EAS595 HW5

Abhishek Kumar

Prof. Ehsan Efsahani

TEXT BOOK.

6.2. P(Dave fail quiz) = $\frac{1}{4}$

a) Prob that Dave fails exactly 2 of next 6 exams (Binomial)

$$= \binom{6}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^4 = \frac{\overset{3}{\cancel{6}}\times 5}{\cancel{2}} \times \frac{1}{4}\times\frac{1}{4}\times\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4} \approx \frac{1215}{4096}$$

b) E[No. of quizes upto 3rd failure] = $E[Y_k] = \frac{k}{P}$

$$= \frac{3}{1/4} = 12$$

∴ No. of test he passes before 3rd failure = 12 - 3 = 9

c) P(Dave fails 2nd time when he takes 8th quiz)
Event A

$$P_1 = P_Y(k) = \binom{t-1}{k-1} p^k (1-p)^{t-k} \qquad t = k, k+1, \ldots$$

$$= \binom{7}{1}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^6$$

P_2 = P(Dave fails 3rd time when he takes 9th quiz | A) = $\frac{1}{4}$

$\therefore$ P(2nd & 3rd time Dave fails a quiz when he takes 8th & 9th

quizes) $= \binom{7}{1}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^6 \cdot \left(\frac{1}{4}\right)$

$= \binom{7}{1}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^6$

d) Let F indicate that Dave fails a quiz

& let S " passes "

$\therefore$ P(Dave fails 2 quizes in a row before he passes 2 quizes in a row) =

$P(\underbrace{FF \cup SFF}_{\text{independent}} \cup FSFF \cup SFSFF \cup \ldots)$

$= P(FF) + P(SFF) + P(FSFF) + P(SFSFF) + \ldots$

$= \underline{\left(\frac{1}{4}\right)^2} + \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2 + \underline{\frac{1}{4}\cdot\frac{3}{4}\cdot\left(\frac{1}{4}\right)^2} + \frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\cdot\left(\frac{1}{4}\right)^2 + \ldots$

$= \left[\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\frac{3}{4}\cdot\left(\frac{1}{4}\right)^2 + \ldots\right] +$

Geometric infinite Series $\Rightarrow$

$\left[\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2 + \frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\left(\frac{1}{4}\right)^2 + \ldots\right]$

$= \dfrac{\left(\frac{1}{4}\right)^2}{1-\frac{1}{4}\cdot\frac{3}{4}} + \dfrac{\frac{3}{4}\cdot\left(\frac{1}{4}\right)^2}{1-\frac{3}{4}\cdot\frac{1}{4}} = \dfrac{1}{16-3} + \dfrac{3/4}{16-3} = \dfrac{7}{52}$

6.3 $P_I = 1/6$, $P_B = 5/6$, $P_{1/B} = 2/5$ & $P_{2/B} = 3/5$

For every slot P(task from user 1) $= P_1 = \frac{5}{6} \cdot \frac{2}{5} = 1/3$

ⓐ P(task from user 1 is executed for the 1^{st} time during

4^{th} slot) $= P_1 (1-P_1)^3 = \frac{1}{3} \times \left(\frac{2}{3}\right)^3$

ⓑ Given 5 out of 1^{st} 10 slot were idle, we need to find

the probability that the 6^{th} idle slot is 12. = Event A

Since events related to different slots are independent,

previous slot does not make any difference in the

next idle

$\therefore$ P(A) = 11^{th} slot not idle $\times$ 12^{th} slot idle

$= 5/6 \times \frac{1}{6} = \frac{5}{36}$

ⓒ Expected no. of slots upto & including the 5^{th} task

from user 1 $= E[Y_K] = \frac{K}{P_1}$

where $P_1 = P_B \times P_{1/B} = \frac{5}{6} \times \frac{2}{5} = \frac{1}{3}$

$\therefore E[Y_K] = \frac{5}{1/3} = 15$

(d) Expected no. of busy slots upto & including the 5^{th} task from user 1

$$E[Y_K] = \frac{K}{P_{1/B}} = \frac{5}{2/5} = \frac{25}{2}$$

(e) This is a pascal distribution.

No. of tasks T from user 2 until the 5^{th} task from user 1 = No. B of busy slots until the 5^{th} task from user 1 minus 5.

