1-p)$$

$$\therefore F_X(x) = \begin{cases} pe^{\lambda x} & x < 0 \\ 1 - e^{-\lambda x}(1-p) & x \geq 0 \end{cases}$$

15 a) Since the area under the semicircle is $\frac{\pi r^2}{2}$ and the the point has a uniform PDF, so



$$f_{X,Y}(x,y) = \begin{cases} \frac{2}{\pi r^2} & \text{for } x,y \text{ in the semicircle} \\ 0 & \text{otherwise.} \end{cases}$$

b) $$f_Y(y) = \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{2}{\pi r^2}\,dx$$

$$= 2\times 2 \int_0^{\sqrt{r^2-y^2}} \frac{1}{\pi r^2}\,dx = \frac{4}{\pi r^2}\cdot\sqrt{r^2-y^2}$$

$$\therefore f_Y(y) = \begin{cases} \frac{4}{\pi r^2}\sqrt{r^2-y^2} & 0 \leq y \leq r, \\ 0, & \text{otherwise} \end{cases}$$

$$\therefore E[Y] = \int_0^r y\cdot\frac{4}{\pi r^2}\sqrt{r^2-y^2}\,dy = \frac{4}{\pi r^2}\int_0^r \sqrt{r^2-y^2}\,y\,dy$$

Let $r^2-y^2 = z^2 \Rightarrow -2y\,dy = 2z\,dz$

$$\therefore E[Y] = \frac{4}{\pi r^2}\int_r^0 z\,(-2dz) = \frac{4}{\pi r^2}\int_0^r z^2\,dz$$

$\therefore \; E[Y] = \dfrac{4r}{3\pi}$

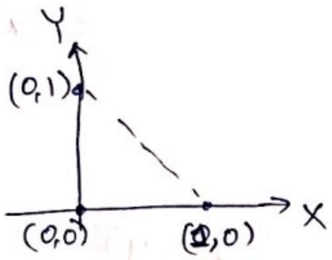
c) Without using marginal PDF of y :

$$E[Y] = \int_0^{r} \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2+y^2}} f_{X,Y}(x,y) \cdot y \, dx \, dy$$

$$= \int_0^{r} \frac{4}{\pi r^2} \sqrt{r^2-y^2} \cdot dy$$

$$= \frac{4r}{3\pi}$$

(23) (a) Area under the triangle is $\frac{1}{2}$ and the point has uniform PDF over the triangle, so



$$f_{X,Y}(x,y) = \begin{cases} \cancel{1}2. & \text{over the } \Delta \\ 0 & \text{otherwise} \end{cases}$$

(b) $f_Y(y) = \int_0^{1-y} 2 \cancel{\frac{1}{2}} \, dx$ | $y = -x + 1$

$$f_Y(y) = \begin{cases} \frac{(1-y)2}{\cancel{2}} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(c) $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)} = \dfrac{2}{2(1-y)} = \dfrac{1}{y-1}$

$$= \begin{cases} \frac{1}{1-y} & 0 \le x \le 1-y \\ 0 & \text{otherwise.} \end{cases}$$

(d) $E[X|Y=y] = \int_0^{1-y} x \cdot \underbrace{\frac{1}{(1-y)}}_{f_{X|Y}(x|y)} dx = \frac{1}{2(1-y)} x^2 \Big|_0^{1-y}$

$$= \frac{(1-y)^2}{2(1-y)} = \frac{1-y}{2}$$

$$\therefore E[X|Y=y] = \begin{cases} \frac{1-y}{2} & 0 \le y \le 1 \;\; \& \;\; y \neq 1 \end{cases}$$

Using total Expectation theorem, we have

$$E[X] = \int_0^1 E[X|Y=y] \cdot \cancel{E[Y]} \, f_Y(y) \, dy$$

$$= \int_0^1 \frac{1-y}{2} \cdot f_Y(y) \, dy$$

$$\Rightarrow E[X] = \frac{1}{2} - \frac{1}{2}\int_0^1 y \cdot f_Y(y)\,dy$$

$$= \frac{1}{2} - \frac{1}{2} E[Y]$$

(e) Due to symmetry, we must have $E[X] = E[Y]$

$\therefore$ We should have

$$E[X] = \frac{1}{2} - \frac{1}{2} E[X]$$

$$\Rightarrow \frac{3}{2} E[X] = \frac{1}{2} \quad \Rightarrow E[X] = 1/3$$

(34) Let X be the event that a toss results in heads.

