Unit - IV

Graph Theory

Introduction

The development of graph theory is very much smaller to the development of probability theory.
The original work of graph theory was motivated by constant efforts to understand or solve real life problems.
The graph theory is very imp area of applied mathematics.

Q. Define Graphs? Explain representation of graph.

Graphs

— A graphs is an ordered pair $(V(G), E(G))$ where

i] $V(G)$ is non empty finite set of elements known as vertices or nodes. $V(G)$ is called the vertex set.

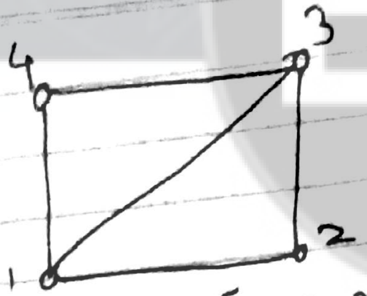
ii] $E(G)$ is family of unordered pairs (not necessarily distinct) of elements of V, known as edges or arc or branches of G.
$E(G)$ is known as edge set.

→ Graphs are so named because can be represented diagrammatically in the plane. It is denoted by $G(V, E)$.

a] Each vertex of G is represented by a point or small circle in the plane.

b] Every edge is represented by a continuous curve or straight line segment. Edges may be the route among states or cities or relation among obj. etc.

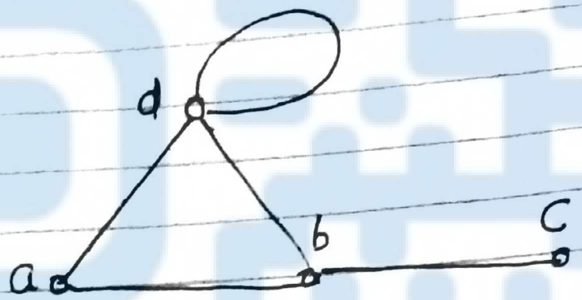
1]



$V(G_1) = \{1, 2, 3, 4\}$

$E(G_1) = \{(1,2), (1,3), (1,4), (2,3), (3,4)\}$

2]



$V(G_2) = \{a, b, c, d\}$

$E(G_2) = \{(a,b), (a,d), (b,c), (b,d), (d,d)\}$

i] If x and y are two vertices of a graph G and unordered pair $(x, y) =$ $(x, y) = e$ is an edge then we can say that edge e joins x and y or e is incident to both vertices x and y

In this case, vertices x and y are said to be incident one eq. In eq. 1

~~If~~ $e = \{2, 3\}$

$\therefore$ e is incident at 2 and 3 and vertice 2, 3 are one incident on $e = (2, 3)$

ii] Two vertices x and y are said to be adjacen to each other if the pair (x, y) is an edge of G.

Basic definations

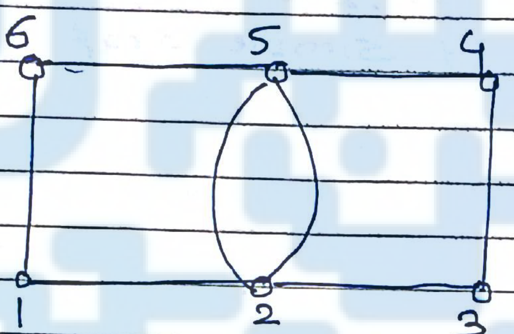
Q. Explain Types of graph.?

1] Multigraphs :-

— A graph in which a pair of vertices is joined by two or more edges is called a multigraph or multiple graph.

i.e. A graph having multiple edges is called a multigraph.

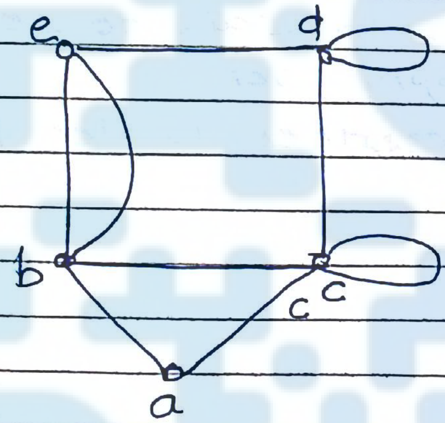
1]



$V(G) = \{1, 2, 3, 4, 5, 6\}$

$E(G) = \{(1,2), (1,6), (2,3), (2,5), (2,5), (3,4), (4,5), (5,6)\}$

2]

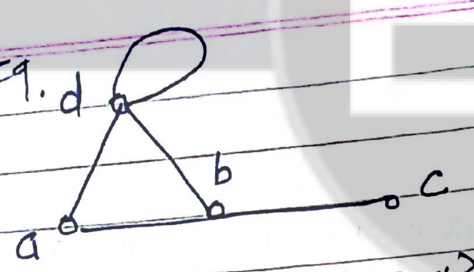


$V(G) = \{a, b, c, d\}$

$E(G) = \{(a,b), (a,c), (b,e), (b,e), (b,c), (c,c), (c,d), (d,d), (d,e)\}$

2] Pseudograph :-

A graph having loops but no multiple edges is called a pseudograph eg.



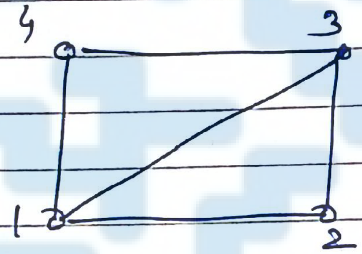
$V(G) = \{a, b, c, d\}$

$E(G) = \{(a,b), (a,d), (b,c), (b,d), (d,d)\}$

3] Simple graph :→

A graph without loops and multiple edges is called a simple graph.

eg.

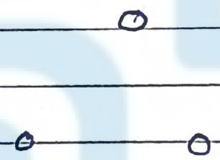


4] Null graph :→

A graph $G(V, E)$ is said to be null graph if E is a empty set.

Null graph on n vertices is denoted by N_n

eg.



N_1

5] Finite graph :→

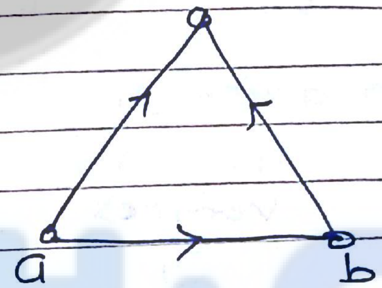
A graph $G(V, E)$ in which $V(x)$ & $E(x)$ are finite sets is called a finite graph.

6] Directed graph :→

A graph $G(V,E)$ is said to be directed graph if the elements of E are an ordered pairs of vertices.

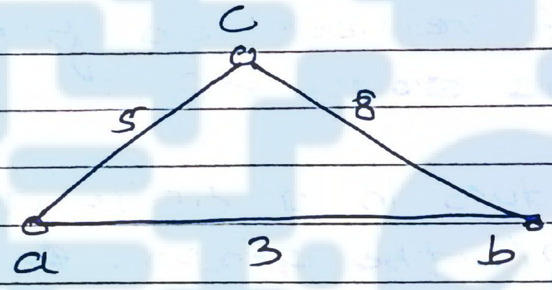
$E = \{(a,b), (b,c), (a,c)\}$

here
$(a,c) \neq (c,a)$,
$(c,a) \notin E(G)$.



7] Weighted graph :→

A graph $G(V,E)$ in which some weight is assigned to every edge of G. is called weighted.



Weighted graph.

8] Degree of vertex :-

Indegree

a] In a directed graph G the number of edges ending at vertex v is called the Indegree of v.

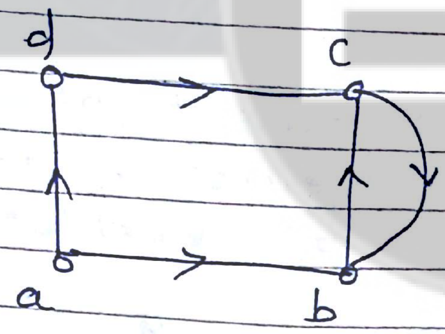
It is denoted by $\deg G^{+}(v)$ or $d^{+}(v)$

b] Outdegree :-

In a directed graph G, the number of edges begining at vertex v is called the outdegree of v.

