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HW3. EAS595

② Let $Z = e^X$, $f_Z(z) = ?$

$$F_Z(z) = P(Z < z) = P(e^X < z) = P(\log e^X < \log z) \begin{cases} \text{if } z > 0 \\ \text{f.i.t.s.} \end{cases}$$

$$= P(X < \log z) = F_X(\log z)$$

$$\therefore f_Z(z) = \frac{d F_Z(z)}{dz} = \frac{d F_X(\log z)}{dz}$$

$$f_Z(z) = \begin{cases} \frac{1}{z} f_X(\log z) & z > 0 \\ 0 & \text{otherwise} \end{cases}$$

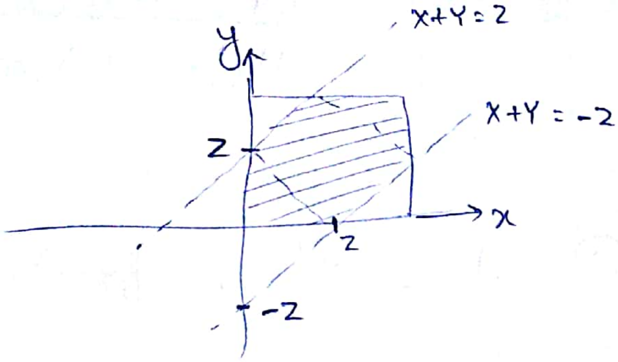
When X is $U[0,1]$

$$\therefore f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

We'll have

$$f_Z(z) = \begin{cases} \frac{1}{z} & \text{, if } 1 < z < e \\ 0 & \text{, otherwise.} \end{cases}$$

(5) Let $Z = |X + Y|$



$F_Z(z) = P(|X-Y| < z)$

$= P(-z < X-Y < z)$ = area under the shaded region

where X, Y are U[0, 1]

$\therefore F_Z(z) = \frac{1}{2} z \cdot z \times 2 + \sqrt{2} z \cdot \sqrt{2}(1-z) \qquad 0 < z < 1$

$= z^2 + 2z(1-z)$

$= z^2 + 2z - 2z^2$

$= 2z - z^2$

$\therefore F_Z(z) = \begin{cases} 2z - z^2 & 0 < z < 1 \\ 1 & \cancel{\text{otherw}}\ z > 1 \end{cases}$

$\therefore f_Z(z) = \begin{cases} 2 - 2z & 0 < z < 1 \\ 0 & \cancel{\text{otherwise}}\ z > 1 \end{cases}$

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(7) The distance b/w two points ~~can be~~ is $|X-Y|$ if X & Y are random variables:

The PDF of the R.V $|X-Y|$ is calculated in problem (5), which is

$$f_Z(z) = \begin{cases} 2(1-z) & 0 < z < 1 \\ 0 & z > 1 \end{cases}$$

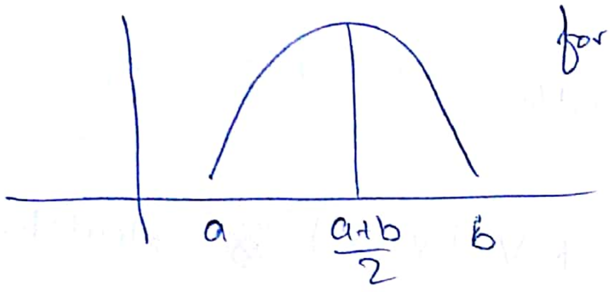
$$\therefore E[Z] = \int_0^1 z \cdot 2(1-z)\,dz$$

$$= \int_0^1 2z\,dz - \int_0^1 2z^2\,dz$$

$$= \frac{2z^2}{2}\Big|_0^1 - \frac{2z^3}{3}\Big|_0^1$$

$$= 1 - \frac{2}{3} = \frac{1}{3}$$

(13)



for X & Y.

$\therefore$ X + Y will have a mean of a+b

Hence for X−Y we will shift PDF by (a+b) to left.