(a) $\therefore P(X) = \int_0^1 P(X \mid P = p) \cdot f_P(p)\,dp$

$$= \int_0^1 p \cdot pe^p\,dp = \int_0^1 p^2 e^p\,dp$$

$$= p^2 e^p \Big|_0^1 - \int_0^1 2p \cdot e^p\,dp$$

$$= e - \left[2pe^p \Big|_0^1 - \int_0^1 2e^p\,dp \right]$$

$$= e - \left[2e - \left(2e^p \Big|_0^1 \right) \right] = e - [2e - (2e - 2)]$$

$$= e - \cancel{2e} + \cancel{2e} - 2 = e - 2$$

(b) From Bayes rule, we have

$$\cancel{f_{P|X}(p) = \frac{P(P=p, X)}{P(X)}}$$

$$f_{P|X}(p) = \frac{P(X \mid P=p)\, f_P(p)}{P(X)} \quad \text{for } 0 \le p \le 1$$

$$= \frac{pe^p \cdot p}{e-2} = \frac{p^2 e^p}{e-2}$$

$$\therefore f_{P|X}(p) = \begin{cases} \frac{p^2 e^p}{e-2} & 0 \le p \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(c) Let X_1 = head at first toss &

X_2 = head at 2^{nd} toss

$$\therefore P(X_2 \mid X_1) = \int_0^1 P(X_2 \mid P=p, X_1) \cdot f_{P|X_1}(p)\, dp$$

~~As X_1 & X_2 are independent~~

$$P(X_2 \mid X_1) = \int_0^1 P(X_2 \mid P=p)\, f_{P|X_1}(p)\, dp$$

$$= \int_0^1 p \cdot \frac{p^2 e^p}{e-2}\, dp$$

$$= \frac{1}{e-2}\int_0^1 p^3 e^p \, dp$$

$$= \frac{1}{e-2}\left\{ p^3 e^p \Big|_0^1 - \int_0^1 3p^2 e^p \, dp \right\}$$

$$= \frac{1}{e-2}\left\{ e - \left[3p^2 e^p \Big|_0^1 - \int_0^1 6p e^p \, dp \right] \right\}$$

$$= \frac{1}{e-2}\left\{ e - \left[3e - \left\{ 6pe^p \Big|_0^1 - \int_0^1 6e^p \, dp \right\} \right] \right\}$$

$$= \frac{1}{e-2}\left\{ e - \left[3e - \left\{ 6e - 6e^p \Big|_0^1 \right\} \right] \right\}$$

$$= \frac{1}{e-2}\left\{ e - \left[3e - \left\{ 6e - (6e-6) \right\} \right] \right\}$$

$$= \frac{1}{e-2}\left\{ e - 3e + \left\{ 6e - 6e + 6 \right\} \right\}$$

$$= \frac{-2e+6}{e-2}$$

$$= \frac{6-2e}{e-2} \simeq 0.786$$

OTHER QUESTIONS

1.) Let T be the time at which the police catches the suspect

and $A_x \rightarrow$ event that police & suspect are x unit apart

$B_x \rightarrow$ event that after 1 second police & suspect are x unit apart

for $x > 1$, we have:

$$A_x = \underbrace{(A_x \cap B_x)}_{\substack{\text{if suspect} \\ \text{moves away from} \\ \text{police}}} \cup \underbrace{(A_x \cap B_{x-1})}_{\substack{\text{if suspect} \\ \text{does not} \\ \text{move}}} \cup \underbrace{(A_x \cap B_{x-2})}_{\substack{\text{if suspect moves} \\ \text{towards the police}}}$$

for $x = 1$, we have:

$$A_1 = (A_1 \cap B_1) \cup (A_1 \cap B_0)$$

Now, let's use total expectation theorem:

$$\left.\begin{aligned} E[T \mid A_x] = {} & P(B_x \mid A_x) \cdot E[T \mid A_x \cap B_x] \\ & + P(B_{x-1} \mid A_x) \cdot E[T \mid A_x \cap B_{x-1}] \\ & + P(B_{x-2} \mid A_x) \cdot E[T \mid A_x \cap B_{x-2}] \end{aligned}\right\} \begin{matrix} \text{when} \\ x > 1 \\ \text{-eq 1} \end{matrix}$$

and for $x=1$, we have

$$E[T|A_1] = P(B_1|A_1)\,E[T|A_1 \cap B_1] + P(B_0|A_1)\,E[T|A_1 \cap B_0] \quad \text{-②}$$