It is denoted by $\deg G^{-}(v)$ or $d^{-}(v)$

eg.



In graph G1

vertices	Indegree	outdegree
a	0	2
d	1	1
c	2	1
b	2	1

9] order and size of graph :-

The number of vertices in a finite graph G is called the order of G

The number of edges in a finite graph G is called size of the graph.

If G is a (P, Q) graph then G has p vertices and q edges.

10] Degree sequence of a graph.

— Let G be a graph with vertex set $V = \{v_1, v_2 \dots v_n\}$ & $d_i = \deg(v_i)$ then the sequence $(d_1, d_2 \dots d_n)$ in any order is called the degree sequence of G.

Q. Define Handshaking lemma theorem?

Theorem :→

Handshaking lemma

Let $G(V, E)$ be any graph then

$$\sum_{v \in V} d(v) = 2q$$

where q denotes the number of edges of G.

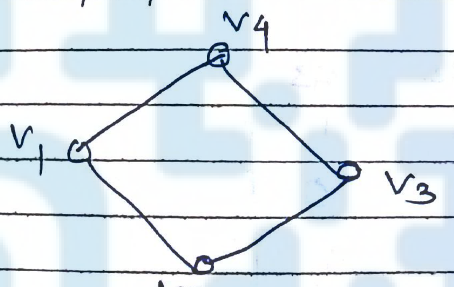
Q. How to represent graph in matrix form?

* Matrix Representation of a Graph

→ we saw that graphs can be represented either set theoretically or diagrammatically.

- Graphs can also be represented by matrices.
- It is very much useful to store graphs in a computer.

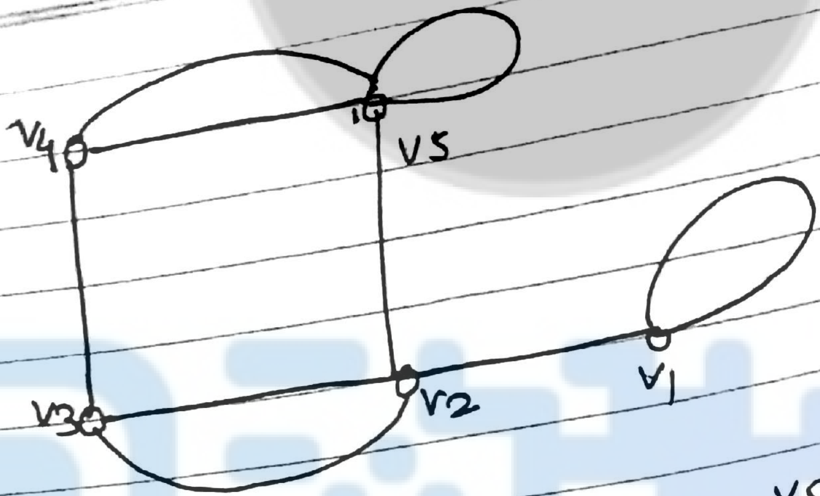
eg. Adjacent matrices of the following graphs are



G_1

$$A(G_1) = \begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \quad 4 \times 4$$

2]

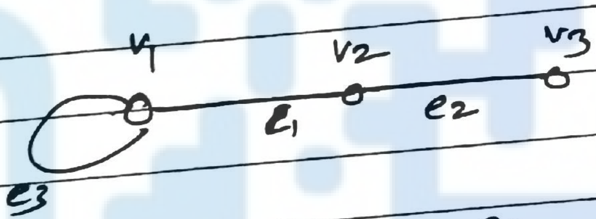


$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 & 1 \end{bmatrix} \end{array}$$

Q. Define Incidence Matrix :-

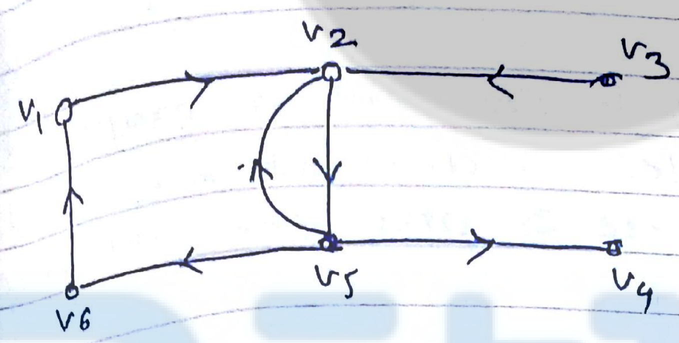
* Incidence Matrix :->

Let G be a graph with n vertices and m edges without self loops.



$$X(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Adjacency Matrix of Diagraph (Directed graph)



~~A(D)=~~

$$A(D) = \begin{array}{c|cccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ \hline v_1 & 0 & 1 & 0 & 0 & 0 & 0 \\ v_2 & 0 & 0 & 0 & 0 & 1 & 0 \\ v_3 & 0 & 1 & 0 & 0 & 0 & 0 \\ v_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_5 & 0 & 1 & 0 & 1 & 0 & 1 \\ v_6 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}$$

eg.

Determine the number of edges in a graph with 6 nodes, 2 of degree 4 and 4 of degree 2. Draw such graphs.

→ Let G be a required graph with 6 node and m edges.

∴ by handshaking lemma.

$$\sum_{v \in G} d(v) = 2m$$

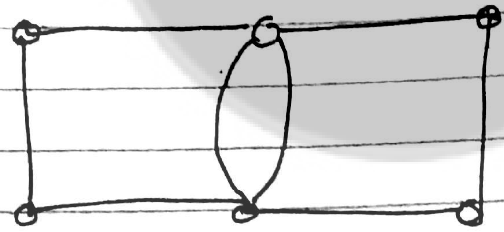
$$\therefore d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) + d(v_6) = 2m$$

$$4 + 4 + 2 + 2 + 2 + 2 = 2m$$

$$2m = 16$$

$$m = 8$$

hence 8 edges are required.



2] Is it possible to Construct a graph with 12 nodes such that 2 of the nodes have degree 3 and the remaining have degree 4

→ Let G be the Required graph with 12 vertices

by handshaking lemma

$$\sum_{v \in V(G)} d(v) = 2m$$

$$(3+3)+(4+4+4+4+4+4+4+4+4+4) = 2m$$

$$6+40 = 2m$$

$$m = 23$$

∴ It is possible to Construct the graph.

Some Important Definitions.

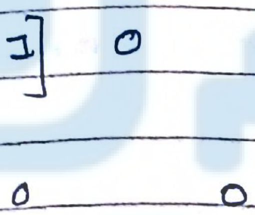
Q. Define Regular graph?

I] Regular Graph :- A graph G is said to be r-regular graph if every vertex of G has degree r

i] Regular graph of degree zero is called null graph.

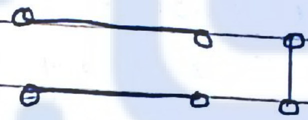
ii] Regular graph of degree 3 is called Cubic graph

eg. I]



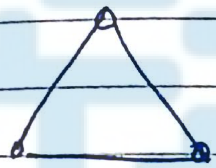
1- regular graph

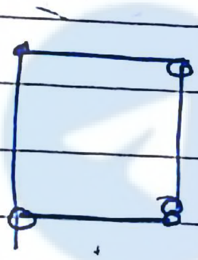
ii]



1- regular graph

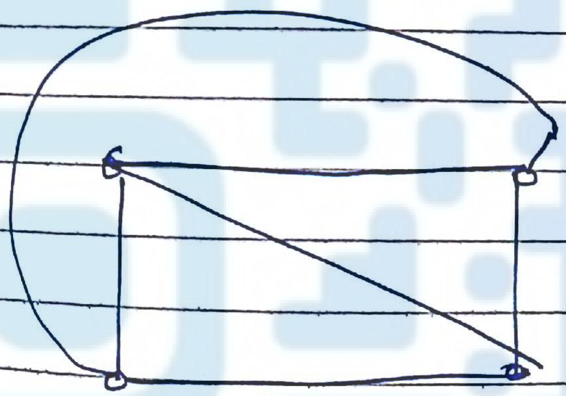
iii]





2- regular graphs.

iv]



3- regular graph.