If we see the X+Y & X−Y PDF and combine X+Y & X−Y, = 2X

Its mean is twice of mean of X & the PDF is also double of X.

So, we should have X−Y PDF moved to left by (a+b)

(18) $\rho(R,S) = \dfrac{Cov(R,S)}{\sqrt{Var(R).Var(S)}}$

$\because$ $Cov(R,S) = E[RS] - E[R]E[S]$

$= E[(W+X)(X+Y)] - E[W+X]\,E[X+Y]$

$= E[WX + WY + X^2 + XY] - (E[W]+E[X])(E[X]+E[Y])$

$= \cancel{E[WX]}^{0} + \cancel{E[WY]}^{0} + E[X^2] + \cancel{E[XY]}^{0}$
$- (\cancel{E[W]}^{0} + \cancel{E[X]}^{0})(\cancel{E[X]}^{0} + \cancel{E[Y]}^{0})$

$= E[X^2]$

and $E[X^2] = Var(X) + \cancel{E[X]}^{0\,2}$

$\therefore$ $E[X^2] = Var[X] = 1$

$\therefore$ $Cov(R,S) = 1$

And $Var(R) = Var(W+X) = Var(W) + Var(X) = 1+1 = 2$

Similarly, $Var[S] = 2$

$\therefore$ $\rho(R,S) = \dfrac{1}{\sqrt{2.2}} = \dfrac{1}{2}$

and $\rho(R,T) = \dfrac{Cov(R,T)}{\sqrt{Var(R) \cdot Var(T)}}$

And, $Cov(R,T) = E[RT] - E[R]E[T]$

$$= E[(W+X)(Y+Z)] - E[W+X]\,E[Y+Z]$$

$$= E[WY] + E[WZ] + E[XY] + E[XZ]$$
$$- (E[W]+E[X])(E[Y]+E[Z])$$

$$= 0$$ (as all are pairwise uncorreleted or have Expectation of 0.)

$\therefore \rho(R,T) = 0$

Q4 (a) Let,

X = amount of time that professor devotes to the task

and, Y = length of the time interval b/w 9am & the time of professor's arrival.

$\therefore E[X] = E[E[X|Y]]$

$f(x|Y=y) = \lambda e^{-\lambda x}$, where $\lambda = \frac{1}{5-y}$

is an exponential distribution and so its mean is $\frac{1}{\lambda}$

$\Rightarrow E[X|Y] = \frac{1}{\lambda} = 5-Y$

$\therefore E[X] = E[5-Y] = 5 - \underbrace{E[Y]}^{=2} = 5-2 = 3$

(b) Let $Z = X+Y$, be the length of time from 9 am until the professor completes his task.

$\therefore E[Z] = E[X+Y] = E[X] + E[Y] = 2+3 = 5$

$\therefore$ The expected time at which professor finishes his task is ~~9+5~~ 9am + 5hrs = 2pm.

(c) Let

W = length of time b/w 9 am and arrival of Ph.D student

R = amount of time student spends with the professor, given he finds the professor.

& T = amount of time professor spends with the student

F = probability that the student finds the professor

∴ We can write

$$E[T] = P(F) \cdot E[T|F]$$

given:

$$E[T|F] = E[R] = \frac{1}{2}$$

$$\therefore E[T] = \frac{1}{2} \cdot P(F)$$

Also,

$$P(F) = P(Y \leq W \leq X+Y)$$

$$= 1 - P(P(W<Y) + P(W > X+Y))$$

where,

$$P(W<Y) = \int_0^4 \frac{1}{4} \int_0^y \frac{1}{8}\, dw\, dy = \int_0^4 \frac{1}{4} \frac{y}{8}\, dy = \frac{16}{2\times4\times8} = \frac{1}{4}$$

and $P(W>X+Y) = \int_0^4 P(W>X+Y \mid Y=y)\, f_Y(y)\, dy$

$$= \int_0^4 P(X<W-Y \mid Y=y)\, f_Y(y)\, dy$$

$$= \int_0^4 \int_0^8 F_{X|Y}(w-y)\, f_W(w)\, f_Y(y)\, dw\, dy$$

$$= \int_0^4 \frac{1}{4} \int_y^8 \frac{1}{8} \int_0^{w-y} \frac{1}{5-y} e^{-\frac{x}{5-y}}\, dx\, dw\, dy$$