This is a pascal random variable of order 5 with parameter $P_{1/B} = \frac{2}{5}$.

$$\therefore P_B(t) = \binom{t-1}{4}\left(\frac{2}{5}\right)^5\left(1-\frac{2}{5}\right)^{t-5}, \quad t = 5, 6, \ldots$$

$$\therefore E[\text{No. of slots for user 2}] = 15 \times \frac{1}{2} = 7.5$$

$$\text{Var}[\quad " \quad] = \frac{5(1-2/5)}{(2/5)^2} = \frac{3/25}{4} = 18.75$$

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6.10

(a) P(fisherman stays more than 2 hrs) = P(no fish caught in 2 hrs)

$$\boxed{\lambda T = 0.6 \times 2 = 1.2}$$

$$= \frac{(1.2)^0 e^{-1.2}}{0!} = e^{-1.2}$$

(b) For the fisherman to spend b/w 2-5 hours, he should not catch fish b/w 0-2 hours and should catch at least 1 fish b/w 2-5 hrs

$$= \frac{(1.0)^2 \times e^{-1.2}}{0!} = e^{-1.2}$$

$$\Rightarrow e^{-1.2} \times (1 - \text{No fish in 2 hrs})$$

$$= e^{-1.2} \times (1 - e^{-1.8})$$

$$= 0.25097$$

(c) Prob. that he catches atleast 2 fish

$$= 1 - p(\text{no fish}) - P(1 \text{ fish caught})$$

$$= 1 - e^{-1.2} - \left(\frac{(1.2)^1 \times e^{-1.2}}{1!}\right) = 0.337$$

(d) The expected no. of fish that he catches =

No. of fish caugh in 1st 2 hrs + ~~No~~ No of fish caught

during 1st 2 hrs & catches a fish after 2 hrs

$= 2\times0.6 + P(\text{No fish caught in 1st 2 hrs})$

$= 2\times0.6 + e^{-1.2}$

$= 1.5$

(e) It means that he didn't catch any fish in the 1st 4 hrs.

So the expected time to catch 1st fish and quits is $= \frac{1}{\lambda}$

$= \frac{1}{0.6} = 1.667$

So, total expected time for fishing $= 4 + 1.667 = 5.667$

6.14

a) Type - A bulb $f_X(x) = \begin{cases} e^{-x} & , \text{ if } x \geq 0 \\ 0 & , \text{ otherwise} \end{cases}$

Type B bulbs $f_X(x) = \begin{cases} 3e^{-3x} & , \text{ if } x \geq 0 \\ 0 & , \text{ otherwise} \end{cases}$

(a) $E[1^{st} \text{ failure}] = \frac{1}{2}\int_0^{\infty} xe^{-x} + \frac{1}{2}\int_0^{\infty} 3xe^{-3x}$

$= \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} = \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = 2/3$

(b) No failure before time t

$= \frac{1}{2}\int_0^{t} e^{-x}\,dx + \frac{1}{2}\int_0^{t} 3e^{-3x}\,dx$

$= \frac{1}{2}\left(1 - e^{-t} + 1 - e^{-3t}\right)$

$= 1 - \frac{1}{2}\left(e^{-3t} + e^{-t}\right)$

(c) $P(\text{Bulb of type A} \mid \text{no failure}) = \dfrac{\frac{1}{2}e^{-t}}{\frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t}}$

$$= \frac{1}{1+e^{-2t}}$$

(d) $Var(x) = E[x^2] - (E[x])^2$

$$= \frac{10}{9} - \left(\frac{2}{3}\right)^2 = \frac{2}{3}$$

(e) Prob. that out of 12 bulbs, 3 were in 1st 11 and the 12th bulb was of type A, $n=12$, $k=4$, $p=\frac{1}{2}$

$$\binom{12-1}{4-1}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^{12-3} = \binom{11}{3}\left(\frac{1}{2}\right)^{12}$$

(f) $n=12$, $p=\frac{1}{2}$, $k=4$

$$\binom{12}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{12}$$

(g) $P(\text{failure}) = P(\text{Bulb A})\,P(\text{failure}/\text{Bulb A}) + P(\text{Bulb B}) \cdot P(\text{failure}/\text{bulb B})$

$$= \frac{1}{2}e^{-x} + \frac{1}{2}3e^{-3x}$$

$$= \frac{1}{2}\left(e^{-x} + 3e^{-3x}\right)$$

h) Using Erlang pdf with $\lambda = 3$, $k = 2$, we have

$$f_Y(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!} = 3^2 \times y \times e^{-3y} = 9ye^{-3y}$$

$$P(T < y / Y = y) = 1 - e^{-y}, \quad y \geq 0$$

$$P(T < y) = \int_0^\infty f_Y(y)\, P(T < y / Y = y)\, dy$$

$$= \int_0^\infty 9ye^{-3y}(1 - e^{-y})\, dy$$

$$= \int_0^\infty 9ye^{-3y} dy - \int_0^\infty 9ye^{-4y} dy$$

$$= \frac{7}{16}$$

i) $E[N] = 6$, $Var[N] = 12 \times \frac{1}{4} = 3$

$$E[V] = E[N]\, E[X_i] = 2$$

$$Var(V) = Var(X_i) E[N] + E[X_i]^2 Var(N)$$

$$= \frac{1}{9} \cdot 6 + \frac{1}{9} \cdot 3 = 1$$

j) $E[T/D] = t + E[T-t / D\cap A] \times P(A/D) + E[T-t / D\cap B] \times P(B/D)$

$$= t + 1\cdot\frac{1}{1+e^{-2t}} + \frac{1}{3}\left(1 - \frac{1}{1+e^{-2t}}\right)$$

$$= t/3 + \frac{2}{3}\left(\frac{1}{1+e^{-2t}}\right)$$

OTHER QUESTIONS.