Q. Define Complete Graph?

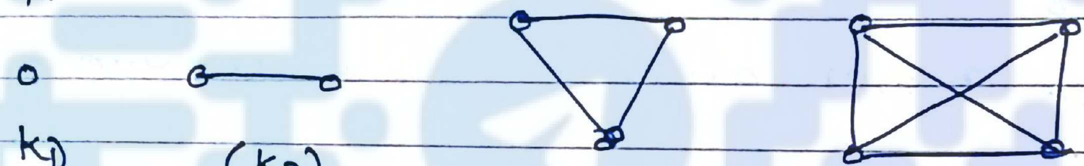
A] Complete Graph :→

A simple graph G in which every pair of distinct vertices are adjacent is called a complete graph.

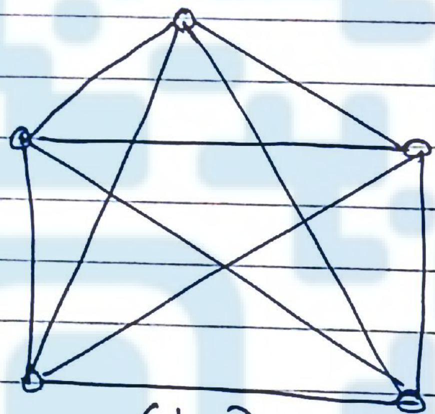
If G is a complete graph on n vertices then it is denoted by K_n

In a complete graph, there is an edge betn every pair of distinct vertices.

eg.



(K_1) (K_2) (K_3) (K_4)



(K_5)

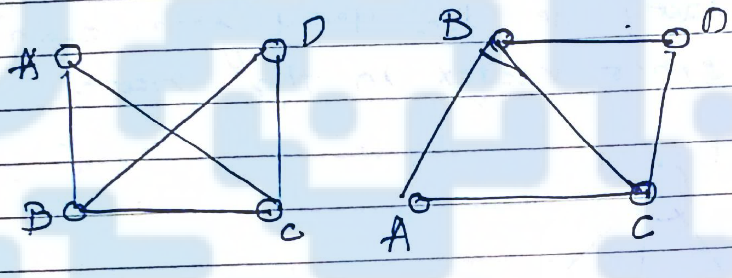
III] Bipartite graph

IV] Complete Bipartite graph

Q. Define Isomorphic graph.

→ Isomorphic Graph :→

Two graphs are isomorphic if there is one to one matching betn vertices of two graphs with the property that whenever there is an edge betn two vertices of either one of the graphs, there is an edge betn the corresponding vertices of the other graph.



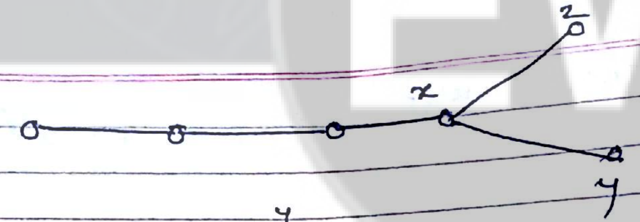
Defn :- Let $G_1(V_1, E_1)$ & $G_2(V_2, E_2)$ be two graphs. G_1, G_2 are said to be isomorphic graphs if

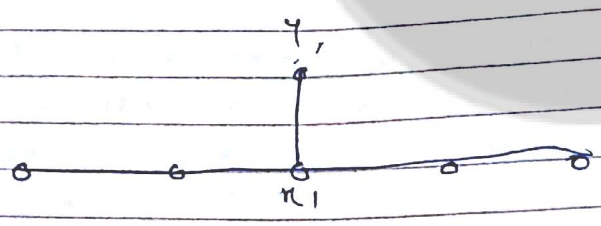
1] There exists a bijection function
$\phi : V_1 \rightarrow V_2$

2] There exists a bijection function
$\psi : E_1 \rightarrow E_2$ such that $e = (x, y)$ is an edge in G_1 iff $(\phi(x), \phi(y))$ is an edge in G_2

The pair of function ϕ and ψ is called an Isomorphism of G_1 & G_2. it is denoted by
$G_1 \cong G_2$

∴ G_1 & G_2 have same No of vertices
G_1 & G_2 must have same no. of edges
G_1 & G_2 must have equal no. of vertices with the same degree.
G_1 & G_2 have equal No of loops
G_1 & G_2 have same no. of pendent.

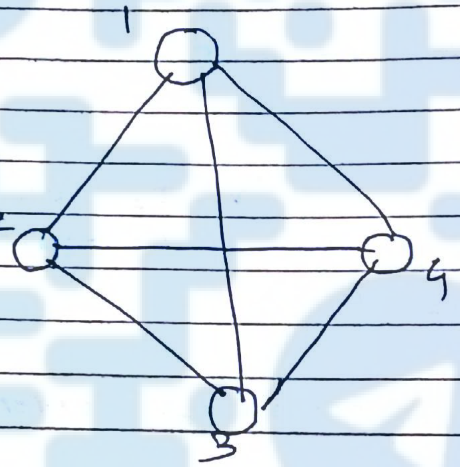




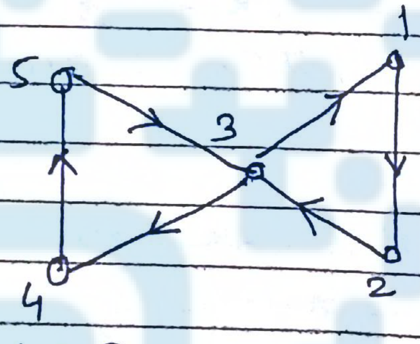
Q. Define Complete graph ?

→ Complete graph :-

A simple graph also in which there is exactly one edge going from each vertex to each other vertex in the graph



* Linked Representation of Graphs.



1 : 2

2 : 3

3 : 1, 4

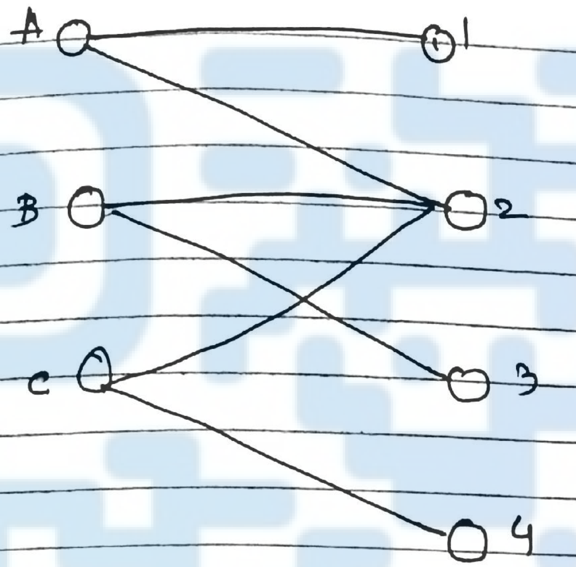
4 : 5

5 : 3

Q. Define bipartite graph?

Bipartite graph :-

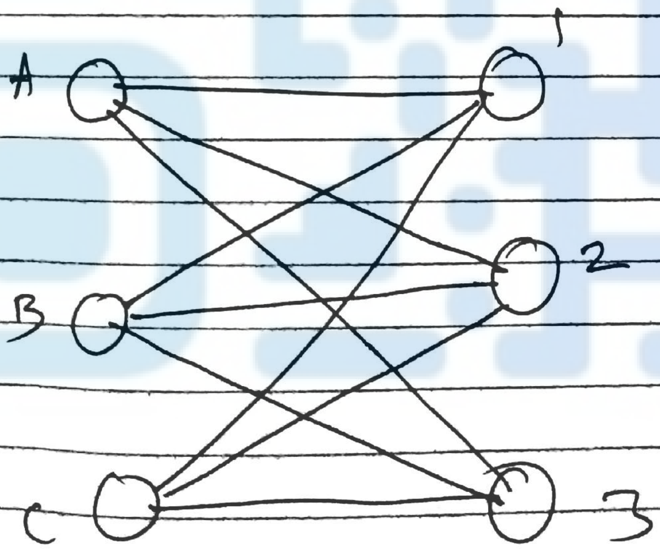
— A graph whose vertices can be divided into two classes. Edges exist only betⁿ vertices that belongs to different classes.