$$= \frac{12}{32} + \frac{1}{32} \int_0^4 (5-y)\, e^{-\frac{8-y}{5-y}}\, dy$$

$$= \frac{12}{32} + \frac{1}{32} \times 1.7584$$

$$= 0.6799$$

Thus,

$$P(Y \leq W \leq X+Y) = 1 - \left(P(W<Y) + P(W>X+Y)\right)$$

$$= 1 - 0.68 = 0.32$$

$\therefore E[T] = \frac{1}{2} \times P(F) = \frac{0.32}{2} = 0.16 = \underline{9.6 \text{ mins.}}$

Let Z = length of time from 9.am until the professor leaves his office

$\therefore E[Z] = P(F)E[Z|F] + P(F^c)E[Z|F^c] = P(F)E[X+Y+R] + P(F^c)E[X+Y]$

We have $E[X+Y+R] = E[X+Y] + E[R] = 5 + \frac{1}{2} = \frac{11}{2}$

$\therefore E[Z] = 0.68 \times \frac{1}{2} + 0.32 \times \frac{11}{2} = 5.16$. $\therefore$ He leave at 9am + 5.16 hrs.

(30) For std normal distribution:

$$M_x(s) = \int e^{sx} \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$= \int \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2} + sx} = \int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-s)^2 + \frac{s^2}{2}}$$

$$= e^{s^2/2} \int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-s)^2}$$

$$= e^{s^2/2} . 1$$

$$\therefore M_x(s) = e^{s^2/2}$$

$$\frac{dM}{ds} = \frac{1}{2} \cdot 2s \cdot e^{s^2/2} = s e^{s^2/2}$$

$$\frac{d^2M}{ds^2} = e^{s^2/2} + s \cdot \frac{2s}{2} e^{s^2/2} = e^{s^2/2} + s^2 e^{s^2/2}$$

$$\frac{d^3M}{ds^2} = \frac{\cancel{2}s}{\cancel{2}} \cdot e^{s^2/2} + 2s\, e^{s^2/2} + s^2 . \frac{\cancel{2}s}{\cancel{2}} e^{s^2/2} = 3s e^{s^2/2} + s^3 e^{s^2/2}$$

$$\frac{d^4M}{ds^4} = 3e^{s^2/2} + 3s \cdot \frac{1}{\cancel{2}} \cdot \cancel{2}s e^{s^2/2} + 3s^2 e^{s^2/2} + s^3 \cdot \frac{1}{\cancel{2}} \cdot \cancel{2}s \cdot e^{s^2/2}$$

$$= 3e^{s^2/2} + 3s^2 e^{s^2/2} + 3s^2 e^{s^2/2} + s^4 e^{s^2/2}$$

$$\therefore E[x3] = \frac{d^3M}{ds3}\Big|_{s=0} = 0$$

and.

$$E[x4] = \frac{d^4M}{ds4}\Big|_{s=0} = 3$$

(34) Let $Z = X_1 + X_2$, where X_i is succes for i^{th} player

$\therefore$ Convolution of X_1 & X_2 given

$$P_Z(z) = \begin{cases} (1-P_1)(1-P_2) & \text{if } z=0 \\ P_1(1-P_2) + P_2(1-P_1) & \text{if } z=1 \\ P_1 \cdot P_2 & \text{if } z=2 \\ 0 & \text{otherwise} \end{cases}$$

Now, to get X, we convolve Z & X_3

$\therefore$ $X = Z + X_3$, we get

$$P_X(x) = \begin{cases} (1-P_1)(1-P_2)(1-P_3) & \text{if } z=0 \\ P_1(1-P_2)(1-P_3) + P_2(1-P_1)(1-P_3) + P_3(1-P_2)(1-P_1) & \text{if } z=1 \\ P_1P_2(1-P_3) + P_1P_3(1-P_2) + P_2P_3(1-P_1) & \text{if } z=2 \\ P_1P_2P_3 & \text{if } z=3 \\ 0 & \text{otherwise} \end{cases}$$