① Let $\{N(t), t \in [0, \infty)\}$ be a poisson process with $\lambda = 0.4$

ⓐ Y, the no. of arrivals in (2, 4) is a poisson process with $\mu = 0.4 \times 2$,

$\therefore \lambda t = 0.4 \times 2 = 0.8$

$$P(k, T) = \frac{(\lambda T)^k e^{-\lambda T}}{k!}, \quad k = 0, 1, 2, \ldots$$

$$P(0, 2) = \frac{(0.8)^0 e^{-0.8}}{0!} \Rightarrow e^{-0.8} \Rightarrow 0.449$$

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(b) Let Y_1 be the no. of arrivals in $(0,1)$, then

$Y_1 \sim \text{Poisson}(\mu = 1 \times 0.4)$ therefore

$\lambda t = 0.4 \times 1 = 0.4$

$$P(x,T) = \frac{(\lambda T)^k e^{-\lambda T}}{k!} \quad , k = 0, 1, 2$$

$$P(0,1) = \frac{(0.4)^1 \times e^{-0.4}}{1!} = 0.597$$

$Y_2 \sim \text{Poisson}(\mu = 2 \times 0.4)$

$\therefore \lambda t = 0.4 \times 2 = 0.8$

$$P(k,T) = \frac{(0.8)^1 \times e^{-0.8}}{1} = 0.36$$

(c) $P(1 \text{ arrival in } (0,1))$

$$P(0,1) = \frac{(0.4)^1 \times e^{-0.4}}{1!} = 0.597$$

Since both the intervals are overlapping we can split the intervals into two disjoint intervals.

$P(3 \text{ arrivals in } (0,5)) = P(1 \text{ arrival in } (0,1))$

$\cancel{\lambda t = 4(0.4) = 1.6} + P(2 \text{ arrival in } (1,5))$

$P(3 \text{ arrivals in } (0,5)) = \frac{(1.6)^1 \times e^{-1.6}}{1!} + \frac{(1.6)^2 \times e^{-1.6}}{2!}$

$\simeq 0.323 + 0.258$

$\simeq 0.581$

② Let $N_1(t)$ & $N_2(t)$ be two independent Poisson process with $\lambda_1 = 1$ & $\lambda_2 = 2$.

ⓐ $P(N(1)=2, N(2)=5) = P($two arrivals in $(0,1]$ & 3 arrivals in $(1,2])$

$P(2 \text{ arrivals in } (0,1]) = \frac{(3\times1)^2 \cdot e^{-3\cdot1}}{2!} = \frac{9\times e^{-3}}{2} = 0.22404$

$P(3 \text{ arrivals in } (1,2]) = \frac{(3\times1)^3 \times e^{-3}}{3!} = \frac{\cancel{27}^{9} \cdot e^{-3}}{\cancel{6}2} = 0.22404$

$\therefore P(N(1)=2, N(2)=5) = \left(\frac{9\times e^{-3}}{2}\right) \cdot \left(\frac{27\times e^{-3}}{6}\right)$

$P(N(1)=2, N(2)=5) = 0.0501$

b) $P(N_1(1)=1 \mid N(1)=2) = \dfrac{P(N_1(1)=1, N(1)=2)}{P(N(1)=2)}$

$$= \frac{P(N_1(1)=1)\cdot P(N_2(1)=1)}{P(N(1)=2)}$$

$$= \frac{P(N_1(1)=1)\cdot P(N_2(1)=1)}{P(N(1)=2)}$$ Since $N(1)=N_1(1)+N_2(1)$

$$P(N_1(1))=1 \Rightarrow \frac{(\lambda_1 T)^k e^{-\lambda T}}{k!}$$

$$P(1,1)=\frac{(1)(1)^1 e^{-1(1)}}{1!}=e^{-1}$$ for $N_1(1)=1$

for $N_2(1)=1$, $P(1,1)=\dfrac{(\lambda_2 T)^k e^{-\lambda T}}{k!} \Rightarrow \dfrac{2^k e^{-2(1)}}{1!} = \dfrac{2\cdot e^{-2}}{1!}$

$\Rightarrow 2e^{-2}$ for $N_2(1)=1$

$$P(N(1)=2) \Rightarrow P(2,1)=\frac{(3\times 1)^2 e^{-3}}{2!} = \frac{9\times e^{-3}}{2} = 0.22404$$

$$P(N_1(1)=1 \mid N(1)=2) = \frac{e^{-1}.2.e^{-2}}{0.22404} = \frac{0.367\times 2\times 0.1353}{0.22404}$$

$$= 0.4432$$