Counting

Unit - 3

Q. Define Combination

→ Combination :→

A combinatination is a selection of some or all objects from a set of given objects where the order of the object does not matter

The number of combinations of n different objects taken r at a time is given by ^nC_r & defined as

$$^nC_r = \frac{n!}{r!\,(n-r)!} \; ; \; r \le n$$

Q. What is permutation?

Permutations

An arrangement in a sequence of elements of a set is called a permutation of elements.

Depending upon the nature of arrangements, there are 3 types of permutation

1] Permitations when all obj are distinct

2] Permitations when all obj are not distinct

$$\frac{n!}{r_1!\, r_2! \ldots r_k!}$$

3] Permutations with repeated obj's

$$n^r$$

i] suppose repetitions are not permittable, how many four digit numbers can be formed from six digits 1, 2, 3, 5, 7, 8?

→ Out of 6 numbers, 4 digit numbers can be formed in $6P4$ ways

$\therefore$ No. of ways $= \frac{6!}{(6-4)!} = \frac{6!}{2!} = 360$

ii] How many of such numbers are less than 4000?

→

1st number chosen from 3 way i.e 1, 2, 3
2nd digit can be any one from remaining 5 digits.

3rd digit can be any of the remaining 4
& 4th digit is any one of the remaining 3 digits.

$\therefore$ Total no. of ways $3 \times 5 \times 4 \times 3 = 180$

iii] How many are even

last digit can be chosen in 2 ways.
2 or 8 $= 2$
The first digit can be chosen in any one of the remaing 5 digits $= 5$
2nd in any of 4 digit $= 4$
3rd in any of the 3 digit $= 3$

$(5 \times 4 \times 3) \times 2$

v] How many contains 3 & 5

The 3 can occupy any 4 positions & remaining 3 positions will be occupied by the digit 5

So 4×3 i.e. 12

remaining two position can be occupied in 4 diff. ways & the remaining position will be occupied in 3 diff. ways.

$\therefore$ $12 \times 4 \times 3$

vi] How many are divisible by 10

→ Not even a single no. is divisible by 10 as there is no zero at unit's place.

Q. binomial theorem

→

The binomial coefficient $\binom{n}{k}$ is the no. of ways of picking k unordered outcomes from n possibilities also known as combination or combinatorial number.

The symbol nC_k and $\binom{n}{k}$ are used to denote a binomial coefficient, and are sometimes read as "n choose k"

$$nC_k = \binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

let x, y be any variables and n be a non-ve integer then

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \ldots \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

$$= \binom{n}{n} x^n y^0 + \binom{n}{n-1} x^{n-1} y + \binom{n}{n-2} x^{n-2} y^2 + \ldots + \binom{n}{1} x y^{n-1} + \binom{n}{0} x^0 y^n$$

$$= \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} \quad (\because nC_r = nC_{n-r})$$

$$(x+y)^1 = x^1 + y^1 = \binom{1}{0}x + \binom{1}{1}y$$

$$(x+y)^2 = x^2 + 2xy + y^2 = \binom{2}{0}x^2 + \binom{2}{1}xy + \binom{2}{2}y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$= \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$$

$$= \binom{3}{3}x^3 + \binom{3}{2}x^2y + \binom{3}{1}xy^2 + \binom{3}{0}y^3$$

4] $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$$= \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 +$$

$$\binom{4}{3}xy^3 + \binom{4}{4}x^0y^4$$

Q. Explain Generalised permutation and combination

→ Generation of permutation

Suppose we want to generate $n!$ permutations of n distinct objects, for $n = 1, 2, 3$. it is simple but when n is large it is difficult to keep track of what we have written & make sure that we shall write down all permutations with no repetition or omissions

- Suppose from the initial permultation 1, 2, 3, ... n by using the next permutation procedure repeatedly we shall obtain all the permutations of 1, 2, 3 ... n, the last permultation is $n, (n-1)(n-2) \ldots \ldots 4, 3, 2, 1$.

procedure for next permutation

Step 1: Given a permutation $a_1, a_2, a_3 \ldots a_n$ of 1, 2, 3, ... n

Step 2: Scan from right to left (L ← R). Find the m such that $a_m < a_{m+1}$

Step 3: $a = \min \{a_k \mid k = m+1, m+2, m+3, \ldots n, a_k > a_m\}$

Step 4: The next permutation is $a_1\, a_2\, a_3 \ldots a_{m-1}, a, x, x, x \ldots$ where $x, x, x \ldots$ are the remaining numbers arranged in the increasing order.

Let $n=6$ and given permutation is 125364
find the next permutation

→ Step 1. Given permutation is 125364

Step 2: Scan from right to left & find the first m such that $a_m < a_{m+1}$

as $3 < 6$ and position of 3 is 4^{th} $\therefore m=4$

Step 3 = $a = \min\{a_k \mid a_k > a_m\} = \min\{6,4\} = 4$

Step 4: Replace element a_m by a i.e 3 by 4, Keep previous elements as it is. i.e. 125 & write all remaining elements in increasing order i.e. 36.