The transform of X is the product of the transforms of X_is.

$$\therefore M_X(s) = \left((1-P_1) + P_1e^s\right)\left((1-P_2) + P_2e^s\right)\left((1-P_3) + P_3e^s\right)$$

$$M_X(s) = P_1P_2P_3e^{3s} + \left(P_1P_2^{\times(1-P_3)} + P_2P_3(1-P_1) + P_1P_3(1-P_2)\right)e^{2s} +$$

$$\{P_1(1-P_2)(1-P_3) + P_2(1-P_1)(1-P_3) + P_3(1-P_1)(1-P_2)\}e^s + (1-P_1)(1-P_2)(1-P_3)$$

and thus, the PMF is

$$f_X(x) = \begin{cases} (1-P_1)(1-P_2)(1-P_3) & \text{if } z=0 \\ P_1(1-P_2)(1-P_3) + P_2(1-P_1)(1-P_3) + P_3(1-P_1)(1-P_2) & \text{if } z=1 \\ P_1P_2(1-P_3) + P_1P_3(1-P_2) + P_2P_3(1-P_1) & \text{if } z=2 \\ P_1P_2P_3 & \text{if } z=3 \\ 0 & \text{otherwise} \end{cases}$$

The answer conforms with the one using convolution.

Extra problems:

1.1 $Y = X^2$, $\quad X \sim U(0,1)$

$\therefore F_Y(y) = P(Y < y) = P(X^2 < y) = P(X < \sqrt{y})$ for $0 < y < 1$

$\therefore F_Y(y) = P(X < \sqrt{y}) = F_X(\sqrt{y})$

$\therefore f_Y(y) = \dfrac{d F_X(\sqrt{y})}{dy} = f_X(\sqrt{y}) \cdot \dfrac{1}{2\sqrt{y}} = \dfrac{1}{2\sqrt{y}}$

$$\therefore f_Y(y) = \begin{cases} \dfrac{1}{2\sqrt{y}} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

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1.2. $\rho(X,Y) = \rho(X,X^2) = \dfrac{Cov(X,X^2)}{\sqrt{Var(X)\,Var(X^2)}}$

$E[X] = \frac{1}{2}$

$E[X^2] = \int_0^1 x^2\,dx = \frac{1}{3}$

$E[X^3] = \int_0^1 x^3\,dx = \frac{1}{4}$

$E[X^4] = \int_0^1 x^4\,dx = \frac{1}{5}$

$\therefore Cov(X,X^2) = E[X.X^2] - E[X]E[X^2]$

$= \frac{1}{4} - \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$

$Var(X) = E[X^2] - E[X]^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

$Var(X^2) = E[X^4] - E[X^2]^2 = \frac{1}{5} - \left(\frac{1}{3}\right)^2 = \frac{1}{5} - \frac{1}{9} = \frac{4}{45}$

$\therefore \rho(X,Y) = \dfrac{1/12}{\sqrt{\frac{1}{3\cancel{12}}\cdot\frac{\cancel{4}}{45}}} = \dfrac{\frac{1}{12}}{\sqrt{\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{15}}} = \frac{1}{12}\times\frac{3\sqrt{15}}{\cancel{1}} = \frac{\cancel{1}}{4}\sqrt{15}$

$= \frac{\sqrt{15}}{4}$

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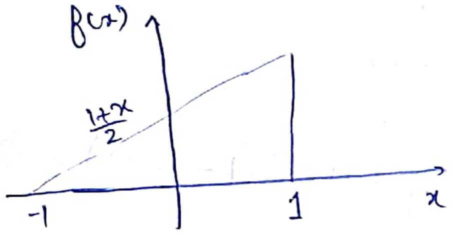
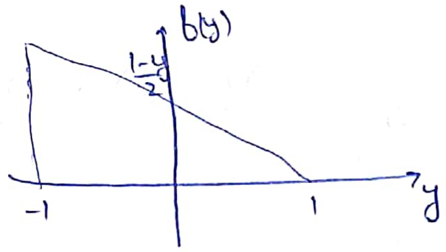
② $f_X(x) = \begin{cases} \frac{1+x}{2} & -1 \le x < 1 \\ 0 & \text{other} \end{cases}$