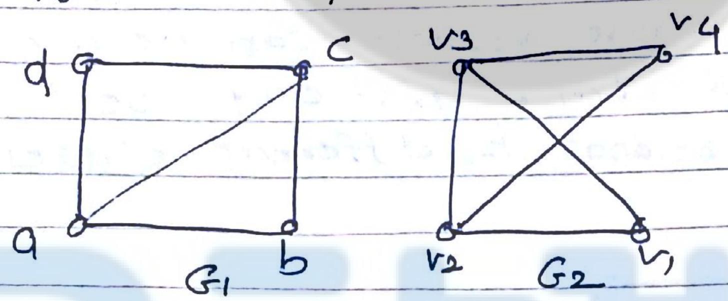
Q. Define Complete bipartite graph

→ Complete bipartite graph

— A graph whose vertices can be divided into two classes & edges exist betⁿ ~~one class~~ each & every vertex of ~~each~~ one class to each & every vertex of other class.

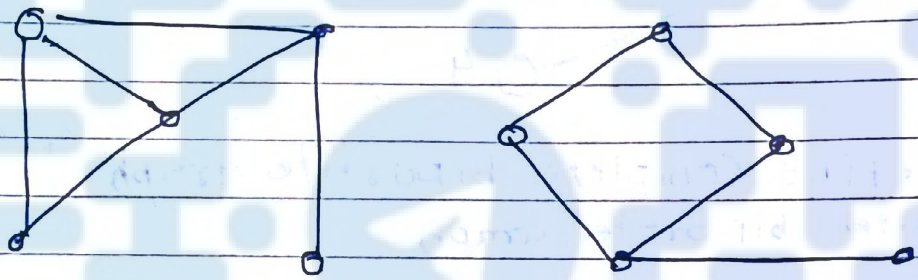


find wheather the following pair of graphs are isomorphic or not.



→ both graphs have 4 vertices & 5 edges
both have 2 vertices of degree 3 & 2 vertices of degree 2

∴ 2



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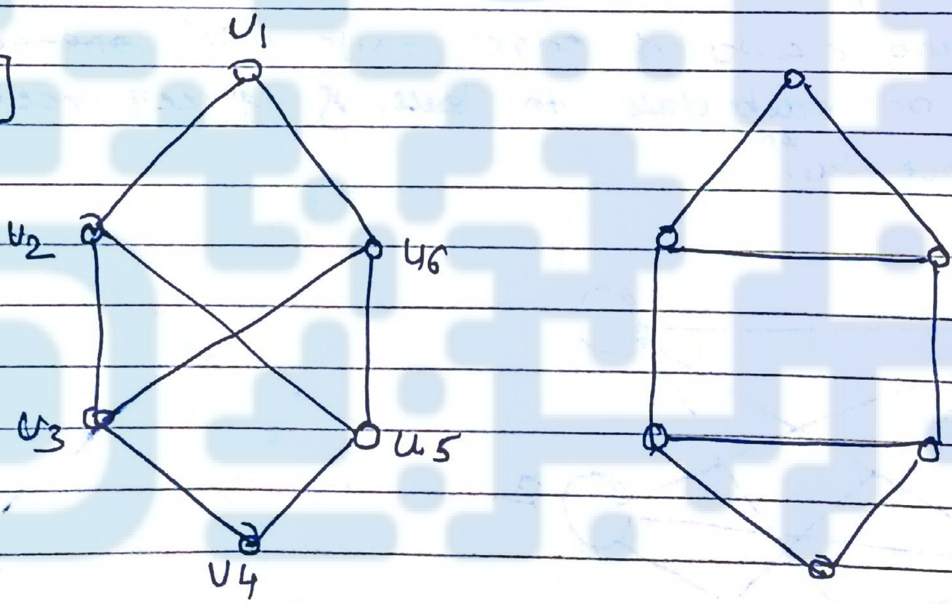
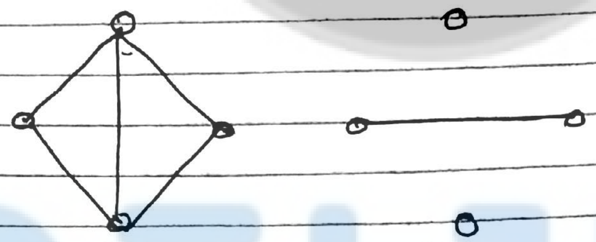


G_1 & G_2 have 6 vertice & 8 edges

Q. Give the eq. of complement of graph
complement of graph.

1) 

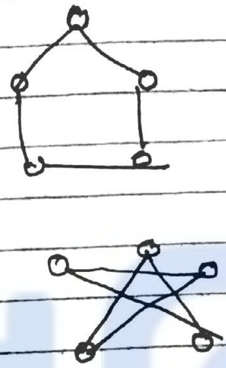
G $\quad$ $\bar{G}$

2) 

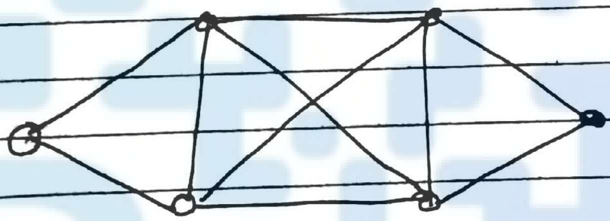
K $\quad$ $\bar{K}$

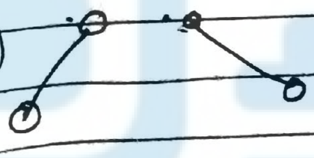
3) 

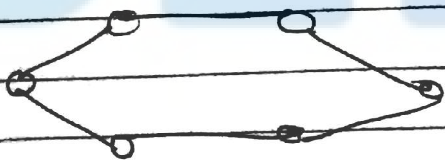
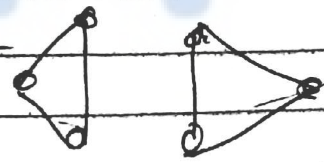
find complementary graph.



find the 1-factor, 2 factor graph of the following graph

1] 

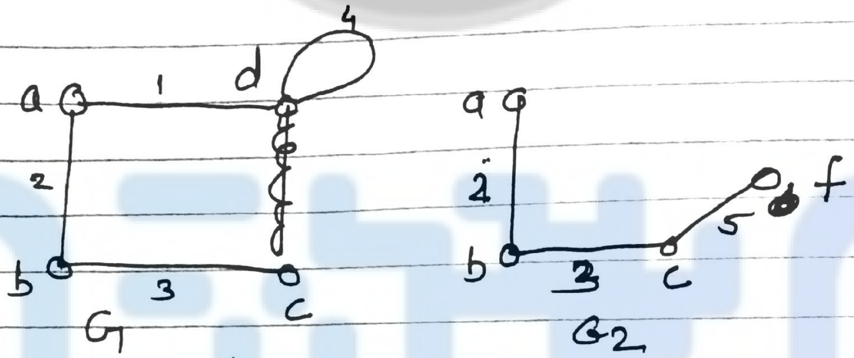
1) 

