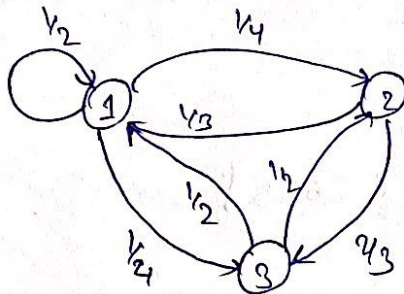Extra Prob 1.

$$A = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.5 \end{bmatrix} \qquad P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

State transition diagram.

@ $P(x1=3, x2=2, x3=1) = P(x1=3) \cdot P(3\to2) \cdot P(2\to1)$

$$P(x1=3) = \begin{bmatrix} 0.25 & 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0.458 & 0.3125 & 0.229 \end{bmatrix}$$

$\therefore P(x1=3) = 0.229$

$\therefore P(x1=3, x2=2, x3=1) = 0.229 \times 0.5 \times \frac{1}{3} = 0.038$

P(x3=2)

$$x3 := AP^3 = \begin{bmatrix} 0.25 & 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}^3$$

$$= \begin{bmatrix} 0.25 & 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} 0.458 & 0.26 & 0.28 \\ 0.43 & 0.167 & 0.40 \\ 0.48 & 0.33 & 0.187 \end{bmatrix}$$

$$= \begin{bmatrix} 0.462 & 0.272 & 0.2635 \end{bmatrix}$$

$$\therefore P(x3 = 2) = 0.272$$

At steady state

$$AP = A \quad , \quad \text{let } A = \begin{bmatrix} a & b & c \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix}$$

$$\Rightarrow \frac{a}{2} + \frac{b}{3} + \frac{c}{2} = a \quad \Rightarrow 3a + 2b + 3c = 6a \quad \Rightarrow 3a = 2b + 3c \quad \text{①}$$

$$\Rightarrow \frac{a}{4} + \frac{2b}{3} = b \quad \Rightarrow 3a + 8c = 12b \quad \text{②}$$

$$\Rightarrow \frac{a}{4} + \frac{c}{2}$$

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$$\Rightarrow \quad \frac{a}{4} + \frac{c}{2} = b \quad \Rightarrow \quad a + 2c = 4b \qquad - ②$$

$$\Rightarrow \quad \frac{a}{4} + \frac{2b}{3} = c \quad \Rightarrow \quad 3a + 8b = 12c \qquad - ③$$

from ① & ③

$$(2b + 3c) + 8b = 12c$$

$$\Rightarrow \quad 10b = 9c \qquad \Rightarrow \quad b = \frac{9}{10}c$$

$$a + 2c = \frac{4 \times 9}{10}c \quad \Rightarrow \quad a + 2c = 3.6c \quad \Rightarrow \quad a = 1.6c$$

We know that

$$a + b + c = 1$$

$$\Rightarrow \quad 1.6c + 0.9c + c = 1 \quad \Rightarrow \quad 3.5c = 1 \quad \Rightarrow \quad c = \frac{1}{3.5}$$

∴ steady state prob = $\begin{bmatrix} 0.457 & 0.257 & 0.286 \end{bmatrix}$

Extra Problem 2.

The mean 1st passage times t_i to reach state 1 starting from j, are given by

$$t_1 = 0, \quad t_i = 1 + \sum_{j=1}^{m} P_{ij} \, t_j \qquad \text{for all } j \neq 1$$

$$\therefore \ t_2 = 1 + \frac{1}{3} t_1^{\,0} + \frac{2}{3} t_3 \qquad \Rightarrow \ t_2 = 1 + \frac{2}{3} t_3 \qquad \textcircled{a}$$

$$t_3 = 1 + \frac{1}{2} t_1^{\,0} + \frac{1}{2} t_3 \qquad \Rightarrow \ t_3 = 2, \ t_2 = \frac{7}{3}$$

$E[R \mid X0 = 1]$ is the mean return time to state 1.

It is given by

$$t_1^{*} = 1 + P_{11} t_1 + P_{12} t_2 + P_{13} t_3$$

$$= 1 + \frac{1}{2} \times 0 + \frac{1}{4} \times \frac{7}{3} + \frac{1}{4} \times 2 = 1 + \frac{7}{6} + \frac{1}{2} \ \Rightarrow \ \frac{12 + 7 + 6}{12} = \frac{25}{12}$$

$$= \frac{16}{6} = \frac{8}{3}$$