$\therefore$ The next permutation is 125436

* Find the next two permutations of 125436

→ Given that 125436
$m=5$, $a=\min\{6\}=6$

$\therefore$ next permutation is 125463

~~m=4~~

ii) Find the next permutation of 125463

$m=4$, $a=\min\{6\}=6$

$\therefore$ the next permutation is 125634

Q. What is coefficient of $x^{13}y^7$ in the expansion of $(x+y)^{20}$?

→ we know that the coefficient of $x^{n-r}y^r$ is the expantion of $(x+y)^n$

$${}^nC_r = \binom{n}{r}$$

$$\therefore n = 20, \quad r = 7, \quad n-r = 13$$

∴ The required coefficient is

$$\binom{20}{7} = \frac{20!}{7! \times 13!} = 77520$$

* what is the coefficient of $x^{12}y^{13}$ in the expansion of $(x-y)^{25}$

$$\rightarrow -\binom{25}{12} = \frac{-25!}{12!\,13!} = -5,200800$$

* What is the coefficient of x^8y^{12} in the expansion of $(2x-3y)^{20}$?
Hence find the coefficient of $x^{12}y^8$

$$(2x-3y)^{20} = \sum_{r=0}^{20} \binom{20}{r} (2x)^{20-r} (-3y)^r$$

$$\therefore r = 12$$

∴ The coefficient of x^8y^{12} is obtained by putting $r=12$ in R.H.S. of above formula.
Hence the required coefficient $= \binom{20}{12} (2)^8 (-3)^{12}$

$$= 125970\,(2)^8(-3)^{12}$$

what is the coefficient of x^{10} in the exponsion of $\left(x+\frac{1}{x}\right)^{100}$?

$\longrightarrow$

We have

$$\left(x+\frac{1}{x}\right)^{100} = \sum_{r=0}^{100} \binom{100}{r} x^{r} \left(\frac{1}{x}\right)^{100-r}$$

$$= \sum_{r=10}^{100} \binom{100}{r} x^{2r-100}$$

put $2r-100 = 10$

$2r = 110$

$r = 55$

$\therefore$ the coefficient of x^{10} is $\binom{100}{55}$

* In a box, there are 40 floppy disks of which 4 are defective. Determine In how many ways we can select five floppy disks?

→ i] In how many ways we can select five floppy disks?

ANS → there are 40 floppy disks out of which we have to select 5 floppy disks in $^{40}C_5$ ways.

$$\therefore {}^{40}C_5 = \frac{40!}{5! \times (40-5)!} = \frac{40 \times 39 \times 38 \times 37 \times 36}{5 \times 4 \times 3 \times 2 \times 1}$$

$$= 658008.$$

ii] In how many ways we can select five from non-defective floppy disks?

Ans →

There are $40-4=36$ non defective floppy disks out of which we have to select 5.

This can be done in $^{36}C_5$ ways

$$\therefore {}^{36}C_5 = \frac{36!}{5! \times 31!} = \frac{36 \times 35 \times 34 \times 33 \times 32}{5 \times 4 \times 3 \times 2 \times 1} = 376992$$

iii] To select exactly three defective floppy disks out of 4 disks, we have $^4C_3$ ways.

& remaining two floppy disks can be selected from 36 disks in $^{36}C_2$ ways

$\therefore$ the required no. of ways $= {}^4C_3 \times {}^{36}C_2$

$= 2520$

iv] There are 4 defective disks out of which at least one must be selected. We know that the total no. of ways to select 5 disks from 40 disks is $^{40}C_5$

Also the no. of way to select 5 floppy disks with no defective is $^{36}C_5$ ways

$\therefore$ The required no. of ways $= {}^{40}C_5 - {}^{36}C_5 = 281016$

11] Combination with Repetitions :-

$r =$ combinations

$n =$ categories.

then

$C(n+r-1, r)$ $\quad {}^{n+r-1}C_r$

Q. How many 4 combinations of $\{1,2,3,4,5,6\}$ are there with unlimited repetition

→ we have $r = 4$, $n = 6$.

$\therefore$ the no. of 4 combinations of $\{1,2,3,4,5,6\}$ are

$$C(6+4-1, 4) = C(9,4)$$
$$= \frac{9!}{4!(5!)} = 126.$$

* The number of non -ve integer solutions to $x_1 + x_2 + x_3 + x_4 = 20$

→

we have, $r = 20$, $n = 4$

$$= C(4+20-1, 20) = C(23,20) = \frac{23!}{20! \times 3!} = 1771$$

* The number of way of placing 8 similar balls in 5 numbered boxes

→ The no. of way of placing $r = 8$ similar balls in $n = 5$ boxes is

$$C(5+8-1, 8) = C(12,8) = 495$$

In how many ways can 25 late admitted students be assigned to 3 practicals batches if the first batch can accommodate 10 students, the second 8 and third only 7

→ The first batch can be assigned 10 students in $^{25}C_{10}$ ways.
the 2nd batch can be assigned 8 students in $^{15}C_8$ ways
3rd Batch can assigned $^{7}C_7 = 1$ ways

∴ by product rule
$^{25}C_{10} \times {}^{15}C_8$