2]  or 

Q. What are different operations on graphs.

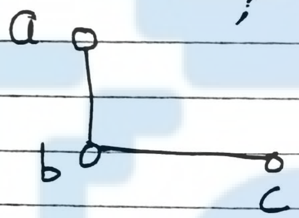
operations on graphs

A] Intersection of two graphs

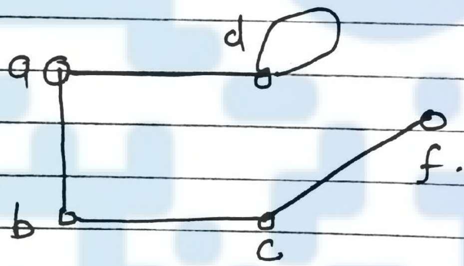


⇓

$G_1 \cap G_2$

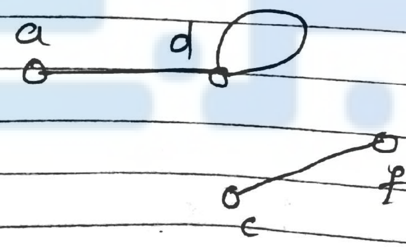


B] Union of two graphs :-



C] Ring sum of two graphs

$(E_1 \cup E_2) - (E_1 \cap E_2)$

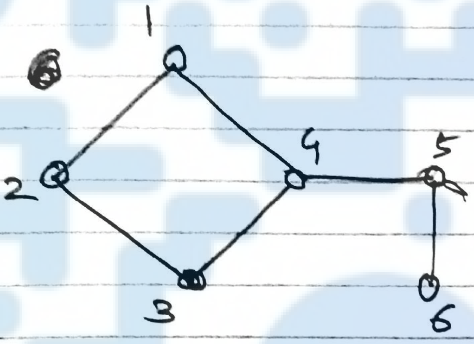


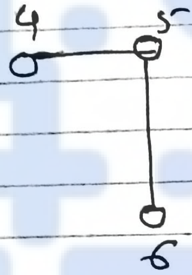
Q. Define subgraph

* Subgraph :→

A graph consisting of some of the vertices of the original graph & some of the original edges betⁿ those vertices is called subgraph

→ Let $G(V, E)$ be any graph
$H(V_1, E_2)$ is said to be a subgraph of G if $V_1 \subseteq V$ & $E_1 \subseteq E$



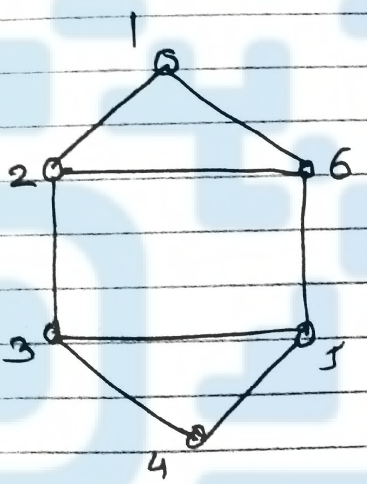


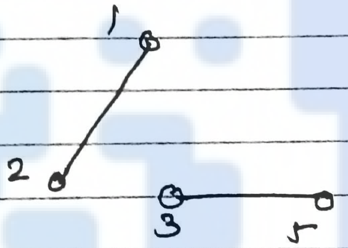
H

G

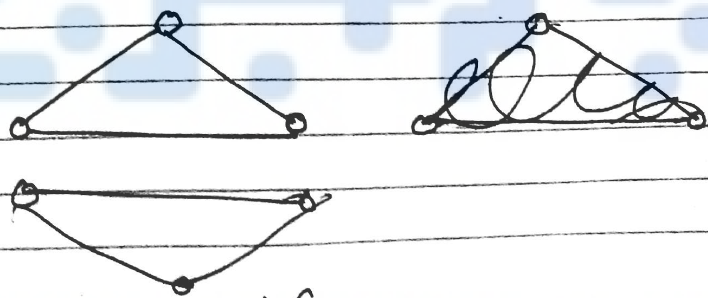
Q. What are factors of graph?

* factors of graph:-





1-factor of G.



2-factor of G.

Q Define, cyclic graph, Acyclic graph, circuit, Hamilton path, Hamilton cycle

Cyclic graph :-

A directed graph with cycle

i.e {0, 3, 6, 0}

Acyclic graph:->

Path :.

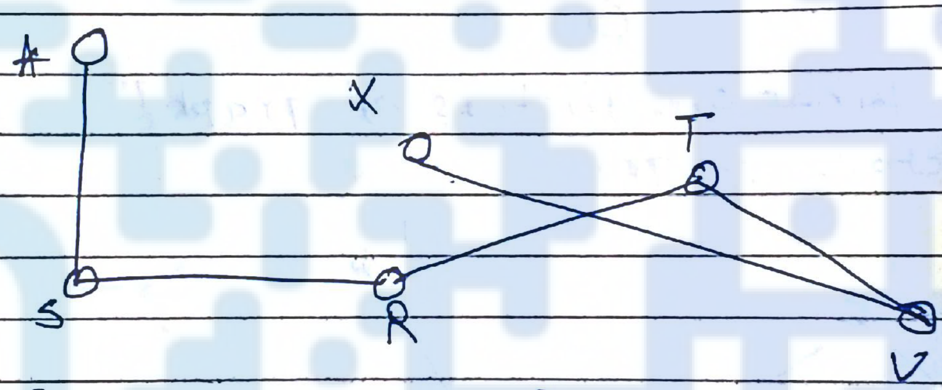
A path in a graph is a walk that uses no edge more than once

Circuit :.

- A circuit in a graph is path that begins and ends at the same vertex

Hamilton path

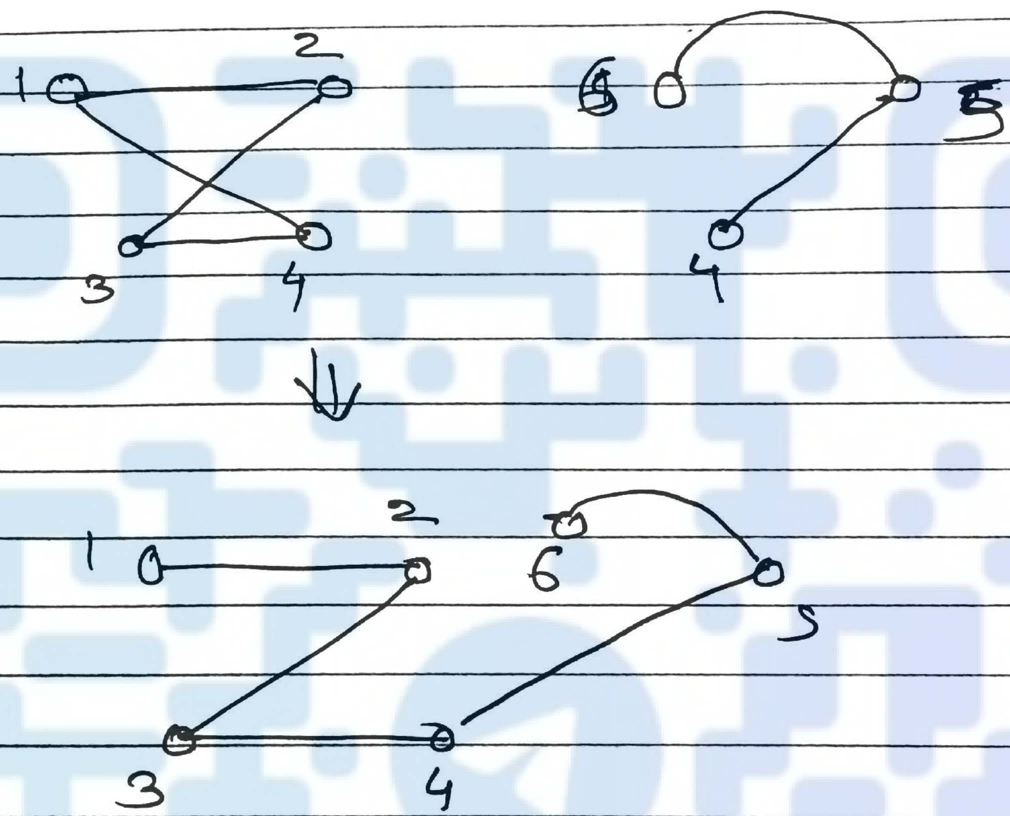
- A path that spans all the vertices in a diagraph.



eg. {A, S, R, T, V, X}

Hamilton cycle:. A cycle that covers all the vertices in a graph

* Sum of two graphs.