And from the question we can infer:

$$P(B_1|A_1) = 1-p, \quad \text{and} \quad P(B_0|A_1) = p$$

Now,

$$E[T|A_1 \cap B_1] = 1 + E[T|A_1], \quad \text{- eq 3}$$

$$E[T|A_1 \cap B_0] = 1 \quad \text{- eq 4}$$

Now, putting values of eq 3 & eq 4 in eq 2, we get

$$E[T|A_1] = (1-p)(1+E[T|A_1]) + p$$

$$= 1 - \cancel{p} + (1-p)E[T|A_1] + \cancel{p}$$

$$= 1 + (1-p)E[T|A_1]$$

$$E[T|A_1] = \frac{1}{p}$$

Now, for $d=2$, we have

$$E[T|A_2] = \left(\frac{1-p}{2}\right)E[T|A_2 \cap B_2] + p\,E[T|A_2 \cap B_1] + \left(\frac{1-p}{2}\right)E[T|A_2 \cap B_0] \quad \text{-eq 5}$$

$\Rightarrow$ Also, we have

$$E[T|A_2 \cap B_0] = 1$$

$$E[T|A_2 \cap B_1] = 1 + E[T|A_1]$$

$$E[T|A_2 \cap B_2] = 1 + E[T|A_2]$$

Now, substituting the above in eq 5, we get

$$E[T|A_2] = \left(\frac{1-p}{2}\right)\left(1 + E[T|A_2]\right) + p\left(1 + E[T|A_1]\right) + \left(\frac{1-p}{2}\right)$$

$$\Rightarrow E[T|A_2]\left(1 - \left(\frac{1-p}{2}\right)\right) = \frac{1-p}{2} + p\left(1 + \frac{1}{p}\right) + \frac{1-p}{2}$$

$$\Rightarrow E[T|A_2] = \frac{1-\cancel{p} + \cancel{p} + 1}{\left(\frac{2-1+p}{2}\right)} = \frac{2 \times 2}{1+p} = \frac{4}{1+p}$$

So, we can generalize these for $x > 2$

$$E[T|A_x] = \left(\frac{1-p}{2}\right)\left(1 + E[T|A_x]\right) + p\left(1 + E[T|A_{x-1}]\right) + \left(\frac{1-p}{2}\right)\left(1 + E[T|A_{x-2}]\right)$$

And, thus we can generate $E[T|A_x]$ recursively for and distance x, using $E[T|A_1]$ & $E[T|A_2]$

And, the expected value of T can be obtained using $P_X(x)$ for the initial distance X and the total expectation theorem as:

$$E[T] = \sum_x P_X(x) E[T|A_x]$$

② ~~fx~~ The area of the triangle is $\frac{1}{2} \times 1 \times \frac{1}{2} = 1/4$ and

X, Y have uniform distribution over the triangle, therefore

$$f_{X,Y}(x,y) = \begin{cases} 4 & \text{over the triangle.} \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_y^{1-y} 4\,dx \qquad 0 \leq y \leq \frac{1}{2}$$

$$= 4(1-y-y)$$

$$f_Y(y) = \begin{cases} 4(1-2y) & 0 \leq y \leq 1/2 \\ 0 & \text{otherwise} \end{cases}$$

. And $f_{X|Y} = \dfrac{f_{X,Y}(x,y)}{f_Y(y)} = \dfrac{4}{4(1-2y)} = \dfrac{1}{1-2y}$

$$f_{X|Y} = \begin{cases} \dfrac{1}{1-2y} & y \leq x \leq 1-y \\ 0 & \text{otherwise} \end{cases}$$