$f_Y(y) = \begin{cases} \frac{1+x}{2} & -1 \le x < 1 \\ 0 & \text{other} \end{cases}$



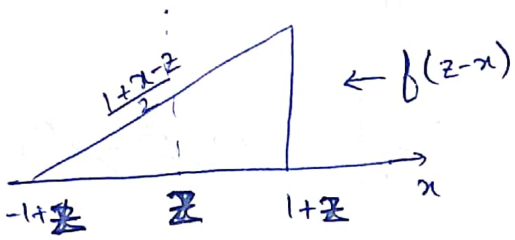
Case 1, $-1 < 1+z < 1$

$\Rightarrow -2 < z < 0$



$$f(z) = \int_{-1}^{1+z} \frac{1+x}{2} \cdot \frac{1+x-z}{2} \, dx$$

$$= \frac{1}{4} \int_{-1}^{1+z} (1 - z + 2x - xz + x^2) \, dx$$



$$= \frac{1}{4} \int_{-1}^{1+z} \left((1-z) + (2-z)x + x^2 \right) dx$$

$$= \frac{1}{4} \left[(1-z)(1+z-(-1)) + \frac{2-z}{2} x^2 \Big|_{-1}^{1+z} + \frac{1}{3} \cdot \left((1+z)^3 - (-1)^3 \right) \right]$$

$$= \frac{1}{4} \left[(1-z)(2+z) + \frac{2-z}{2} \cdot (\cancel{1} + z^2 + 2z \cancel{-1}) + \frac{1}{3}\left(1 + z^3 + 3z^2 + 3z + 1\right) \right]$$

$$= \frac{1}{4} \left[\frac{-z^3 + 12z + 16}{6} \right]$$

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Case 2. When $-1<-1+z<1$

$\Rightarrow 0<z<2$

$$f(z)=\int_{-1+z}^{1} \frac{1+x}{2}\cdot\frac{1+x-z}{2}\,dx$$

$$=\frac{1}{4}\int_{-1+z}^{1}\left((1-z)+(2-z)x+x^2\right)dx$$

$$=\frac{1}{4}\left[(1-z)(1-(z-1))+(2-z)\frac{x^2}{2}\Big|_{z-1}^{1}+\frac{x^3}{3}\Big|_{z-1}^{1}\right]$$

$$=\frac{1}{4}\left[(1-z)(2-z)+\frac{(2-z)}{2}\left(1-(z^2+1-2z)\right)+\frac{1}{3}\left(1-(z^3-1-3z^2+3z)\right)\right]$$

$$=\frac{1}{4}\left[2-3z+z^2+\left(1-\frac{z}{2}\right)\left(\cancel{1}-z^2\cancel{-1}+2z\right)+\frac{1}{3}\left(1-z^3+1+3z^2-3z\right)\right]$$

$$=\frac{1}{4}\left[2-3z+\cancel{z^2}-\cancel{z^2}+2z+\frac{z^3}{2}-z^2+\frac{1}{3}\left(2-3z+3z^2-z^3\right)\right]$$

$$=\frac{1}{4}\left[\frac{12-6z-\cancel{6z^2}+3z^3+4-6z+\cancel{6z^2}-2z^3}{6}\right]$$

$$=\frac{z^3-12z+16}{24}$$

$$\therefore\ f_Z(z)=\begin{cases}\dfrac{z^3-12z+16}{24} & 0\le z\le 2\\[2ex] \dfrac{-z^3+12z+16}{24} & -2\le z<0\\[2ex] 0 & \text{otherwise.}\end{cases}$$