Q. Define simple path, Elementary path.

* Simple Path:-

A path in a graph G is said to be a simple path if the edges do not repeat in the path. Vertices may be repeated

i] $v_1 - e_1 - v_2 - e_2 - v_3 - e_{10} - v_5$

* Elementary Path

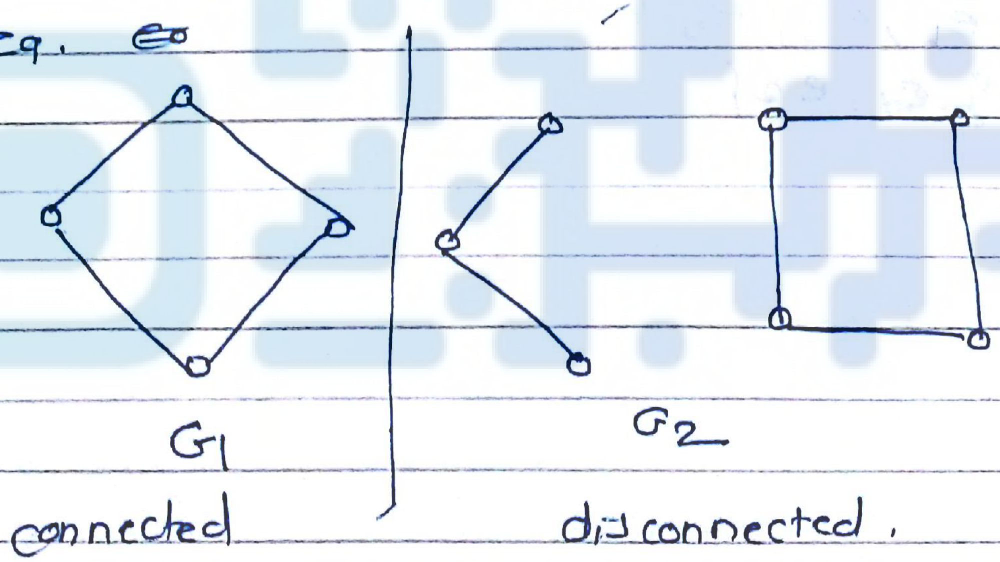
A path in a graph G is said to be elementary path if vertices do not repeat in the path. Every elementary path is a simple path

Q. Define connected & disconnected graph.

* Connected graph: & Disconnected Graph

A graph G is said to be connected graph if there exists a path betn every pair of vertices. A graph which is not connected is called the disconnected graph

eg. Eo



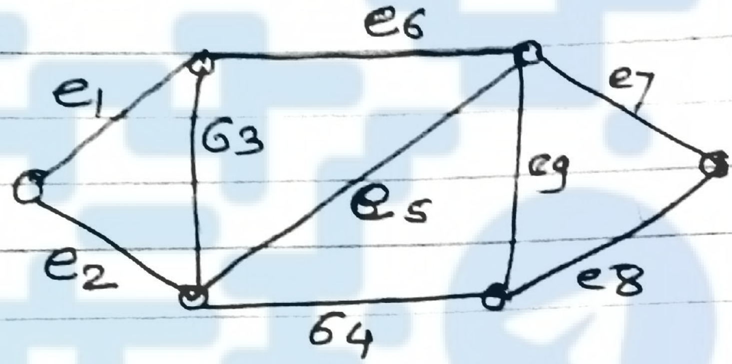
Q. define connectivity of graph:—

* Connectivity of graph:
* Edge connectivity.

A set of edges of a connected graph G whose removal disconnects G is called a disconnecting set of G.

Q. Define cutset.

A cutset is defined as a minimal disconnecting set i.e. A minimal set of edges whose removal from G gives a disconnected graph is called a cutset.



G.

cutsets of G are as follows.

i] $\{e_4, e_5, e_6\}$

ii] $\{e_1, e_3, e_6\}$

iii] $\{e_1, e_2\}$

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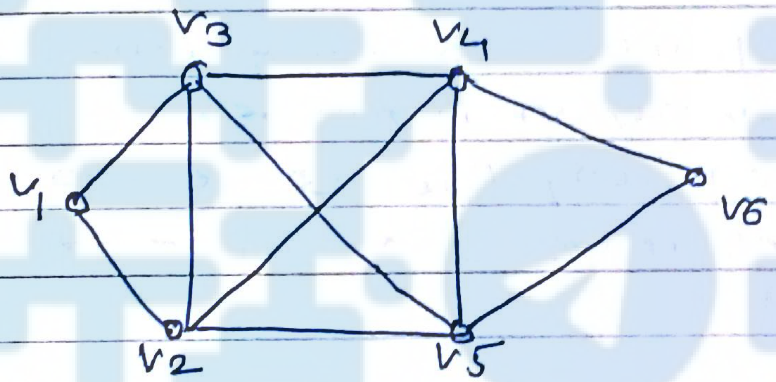
Q. Define vertex connectivity

* vertex connectivity.

* The vertex connectivity $K(G)$ of a simple connected graph G is defined as the smallest number of vertices whose removal disconnects the graph.



In graph G, the sets $\{v_2, v_5, v_4\}$, $\{v_2, v_3, v_4, v_5\}$, $\{v_2, v_3\}$ disconnected graph G.

The smallest set is $\{v_2, v_3\}$

$K(G) = 2$

Q. Explain shortest path algorithm

→ * Shortest Path algorithm

- weighted graph is a graph in which each edge has a weight.

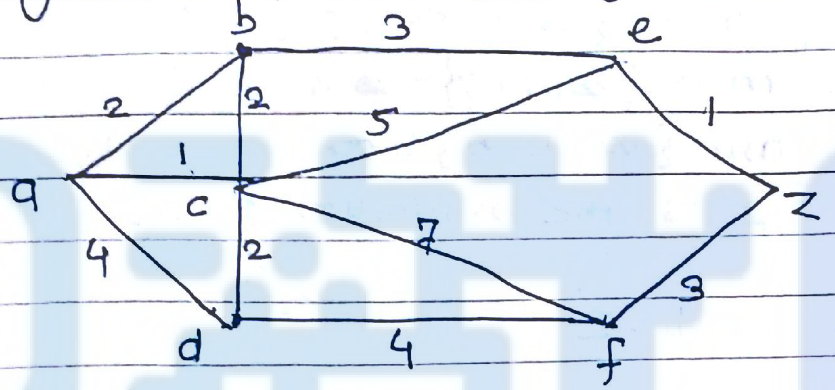
- The weight of graph G is the sum of weight of all edges of G.

- Weighted graph has many applications in communication networks.

Given a railway n/w connecting several cities. determine a shortest route bet^n two cities.

The algorithm to find the shortest path from source to destination is a Dijkshtra's algorithm

* Find the shortest path from a–z in the given graph using Dijkstra's algo.



step-1: set $P = \phi$, $T = \{a, b, c, d, e, f, z\}$

$L\{a\} = 0$

$L\{x\} = \infty,\ \forall x \in T,\ x \neq a$

step2: $v = a$, the permanent label of a is 0

$P = \{a\}$, $T = \{b, c, d, e, f, z\}$

$L\{b\} = \min\{\text{old } L(b),\ L(a) + w(a, b)\}$

$= \min(\infty, 0 + 2) = 2$

$L\{c\} = \min\{\infty, 0 + 1\} = 1$

$L\{d\} = \min\{\infty, 0 + 4\} = 4$

$L\{e\} = \min\{\infty, 0 + \infty\} = \infty$

$L\{f\} = \min\{\infty, 0 + \infty\} = \infty$

$L\{z\} = \min\{\infty, 0 + \infty\} = \infty$

$\therefore L\{c\} = 1$ is the minimum label.

step3: $v = c$, c is 1

$P = \{a, c\}$, $T = \{b, d, e, f, z\}$

$L\{b\} = \min\{2, 1+2\} = 2$

$L\{d\} = \min\{4, 1+2\} = 3$

$L\{e\} = \min\{\infty, 1+5\} = 6$

$L\{f\} = \min\{\infty, 1+7\} = 8$

$L\{z\} = \min\{\infty, 1+\infty\} = \infty$

$\therefore L\{b\} = 2$ is the minimum label.

step 4: $v = b$, $b = 2$

$P = \{a, c, b\}$, $T = \{d, e, f, z\}$

$L\{d\} = \min\{3, 2+\infty\} = 3$

$L\{e\} = \min\{6, 2+3\} = 5$

$L\{f\} = \min\{8, 2+\infty\} = 8$

$L\{z\} = \min\{\infty, 2+\infty\} = \infty$

$\therefore L\{d\} = 3$ is min label.

Step 5: $v = d$, $d = 3$

$P = \{a, c, b, d\}$, $T = \{e, f, z\}$

$L\{e\} = \min\{5, 3+\infty\} = 5$

$L\{f\} = \min\{8, 3+4\} = 7$

$L\{z\} = \min\{\infty, 3+\infty\} = \infty$

$\therefore L\{e\} = 5$ is the minimum label.

step 6: $v = e$, the permanent label of e is 5

$P = \{a, c, b, d, e\}$, $T = \{f, z\}$

$L\{f\} = \min\{7, 5+\infty\} = 7$

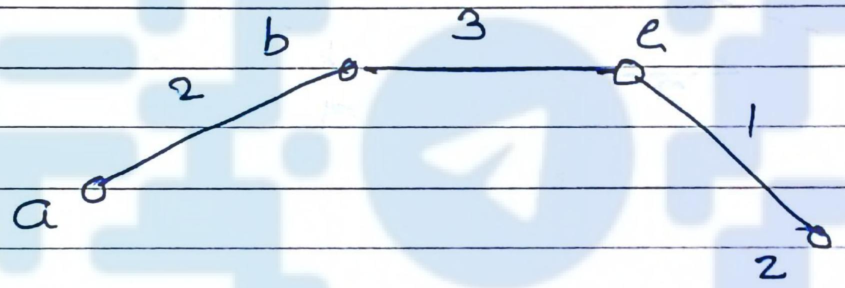
$L\{z\} = \min\{\infty, 5+1\} = 6$

$\therefore L\{z\} = 6$ is the minimum label.

step 7 $v = z,$ $z = 6$

∴ Hence the length of shortest path from a to z is 6

path is a b e z



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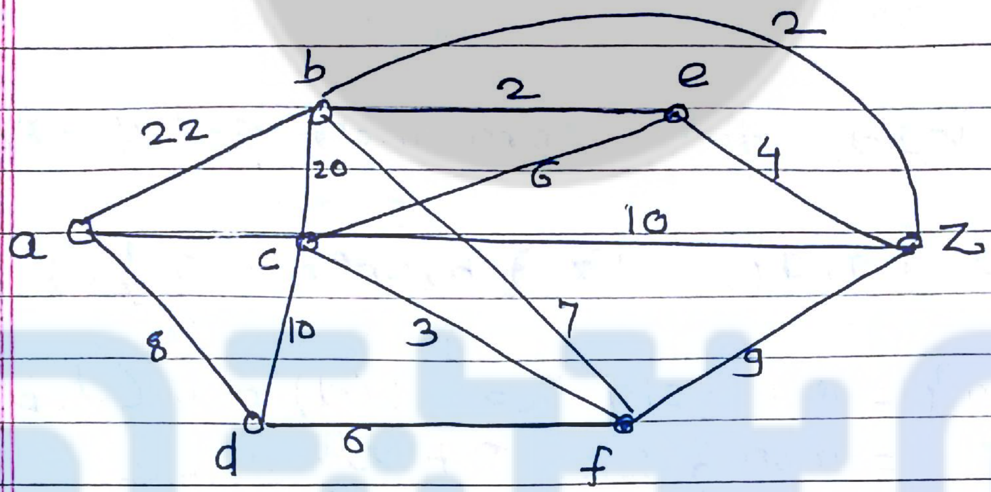
* Use Dijkstra's algorithm to find the shortest path betⁿ a & z.



→

step 1 : Let P be the set of those vertices which have permanent labels & T = set of all vertices of G.

$\therefore$ $P = \phi$, $T = \{a, b, c, d, e, f, z\}$

$\therefore$ $L\{a\} = 0$

$L\{x\} = \infty$, $\forall x \in T$, $x \neq a$

Step 2: $v = a$, the permanent label of a is 0

$P = \{a\}$, $T = \{b, c, d, e, f, z\}$

$L\{b\} = \min\{\text{old } L(b), L(a) + \omega(a,b)\}$
$= \min\{\infty, 0 + 22\} = 22$

$L(c) = \min\{\infty, 0 + 16\} = 16$

$L(d) = \min\{\infty, 0 + 8\} = 8$

$L(e) = \min\{\infty, 0 + \infty\} = \infty$

$L(f) = \min(\infty, 0 + \infty) = \infty$

$L(z) = \min(\infty, 0 + \infty) = \infty$

$\therefore L\{d\} = 8$ is the minimum label.

step3 :- $r = d$, the permanent label of d is 8

$P = \{a, d\}$, $T = \{b, c, e, f, z\}$

$L\{b\} = \min\{\text{old } L(b), L(d) + \omega(d, b)\}$
$= \min\{22, 8 + \infty\} = 22$

$L\{c\} = \min\{16, 8 + 10\} = 16$

$L(e) = \min(\infty, 8 + \infty) = \infty$

$L(f) = \min(\infty, 8 + 6\} = 14$

$L(z) = \min(\infty, 8 + \infty\} = \infty$

$\therefore L\{f\} = 14$ is the minimum label.

step4 $r = f$. the permanent label of f is 14

$P = \{a, d, f\}$, $T = \{b, c, e, z\}$

$L\{b\} = \min\{\text{old } L(b), L(f) + \omega(b, f)\}$

$= \min\{22, 14 + 7\} = 21$

$L\{c\} = \min\{16, 14 + 3\} = 16$

$L\{e\} = \min(\infty, 14 + \infty\} = \infty$

$L\{z\} = \min\{\infty, 14 + 9\} = 23$

$\therefore L\{c\} = 16$ is the minimum label

$\therefore$

Step 5 $v = c$, the permanent label of c is 16

$P = \{a, d, f, c\}$, $T = \{b, e, z\}$

$L\{b\} = \min\{\text{old } L(b), L(f) + \omega(f, b)\}$

$= \min\{21, 16 + 20\} = 21$

$L\{e\} = \min\{\infty, 16 + 6\} = 22$

$L\{z\} = \min\{23, 16 + 10\} = 23$

$\therefore L\{b\} = 21$ is the minimum label.

Step 6 $v = b$, the permanent label of b is 21

$P = \{a, d, f, c, b\}$, $T = \{e, z\}$

$L\{e\} = \min\{\text{old } L(e), L(b) + \omega(e, b)\}$

$= \min\{22, 21, 2\} = 22$

$L\{z) = \min\{23, 21 + 2\} = 23$

$\therefore L\{e\} = 22$ is the minimum label

Step 7 $v = e$, the permanent label of e is 22

$\therefore P = \{a, d, f, c, b, e\}$, $T\{z\}$

$L(z) = \min\{\text{old } L(z), L(e) + \omega(e, z)\}$

$= \min\{23, 22 + 4\} = 23$ which is the minimum label

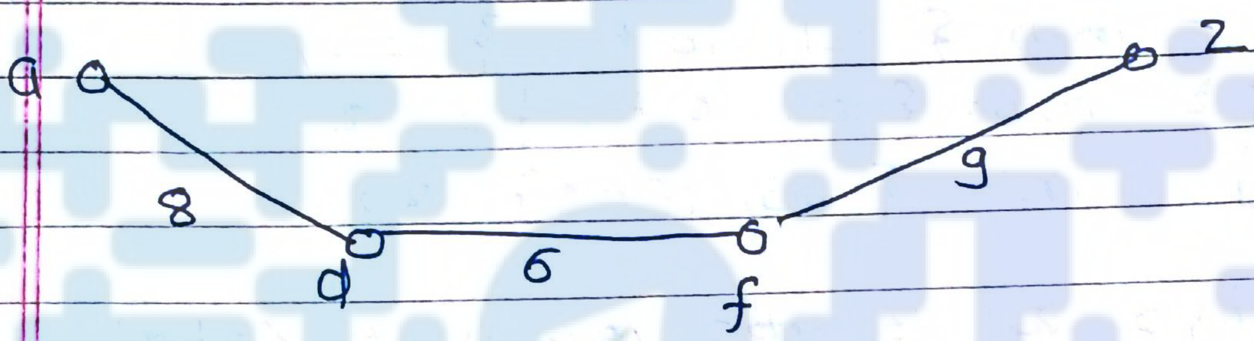
Step 8. $v = z$, the permanent label of z is 23

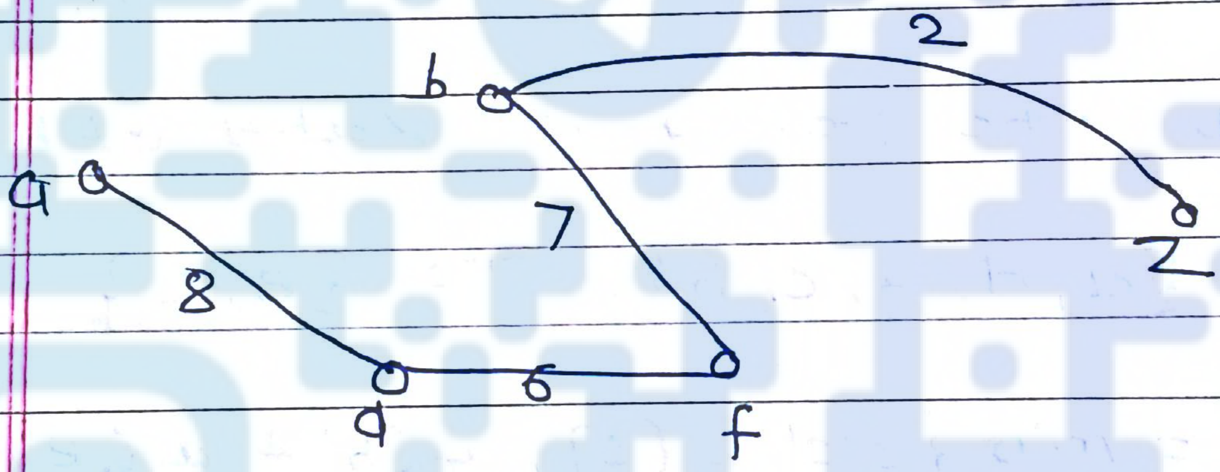
$\therefore$ Hence the length of the shortest path from a to z is 23

$= 7 + 6 - 2 = 11 = e$

∴ The shortest path is

adfz or adfbz



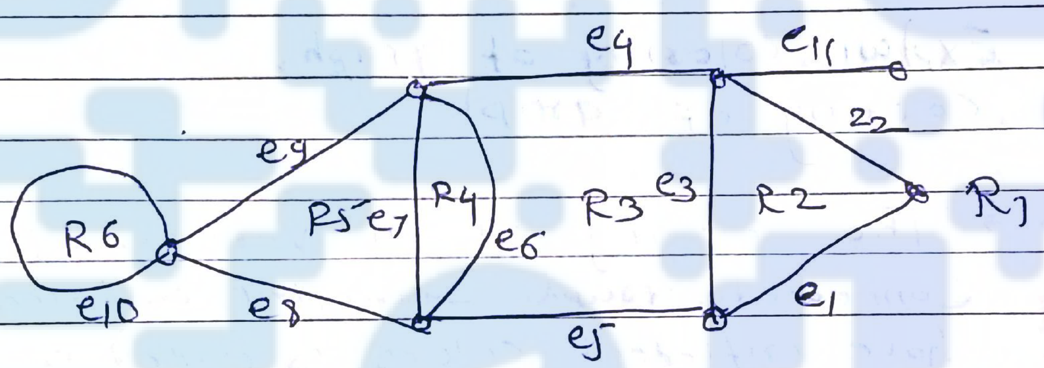


Q. What is regions in graph?

→ Regions :

* A plane representation of a graph divide the plane into parts or regions.
They are also known as ~~regions~~ faces or windows or meshes.

A region or face is characterised by the set edges forming its boundary



G

The graph G has 6 regions, 7 vertices & 11 edges, Region R_1 is an infinite region known as exterior region

$R_2 = \{e_1, e_2, e_3\}$
$R_3 = \{e_3, e_4, e_5, e_6\}$
$R_4 = \{e_6, e_7\}$
$R_5 = \{e_7, e_8, e_9\}$
$R_6 = \{e_{10}\}$

It is observed that

$n = 7, e = 11, r = 6$

$\therefore n + r - 2$

$= 7 + 6 - 2 = 11 = e$

Q. ~~Explain~~ Define Euler's formula?

Euler's Formula

:- For any connected plannet graph G, with v number of vettices, e no of edge & r number of regions

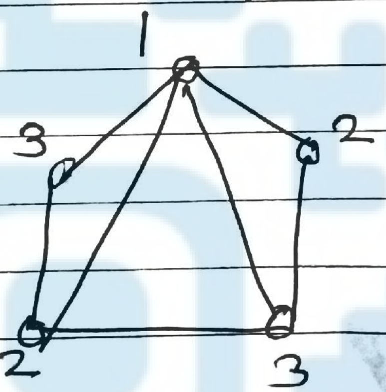
$$v - e + r = 2$$

or $v + r - 2 = e$

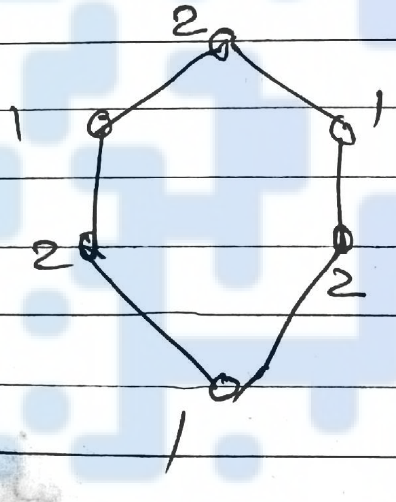
Q. Explain coloring of graph.

* Coloring of graph

→ The coloring of all vettices of a Connected graph such that adjacent vettices have diffeeent colors is called a propee coloring or vettex coloring or simply a coloring of graphs.



proper coloring



Q. Define cromatic number of graph

→ The chromatic number of a graph G is denoted by $\chi(G)$ & defined as the minimum number of colors required to color the vertices of G so that the adjacent vertices get diff colors.